

Phil/LPS 31 Introduction to Inductive Logic

Lecture 5

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Topics

- ▶ Inference in Quantified Relational Logic
- ▶ Models
- ▶ Witnesses and Counterexamples
- ▶ The Rules of Quantified Relational Logic are truth-preserving
- ▶ Tying things up

Inference in Quantified Relational Logic

- ▶ The goal is to say why this argument in quantified relational logic is valid.

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2. Ruth Barcan Marcus is a logician.

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- ▶ So we need to discuss the mechanics of inference in quantified relational logic.

Inference in Quantified Relation Logic

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1. Hypatia is wise. Wh
- \therefore 2. Someone is wise. $\exists xWx$ (From 1 and By Existential Generalization)

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Rule 4. From $\exists x\mathcal{F}$, you may suppose that $\mathcal{F}(a)$ and use $\mathcal{F}(a)$ together with other premises to infer a **sentence** G using other truth preserving rules of inference. Here $\mathcal{F}(a)$ is the result of uniformly replacing every free occurrence of x in \mathcal{F} with an arbitrary constant a , **which has not occurred before**. This rule is known as **Existential Supposition**.

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- ▶ These are the only truth-preserving rules of inference that we will avail for ourselves in quantified relational logic but other rules of inference can be derived from these.

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- ▶ So combining the first and second arguments we can conclude that something is not wise if and only if not everything is wise, i.e., $(\exists x \neg Wx \leftrightarrow \neg \forall x Wx)$

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 3. Infer $\forall x \mathcal{F}$ using the Useful Equivalence.

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 6. So $\neg\neg\forall x(Tx \rightarrow Sx)$, which is logically equivalent to $\forall x(Tx \rightarrow Sx)$.

Models: Sentential Logic

- ▶ We know why the truth-preserving rules of sentential logic are truth-preserving. But since \exists and \forall are **not truth-functional** we cannot use truth tables to check whether the rules of inference given above are truth-preserving.

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- ▶ This will require a tool which we will call a **model** or **intended interpretation** of some **sentences** in quantified relational logic.
- ▶ I emphasize the word **sentence** because truth and falsity are properties of sentences, not formulas. A sentence in quantified relational logic is a formula with **no free variables** (Definition).

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- ▶ In the model, the truth value of p is 1 if p is true and 0 if p is false. Similarly in the model, the truth value of $\neg p$ is 0 if p is true and 1 if p is false. And so on for the other formulas formed using the rest of the truth functional connectives.
- ▶ Thus, the model for sentential logic can be visualized as a table, which specifies **the truth value conditions** of the sentences in sentential logic. This is the powerful idea behind **Tarski's definition of truth for formalized languages**, which influenced **truth conditional semantics** in philosophy, where we think of **the actual world** as a model for a natural language like English.

Models: Sentential Logic

Formula	Truth value conditions
(F)	1 if F is true; 0 otherwise.
$(\neg F)$	1 if F is false, 0 otherwise.
$(F \wedge G)$	1 if both F and G are true, 0 otherwise.
$(F \vee G)$	0 if both F and G are false, 1 otherwise.
$(F \rightarrow G)$	1 if either F is false or G is true; 0 otherwise.
$(F \leftrightarrow G)$	1 if truth value of $F =$ truth value of G ; 0 otherwise.

Models: Quantified Relational Logic

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- ▶ But the idea is the same as the model for sentential logic. We will specify the truth value conditions for sentences in quantified relational logic. Importantly, the truth conditions **for truth functional connectives** in sentential logic **carry over**.
- ▶ But! The **crucial difference** now is that we add more structure to the model in order to account for (1) terms, (2) predicates and relations, (3) the order in which the terms stand in various relations and (4) quantifiers.

Models and Counterexamples: Quantified Relational Logic

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- ▶ To account for relations we introduce the notion of an **extension**.
- ▶ If P is a predicate symbol in quantified relational logic, the extension of P is a **set** of those terms to which the predicate applies. We write this set as $[P] = \{a \in \mathcal{U} \mid Pa\}$ “The set of those a 's in \mathcal{U} that are P ”

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- ▶ If R is a relation, the extension of R is a set of **ordered pairs** of terms that stand in that relation to each other or themselves. We write this set $[R] = \{ \langle a, b \rangle \text{ for } a, b \in \mathcal{U} \mid Rab \}$ “The set of those a 's and b 's in \mathcal{U} such that Rab ”

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- ▶ To account for $\forall x\mathcal{F}$ we consider **every** x -variant \mathcal{F}' of \mathcal{F} and to account for $\exists x\mathcal{F}$ we consider **some** x -variant \mathcal{F}' of \mathcal{F} (Soon to be made precise).

Models: Quantified Relational Logic

Finally we can specify **the form of an interpretation or model** of quantified relational logic. The form of every model of some sentences in quantified relational logic will have two things: (1) the **interpretation of symbols** and (2) the **truth conditions** of sentences in the model.

Symbols	Interpretation
Constant	Term in the universe of discourse
Predicate, P	$[P] = \{a \in \mathcal{U} \mid Pa\}$
Relation, R	$[R] = \{ \langle a, b \rangle \text{ for } a, b \in \mathcal{U} \mid Rab \}$

Sentence	Truth value conditions in the model
Pa	1 if $a \in [P]$, 0 otherwise.
Rab	1 if $\langle a, b \rangle \in [R]$, 0 otherwise.
$\exists x \mathcal{F}$	1 if for some x-variant \mathcal{F}' of \mathcal{F} , the truth value of \mathcal{F}' is 1.
$\forall x \mathcal{F}$	1 if for every x-variant \mathcal{F}' of \mathcal{F} , the truth value of \mathcal{F}' is 1.

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1. Lab "a is larger than b."
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6. $\forall x Lax$ "a is larger than every x"
 7. $\exists y Wyb$ "Some y is west of b"
 8. $\forall x \exists y (Lxy \wedge Wyx)$ "For every x some y is larger than x and y is west of x."

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 8. $\forall x \exists y (Lxy \wedge Wyx)$ "For every x some y is larger than x and y is west of x."
 9. $\exists x \forall y (Lxy \wedge Wyx)$ "There is an x such that x is larger than every y and y is west of x."

Models: Witnesses and Counterexamples

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- Consider (4) $\exists x Lbx$. We said that (4) is true because for some x -variant of Lbx , $\langle b, c \rangle \in [L]$, i.e., “Los Angeles is larger than Chicago”. The term $c = \text{Chicago}$ here is called a **witness** of the **existentially quantified** sentence $\exists x Lbx$.

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- ▶ Find a witness for (5) $\exists y Wyc$.
 - ▶ Is there a witness for (7) $\exists y W_yb$?

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- ▶ Find a witness for (5) $\exists y W_{yc}$.
 - ▶ Is there a witness for (7) $\exists y W_{yb}$?
 - ▶ Is there a witness for (9) $\exists x \forall y (L_{xy} \wedge W_{yx})$?

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- Consider (5) $\forall x(Lax)$. We said this sentence is false because for some x -variant of Lax , $\langle a, a \rangle \notin [L]$, i.e., “New York is not larger than New York”. The term $a = \text{New York}$ here is called a **counterexample** of the **universally quantified** sentence $\forall xLax$.

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- ▶ Find a counterexample for $\forall x Wxa$
 - ▶ Is there a counterexample for (8) $\forall x \exists y (Lxy \wedge Wyx)$ "For every x there is some y that is larger than x and y is west of x ."?

Models: Witnesses and Counterexamples

- ▶ The case of (8) $\forall x \exists y (Lxy \wedge Wyx)$ and (9) $\exists x \forall y (Lxy \wedge Wyx)$ illustrate something very important. (8) is true, but (9) is false in the model. So, be careful. The **order of quantifiers matters!** (See LPS 30!)

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- ▶ In general a **witness** to an **existentially** quantified sentence is a term in the domain that makes the sentence **true**.
- ▶ In general, for a **non-empty** domain \mathcal{U} , a counterexample to an **universally** quantified sentence is a term in \mathcal{U} that makes the sentence **false**.
- ▶ Note that in the case of counterexamples, the domain has to be non-empty. Think about it. If the domain is empty, then there is no counterexample!

Models: The Rules of Quantified Relational Logic are Truth-preserving.

- ▶ Finally, we can say why the rules of quantified relational logic are truth-preserving. We will not give a formal proof here. But we will reason in terms of **witnesses** and **counterexamples**.

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- ▶ Consider Rule 2. From an instance \mathcal{F} , infer $(\exists x\mathcal{F})$. This rule is known as **Existential Generalization**.

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- ▶ Finally, we can say why the rules of quantified relational logic are truth-preserving. We will not give a formal proof here. But we will reason in terms of **witnesses** and **counterexamples**.
- ▶ Consider Rule 2. From an instance \mathcal{F} , infer $(\exists x\mathcal{F})$. This rule is known as **Existential Generalization**.
- ▶ This rule is truth-preserving because the instance \mathcal{F} automatically acts as a witness to the existentially quantified sentence $(\exists x\mathcal{F})$. So the sentence $(\mathcal{F} \rightarrow \exists x\mathcal{F})$ is a tautology in the model.

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- ▶ Now consider Rule 3. From $\forall x\mathcal{F}$ you may infer **any** of its instances. This rule is known as **Universal Instantiation**.

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- ▶ Now consider Rule 3. From $\forall x\mathcal{F}$ you may infer **any** of its instances. This rule is known as **Universal Instantiation**.
- ▶ This rule is truth-preserving because if \mathcal{U} is not empty and if $\forall x\mathcal{F}$ is true, then there is no counterexample. So the sentence $\forall x\mathcal{F} \rightarrow \mathcal{F}$, where \mathcal{F} is any instance of $(\forall x\mathcal{F})$, is a tautology in the model.

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- ▶ Finally consider Rule 4. From $\exists x\mathcal{F}$, you may suppose that $\mathcal{F}(a)$ and use $\mathcal{F}(a)$ together with other premises to infer a **sentence** G using other truth preserving rules of inference. Here $\mathcal{F}(a)$ is the result of uniformly replacing every free occurrence of x in \mathcal{F} with an arbitrary constant a , **which has not occurred before**.

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- ▶ This rule is truth-preserving for the following reason. Suppose \mathcal{U} is not empty. Either there is a witness for $\exists x\mathcal{F}$ or there is no witness for $\exists x\mathcal{F}$. If there is a witness, then $(\exists x\mathcal{F} \rightarrow G)$ is a tautology in the model because the inference to G is arrived at using other truth preserving rules of inference. If there is no witness, then $(\exists x\mathcal{F} \rightarrow G)$ is a tautology in the model because the antecedent is never satisfied.

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- ▶ This rule is truth-preserving for the following reason. Suppose \mathcal{U} is not empty. Either there is a witness for $\exists x\mathcal{F}$ or there is no witness for $\exists x\mathcal{F}$. If there is a witness, then $(\exists x\mathcal{F} \rightarrow G)$ is a tautology in the model because the inference to G is arrived at using other truth preserving rules of inference. If there is no witness, then $(\exists x\mathcal{F} \rightarrow G)$ is a tautology in the model because the antecedent is never satisfied.
- ▶ In either case the inference from $\exists x\mathcal{F}$ to G is truth preserving. So, $(\exists x\mathcal{F} \rightarrow G)$ is a tautology in the model.

Models: Tying Everything Up.

- ▶ We can now say why the argument we started the discussion of quantified relational logic with is valid.

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 1. All logicians are wise.

Models: Tying Everything Up.

- ▶ We can now say why the argument we started the discussion of quantified relational logic with is valid.
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 - ∴ 3. Wa
- ▶ But it is not immediately obvious why it is valid.

Models: Tying Everything Up

- We can now say that his argument is valid because we can infer (Wa) using truth-preserving rules of inference as follows:

1.	$\forall x(Lx \rightarrow Wx)$	Premise
2.	La	Premise
3.	$(La \rightarrow Wa)$	From 1. and Universal Instantiation
4.	Wa	From 2. and 3. and Modus Ponens.

Models: Tying Everything Up

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- ▶ See **Homework 3** and **Homework 4** for more exercises.

Models: Tying Everything Up

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- ▶ See **Homework 3** and **Homework 4** for more exercises.
- ▶ We have now covered enough of deductive logic to understand its strengths. Next time we will begin discussing its limitations using Hume's Problem of Induction and Goodmans's Riddle of Induction in order to motivate inductive logic.