Phil/LPS 31 Introduction to Inductive Logic
Lecture 4

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Topics

- Quantified Relational Logic: Motivation
- Quantified Relational Logic: Variables and Quantifiers
- Quantified Relational Logic: Predicates and Relations
- Paraphrasing sentences using Quantified Relational Logic

Quantified Relational Logic: Motivation

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- But full grammatical sentences in English also have proper names for individuals, who stand in certain relations to other individuals, themselves, or concrete and abstract objects.

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- But full grammatical sentences in English also have proper names for individuals, who stand in certain relations to other individuals, themselves, or concrete and abstract objects.
- So we need a logic with more expressive power to represent not only the truth-functional structure of sentences in a natural language like English, but also their internal or grammatical structure. This logic is called quantified relational logic or first order logic.

Quantified Relational Logic: Variables and Quantifiers

Consider the following sentences:

(1) Hypatia is wise.

(2) Cavendish is wise.

(3) Hypatia is female.

(4) Cavendish is female.

- Fundamental to quantified predicate logic is the notion of a variable and a quantifier, which we use to generalize.

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- Look at (1) and (2), if we want to generalize from these two sentences, we can drop the proper names ‘Hypatia’ and ‘Cavendish’ and simply say that:
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- If the universe of discourse involves only Hypatia and Cavendish we generalize and say that:

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- If the universe of discourse involves only Hypatia and Cavendish we generalize and say that:
  (7) Everyone is wise.

Quantified Relational Logic: Variables and Quantifiers

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- In quantified relational logic, we paraphrase (6) and (7) as:

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- In quantified relational logic, we paraphrase (6) and (7) as:
  (8) $\exists x$ (x is wise) Read as "There exists an x such that x is wise."

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- In quantified relational logic, we paraphrase (6) and (7) as:
  (8) $\exists x$ (x is wise)   Read as "There exists an x such that x is wise."
  (9) $\forall x$ (x is wise)   Read as "For all x, x is wise." or "Every x is wise."

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  (10) $4^2 - 2^2 = (4-2)(4+2)$

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- In fact, (10) is true not just for some two numbers but any two numbers. So we can generalize even further and say:
  (12) $\forall x \forall y (x^2 - y^2 = (x - y)(x + y))$
- We will not have to represent anything crazy like (12) and (13) for this class, but you will need to have an idea similar to what happens in high school algebra to understand what is going on.

Quantified Relational Logic: Variables and Quantifiers

Let us go back to some of our original sentences:

(1) Hypatia is wise.

(2) Cavendish is wise.

We said we can represent (1) and (2) as either as:

(6) Someone is wise, or

(7) Everyone is wise.

Or symbolically as:

(8) $\exists x$ (x is wise)

(9) $\forall x$ (x is wise)

- The symbols $\exists$ and $\forall$ are called quantifiers. x is a called a variable that is bound by that quantifier.

Quantified Relational Logic: Variables and Quantifiers

Using quantifiers and variables generalize the other two sentences we started with assuming that the universe of discourse includes only Hypatia and Cavendish.

(3) Hypatia is female.

(4) Cavendish is female.

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Quantified Relational Logic: Predicates and Relations

Now consider all four of our original sentences:

(1) Hypatia is wise.

(2) Cavendish is wise.

(3) Hypatia is female.

(4) Cavendish is female.

- Notice that there is variation in proper names ("Hypatia", "Cavendish") and what comes after the proper name ("is wise", "is female"), which are both one place relations.

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- Notice that there is variation in proper names ("Hypatia", "Cavendish") and what comes after the proper name ("is wise", "is female"), which are both one place relations.
- But! what is is common between all four sentences is their logical form. They are all of the logical form:

  $$<\text{subject}> + <\text{one place relation}>$$

Quantified Relational Logic: Predicates and Relations.

(1) Hypatia is wise.

(2) Cavendish is wise.

(3) Hypatia is female.

(4) Cavendish is female.

- We have already seen how to generalize when the subject of a sentence is a proper name. We introduced symbols for quantifiers ($\exists$ and $\forall$) and variables (x, y).

Quantified Relational Logic: Predicates and Relations.

(1) Hypatia is wise.

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- We have already seen how to generalize when the subject of a sentence is a proper name. We introduced symbols for quantifiers ($\exists$ and $\forall$) and variables (x, y).
- How do we generalize sentences which have the same logical form:

$$<\text{subject}> + <\text{one place relation}> \ ?$$

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- We introduce the relational symbol W for the relation "is wise" and paraphrase write (1) and (2) as:

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  (14) Wx Read as "x is W"

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  (15) Wy Read as "y is W"

Quantified Relational Logic: Predicates and Relations.

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- Introduce symbols the relational symbol F for the relation "is female" and paraphrase (3) and (4):

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- So the structure of (1), (2) can be represented most generally using quantified relational logic as either:

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  (16) $\exists x W x$ Read as "There exists an x and x is W", or

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  (16) $\exists x W x$ Read as "There exists an x and x is W", or
  (17) $\forall x W x$ Read as "Every x is a W".

Quantified Relational Logic: Predicates and Relations

Using appropriate symbols for quantifiers, variables and relations, paraphrase or formalize the following sentences in quantified predicate logic:

(3) Hypatia is female.

(4) Cavendish is female.

Quantified Relational Logic: Predicates and Relations

Now consider the following sentences:

(1) Hypatia is wise.

(18) Hypatia possesses wisdom.

(19) Ottoline met Russell.

- While (1) and (18) appear to mean the same thing, they have different logical form. (1) is of the form:

  <subject> + <one place relation>

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- The numbers one and two that tell us how many subjects or objects (“one place” or “two place”) a relation needs in a full grammatical sentence of English are called the arity of the relation.

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There is a special name for relations whose arity is one. They are called predicates. "is wise", "is female" are all predicates.

To paraphrase sentences like:

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we introduce symbols for 2-place relations and use different variables for all the distinct subjects or objects in the relation.

- So (18) can be paraphrased as:

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  Here we chose the variable x to stand for Hypatia and the variable y to stand for wisdom and the relational symbol P to stand for the two place relation $< x$ possesses $y >$.

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- Once we made this choice, the correct paraphrase was Pxy not Pyx! So with relations order matters!

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- Once we have paraphrased (18) as Pxy we can use quantifiers and variables to generalize (18) as:

$$\exists x \exists y Pxy$$

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Using the symbol M for the relation $< x$ met $y >$ paraphrase (19).

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  - Upper case letters $(P, Q, R, S, T, \ldots, A, B, C, D, \ldots)$ of the English alphabet as symbols for relations (one or two place only).

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  - Quantifiers: $\forall$, $\exists$
  - Brackets: (for left bracket and) for right bracket.

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  3. If Rxy is a formula, then $\exists x Rxy$ is a formula and $\forall x Rxy$ is a formula. Here y is a free variable but x is bound.

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  3. If Rxy is a formula, then $\exists x Rxy$ is a formula and $\forall x Rxy$ is a formula. Here y is a free variable but x is bound.
  4. If $\mathcal{F}$ is formula by rule 3, then the result of binding any free variable in $\mathcal{F}$ is a formula.

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  4. If $\mathcal{F}$ is formula by rule 3, then the result of binding any free variable in $\mathcal{F}$ is a formula.
  5. A formula $\mathcal{F}$ formed by either rule 1, 2, 3 or 4 is called an atomic formula. If $\mathcal{F}$ and $\mathcal{G}$ are atomic formulas, then $(\mathcal{F} \vee \mathcal{G})$, $(\mathcal{F} \wedge \mathcal{G})$, $\neg\mathcal{F}$, $(\mathcal{F} \rightarrow \mathcal{G})$ are molecular formulas.

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  5. A formula $\mathcal{F}$ formed by either rule 1, 2, 3 or 4 is called an atomic formula. If $\mathcal{F}$ and $\mathcal{G}$ are atomic formulas, then $(\mathcal{F} \vee \mathcal{G})$, $(\mathcal{F} \wedge \mathcal{G})$, $\neg\mathcal{F}$, $(\mathcal{F} \rightarrow \mathcal{G})$ are molecular formulas.
  6. If $\mathcal{H}$ is a molecular formula, then the result of binding any free variable in $\mathcal{H}$ is a molecular formula.

Quantified Relational Logic: The Logic Itself

- The Closure Condition says that a formula of quantified relational logic is either an atomic formula or a molecular formula built from atomic formulas by finite applications of rule 5 and 6.