Phil/LPS 31 Introduction to Inductive Logic
Lecture 5

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Topics

- Inference in Quantified Relational Logic
- Models
- Witnesses and Counterexamples
- The Rules of Quantified Relational Logic are truth-preserving
- Tying things up

Inference in Quantified Relational Logic

- The goal is to say why this argument in quantified relational logic is valid.

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2. Ruth Barcan Marcus is a logician.

∴ 3. Ruth Barcan Marcus is wise.

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- So we need to discuss the mechanics of inference in quantified relational logic.

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$\therefore$2. Someone is wise. $\exists x Wx$ (From 1 and By Existential Generalization)

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  $\therefore$2. If Ruth Barcan Marcus is a logician, then she is wise. $(La \rightarrow Wa)$ (From 1 and By Universal Instantiation)

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Rule 4. From $\exists x \mathcal{F}$, you may suppose that $\mathcal{F}(a)$ and use $\mathcal{F}(a)$ together with other premises to infer a sentence G using other truth preserving rules of inference. Here $\mathcal{F}(a)$ is the result of uniformly replacing every free occurrence of x in $\mathcal{F}$ with an arbitrary constant a, which has not occurred before. This rule is known as Existential Supposition.

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- These are the only truth-preserving rules of inference that we will avail for ourselves in quantified relational logic but other rules of inference can be derived from these.

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- So combining the first and second arguments we can conclude that something is not wise if and only if not everything is wise, i.e, $(\exists x \neg Wx \leftrightarrow \neg \forall x Wx)$

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  3. Infer $\forall x\mathcal{F}$ using the Useful Equivalence.

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  6. So $\neg\neg\forall x(Tx \rightarrow Sx)$, which is logically equivalent to $\forall x(Tx \rightarrow Sx)$.

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- This will require a tool which we will call a model or intended interpretation of some sentences in quantified relational logic.
- I emphasize the word sentence because truth and falsity are properties of sentences, not formulas. A sentence in quantified relational logic is a formula with no free variables (Definition).

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- Thus, the model for sentential logic can be visualized as a table, which specifies the truth value conditions of the sentences in sentential logic. This is the powerful idea behind Tarski's definition of truth for formalized languages, which influenced truth conditional semantics in philosophy, where we think of the actual world as a model for a natural language like English.

Models: Sentential Logic

Formula	Truth value conditions
(F)	1 if F is true; 0 otherwise.
$(\neg F)$	1 if F is false, 0 otherwise.
$(F \wedge G)$	1 if both F and G are true, 0 otherwise.
$(F \vee G)$	0 if both F and G are false, 1 otherwise.
$(F \rightarrow G)$	1 if either F is false or G is true; 0 otherwise.
$(F \leftrightarrow G)$	1 if truth value of F $=$ truth value of G; 0 otherwise.

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- But the idea is the same as the model for sentential logic. We will specify the truth value conditions for sentences in quantified relational logic. Importantly, the truth conditions for truth functional connectives in sentential logic carry over.

Models: Quantified Relational Logic

- Since quantified relational logic formalizes natural language to reveal more grammatical or internal structure, namely, proper names for individuals or terms and relations between these terms (in the right order); a model for quantified relational logic will necessarily be more complex.
- But the idea is the same as the model for sentential logic. We will specify the truth value conditions for sentences in quantified relational logic. Importantly, the truth conditions for truth functional connectives in sentential logic carry over.
- But! The crucial difference now is that we add more structure to the model in order to account for (1) terms, (2) predicates and relations, (3) the order in which the terms stand in various relations and (4) quantifiers.

Models and Counterexamples: Quantified Relational Logic

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- If P is a predicate symbol in quantified relational logic, the extension of P is a set of those terms to which the predicate applies. We write this set as $[P] = \{a \in \mathcal{U} \mid Pa\}$ "The set of those a's in $\mathcal{U}$ that are P"

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- If R is a relation, the extension of R is a set of ordered pairs of terms that stand in that relation to each other or themselves. We write this set $[R] = \{< a, b > \text{ for } a, b \in \mathcal{U} \mid Rab\}$ "The set of those a's and b's in $\mathcal{U}$ such that Rab"

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- To account for $\forall x\mathcal{F}$ we consider every x-variant $\mathcal{F}'$ of $\mathcal{F}$ and to account for $\exists x\mathcal{F}$ we consider some x-variant $\mathcal{F}'$ of $\mathcal{F}$ (Soon to be made precise).

Models: Quantified Relational Logic

Finally we can specify the form of an interpretation or model of quantified relational logic. The form of every model of some sentences in quantified relational logic will have two things: (1) the interpretation of symbols and (2) the truth conditions of sentences in the model.

Symbols	Interpretation
Constant	Term in the universe of discourse
Predicate, P	$[P] = \{a \in \mathcal{U} \mid Pa\}$
Relation, R	$[R] = \{< a, b > \text{ for } a, b \in \mathcal{U} \mid Rab\}$

Sentence	Truth value conditions in the model
Pa	1 if $a \in [P]$, 0 otherwise.
Rab	1 if $< a, b > \in [R]$, 0 otherwise.
$\exists x \mathcal{F}$	1 if for some x-variant $\mathcal{F}'$ of $\mathcal{F}$, the truth value of $\mathcal{F}'$ is 1.
$\forall x \mathcal{F}$	1 if for every x-variant $\mathcal{F}'$ of $\mathcal{F}$, the truth value of $\mathcal{F}'$ is 1.

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  4. $\exists xLbx$ "There exists an *x* such that *b* is larger than *x*."

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- Determine whether the following sentences are true in the model:
  1. Lab "a is larger than b."
  2. Lbc "b is larger than c."
  3. Wca "c is west of a."
  4. $\exists xLbx$ "There exists an x such that b is larger than x."
  5. $\exists yWyc$ "Some y is west of c."

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   6. $\forall x Lax$ "a is larger than every x"
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   8. $\forall x \exists y (Lxy \wedge Wyx)$ "For every x some y is larger than x and y is west of x."

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    8. $\forall x \exists y (Lxy \wedge Wyx)$ "For every x some y is larger than x and y is west of x."
    9. $\exists x \forall y (Lxy \wedge Wyx)$ "There is an x such that x is larger than every y and y is west of x."

Models: Witnesses and Counterexamples

The model is:

1. $\mathcal{U}$ = { New York, Los Angeles, Chicago }
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- Consider (4) $\exists xLbx$. We said that (4) is true because for some x-variant of Lbx, $< b, c > \in [L]$, i.e., "Los Angeles is larger than Chicago". The term c = Chicago here is called a witness of the existentially quantified sentence $\exists xLbx$.

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- Is there a witness for (7) $\exists y Wyb$?

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- Find a witness for (5) $\exists y Wyc$.
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- Is there a witness for (9) $\exists x \forall y (Lxy \wedge Wyx)$?

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- Consider (5) $\forall x(Lax)$. We said this sentence is false because for some x-variant of Lax, $< a, a > \notin [L]$, i.e., "New York is not larger than New York". The term a = New York here is called a counterexample of the universally quantified sentence $\forall xLax$.

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- Find a counterexample for $\forall x Wxa$
- Is there a counterexample for (8) $\forall x \exists y (Lxy \land Wyx)$ "For every x there is some y that is larger than x and y is west of x."?

Models: Witnesses and Counterexamples

- The case of (9) $\exists x \forall y(Lxy \wedge Wyx)$ and (10) $\exists x \exists y(Lxy \wedge Wyx)$ illustrate something important. (10) is true because we just need to find at least one witness; $(Lab \wedge Wba)$ is a witness. But (9) is false because we have to verify that there are no counterexamples. So we had to check every y-variant of $(Lay \wedge Wya)$ including $(Laa \wedge Waa)$, which is a counterexample.

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- In general a witness to an existentially quantified sentence is a term in the domain that makes the sentence true.
- In general, for a non-empty domain $\mathcal{U}$, a counterexample to an universally quantified sentence is a term in $\mathcal{U}$ that makes the sentence false.
- Note that in the case of counterexamples, the domain has to be non-empty. Think about it. If the domain is empty, then there is nothing to check. So there is trivially no counterexample!

Models: The Rules of Quantified Relational Logic are Truth-preserving.

- Finally, we can say why the rules of quantified relational logic are truth-preserving. We will not give a formal proof here. But we will reason in terms of witnesses and counterexamples.

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- Finally, we can say why the rules of quantified relational logic are truth-preserving. We will not give a formal proof here. But we will reason in terms of witnesses and counterexamples.
- Consider Rule 2. From an instance $\mathcal{F}$, infer $(\exists x \mathcal{F})$. This rule is known as Existential Generalization.
- This rule is truth-preserving because the instance $\mathcal{F}$ automatically acts as a witness to the existentially quantified sentence $(\exists x \mathcal{F})$. So the sentence $(\mathcal{F} \rightarrow \exists x \mathcal{F})$ is a tautology in the model.

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- Now consider Rule 3. From $\forall x \mathcal{F}$ you may infer any of its instances. This rule is known an Universal Instantiation.

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- This rule is truth-preserving because if $\mathcal{U}$ is not empty and if $\forall x \mathcal{F}$ is true, then there is no counterexample. So the sentence $\forall x \mathcal{F} \rightarrow \mathcal{F}$, where $\mathcal{F}$ is any instance of $(\forall x \mathcal{F})$, is a tautology in the model.

Models: The Rules of Quantified Relational Logic are Truth-preserving.

- Finally consider Rule 4. From $\exists x \mathcal{F}$, you may suppose that $\mathcal{F}(a)$ and use $\mathcal{F}(a)$ together with other premises to infer a sentence G using other truth preserving rules of inference. Here $\mathcal{F}(a)$ is the result of uniformly replacing every free occurrence of x in $\mathcal{F}$ with an arbitrary constant a, which has not occurred before.

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- This rule is truth-preserving for the following reason. Suppose $\mathcal{U}$ is not empty. Either there is a witness for $\exists x \mathcal{F}$ or there is no witness for $\exists x \mathcal{F}$. If there is a witness, then $(\exists x \mathcal{F} \rightarrow G)$ is a tautology in the model because the inference to G is arrived at using other truth preserving rules of inference. If there is no witness, then $(\exists x \mathcal{F} \rightarrow G)$ is a tautology in the model because the antecedent is never satisfied.

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- This rule is truth-preserving for the following reason. Suppose $\mathcal{U}$ is not empty. Either there is a witness for $\exists x \mathcal{F}$ or there is no witness for $\exists x \mathcal{F}$. If there is a witness, then $(\exists x \mathcal{F} \rightarrow G)$ is a tautology in the model because the inference to G is arrived at using other truth preserving rules of inference. If there is no witness, then $(\exists x \mathcal{F} \rightarrow G)$ is a tautology in the model because the antecedent is never satisfied.
- In either case the inference from $\exists x \mathcal{F}$ to G is truth preserving. So, $(\exists x \mathcal{F} \rightarrow G)$ is a tautology in the model.

Models: Tying Everything Up.

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  $\therefore$ 3. Ruth Barcan Marcus is wise.
- This argument has the following form.
  1. $\forall x(Lx \to Wx)$
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  $\therefore$ 3. Wa
- But it is not immediately obvious why it is valid.

Models: Tying Everything Up

- We can now say that his argument is valid because we can infer (Wa) using truth-preserving rules of inference as follows:

1.	$\forall x(Lx \rightarrow Wx)$	Premise
2.	La	Premise
3.	$(La \rightarrow Wa)$	From 1. and Universal Instantiation
4.	Wa	From 2. and 3. and Modus Ponens.

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3.	$(La \rightarrow Wa)$	From 1. and Universal Instantiation
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- See Homework 4 for more exercises.

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2.	La	Premise
3.	$(La \rightarrow Wa)$	From 1. and Universal Instantiation
4.	Wa	From 2. and 3. and Modus Ponens.

- See Homework 4 for more exercises.
- We have now covered enough of deductive logic to understand its strengths. Next time we will begin discussing its limitations using Hume's Problem of Induction and Goodmans's Riddle of Induction in order to motivate inductive logic.