Phil/LPS 31 Introduction to Inductive Logic
Lecture 10

David Mwakima
dmwakima@uci.edu
University of California, Irvine

May 5th 2023

Topics

- The logic of sets
- Fields
- Kolmogorov Axioms

The logic of sets

- To study the logic of sets, we will use a useful tool that I will call a "presence table", following an idea due to Professor Brian Skyrms in LPS. A presence table will be like a truth table, except we will use it to study the logic of sets.

The logic of sets

- To study the logic of sets, we will use a useful tool that I will call a "presence table", following an idea due to Professor Brian Skyrms in LPS. A presence table will be like a truth table, except we will use it to study the logic of sets.
- The logic of sets will be like the logic of sentential logic. Recall that in sentential logic, we study the logic of combining simple sentences to form complex sentences using truth-functional connectives.

The logic of sets

- To study the logic of sets, we will use a useful tool that I will call a "presence table", following an idea due to Professor Brian Skyrms in LPS. A presence table will be like a truth table, except we will use it to study the logic of sets.
- The logic of sets will be like the logic of sentential logic. Recall that in sentential logic, we study the logic of combining simple sentences to form complex sentences using truth-functional connectives.
- Now we are going to use set operations (union $\cup$, intersection $\cap$, complement, $\cdot^c$, powerset $\mathcal{P}$) to combine simple sets to form more complex sets using these operations.

The logic of sets

- To study the logic of sets, we will use a useful tool that I will call a "presence table", following an idea due to Professor Brian Skyrms in LPS. A presence table will be like a truth table, except we will use it to study the logic of sets.
- The logic of sets will be like the logic of sentential logic. Recall that in sentential logic, we study the logic of combining simple sentences to form complex sentences using truth-functional connectives.
- Now we are going to use set operations (union $\cup$, intersection $\cap$, complement, $\cdot^c$, powerset $\mathcal{P}$) to combine simple sets to form more complex sets using these operations.
- Then we are going to use the subset relation, $\subset$, and the identity relation, $=$, to study the relationships between sets.

The logic of sets

- Here's the first example of a presence table. Recall $A \subset B$ if whenever $x \in A$, then $x \in B$. Think of the conditional $\rightarrow$.

The logic of sets

- Here's the first example of a presence table. Recall $A \subset B$ if whenever $x \in A$, then $x \in B$. Think of the conditional $\rightarrow$.
- Here is the presence table for $A \subset B$:

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

The logic of sets

- Here's the presence table for $A \subset B$ for reference.

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

The logic of sets

- Here's the presence table for $A \subset B$ for reference.

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

- In the first column, P means some x is present in A and A (for "absent") means x is not in A. 1 under the column for $A \subset B$ means the sentence "A is a subset of B" is true. 0 under the column for $A \subset B$ means that on that row the sentence "A is a subset of B" is false. So we see that $A \subset B$ is false if and only if $x \in A$, i.e., present in A; but $x \notin B$, i.e., absent in B.

The logic of sets

- Here's the presence table for $A \subset B$ for reference.

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

- In the first column, P means some x is present in A and A (for "absent") means x is not in A. 1 under the column for $A \subset B$ means the sentence "A is a subset of B" is true. 0 under the column for $A \subset B$ means that on that row the sentence "A is a subset of B" is false. So we see that $A \subset B$ is false if and only if $x \in A$, i.e., present in A; but $x \notin B$, i.e., absent in B.

- Exercise. Construct the presence table for:

The logic of sets

- Here's the presence table for $A \subset B$ for reference.

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

- In the first column, P means some x is present in A and A (for "absent") means x is not in A. 1 under the column for $A \subset B$ means the sentence "A is a subset of B" is true. 0 under the column for $A \subset B$ means that on that row the sentence "A is a subset of B" is false. So we see that $A \subset B$ is false if and only if $x \in A$, i.e., present in A; but $x \notin B$, i.e., absent in B.

- Exercise. Construct the presence table for:
  - $(A \cup B)$

The logic of sets

- Here's the presence table for $A \subset B$ for reference.

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

- In the first column, P means some x is present in A and A (for "absent") means x is not in A. 1 under the column for $A \subset B$ means the sentence "A is a subset of B" is true. 0 under the column for $A \subset B$ means that on that row the sentence "A is a subset of B" is false. So we see that $A \subset B$ is false if and only if $x \in A$, i.e., present in A; but $x \notin B$, i.e., absent in B.
- Exercise. Construct the presence table for:
  - $(A \cup B)$
  - $(A \cap B)$

The logic of sets

- Here's the presence table for $A \subset B$ for reference.

A	B	$A \subset B$
P	P	1
P	A	0
A	P	1
A	A	1

- In the first column, P means some x is present in A and A (for "absent") means x is not in A. 1 under the column for $A \subset B$ means the sentence "A is a subset of B" is true. 0 under the column for $A \subset B$ means that on that row the sentence "A is a subset of B" is false. So we see that $A \subset B$ is false if and only if $x \in A$, i.e., present in A; but $x \notin B$, i.e., absent in B.

- Exercise. Construct the presence table for:
  - $(A \cup B)$
  - $(A \cap B)$
  - $(A^c \cap B)$, where $B \subset \mathcal{U}$ and $\mathcal{U}$ is some universe of discourse or universal set in a given context.

The logic of sets

- Once we know the presence tables for these basic set operations, we can study the relationships between more complex combinations of sets and certain equivalences between sets.

The logic of sets

- Once we know the presence tables for these basic set operations, we can study the relationships between more complex combinations of sets and certain equivalences between sets.
- For example, we can use presence tables to verify that $(A \cap B) \subset A$.

The logic of sets

- Once we know the presence tables for these basic set operations, we can study the relationships between more complex combinations of sets and certain equivalences between sets.
- For example, we can use presence tables to verify that $(A \cap B) \subset A$.
- In addition, we can use presence tables verify that $(A = B)$. Recall that to do this, we need to check that: (1) $A \subset B$ and (2) $B \subset A$.

The logic of sets

- Once we know the presence tables for these basic set operations, we can study the relationships between more complex combinations of sets and certain equivalences between sets.
- For example, we can use presence tables to verify that $(A \cap B) \subset A$.
- In addition, we can use presence tables verify that $(A = B)$. Recall that to do this, we need to check that: (1) $A \subset B$ and (2) $B \subset A$.
- If we use a presence table to verify $A = B$, we need to check whether the sentence "$(((A \subset B) \wedge (B \subset A)) \rightarrow A = B)$" is a tautology.

The logic of sets

- Once we know the presence tables for these basic set operations, we can study the relationships between more complex combinations of sets and certain equivalences between sets.
- For example, we can use presence tables to verify that $(A \cap B) \subset A$.
- In addition, we can use presence tables verify that $(A = B)$. Recall that to do this, we need to check that: (1) $A \subset B$ and (2) $B \subset A$.
- If we use a presence table to verify $A = B$, we need to check whether the sentence "$(((A \subset B) \wedge (B \subset A)) \rightarrow A = B)$" is a tautology.
- In fact, we can show that $A = B \equiv ((A \subset B) \wedge (B \subset A))$ by showing that $(((A \subset B) \wedge (B \subset A)) \leftrightarrow A = B)$ is a tautology. The symbol $\equiv$ has been introduced in Homework 4.

The logic of sets

- Verify the following relationships between sets.

The logic of sets

- Verify the following relationships between sets.
  - $(B^c \cap A) \subset A$

The logic of sets

- Verify the following relationships between sets.
  - $(B^c \cap A) \subset A$
  - $A \subset (A \cup B)$

The logic of sets

- Verify the following relationships between sets.
  - $(B^c \cap A) \subset A$
  - $A \subset (A \cup B)$
  - $(A \cup B) = (B \cup A)$

The logic of sets

- Verify the following relationships between sets.
  - $(B^c \cap A) \subset A$
  - $A \subset (A \cup B)$
  - $(A \cup B) = (B \cup A)$
  - $A \cup (B \cup C) = (A \cup B) \cup C$

The logic of sets

- Verify the following relationships between sets.
  - $(B^c \cap A) \subset A$
  - $A \subset (A \cup B)$
  - $(A \cup B) = (B \cup A)$
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

The logic of sets

- Verify the following relationships between sets.
  - $(B^c \cap A) \subset A$
  - $A \subset (A \cup B)$
  - $(A \cup B) = (B \cup A)$
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- See Homework 5 for more exercises.

Fields

- We are now ready to describe the logic of sets that we can use to define probability functions.

Fields

- We are now ready to describe the logic of sets that we can use to define probability functions.
- The step we are taking here is analogous to the step we took in defining the formal system of sentential logic.

Fields

- We are now ready to describe the logic of sets that we can use to define probability functions.
- The step we are taking here is analogous to the step we took in defining the formal system of sentential logic.
- We will call the formal system of the logic of sets that we will use to define probability functions a field.

Fields

- We are now ready to describe the logic of sets that we can use to define probability functions.
- The step we are taking here is analogous to the step we took in defining the formal system of sentential logic.
- We will call the formal system of the logic of sets that we will use to define probability functions a field.
- We define a field $\mathcal{F}$ as a non-empty collection of sets Ω that is closed under finite applications of set-theoretic operations of taking unions and complements.

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:
  (1) $\Omega \in \mathcal{F}$

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:
  (1) $\Omega \in \mathcal{F}$
  (2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:
  (1) $\Omega \in \mathcal{F}$
  (2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
  (3) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:
  (1) $\Omega \in \mathcal{F}$
  (2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
  (3) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$
- This is a good time to compare what a field is with the definition of the formal system of sentential logic. Do that!

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:
  (1) $\Omega \in \mathcal{F}$
  (2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
  (3) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$
- This is a good time to compare what a field is with the definition of the formal system of sentential logic. Do that!
- (1), (2) and (3) characterize what a field is completely. Other authors add that a field is closed under finite applications of the operation of taking intersections.

Properties of Fields

- To say that a field $\mathcal{F}$ is closed under finite applications of set theoretic operations means:
  (1) $\Omega \in \mathcal{F}$
  (2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
  (3) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$

- This is a good time to compare what a field is with the definition of the formal system of sentential logic. Do that!

- (1), (2) and (3) characterize what a field is completely. Other authors add that a field is closed under finite applications of the operation of taking intersections.

- In your Homework 5 you will show that taking the complement of a union of two sets gives you the intersection of their complements. So we can define intersections in terms of complements and unions.

Kolmogorov Axioms

- The Kolmogorov Axioms characterize what a probability function P defined on a field is as follows:

Kolmogorov Axioms

- The Kolmogorov Axioms characterize what a probability function P defined on a field is as follows:
  - $P(\Omega) = 1$ (Normalized)

Kolmogorov Axioms

- The Kolmogorov Axioms characterize what a probability function P defined on a field is as follows:
  - $P(\Omega) = 1$ (Normalized)
  - $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$ (Non-negative)

Kolmogorov Axioms

- The Kolmogorov Axioms characterize what a probability function P defined on a field is as follows:
  - $P(\Omega) = 1$ (Normalized)
  - $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$ (Non-negative)
  - If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$ (Additive)

Kolmogorov Axioms

- The Kolmogorov Axioms characterize what a probability function P defined on a field is as follows:
  - $P(\Omega) = 1$ (Normalized)
  - $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$ (Non-negative)
  - If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$ (Additive)
- A probability function is a normalized, non-negative and additive real-valued set function defined on a field.

Kolmogorov Axioms

- The Kolmogorov Axioms characterize what a probability function P defined on a field is as follows:
  - $P(\Omega) = 1$ (Normalized)
  - $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$ (Non-negative)
  - If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$ (Additive)
- A probability function is a normalized, non-negative and additive real-valued set function defined on a field.
- We will pick up from here next time!