Phil/LPS 31 Introduction to Inductive Logic
Lecture 10

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Topics

- The logic of sets
- Fields
- Kolmogorov Axioms

The logic of sets

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- Then we are going to use the subset relation, $\subset$, and the identity relation, $=$, to study the relationships between sets.

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- Exercise. Construct the presence table for:
  - $(A \cup B)$
  - $(A \cap B)$
  - $(A^c \cap B)$, where $B \subset \mathcal{U}$ and $\mathcal{U}$ is some universe of discourse or universal set in a given context.

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- If we use a presence table to verify $A = B$, we need to check whether the sentence "$(((A \subset B) \wedge (B \subset A)) \rightarrow A = B)$" is a tautology.
- In fact, we can show that $A = B \equiv ((A \subset B) \wedge (B \subset A))$ by showing that $(((A \subset B) \wedge (B \subset A)) \leftrightarrow A = B)$ is a tautology. The symbol $\equiv$ has been introduced in Homework 4.

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- See Homework 5 for more exercises.

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- We define a field $\mathcal{F}$ as a non-empty collection of sets Ω that is closed under finite applications of set-theoretic operations.
- Compare with the definition of the formal system of sentential logic.

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- (1), (2) and (3) characterize what a field is completely. In your Homework 5 you will verify additional properties of fields based on just these three properties.

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- We will pick up from here next time!