1

Bayesian Robustness

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ABSTRACT An overview of the robust Bayesian approach is presented, primarily focusing on developments in the last decade. Examples are presented to motivate the need for a robust approach. Common types of robustness analyses are illustrated, including global and local sensitivity analysis and loss and likelihood robustness. Relationships with other approaches are also discussed. Finally, possible directions for future research are outlined.

Key words: prior/likelihood/loss robustness, algorithms, applications.

1.1 Introduction

Robust Bayesian analysis is concerned with the sensitivity of the results of a Bayesian analysis to the inputs for the analysis. The early 90s was the golden age of robust Bayesian analysis, in that many statisticians were highly active in research in the area, and rapid progress was being achieved. It was a major topic in many meetings, even some non-Bayesian meetings, and there were three meetings explicitly dedicated to the topic: the Workshop on Bayesian Robustness, held at Purdue University in 1989; and two International Workshops on Bayesian Robustness, held in Italy (Milano, 1992, and Rimini, 1995). The proceedings of the latter two conferences were published, respectively, in the *Journal of Statistical Planning and Inference* (vol. 40, no. 2 & 3), and in Berger et al. (1996), as an IMS Lecture Notes volume.

An issue of the journal *Test* (vol. 3, 1994) was mainly devoted to a review paper by Berger on robust Bayesian analysis, followed by discussions from many robust Bayesians. Several of these discussions were themselves extensive reviews of particular aspects of Bayesian robustness, so that the volume serves as a highly effective summary of the state of Bayesian robustness in 1994. Earlier reviews of Bayesian robustness were given in Berger (1984, 1985, 1990); these include considerable philosophical discussion of the approach, together with discussion of its history, including its origins in papers by Good in the 50s and, later, by Kadane and Chuang (1978). Another good source for information on Bayesian robustness is Walley (1991), although the book takes a somewhat different slant on the problem.

In the early 90s, there was an explosion of publications focusing on studying sensitivity to the prior distribution, in part because non-Bayesians often view this sensitivity to be the major drawback of Bayesian analysis. This work focused on replacing a single prior distribution by a class of priors and developing methods of computing the range of the ensuing answers as the prior varied over the class. This approach, called "global robustness", was soon supplemented by "local robustness" techniques, which focused on studying local sensitivity (in the sense of derivatives) to prior distributions. Interest naturally expanded into study of robustness with respect to the likelihood and loss function as well, with the aim of having a general approach to sensitivity towards all the ingredients of the Bayesian paradigm (model/prior/loss). Practical implementations of robust Bayesian ideas began to appear, including Godsill and Rayner (1996), Greenhouse and Wasserman (1996), Sargent and Carlin (1996) and Ríos Insua et al. (1999).

In the last half of the 90s, robust Bayesian analysis shifted from being a hot topic to being a mature field within Bayesian analysis, with continued gradual development, but with less of a sense of urgency. This change came about for several reasons. First, the initial flurry of fundamental theoretical advances naturally slowed, as will happen with any new field. Second, few Bayesians continued to question the need to view robustness or sensitivity as a serious issue, so that the philosophical excitement with the idea waned. Indeed, the consensus was that the time was now ripe to develop user-friendly implementations of the existing robust Bayesian methodology. The timing was not great, however, in that, coincidentally, this period also marked the explosion of interest in MCMC computational methods for Bayesian statistics. This had two serious effects on the field of Bayesian robustness. First, many of the researchers in robust Bayesian methods shifted their research into the MCMC arena. Second, the MCMC methodology was not directly compatible with many of the robust Bayesian techniques that had been developed, so that it was unclear how formal robust Bayesian analysis could be incorporated into the future "Bayesian via MCMC" world.

Paradoxically, MCMC has dramatically increased the need for consideration of Bayesian robustness, in that the modeling that is now routinely utilized in Bayesian analysis is of such complexity that inputs (such as priors) can be elicited only in a very casual fashion. It is now time to focus again on Bayesian robustness and to attempt to bring its ideas into the Bayesian mainstream.

This book aims at collecting contributions on the different aspects of Bayesian robustness, allowing statisticians and practitioners to access the wealth of research and the applied possibilities offered by the approach. New opportunities are offered by developments in algorithms (see Betrò and Guglielmi, 2000, Lavine et al., 2000, and MacEachern and Müller, 2000), the possibility of using MCMC methods and the need for sensitivity analysis in other fields, such as Bayesian model selection and Bayesian nonparametrics (see the papers by Liseo, 2000, and Basu, 2000). Bayesian robustness is playing a relevant role in SAMO (Sensitivity Analysis of Model Output), a group interested in investigating the relative importance of model input parameters on model predictions (see Saltelli et al., 2000) in many applied areas, from chemical engineering to econometrics.

In Section 2 of this paper, we present some motivating examples that indicate the importance of consideration of Bayesian robustness. Different approaches to Bayesian robustness are illustrated in Section 3. Description of "typical" robust analyses and ways to achieve robustness are presented, respectively, in Sections 4 and 5. Section 6 discusses connections between Bayesian robustness and other approaches, such as the frequentist approach of Huber (1973) and the approach based on lower probabilities of Walley (1991). Finally, directions for future research are discussed in Section 7.

1.2 Motivating examples

This section gives several examples and ideas to indicate the importance and uses of robustness in a Bayesian setting. The first example, concerned with p-values, demonstrates the use of robust Bayesian methods in identifying inadequacies in common statistical procedures. The next example considers the issue of imprecision in beliefs and preference assessment, which, in turn, points to foundational motivations for robust Bayesian analysis. We then give some examples that indicate potential dangers in approaching the problem from an overly simplified perspective.

1.2.1 Interpreting p-values

Robust Bayesian methods can be used to show that the common interpretations of certain classical procedures are not correct. Perhaps the most important example concerns p-values in testing a point-null hypothesis. Most statistical users believe that when the p-value is 0.05, the point null is very likely to be wrong, and when it is 0.01, it is almost certain to be wrong. However, robust Bayesian computations in Berger and Sellke (1987) show that these interpretations are not valid.

As an illustration, consider a long series of standard normal tests of, say, $H_0\colon \theta_i = 0$ versus $H_1\colon \theta_i \neq 0$, $i = 1, 2, \ldots$. (Similar results hold if the null hypotheses are small intervals, as discussed in Berger and Delampady, 1987.) Suppose 50% of the null hypotheses in this series are true. For each test, data is collected and the p-value computed, resulting in a corresponding series of p-values. Now consider the subset of these for which, say, $0.04 < p < 0.05$ (or $0.009 < p < 0.01$).

Fact: Among all the tests for which $0.04 < p < 0.05$ ($0.009 < p < 0.01$), at least 24% (7%) of the null hypotheses will be true.

The results are qualitatively similar for any initially specified proportion of true nulls; among those tests for which $0.04 < p < 0.05$ or $0.009 < p < 0.01$, a surprisingly large fraction (compared with the initial fraction) of the null hypotheses will be true.

This result is established by a robust Bayesian argument in which the parameters θ_i in the alternatives are allowed to assume any possible values (or, equivalently, any prior distribution for them is allowed). Then, conditional on the p-value being near 0.05, 24% is the *lower bound* on the proportion of true nulls (or posterior probability of the null), the lower bound being over all possible choices of the alternative hypotheses. Thus, in practice, one can expect the actual proportion of true nulls to be much higher than 24% when the p-value is 0.05 (and if 50% of the nulls were originally true). Further discussion of this phenomenon, from both parametric and nonparametric robust Bayesian perspectives, can be found in Sellke et al. (1999), which also gives a simple simulation program for explicitly exhibiting the phenomenon.

1.2.2 Elicitation in action

As suggested in many of the papers in this volume, the usual practical motivation underlying robust Bayesian analysis is the difficulty in assessing the prior distribution. Consider the simplest case in which it is desired to elicit a prior over a finite set of states $\theta_i, i = 1, \ldots, I$. A common technique to assess a precise $\pi(\theta_i) = p_i$, with the aid of a reference experiment, proceeds as follows: one progressively bounds $\pi(\theta_i)$ above and below until no further discrimination is possible and then takes the midpoint of the resulting interval as the value of p_i. Instead, however, one could directly operate with the obtained constraints $\alpha_i \leq \pi(\theta_i) \leq \beta_i$, acknowledging the cognitive limitations. Constraints might also come in different forms, such as knowledge that $\pi(\theta_i) \geq \pi(\theta_j)$.

The situation is similar in the continuous case; a common elicitation technique is to discretize the range Θ of the random variable and to assess the probabilities of the resulting sets. Again, however, one actually ends up only with some constraints on these probabilities. This is often phrased in terms of having bounds on the quantiles of the distribution. A related idea is having bounds on the distribution function corresponding to the random variable. Once again, qualitative constraints, such as unimodality, might also be added. Perhaps most common for a continuous random variable is to assess a parameterized prior, say a conjugate prior, and place constraints on the parameters of the prior; while this can be a valuable analysis, it does not serve to indicate robustness with respect to the specified form of the prior.

A similar situation holds in modeling preferences. One might, say, assess the loss of some consequences through the certainty equivalent method, and then fit a loss function. However, in reality, one only ends up with upper and lower constraints on such losses, possibly with qualitative features such as monotonicity and convexity, if preferences are increasing and risk averse. These constraints can often be approximated by an upper and a lower loss function, leading to the consideration of all loss functions that lie between these bounds (and that possibly satisfy the other qualitative constraints). If a parameterized loss function is assessed, say an exponential loss, the constraints are typically placed on the parameters of the loss, say the risk aversion coefficient.

In developing the model for the data itself there is typically great uncertainty, and a need for careful study of robustness or sensitivity. As this is appreciated by virtually all statisticians, we need not consider specific examples. It should be noted, however, that uncertainty in the model is typically considerably more important than uncertainty in the prior or loss.

A final comment concerning the limits of elicitation concerns the situation in which there are several decision makers (DMs) and/or experts involved in the elicitation. Then it is not even necessarily possible theoretically to obtain a single model, prior, or loss; one might be left with only classes of each, corresponding to differing expert opinions.

1.2.3 Foundations for robust analysis

One of the main arguments for the Bayesian approach is that virtually any axiomatic system leads to the subjective expected loss model, under which we are essentially led to the conclusions that: i) beliefs should be modeled with a probability distribution which, in the presence of additional information is updated through Bayes formula; ii) preferences and risk attitudes over consequences should be modeled with a loss function; and iii) preferences over alternatives should be modeled according to the minimum (subjective) expected loss principle. This framework provides the foundational underpinning for standard Bayesian inference and decision analysis. It is of considerable interest to understand the changes in this framework, and its conclusions, that result from the type of robust Bayesian considerations mentioned above.

We have argued that, in practical settings, it is not possible to precisely assess the prior, the model and the loss function. This contradicts the assumption, in the common axiom systems, of completeness in preferences and beliefs. Interestingly, however, relaxation of this assumption does not dramatically alter the conclusions of the foundations. Indeed, as outlined in Ríos Insua and Criado (2000) in this volume, one is essentially led to the same conclusion: imprecise beliefs and preferences can be modeled by a class of priors and a class of loss functions, so that preferences among alternatives can be represented by inequalities on the corresponding posterior expected losses. The web page `http://ippserv.rug.ac.be` contains considerable additional information on this subject.

1.2.4 *Some views on simplified robust analyses*

Before fully exploring key concepts in robust Bayesian analysis, it is useful to discuss what could be termed simplified robust analyses. The first, and most common, of these simplified analyses is informal sensitivity study, which consists of merely trying a few different inputs and seeing if the output changes significantly. A typical argument for this informal approach would be that Bayesian analysis in complex problems typically entails messy computations, and one cannot afford the additional computational burden that would be imposed by a formal robustness analysis. While informal analysis is certainly an important tool (and far better than ignoring robustness), a simple example shows its limitations and the need for undertaking more formal analyses.

Consider the no-data problem given by the decision table:

	θ_1	θ_2
a	c_1	c_2
b	c_2	c_3

Let $\pi(\theta_1) = p_1 = 1 - \pi(\theta_2)$. Assessment of this probability and the losses involved in the decisions results in the intervals $0.4 \le p_1 \le 0.6$, $-0.5 \le L(c_1) \le 0$, $-0.25 \le L(c_2) \le -.75$, and $-0.5 \le L(c_3) \le -1$. Consider the following four loss probability pairs associated with the bounds on losses and probabilities:

p_1	loss (l)	Exp. loss a	Exp. loss b
0.4	$L(c_1) = 0.,\ L(c_2) = -0.25,\ L(c_3) = -0.50$	-0.15	-0.40
0.4	$L(c_1) = -0.5,\ L(c_2) = -0.75,\ L(c_3) = -1$	-0.65	-0.90
0.6	$L(c_1) = 0.,\ L(c_2) = -0.25,\ L(c_3) = -0.50$	-0.10	-0.35
0.6	$L(c_1) = -0.5,\ L(c_2) = -0.75,\ L(c_3) = -1$	-0.60	-0.85

In all four cases, the expected loss of a is bigger than that of b, so this "simple" robustness analysis might lead one to conclude that alternative b is preferred to alternative a. Note, however, that this is not necessarily true: for instance, if $p_1 = 0.6$, $L(c_1) = -0.5$, $L(c_2) = -0.25$, and $L(c_3) = -0.5$, then the expected losses of a and b are, respectively, $-.4$ and $-.35$, so that a would be preferred.

Another type of simplified robustness analysis that is often encountered is the study of sensitivity of the outcome to one input of the problem, with the other inputs remaining fixed. It can happen, however, that a problem is insensitive to changes in only one input at a time, while being sensitive to simultaneous changes in inputs. This can also be illustrated using the above example.

Fix $p_1 = 0.5$ (the center of the interval) in the example above. Note that the difference in the expected loss of alternatives a and b is

$$.5L(c_1) + .5L(c_2) - .5L(c_2) - .5L(c_3) = .5(L(c_1) - L(c_3)).$$

Since $L(c_1) \geq L(c_3)$ for all values of L, it can be concluded that the problem is robust to changes only in L. Analogously, if we consider $L(c_1) = -.25$, $L(c_2) = -.5$ and $L(c_3) = -.75$ (the centers of the various intervals for the $L(c_i)$), the difference in the expected loss of a and b is

$$-p_1.25 - (1 - p_1).5 + p_1.5 + (1 - p_1).75 = .25.$$

Thus there is robustness against only changes in probabilities. However, as seen above, the problem is not robust with respect to joint changes in probabilities and losses.

1.3 Main issues in Bayesian robustness

In this section, we briefly outline the central issues in Bayesian robustness. Further discussion can be found in Berger (1984, 1985, 1990 and 1994), Wasserman (1992) and the papers in this volume.

1.3.1 The goal

As discussed earlier, robustness with respect to the prior stems from the practical impossibility of eliciting a unique distribution. Similar concerns apply to the other elements (likelihood and loss function) considered in a Bayesian analysis. The main goal of Bayesian robustness is to quantify and interpret the uncertainty induced by partial knowledge of one (or more) of the three elements in the analysis. Also, ways to reduce the uncertainty are studied and applied until, hopefully, robustness is achieved.

As an example, suppose the quantity of interest is the posterior mean; the likelihood is considered to be known, but the prior distribution is known only to lie in a given class. The uncertainty might be quantified by specifying the range spanned by the posterior mean, as the prior varies over the class. One must appropriately interpret this measure; for instance, one might say that the range is "small" if it spans less than 1/4 of a posterior standard deviation (suitably chosen from the range of possible posterior standard deviations). If the range is small, robustness is asserted, and the analysis would be deemed to be satisfactory. If, however, the range is not small, then some way must be found to reduce the uncertainty; narrowing the class of priors or obtaining additional data would be ideal.

1.3.2 Different approaches

We discuss the different possible approaches to Bayesian robustness for the simple situation in which the posterior mean is the quantity of interest and the uncertainty is in the prior distribution. Virtually any quantity of interest and any type of input uncertainty could have been used instead.

There are three main approaches to Bayesian robustness. The first is the *informal approach* already mentioned, in which a few priors are considered and the corresponding posterior means are compared. The approach is appealing because of its simplicity and can help, but it can easily "miss" priors that are compatible with the actually elicited prior knowledge and yet which would yield very different posterior means.

The second approach is called *global robustness* (see Moreno, 2000, for a thorough illustration). One (ideally) considers the class of all priors compatible with the elicited prior information and computes the range of the posterior mean as the prior varies over the class. This range is typically found by determining the "extremal" priors in the class that yield the maximum and minimum posterior means. Such computations can become cumbersome in multidimensional problems.

The third approach, called *local robustness*, is described by Gustafson (2000) and Sivaganesan (2000). It is interested in the rate of change in inferences, with respect to changes in the prior, and uses differential techniques to evaluate the rate. Local sensitivity measures are typically easier to compute in complicated situations than are global measures, but their interpretation (and calibration) is not always clear.

1.3.3 Uncertainty modeling

Classes of priors/likelihoods/loss functions are specified according to the available (partial) knowledge (e.g., quantiles or unimodality of the prior or convexity of the loss function). They can be classified into neighborhood or near-ignorance classes, using the classification proposed by Pericchi and Walley (1991). In the former case, one imagines that a "standard" Bayesian elicitation has yielded a specific prior (likelihood, loss), and considers neighborhoods of this prior (likelihood, loss) to study robustness. Various examples of such neighborhoods will be given later (and in other papers in this volume). Note, also, that the local approach to robustness implicitly operates on small perturbations within neighborhoods.

The near-ignorance class contains no baseline element and, instead, is based on some (few) features, such as quantiles of the prior distribution or values of the loss function at some points; see Martin et al. (1998). The names in this classification should not be taken at face value, since it is possible to have a near-ignorance class that is much smaller than a neighborhood class.

Desirable, but competing, properties of a class of priors include the following: computation of robustness measures should be as easy as possible; all reasonable priors should be in the class and unreasonable ones (e.g., discrete distributions in many problems) should not; and the class should be easily specified from elicited prior information. Similar properties can be defined for classes of losses and models.

1.3.4 Robustness measures

The most commonly used measure in global robustness is the range, that is, the difference between upper and lower bounds on the quantity of interest. Its value measures the variation caused by the uncertainty in the prior/likelihood/loss function. Its value may be deemed "satisfactorily small" or not, according to context-dependent criteria. It should always be recognized that the range depends strongly on the size of the classes used to represent the uncertain inputs. Thus a small range with an excessively small class is not really comforting, and a large range with an excessively large class is not necessarily damning.

When the range is too large, one ideally narrows the class further (through additional elicitation) or obtains additional data, hopefully resulting in a satisfactory range. If neither is possible, one must resort to an ad-hoc approach and should remain somewhat suspicious of the answer. A natural ad-hoc approach is to replace the class by a single input, obtained by some type of averaging over the class. For instance, if the class is a class of priors, one might choose a hyperprior on the class (perhaps using some default or noninformative-prior Bayesian method) and perform an ordinary Bayesian analysis. Numerous other ad-hoc criteria can be considered, however. An example is the Γ–minimax approach, described in Vidakovic (2000) for the problem of estimation. This typically conservative approach consists of choosing estimators minimizing the supremum, over the class, of either the posterior expected loss or the posterior regret.

While interpretation of the size of the range is usually done within the specific applied context, certain generic measures can be usefully introduced. For instance, Ruggeri and Sivaganesan (2000) consider a scaled version of the range, for a target quantity of interest $h(\theta)$, called *relative sensitivity* and defined as

$$R_\pi = \frac{(\rho_\Pi - \rho_0)^2}{V^\Pi},$$

where ρ_Π and ρ_0 equal $\mathbb{E}(h(\theta)|x)$ under Π and the baseline prior Π_0, respectively, and V^Π is the posterior variance of $h(\theta)$ with respect to Π. The idea is that, in estimation of $h(\theta)$, the posterior variance indicates the accuracy that is attainable from the experiment, so that one will often not worry if the (squared) range is small with respect to this posterior variance.

In the local sensitivity approach to Bayesian robustness, techniques from differential calculus are used and robustness is measured by the supremum of (functional) derivatives over a class.

1.3.5 Algorithms

Computation of robustness measures over a class is usually performed by considering the subset of extremal elements of the class and performing the computations over this subset. For example, in most robustness problems, the extremal elements of the class of all distributions are the Dirac (point-mass) measures, whereas the extremal elements of the class of all symmetric unimodal densities are the uniform densities. One of the first references to employ such variational mathematics in Bayesian robustness was Sivaganesan and Berger (1989).

Among the very large number of techniques that have been developed for robustness computations, the *linearization algorithm* deserves special mention because of its utility and generality of application. Introduced into the field by Lavine (1991), it allows determination of the range of a posterior expectation through a sequence of linear optimizations. A recent algorithmic development for generalized moment constraint classes can be found in Betrò and Guglielmi (2000).

1.4 A guided tour through Bayesian robustness

We illustrate typical steps in a robust Bayesian analysis in a "textbook" case, using one of the reference examples in the literature, due to Berger and Berliner (1986). We assume familiarity with Bayesian analysis; otherwise, we refer the reader to general textbooks, such as Berger (1985) or French and Ríos Insua (2000).

We analyze sensitivity to the prior in the example, following each of the three approaches presented above. (Later examples will illustrate likelihood and loss robustness.) The quantities of interest will be taken to be posterior means and posterior probabilities of credible sets.

Suppose a single observation, x, is observed from a normal distribution with unknown mean θ and variance 1, to be denoted $\mathcal{N}(\theta, 1)$. The classic (old) textbook Bayesian analysis would suggest selection of a conjugate, normal prior for θ, chosen to match some features determined by the elicitation process. Suppose, for instance, that $-.954$, 0, and $.954$ are the elicited quartiles of the distribution of θ. The unique, normal prior with these quartiles is $\Pi \sim \mathcal{N}(0, 2)$. The Bayes estimate of θ under squared error loss would be the posterior mean, given by $\delta_{\mathcal{N}}(x) = 2x/3$.

It is evident that the choice of a normal prior is mathematically convenient but otherwise arbitrary. It is thus natural to consider classes of priors that are compatible with the elicited quartiles, but are otherwise more general.

1.4.1 *Informal approach*

In the informal approach to robustness, one would try a few different priors, compatible with the elicited quartiles, and compare the answers. Three such priors are the normal $\mathcal{N}(0,2)$, double exponential $\mathcal{DE}(0, \log 2/.954)$ and Cauchy distribution $\mathcal{C}(0, .954)$. Resulting posterior means, for different data x, are presented in Table 1.

x	.5	1	1.5	2	2.5	3	3.5	4	10
$\mathcal{N}$	0.333	0.667	1.000	1.333	1.667	2.000	2.333	2.667	6.667
$\mathcal{DE}$	0.292	0.606	0.960	1.362	1.808	2.285	2.776	3.274	9.274
$\mathcal{C}$	0.259	0.540	0.866	1.259	1.729	2.267	2.844	3.427	9.796

TABLE 1. Posterior means for different priors

For moderate or large values of x in Table 1, the posterior mean is reasonably consistent for the Cauchy and double exponential distributions but changes dramatically for the normal distribution. For small values of x, the posterior means are all reasonably consistent. The conclusion of this informal analysis would be that robustness likely obtains for smaller values of x, but not for larger values.

1.4.2 *Global robustness*

We will consider different neighborhood classes of priors, centered at the normal prior, $\mathcal{N}(0,2)$.

1.4.2.1 ε-contamination neighborhoods

When a baseline prior Π_0 (often conjugate) is elicited by usual methods, a natural class of priors for studying sensitivity to the prior is the *ε-contamination* class

$$\Gamma_\varepsilon = \{\Pi : \Pi = (1-\varepsilon)\Pi_0 + \varepsilon Q,\ Q \in \mathcal{Q}\},$$

where $\mathcal{Q}$ is called the class of contaminations. (This class was first utilized in classical robustness studies by Huber, 1973.) The popularity of the class arises, in part, from the ease of its specification and, in part, from the fact that it is typically easily handled mathematically.

A variety of choices of $\mathcal{Q}$ have been considered in the literature. The most obvious choice is to let $\mathcal{Q}$ be the class of all distributions, as was considered in Berger and Berliner (1986) for the above example, with $\varepsilon = 0.2$. (The resulting class for the example is denoted Γ_A below.) Unfortunately, this class is often too large to yield useful robustness bounds.

One important class of refinements of the contaminating class involves the addition of shape constraints, such as unimodality and symmetry. Such constraints can often be readily elicited and can very significantly reduce the range of posterior quantities of interest.

Another class of refinements that is often considered is the addition of quantile constraints, since specification of probabilities of sets is typically easier than other elicitations. In the above example, for instance, Moreno and Cano (1991) considered the class Γ_A, but with, respectively, the median fixed to be that of Π_0 (class Γ_2) and the quartiles fixed at those of Π_0 (class Γ_4). An alternative quantile specification was considered in Betrò et al. (1994): because it is often easier to elicit features of the marginal distribution of X (i.e., features of the distribution of observables) than features of the prior, it is natural to consider Γ_{2m} and Γ_{4m}, the classes of priors in Γ_A which yield marginals for X that have, respectively, the same median and the same quartiles as the marginal distribution of X under Π_0.

Computation of the range of posterior functionals for these ε–contamination classes has been studied by many authors, and prototype computer codes are available (see Betrò and Guglielmi, 2000, for a review). The upper and lower bounds are actually achieved at discrete distributions, which greatly simplifies the analysis.

As in Betrò et al. (1994), we consider the robustness of the standard 95% credible interval, $C(x)$, computed assuming the conjugate normal prior Π_0 were true. For Γ_A and the various quantile classes given above, Table 2 gives the range of the posterior probability of $C(x)$ as the priors vary over the classes.

x	min	max	min	max	min	max	min	max	min	max
	$\Pi \in \Gamma_A$		$\Pi \in \Gamma_2$		$\Pi \in \Gamma_4$		$\Pi \in \Gamma_{2m}$		$\Pi \in \Gamma_{4m}$	
0.5	.818	.966	.842	.965	.906	.963	.844	.964	.906	.960
1.0	.773	.967	.823	.965	.876	.962	.834	.962	.907	.959
1.5	.707	.969	.787	.965	.861	.963	.810	.961	.892	.956
2.0	.615	.973	.728	.966	.827	.963	.746	.965	.856	.959
2.5	.496	.978	.640	.970	.755	.964	.652	.969	.795	.961
3.0	.363	.983	.522	.975	.666	.968	.525	.975	.735	.962
3.5	.235	.988	.377	.980	.534	.973	.377	.981	.624	.962
4.0	.135	.993	.237	.988	.377	.980	.237	.988	.469	.975

TABLE 2. Upper and lower bounds on the posterior probability of the 95% credible interval from the $\mathcal{N}(0,2)$ prior

As could be expected, the least robust situation is when $x = 4$. For the original Γ_A, the posterior probability of $C(4)$ varies over an enormous range. Utilizing the refined classes is of some help in reducing the range but, even for Γ_{4m}, the range would typically be viewed as unacceptably large. One could add additional constraints on the class, such as unimodality (as in Moreno and González, 1990), but even this would not help here; the basic problem is that the credible set from the conjugate prior is simply not a robust credible set when rather extreme data is observed. (For results concerning optimally robust credible sets, see Sivaganesan et al., 1993.)

A "nice" case in Table 2 is when $x = 1$ and Γ_{4m} is used; one might well be satisfied with the range $(.907, .959)$. We focused here on the range of the posterior probability of a credible interval, but we could have considered the posterior mean and variance, as in Sivaganesan and Berger (1989) and Sivaganesan (1988, 1989), where the prior Π_0 was contaminated by many classes of distributions: arbitrary, symmetric unimodal, unimodal,... Related results on hypothesis testing can be found in Moreno (2000). Finally, as indicated above in the comment about optimally robust credible sets, one can also search for the most robust posterior estimators among those commonly used, such as the mean, median and mode. See Sivaganesan (1991) for comparisons under the contaminating class of all bounded unimodal distributions having the same mode and median as Π_0.

1.4.2.2 Other neighborhood classes

One drawback of ε–contamination classes is that they are not true neighborhood classes in a topological sense. A variety of other classes have been considered that do have a formal interpretation in this sense, such as the class based on a concentration function and the class based on distribution bands. The former class consists of those distributions for which with the probability of each set is bounded by a given function of the set's probability under a baseline distribution. The example of Berger and Berliner (1986) has been investigated for concentration function neighborhoods; see Fortini and Ruggeri (2000) and references therein.

The distribution band class, described in Basu and DasGupta (1995) and Basu (1995), is defined às

$$\Gamma_{BDG} = \{F : F \text{ is a cdf and } F_L(\theta) \leq F(\theta) \leq F_U(\theta), \forall\theta\},$$

where F_L and F_U are given cdfs, with $F_L(\theta) \leq F_U(\theta)$. The class includes, as special cases, well-known metric neighborhoods of a fixed cdf, such as Kolmogorov and Lèvy neighborhoods. This class has the additional desirable property of being comparatively easy to elicit. For instance, in the normal example above one might specify the upper and lower cdfs to be those corresponding to the specified normal and Cauchy priors, if it were felt that the main uncertainty was in the tail of the prior. Additional "shape" requirements, such as symmetry and unimodality of the distributions, can also be added, as in Basu (1994).

1.4.2.3 Quantile classes

Often, there will be no baseline prior and the class of priors will simply be those that satisfy the features that are elicited. A particularly tractable class of features are generalized moment conditions (see Betrò and Guglielmi, 2000). The most common example of a generalized moment condition is specification of a quantile.

An example of a quantile class is that considered in Berger and O'Hagan (1988), O'Hagan and Berger (1988) and Moreno and Cano (1989). The elicitation proceeds by determining the prior probabilities of each of the intervals I_i, $i = 1, 6$, in Table 3. The probabilities indicated therein are the elicited values, and the quantile class, Γ_Q, consists of all distributions that are compatible with these assessments. Note that the $\mathcal{N}(0, 2)$ distribution is one such prior.

I_i	$(-\infty, -2]$	$(-2, -1]$	$(-1, 0]$	(0,1]	(1,2]	$(2, \infty)$
p_i	.08	.16	.26	.26	.16	.08

TABLE 3. Intervals and prior probabilities

Computation of ranges of a posterior quantity of interest is rather simple for a quantile class, since upper and lower bounds are achieved for discrete distributions giving mass to one point in each interval I_i (see, e.g., Ruggeri, 1990). But discrete distributions are typically quite unnatural, so that more realistic classes are often obtained by considering Γ_U, the unimodal priors in Γ_Q, or Γ_{BU}, the priors in Γ_U with known mode a_k and such that the density at a_k does not exceed a given threshold. Computations for Γ_U and Γ_{BU} are, however, quite involved. As a compromise, O'Hagan and Berger (1988) proposed the class Γ_{QU} of quasiunimodal priors, with $\Gamma_{BU} \subset \Gamma_U \subset \Gamma_{QU} \subset \Gamma_Q$, for which computations are relatively simple.

As an illustration of the effect of the various restrictions on the quantile class, Table 4, taken from Berger and O'Hagan (1988), considers the range of the posterior probability of the intervals I_i for the various classes, in a specific situation. Also reported is the posterior probability, p_i^*, for each I_i, under the single prior $\mathcal{N}(0, 2)$.

The main interest here is that imposing any of the shape constraints results in a considerable reduction in the ranges of the posterior probabilities, compared with the ranges for the raw quantile class, Γ_Q.

1.4.3 Local sensitivity

Local sensitivity analysis considers the behavior of posterior quantities of interest under infinitesimal perturbations from a specified baseline prior Π_0. Different approaches to local sensitivity are discussed in, e.g., Gustafson (2000) and Sivaganesan (2000), along with a discussion of their properties.

I_i	p_i^*	Γ_Q	Γ_{QU}	Γ_U	Π_{BU}
$(-\infty,-2]$	.0001	(0,.001)	(0,.0002)	(0,.0002)	(0,.0002)
$(-2,-1]$	.0070	(.001,.029)	(.006,.011)	(.006,.011)	(.006,.010)
$(-1,0]$	.1031	(.024,.272)	(.095,.166)	(.095,.166)	(.095,.155)
$(0, 1]$	.3900	(.208,.600)	(.320,.447)	(.322,.447)	(.332,.447)
$(1, 2]$	.3900	(.265,.625)	(.355,.475)	(.357,.473)	(.360,.467)
$(2,\infty)$	.1102	(0,.229)	(0,.156)	(0,.156)	(0,.154)

TABLE 4. Posterior ranges for I_i

As an illustration of the approach, consider the following example, studied in Basu (1996), which is similar to that analyzed in Section 1.4.2. Consider an $\mathcal{N}(\theta, 1)$ model, for the data x, and a baseline prior Π_0, chosen to be $\mathcal{N}(0, 2.19)$; this prior has median zero and quartiles at -1 and 1. Consider "small" perturbations

$$\Pi_Q = (1-\varepsilon)\Pi_0 + \varepsilon Q = \Pi_0 + \varepsilon(Q - \Pi_0)$$

of Π_0, in the direction $Q-\Pi_0$, with Q in a class Γ of priors. As a sensitivity measure, consider

$$\overline{G}_{\rho_0} = \sup_{Q\in\Gamma} G_{\rho_0}(Q - \Pi_0),$$

where

$$G_{\rho_0}(Q - \Pi_0) = \frac{d}{d\varepsilon}\rho((1-\varepsilon)\Pi_0 + \varepsilon Q)_{\varepsilon=0} \qquad (1)$$

is the Gâteaux derivative of the functional $\rho(\cdot)$ at Π_0 in the direction $Q-\Pi_0$.

We consider three classes (out of the five considered by Basu): Γ_A, Γ_U and Γ_{SU}, corresponding to arbitrary, unimodal (around 0) and symmetric, unimodal (around 0) distributions. We present Basu's results for the functionals ρ_1 (posterior mean), ρ_2 (posterior median) and ρ_4 (posterior variance), omitting the posterior mode, ρ_3, whose $\overline{G}_{\rho_0}$ is always ∞ for the three classes we consider when $x \neq 0$. This subset of results, from Basu (1996), is presented in Table 5.

As pointed out in Basu (1996), for the samples $x = 0$ and $x = 1$, the centers of Π_0 and the likelihood are nearly matching and the posterior mean is less sensitive than the posterior median to infinitesimal perturbations. The opposite happens for larger values of x. The table gives clear indications of the negative and positive aspects of local sensitivity. Observe that $\overline{G}_{\rho_0}$ is not scale invariant, so that interpreting its magnitude can be tricky; as an example, 0.5 could be large or small depending on the problem and the order of magnitude of the functionals. Scale-invariant measures could be

	$x=0$			$x=1$			$x=2$		
	ρ_1	ρ_2	ρ_4	ρ_1	ρ_2	ρ_4	ρ_1	ρ_2	ρ_4
Γ_A	1.08	1.86	1.23	1.69	2.17	1.79	3.47	3.47	4.33
Γ_U	0.81	1.86	1.23	0.78	1.73	1.06	1.25	1.05	1.08
Γ_{SU}	0.00	0.00	0.21	0.22	0.20	0.27	0.58	0.50	0.59
	$x=3$			$x=4$			$x=5$		
	ρ_1	ρ_2	ρ_4	ρ_1	ρ_2	ρ_4	ρ_1	ρ_2	ρ_4
Γ_A	9.38	7.59	13.83	33.72	22.65	57.84	161.88	92.41	318.92
Γ_U	3.17	2.48	3.66	10.64	7.45	14.93	46.92	29.05	78.55
Γ_{SU}	1.59	1.24	1.84	5.32	3.73	7.47	23.46	14.52	39.28

TABLE 5. $\overline{G}_{\rho_0}$ for the posterior mean (ρ_1), median (ρ_2) and variance (ρ_4)

obtained, for example by adapting the scaled range in Ruggeri and Sivaganesan (2000), but calibration remains a troublesome issue, as discussed in Sivaganesan (2000). Nonetheless, we can use $\overline{G}_{\rho_0}$ either to compare "similar" functionals, such as the mean and the median, in the same problem (same sample and class of priors), or to quantify the relative decrease in sensitivity when narrowing the class of priors and considering the same functional.

1.4.4 Interactive robustness

An effective robust Bayesian analysis should be interactive, asking for new information from different sources until robustness is achieved, for instance, the range of a posterior functional is below a fixed threshold. We have presented, in the previous sections, different ways to reduce the width of the class of priors, such as introduce new quantiles, unimodality and symmetry. Since the acquisition of new information has a cost, optimal ways of obtaining new information should be pursued.

Suppose, for instance, that one is developing a quantile class of priors. As suggested in Berger (1994), one useful way to proceed, when robustness has not been obtained for the current class, is to determine the "optimal" additional quantile to elicit, in the sense of optimally reducing the range of the posterior quantity of interest. Different implementations of this idea have been proposed in Liseo et al. (1996), Moreno et al. (1996) and Ríos Insua et al. (1999).

Moreno et al. (1996) consider, for each $j = 1, \dots, n$, the class

$$\Gamma_j = \{\pi : \pi(\theta) = \pi_0(\theta) + \varepsilon_j \mathbf{1}(\theta)(A_j)(q(\theta) - \pi_0(\theta)), q \in \mathcal{Q}\}.$$

This class differs from the original class in that the contamination is allowed only on one set A_j at a time. Computing the ranges of the posterior functional over each class Γ_j, Moreno et al. select the interval leading to the largest range as that in which the new quantile is to be determined and they choose, as the splitting point of the interval, the minimizer of the range over an associated class.

Liseo et al. (1996) considered the quantity

$$\delta(A_j,\Gamma) = \sup_{\Pi\in\Gamma} \int_{A_j} h(\theta)l(\theta)\Pi(d\theta) - \inf_{\Pi\in\Gamma} \int_{A_j} h(\theta)l(\theta)\Pi(d\theta),$$

where h is the function of interest, l is the likelihood and $A_j, j = 1,\dots,n$ and Γ are, respectively, the intervals and the class of priors determined by the quantiles. They computed $\delta(A_j,\Gamma)$ for all $j = 1,\dots,n$, and split the interval A_j, with the largest of such values, into two intervals having (subjectively) the same probability content. The authors proved that their method leads to ranges as small as one wishes. The method can be easily implemented and its understanding for a practitioner is easy; new quantiles could be obtained by applying, for example, a betting scheme. The major drawback might be the slowness in the reduction of the range.

Ríos Insua et al. (1999) considered contaminations of a baseline prior, Π_0, with the signed measures $\delta = \varepsilon(\Pi - \Pi_0)$, Π being another probability measure and $\varepsilon \le \varepsilon_0$, for a given ε_0. They computed the supremum of the Fréchet derivative, of the posterior functional of interest, over all δ's such that Π belongs to the quantile class under investigation. It can be proved (Ruggeri and Wasserman, 1993, and Martín and Ríos Insua, 1996) that the supremum is given by

$$\frac{\varepsilon_0}{\int l(\theta)\Pi_0(d\theta)} \max\{\sum_j p_j\overline{h_j}, -\sum_j p_j\underline{h_j}\},$$

where $\overline{h_j}$ ($\underline{h_j}$) is the supremum (infimum) of the function h over the interval A_j, $j = 1,\dots,n$, associated with the quantiles, and p_j is the corresponding probability.

Ríos Insua et al. proposed splitting the interval which gives the largest contribution; in their paper the split was based on the experts' opinions. Pros and cons of this approach are similar to the ones for the method proposed by Liseo et al. (1996). In fact, the two approaches rely on very similar measures to choose the interval to split, although they are based, respectively, on local and global robustness measures. Other ideas may be seen in Ríos Insua (1990).

1.4.5 Likelihood and loss robustness

In this section, we briefly illustrate some issues regarding sensitivity with respect to the likelihood and loss function.

1.4.5.1 Likelihood robustness

An informal approach, similar to that discussed in Section 1.4.1, can be followed in analyzing likelihood robustness; one can simply try several models and see if the answer changes significantly. It is difficult to develop more formal approaches to likelihood robustness, for several reasons, one of which is that parameters often change meanings in going from one model to another; hence changing models also often requires changing the prior distributions. (This is one argument for performing prior elicitations for the marginal distribution of observables; such elicitations need only be done once and will induce appropriate priors for each of the models under consideration.)

Sometimes a parameter has a clear meaning, regardless of the model under consideration, and more formal methods can be employed. An example is that of the median of a symmetric distribution. For instance, Shyamalkumar (2000) discusses the situation in which there are two competing models (normal $\mathcal{N}(\theta, 1)$ and Cauchy $\mathcal{C}(\theta, 0.675)$), with classes of priors on the (well-defined) common parameter θ. The scale parameters of the two models are chosen to match the interquartile range. The priors on θ are specified to be in ε–contamination neighborhoods, where the contaminating classes are either Γ_A or Γ_{SU}, containing, respectively, arbitrary and symmetric, unimodal (around zero) distributions. Shyamalkumar considered $\varepsilon = 0.1$ and computed upper and lower bounds on the posterior mean $\mathbb{E}(\theta|x)$ of θ, as shown in Table 6.

Data	Likelihood	Γ_A		Γ_{SU}	
		$\inf \mathbb{E}(\theta\|x)$	$\sup \mathbb{E}(\theta\|x)$	$\inf \mathbb{E}(\theta\|x)$	$\sup \mathbb{E}(\theta\|x)$
$x = 2$	Normal	0.93	1.45	0.97	1.12
	Cauchy	0.86	1.38	0.86	1.02
$x = 4$	Normal	1.85	4.48	1.96	3.34
	Cauchy	0.52	3.30	0.57	1.62
$x = 6$	Normal	2.61	8.48	2.87	5.87
	Cauchy	0.20	5.54	0.33	2.88

TABLE 6. Bounds on posterior mean for different models and classes of priors

As noticed by Shyamalkumar, even though the widths of the ranges of the posterior means are similar for the two likelihoods, the centers of the ranges are quite different when x is larger.

As a further extension, one might consider a parametric class of models, such as that of Box and Tiao (1962), given by

$$\Lambda_{BT} = \{f(y|\theta, \sigma, \beta) = \frac{\exp\{-\frac{1}{2}\left|\frac{y-\theta}{\sigma}\right|^{\frac{2}{1+\beta}}\}}{\sigma 2^{(1.5+0.5\beta)}\Gamma(1.5+0.5\beta)};\ \text{any } \theta, \sigma > 0, \beta \in (-1, 1]\}. \tag{2}$$

An application of this class is given in Shyamalkumar (2000), which reviews, as well, other current approaches to likelihood robustness.

1.4.5.2 Loss robustness

One of the motivations for comparing the robustness of the posterior mean and median, as done in Sivaganesan (1991), is that the two estimators are optimal Bayesian estimators for, respectively, the squared error and absolute error loss functions. More generally, a group of decision makers could have different ideas about the features of an optimal estimator, leading to different loss functions. We could be interested, for example, in quantifying the changes either in the posterior expected loss (e.g. in Dey et al., 1998) or in the optimal action (e.g., mean and median in Basu and DasGupta, 1995).

Various loss robustness measures have been proposed, most of which are discussed in Dey et al. (1998), which explores robustness properties of classes of LINEX losses under exponential and discrete power series families of distributions using conjugate priors. In particular, they consider the class of LINEX losses defined as

$$\Lambda = \{L_b : L_b(\theta, a) = \exp\{b(a-\theta)\} - b(a-\theta) - 1, b \neq 0, b_0 < b < b_1\},$$

where b_0 and b_1 are fixed. They consider a normal model for the observation and a conjugate prior for its mean, with specified variance, and compute the range of the posterior expected loss and the posterior regret (see Berger, 1985, for definition and properties). These results are reviewed in Dey and Micheas (2000).

Other approaches, based upon stability of decisions and nondominated actions, are thoroughly reviewed in Kadane et al. (2000) and Martín and Arias (2000), respectively.

1.4.5.3 Loss and prior robustness

Research on sensitivity jointly with respect to the prior and the loss is not very abundant. Note that, as indicated in Section 2, it is possible that the problem is robust with respect to the prior only and the loss only, but rather sensitive when both elements are jointly considered.

An outline of such general sensitivity analyses is given in Ríos Insua et al. (2000). Basu and DasGupta (1995) jointly considers priors in a distribution band and a finite number of loss functions. Martín and Ríos Insua (1996) used Fréchet derivatives to investigate local sensitivity for small perturbation in both the prior distribution and the loss function.

1.5 Inherently robust procedures

In the previous sections, we have considered ways to increase robustness by narrowing classes of priors and/or likelihood and/or loss functions. Now we focus on choices of priors/likelihood/loss which should be less sensitive to changes in the other components.

1.5.1 *Robust priors*

As thoroughly discussed in Berger (1985, 1994), flat-tailed distributions tend to be more robust than standard conjugate choices (e.g., Cauchy vs. normal distributions). The construction of a class of priors should take into account tail behavior in addition to symmetry, unimodality, etc. Tail modeling is straightforward for certain classes of priors, for instance the *density bounded* class

$$\Gamma_{DB} = \{\pi : L(\theta) \leq \pi(\theta) \leq U(\theta), \forall\theta\},$$

used by Lavine (1991). By appropriate choice of L and U, one can avoid priors with too-sharp tails. For instance, choosing L and U to be different multiples of a Cauchy density would preclude priors with normal-like tails. Additional constraints, such as unimodality, can be added to density bounded classes to further ensure that only reasonable prior densities are being considered.

Hierarchical priors are typically inherently robust in several senses, as discussed in Berger (1985). In part, this is because they are typically flat-tailed distributions. As an example, Moreno and Pericchi (1993) consider an ε-contamination neighborhood of an $\mathcal{N}(\mu, \tau)$ prior and show that, when the prior information concerning τ is vague, the range of the desired posterior quantity is large. However, if one places a prior on τ, which is equivalent to considering a hierarchical model, then the posterior range is drastically reduced.

1.5.2 *Robust models*

Many robust procedures in the literature are primarily concerned with obtaining resistance to outliers. For instance, the class (2), defined by Box and Tiao (1962), has been widely depicted as a robust class and its application for this purpose has been extensively discussed; see, e.g., Box and Tiao (1973). Another class of models, the extended power family, has been proposed by Albert et al. (1991). It can be viewed as a smooth alternative to the Box and Tiao power-series family, since its tails have similar functional forms but, unlike the power-series family, it is everywhere differentiable.

1.5.3 Robust estimators

As already discussed, different loss functions can lead to different Bayesian estimates. In particular, squared error and absolute error loss yield, respectively, the posterior mean and the posterior median as the optimal Bayesian estimator. It is interesting to study which such common estimators are more inherently robust. Sivaganesan (1991) investigated the behavior of the posterior mean, median and mode when the prior is in the ε–contamination class, restricted to all bounded unimodal distributions with the same mode and median as the baseline Π_0. He concludes that the posterior mean and posterior median are preferable to the posterior mode. Moreover, the posterior median seems to be worse, though only slightly, than the posterior mean.

Basu (1996) also studied this issue, for the example presented in Section 1.4.3. He found that the posterior mean is more (less) sensitive than the posterior median for large (small) samples.

1.5.4 Other robust procedures

Marin (2000) considers a robust version of the dynamic linear model which is less sensitive to outliers. This is achieved by modeling both sampling errors and parameters as multivariate exponential power distributions, instead of normal distributions.

Another possibility to enhance robustness is offered by the Bayesian non parametric approach; see Berger (1994). Parametric models can be embedded in nonparametric models, presumably achieving a gain in robustness. An example is discussed in MacEachern and Müller (2000), where mixtures of Dirichlet process are considered and an efficient MCMC scheme is presented.

1.6 Comparison with other approaches

Bayesian robustness has had interactions with other approaches, such as classical robustness, illustrated in Huber (1981), and upper and lower probabilities, presented in Walley (1991). We briefly review common points and differences.

1.6.1 Frequentist robustness

Certain notions developed in classical robustness have been utilized in Bayesian robustness. We have already mentioned one, ε–contamination classes of distributions, that were introduced in Huber (1973) for classical robustness problems. In the classical context, they are primarily used to model possible outliers.

The local sensitivity approach, discussed in Gustafson (2000), borrows the notion of an *influence function* from classical robustness theory (Hampel et al., 1986). Gustafson observes that the Gâteaux derivative can be written as

$$G_{\rho_0}(Q - \Pi_0) = \int I(z) d[Q - \Pi_0](z),$$

where $I(z)$ is the influence function. Thus plots of the influence function can be very useful in visually assessing sensitivity. For most posterior quantities, the influence function can be easily determined.

Peña and Zamar (1996) present an asymptotic approach to Bayesian robustness that allows use of classical robustness tools, such as the influence function and the maximum bias function. We refer to their paper for an illustration of the approach.

1.6.2 Imprecise probability

Imprecise probability is a generic term used to describe mathematical models that measure uncertainty without precise probabilities. This is certainly the case with robust Bayesian analysis, but there are many other imprecise probability theories, including upper and lower probabilities, belief functions, Choquet capacities, fuzzy logic, and upper and lower previsions; see Walley (1991) and the web page `http://ippserv.rug.ac.be`.

Some of these theories, such as fuzzy logic and belief functions, are only tangentially related to robust Bayesian analysis. Others are intimately related; for example, some classes of probability distributions that are considered in robust Bayesian analysis, such as distribution band classes, can also be interpreted in terms of upper and lower probabilities. Also, classes of probability distributions used in robust Bayesian analysis will typically generate upper and lower previsions as their upper and lower envelopes. Walley (1991) describes the connection between robust Bayesian analysis (in terms of sensitivity to the prior) and the theory of coherent lower previsions.

In a rough sense, the major difference between robust Bayesian analysis and these alternative theories is that robust Bayesian analysis stays with ordinary Bayesian intuition, considering, as ideal, classes (of priors, say) that consist only of those priors that are individually compatible with prior beliefs. In contrast, the alternative theories view the classes themselves (not the individual priors) as the basic elements of the theory. We do not take a philosophical position on this issue, but it is useful to note that the robust Bayesian approach is directly compatible with ordinary Bayesian analysis, and hence is more immediately accessible to Bayesian practitioners.

1.6.3 Hierarchical approaches

We have already mentioned that hierarchical modeling has certain inherent robustness properties. We further mentioned that one approach to dealing with a lack of robustness (with respect to, say, the prior) is to place a hyperprior on the class of priors, which is a type of hierarchical analysis. Indeed, if there were no possibility of obtaining additional information to deal with the lack of robustness, we would recommend this technique, with the hyperprior being chosen in some default fashion. Note, however, that this effectively means one would be working with a single prior (that arising from marginalizing over the hyperprior), so real robustness could not be claimed; in a sense, what has been accomplished is simply to use the inherent robustness of hierarchical modeling to develop a single prior that is hopefully reasonably robust. One could, of course, also embed the hyperprior in a class of priors and deal with robustness in that fashion; this is sensible if the hyperprior has clear intuitive meaning but otherwise can offer no real gain.

1.6.4 Reference and objective Bayes approaches

The primary applied form of Bayesian analysis, for over 200 years, has been the objective Bayes approach (often called the reference approach today), in which a single prior distribution is chosen in a default or noninformative fashion. A nice review of this approach is given in Kass and Wasserman (1996).

The objective Bayes approach is *not* related to the robust Bayesian approach through any type of formal attempt at being noninformative. (After all, if taken literally, *noninformative* would suggest that one consider the class of *all* possible prior distributions.) Rather, the objective Bayes approach is related in that it can be viewed as proposing a particular robust prior for default use of Bayesian methods.

There are two further relationships that should be mentioned. One has already been discussed, namely that, when robustness is not obtained, a good generic way to proceed is to determine a default hyperprior over the class of priors, and perform the (now single-prior) Bayesian analysis. Objective Bayesian techniques can be applied to determine a suitable default hyperprior. The second interesting relationship is the reverse: often there exist a number of possible default priors, and it is then natural to consider a Bayesian robustness analysis with respect to the class of default or noninformative priors.

1.7 Future work

We finish this paper with a brief discussion of potential topics for further research. Considerably more discussion of this issue can be found in the papers in the volume.

The most important challenge for the field of Bayesian robustness is to increase its impact on statistical practice; indeed, to make it a routine component of applied work. One avenue for doing so is simply to show, by example, the power of the methods. Thus we have included several papers in this volume that illustrate the uses of Bayesian robustness in applied settings. Marin (2000) applies a robust version of dynamic linear models to forecast unemployment and activity ratios. Carlin and Perez (2000) illustrates forward and backward robustness methods in epidemiological settings. Bielza et al. (2000) demonstrates the use of simulation methods and the expected value of perfect information to perform sensitivity analysis in a complex medical decision making problem. Cagno et al. (2000) applies robust methods to predict failures in gas pipelines, and Wilson and Wiper (2000) applies them in software reliability problems.

Perhaps the most important way to bring robust Bayesian methods to practitioners is to have these methods available in standard statistical software. There are several ongoing efforts to develop general Bayesian statistical software, and incorporation of robust Bayesian methodology into such software is clearly desirable. Indeed, as indicated in Section 1.4.4, Bayesian robustness can perhaps best be viewed as a component of general Bayesian software. Ideally, such software would proceed as follows:

- Elicit preferences and beliefs from the decision maker.
- Conduct a robustness study with classes compatible with the information elicited, and certain representative quantities of interest.
- If robustness appears to hold, conduct a standard Bayesian analysis with a representative prior/model/loss from the classes (or with respect to some hyperprior average); in part, performing the reported analysis with a single prior/model/loss would be for ease in communication.
- If robustness does not hold, elicit additional information to further constrain the classes (with the software perhaps suggesting which additional information would be best to elicit), or obtain additional data, and repeat.

The least understood aspect of this program is the possibility of interactive elicitation and is a topic that deserves much more research. Another aspect that requires clarification is the extent to which one can automate the process of determining whether or not robustness holds. Frequently, this will be a context-dependent decision, but generic calibration of robustness measures is often possible – some examples are given in McCulloch (1989) and Ríos Insua (1990) – and further efforts in this direction are clearly of interest.

In the introduction, we referred to the enormous impact that MCMC methods have had on Bayesian analysis, and we mentioned that Bayesian robustness methodology will need to be compatible with MCMC methods to become widely used. There are some useful results in this direction. For instance, once one has obtained an MCMC sample from the posterior associated with a baseline prior, obtaining an MCMC sample from the posterior of a "close" prior can be done via importance sampling, see for example Smith and Gelfand (1992). But much additional work needs to be done to more closely relate robust Bayesian techniques with MCMC.

Another area of enduring importance to the field is foundations. As will be seen in later papers, there is still interest in axiomatic models that lead to robust Bayesian analysis. For instance, a direct implication of most axiomatic systems is that nondominated actions or estimators should be considered. Indeed, a topic of considerable importance is that of further developing methods for approximating the set of nondominated estimators or actions, as explained in Martin and Arias (2000). On the other hand, axiomatics such as in Nau (1995), lead to somewhat different models that would require new computational developments.

To summarize, there are many challenging problems in robust Bayesian analysis from methodological, foundational and computational points of view. The field has a very exciting future.

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