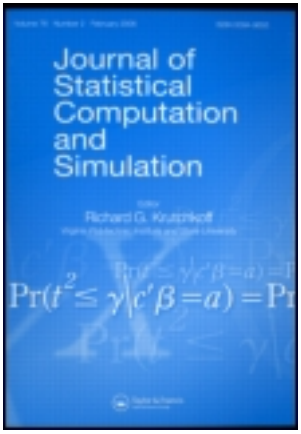

Journal of Statistical Computation and Simulation
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gscs20

Sensitivity analysis in statistical decision theory: A decision analytic view

David Rios. Insua [a b], Jacinto. Martin [a], Les. Proll [c], Simon. French [c] & Abdellah. Salhi [d]

[a] Decision Analysis Group, Madrid Technical University, Spain

[b] CNR-IAMI, Italy

[c] School of Computer Studies, University of Leeds, UK

[d] Department of Electronics and Computer Science, University of Southampton, UK

Published online: 20 Mar 2007.

To cite this article: David Rios. Insua , Jacinto. Martin , Les. Proll , Simon. French & Abdellah. Salhi (1997) Sensitivity analysis in statistical decision theory: A decision analytic view, Journal of Statistical Computation and Simulation, 57:1-4, 197-218, DOI: 10.1080/00949659708811808

To link to this article: http://dx.doi.org/10.1080/00949659708811808

J. Statist. Comput. Simul., 1997, Vol. 57, pp. 197–218
Reprints available directly from the publisher
Photocopying permitted by license only

SENSITIVITY ANALYSIS IN STATISTICAL DECISION THEORY: A DECISION ANALYTIC VIEW*

DAVID RIOS INSUA[a,b,*], JACINTO MARTIN[a], LES PROLL[c], SIMON FRENCH[c] and ABDELLAH SALHI[d]

[a]*Decision Analysis Group, Madrid Technical University, Spain;*
[b]*CNR-IAMI, Italy;* [c]*School of Computer Studies, University of Leeds, UK;*
[d]*Department of Electronics and Computer Science, University of Southampton, UK*

(Received 9 September 1995; In final form 3 August 1996)

Sensitivity analysis provides a way to mitigate traditional criticisms of Bayesian statistical decision theory, concerning dependence on subjective inputs. We suggest a general framework for sensitivity analysis allowing for perturbations in both the utility function and the prior distribution. Perturbations are constrained to classes modelling imprecision in judgements. The framework discards first definitely bad alternatives; then, identifies alternatives that may share optimality with a current one; and, finally, detects least changes in the inputs leading to changes in ranking. The associated computational problems and their implementation are discussed.

Keywords: Bayesian statistics; statistical decision theory; sensitivity analysis; classes of priors; classes of utilities

1. INTRODUCTION

Bayesian Statistics has been frequently criticised for two related issues. First, its foundations require precise inputs (preferences, beliefs) to an analysis and, very frequently, experts are not able to provide enough information to assess them sufficiently finely. Second, the output of a

*Corresponding author.

Bayesian analysis may depend critically on those inputs. As a consequence, there have been significant developments in Bayesian sensitivity analysis recently, see Berger (1994), for a fine review.

In this context, sensitivity analysis consists of undertaking a class of Bayesian analyses and basing conclusions on their common ground. If the conclusions agree, the analysis seems robust, suggesting that no further modelling is required. For the other case, there are tools which suggest features requiring additional modelling efforts. Two types of analyses are considered: *global analysis*, in which upper and lower bounds of quantities of interest (say a posterior probability or expected utility) are computed when the prior and/or the utility ranges over a class of interest, and *local analysis*, in which we evaluate changes in the quantities of interest when the prior and/or the utility are slightly perturbed.

However, for reasons described in Berger (1994), most work on Bayesian sensitivity analysis has concentrated on investigations about sensitivity with respect to the prior. A possible reason is that Bayesian Statistics is usually viewed in isolation from statistical decision theory, see French (1995). Note though that we could argue that since, ultimately, inference is made to support decision making, it should conform to the principles of decision analysis. Consequently, we should be interested in performing more general sensitivity analyses with respect to both the prior and the utility function. A more extreme view, leading to the same conclusion, suggests that all inference problems may be framed as decision theoretical problems, see Bernardo and Smith (1994). Finally, on a more pragmatic level, performing sensitivity analysis on the prior only may lead to wrong conclusions concerning robustness, as we illustrate with our example. See Ríos Insua and Martín (1994) for additional discussion.

In this paper, we shall propose and describe a general framework for sensitivity analysis in Bayesian Statistics, in which we study the effect of perturbations in both the utility and the prior. The approach proposed stems from our related work in multicriteria decision making, see Ríos Insua and French (1991) and Proll, Ríos Insua and Salhi (1993a). Here we describe how we can extend this work to statistical problems, with emphasis on recent developments and computational issues. We illustrate our proposal with a multiple hypothesis testing problem and a medical example.

In Section 2, we describe the basic framework, notation, and the main problems we address. Given preferences modelled with a utility function and beliefs with a probability distribution, the Bayesian approach suggests the alternative with maximum expected utility as optimal. For reasons mentioned above, we need tools to analyse the behaviour of that optimal alternative against changes in the utility and the prior. We suggest that these perturbations can be constrained to classes of distributions and utility functions, modelling imprecision in beliefs and preferences. Section 3 provides concepts allowing us to identify the alternatives on which the expert should concentrate his attention. In addition, it provides measures of the loss of optimality which help to decide whether further analysis is required. If this is the case, Section 4 provides tools to facilitate additional modelling. Section 5 suggests how the analysis extends to continuous problems, through a convenient discrete approximation. Section 6 describes implementation issues.

2. THE BASIC FRAMEWORK

We adopt a decision analytic view of Bayesian analysis. We assume that we have to choose among a set $\mathscr{A}$ of alternatives a. Until Section 5, we assume that $\mathscr{A}$ is finite. Typical statistical problems include multiple hypothesis testing and classification problems, see e.g., Berger (1985).

We compare alternatives according to their posterior expected utility. For that, we assess the prior beliefs on a state variable $\theta \in \Theta$ in a prior distribution π_0. They are updated to the posterior $\pi_0(\cdot|x)$, where x is the result of an experiment with likelihood $l(x|\theta)$ over a sample space X. We associate a consequence $c \in C$, to each pair (a,θ). Preferences over consequences are modelled with a utility function u_0, and we associate with each alternative a its posterior expected utility:

$$T(u_0,\pi_0,a) = \frac{\int_\Theta u_0(a,\theta) l(x|\theta) d\pi_0(\theta)}{\int_\Theta l(x|\theta) d\pi_0(\theta)} = \frac{N(u_0,\pi_0,a)}{D(\pi_0)}.$$

We maximize $T(u_0,\pi_0,a)$ with respect to a, as a way of obtaining the optimal alternative a_*. However, the assessment of u_0 and π_0 is far

from simple, and we shall need tools to check their impact on the optimal alternative.

For that, we shall perturb the assessment u_0, π_0 over classes $\mathcal{U}$ of utility functions and Γ of prior distributions. Note first that it is possible to provide foundations for this model, see e.g., Nau, Ríos Insua and Martín (1996). Essentially, they lead to modelling a preference relation $\preceq$ over $\mathcal{A}$ through a representation of the type

$$a \preceq b \Leftrightarrow (T(u,\pi,a) \leqslant T(u,\pi,b), \forall (u,\pi) \in \mathcal{U} \times \Gamma), \tag{1}$$

for $a,b \in \mathcal{A}$. In words, we model the expert's beliefs with a class Γ of prior distributions π and his preferences with a class $\mathcal{U}$ of utility functions u, and rank alternatives (where $a \preceq b$ means a is at most as preferred as b) according to the corresponding class of posterior expected utilities. Important classes of priors are described in Berger (1994). Important classes of utility functions in statistical contexts are described in Martín, Ríos Insua and Ruggeri (1996). Note that we may actually rewrite (1) as

$$a \preceq b \Leftrightarrow (N(u,\pi,a) \leqslant N(u,\pi,b), \forall (u,\pi) \in \mathcal{U} \times \Gamma), \tag{2}$$

since the denominators $D(\pi)$ cancel out. We shall use (1) or (2) as convenient.

Model (1) provides an appropriate framework for sensitivity analysis in Statistical Decision Theory. First, we may explore which alternatives are worth retaining in the analysis, given the classes assessed. This is the topic of Section 3, which helps to detect whether further analysis and modelling is required. For that purpose, we provide in Section 4 several tools which suggest where to concentrate additional modelling efforts: essentially how to reduce the imprecision in the current assessments Γ and $\mathcal{U}$. We shall use a simple example to illustrate our ideas.

Example 1 We formulate a multiple hypothesis testing problem. Assume we have six competing hypothesis $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Let $\Theta = \{\theta_1, \ldots, \theta_4\}$ be the set of states and $\mathcal{C} = \{c_1, \ldots, c_7\}$ be the set of consequences so that:

TABLE I Consequences of chosen hypotheses given the states

	θ_1	θ_2	θ_3	θ_4
a_1	c_3	c_4	c_5	c_6
a_2	c_2	c_5	c_5	c_5
a_3	c_1	c_3	c_5	c_7
a_4	c_2	c_3	c_3	c_7
a_5	c_1	c_2	c_6	c_7
a_6	c_3	c_3	c_6	c_6

Suppose we have the following initial assessments for the utility function and prior distribution:

$$u_0(c_1) = 0 \quad u_0(c_2) = 0.5 \quad u_0(c_3) = 0.7 \quad u_0(c_4) = 0.8$$

$$u_0(c_5) = 0.85 \quad u_0(c_6) = 0.9 \quad u_0(c_7) = 1$$

$$\pi_0(\theta_1) = 0.3 \quad \pi_0(\theta_2) = 0.3 \quad \pi_0(\theta_3) = 0.2 \quad \pi_0(\theta_4) = 0.2$$

For simplicity, suppose that $l(x|\theta_i)$ is constant for the x obtained. Then, the ranking and posterior expected utility of the hypothesis is :

TABLE II Ranking according to expected utilities of hypotheses

a_1	a_6	a_2	a_4	a_3	a_5
0.800	0.780	0.745	0.700	0.580	0.530

a_1 is the current optimal hypothesis, but we want to check the sensitivity of this conclusion to changes in the prior and the utility.

In this context, typical constraints for the utility function would include upper and lower bounds for the values of the utilities of the consequences and shape constraints. Let us assume for example, that

$$\mathcal{U} = \{u: \text{nondecreasing}, u(c_1) = 0, 0.3 \leqslant u(c_2) \leqslant 0.6, 0.6 \leqslant u(c_3) \leqslant 0.75,$$

$$0.75 \leqslant u(c_4) \leqslant 0.85, 0.8 \leqslant u(c_5) \leqslant 0.9, 0.85 \leqslant u(c_6) \leqslant 0.95, u(c_7) = 1\}.$$

Similarly, typical constraints for the prior would include upper and lower bounds for some of the probabilities and comparisons among

some of them, for example

$$\Gamma = \Big\{\pi : 0.2 \leqslant \pi(\theta_1) \leqslant 0.5, 0.2 \leqslant \pi(\theta_2) \leqslant 0.4, 0.1 \leqslant \pi(\theta_3) \leqslant 0.4,$$

$$0.1 \leqslant \pi(\theta_4) \leqslant 0.4, \pi(\theta_1) + \pi(\theta_2) \geqslant \pi(\theta_3) + \pi(\theta_4), \pi(\theta_2) \geqslant \pi(\theta_3)$$

$$\pi(\theta_3) + \pi(\theta_2) \geqslant \pi(\theta_4), \sum_{i=1}^{4} \pi(\theta_i) = 1\Big\}$$

Note that $u_0 \in \mathscr{U}$ and $\pi_0 \in \Gamma$. ◁

To provide additional insights, we give a description of a real case. Some of the ideas described below are illustrated with it, and other cases, in Martín and Müller (1996). Other examples may be seen in Berger *et al.* (1996).

Example 2 Palmer and Müller (1994) consider optimal apheresis decisions for cancer patients undergoing high-dose chemotherapy. Between a pretreatment and start of the chemotherapy, stem cells are collected to allow later reconstitution of white blood cell components. Depending on the pretreatment, the first stem cell collection process (apheresis) is scheduled on the fifth or seventh day after pretreatment. We have to decide for which of the remaining days further aphereses should be scheduled. The optimal solution should propose stem cell collections on days with high predicted stem cell concentrations. The prediction is based on observations of stem cell levels (represented by CD34 antigen levels) from past patients.

Let y_{ij}, $i = 1,\ldots,I$ and $j = 1,\ldots,n_i$ denote the observed CD34 count for patient i on day t_{ij}. Let $y_i = (y_{i1},\ldots,y_{i,n_i})$ and $y = (y_1,\ldots,y_I)$ denote the ith patient's data and combined data vector. Palmer and Müller (1994) specify a hierarchical model for the data. Patients undergo one of two possible pre-treatments. A covariate $x_i = 1$ or 2 records each patient's pre–treatment. Then,

$$y_{ij} = z_i g(t_{ij}; e_i, s_i) + \varepsilon_{ij}, i = 1,\ldots,I, j = 1,\ldots,n_i \; \varepsilon_{ij} \sim N(0,\sigma^2),$$

where $g(t;e,s) = 1/m\Gamma(t;a,b)$ is a Gamma function with parameters $b = e/s^2$, and $a = e \cdot b$ is chosen to have mean and variance matching e and s^2 and rescaled by $m = [(a-1)/b]^{a-1}\exp(-(a-1))$, so that $\sup(g) = 1.0$.

The prior has two stages, the first one depending on the covariates; the second one, common.

$$\theta_i \sim N(\eta_k, V),$$

$$\theta_i = (e_i, s_i, z_i),$$

$$k = x_i,$$

$$\sigma^{-2} \sim \Gamma(a_0/2, b_0/2).$$

For a new patient $h = I + 1$, denote with $y_h = (y_{h1},\ldots,y_{h,n_h})$ the (unknown) stem cell counts on the remaining days $t_{h1},\ldots,t_{h,n_h}$. Denote with $a = (d_1,\ldots,d_{n_h})$ a vector of indicators $d_j = 1(0)$ if a stem cell collection is (not) to be scheduled for day t_{hj}. We have a count y_{ho} for a first day t_0. Let $n_a = 1 + \Sigma d_j$ denote the number of scheduled aphereses, with the added one because of the already realized apheresis at day t_0. Let A be the event of failing to collect a target number y^* of stem cells: $A = \{\sum_j d_j\, y_{hj} < y^*\}$. The utility function includes a cost for sampling and a penalty associated with event A, with weights c and p, respectively:

$$u(a,\theta,y) = -c\, n_a - p\Pr(A|a,\theta).$$

The decision problem is then:

$$a^* = \arg\max_a \int u(a,\theta)\, dp(\theta|y),$$

where $p(\theta|y) \propto l(y|\theta)\pi(\theta)$ is the posterior distribution on the unknown model parameters given the data $y = (y_1,\ldots,y_I,y_{ho})$.

The assessment of prior and penalty parameters is complicated, so we would check their impact in the conclusions of the analysis. For that, we could use the parametric classes:

$$\Gamma_\Upsilon = \{\pi : \pi \sim N(\eta_k, V), \eta_k \in \Upsilon_k,\ k = x_i\},$$

$$\mathcal{U}_P = \{u : u = -c\, n_a - p\Pr(A|a,\theta), c \in C, p \in P\}$$ ◁

3. FILTERING BAD ALTERNATIVES

Our starting point is the set of alternatives $\mathscr{A}$, ranked according to the order defined in (1), given the assessments $\mathscr{U}$ and Γ. We shall define two filters of $\mathscr{A}$ which allow us, on the one hand, to discard bad alternatives and define the set of alternatives on which the expert should focus his attention, and, on the other hand, to provide estimates of maximum differences in posterior expected utilities.

3.1. Nondominated Solutions

Nondominated solutions are the natural solutions associated with (1).

DEFINITION 1 $a \in \mathscr{A}$ is nondominated if there is no other alternative $b \in \mathscr{A}$ such that $a \prec b$.

To compute the set $ND(\mathscr{A})$ of nondominated alternatives in $\mathscr{A}$, we need first to provide a way to check whether an alternative a dominates another alternative b. Conceptually, we could solve the optimisation problem

$$\begin{array}{ll} \min & T(u,\pi,a) - T(u,\pi,b) = \Delta_{ab} \\ \text{s.t.} & (u,\pi) \in \mathscr{U} \times \Gamma \end{array} \tag{3}$$

since

- If $\Delta_{ba} > 0$, $a \prec b$.
- If $\Delta_{ba} = 0$,
  If $\Delta_{ab} < 0$, $a \prec b$
  If $\Delta_{ab} = 0$, $b \sim a$
- If $\Delta_{ba} < 0$, if $\Delta_{ab} = 0$, $b \prec a$

In such a way, we may discard the dominated alternatives and retain a single representative of each set of equivalent alternatives. Note that we do not actually need to solve problems of the form (3) to full optimality, since we are essentially interested in the computationally cheaper problem of finding the sign of the infimum. However, solving them provides the bounds Δ_{ab} on the differences of expected utilities, which may suggest when to terminate the analysis as discussed below. Problems of type (3) have been solved in many settings.

For example, if (u,π) are defined parametrically and we have a primitive for $T(u,\pi,a)$, we may solve the corresponding nonlinear programming problem, as in Proll *et al.* (1993a); for nonparametric settings, some important cases when there is imprecision in the prior only have been dealt with in Martín and Ríos Insua (1995a); when there is imprecision in the utility, some important cases are in Martín *et al.* (1996); the nonparametric case with imprecision in both the utility and the prior is more difficult, but some cases are described in Martín (1995).

Given a procedure for checking dominance between alternatives, a pairwise comparison based procedure, as in Ríos Insua and French (1991), gives us the nondominated set $ND(\mathscr{A})$:

1. Let $d(i)=0, \forall i$
2. For $i=1$ to $n-1$
   If $d(i)=0$
   For $j=i+1$ to n
   If $d(j)=0$
   If $e_j \preceq e_i$ then $d(j)=1$
   Else, if $[T(u_0,\pi_0,e_i)=T(u_0,\pi_0,e_i)$ and $e_i \preceq e_j]$
   then $d(i)=1$ and increment i.
3. Let $ND(\mathscr{A})=\{e_i: d(i)=0\}$.

To alleviate the computational effort we could start by relabeling the alternatives as $(e_1,\ldots,e_n)$, according to their evaluation in (u_0,π_0) so that $T(u_0,\pi_0,e_{i+1}) \leqslant T(u_0,\pi_0,e_i), \forall_i$

3.2. Potentially Optimal Alternatives

Another natural solution concept is that of potential optimality.

DEFINITION 2 a is potentially optimal if it maximises the posterior expected utility for some pair $(u,\pi) \in \mathscr{U} \times \Gamma$.

To check whether an alternative a_j is potentially optimal, we need to solve the optimisation problem

$$\begin{array}{ll} \min & z_j \\ \text{s.t} & T(u,\pi,a_j) - T(u,\pi,a) + z_j \geqslant 0,\ a \neq a_j, \\ & (u,\pi) \in \mathscr{U} \times \Gamma \end{array} \tag{4}$$

Then, a_j is potentially optimal iff the optimal value z_j is less than or equal 0. Discussion of the solution of these problems may be seen in Proll *et al.* (1993a), when (u,π) are defined parametrically and we have a primitive for $T(u,\pi,a)$, therefore having a nonlinear programming problem. In the nonparametric case, we may relate the problem to a generalized moment problem, see Betró and Guglielmi (1996). Note again that we are mainly interested in finding the sign of the optimal value, rather than the value itself. However, solving problem (4) to full optimality again provides estimates of maximum losses of expected utility, when suggesting a_j as optimal.

Example 1 (cont.) In our hypothesis testing problem, we have, for example, that $T(u,\pi,a_1) = u(c_3)\pi(\theta_1) + u(c_4)\pi(\theta_2) + u(c_5)\pi(\theta_3) + u(c_6)\pi(\theta_4)$, and problems (3) and (4) may be solved by nonlinear programming.

The set of nondominated alternatives is $\{a_1,a_2,a_6,a_4\}$. The optimal values of problems (4) are:

TABLE III Optimal values for checking potential optimality

a_6	-0.060
a_2	-0.060
a_4	-0.035

Hence, the set of nondominated potentially optimal alternatives is $\{a_1,a_2,a_6,a_4,\}$. We may also say, for example, that the maximum difference in expected utility when a_6 is optimal is 0.060. These differences may suggest that alternatives are very similar in expected utility and, consequently, no additional modelling is required. ◁

4. EFFECTS OF CHANGES IN PRIOR AND UTILITY

The above procedures serve both as filters for uninteresting alternatives and as measures of possible losses in expected utility due to the adoption of a_* as the optimal alternative. With this information we may decide not to conduct further analysis, typically when the set of nondominated alternatives is a singleton or its elements are similar in expected utility. If not, we need tools to suggest how to undertake additional analysis, some of which we describe below.

4.1. Adjacent Potentially Optimal Alternatives

First, since a_* is suggested as the current optimal alternative, it seems natural to look for those alternatives which may share optimality with it. This leads to the idea of adjacent potential optimality.

DEFINITION 3 a is adjacent potentially optimal (apo) to a_* if there is $(u,\pi) \in \mathcal{U} \times \Gamma$ such that $T(u, \pi, a) = T(u, \pi, a_*) \geqslant T(u, \pi, b)$, $\forall b \in \mathcal{A}$.

An alternative a will be apo to a_* iff the optimal value of problem (5) is 0:

$$\begin{aligned} \max \quad & T(u,\pi,a) - T(u,\pi,a_*) \\ \text{s.t} \quad & T(u,\pi,b) - T(u,\pi,a_*) \leqslant 0, b \in \mathcal{A} \\ & (u,\pi) \in \mathcal{U} \times \Gamma \end{aligned} \tag{5}$$

The solution of this problem in various important cases is described in Proll *et al.* (1993a) again as nonlinear programming problems in the parametric case. Nonparametric problems lead to generalized moment problems, as indicated before. After solving problems (5), we identify the alternatives that may share optimality with the current optimal a_*, together with pairs (u,π) for which optimality is shared.

We could be more informative, trying to find out least changes leading to changes in ranking. A natural starting point is the comparison of a_* with alternatives a, when a varies in the set of interesting alternatives, say those that are apo. The idea then would be to define neighbourhoods of the current assessment, in which the optimality of a_* holds. If neighbourhoods are defined by some sensitivity measure ε, we could say that a_* *is* ε'-robust, with ε' the minimum value of ε obtained for various alternatives. We can then obtain the most competitive alternative and the most critical conditions under which there will be a change in ranking. This information may be used to identify the precision needed in the decision maker's assessments. Typically, this information can be used to judge how much we can perturb the initial assessment so as to preserve the optimality of a_*, and to identify specially sensitive directions, such that perturbations of the initial assessments u_0 and π_0 in those directions lead faster to outranking of a_* by another alternative.

4.2. Distance Based Analysis

In some cases, it may be possible to solve for the closest pair (utility, prior) to the current assessment leading to changes in the ranking, where closest is defined in terms of an appropriate distance. This amounts to solving for each a of interest

$$\begin{aligned} \min \quad & d((u,\pi),(u_0,\pi_0)) = d_a \\ \text{s.t} \quad & T(u,\pi,a) - T(u,\pi,a_*) \geqslant 0, \\ & (u,\pi) \in \mathscr{U} \times \Gamma \end{aligned} \tag{6}$$

Then, we can find $d_{a'} = \min d_a$ and suggest the minimising alternative a' as the closest competitor to the current optimal. In general, it will be difficult to interpret this distance. We could find the maximum perturbation ρ allowed, given the constraints

$$\begin{aligned} \max \quad & d((u,\pi),(u_0,\pi_0)) = \rho \\ \text{s.t.} \quad & (u,\pi) \in \mathscr{U} \times \Gamma \end{aligned} \tag{7}$$

and use $r = d_{a'}/\rho$ as a sensitivity measure. Note that we would aim at improving this measure, which suggests obvious ways of continuing the analysis. One possibility to increase r would be to attempt direct comparisons between a' and a_*. Another possibility would be to reduce the class, leading therefore to a reduction of the denominator. In both cases, we have the optimal solutions of the corresponding optimisation problems which help in those tasks. The solution of the problems for parametric cases are illustrated as nonlinear programing problems in Proll *et al.* (1993a). In nonparametric cases, we have, again, generalised moment problems.

Other calibration procedures are suggested in McCulloch (1989).

4.3. Differential Analysis

For the second notion, we rely on linear perturbations of the current assessment, within the given classes, as specified in the definition of the

ε-robust neighbourhood. This requires that the classes are convex, without loss of generality, see Nau *et al.* (1996):

DEFINITION 4 (u_0,π_0) is ε-robust for $a \preceq a_*$ within $\mathscr{U} \times \Gamma$, if $T(tu+1-t)u_0,\ t\pi+(1-t)\pi_0, a) \leqslant T(tu+(1-t)u_0,\ t\pi+(1-t)\pi_0, a_*)$ whenever $t<\varepsilon$, $(u,\pi)\in\mathscr{U}\times\Gamma$.

Computation of the corresponding neighbourhoods is fairly simple when we consider perturbations only on the utility function or the probability distribution. For example, we have the following simple result, where $T^a(u) = T(u,\pi_0,a_*) - T(u,\pi_0,a)$, for an alternative a of interest.

PROPOSITION 1 *u_0 is ε_u-robust within $\mathscr{U}$, where*

$$\varepsilon_u = \min\left\{1, \frac{T^a(u_0)}{T^a(u_0) - \inf_{u\in\mathscr{U}} T^a(u)}\right\}.$$

A similar result holds for the case of imprecision only in the prior with the operator

$$N^a(u,\pi) = N(u,\pi,a_*) - N(u,\pi,a),$$

with u fixed at u_0.

PROPOSITION 2 *π_0 is ε_π-robust within Γ, where*

$$\varepsilon_\pi = \min\left\{1, \frac{N^a(u_0,\pi_0)}{N^a(u_0,\pi_0) - \inf_{u\in\mathscr{U}} N^a(u,\pi_0)}\right\}.$$

The problems leading to ε_u and ε_π may be solved with tools from global robust Bayesian analysis, appropriate for finding the infima involved.

Computations of ε-robust neighbourhoods are difficult in the general case of imprecision in both the utility and the prior. However, we may take linear approximations of the expected utilities, with the aid of Fréchet derivatives, and provide useful bounds. Fréchet derivatives were introduced in Bayesian analysis by Diaconis and Freedman (1986), and developed, mainly, by Ruggeri and Wasserman (1993) for

the case of imprecision in beliefs in inference problems, and by Martín and Ríos Insua (1996) for the case of imprecision in both preferences and beliefs. Here we describe and interpret two results from Martín and Ríos Insua (1995b) to bound the ε-robustness for two alternatives a and a_*.

We provide first a lower bound for the ε-robustness, that is, we suggest at least how much we may contaminate the initial assessment (u_0,π_0) before the posterior expected utility of a is bigger than that of a_*. When the lower bound is 1, there is dominance, but we are assuming that we undertake this analysis only for nondominated alternatives.

PROPOSITION 3 *(u_0,π_0) is, at least, ε_1-robust for $a \underline{\prec} a_*$ within $\mathcal{U} \times \Gamma$ with*

$$\varepsilon_1 = \min\left\{1, \frac{\inf_{(u,\pi)\in\mathcal{U}\times\Gamma}\hat{N}(u,\pi) + \sqrt{[\inf_{(u,\pi)\in\mathcal{U}\times\Gamma}\hat{N}(u,\pi)]^2 + 4KN^a(u_0,\pi_0)}}{2K}\right\},$$

$$K = 4\sup l(x|\theta), \text{ and}$$

$$\hat{N}(u,\pi) = N^a(u,\pi_0) + N^a(u_0,\pi) - 2N^a(u_0,\pi_0).$$

We provide now an upper bound of the ε-robustness.

PROPOSITION 4 *With the notation of Proposition 3, if*

$$-\inf_{(u,\pi)\in\mathcal{U}\times\Gamma} \hat{N}(u,\pi) > 2\sqrt{KN^a(u_0,\pi_0)},$$

then (u_0,π_0) is, at most, ε_2-robust for $a \underline{\prec} a_*$ within $\mathcal{U} \times \Gamma$, with

$$\varepsilon_2 = \min\left\{1, \sqrt{\frac{N^a(u_0,\pi_0)}{K}}, \varepsilon_u, \varepsilon_\pi\right\}.$$

with ε_u and ε_π defined as above.

If the inequality does not hold, then

$$\varepsilon_2 = \min\{1, \varepsilon_u, \varepsilon_\pi\}.$$

Note that the computations required in Propositions 3 and 4 are amenable, since computing K requires solving a nonlinear programming problem and the infimum of $\hat{N}$ may be computed with robust Bayesian analysis methods, see Berger (1994).

Jointly, the above results allow us to say that the ε-robustness for the preference $a \preceq a_*$ and the current assessment is between ε_1 and ε_2, with obvious interpretation when any of these values is 1. Clearly, the tighter the interval, the more informative it is.

By undertaking the above approach for various alternatives of interest, we may get an idea of the ε-robustness of the current optimal alternative a_*, together with an idea of the critical judgements, obtained as the optima in the above problems.

Example 1 (cont) It is easily verified that a_6, a_2 and a_4 are adjacent potentially optimal alternatives to a_1. For these alternatives the results of the distance analyses for l_1 and l_∞ distances are

TABLE IV Distance based sensitivity analysis

	d_6	d_2	d_4	ρ	r
l_1	0.06	0.15	0.47	0.92	0.07
l_∞	0.01	0.03	0.08	0.2	0.08

The analysis suggests that small changes in the assessment lead to a change in the ranking, a_6 being the closest competitor to a_1. Recall, though, that differences in posterior expected utility between these two alternatives were not too big

Table V provides the ε-robustness with respect to the utility function, ε_u; with respect to the probability distribution, ε_π and bounds ε_1, ε_2 for joint sensitivity with respect to the utility and prior.

TABLE V Local sensitivity analysis

	ε_u	ε_π	ε_1	ε_2
a_6	0.4	1	0.38	0.4
a_2	0.55	1	0.28	0.55
a_3	1	1	0.43	1

Note that, for example, if we study only sensitivity with respect to the probability distribution, the conclusion is that the model is robust,

in the sense that a_1 remains optimal since ε_π is always 1. Moreover, if we consider only sensitivity with respect to the utility function, alternative a_1 is preferred to a_4, for any u. But, this does not hold if we consider imprecision in both the prior and the utility.

Our local analysis suggests that the most competitive alternatives are a_6 and a_2. The bounds allow us to say that the ε-robustness of a_4 is greater than the ε-robustness of a_6. But we do not know whether a_6 is more competitive than a_2. Finally, we can say that a_1 is, at least, 0.28-robust and at most 0.4-robust. ◁

5. APPROXIMATIONS IN THE CONTINUOUS CASE

The previous sections assumed that the set $\mathscr{A}$ of alternatives is finite. Our framework could be actually applied in the continuous case to a discrete approximation to $\mathscr{A}$. Note that since the optimal alternative will be among the nondominated ones, we need only apply the above to an approximation of the nondominated set of $\mathscr{A}$.

To obtain that approximation, we shall assume that we are able to sample from $\mathscr{A}$, a typical example being when $\mathscr{A}$ may be described as a subset in R^n. We assume that we do this in such a way that all points in $\mathscr{A}$ are in the support of the underlying sampling distribution. The simplest case is the uniform distribution on $\mathscr{A}$, which can be generated by the rejection method, although this may be inefficient. Occasionally we may have additional information which may be reflected in the sampling distribution used.

The approach proceeds then by obtaining a sample $\mathscr{A}_n = \{a_1,\ldots,a_n\} \subset \mathscr{A}$ and finding, with the procedure in section 3, the nondominated set N_n of $\mathscr{A}_n$. We can prove easily that the procedure converges to the true nondominated set as n increases, under our assumption that all points are in the support of the sampling distribution:

LEMMA 1 *$\forall a \in \mathscr{A}$ almost surely there is a subsequence (a_{n_j}) which converges to a.*

In particular, this holds $\forall a \in ND(\mathscr{A})$. Then,

PROPOSITION 5 *$\forall a \in ND(\mathscr{A})$, almost surely there is a convergent subsequence (a_{n_j}), with $a_{n_j} \in N_{n_j}$ such that $a_{n_j} \to b$ and $b \sim a$ (i.e., $b \preceq a$ and $a \preceq b$).*

The result implies that, for the b found in Proposition 5, $T(u,\pi,a) = T(u,\pi,b)$, $\forall (u,\pi) \in \mathcal{U} \times \Gamma$. Under identifiability conditions, i.e., when $T(u,\pi,a) = T(u,\pi,b)$, $\forall\, (u,\pi) \in \mathcal{U} \times \Gamma \Rightarrow a = b$, we have

COROLLARY 1 *$\forall a \in ND(\mathcal{A})$, almost surely there is a sequence (a_{n_j}), with $a_{n_j} \in N_{n_j}$ such that $a_{n_j} \to a$.*

The above results suggest using N_n as a discrete approximation to $ND(\mathcal{A})$.

For implementation,we need a stopping rule. This problem is related to that of providing stopping rules for optimisation algorithms. One possible criteria is to appeal to the stability of the discrete approximation. Specifically, the approach that we use is to stop when the discrete approximation changes little (say 5%) from one iteration to another. Then, the procedure is:

1. Initialization $l = 1$, $N_0 = \emptyset$.
2. Generate a sample $\mathcal{A}_l$ from $\mathcal{A}$.
3. $N_l = ND\ (\mathcal{A}_l \cup N_{l-1})$.
4. If stopping rule not satisfied, $l = l + 1$, go to 2.

The approach we follow is to apply our framework to the discrete approximation of the nondominated set. We compute the potentially optimal alternatives, the adjacent potentially optimal alternatives and undertake distance and/or differential analyses. This information is then fedback to the expert in the hope of stimulating him to further thinking about the problem.

6. IMPLEMENTATION

We describe here our implementations of the previous framework.

6.1. The Finite Case

Each of the filtering phases and the distance analysis phase require the solution of a number of mathematical programmes of various types. The classes of mathematical programme requiring solution vary between the phases and depend on the form of the posterior expected utility function for the decision model adopted, the nature of the set

$\mathscr{U} \times \Gamma$, and, in the distance analysis phase, on the metric chosen. The number of optimisation problems to be solved depends principally on $\mathscr{A}$, e.g., in the non-dominance phase we may need to solve $|\mathscr{A}|*(|\mathscr{A}|-1)/2$ problems, and their size on $|\Theta|$ and $|\mathscr{C}|$. Some of the problems are nonconvex and hence global optimisation is necessary to provide reliable sensitivity information. In addition the framework is embedded naturally in a cycle of modelling, optimisation and sensitivity analysis until the model is requisite, see Phillips (1984). This implies that the mathematical programmes may need to be generated and solved several times. Thus implementation of the framework described above requires care since the computational load is potentially heavy.

We have undertaken a number of implementations of the framework for three types of decision model:

- linear-in which $T(u,\pi,a)$ is linear in u and π and $\mathscr{U} \times \Gamma$ is defined by linear constraints. Such models may arise, for example, in multiattribute value methods when there is imprecision in the weights.
- bilinear-in which $T(u,\pi,a)$ is bilinear in u and π and $\mathscr{U} \times \Gamma$ is defined by linear constraints. Such models may arise, for example, in multiattribute value methods when there is imprecision in both the weights and the scores.
- general-in which $T(u,\pi,a)$ and $\mathscr{U} \times \Gamma$ have general form. Such models may arise, for example, from decision trees, influence diagrams and multiattribute utility models with imprecision in both utilities and probabilities.

An initial motivation for this work was to provide an aid to DMs' understanding of the implications of, and possible inconsistencies in, their judgements during decision conferences, see French (1992). The nature of a decision conference implies that the models developed are relatively simple but that results must be obtained in near real-time, usually on a PC. Thus our concentration has been on linear and bilinear models, which give rise to somewhat simpler mathematical programmes than do general models and hence pose less of a problem computationally. For these two cases, we have developed general purpose packages in FORTRAN 77 which allow distance analysis to be performed in the l_1, l_2 and l_∞ metrics and which require only data input from the analyst. For the general model, it is necessary to write

problem-specific software describing $T(u,\pi,a)$ and $U \times \Gamma$. Improvements to the prototype implementation described in Ríos Insua (1990) have included changes to the high-level sensitivity analysis algorithm, to the detailed algorithms used to solve the various subproblems and to the formulation of those subproblems, details are given in Proll *et al.* (1996). Global optimisation is performed by a variant of simulated annealing, Proll *et al.* (1993c). Our results suggest that a PC-based implementation is viable for realistic sized linear and bilinear discrete MCDA models.

We have also investigated a coarse-grain parallel approach, using the processor farm model, in which we exploit the independence of the problems to be solved within each phase, allowing complete mathematical programmes to be solved on a single processor, see Proll *et al.* (1993b) for fuller details. Although useful speed-up was observed, we did not achieve the linear speed-up which initial discussions with colleagues and software suppliers had led us to anticipate. This is probably due to the relatively large communication overhead in our application since the subproblems were of relatively small size, the inefficiency of the scheduling software used and the inherent sequential nature of the sensitivity analysis algorithm. Typically a two-fold speed-up on the linear model was achieved and a three-fold speed-up on the bilinear model using a network of five transputers mounted on a PC. More recent developments in both the technology and knowledge-base of parallel processing will probably allow this approach to be improved. Such improvements would particularly benefit solution of the general model.

The above discussion assumes a parametric form and explicit expressions for the expected utilities. Nonparametric problems, as mentioned above, are related to generalized moment problems.

6.2. The Continuous Case

Our implementations in the continuous case have been more case based. The basic strategy is to obtain an approximation of the nondominated set and then apply the finite framework.

For the discrete approximation there are three key steps. First, sampling from the set of alternatives. For the cases we have considered, we have always embedded the set $\mathscr{A}$ of alternatives in a subset of

$\mathbf{R}^n$ and used rejection sampling. Second, computing the nondominated set of the discrete approximation. We do this by applying the methods in the finite case. Third, the stopping rule. Besides the rule based on stability from Section 5, in some cases we know that preferences are monotonic and we may take advantage of this information. For example, we may fit, by regression, a surface to the discrete approximation of the nondominated set. Then, because of monotonicity, alternatives lying above the surface seem promising as nondominated alternatives. Therefore, an alternative rule we use is to stop when a small fraction of a new sample lies above the regression surface.

The rest of the framework leads to problems which may be viewed as generalised moment problems, for which robust Bayesian analysis provides convenient tools, see Berger (1994).

7. DISCUSSION

Sensitivity analysis in Bayesian Statistics is typically confined to checking only the impact of the probability model on the output of the analysis. Our example shows that this may be misleading when undertaking a decision analytic view, in which case we need to check sensitivity with respect to the preference model as well.

We have provided a framework to undertake this task. Essentially, it entails identifying appropriate classes for the utility and the prior; discarding bad alternatives (dominated or not potentially optimal) and identifying losses in expected utility; identifying alternatives that may share optimality with the current optimal one; identifying least changes in inputs leading to changes in ranking. The framework seems most useful in guiding the modelling process. However, it implies challenging computational problems that we have described and illustrated. We have already provided solutions in simpler cases. Solutions for the more general continuous case are on their way.

Acknowledgements

This work was partially supported by grants CICYT TIC95-0028, NATO CRG910546, NATO CRG950214 and British Council Acciones Integradas 667. We are grateful to discussions with Concha

Bielza, Peter Müller and participants at SAMO'95. The work was completed while David Ríos Insua was visiting CNR-IAMI.

References

[1] Berger, J. (1985). *Statistical Decision Theory and Bayesian Analysis*, Springer.

[2] Berger, J. (1994). An overview of robust Bayesian analysis (with discussion), *Test*, **3**, 5–124.

[3] Berger, J., Betrò, B., Moreno, E., Pericchi, L., Ruggeri, F., Salinetti, G. and Wasserman, L. (eds) (1996). *Bayesian Robustness*, IMS.

[4] Bernardo, J. M. and Smith, A. (1994). *Bayesian Theory*, Wiley.

[5] Betrò, B. and Guglielmi, A. (1996). Numerical robust Bayesian analysis under generalized moment conditions, in Berger *et al.* (eds) *Bayesian Robustness*, IMS.

[6] Diaconis, P. and Freedman, D. (1986). On the consistency of Bayes estimates, *Annals of Statistics*, **14**, 1–67.

[7] French, S. (1992). Strategic decision analysis and group decision support. In P. DeWilde and J. Vandewalle, editors, *Computer Systems and Software Engineering*, 217–230. Kluwer Academic Publishers, Dordrecht.

[8] French, S. (1995). The framing of statistical decision theory: a decision analytic view, in Bernardo, Berger, Dawid, Smith (eds) *Bayesian Statistics*, **5**, Oxford U.P.

[9] Martin, J. (1995). *Robustez Bayesiana con imprecision en preferencia y creencias*, Ph.D. Thesis, Univ. Polit. Madrid.

[10] Martín, J. and Müller, P. (1996). Sensitivity analysis in medical decision problem, *Tech. Rep.*, UPM.

[11] Martín, J. and Ríos Insua, D. (1995a). Checking dominance for classes of priors, *Tech. Rep.*, UPM.

[12] Martín, J. and Ríos Insua, D. (1995b). Sensitivity of preferences, *Tech. Rep.*, UPM.

[13] Martín, J. and Ríos Insua, D. (1996). Local sensitivity in Bayesian Decision Theory with discussion, in Berger *et al.* (eds) *Bayesian Robustness*, IMS, **29**, 121–137.

[14] Martín, J., Ríos Insua, D. and Ruggeri, F. (1996). *Bayesian Loss Robustness*, *Tech. Rep.*, UPM.

[15] McCulloch, R. (1989). Local model influence, *Journal American Statistical Association*, **84**, 473–478.

[16] Nau, R., Ríos Insua, D. and Martín, J. (1996). Incompleteness of preferences and the foundations of robust Bayesian statistics, *Tech. Rep. Fuqua School of Business*, Duke University.

[17] Palmer, J. and Müller, P. (1994). Bayesian optimal design in population models of hematologic data, *Tech. Rep., Duke University*.

[18] Phillips, L. D. (1984). A theory of requisite decision models, *Acta Psychologica*, **56**, 29–48.

[19] Proll, L. G., Ríos Insua, D. and Salhi, A. (1993a). Mathematical programming and the sensitivity of multi-criteria decisions, *Annals of Operations Research*, **43**, 109–122.

[20] Proll, L. G., Ríos Insua, D. and Salhi, A. (1993b). a parallel implementation of a mathematical programming framework for sensitivity analysis in MCDM, presented at APMOD93 *Applied Mathematical Programming and Modelling*, Budapest, January 1993.

[21] Proll, L. G., Salhi, A., Jiménez, JIM., Proll, L. G. and Ríos Insua, D. (1993c). A comparison of two stochastic approaches to global optimisation, presented at APMOD93 *Applied Mathematical Programming and Modeling*, Budapest, January 1993.

[22] Proll, L. G., Ríos Insua, D. and Salhi, A. (1996). An improved implementation of an algorithm for sensitivity analysis in MCDA *Res, Rep. School of Computer Studies*, University of Leeds.
[23] Ríos Insua, D. (1990). *Sensitivity Analysis in Multiobjective Decision Making*, Springer.
[24] Ríos Insua, D. and French, S. (1991). A framework for sensitivity analysis in discrete multiobjective decision making, *European Journal Operational Research*, **54**, 176–190.
[25] Ríos Insua, D. and Martín, J. (1994). Robustness issues under imprecise beliefs and preferences, *J. Statist. Planning and Inference*, **40**, 383–389.
[26] Ruggeri, F. and Wasserman, L. (1993). Infinitesimal sensitivity of posterior distributions, *Can. Jour. Stats.*, **21**, 195–203.