Davy Nolan CS1003 Homework III

Q1

$$\begin{pmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = A$$

matrix of cofactors $= \tilde{A}$

$$\tilde{A} = \begin{pmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} -5 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -5 & -2 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -5 & -2 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} -5 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} +((1)(1)-0) & -((2)(1)-0) & +(0-1) \\ -((-2)(1)-0) & +(-5(1)-(-2)(1)) & -(0-(-2)(1)) \\ +(0-(-2)(1)) & -(0-(-2)(2)) & +((-5)(1)-(-2)(2)) \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & -2 \\ 2 & -4 & -1 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)}\left(\tilde{A}\right)^T}$$

$$A = \begin{pmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det(A) = +(-5)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad -(-2)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \quad +(-2)\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -5((1)-(0)) \quad +2(2-0) \quad -2(0-1)$$

$$= -5 \ +4 \ +2$$

$$\det(A) = 1$$

$$(\tilde{A})^T = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & -2 \\ 2 & -4 & -1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ -1 & -2 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ -1 & -2 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ -1 & -2 & -1 \end{pmatrix}$$

Q2

$$\begin{pmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 3 \end{pmatrix} = A$$

$$A\vec{v} = \lambda \vec{v}$$

$$\det(A - \lambda I) = 0$$

$$\lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

* **Rule of Sarrus**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$+(a_{11})(a_{22})(a_{33}) + (a_{12})(a_{23})(a_{31}) + (a_{13})(a_{21})(a_{32})$$
$$-(a_{13})(a_{22})(a_{31}) - (a_{11})(a_{23})(a_{32}) - (a_{12})(a_{21})(a_{33})$$

$$\left|\begin{matrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{matrix}\right| \begin{matrix} 4-\lambda & 2 \\ 2 & 4-\lambda \\ -1 & 1 \end{matrix}$$

$$+(4-\lambda)(4-\lambda)(3-\lambda)+(2)(1)(-1)+(-1)(2)(1)$$
$$-(-1)(4-\lambda)(-1) \quad -(4-\lambda)(1)(1) \quad -(2)(2)(3-\lambda)$$

$$= -\lambda^3+11\lambda^2-40\lambda+48 -2 \quad -2 \quad +\lambda-4$$
$$+\lambda-4 \quad +4\lambda-12$$

$$= -\lambda^3+11\lambda^2-34\lambda+24$$

$$-\lambda^3+11\lambda^2-34\lambda+24=0$$
$$\lambda^3-11\lambda^2+34\lambda-24=0$$

Trial and error for values of λ using factors of -24:

$\lambda=1 \qquad (1)^3-11(1)^2+34(1)-24=0$

$$1-11+34-24=0$$
$$0=0 \quad \checkmark$$

Therefore, $\lambda=1$ is a root

$\lambda-1$ is a factor

$$
\begin{array}{r}
\lambda^2 - 10\lambda + 24 \\
\lambda - 1 \,\big)\overline{\lambda^3 - 11\lambda^2 + 34\lambda - 24} \\
\underline{-\lambda^3 \pm \lambda^2} \qquad\qquad \\
-10\lambda^2 + 34\lambda \qquad \\
\underline{\pm 10\lambda^2 \mp 10\lambda} \qquad \\
24\lambda - 24 \\
\underline{-24\lambda \pm 24} \\
0 \checkmark
\end{array}
$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda - 6)(\lambda - 4)$$

$\boxed{\lambda = 6}$ $\boxed{\lambda = 4}$ $\boxed{\lambda = 1}$ eigenvalues

Finding the eigenvectors

① Using: $\lambda = 6$

$\lambda = 6$ gives us $A - \lambda I = \begin{pmatrix} -2 & 2 & -1 \\ 2 & -2 & 1 \\ -1 & 1 & -3 \end{pmatrix}$

$$\begin{array}{c} \left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 2 & -2 & 1 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right) \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

① Multiply R1 by $-\frac{1}{2}$

$1 \quad -1 \quad \frac{1}{2} \quad 0$

$$\begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 2 & -2 & 1 & 0 \\ -1 & 1 & -3 & 0 \end{pmatrix}$$

② Add -2R1 to R2

R2: $2 \quad -2 \quad 1 \quad 0$

-2R1: $-2 \quad 2 \quad -1 \quad 0$

$0 \quad 0 \quad 0 \quad 0$

$$\begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -3 & 0 \end{pmatrix}$$

③ Add R1 to R3:

R3: $-1 \quad 1 \quad -3 \quad 0$

R1: $1 \quad -1 \quad \frac{1}{2} \quad 0$

$0 \quad 0 \quad \frac{-5}{2} \quad 0$

$$\begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-5}{2} & 0 \end{pmatrix}$$

④ Swap R2 and R3

$$\begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

⑤ Multiply R2 by $-\frac{2}{5}$

$0 \quad 0 \quad 1 \quad 0$

$$\begin{pmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_2 + \frac{1}{2}x_3 = 0$

$x_3 = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

$x = t$

$x_2 = t$

$x_3 = 0$

Eigenvector when $\lambda = 6$

$$\therefore x = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

② Using $\lambda = 4$

$$\begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right) \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

① Swap R1 and R2

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

② Multiply R1 by $\frac{1}{2}$

$1 \quad 0 \quad \frac{1}{2} \quad 0$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

③ Add R1 to R3

R3: $-1 \quad 1 \quad -1 \quad 0$

R1: $1 \quad 0 \quad \frac{1}{2} \quad 0$

$0 \quad 1 \quad -\frac{1}{2} \quad 0$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \end{pmatrix}$$

④ Multiply R2 by $\frac{1}{2}$

$0 \quad 1 \quad -\frac{1}{2} \quad 0$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \end{pmatrix}$$

~~⑤ Add R1 to R3~~

5 Add $-R2$ to $R3$:

$$\begin{array}{lcccc} R3: & 0 & 1 & -\frac{1}{2} & 0 \\ -R2: & 0 & -1 & \frac{1}{2} & 0 \\ \hline & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 + \frac{1}{2}x_3 = 0$$

$$x_2 - \frac{1}{2}x_3 = 0$$

$$x_1 = -\frac{1}{2}x_3$$
$$x_2 = \frac{1}{2}x_3$$

$$x_1 = -\frac{1}{2}t$$
$$x_2 = \frac{1}{2}t$$
$$x_3 = t$$

$$\therefore x = t\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Eigenvector when $\lambda = 4$

BN

Using $\lambda = 1$

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\begin{array}{c} \begin{pmatrix} 3 & 2 & -1 & 0 \\ 2 & 3 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{pmatrix} \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

① Add $-\frac{2}{3}R1$ to $R2$

$$\begin{array}{lcccc} R2: & 2 & 3 & 1 & 0 \\ -\frac{2}{3}R1: & -2 & -\frac{4}{3} & \frac{2}{3} & 0 \\ \hline & 0 & \frac{5}{3} & \frac{5}{3} & 0 \end{array} \qquad \begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & \frac{5}{3} & \frac{5}{3} & 0 \\ -1 & 1 & 2 & 0 \end{pmatrix}$$

② Add $\frac{1}{3}R1$ to $R3$:

$$\begin{array}{lcccc} R3: & -1 & 1 & 2 & 0 \\ \frac{1}{3}R1: & 1 & \frac{2}{3} & -\frac{2}{3} & 0 \\ \hline & 0 & \frac{5}{3} & \frac{5}{3} & 0 \end{array} \qquad \begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & \frac{5}{3} & \frac{5}{3} & 0 \\ 0 & \frac{5}{3} & \frac{5}{3} & 0 \end{pmatrix}$$

③ Add $-R2$ to $R3$:

$$\begin{array}{lcccc} R3: & 0 & \frac{5}{3} & \frac{5}{3} & 0 \\ -R2: & 0 & -\frac{5}{3} & -\frac{5}{3} & 0 \\ \hline & 0 & 0 & 0 & 0 \end{array} \qquad \begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & \frac{5}{3} & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

④ Mult. $R2$ by $\frac{3}{5}$

$$\begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(5) Add $-2R2$ to $R1$

$$\begin{array}{lcccc} R1: & 3 & 2 & -1 & 0 \\ -2R2: & 0 & -2 & -2 & 0 \\ \hline & 3 & 0 & -3 & 0 \end{array}$$

$$\begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(6) Mult. $R1$ by $\frac{1}{3}$:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$1x_1 - 1x_3 = 0$$
$$1x_2 + 1x_3 = 0$$

$$x_1 = x_3$$
$$x_2 = -x_3$$

$$x_1 = t$$
$$x_2 = -t$$
$$x_3 = t$$

$$x = t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvector when $\lambda = 1$