Faculty of Engineering, Mathematics and Science

School of Computer Science and Statistics

SF Integrated Computer Science **Sample Exam 2019**
SF CSL

MA2C03: Discrete Mathematics

DATE VENUE TIME

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Instructions to Candidates:

Credit will be given for the best 6 questions answered.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let A and B be sets, and let $f : A \to B$ be a function. Prove that the inverse f^{-1} exists if and only if f is bijective.

(8 points)

(b) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 3y).$$

Determine the following:

(i) Whether or not the relation Q is *reflexive*;

(ii) Whether or not the relation Q is *symmetric*;

(iii) Whether or not the relation Q is *anti-symmetric*;

(iv) Whether or not the relation Q is *transitive*;

(v) Whether or not the relation Q is an *equivalence relation*;

(vi) Whether or not the relation Q is a *partial order*.

Give appropriate short proofs and/or counterexamples to justify your answer.

(12 points)

(End of Question)

2. Let A consist of all 2 by 2 matrices with coefficients in the set of real numbers $\mathbb{R}$ with the operation of matrix addition.

   (a) Is $(A, +)$ a semigroup? Justify your answer.

   (5 points)

   (b) Is $(A, +)$ a monoid? Justify your answer.

   (5 points)

   (c) Is $(A, +)$ a group? Justify your answer.

   (5 points)

   (d) Is A finite, countably infinite, or uncountably infinite? Justify your answer.

   (5 points)

(End of Question)

3. Let L be the language over the alphabet $A = \{a, l, p\}$ consisting of all words containing at least one of the substrings pl or la.

   (a) Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions.

   (5 points)

   (b) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules.

   (5 points)

   (c) Write down a regular expression that gives the language L and justify your answer.

   (5 points)

   (d) Let M be the set of languages L' over the alphabet $A = \{a, l, p\}$ satisfying that $L \cap L' \neq \emptyset$. Is M finite, countably infinite or uncountably infinite? Justify your answer.

   (5 points)

   (End of Question)

4. In this question, all graphs are undirected graphs.

   (a) Let (V, E) be a graph such that $\deg v \geq 2$ for all vertices $v \in V$. Prove that (V, E) contains at least one simple circuit.

   (7 points)

   (b) Let (V, E) be the graph with vertices a, b, c, d, e, f, and g, and edges ab, ac, cd, bd, de, dg, ef and fg.

   (i) Draw this graph. Write down its incidence table and its adjacency table.

   (ii) Is this graph complete? Justify your answer.

   (iii) Is this graph bipartite? Justify your answer.

   (iv) Does this graph have an Eulerian circuit? Justify your answer.

   (v) Does this graph have a Hamiltonian trail? Justify your answer.

   (vi) Is this graph a tree? Justify your answer.

   (8 points)

   (c) Let (V, E) be the graph defined in part (b). Give an example of an isomorphism $\varphi : V \to V$ from the graph (V, E) to itself that satisfies $\varphi(b) = e$.

   (5 points)

   (End of Question)

5. (a) Consider the connected undirected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L, and with edges listed with associated costs in the following table:

JK	EF	BH	DE	AD	IJ	BL	CE	HG	FH
1	2	2	3	3	4	5	5	6	7
AB	FJ	GK	EI	EJ	CD	CF	HL	AC	BC
7	8	8	9	10	11	12	12	13	14

Draw the graph labelling each edge with its cost, then determine the minimum spanning tree of this graph generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

(11 points)

(b) How many distinct directed graphs with four vertices $V = \{a, b, c, d\}$ but no loops are there? Justify your answer.

(5 points)

(c) Prove that every complete undirected graph K_n for $n \geq 3$ has a Hamiltonian circuit.

(4 points)

(End of Question)

6. (a) Is the set of all integers divisible by 11 finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(b) Is $\{(x, y) \in \mathbb{R}^2 \mid y = e^x\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(c) Is $\{z \in \mathbb{C} \mid z^8 + 5z^6 - 3z^2 + 1 = 0\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(3 points)

(d) Let A be a finite alphabet. Prove that the set of all Turing-decidable languages over A is countably infinite.

(7 points)

(End of Question)

7. (a) Give the definition of a regular language.

(6 points)

(b) Consider the language L over the alphabet $A = \{a, l, p\}$ consisting of all words of the form $ppa^m l^{3m}$ for $m \in \mathbb{N}^*$. Use the Pumping Lemma to show the language L is not regular.

(7 points)

(c) Prove that any language generated by a regular grammar may be generated by a regular grammar in normal form.

(7 points)

8. (a) Consider the language over the binary alphabet $A = \{0, 1\}$ given by

$$L = \{0^m 1^{m+1} \mid m \in \mathbb{N},\ m \geq 0\}.$$

Write down the algorithm of a Turing machine that recognizes L. Process the following strings according to your algorithm: 1, 01, 011, and 010.

(8 points)

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

(6 points)

(c) Give an example of a language that fails to be Turing-recognizable or provide a proof that not all languages are Turing-recognizable.

(6 points)

(End of Question)