CSU33081 Exam Paper 2020 Answer Explanations

Student Name: Davy Nolan
Student Number: 17330208

Question 1
Represent the summation of the following equations in MATLAB:

$$2x^2 + 2x - 6$$

And

$$x^3 + 2x - 4$$

Rewrite equations in same form:

$$0x^3 + 2x^2 + 2x - 6$$
$$x^3 + 0x^2 + 2x - 4$$

Take the coefficients in that order and enter them into vectors:

$$[0\ 2\ 2\ -6] + [1\ 0\ 2\ -4]$$

$\therefore Answer = C$

Question 2

Taking the MATLAB commands line by line:

A=eye(3,3);

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For x=1:2:3

$$for(int\ x = 1; x \leq 3; x = x + 2)$$

Iteration 1:

x=1;

A(1,1) = 1;

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Iteration 2:

x=3;

A(1,3)=1;

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore Answer = B$

Question 3

Taking the MATLAB commands line by line:

x=[6:8;-1:1;5 6 7];

$$x = \begin{pmatrix} 6 & 7 & 8 \\ -1 & 0 & 1 \\ 5 & 6 & 7 \end{pmatrix}$$

y=x(:,3);

$$x = \begin{pmatrix} 6 & 7 & 8 \\ -1 & 0 & 1 \\ 5 & 6 & 7 \end{pmatrix} \rightarrow y = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$$

size(y')

$$\begin{array}{cc} & 1 \\ y = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix} & 3 \end{array} \rightarrow 1 \times 3$$

$\therefore Answer = C$

Question 4

Calculate the Truncation Error, $f(x) - P_2(x)$ at $x = 2.5$

$$f(x) = 3 - 17x^3$$

Taylor Series Formula:

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$
$$+ \frac{f^n(x_0)}{n!}(x - x_0)^n$$

Using the Taylor Series Polynomial approximation of degree two, $P_2(x)$ expanded about the point $x_0 = 2.0$:

$$T_2(x) = f(2.0) + f'(2.0)(x - 2.0) + \frac{f''(2.0)}{2!}(x - 2.0)^2$$

$$f(x) = 3 - 17x^3$$
$$f'(x) = -51x^2$$
$$f''(x) = -102x$$

$$T_2(x) = (3 - 17(2.0)) + (-51(2.0)^2)(x - 2.0) + \frac{(-102(2.0))}{2!}(x - 2.0)^2$$
$$= -133 - 204x + 408 - 102(x^2 - 4x + 4)$$
$$= 275 - 204x - 102x^2 + 408x - 408$$
$$= -102x^2 + 204x - 133$$

Calculating Truncation Error:

$$f(x) - P_2(x) \text{ at } x = 2.5$$

$$f(2.5) - P_2(2.5)$$
$$= (3 - (172.5)^3) - (-102(2.5)^2 + 204(2.5) - 133)$$
$$= -262.625 + 260.5$$
$$= -2.125$$

$\therefore Answer = E \rightarrow\ -2.125$

Question 5

Use the Secant Method to find a root of the function:

$$f(x) = 16x^5 - 73x^2 - 133$$

Accurate to within an error of $\epsilon = x_n - x_{x-1} = 0.001$ where x_n is the value of x at the n^{th} iteration.

Starting points:

$$x_0 = 3,\ x_1 = 2.5$$

Secant Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{\left(\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right)}$$

$$x_{n+1} = x_n - \frac{16{x_n}^5 - 73{x_n}^2 - 133}{\left(\frac{(16{x_n}^5 - 73{x_n}^2 - 133) - (16{x_{n-1}}^5 - 73{x_{n-1}}^2 - 133)}{x_n - x_{n-1}}\right)}$$

Iteration 1

$$x_0 = 3,\ x_1 = 2.5$$

$$x_2 = x_1 - \frac{16{x_1}^5 - 73{x_1}^2 - 133}{\left(\frac{(16{x_1}^5 - 73{x_1}^2 - 133) - (16{x_0}^5 - 73{x_0}^2 - 133)}{x_1 - x_0}\right)}$$

$$x_2 = 2.5 - \frac{16(2.5)^5 - 73(2.5)^2 - 133}{\left(\frac{(16(2.5)^5 - 73(2.5) - 133) - (16(3)^5 - 73(3)^2 - 133)}{2.5 - 3}\right)}$$

$$x_2 = 2.5 - \frac{973.25}{\left(\frac{973.25 - 3098}{-0.5}\right)}$$

$$x_2 = 2.27097$$

Iteration 2

$$x_1 = 2.5,\ x_2 = 2.27097$$

$$x_3 = x_2 - \frac{16{x_2}^5 - 73{x_2}^2 - 133}{\left(\frac{(16{x_2}^5 - 73{x_2}^2 - 133) - (16{x_1}^5 - 73{x_1}^2 - 133)}{x_2 - x_1}\right)}$$

$$x_3 = 2.27097 - \frac{16(2.27097)^5 - 73(2.27097)^2 - 133}{\left(\frac{(16(2.27097) - 73(2.27097) - 133) - (16(2.5)^5 - 73(2.5)^2 - 133)}{2.27097 - 2.5}\right)}$$

$$x_3 = 2.27097 - \frac{456.96685}{\left(\frac{456.96685 - 973.25}{-22903}\right)}$$

$$x_3 = 2.27097 - 0.20271$$

$$x_3 = 2.06826$$

Iteration 3

$$x_2 = 2.27097,\ x_3 = 2.06826$$

$$x_4 = x_3 - \frac{16{x_3}^5 - 73{x_3}^2 - 133}{\left(\frac{(16{x_3}^5 - 73{x_3}^2 - 133) - (16{x_2}^5 - 73{x_2}^2 - 133)}{x_3 - x_2}\right)}$$

$$x_4 = 2.06826 - \frac{16(2.06826)^5 - 73(2.06826)^2 - 133}{\left(\frac{(16(2.06826) - 73(2.06826) - 133) - (16(2.27097)^5 - 73(2.27097)^2 - 133)}{2.06826 - 2.27097}\right)}$$

$$x_4 = 2.06826 - \frac{160.27099}{\left(\frac{160.27099 - 456.96685}{-0.20271}\right)}$$

$$x_4 = 2.06826 - 0.10950$$

$$x_4 = 1.958756$$

Iteration 4

$$x_3 = 2.06826,\ x_4 = 1.958756$$

$$x_5 = x_4 - \frac{16{x_4}^5 - 73{x_4}^2 - 133}{\left(\frac{(16{x_4}^5 - 73{x_4}^2 - 133) - (16{x_3}^5 - 73{x_3}^2 - 133)}{x_4 - x_3}\right)}$$

$$x_5 = 1.958756$$

$$- \frac{16(1.958756)^5 - 73(1.958756)^2 - 133}{\left(\frac{(16(1.958756) - 73(1.958756) - 133) - (16(2.06826)^5 - 73(2.06826)^2 - 133)}{1.958756 - 2.06826}\right)}$$

$$x_5 = 1.958756 - \frac{48.25973}{\left(\frac{48.25973 - 160.27099}{-0.10950}\right)}$$

$$x_5 = 1.958756 - 0.04718$$

$$x_5 = 1.911576$$

Iteration 5

$$x_4 = 1.958756, \; x_5 = 1.911576$$

$$x_6 = x_5 - \frac{16{x_5}^5 - 73{x_5}^2 - 133}{\left(\frac{(16{x_5}^5 - 73{x_5}^2 - 133) - (16{x_4}^5 - 73{x_4}^2 - 133)}{x_5 - x_4}\right)}$$

$$x_6 = 1.911576 - \frac{16(1.911576)^5 - 73(1.911576)^2 - 133}{\left(\frac{(16(1.911576) - 73(1.911576) - 133) - (16(1.958756)^5 - 73(1.958756)^2 - 133)}{1.911576 - 1.958756}\right)}$$

$$x_6 = 1.911576 - \frac{8.64233}{\left(\frac{8.64233 - 48.25973}{-0.04718}\right)}$$

$$x_6 = 1.911576 - 0.010292$$

$$x_6 = 1.901285$$

Iteration 6

$$x_5 = 1.911576, \; x_6 = 1.901285$$

$$x_7 = x_6 - \frac{16{x_6}^5 - 73{x_6}^2 - 133}{\left(\frac{(16{x_6}^5 - 73{x_6}^2 - 133) - (16{x_5}^5 - 73{x_5}^2 - 133)}{x_6 - x_5}\right)}$$

$$x_7 = 1.901285 - \frac{16(1.901285)^5 - 73(1.901285)^2 - 133}{\left(\frac{(16(1.901285) - 73(1.901285) - 133) - (16(1.911576)^5 - 73(1.911576)^2 - 133)}{1.901285 - 1.911576}\right)}$$

$$x_7 = 1.901285 - \frac{0.630803}{\left(\frac{0.630803 - 8.64233}{-0.010291}\right)}$$

$$x_7 = 1.901285 - 0.00081$$

$$x_7 = 1.900475$$

Check:

$$\epsilon = x_n - x_{x-1}$$
$$\epsilon = 1.900475 - 1.901285$$
$$\epsilon = -0.00081$$

Which is within

$$\epsilon = 0.001$$

$\therefore Answer = C$

Question 6

Find the upper triangular matrix [U] in the [L][U] decomposition of the matrix:

$$A = \begin{pmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{pmatrix} = [L][U]$$

Lower and Upper triangular matrices have the forms:

$$[L] = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \qquad [U] = \begin{pmatrix} U_{11} & U_{11} & U_{11} \\ 0 & U_{11} & U_{11} \\ 0 & 0 & U_{11} \end{pmatrix}$$

Where:

$$A = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{pmatrix} = \begin{pmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{pmatrix}$$

Row 1

$$U_{11} = 25,\ U_{12} = 5,\ U_{13} = 4$$

Row 2

$$L_{21}U_{11} = 10$$
$$L_{21}(25) = 10$$
$$L_{21} = 0.4$$

$$L_{21}U_{12} + U_{22} = 8$$
$$(0.4)(5) + U_{22} = 8$$
$$U_{22} = 6$$

$$L_{21}U_{13} + U_{23} = 16$$
$$(0.4)(4) + U_{23} = 16$$
$$U_{23} = 14.4$$

Row 3

$$L_{31}U_{11} = 8$$
$$L_{31}(25) = 8$$
$$L_{31} = 0.32$$

$$L_{31}U_{12} + L_{32}U_{22} = 12$$
$$(0.32)(5) + L_{32}(6) = 12$$
$$L_{32} = 1.733333$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 22$$
$$(0.32)(4) + (1.733333)(14.4) + U_{33} = 22$$
$$U_{33} = -4.24$$

The Upper triangular matrix can now be constructed:

$$[U] = \begin{pmatrix} U_{11} & U_{11} & U_{11} \\ 0 & U_{11} & U_{11} \\ 0 & 0 & U_{11} \end{pmatrix} = \begin{pmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.4 \\ 0 & 0 & -4.24 \end{pmatrix}$$

$\therefore Answer = C$

Question 7

Using $x_1 = 1,\ x_2 = 3,\ x_3 = 5$as an initial guess at the solution, determine the values of x_1, x_2 and x_3 that result from three iterations of the Gauss-Seidel method applied to this matrix equation:

$$\begin{pmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$$

Creating algebraic equations from the matrix equations:

1. $12x_1 + 7x_2 + 3x_3 = 2$
2. $x_1 + 5x_2 + x_3 = -5$
3. $2x_1 + 7x_2 - 11x_3 = 6$

Iteration 1

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12} = \frac{2 - 7(3) - 3(5)}{12} = -2.833333$$

$$x_2 = \frac{-5 - x_1 - x_3}{5} = \frac{-5 - (-2.83333) - (5)}{5} = -1.43333$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11} = \frac{6 - 2(-2.833333) - 7(-1.43333)}{-11} = -1.9727$$

Iteration 2

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12} = \frac{2 - 7(-1.43333) - 3(-1.9727)}{12} = 1.4959$$

$$x_2 = \frac{-5 - x_1 - x_3}{5} = \frac{-5 - (1.4959) - (-1.9727)}{5} = -0.90464$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11} = \frac{6 - 2(1.4959) - 7(-0.90464)}{-11} = -0.84914$$

Iteration 3

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12} = \frac{2 - 7(-0.90464) - 3(-0.84914)}{12} = 0.90666$$

$$x_2 = \frac{-5 - x_1 - x_3}{5} = \frac{-5 - (0.90666) - (-0.84914)}{5} = -1.0115$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11} = \frac{6 - 2(0.90666) - 7(-1.0115)}{-11} = -1.0243$$

$\therefore Answer = C$

Question 8

Calculate the dominant eigenvalue and an associated eigenvector using the Power Method for the following matrix. Perform four iterations beginning with an initial estimate of:

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}$$

Iteration 1

$$Ax_0 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$$

$$n_0 = 11$$

$$x_1 = \begin{bmatrix} \frac{9}{11} \\ \frac{11}{11} \end{bmatrix} = \begin{pmatrix} 0.8182 \\ 1 \end{pmatrix}$$

Iteration 2

$$Ax_1 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 0.8182 \\ 1 \end{pmatrix} = \begin{pmatrix} 8.2727 \\ 9.9091 \end{pmatrix}$$

$$n_1 = 9.9091$$

$$x_2 = \begin{pmatrix} \frac{8.2727}{9.9091} \\ \frac{9.9091}{9.9091} \end{pmatrix} = \begin{pmatrix} 0.8349 \\ 1 \end{pmatrix}$$

Iteration 3

$$Ax_2 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0.8349 \\ 1 \end{pmatrix} = \begin{pmatrix} 8.3394 \\ 10.0092 \end{pmatrix}$$

$$n_2 = 10.0092$$

$$x_3 = \begin{pmatrix} \frac{8.2727}{10.0092} \\ \frac{10.0092}{10.0092} \end{pmatrix} = \begin{pmatrix} 0.8332 \\ 1 \end{pmatrix}$$

Iteration 4

$$Ax_3 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0.8332 \\ 1 \end{pmatrix} = \begin{pmatrix} 8.3327 \\ 9.999 \end{pmatrix}$$

$$n_3 = 9.999$$

$$x_4 = \begin{pmatrix} \frac{8.3327}{9.999} \\ \frac{9.999}{9.999} \end{pmatrix} = \begin{pmatrix} 0.8333 \\ 1 \end{pmatrix}$$

$$\therefore Answer = C$$

Question 9

For the function :

$$f(x) = x^2 log_2(x)$$

And the points:

$$x_0 = 2,\ x_1 = 3,\ x_2 = 7$$

Calculate Newton's second divided difference $f[x_2, x_1, x_0]$

$$f[x_2, x_1, x_0] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{f(3) - f(2)}{3 - 2}$$

$$= \frac{14.264663 - 4}{1}$$

$$f[x_0, x_1] = 10.264663$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(7) - f(3)}{7 - 3}$$

$$= \frac{137.560391 - 14.264663}{4}$$

$$f[x_1, x_2] = 30.823932$$

$$\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{30.823932 - 10.264663}{7 - 2}$$

$$= 4.111854$$

$\therefore Answer = D$

Question 10

Evaluate the following integral using three-point Gaussian Quadrature:

$$\int_{0}^{2\pi} \frac{1}{2+cosx}\,dx$$

Three point rule:

$$\int_{a}^{b} f(x)\,dx \cong c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

Where:

$$\boxed{\begin{array}{c} c_1 = 0.5556,\ c_2 = 0.8889,\ c_3 = 0.5556 \\ \\ x_1 = -0.7746,\ x_2 = 0,\ x_3 = 0.7746 \end{array}}$$

$$\int_{a}^{b} f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\,dx$$

The following was calculated in radians:

$$\int_{0}^{2\pi} \frac{1}{2+cosx}\,dx \cong \frac{2\pi-0}{2}\int_{0}^{2\pi} \frac{1}{2+cosx}\,dx$$

$$\cong \pi(0.5556\left(\frac{1}{2+\cos\left(\frac{2\pi-0}{2}(-0.7746)+\frac{2\pi+0}{2}\right)}\right)$$
$$+ 0.8889\left(\frac{1}{2+\cos\left(\frac{2\pi-0}{2}(0)+\frac{2\pi+0}{2}\right)}\right)$$
$$+ 0.5556\left(\frac{1}{2+\cos\left(\frac{2\pi-0}{2}(0.7746)+\frac{2\pi+0}{2}\right)}\right)) \cong 4.05745$$

$\therefore Answer = A$