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  Sorites : heap (minus one grain)

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- computability
  Turing: Halting Problem

Tolerance and Sorites chains

A unary relation P is *tolerant up to near$_P$* if

$$P(x) \ \text{ whenever } \mathit{near}_P(x,y) \text{ and } P(y).$$

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Example 3. $P(x)$ is $\textit{young}(x)$, $\textit{sunny}(x)$,
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A unary relation P is *tolerant up to* $near_P$ if

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A *Sorites chain* is a sequence $y_1, \ldots, y_n$ such that P holds of y_1 but not y_n, even though $near_P(y_i, y_{i+1})$ for $1 \leq i < n$.

The Halting Problem

Given a program P and data D, return either 0 or 1 (as output), with 1 indicating that P halts on input D

$$\mathrm{HP}(P, D) \quad := \quad \begin{cases} 1 & \text{if } P \text{ halts on } D \\ 0 & \text{otherwise} \end{cases}$$

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Theorem (Turing) *No TM computes HP.*

The proof is similar to the Liar's Paradox distributed as follows

H: 'L speaks the truth'
L: 'H lies'

with a spoiler L (exposing H as a fraud).

Proof of uncomputability

Given a TM P that takes two arguments, we show P does not compute HP by defining a TM $\overline{P}$ such that

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Let

$$\overline{P}(D) \quad :\simeq \quad \begin{cases} 1 & \text{if } P(D, D) = 0 \\ \text{loop} & \text{otherwise.} \end{cases}$$

and notice

$$\begin{aligned} \mathrm{HP}(\overline{P}, \overline{P}) \quad &= \quad \begin{cases} 1 & \text{if } \overline{P} \text{ halts on } \overline{P} \\ 0 & \text{otherwise} \end{cases} && \text{(def of HP)} \\ &= \quad \begin{cases} 1 & \text{if } P(\overline{P}, \overline{P}) = 0 \\ 0 & \text{otherwise} \end{cases} && \text{(def of } \overline{P}\text{)} \end{aligned}$$

$\square$

Semi-solvability of HP

There is a TM that meets the positive part of HP (looping exactly when HP asks for 0), in view of the existence of a

Universal Turing Machine: a TM U that runs P on D

$$U(P, D) \quad \simeq \quad P(D)$$

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A Turing machine can be quite complex.
We begin with some simple instances for which halting is no problem — finite state machines (viz., Turing machines that cannot move left or write, and terminate as soon as they've read the entire input).