# Terminology

Fuzzy sets are sets whose elements have degrees of membership of the sets. They are an extension of the classical set.

Membership of a set governed by classical set theory is described accordin to a bivalent condition.

## Membership Function

Consider the *tallness* membership function. In the Western world, men who are 1.9m tall is regarded as definitely *tall*. Men of height 1.2m are regarded as *not-tall*. Men of heigher around 1.5m to 1.7m are regarded of *medium* height.

$$tall(h) = 1 \text{ if } h \geq 1.9$$ $$tall(h) = 0 \text{ if } h \leq 1.2$$

## Fuzzy System

A fuzzy system can be contrasted with a conventional system in three main ways:

1. A linguistic variable is defined as a variable whose values are sentences in a natural or artificial language
2. Fuzzy control statements are expressions in the form of `if A then B` where A and B have fuzzy meaning
3. A fuzzy algorithm is an ordered sequence of instructions which may contain fuzzy assignment and conditional statements
    - The execution of such instructions is governed by the compositional rule of inference and the rule of the preponderant alternative

## Fuzzy Restriction

A fuzzy relation which acts as an elastic constraint on the values that may be assigned to a variable.

Calculus of Fuzzy Restrictions is essentially a body of concepts and techniques for dealing with fuzzy restrictions in a systematic way: to furnish a conceptual basis for approximate reasoning - neither exact now inexact reasoning.

# Properties

Elements of a fuzzy set may belong to the set, may not belong to the set, or may belong to a degree. Their membership is *elastic*.

Membership of a crisp set is described by a bivalent condition; the membership of a fuzzy set is described by a multi-valent condition.

# Fuzzy Sets

| 'Large Number' | Comment | Degree of Membership |
|---:|---|---:|
| 10 | Surely | 1 |
| 9 | Surely | 1 |
| 8 | Quite possibly | 0.8 |
| 7 | Maybe | 0.5 |
| 6 | Sometimes | 0.2 |
| 5, 4, 3, 2, 1 | Definitely not | 0 |

We can denote the notion of "large number" by the fuzzy set $$A= \frac{0}{1} + \frac{0}{2} + \frac{0}{3} + \frac{0}{4} + \frac{0}{5} + \frac{0.2}{6} + \frac{0.5}{7} + \frac{0.8}{8} + \frac{1}{9} + \frac{1}{10}$$

**Membership Functions:**
For the sake of convenient, a fuzzy set is usually denoted as $$A = \frac{\mu_{A}(x_{i})}{x_{i}} + \dots + \frac{\mu_{A}(x_{n})}{x_{n}}$$ that belongs to a finite universe of discourse: $$A \underset{\sim}{\subset} {x_{1}, x_{2}, \dots, x_{n} } \equiv X$$ where $\mu_{A} \frac{(x_{i})}{x_{i}}$ (a singleton) is a pair "grade of membership element".

| 'Large Number' | $\mu(.)$ |
|---:|---|
| 10 | $\mu_{A}(10)=1$ |
| 9 | $\mu_{A}(9)=1$ |
| 8 | $\mu_{A}(8)=0.8$ |
| 7 | $\mu_{A}(7)=0.5$ |
| 6 | $\mu_{A}(6)=0.2$ |
| 5, 4, 3, 2, 1 | $\mu_{A}(5)=\mu_{A}(4)=\mu_{A}(3)=\mu_{A}(2)=\mu_{A}(1)=0$ |

## Fuzzy Subsets

We can define from this

- "Small number": $\mu_{B}(.) = 1-\mu_{A}(.)$
- "Very Large number": $\mu_{C} = \mu_{A}(.) \times \mu_{A}(.)$
- "Largish number": $\mu_{D} = \sqrt{\mu_{A}(.)}$

**Membership Functions:**
Let $X={x}$ be a universe of discourse, i.e. a set of all possible (feasible or relevant) elements with regard to a fuzzy (vague) concept (property). Then $$A \underset{\sim}{\subset} X(A \text{ of } X)$$ denotes a fuzzy subset, or loosely fuzzy set, a set of ordered pairs ${(x, \mu_{A}(x))}$ where $X \in x$.

- $\mu_{A}: X \rightarrow [0, 1]$ the membership function of A
- $\mu_{A}(x) \in [0, 1]$ the grade of membership of x in A

Many authors denote $$\mu_{A}(x) \equiv A(x)$$

A fuzzy set is often denoted by its membership function. If $[0, 1]$ is replaced by ${0, 1}$: This definition coincides with the characteristic function based on the definition of an ordinary, i.e. non-fuzzy set.

## Properties

| Properties | Definition |
| --- | --- |
| $P_1$ | Equality of two fuzzy sets |
| $P_2$ | Inclusion of one set into another fuzzy set |
| $P_3$ | Cardinality of a fuzzy set |
| $P_4$ | An empty fuzzy set |
| $P_5$ | $\alpha$-cuts |

More formally
Let $X$ be some universe of discourse
Let $S$ be a subset of $X$

Then, we define a characteristic function or membership function, $\mu$. $$\mu_{S}: X \rightarrow {0, 1}$$ Such that for any element $x \in X$.

If $\mu_{s} (x) = 1$ then $x$ is a member of the set $S$
If $\mu_{s} (x) = 0$ then $x$ is not a member of the set $S$

> Remember the curly brackets are used to refer to binary value.

For a fuzzy subset $(A)$ we use square brackets to indicate the existence of a *unit interval*. For each $x$ in the universe of discourse $X$, the function $\mu_{A}$ is associated with the fuzzy subset $A$. $$\mu_{A}: X \rightarrow [0, 1]$$ $\mu_{A}(x)$ indicates to the degree to which $x$ belongs to the fuzzy subset $A$.

A fuzzy subset of $X$ is called *normal* if there exists at least one element $x \in X$ such that $\mu_{A}(x) = 1$. A fuzzy subset that is not normal is called subnormal. All crisp subsets except for the null set are normal. In fuzzy set theory, the concept of nullness essentially generalises to subnormality.

The *height* of a fuzzy subset $A$ is the largest membership grade of an element in $A$ $$height(A) = \max_{x}(\mu_{A}(x))$$

Assume $A$ is a fuzzy subset of $X$;

- The *support* of $A$ is the crisp subset of $X$ whose elements all have non-zero membership grades in $A$: $$supp (A) = { x \mid \mu_{A}(x) > 0 \land x \in X }$$
- The *core* is the crisp subset of $X$ consisting of all elements with membership grade 1: $$core (A)

= { x \mid \mu_{A}(x) = 1 \land x \in X }$$

A normal fuzzy subset has a non-null core while a subnormal fuzzy subset has a null core.

Consider two fuzzy subsets of the set $X = { a, b, c, d, e }$ referred to as $A = { \frac{1}{a}, \frac{0.3}{b}, \frac{0.2}{c}, \frac{0.8}{d}, \frac{0}{e} }$ and $B = { \frac{0.6}{a}, \frac{0.9}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0.2}{e} }$

$A$:

- Is a normal fuzzy set (Element $a$ has unit membership)
- $height(A) = 1$
- $supp(A) = {a, b, c, d }$
- $core(A) = {a}$
- $card(A) = 1+0.3+0.2+0.8+0=2.3$

$B$:

- Is a subnormal fuzzy set
- $height(B) = 0.9$
- $supp(B) = {a, b, c, d, e}$
- $core(B) = \emptyset$
- $card(B) = 0.9+0.6+0.1+0.3+0.2=2.1$

# Operations

| Operations | Definition |
| --- | --- |
| $O_1$ | Complementation |
| $O_2$ | Intersection |
| $O_3$ | Union |
| $O_4$ | Bounded Sum |
| $O_5$ | Bounded Difference |
| $O_6$ | Concentration |
| $O_7$ | Dilation |

## Union \& Intersection

The union of fuzzy subsets $A$ and $B$ of the set $X$ is denoted as the fuzzy subset $C$ of $X$. $$C = A \cup B \text{ such that for each } x \in X, \mu_{C}(x)=\max [\mu_{A}(x), \mu_{B}(x)]$$

The intersection of the fuzzy subsets $A$ and $B$ is denoted as the fuzzy subset $D$ of $X$. $$D = A \cap B \text{ such that for each } x \in X, \mu_{D}(x)=\min [\mu_{A}(x), \mu_{B}(x)]$$

Min and max play a fundamental role in fuzzy set theory and are usually computed from the following formulae: $$\max (a, b) = \frac{a+b+\mid a - b \mid}{2}$$ $$\min (a, b) = \frac{a+b-\mid a-b \mid}{2}$$

Recall $A = { \frac{1}{a}, \frac{0.3}{b}, \frac{0.2}{c}, \frac{0.8}{d}, \frac{0}{e} }$ and $B = { \frac{0.6}{a}, \frac{0.9}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0.2}{e} }$. The union of $A$ and $B$ is $$A \cup B = { \frac{1}{a}, \frac{0.9}{b}, \frac{0.2}{c}, \frac{0.8}{d}, \frac{0.2}{e} }$$ and the intersection is $$A \cap B = { \frac{0.6}{a}, \frac{0.3}{b}, \frac{0.1}{c}, \frac{0.3}{d}, \frac{0}{e} }$$

## Bounded Sum \& Bounded Difference

The bounded sum of fuzzy subsets $A$ and $B$ of the set $X$ is denoted as the fuzzy subset $C$ of $X$. $$C = A+B \text{ such that for each } x \in X, \mu_{C}(x) = \min [1, (\mu_{A}(x) + \mu_{B}(x))]$$

The bounded difference of the fuzzy subsets $A$ and $B$ is denoted as the fuzzy subset $D$ of $X$. $$D = A-B \text{ such that for each } x \in X, \mu_{D}(x) = \max [0, (\mu_{A}(x)+\mu_{B}(x)-1)]$$

## Complement

The complement of a fuzzy subset $A$ of $X$ is denoted by $\bar{A} = X - A$ and the membership function of the complement is given by $$\mu_{\bar{A}}(x) = 1 - \mu_{A}(x)$$

Generally the intersection of a fuzzy subset and its complement is not the null set. $$E = \bar{A} \cap A \neq \emptyset$$ The distinction between a fuzzy set and its complement, especially when compared with the distinction between a crisp set and its complement, is not as clear cut.

## Concentration \& Dilation

If $A$ is a fuzzy subset of $X$ and $\alpha$ is any non-negative number, then $A^{\alpha}$ is the fuzzy subset $B$ such that $$\mu_{B}(x) = (\mu_{A}(x))^{\alpha}$$

$A = { \frac{1}{a}, \frac{0.6}{b}, \frac{0.3}{c}, \frac{0}{d}, \frac{0.5}{e} }$

$A^{2} = { \frac{1}{a}, \frac{0.36}{b}, \frac{0.09}{c}, \frac{0}{d}, \frac{0.25}{e} }$

**To concentrate**: to contract, condense; to intensify.
If $\alpha > 1$ then $A^{\alpha} \subset A \rightarrow \text{decreases membership}$

**To dilate**: to expand, amplify, enlarge.
If $\alpha < 1$ then $A^{\alpha} \supset A \rightarrow \text{increases membership}$

> If $A$ is a crisp subset and $a>0$ then $A^{\alpha} = A$

#### Level Set

If $A$ is a fuzzy subset of $X$ and $0 \leq \alpha \leq 1$ then we can define another fuzzy subset $F$ such that $$F = \alpha A;\hspace{0.5cm} \mu_{F}(x) = \alpha \mu_{A}(x) \hspace{0.5cm} x \in X$$

The $\alpha$-level set of the fuzzy subset $A$ (of $X$) is the *crisp* subset of $X$ consisting of all the elements in $X$ such that $$A_{\alpha} = { x \mid \mu_{A}(x) \geq \alpha, x \in X}$$

Example:
$A = { \frac{1}{a}, \frac{0.3}{b}, \frac{0.2}{c}, \frac{0.8}{d}, \frac{0}{e} }$
$A_{\alpha} = {a, b, d}, \hspace{1cm} 0.2 < \alpha \leq 0.3$

# Triangular norms and conorms

A triangular norm (t-norm) and conorm (t-conorm or s-norm) are a binary operations $T$ or $S$ on the interval $[0, 1]$ satisfying the following conditions:

- Commutivity: $T(x, y) = T(y, x)$
- Associativity: $T(x, T(y, z)) = T(T(x, y), z)$
- Monotonicity: $y \leq z \rightarrow T(x, y) \leq T(x, z)$
- Identity element 1: $T(x, 1) = x$

Operations of T-norms:

- $T_{min}(x, y) = \min{(x, y)} = x \cap y$
- $T_{Algebraic Product}(x, y) = x \times y$
- $T_{Bounded Difference}(x, y) = \max(0, x+y-1) = 0 \cup (x+y-1)$

Operations of S-norm:

- $S_{max}(x, y) = \max{(x, y)} = x \cup y$
- $S_{Algebraic Sum}(x, y) = x+y - x \times y$
- $S_{Bounded Sum}(x, y) = \min(1, x+y) = 1 \cap x+y$

T-norm specifies the intersection/conjunction of a fuzzy subset, where the following inequalty holds:
$$T_{BD} \leq T_{AP} \leq T_{min}$$

S-norm specifies the union/disjunction of a fuzzy subset, where the following inequality holds:
$$S_{max} \leq S_{AS} \leq S_{BS}$$

# Membership Functions

## Triangular

$$trimf(x; a, b, c) = \max{(\min{(\frac{x-a}{b-a}, \frac{c-x}{c-b})}, 0)}$$

## Trapezoidal

$$trapmf(x; a, b, c, d) = \max{(\min{(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c})}, 0)}$$

## Gaussian

$$gaussmf(x; a, b, c) = e^{-\frac{1}{2}(\frac{x-c}{\sigma})^{2}}$$

## Generalised Bell

$$gbellmf(x; a, b, c) = \frac{1}{1+ \left\vert \frac{x-c}{b} \right\vert^{2b}}$$

# Fuzzy Relationships

The cartesian or cross product of fuzzy subsets $A$ and $B$, of sets $X$ and $Y$ respectively is denoted as $A \times B$. This cross product relationships $T$ on the set $X \times Y$ is denoted as $T = A \times B$. $$\mu_{T} (x, y) = \min [(\mu_{A} (x), \mu_{B}(y))]$$

$A = { \frac{1}{a_{1}}, \frac{0.6}{a_{2}} }$
$B = { \frac{0.6}{b_{1}}, \frac{0.9}{b_{2}} }$
$A \times B = { \frac{0.6}{(a_{1}, b_{1})}, \frac{0.9}{(a_{1}, b_{2})}, \frac{0.6}{(a_{2}, b_{1})}, \frac{0.6}{(a_{2}, b_{2})} }$

More generally, if $A_{1}, A_{2}, \dots, A_{n}$ are fuzzy subsets of $X_{1}, X_{2}, \dots, X_{n}$, then their cross product $$A_{1} \times A_{2} \times \dots \times A_{n}$$ is a fuzzy subset of $$X_{1} \times X_{2} \times \dots \times X_{n}$$ and $$\mu_{T}(x_{1}, x_{2}, \dots, x_{n} = \underset{i}{\min} [\mu_{A_{i}}(x_{i})]$$

## Example

Electric motors are a examples of good control systems that run on simple heuristics relating to the speed of the rotor in the motor: change the strength of the magnetric field to adjust the speed at which the rotor is moving.

- If the motor is too slow, speed it up
- If the motor speed is right, don't change it
- If the motor is too fast, slow it down.

In order to change speed, an operator of a control plant will have to apply more or less voltage: there are three reference fuzzy sets representing the linguistic values:

- `increase voltage(speed up)`
- `no change(do nothing)`
- `decrease voltage(slow down)`

Consider an air conditioner which has five control switches: `cold`, `cool`, `pleasant`, `warm`, `hot`. The corresponding speeds of the motor controlling the air conditioner has the graduations: `minimal`, `slow`, `medium`, `fast`, `blast`.

The rules governing the air conditioner are as follows:

1. If *temperature* is `cold` then *speed* is `minimal`
2. If *temperature* is `cool` then *speed* is `slow`
3. If *temperature* is `pleasant` then *speed* is `medium`
4. If *temperature* is `warm` then *speed* is `fast`
5. If *temperature* is `hot` then *speed* is `blast`

The rules can be expressed as the cross product $$\text{control} = \text{temperature} \times \text{speed}$$ where $$\text{temperature} = {cold, cool, pleasant, warm, hot}$$ $$\text{speed} = {minimal, slow, medium, fast, blast}$$

$$\mu_{control} (T, V) = \min [\mu_{temp}(T), \mu_{speed}(V)]$$

| Temp | Cold | Cool | Pleasant | Warm | Hot |
|---|---|---|---|---|---|
| 0 | Y* | N | N | N | N |
| 5 | Y | Y | N | N | N |
| 10 | N | Y | N | N | N |
| 12.5 | N | Y* | N | N | N |
| 15 | N | Y | N | N | N |
| 17.5 | N | N | Y* | N | N |
| 20 | N | N | N | Y | N |
| 22.5 | N | N | N | Y* | N |
| 25 | N | N | N | Y | N |
| 27.5 | N | N | N | N | Y |
| 30 | N | N | N | N | Y* |

| RPM | Minimal | Slow | Medium | Fast | Blast |
|---|---|---|---|---|---|
| 0 | Y* | N | N | N | N |
| 10 | Y | N | N | N | N |
| 20 | Y | Y | N | N | N |
| 30 | N | Y* | N | N | N |
| 40 | N | Y | N | N | N |
| 50 | N | N | Y* | N | N |
| 60 | N | N | N | Y | N |
| 70 | N | N | N | Y* | N |
| 80 | N | N | N | Y | Y |
| 90 | N | N | N | N | Y |
| 100 | N | N | N | N | Y* |

These can be expressed using the triangular membership function.

# Fuzzy Patch

A fuzzy patch is defined by a fuzzy rule: a patch is a mapping of two membership functions, it is a product of two geometrical objects, line segments, triangles, squares, etc.

In a fuzzy controller, a rule in the rule set of the controller can be visualised as a 'device' for generating the product of the input/output fuzzy sets.

Geometrically a patch is an area that represents the casual association between the cause (inputs- and the effect (outputs). The size of the patch indicates the vagueness implicit in the rule as expressed through the membership functions of the inputs and outputs.

The total area occupied by a patch is an indication of the vagueness of a given rule that can be used to generate the patch. Consider a one-input-one-output rule: if the input is crisp and the output is fuzzy then the patch becomes a line. If both are crisp sets then the patch is vanishingly small - a point.

# Linguistic Terms and Variables

Informally, a linguistic variable is a variable whose values are words or sentences in a natural or artificial language. For example if age is interpreted as a linguistic variable, then its term-set $T()$, that is, the set of its linguistic values, might be $$T(age) = \text{young}+\text{old}+\text{very young}+\text{not young}+\text{very old}+\dots$$

A primary fuzzy set, that is, a term whose meaning must be defined a priori, and which serves as a basis for the computation of the meaning of the non-primary terms in $T()$.

The primary terms here are young and old, whose meanings might be defined by the respective membership functions $\mu_{young}$ and $\mu_{old}$.

Non primary membership functions: $\mu_{\text{very young}}, (\mu_{\text{very young}})^{2}$, etc.

The association of a fuzzy set to a linguistic term offers the principal advantage in that human experts usually articulate their knowledge through the use of linguistic terms.

Two contrasting points about a linguistic variable are that it is a variable whose value can be interpreted **quantatitvely** using a corresponding membership function, and **qualitatively** using an expression involving linguistic terms. The notion of linguistic variables has led to a uniform framework where both qualitative and quantitative variables are used: some attribute the creation and refinement of this framework to be the reason that fuzzy logic is so popular as it is.