Overview

- More counting
- Permutations
- Permutations of repeated distinct objects
- Combinations

More counting

We'll need to count the following again and again, let's do it once:

- Counting the number of ways to generate an ordered subset of size k from a set of n distinguishable objects (Permutation)
- Counting the number of ways to generate an unordered subset of size k from a set of n distinguishable objects (Combination)

We'll see that counting permutations is based on the product rule of counting and counting combinations is based on permutations.

More counting

Examples:

- Counting the number of distinct pizzas we can create by selecting 4 toppings from 6 available.
- How many distinct lottery numbers when choose 6 in range 1-47.
- How many ways can 3 bit errors occur in a string of 8 bits.
- How many ways can I allocate 50 servers from a pool of 100 servers.
- How many routes are there between two points in a network.

Permutations

Permutation: An ordered arrangement

- Example. How many ways can we arrange the letters in the word "abc" ?

a	b	c
b	a	c
c	b	a
a	c	b
b	c	a
c	a	b

  The first letter can be chosen from any of 3, the second from any of 2, the third from 1. So by the product rule there are $3 \times 2 \times 1 = 6$ possible permutations
- Recall $n! = n(n-1)(n-2) \cdots 3 \times 2 \times 1$. So $3! = 3 \times 2 \times 1 = 6$
- In general, number of permutations of n objects in $n!$ – by direct application of product rule.

Permutations

How many ways can we arrange the letters in the word "moo" ?

- Label the letters uniquely mo_1o_2. Then we have $3! = 6$ permutations, same as "abc".

$$\begin{array}{ccc} \mathrm{m} & o_1 & o_2 \\ \mathrm{m} & o_2 & o_1 \\ o_1 & \mathrm{m} & o_2 \\ o_2 & \mathrm{m} & o_1 \\ o_1 & o_2 & \mathrm{m} \\ o_2 & o_1 & \mathrm{m} \end{array}$$

- But if treat the two o's as the same we get only 3 distinct arrangements:

$$\begin{array}{ccc} \mathrm{m} & \mathrm{o} & \mathrm{o} \\ \mathrm{o} & \mathrm{m} & \mathrm{o} \\ \mathrm{o} & \mathrm{o} & \mathrm{m} \end{array}$$

- Take mo_2o_1, If we permute the o's we get mo_1o_2 but it still reads moo. There are $2! = 2$ ways to permute the two o's. So we need to divide 3! by 2!, which gives us $6/2 = 3$ permutations

Permutations

A slightly harder example: "pepper".

- Three p's, two e's and one r. Label as $p_1 e_1 p_2 p_3 e_2 r$.
- Consider one permutation e.g. ppeper. How many equivalent ways can we write this ?

p_1	p_2	e_1	p_3	e_2	r	p_1	p_2	e_2	p_3	e_1	r
p_1	p_3	e_1	p_2	e_2	r	p_1	p_3	e_2	p_2	e_1	r
p_2	p_1	e_1	p_3	e_2	r	p_2	p_1	e_2	p_3	e_1	r
p_2	p_3	e_1	p_1	e_2	r	p_2	p_3	e_2	p_1	e_1	r
p_3	p_1	e_1	p_2	e_2	r	p_3	p_1	e_2	p_2	e_1	r
p_3	p_2	e_1	p_1	e_2	r	p_3	p_2	e_2	p_1	e_1	r

- Can arrange p_1, p_2, p_3 in 3! different orders. Can arrange e_1, e_2 in 2! different orders. Can arrange r in $1! = 1$ different ways (trivially)
- So $3!2! = 12$ ways to write ppeper.
- 6! ways to arrange $p_1 e_1 p_2 p_3 e_2 r$. So $\frac{6!}{3!2!1!} = 60$ possible letter arrangements.

Permutations

With **permutations of repeated distinct objects** in general we have the following. Permuting n objects with k groups (first group has n_1 objects, second n_2 objects etc):

- Consider all of the n objects to be distinct at first and compute $n!$
- For the first distinct group with n_1 objects, divide $n!$ by the permutations of this group $n_1!$ Repeat for the second group with n_2 objects, and so on.
- Number of permutations is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

- In the special case when $k = n$, $n_1 = 1 = n_2 = \cdots = n_n$ then we get back to $\frac{n!}{1!} = n!$.

Combinations

Interested in counting the number of different groups of k objects that can be formed from a total of n objects. Now order does not matter.

- Example: How many groups of 3 letters could be selected from the set of 5 letters $\{A, B, C, D, E\}$?
- There are 5 ways to select the first letter, 4 ways to select the second letter, 3 ways to select the third letter. So $5 \times 4 \times 3 = 60$ ways of selecting a group when the order matters.
- What about when the order doesn't matter ?
- Each group containing letters A, B, C is counted in the 60. There are 6 such groups: ABC, ACB, BAC, BCA, CAB and CBA. Lumping these together we need to divide 60 by 6 to get number of groups when don't care about letter order.
- Frame it as a repeated permutation problem ... for each group of 3 letters there are $3! = 3 \times 2 \times 1 = 6$ permutations, so number of unordered groups is $\frac{5\times4\times3}{3\times2\times1}$

Combinations

- In general there are $n(n-1)(n-2)\cdots(n-k+1)$ ways that a group of k items can be selected from n items, when order matters.
- Each group of k items will be counted $k!$ times in this count, so we need to divide by this to get number of unordered groups. That is, number of different groups of k objects that can be formed from a total of n objects is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

- Notation: for $0 \leq k \leq n$ define $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  We say that $\binom{n}{k}$ is number of possible combinations of n objects taken k at a time. Say "n choose k".
- Note that $0! = 1$ by convention. So $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

Combinations

Example:

- How many ways can 3 bit errors occur in a string of 8 bits. $\binom{8}{3} = 56$.
- How many ways can I allocate 50 servers from a pool of 100 servers. $\binom{100}{50} \approx 10^{29}$.
- Number of distinct pizzas we can create by selecting 4 toppings from 6 available. $\binom{6}{4} = 15$.
- How many distinct lottery numbers when choose 6 in range 1-47. $\binom{47}{6} = 10,737,573$

Combinations

Pizza toppings:

- Gorgonzola
- Olives
- Peppers
- Mushrooms
- Artichokes
- Epoisses de Bourgogne[1]

How many different combinations ? $\binom{6}{4} = 15$. But can't use Gorgonzola and Epoisses together as just too stinky. How many different combinations now ?

[1] Apparently banned from public transport in Paris, Napoleon's favourite

Combinations

Solution 1:

- Case 1: Gorgonzola and 3 other toppings (excluding Epoisses). $\binom{4}{3}$
- Case 2: Epoisses and 3 other toppings (excluding Gorgonzola). $\binom{4}{3}$
- Case 3: 4 toppings that aren't Gorgonzola or Epoisses. $\binom{4}{4}$
- Total is $\binom{4}{3} + \binom{4}{3} + \binom{4}{4} = 9$

Solution 2:

- All combinations. $\binom{6}{4}$
- Gorgonzola + Epoisses + 2 other toppings. $\binom{4}{2}$
- Remainder: $\binom{6}{4} - \binom{4}{2} = 9$

Power Sets

- **Power set of** S: the set of all subsets of S, including the empty set and S itself. Sometimes written 2^S.
- Example: $S = \{A, B, C\}$,

$$2^S = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B\ C\}\}$$

- Note that in a set the elements are unordered i.e. set $\{A, B\}$ is same as set $\{B, A\}$

$$\begin{aligned} |2^S| &= \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \\ &= 1 + 3 + 3 + 1 = 8 \end{aligned}$$

Power Sets

- Let $|S| = n$. In general,

$$|2^S| = \sum_{k=0}^{n} \binom{n}{k}$$

- **Binomial Theorem**: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ (see book for proof)
- Example:

$$|2^S| = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

- So $|2^S| = 2^{|S|} = 2^n$

Basket Data

- Basket data also called transaction data.
- Plenty of it.
- Example:

ID	apples	beer	cheese	eggs	ice cream
1	1	1			1
2			1	1	
3		1	1		
4		1			1
5				1	
6	1	1	1		
7		1			1
8				1	

Basket Data

ID	apples	beer	cheese	eggs	ice cream
1	1	1			1
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3		1	1		
4		1			1
5				1	
6	1	1	1		
7		1			1
8				1	

Discovering "rules".

- A rule is something like this: *If a basket contains beer then it also contains ice cream*
- Accuracy: when the *if* part is true, how often is the *then* part true.
- Coverage: how much of the database contains the *if* part
- 5 out of 8 entries contain beer (coverage is $\frac{5}{8} = 0.625$). Of these 3 also contain ice cream (accuracy is $\frac{3}{5} = 0.6$).
- Is this rule interesting/surprising i.e. do beer and ice cream appear in same basket more than we would expect by chance ?

Basket Data

ID	apples	beer	cheese	eggs	ice cream
1	1	1			1
2			1	1	
3		1	1		
4		1			1
5				1	
6	1	1	1		
7		1			1
8				1	

- $\frac{5}{8} = 0.625$ of baskets contain beer, $\frac{3}{8} = 0.375$ contain ice cream. So if these are independent and we pick a basket uniformly at random we expect $0.625 \times 0.375 \approx 0.23$ of baskets to contain both.
- Is observed fraction 0.6 with beer and ice cream interestingly larger than 0.23 ?
- Depends on the amount of data (only 8 baskets, but what if had 1M baskets ? Or 100M ?). Depends on our assumptions e.g. independence.
- For large data sets, can't enumerate all possible "rules". Smart algorithms for enumerating rules with specified minimum coverage, see https://en.wikipedia.org/wiki/Apriori_algorithm.

Prediction: Regression

We have some data e.g. scores in ST3009 tutorials and in final exam:

3	7	2	9	1	75
5	8	2	9	2	85
4	1	1	1	3	25
6	8	2	1	4	55

We get some new data:

3	6	1	8	1	?

- Can we accurately predict the final exam score with high probability ?
- E.g. picking a number between 0 and 100 uniformly at random is certainly a prediction, but hopefully a poor one.
- Expect that quality of prediction depends on the amount of data and on our assumptions

Prediction: Classification

We have some data which is labelled A or B e.g. has passed ST3009 exam:

3	7	2	9	1	A
5	8	2	9	2	A
4	1	1	1	3	B
6	8	2	1	4	A

We get some new data:

3	6	1	8	1	?

- Can we accurately predict the label A or B with high probability ?