Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Computer Science and Statistics

SF Integrated Computer Science
SF CSL

Trinity Term 2018

MA2C03: Discrete Mathematics

Wednesday, May 2 RDS 09:30 — 12:30

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Instructions to Candidates:

Credit will be given for the best 6 questions answered.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let $f : [0, 2] \to \mathbb{R}$ be the function defined by $f(x) = e^x$ for all $x \in [0, 2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

(8 points)

(b) For $x, y \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, xRy if and only if $\dfrac{x}{y}$ is an odd integer. Determine the following:

(i) Whether or not the relation R is *reflexive*;

(ii) Whether or not the relation R is *symmetric*;

(iii) Whether or not the relation R is *anti-symmetric*;

(iv) Whether or not the relation R is *transitive*;

(v) Whether or not the relation R is an *equivalence relation*;

(vi) Whether or not the relation R is a *partial order*.

Give appropriate short proofs and/or counterexamples to justify your answer.

(12 points)

(End of Question)

2. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ with the operation of addition given by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

   (a) Is $(A, \cdot)$ a semigroup? Justify your answer.

   (5 points)

   (b) Is $(A, \cdot)$ a monoid? Justify your answer.

   (5 points)

   (c) Is $(A, \cdot)$ a group? Justify your answer.

   (5 points)

   (d) Is A finite, countably infinite, or uncountably infinite? Justify your answer.

   (5 points)

   (End of Question)

3. Let L be the language over the alphabet $A = \{a, l, p\}$ consisting of all words that involve exactly TWO of the letters a, l, and p. For example, $allaaal$ is a valid string whereas lll and $alpal$ are not.

   (a) Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions.

   (5 points)

   (b) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules.

   (5 points)

   (c) Write down a regular expression that gives the language L and justify your answer.

   (5 points)

   (d) Consider the language L' over the alphabet $A = \{a, l, p\}$ consisting of all words of the form $ppa^m l^{3m}$ for $m \in \mathbb{N}^*$. Use the pumping lemma to show the language L' is not regular.

   (5 points)

   (End of Question)

4. In this question, all graphs are undirected graphs.

   (a) (i) What is meant by saying that a graph is *complete*?

   (ii) What is meant by saying that a graph is *regular*?

   (iii) What is the formula that relates the number of vertices of a tree with its number of edges?

   (iv) What is a spanning tree of a graph (V, E)?

(7 points)

   (b) Let (V, E) be the graph with vertices a, b, c, d, e, f, and g, and edges ab, bc, cd, de, ae, cf, and dg.

   (i) Draw this graph. Write down its incidence table and its adjacency table.

   (ii) Is this graph connected? Justify your answer.

   (iii) Is this graph bipartite? Justify your answer.

   (iv) Does this graph have an Eulerian trail? Justify your answer.

   (v) Does this graph have a Hamiltonian circuit? Justify your answer.

   (vi) Is this graph a tree? Justify your answer.

(8 points)

   (c) Let (V, E) be the graph defined in part (b). Give an example of an isomorphism $\varphi : V \to V$ from the graph (V, E) to itself that satisfies $\varphi(c) = d$.

(5 points)

(End of Question)

5. Consider the connected undirected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L, and with edges listed with associated costs in the following table:

AD	BC	EI	CF	JK	IJ	BL	CE	HG	FH
2	2	2	3	4	5	6	6	6	7
AB	FJ	GK	BH	EJ	CD	DE	HL	AC	EF
8	8	9	9	10	10	11	11	12	13

(a) Draw the graph and label each edge with its cost.

(2 points)

(b) Determine the minimum spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. Draw the graph corresponding to each step of the algorithm.

(9 points)

(c) Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex I, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, list the edge that is added and draw the graph.

(9 points)

(End of Question)

6. (a) Is the set of odd integers finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(b) Is $\{(x, y) \in \mathbb{R}^2 \mid y = 3x^2 + 2\}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(c) Give the definition of a regular expression.

(3 points)

(d) Let A be a finite alphabet. Prove that the set of all regular languages over the alphabet A is countably infinite.

(7 points)

(End of Question)

7. (a) How many distinct directed graphs with three vertices $V = \{a, b, c\}$ are there? Justify your answer.

(8 points)

(b) Using the one-to-one correspondence between directed graphs and relations, draw a directed graph that corresponds to a relation on $V = \{a, b, c\}$ that is reflexive but neither symmetric nor transitive. Justify your answer.

(4 points)

(c) Using the one-to-one correspondence between directed graphs and relations, draw a directed graph that corresponds to a relation on $V = \{a, b, c\}$ that is symmetric but neither reflexive nor transitive. Justify your answer.

(4 points)

(d) Using the one-to-one correspondence between directed graphs and relations, draw a directed graph that corresponds to a relation on $V = \{a, b, c\}$ that is transitive but not reflexive nor symmetric. Justify your answer.

(4 points)

8. (a) Consider the language over the binary alphabet $A = \{0, 1\}$ given by

$$L = \{10^n 1^n \mid n \in \mathbb{N},\, n \geq 1\}.$$

Write down the algorithm of a Turing machine that recognizes L. Process the following strings according to your algorithm: 1, 11, 101, and 1011.

(8 points)

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

(6 points)

(c) Show that not all languages are Turing-recognizable by a countability argument.

(6 points)

(End of Question)