

CSU33081 Exam Paper 2020 Answer Explanations

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Question 1

Represent the summation of the following equations in MATLAB:

$$2x^2 + 2x - 6$$

And

$$x^3 + 2x - 4$$

Rewrite equations in same form:

$$\begin{aligned} 0x^3 + 2x^2 + 2x - 6 \\ x^3 + 0x^2 + 2x - 4 \end{aligned}$$

Take the coefficients in that order and enter them into vectors:

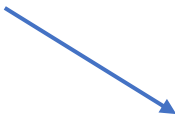
$$[0 \ 2 \ 2 \ -6] + [1 \ 0 \ 2 \ -4]$$

$\therefore \text{Answer} = C$

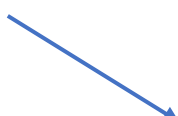
Question 2

Taking the MATLAB commands line by line:

`A=eye(3,3);`


$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

`For x=1:2:3`

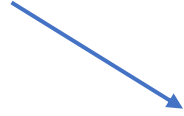


`for(int x = 1; x ≤ 3; x = x + 2)`

Iteration 1:

`x=1;`

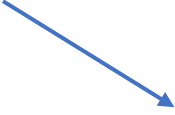
`A(1,1) = 1;`


$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Iteration 2:

`x=3;`

`A(1,3)=1;`

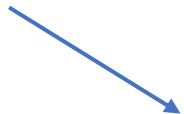

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore \text{Answer} = B$

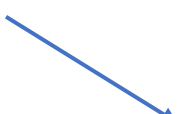
Question 3

Taking the MATLAB commands line by line:

`x=[6:8;-1:1;5 6 7];`

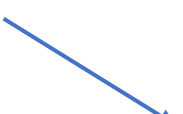

$$x = \begin{pmatrix} 6 & 7 & 8 \\ -1 & 0 & 1 \\ 5 & 6 & 7 \end{pmatrix}$$

`y=x(:,3);`


$$x = \begin{pmatrix} 6 & 7 & 8 \\ -1 & 0 & 1 \\ 5 & 6 & 7 \end{pmatrix} \rightarrow y = \begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}$$

The third column of matrix x is circled in green, and a green arrow points from it to the vector y.

`size(y')`


$$y = \begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix} \xrightarrow{1 \times 3} 1 \times 3$$

The number 1 is written in green above the vector y, and the number 3 is written in green to the left of the arrow pointing to '1 x 3'.

$\therefore \text{Answer} = C$

Question 4

Calculate the Truncation Error, $f(x) - P_2(x)$ at $x = 2.5$

$$f(x) = 3 - 17x^3$$

Taylor Series Formula:

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^n(x_0)}{n!}(x - x_0)^n$$

Using the Taylor Series Polynomial approximation of degree two, $P_2(x)$ expanded about the point $x_0 = 2.0$:

$$T_2(x) = f(2.0) + f'(2.0)(x - 2.0) + \frac{f''(2.0)}{2!}(x - 2.0)^2$$

$$f(x) = 3 - 17x^3$$

$$f'(x) = -51x^2$$

$$f''(x) = -102x$$

$$\begin{aligned} T_2(x) &= (3 - 17(2.0)) + (-51(2.0)^2)(x - 2.0) + \frac{(-102(2.0))}{2!}(x - 2.0)^2 \\ &= -133 - 204x + 408 - 102(x^2 - 4x + 4) \\ &= 275 - 204x - 102x^2 + 408x - 408 \\ &= -102x^2 + 204x - 133 \end{aligned}$$

Calculating Truncation Error:

$$f(x) - P_2(x) \text{ at } x = 2.5$$

$$\begin{aligned} &f(2.5) - P_2(2.5) \\ &= (3 - (17(2.5)^3)) - (-102(2.5)^2 + 204(2.5) - 133) \\ &= -262.625 + 260.5 \\ &= -2.125 \end{aligned}$$

$\therefore \text{Answer} = E \rightarrow -2.125$

Question 5

Use the Secant Method to find a root of the function:

$$f(x) = 16x^5 - 73x^2 - 133$$

Accurate to within an error of $\epsilon = x_n - x_{n-1} = 0.001$ where x_n is the value of x at the n^{th} iteration.

Starting points:

$$x_0 = 3, x_1 = 2.5$$

Secant Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{\left(\frac{f(x_n) - f(x_{n-1}))}{x_n - x_{n-1}}\right)}$$

$$x_{n+1} = x_n - \frac{16x_n^5 - 73x_n^2 - 133}{\left(\frac{(16x_n^5 - 73x_n^2 - 133) - (16x_{n-1}^5 - 73x_{n-1}^2 - 133)}{x_n - x_{n-1}}\right)}$$

Iteration 1

$$x_0 = 3, x_1 = 2.5$$

$$x_2 = x_1 - \frac{16x_1^5 - 73x_1^2 - 133}{\left(\frac{(16x_1^5 - 73x_1^2 - 133) - (16x_0^5 - 73x_0^2 - 133)}{x_1 - x_0}\right)}$$

$$x_2 = 2.5 - \frac{16(2.5)^5 - 73(2.5)^2 - 133}{\left(\frac{(16(2.5)^5 - 73(2.5)^2 - 133) - (16(3)^5 - 73(3)^2 - 133)}{2.5 - 3}\right)}$$

$$x_2 = 2.5 - \frac{973.25}{\left(\frac{973.25 - 3098}{-0.5}\right)}$$

$$x_2 = 2.27097$$

Iteration 2

$$x_1 = 2.5, x_2 = 2.27097$$

$$x_3 = x_2 - \frac{16x_2^5 - 73x_2^2 - 133}{\left(\frac{(16x_2^5 - 73x_2^2 - 133) - (16x_1^5 - 73x_1^2 - 133)}{x_2 - x_1}\right)}$$

$$x_3 = 2.27097$$

$$- \frac{16(2.27097)^5 - 73(2.27097)^2 - 133}{\left(\frac{(16(2.27097)^5 - 73(2.27097)^2 - 133) - (16(2.5)^5 - 73(2.5)^2 - 133)}{2.27097 - 2.5}\right)}$$

$$x_3 = 2.27097 - \frac{456.96685}{\left(\frac{456.96685 - 973.25}{-22903}\right)}$$

$$x_3 = 2.27097 - 0.20271$$

$$x_3 = 2.06826$$

Iteration 3

$$x_2 = 2.27097, x_3 = 2.06826$$

$$x_4 = x_3 - \frac{16x_3^5 - 73x_3^2 - 133}{\left(\frac{(16x_3^5 - 73x_3^2 - 133) - (16x_2^5 - 73x_2^2 - 133)}{x_3 - x_2}\right)}$$

$$x_4 = 2.06826$$

$$- \frac{16(2.06826)^5 - 73(2.06826)^2 - 133}{\left(\frac{(16(2.06826)^5 - 73(2.06826)^2 - 133) - (16(2.27097)^5 - 73(2.27097)^2 - 133)}{2.06826 - 2.27097}\right)}$$

$$x_4 = 2.06826 - \frac{160.27099}{\left(\frac{160.27099 - 456.96685}{-0.20271}\right)}$$

$$x_4 = 2.06826 - 0.10950$$

$$x_4 = 1.958756$$

Iteration 4

$$x_3 = 2.06826, x_4 = 1.958756$$

$$x_5 = x_4 - \frac{16x_4^5 - 73x_4^2 - 133}{\left(\frac{(16x_4^5 - 73x_4^2 - 133) - (16x_3^5 - 73x_3^2 - 133)}{x_4 - x_3}\right)}$$

$$x_5 = 1.958756$$

$$- \frac{16(1.958756)^5 - 73(1.958756)^2 - 133}{\left(\frac{(16(1.958756) - 73(1.958756) - 133) - (16(2.06826)^5 - 73(2.06826)^2 - 133)}{1.958756 - 2.06826}\right)}$$

$$x_5 = 1.958756 - \frac{48.25973}{\left(\frac{48.25973 - 160.27099}{-0.10950}\right)}$$

$$x_5 = 1.958756 - 0.04718$$

$$x_5 = 1.911576$$

Iteration 5

$$x_4 = 1.958756, x_5 = 1.911576$$

$$x_6 = x_5 - \frac{16x_5^5 - 73x_5^2 - 133}{\left(\frac{(16x_5^5 - 73x_5^2 - 133) - (16x_4^5 - 73x_4^2 - 133)}{x_5 - x_4}\right)}$$

$$x_6 = 1.911576$$

$$- \frac{16(1.911576)^5 - 73(1.911576)^2 - 133}{\left(\frac{(16(1.911576)^5 - 73(1.911576)^2 - 133) - (16(1.958756)^5 - 73(1.958756)^2 - 133)}{1.911576 - 1.958756}\right)}$$

$$x_6 = 1.911576 - \frac{8.64233}{\left(\frac{8.64233 - 48.25973}{-0.04718}\right)}$$

$$x_6 = 1.911576 - 0.010292$$

$$x_6 = 1.901285$$

Iteration 6

$$x_5 = 1.911576, x_6 = 1.901285$$

$$x_7 = x_6 - \frac{16x_6^5 - 73x_6^2 - 133}{\left(\frac{(16x_6^5 - 73x_6^2 - 133) - (16x_5^5 - 73x_5^2 - 133)}{x_6 - x_5}\right)}$$

$$x_7 = 1.901285$$

$$- \frac{16(1.901285)^5 - 73(1.901285)^2 - 133}{\left(\frac{(16(1.901285)^5 - 73(1.901285)^2 - 133) - (16(1.911576)^5 - 73(1.911576)^2 - 133)}{1.901285 - 1.911576}\right)}$$

$$x_7 = 1.901285 - \frac{0.630803}{\left(\frac{0.630803 - 8.64233}{-0.010291}\right)}$$

$$x_7 = 1.901285 - 0.00081$$

$$x_7 = 1.900475$$

Check:

$$\begin{aligned}\epsilon &= x_n - x_{n-1} \\ \epsilon &= 1.900475 - 1.901285 \\ \epsilon &= -0.00081\end{aligned}$$

Which is within

$$\epsilon = 0.001$$

$\therefore \text{Answer} = C$

Question 6

Find the upper triangular matrix [U] in the [L][U] decomposition of the matrix:

$$A = \begin{pmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{pmatrix} = [L][U]$$

Lower and Upper triangular matrices have the forms:

$$[L] = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \quad [U] = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

Where:

$$A = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{pmatrix} = \begin{pmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{pmatrix}$$

Row 1

$$U_{11} = 25, U_{12} = 5, U_{13} = 4$$

Row 2

$$L_{21}U_{11} = 10$$

$$L_{21}(25) = 10$$

$$L_{21} = 0.4$$

$$L_{21}U_{12} + U_{22} = 8$$

$$(0.4)(5) + U_{22} = 8$$

$$U_{22} = 6$$

$$L_{21}U_{13} + U_{23} = 16$$

$$(0.4)(4) + U_{23} = 16$$

$$U_{23} = 14.4$$

Row 3

$$L_{31}U_{11} = 8$$

$$L_{31}(25) = 8$$

$$L_{31} = 0.32$$

$$L_{31}U_{12} + L_{32}U_{22} = 12$$

$$(0.32)(5) + L_{32}(6) = 12$$

$$L_{32} = 1.733333$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 22$$

$$(0.32)(4) + (1.733333)(14.4) + U_{33} = 22$$

$$U_{33} = -4.24$$

The Upper triangular matrix can now be constructed:

$$[U] = \begin{pmatrix} U_{11} & U_{11} & U_{11} \\ 0 & U_{11} & U_{11} \\ 0 & 0 & U_{11} \end{pmatrix} = \begin{pmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.4 \\ 0 & 0 & -4.24 \end{pmatrix}$$

$\therefore \text{Answer} = C$

Question 7

Using $x_1 = 1$, $x_2 = 3$, $x_3 = 5$ as an initial guess at the solution, determine the values of x_1 , x_2 and x_3 that result from three iterations of the Gauss-Seidel method applied to this matrix equation:

$$\begin{pmatrix} 12 & 7 & 3 \\ 1 & 5 & 1 \\ 2 & 7 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 6 \end{pmatrix}$$

Creating algebraic equations from the matrix equations:

1. $12x_1 + 7x_2 + 3x_3 = 2$
2. $x_1 + 5x_2 + x_3 = -5$
3. $2x_1 + 7x_2 - 11x_3 = 6$

Iteration 1

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12} = \frac{2 - 7(3) - 3(5)}{12} = -2.833333$$

$$x_2 = \frac{-5 - x_1 - x_3}{5} = \frac{-5 - (-2.833333) - (5)}{5} = -1.433333$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11} = \frac{6 - 2(-2.833333) - 7(-1.433333)}{-11} = -1.9727$$

Iteration 2

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12} = \frac{2 - 7(-1.433333) - 3(-1.9727)}{12} = 1.4959$$

$$x_2 = \frac{-5 - x_1 - x_3}{5} = \frac{-5 - (1.4959) - (-1.9727)}{5} = -0.90464$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11} = \frac{6 - 2(1.4959) - 7(-0.90464)}{-11} = -0.84914$$

Iteration 3

$$x_1 = \frac{2 - 7x_2 - 3x_3}{12} = \frac{2 - 7(-0.90464) - 3(-0.84914)}{12} = 0.90666$$

$$x_2 = \frac{-5 - x_1 - x_3}{5} = \frac{-5 - (0.90666) - (-0.84914)}{5} = -1.0115$$

$$x_3 = \frac{6 - 2x_1 - 7x_2}{-11} = \frac{6 - 2(0.90666) - 7(-1.0115)}{-11} = -1.0243$$

$\therefore \text{Answer} = C$

Question 8

Calculate the dominant eigenvalue and an associated eigenvector using the Power Method for the following matrix. Perform four iterations beginning with an initial estimate of:

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}$$

Iteration 1

$$Ax_0 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 11 \end{pmatrix}$$

$$n_0 = 11$$

$$x_1 = \begin{pmatrix} \frac{9}{11} \\ \frac{11}{11} \end{pmatrix} = \begin{pmatrix} 0.8182 \\ 1 \end{pmatrix}$$

Iteration 2

$$Ax_1 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0.8182 \\ 1 \end{pmatrix} = \begin{pmatrix} 8.2727 \\ 9.9091 \end{pmatrix}$$

$$n_1 = 9.9091$$

$$x_2 = \begin{pmatrix} \frac{8.2727}{9.9091} \\ \frac{9.9091}{9.9091} \end{pmatrix} = \begin{pmatrix} 0.8349 \\ 1 \end{pmatrix}$$

Iteration 3

$$Ax_2 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0.8349 \\ 1 \end{pmatrix} = \begin{pmatrix} 8.3394 \\ 10.0092 \end{pmatrix}$$

$$n_2 = 10.0092$$

$$x_3 = \begin{pmatrix} \frac{8.2727}{10.0092} \\ \frac{10.0092}{10.0092} \end{pmatrix} = \begin{pmatrix} 0.8332 \\ 1 \end{pmatrix}$$

Iteration 4

$$Ax_3 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0.8332 \\ 1 \end{pmatrix} = \begin{pmatrix} 8.3327 \\ 9.999 \end{pmatrix}$$

$$n_3 = 9.999$$

$$x_4 = \begin{pmatrix} \frac{8.3327}{9.999} \\ \frac{9.999}{9.999} \end{pmatrix} = \begin{pmatrix} 0.8333 \\ 1 \end{pmatrix}$$

$\therefore \text{Answer} = C$

Question 9

For the function :

$$f(x) = x^2 \log_2(x)$$

And the points:

$$x_0 = 2, x_1 = 3, x_2 = 7$$

Calculate Newton's second divided difference $f[x_2, x_1, x_0]$

$$f[x_2, x_1, x_0] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{f(3) - f(2)}{3 - 2}$$

$$= \frac{14.264663 - 4}{1}$$

$$f[x_0, x_1] = 10.264663$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(7) - f(3)}{7 - 3}$$

$$= \frac{137.560391 - 14.264663}{4}$$

$$f[x_1, x_2] = 30.823932$$

$$\begin{aligned} \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} &= \frac{30.823932 - 10.264663}{7 - 2} \\ &= 4.111854 \end{aligned}$$

$\therefore \text{Answer} = D$

Question 10

Evaluate the following integral using three-point Gaussian Quadrature:

$$\int_0^{2\pi} \frac{1}{2 + \cos x} dx$$

Three point rule:

$$\int_a^b f(x) dx \cong c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

Where:

$$c_1 = 0.5556, c_2 = 0.8889, c_3 = 0.5556$$
$$x_1 = -0.7746, x_2 = 0, x_3 = 0.7746$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$$

The following was calculated in radians:

$$\begin{aligned} \int_0^{2\pi} \frac{1}{2 + \cos x} dx &\cong \frac{2\pi - 0}{2} \int_0^{2\pi} \frac{1}{2 + \cos x} dx \\ &\cong \pi(0.5556 \left(\frac{1}{2 + \cos\left(\frac{2\pi - 0}{2}(-0.7746) + \frac{2\pi + 0}{2}\right)} \right) \\ &\quad + 0.8889 \left(\frac{1}{2 + \cos\left(\frac{2\pi - 0}{2}(0) + \frac{2\pi + 0}{2}\right)} \right) \\ &\quad + 0.5556 \left(\frac{1}{2 + \cos\left(\frac{2\pi - 0}{2}(0.7746) + \frac{2\pi + 0}{2}\right)} \right)) \cong 4.05745 \end{aligned}$$

$\therefore \text{Answer} = A$