Exercise 2.34

For a given matrix

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \tag{1}$$

we could get its eigenvalues λ From the following equation,

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 9 = 0 \iff \lambda_1 = 7, \lambda_2 = 1 \tag{2}$$

The corresponding eigenvectors are given by

$$A|v_1\rangle = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 7 \begin{pmatrix} a \\ b \end{pmatrix} \iff |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3a}$$

$$A|v_2\rangle = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \iff |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3b}$$

where the coefficient $1/\sqrt{2}$ makes sure the eigenvector is unit vector. Note that A is Hermitian so we can decompose A into

$$A = \sum_i \lambda_i |i\rangle\langle i| = 7 \cdot \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{4}$$

Then we can calculate

- The square root of A,

$$\begin{aligned} \sqrt{A} &= \sum_i \sqrt{\lambda_i} |i\rangle\langle i| = \sqrt{7} \cdot \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{7}+1 & \sqrt{7}-1 \\ \sqrt{7}-1 & \sqrt{7}+1 \end{pmatrix} \end{aligned} \tag{5}$$

- The logarithm of A,

$$\begin{aligned} \log A &= \sum_i \log \lambda_i |i\rangle\langle i| = \log 7 \cdot \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \log 1 \cdot \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \\ &= \frac{\log 7}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned} \tag{6}$$