Exercise 5.1

Consider a transformation F defined as

$$F \equiv \sum_{j=0}^{N-1}\left[\sum_{k=0}^{N-1}\frac{1}{\sqrt{N}}\exp\left(2i\pi\frac{jk}{N}\right)|k\rangle\right]\langle j| = \sum_{j=0}^{N-1}|\tilde{j}\rangle\langle j| \tag{1}$$

where basis $\{|j\rangle\}$ and $\{|k\rangle\}$ are different orthonormal basis set. Above transformation is the quantum Fourier transformation since for any basis vector $|j\rangle$, it equivalently performs

$$|j\rangle \to |\tilde{j}\rangle = \sum_{k=0}^{N-1}\frac{1}{\sqrt{N}}\exp\left(2i\pi\frac{jk}{N}\right)|k\rangle \tag{2}$$

To prove the transformation in eq. (1) is unitary, we have

$$\begin{aligned} F^\dagger F &= \left[\sum_{j=0}^{N-1}|\tilde{j}\rangle\langle j|\right]^\dagger\left[\sum_{n=0}^{N-1}|\tilde{n}\rangle\langle n|\right] \\ &= \left[\sum_{j=0}^{N-1}|j\rangle\langle \tilde{j}|\right]^\dagger\left[\sum_{n=0}^{N-1}|\tilde{n}\rangle\langle n|\right] \\ &= \sum_{n,j}|j\rangle\langle\tilde{j}|\tilde{n}\rangle\langle n| \end{aligned} \tag{3}$$

where $|j\rangle$ and $|n\rangle$ are different notation for same orthonormal basis. Then the inner product $\langle\tilde{j}|\tilde{n}\rangle$ becomes

$$\begin{aligned} \langle\tilde{j}|\tilde{n}\rangle &= \left[\sum_{k=0}^{N-1}\frac{1}{\sqrt{N}}\exp\left(-2i\pi\frac{jk}{N}\right)\langle k|\right]\left[\sum_{m=0}^{N-1}\frac{1}{\sqrt{N}}\exp\left(2i\pi\frac{nm}{N}\right)|m\rangle\right] \\ &= \frac{1}{N}\sum_{m,k}\exp\left(-2i\pi\frac{jk}{N}\right)\exp\left(2i\pi\frac{nm}{N}\right)\langle k|m\rangle \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\exp\left(-2i\pi\frac{jk}{N}\right)\exp\left(2i\pi\frac{nk}{N}\right) \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\exp\left[2i\pi\frac{(n-j)k}{N}\right] \end{aligned} \tag{4}$$

where $|k\rangle$ and $|m\rangle$ are different notations for same orthonormal basis. Note that

$$\begin{aligned} \exp\left[2i\pi\frac{(n-j)(k+1)}{N}\right] &= \exp\left[2i\pi\frac{(n-j)k}{N} + 2i\pi\frac{(n-j)}{N}\right] \\ &= \exp\left[2i\pi\frac{(n-j)k}{N}\right]\exp\left[2i\pi\frac{(n-j)}{N}\right] \end{aligned}$$

so we can use geometry series to compute the sum in eq. (4) with common ratio $r = \exp[2i\pi(n-j)/N]$, then

$$
\begin{aligned}
\langle\tilde{j}|\tilde{n}\rangle &= \frac{1}{N}\sum_{k=0}^{N-1}\exp\left[2i\pi\frac{(n-j)k}{N}\right] \\
&= \frac{1}{N}\frac{1-r^N}{1-r} \\
&= \frac{1}{N}\frac{1-\exp[2i\pi(n-j)]}{1-\exp[2i\pi(n-j)/N]}
\end{aligned}
$$

Then from above calculation,

- for $r=1$, we have $n=j$ and $\langle\tilde{j}|\tilde{n}\rangle = 1$,
- for any $r\neq 1$ we have $n\neq j$ but $n-j$ is a integer. Since

  $$\exp[2i\pi(n-j)] = \cos[2\pi(n-j)] + i\sin[2\pi(n-j)]$$

  For any $n-j$ as integer, $\cos[2\pi(n-j)]$ is always 1 and $\sin[2\pi(n-j)]$ is always 0, so $\exp[2i\pi(n-j)] = 1$ for $\forall n, j$. Meanwhile, $(n-j)/N \neq$ so $\exp[2i\pi(n-j)/N] \neq 1$ and the denominator will not be 1. In this case, $\langle\tilde{j}|\tilde{n}\rangle = 0$.

In conclusion, we have $\langle\tilde{j}|\tilde{n}\rangle = \delta_{jn}$ so $F^\dagger F = \sum_{n,j}\delta_{jn}|j\rangle\langle n|$ which is an identity under the matrix representation of $\{|j\rangle\}$, the basis before transformation. Then we can conclude that the Fourier transformation is unitary.

Question: Is above result basis dependent?