Exercise 2.28

The tensor product is defined as follows. Let $A = \sum_{i,j} a_{ij}|i\rangle\langle j|$ is a $p \times q$ matrix and $B = \sum_{m,n} b_{mn}|m\rangle\langle n|$ is a $r \times s$ matrix, then we have

$$A \otimes B = \sum_{i,m;j,n} a_{ij}b_{mn}(|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{1}$$

If $|i\rangle, |j\rangle, |m\rangle, |n\rangle$ is $|0\rangle = (1 \;\; 0)^t$ or $|1\rangle = (0 \;\; 1)^t$, the result of tensor product eq. (1) can be re-written as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{1q}b_{1s} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{1q}b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}b_{r1} & a_{p2}b_{r2} & \cdots & a_{pq}b_{rs} \end{pmatrix} \tag{2}$$

I will prove the following relations about tensor product,

$$(A \otimes B)^t = A^t \otimes B^t, (A \otimes B)^* = A^* \otimes B^*, (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \tag{3}$$

- For $(A \otimes B)^t$, eq. (2) tell us that $(A \otimes B)^t$ is given by

  $$(A \otimes B)^t = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{21} & \cdots & a_{p1}b_{r1} \\ a_{11}b_{12} & a_{11}b_{22} & \cdots & a_{p2}b_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1q}b_{1s} & a_{1q}b_{2s} & \cdots & a_{pq}b_{rs} \end{pmatrix} \tag{4}$$

  If we write eq. (4) into a form like eq. (1),

  $$(A \otimes B)^t = \sum_{i,m;j,n} a_{ji}b_{nm}(|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{5}$$

  For $A^t \otimes B^t$, we have $A^t = \sum_{i,j} a_{ji}|i\rangle\langle j|$ and $B^t = \sum_{m,n} b_{nm}|m\rangle\langle n|$, then according to eq. (1), we have

  $$A^t \otimes B^t = \sum_{i,m;j,n} a_{ji}b_{nm}(|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{6}$$

  From eq. (5) and (6), we have $(A \otimes B)^t = A^t \otimes B^t$.

- For $(A \otimes B)^*$, according to eq. (1), we have

  $$(A \otimes B)^* = \sum_{i,m;j,n} a_{ij}^*b_{mn}^*(|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{7}$$

  For $A^* \otimes B^*$, I notice that $A^* = \sum_{i,j} a_{ij}^*|i\rangle\langle j|$ and $B^* = \sum_{m,n} b_{mn}^*|m\rangle\langle n|$, then according to eq. (1), we have

$$A^* \otimes B^* = \sum_{i,m;j,n} a_{ij}^* b_{mn}^* (|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{8}$$

From eq. (7) and (8), we have $(A \otimes B)^* = A^* \otimes B^*$.

- For $(A \otimes B)^\dagger$, according to eq. (1), we have

$$(A \otimes B)^\dagger = \sum_{i,m;j,n} a_{ij}^* b_{mn}^* (|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{9}$$

For $A^\dagger \otimes B^\dagger$, I notice that $A^\dagger = \sum_{i,j} a_{ji}^* |i\rangle\langle j|$ and $B^\dagger = \sum_{m,n} b_{nm}^* |m\rangle\langle n|$, then according to eq. (1), we have

$$A^\dagger \otimes B^\dagger = \sum_{i,m;j,n} a_{ji}^* b_{nm}^* (|i\rangle \otimes |m\rangle)(\langle j \otimes \langle n|) \tag{10}$$

From eq. (9) and (10), we have $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.