Properties about the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here are several important properties of Pauli matrices:

- All pauli matrices are hermitian, $\sigma_x = \sigma_x^\dagger, \sigma_y = \sigma_y^\dagger, \sigma_z = \sigma_z^\dagger$.
- All pauli matrices are unitary, $\sigma_x^\dagger \sigma_x = \sigma_y^\dagger \sigma_y = \sigma_z^\dagger \sigma_z = \mathbf{I}$.
- The eigenvectors and eigenvalues of Pauli matrices are shown below.
  - For σ_x, the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, their corresponding eigenvectors are

    $$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

  - For σ_y, the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, their corresponding eigenvectors are

    $$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

  - For σ_z, the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, their corresponding eigenvectors are

    $$|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- The commutation rules for Pauli matrices are given by
  - Pauli matrices commute with themselves,

    $$[\sigma_x, \sigma_x] = [\sigma_y, \sigma_y] = [\sigma_z, \sigma_z] = 0$$

  - One Pauli matrix does not commute with another Pauli matrix,

    $$[\sigma_x, \sigma_y] = 2i\sigma_z, [\sigma_y, \sigma_z] = 2i\sigma_x, [\sigma_z, \sigma_x] = 2i\sigma_y$$

- The anticommutation rules for Pauli matrices are given by
  - Pauli matrices do not anticommute with themselves,

    $$\{X, X\} = \{Y, Y\} = \{Z, Z\} = 2I$$

  - One Pauli matrix anticommute with another Pauli matrix,

    $$\{X, Y\} = \{Y, X\} = \{Y, Z\} = \{Z, Y\} = \{Z, X\} = \{X, Z\} = 0$$

- For state $|0\rangle$ and $|1\rangle$,

  $$\begin{cases} \sigma_x|0\rangle = |1\rangle, & \sigma_x|1\rangle = |0\rangle \\ \sigma_y|0\rangle = i|1\rangle, & \sigma_y|1\rangle = -i|0\rangle \\ \sigma_z|0\rangle = |0\rangle, & \sigma_z|1\rangle = -|1\rangle \end{cases}$$

Properties of raising/lowering Pauli matrix

$$\sigma_+ = |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_- = |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- Both matrices are hermitian
- Both matrices are not unitary
- For state $|0\rangle$ and $|1\rangle$,

$$\begin{cases} \sigma_+|0\rangle = |1\rangle, & \sigma_+|1\rangle = 0 \\ \sigma_-|0\rangle = 0, & \sigma_-|1\rangle = |0\rangle \end{cases}$$