# Exercise 2.35

Suppose that $\vec{v}$ is a 3-dimensional unit vector, $\theta$ is real number, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. We can use Taylor expansion to calculate $\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right)$. The Taylor expansion of $e^x$ is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots \tag{1}$$

Meanwhile, the Taylor expansion of $\sin\theta$ and $\cos\theta$ are given by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \tag{2a}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \tag{2b}$$

According to eq. (1), the Taylor expansion for $\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right)$ is given by

$$\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right) = \sum_{n=0}^{\infty} \frac{(i\theta\vec{v}\cdot\vec{\sigma})^n}{n!} = 1 + i\theta\vec{v}\cdot\vec{\sigma} + \frac{i^2\theta^2(\vec{v}\cdot\vec{\sigma})^2}{2!} + \frac{i^3\theta^3(\vec{v}\cdot\vec{\sigma})^3}{3!} + \frac{i^4\theta^4(\vec{v}\cdot\vec{\sigma})^4}{4!} + \dots \tag{3}$$

From eq. (3) we can find that,

- for $n$ is even number we have

$$n \text{ is even: } 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \frac{A^8}{8!} - \frac{A^{10}}{10!} \dots \tag{4}$$

  where $A = \theta(\vec{v}\cdot\vec{\sigma})$

- for $n$ is odd number we have

$$n \text{ is odd: } iA - \frac{iA^3}{3!} + \frac{iA^5}{5!} - \frac{iA^7}{7!} + \frac{iA^9}{9!} - \frac{iA^{11}}{11!} \dots \tag{5}$$

  where $A = \theta(\vec{v}\cdot\vec{\sigma})$.

From eq. (4) and (5), we could find that eq. (3) can be re-written as

$$\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right) = \cos\left[\theta(\vec{v}\cdot\vec{\sigma})\right] + i\sin\left[\theta(\vec{v}\cdot\vec{\sigma})\right] \tag{6}$$

Note that for $\vec{v}\cdot\vec{\sigma}$, we have

$$\begin{aligned} \vec{v}\cdot\vec{\sigma} &= v_x\sigma_x + v_y\sigma_y + v_z\sigma_z \\ &= \begin{pmatrix} 0 & v_x \\ v_x & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -v_y \\ v_y & 0 \end{pmatrix} + \begin{pmatrix} v_z & 0 \\ 0 & -v_z \end{pmatrix} \\ &= \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \end{aligned} \tag{7}$$

So for $(\vec{v}\cdot\vec{\sigma})^2$, we have

$$
\begin{aligned}
(\vec{v}\cdot\vec{\sigma})^2 &= \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}\begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \\
&= \begin{pmatrix} v_z^2 + (v_x - iv_y)(v_x + iv_y) & v_z(v_x - iv_y) - v_z(v_x - iv_y) \\ v_z(v_x + iv_y) - v_z(v_x + iv_y) & (v_x - iv_y)(v_x + iv_y) + v_z^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{8}
$$

where we get the third identity since $\vec{v}$ is a unit vector. From eq. (7) and eq. (8), we can re-write eq. (4) and eq. (5) as

$$
n \text{ is even: } 1 - I\frac{\theta^2}{2!} + I\frac{\theta^4}{4!} - I\frac{\theta^6}{6!} + I\frac{\theta^8}{8!} - I\frac{\theta^{10}}{10!}\dots \tag{9a}
$$

$$
n \text{ is odd: } i\theta(\vec{v}\cdot\vec{\sigma}) - (\vec{v}\cdot\vec{\sigma})\frac{i\theta^3}{3!} + (\vec{v}\cdot\vec{\sigma})\frac{i\theta^5}{5!} - (\vec{v}\cdot\vec{\sigma})\frac{i\theta^7}{7!} + \dots \tag{9b}
$$

According to eq. (9a) $-$ (9b), we conclude that eq. (6) can be simplified as

$$
\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right) = I\cos\theta + i(\vec{v}\cdot\vec{\sigma})\sin\theta \tag{10}
$$