Exercise 2.11

The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

The eigenvectors, eigenvalues and diagonal representation of Pauli matrices are shown below.

- For σ_x, the eigenvalue λ is the solution of the following equation,

$$|\sigma_x - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{2}$$

  The corresponding eigenvectors are given by

$$\sigma_x|v_1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3a}$$

$$\sigma_x|v_2\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = - \begin{pmatrix} c \\ d \end{pmatrix} \iff |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3b}$$

  The coefficient $1/\sqrt{2}$ is to normalized the eigenvector. From the eigenvalues and eigenvectors, we can find that the diagonalizing matrix D is

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \iff D^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4}$$

  and thus

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = D\Lambda D^{-1} \tag{5}$$

- For σ_y, the eigenvalue λ is the solution of the following equation,

$$|\sigma_y - \lambda I| = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{6}$$

  The corresponding eigenvectors are given by

$$\sigma_y|v_1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{7a}$$

$$\sigma_y|v_2\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = - \begin{pmatrix} c \\ d \end{pmatrix} \iff |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{7b}$$

  The coefficient $1/\sqrt{2}$ is to normalized the eigenvector. From the eigenvalues and eigenvectors, we can find that the diagonalizing matrix D is

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \iff D^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{8}$$

and thus

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = D\Lambda D^{-1} \tag{9}$$

- For σ_y, the eigenvalue λ is the solution of the following equation,

$$|\sigma_z - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{10}$$

The corresponding eigenvectors are given by

$$\sigma_z|v_1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff |v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11a}$$

$$\sigma_z|v_2\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -\begin{pmatrix} c \\ d \end{pmatrix} \iff |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11b}$$

The coefficient $1/\sqrt{2}$ is to normalized the eigenvector. From the eigenvalues and eigenvectors, we can find that the diagonalizing matrix D is

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{12}$$

and thus

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D\Lambda D^{-1} \tag{13}$$