Exercise 4.2

Suppose A is a matrix such that $A^2 = I$ and x is a real number. We can use Taylor expansion to calculate $\exp(iAx)$. The Taylor expansion of e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots \tag{1}$$

Meanwhile, the Taylor expansion of $\sin\theta$ and $\cos\theta$ are given by

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \tag{2a}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \tag{2b}$$

According to eq. (1), the Taylor expansion for $\exp(iAx)$ is given by

$$\exp(iAx) = \sum_{n=0}^{\infty} \frac{(iAx)^n}{n!} = 1 + iAx + \frac{i^2 A^2 x^2}{2!} + \frac{i^3 A^3 x^3}{3!} + \frac{i^4 A^4 x^4}{4!} + \dots \tag{3}$$

From eq. (3) we can find that,

- for n is even number we have

$$n \text{ is even: } 1 - \frac{A^2 x^2}{2!} + \frac{A^4 x^4}{4!} - \frac{A^6 x^6}{6!} + \frac{A^8 x^8}{8!} - \frac{A^{10} x^{10}}{10!} \dots \tag{4}$$

- for n is odd number we have

$$n \text{ is odd: } iAx - \frac{iA^3 x^3}{3!} + \frac{iA^5 x^5}{5!} - \frac{iA^7 x^7}{7!} + \frac{iA^9 x^9}{9!} - \frac{iA^{11} x^{11}}{11!} \dots \tag{5}$$

From eq. (4) and (5), and also note that $A^2 = I$, we could find that eq. (3) can be re-written as

$$\exp(iAx) = I\cos\theta + iA\sin\theta \tag{6}$$