Exercise 2.8

The Gram-Schmidt procedure converts a non-orthonormal basis $|w_1\rangle, \ldots, |w_d\rangle$ in vector space V into an orthonormal basis $|v_1\rangle, \ldots, |v_d\rangle$. I will proof new basis $|v_1\rangle, \ldots, |v_d\rangle$ is orthonormal below. The new vector $|v_{k+1}\rangle$ is given by $|v_1\rangle = |w_1\rangle / |||w_1\rangle||$ and for $1 \leq k \leq d-1$ then $|v_{k+1}\rangle$ is defined by

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle}{|||w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle||} \tag{1}$$

I will prove for $i, j \in [1, d]$ we have $\langle v_i|v_j\rangle = \delta_{ij}$ by induction. That is,

- For $|v_1\rangle$, we have $\langle v_1|v_1\rangle = 1$.
- For $n = 1$ or namely, for $|v_2\rangle$, $\langle v_1|v_2\rangle = 0$ and $\langle v_2|v_2\rangle = 1$.
- If for $n = k$ we have $\langle v_j|v_{k+1}\rangle = \delta_{(k+1)j}$ for $j \leq i+1$, then we have for $n = k+1$, $\langle v_j|v_{k+2}\rangle = \delta_{(k+2)j}$ for $j \leq k+2$.

Here is the detail of proof of each statement.

- For $|v_1\rangle$, we have $|v_1\rangle = |w_1\rangle / |||w_1\rangle||$, then

$$\langle v_1|v_1\rangle = \frac{\langle w_1|w_1\rangle}{|||w_1\rangle||^2} = 1 \tag{2}$$

- For $n = 1$, we have

$$|v_2\rangle = \frac{|w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle}{|||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle||} \tag{3}$$

  Suppose $|a\rangle = |w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle$ then

$$\langle v_2|v_2\rangle = \frac{\langle a|a\rangle}{|||a\rangle||^2} = 1 \tag{4}$$

  Meanwhile, we have

$$\langle v_1|v_2\rangle = \frac{\langle v_1|w_2\rangle - \langle v_1|w_2\rangle\langle v_1|v_1\rangle}{|||w_2\rangle - \langle v_1|w_2\rangle |v_1\rangle||} = 0 \tag{5}$$

- Suppose $n = k$ if we have $\langle v_j|v_{k+1}\rangle = \delta_{(k+1)j}$, then when $n = k+1$, for $j \neq k+2$,

$$\begin{aligned} \langle v_j|v_{k+2}\rangle &= \frac{\langle v_j|w_{k+2}\rangle - \sum_{i=1}^{k+1} \langle v_i|w_{k+2}\rangle\langle v_j|v_i\rangle}{|||w_{k+2}\rangle - \sum_{i=1}^{k+1} \langle v_i|w_{k+2}\rangle |v_i\rangle||} \\ &= \frac{\langle v_j|w_{k+2}\rangle - \langle v_j|w_{k+2}\rangle\langle v_j|v_j\rangle}{|||w_{k+2}\rangle - \sum_{i=1}^{k+1} \langle v_i|w_{k+2}\rangle |v_i\rangle||} \\ &= \frac{\langle v_j|w_{k+2}\rangle - \langle v_j|w_{k+2}\rangle}{|||w_{k+2}\rangle - \sum_{i=1}^{k+1} \langle v_i|w_{k+2}\rangle |v_i\rangle||} = 0 \end{aligned} \tag{6}$$

For $j = k+2$, let $|a\rangle = |w_{k+1}\rangle - \sum_{i=1}^{k+1} \langle v_i|w_{k+2}\rangle |v_i\rangle$, then

$$\langle v_{k+2}|v_{k+2}\rangle = \frac{\langle a|a\rangle}{|||a\rangle||^2} = 1 \tag{7}$$

From eq. (6) and eq. (7), we conclude that for $n = k+1$, $\langle v_j|v_{k+2}\rangle = \delta_{(k+2)j}$ for $j \leq k+2$.