Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, August 22, 2020

The exam is closed book. Make sure that each page is signed with your code number (S-#) and the problem number. Do not write answers to different problems on the same page. Scan solutions to each problem as separate PDF files and upload as instructed before.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems! If you submit more than 3 problems, only the lowest 3 scores will count towards your total score.

The total time allowed **3 hrs**.

Problem II-1. (Classical Mechanics)

The Lagrangian of a particle of mass m moving in one dimension is given by

$$L = \frac{1}{2} m \dot{x}^2 + A \cos(\omega t)\, x \,,$$

where A and ω are both constants.

(i) Find the Hamiltonian which describes the motion of this particle and solve the Hamilton's equations of motion.

(ii) Set up the Hamilton-Jacobi equation for the Hamilton principal function $S(x, P, t)$. Find a complete solution for $S(x, P, t)$.

(iii) Solve for $x(t)$ and $p(t)$ as functions of time, t, using the complete solution for $S(x, P, t)$ found in (ii).

(iv) Assuming the initial conditions at time $t = 0$, $x(0) = 0$ and $\dot{x}(0) = v_0$, show that the solutions found in (i) and (iii) are indeed the same.

Hint: The following integrals might be useful

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C \,, \qquad \int \cos^2 x\, dx = \frac{x}{2} + \sin 2x + C \,.$$

Problem II-2. (Quantum Mechanics)

Consider a particle which moves in one dimension in a potential $V(x)$ that is invariant under an inversion through the origin, i.e. $V(-x) = V(x)$.

(i) Show that each eigenfunction of the Hamiltonian with nondegenerate eigenvalue is also an eigenfunction of the inversion operator. What are the eigenvalues of the inversion operator?

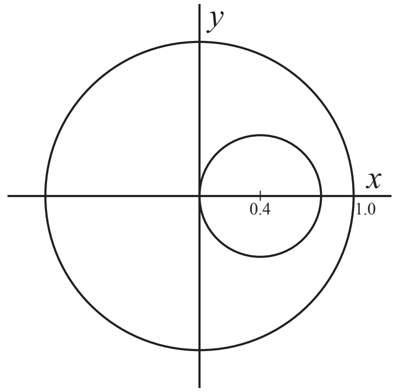
(ii) Consider a state which is represented by the *real unit-norm* wavefunction

$$\Psi(x) = A_1\, \Psi_1(x) + A_2\, \Psi_2(x) \,,$$

where $\Psi_1(x)$ and $\Psi_2(x)$ are *real unit-normalized* eigenfunctions of the Hamiltonian which have different eigenvalues of the inversion operator. What constraints do A_1 and A_2 satisfy?

(iii) Consider $\langle x \rangle$, the expectation value of x in the state Ψ. What are the values of A_1 and A_2 that (a) maximize $\langle x \rangle$, or (b) minimize $\langle x \rangle$?

Problem II - 3. (Electricity and Magnetism)



Consider a pair of long conducting cylinders, with circular cross section, one inside the other, running along the z-direction. They are long enough that you can ignore the effects of the ends. Note that they are *not* concentric cylinders! Let the outer cylinder be of radius 1, while the inner one be of radius 2/5. In the (x, y)-plane the cylinders are placed as in the diagram, where the inner circle passes through the center of the other.

Your task will be to study the potential $\varphi(\boldsymbol{r})$ and the electric field $\boldsymbol{E}(\boldsymbol{r})$ between the two conductors. The outer one has $\varphi = 0$, and the inner one has $\varphi = V$.

Consider the complex coordinate $w = x + iy$. The "complex potential"

$$f(w) = \varphi(x, y) + i\psi(x, y) \ ,$$

is an analytic function that is useful for solving this kind of (effectively two dimensional) problem. Here $\varphi(x, y)$ is the potential and $\psi(x, y)$ is a scalar that can (also) be used to reconstruct the electric field: $E_x = -\partial\psi/\partial y$, $E_y = \partial\psi/\partial x$.

(i) Show using the above information that lines of constant φ (*i.e.* equipotentials) intersect perpendicularly to lines of constant ψ (*i.e.* field lines).

(ii) Consider the map from w to a new complex coordinate $g = u + iv$ given by:

$$g(w) = \frac{2w - 1}{w - 2} \ .$$

Verify (by explicitly mapping at least three points of each circle) that it maps the two non-concentric cylinders to two concentric cylinders, where in the g-plane the outer one has radius 1 and the inner one has radius 1/2.

It can be shown that solutions for complex potentials in the g-plane map to solutions for complex potentials in the w-plane.

(iii) Solve for the complex potential in the g-problem.

(iv) Hence, map back to the w-problem and write the solution for $\varphi(x, y)$.

(v) Prove that equipotentials in the (x, y)-plane are circles. Write an equation for their radii and the locations of their centers.

(vi) Describe and sketch (on the same diagram) sample equipotentals and electric field lines.

Problem II-4. (Mathematical Methods)

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $(e_i)_{i\in\mathbb{N}}$ an orthonormal complete system over $\mathcal{H}$. Moreover, let

$$\sigma : \mathbb{N} \longrightarrow \mathbb{N} \quad / \quad i \longrightarrow \sigma(i)\,,$$

be a bijection of the natural numbers set, $\mathbb{N}$, onto itself. Let's define

$$A_\sigma : \mathcal{H} \longrightarrow \mathcal{H} \quad / \quad v \longrightarrow \sum_{i\in\mathbb{N}} \langle e_i, v\rangle\, e_{\sigma(i)}\,.$$

(i) Show that A_σ is defined on a dense subset of $\mathcal{H}$.

(ii) Find an explicit formula, like the one above, for the adjoint $A_\sigma^\dagger$.

(iii) Show that A_σ is unitary.

(iv) If $\chi \equiv (\chi_i)_{i\in\mathbb{N}}$ is a sequence in $\mathbb{C}$, when is the new operator

$$A_{\sigma,\chi} : v \longrightarrow \sum_{i\in\mathbb{N}} \chi_i\, \langle e_i, v\rangle\, e_{\sigma(i)}\,,$$

still unitary?