Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, August 28, 2021

The exam is closed book. Make sure that each page is signed with your code number (S-#) and the problem number. Do not write answers to different problems on the same page. Scan solutions to each problem as separate PDF files and upload as instructed before.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems! If you submit more than 3 problems, only the lowest 3 scores will count towards your total score.

The total time allowed **3 hrs**.

Problem II-1. (Classical Mechanics)

A particle of mass m and charge q moves with velocity $\boldsymbol{v}$ in a uniform, static magnetic field, $\boldsymbol{B} = B\,\widehat{\boldsymbol{z}}$, and hence is acted on by the Lorentz force

$$\boldsymbol{F} = \frac{q}{c}\,\boldsymbol{v} \times \boldsymbol{B}\,, \tag{1.1}$$

in appropriate units.

(i) Show that the Newton's equations of motion of the particle follow from the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x})\,. \tag{1.2}$$

(ii) Argue that spatial translations in any direction are symmetries of this mechanical system and use the Noether theorem to find the corresponding constant of motion.

(iii) Similarly, show that the rotations about the z-axis are continuous symmetries and find the corresponding constant of motion.

(iv) Find the infinitesimal canonical transformations in the particle's phase space generated by the constants of motion found in (ii).

Problem II-2. (Statistical Physics)

A container with volume V is filled with a fluid at temperature T composed of identical particles with short-range repulsive interactions. The particles can enter and leave the container, at (fixed) chemical potential μ. We use a lattice model to describe this system, in which we divide the container into N cells, each with a volume $v = V/N$ comparable to the particle volume. We assume that unoccupied cells and cells occupied by a single particle have zero energy, that a cell occupied by two particles has energy $\epsilon > 0$, and that a cell cannot contain more than two particles. We denote the number of particles in the container by M.

(i) Construct the canonical partition function of the system, $Z(T, V, M)$, in terms of summations and explain the form of the summations and each term in $Z(T, V, M)$. Hence, show that the grand partition function of this system, $\mathcal{Q}(T, V, \mu)$, can be written as

$$\mathcal{Q}(T, V, \mu) = \left(1 + z + e^{-\beta\epsilon} z^2\right)^N\,, \tag{2.1}$$

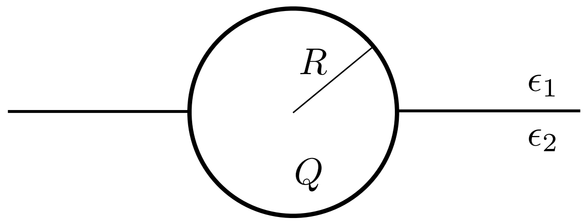
where $\beta = 1/k_B T$ and the fugacity $z = e^{\beta\mu}$.

Hint: $\displaystyle\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n = (1+x)^N$.

(ii) Use Eq. (2.1) to find, at thermodynamic equilibrium, the average number of particles per cell, c, and the pressure P in terms of T and μ.

(iii) Using your results from part (b), find P in terms of T and c in the limit of small c. Provide a physical interpretation of your result.

Problem II-3. (Electricity and Magnetism)



A conducting sphere of radius R and charge Q sits at the origin of the xyz-coordinate system. The space outside the sphere is filled by two different linear homogeneous dielectrics: above the xy-plane, the dielectric constant is ϵ_1, below the xy-plane the dielectric constant is ϵ_2, see the figure.

(i) Find the potential and the electric field everywhere outside the conductor, using the solution of Laplace's equation in spherical coordinates and azimuthal symmetry. Exploiting the fact that a conductor is an equipotential object, you should find that only the $\ell = 0$ term contributes to the series of Legendre polynomials of order ℓ of argument $\cos\theta$, where θ is the polar angle in spherical coordinates. To determine the coefficient of the $\ell = 0$ term, use Gauss's law for the electric displacement $\boldsymbol{D}$.

(ii) Find the distribution of free charge, using Gauss's law for the electric displacement.

(iii) Find the distribution of polarization charge at the surface of the conductor. Pay attention to the direction of the unit vector $\widehat{\boldsymbol{n}}$ in the definition of surface polarization charge, which should go outward from the polarized region.

(iv) Explain why there is no polarization charge on the xy-plane, at the interface between the two dielectric materials.

Problem II - 4. (Quantum Mechanics)

A particle of mass m is confined to stay between two concentric spheres of radii a and b ($a < b$), respectively, but otherwise can move freely within this 3-dimensional volume. Then the potential energy $V(r)$ is zero in the space between the spheres and is infinite elsewhere.

(i) *Briefly* describe (without proof) the steps you would take to reduce the 3-dimensional Schrödinger equation for the particle to an effective one-dimensional equation along the radial direction. Give the radial equation for any angular momentum ℓ in all regions of space (a proof is not required, just state the result). Sketch a plot of the effective potential.

(ii) State what the available energy range is, and describe the classical motion of the particle in words in this energy range. From this, determine what the boundary conditions should be for the radial wavefunction for any angular momentum ℓ.

(iii) Using the uncertainty principle, give a rough estimate of the radial momentum, and then estimate the minimum energy for any ℓ.

(iv) Solve the radial Schrödinger equation for the case of zero angular momentum, $\ell = 0$. Discuss how you select the right solution from the two you should have found.

(v) Using the solution above, compute explicitly the quantized energy eigenvalues, E_n, for all excited states with $\ell = 0$. Give E_n as a function of m, a, b and a quantum number n, and list the possible values of n. How does the ground state energy compare to the estimate in part (iii)?

Hint: If you cannot complete some parts due to lack of time, try to outline the remaining steps to receive some partial credit.