Autumn 2011

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

September 14, 2011

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

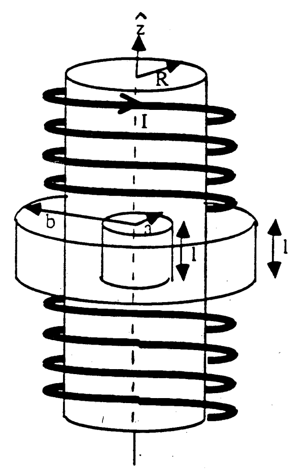
NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Short Answer

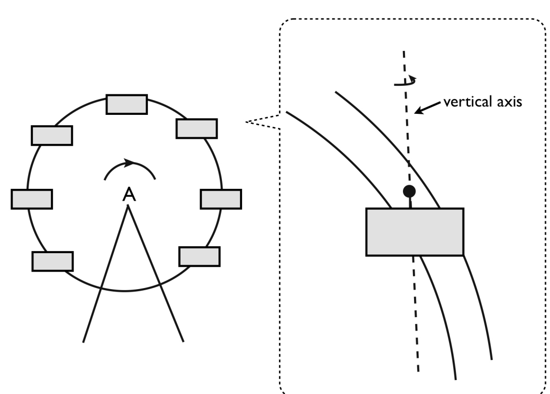
(a) Starting with a single drop (0.05 mL), if you double the amount of water placed in an 15-L bucket every 10 seconds, how long will it take you to fill the bucket?

(b) A collimated beam of monochromatic visible light enters your laboratory through a hole in the wall. Describe explicitly the equipment you would use and what measurements you would make to determine the wavelength of the light.

(c) Consider a very long solenoid carrying a current I as shown. Two long cylindrical shells of length ℓ are suspended coaxially about the solenoid and are free to rotate about their common vertical axis. The inner shell has a uniform surface charge $+Q$ and the outer shell carries a uniform surface charge $-Q$. Assume $\ell \gg b > a$ (the figure is not to scale). As the current is reduced to zero, the cylinders begin to rotate and are left with a net nonzero angular momentum. Since the mechanical angular momentum of the solenoid and the shells is initially zero, qualitatively explain the origin of this angular momentum. [Hint: It is not accounted for by the current in the solenoid.]



(d) On a certain Ferris wheel the gondolas can be rotated by their passengers around vertical axes, as shown in the figure above. Suppose initially the wheel is rotating with a constant angular velocity around its horizontal axis A, but the gondolas are not rotating around their vertical axes. Then all passengers start to rotate the gondolas in the same direction. At that moment, what is the direction of the torque exerted by the support at A on the Ferris wheel?



Short Answer - Solution

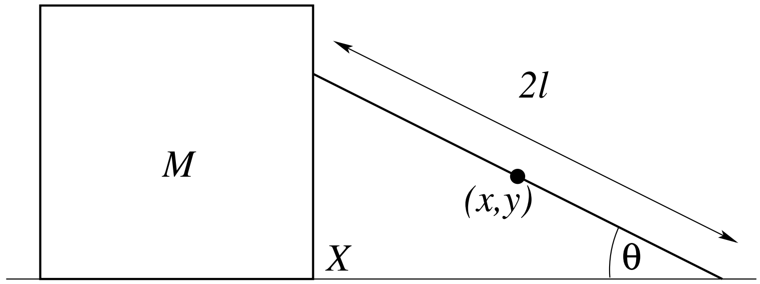
(a) Since we double the amount of water in the bucket every ten seconds, its volume increases as $V = V_1 2^{t/10\,\mathrm{s}}$, where V_1 is the volume of a single drop. Thus, $t = 10\log_2(\frac{V}{V_1})$. For $V = 15$ L we get $V \approx 182\,\mathrm{s}$ which is about 3 min.

(b) Use diffraction gratings. Project the diffraction patterns on a screen a known distance from a grating and measure the separation between the intensity maxima.

(c) The angular momentum of the cylinders come from the angular momentum density ($\sim \vec{r} \times (\vec{E} \times \vec{B})$) stored in the crossed electric and magnetic fields.

(d) Initially the total angular momentum of the Wheel is horizontal. Then because of the rotation of the gondolas, it starts to acquire a vertical component. Therefore the time derivative of the momentum is vertical, and the torque should be in the same direction.

2. Block-and-ladder



A ladder of mass m and length $2l$ leans against a block of mass M as shown in the picture. The friction forces between the ladder, the block and the floor are all negligible and the entire system starts from rest. Choose as generalized coordinates the horizontal position of the right side of the block X, the angle between the ladder and the floor θ, and the position of the ladder's center of mass (x, y).

(a) Write the Lagrangian for this system in terms of the generalized coordinates and their derivatives.

(b) Write the constraints between the generalized coordinates introduced above. Use them to express x and y in terms of X and θ. Exclude x and y from the Lagrangian and write the equations of motion for the system.

(c) Find all of the conserved quantities in the system. Using the conserved quantities, exclude all coordinates except θ and solve for the motion. [You may leave your solution in the form of an integral.]

(d) What (simple) condition is satisfied when the ladder detaches from the block? Assuming that $M \gg m$, find the angle θ_* of the ladder at the moment when it detaches from the block (as a function of the initial angle θ_0).

Block and Ladder - Solution

(a) The kinetic energy of the system consists of three parts: the kinetic energy of the block, the energy of the center of mass motion of the ladder, and the rotational energy of the ladder:

$$K = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2. \tag{1}$$

The moment of inertia of the ladder with respect to its center of mass is

$$I = \frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2. \tag{2}$$

The potential energy of the block is constant, so we only need to consider the potential energy of the ladder $U_{\text{ladder}} = mgy$. This gives the Lagrangian

$$L = K - U = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{6}ml^2\dot{\theta}^2 - mgy. \tag{3}$$

(b) The generalized coordinates are related to each other, as can be seen by considering the ladder's center of mass:

$$x = X + l\cos\theta, \qquad y = l\sin\theta. \tag{4}$$

Using these constraints we can eliminate x and y and their time derivatives from the Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\big[(\dot{X} - l\sin\theta\,\dot{\theta})^2 + l^2\cos^2\theta\,\dot{\theta}^2\big] + \frac{1}{6}ml^2\dot{\theta}^2 - mgl\sin\theta \\ &= \frac{1}{2}(M+m)\dot{X}^2 - ml\sin\theta\,\dot{X}\dot{\theta} + \frac{2}{3}ml^2\dot{\theta}^2 - mgl\sin\theta. \end{aligned} \tag{5}$$

To write the equations of motion, we need the partial derivatives

$$\frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} - ml\sin\theta\,\dot{\theta}, \qquad \frac{\partial L}{\partial X} = 0, \tag{6}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{4}{3}ml^2\dot{\theta} - ml\sin\theta\,\dot{X}, \qquad \frac{\partial L}{\partial \theta} = -mgl\cos\theta - ml\cos\theta\,\dot{\theta}^2. \tag{7}$$

The Euler-Lagrange equations of motion are

$$(M+m)\ddot{X} - ml\sin\theta\,\ddot{\theta} - ml\cos\theta\,\dot{\theta}^2 = 0, \quad \frac{4}{3}ml^2\ddot{\theta} - ml\sin\theta\,\ddot{X} + mgl\cos\theta = 0. \tag{8}$$

(c) We neglect friction, so the total energy is conserved:

$$E = \frac{1}{2}(M+m)\dot{X}^2 - ml\sin\theta\,\dot{X}\dot{\theta} + \frac{2}{3}ml^2\dot{\theta}^2 + mgl\sin\theta = mgl\sin\theta_0. \quad (9)$$

To find the value of the conserved energy we used the fact that in the initial state all the velocities are zero, so only the potential energy of the ladder contributes. Energy conservation also formally follows from the independence of the Lagrangian on time (time translation invariance).

Another conserved quantity follows from equation (6). The variable X is cyclic, and the corresponding generalized momentum is conserved:

$$(M+m)\dot{X} - ml\sin\theta\,\dot{\theta} = 0, \quad (10)$$

where the value of this momentum follows from the initial values of the velocities. This momentum is nothing but the x component of the momentum of the system. Its conservation follows from translational invariance in the x direction.

Using the conservation law (10), we can exclude $\dot{X}$ from the energy. This goes as

$$\dot{X} = \frac{ml}{M+m}\sin\theta\,\dot{\theta}, \quad (11)$$

$$\begin{aligned} E &= \frac{m^2l^2}{2(M+m)}\sin^2\theta\,\dot{\theta}^2 - \frac{m^2l^2}{M+m}\sin^2\theta\,\dot{\theta}^2 + \frac{2}{3}ml^2\dot{\theta}^2 + mgl\sin\theta \\ &= \frac{2}{3}ml^2\dot{\theta}^2 - \frac{m^2l^2}{2(M+m)}\sin^2\theta\,\dot{\theta}^2 + mgl\sin\theta = mgl\sin\theta_0. \end{aligned} \quad (12)$$

Solving this for $\dot{\theta}$, we get

$$\dot{\theta}^2 = \frac{3g}{2l}\frac{\sin\theta_0 - \sin\theta}{1 - \frac{3m}{4(M+m)}\sin^2\theta}. \quad (13)$$

Taking the square root and separating the variables finally gives

$$t = \sqrt{\frac{2l}{3g}}\int_0^\theta \left(\frac{1 - \frac{3m}{4(M+m)}\sin^2\theta}{\sin\theta_0 - \sin\theta}\right)^{1/2} d\theta. \quad (14)$$

This implicitly gives $\theta(t)$.

To find X, we use the consequence (11) of the momentum conservation. It gives

$$X(t) = X_0 - \frac{ml}{M+m}(\cos\theta(t) - \cos\theta_0). \tag{15}$$

Finally, x and y are found from the constraints (4).

(d) When the ladder stops pushing against the block, the block stops accelerating (the normal force that the ladder applies on the block vanishes). Therefore, at the detachment angle θ_*, the system will satisfy the (simple) condition $\ddot{X}(\theta_*) = 0$.

Using this condition, we simplify the Euler-Lagrange equations (8) at the last moment of contact between the block and the ladder:

$$\sin\theta_*\,\ddot{\theta}_* + \cos\theta\,\dot{\theta}_*^2 = 0, \qquad \frac{4}{3}l\ddot{\theta}_* + g\cos\theta_* = 0. \tag{16}$$

Excluding from here $\ddot{\theta}_*$ we obtain

$$\dot{\theta}_*^2 = \frac{3g}{4l}\sin\theta_*. \tag{17}$$

On the other hand, we can use the general equation (13) at the same moment:

$$\dot{\theta}_*^2 = \frac{3g}{2l}\frac{\sin\theta_0 - \sin\theta_*}{1 - \dfrac{3m}{4(M+m)}\sin^2\theta_*}. \tag{18}$$

Equating the two expressions for $\dot{\theta}_*^2$ leads to an equation for the angle θ_*:

$$\frac{3m}{4(M+m)}\sin^3\theta_* - 3\sin\theta_* + 2\sin\theta_0 = 0. \tag{19}$$

In the limit $M \gg m$ we can neglect the first term in this cubic equation, which then becomes linear and gives

$$\theta_* = \arcsin\left(\frac{2}{3}\sin\theta_0\right). \tag{20}$$

3. Charged Incompressible Fluid

You are given one gallon of incompressible fluid with constant charge density. You must choose a configuration for the fluid in order to achieve the largest possible value for the electric field at a given point in space, which we can call the origin.

(a) Should the origin lie inside, outside or on the surface of the fluid?

(b) How much symmetry can you impose on the fluid configuration?

(c) Find the configuration.

Charged Incompressible Fluid - Solution

(a) The origin should lie on the surface. If it lies inside the fluid, there is some cancelation of the electric field created by different fluid elements on the opposite sides of the origin. On the other hand, if the origin is moved away from the fluid, the field at the origin just decays.

(b) Now imagine maximizing the electric field in the z-direction. By imagining moving around fluid elements, you can convince yourself that the shape must be axially-symmetric.

(c) There are at least two solutions. The short one goes as follows:

Imagine a fluid element with the charge dq at some distance r from the origin. The z-component of the electric field produced by this element is $\frac{dq}{r^2}\cos\theta$. The slickest way to find the solution is to note that this fluid element should produce the same z-field at the origin regardless of where it is placed on the surface. So

$$\frac{1}{r^2}\cos\theta = \frac{z}{r^3} = \text{const.}$$

The more detailed one goes as follows:

Choose spherical coordinates with the origin at the point of maximal field, and the z-axis corresponding to the symmetry axis of the body.

Let us assume the charge density of the fluid is one. Then the charge of a fluid element is

$$dq = r^2 dr \sin\theta d\theta d\phi, \tag{21}$$

and the component of the field along the z-axis is

$$dE_z = \frac{dq}{r^2}\cos\theta = \sin\theta\cos\theta d\theta d\phi dr. \tag{22}$$

The shape of the fluid is given by some choice $r = r(\theta)$. The absolute value of the electric field at the origin is given by,

$$E_z \;=\; 2\pi \int_0^{\pi/2} d\theta \, \sin\theta\cos\theta \int_0^{r(\theta)} dr$$

$$= \quad 2\pi \int_0^{\pi/2} d\theta \, r(\theta) \sin\theta \cos\theta. \tag{23}$$

We need to extremize this functional of $r(\theta)$ subject to the constraint of fixed total volume. The total volume is given by

$$\begin{aligned} V &= 2\pi \int_0^{\pi/2} d\theta \, \sin\theta \int_0^{r(\theta)} dr \, r^2 \\ &= \frac{2\pi}{3} \int_0^{\pi/2} d\theta \, r^3(\theta) \sin\theta. \end{aligned} \tag{24}$$

Introducing a Lagrange multiplier λ to account for the volume constraint, we have

$$\frac{\delta}{\delta r(\theta)} (E_z - \lambda V) \quad = \quad 2\pi \left[\sin\theta \cos\theta - \lambda r^2(\theta) \sin\theta\right] = 0, \tag{25}$$

from which we see that

$$r(\theta) = \sqrt{\frac{\cos\theta}{\lambda}}. \tag{26}$$

To fix the Lagrange multiplier λ, we use the volume constraint. The total volume is given by,

$$\begin{aligned} V &= \frac{2\pi}{3\lambda^{3/2}} \int_0^{\pi/2} \cos^{3/2}\theta \sin\theta d\theta \\ &= \frac{2\pi}{3\lambda^{3/2}} \left(-\frac{2}{5} \cos^{5/2}\theta \Big|_0^{\pi/2} \right) \\ &= \frac{4\pi}{15\lambda^{3/2}}, \end{aligned} \tag{27}$$

so

$$\lambda = \left(\frac{4\pi}{15V} \right)^{2/3}. \tag{28}$$

Not that it matters for determining the functional form of the shape but the total volume is one gallon in this problem.

4. Quantum Virial theorem

A particle of mass m is moving in a potential $V(\mathbf{r})$. Consider a dynamical variable $F(\mathbf{r}, \mathbf{p})$, which is a function of the position and momentum of the particle.

(a) Suppose the particle is in an eigenstate $|\psi_n\rangle$ of the Hamiltonian. Show that

$$\frac{d}{dt}\langle\psi_n|F|\psi_n\rangle = 0.$$

(b) Choose a special (simple) observable $F(\mathbf{r}, \mathbf{p})$, and using it derive the following identity:

$$\langle\psi_n|\frac{\mathbf{p}^2}{2m}|\psi_n\rangle = \frac{1}{2}\langle\psi_n|\mathbf{r}\cdot\nabla V(\mathbf{r})|\psi_n\rangle.$$

Hint: use the Heisenberg picture of the time evolution.

(c) Consider the potential $V(\mathbf{r})$ which is homogeneous of degree k, that is, $V(\lambda\mathbf{r}) = \lambda^k V(\mathbf{r})$, for any $\lambda > 0$. Relate the expectation values of the kinetic and the potential energies of the particle in a given stationary state in this case. Find the values of k for the Coulomb potential, a spherical harmonic oscillator, and a ball bouncing vertically near the surface of the Earth.

Quantum Virial theorem - Solution

(a) One can proceed using the Schrodinger or the Heisenberg picture. In the first case the sates evolve while operators do not. Then we have

$$\frac{d}{dt}\langle\psi_n(t)|F|\psi_n(t)\rangle = \frac{d}{dt}\langle\psi_n|\exp\Big(\frac{i}{\hbar}E_nt\Big)F\exp\Big(-\frac{i}{\hbar}E_nt\Big)|\psi_n\rangle$$
$$= \frac{i}{\hbar}\langle\psi_n(t)|E_nF - FE_n|\psi_n(t)\rangle = 0. \tag{29}$$

In the Heisenberg picture we have

$$\frac{d}{dt}\langle\psi_n|F|\psi_n\rangle = \langle\psi_n|\frac{dF}{dt}|\psi_n\rangle, \tag{30}$$

where the time derivative of F is given by the Heisenberg equation

$$\frac{dF}{dt} = \frac{1}{i\hbar}[F,H]. \tag{31}$$

Using $H|\psi_n\rangle = E_n|\psi_n\rangle$, we have

$$\langle\psi_n|\frac{dF}{dt}|\psi_n\rangle = \frac{1}{i\hbar}\langle\psi_n|[F,H]|\psi_n\rangle = \frac{i}{\hbar}\langle\psi_n(t)|E_nF - FE_n|\psi_n(t)\rangle = 0. \tag{32}$$

Therefore, $\frac{d}{dt}\langle\psi_n|F|\psi_n\rangle = 0$.

(b) Here we need a clever choice of the observable F. Consider the case $F(\mathbf{r},\mathbf{p}) = \mathbf{r}\cdot\mathbf{p}$. We have

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{p}) = \frac{d\mathbf{r}}{dt}\cdot\mathbf{p} + \mathbf{r}\cdot\frac{d\mathbf{p}}{dt}. \tag{33}$$

Using the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) \tag{34}$$

and computing commutators we get the time derivatives

$$\frac{d\mathbf{r}}{dt} = \frac{1}{i\hbar}[\mathbf{r},H] = \frac{1}{i\hbar}\frac{1}{2m}[\mathbf{r},\mathbf{p}^2] = \frac{\mathbf{p}}{m}, \quad \frac{d\mathbf{p}}{dt} = \frac{1}{i\hbar}[\mathbf{p},H] = \frac{1}{i\hbar}[\mathbf{p},V(\mathbf{r})] = -\nabla V(\mathbf{r}). \tag{35}$$

Using the conclusion of part 1, we have the desired result

$$\langle\psi_n|\frac{\mathbf{p}^2}{2m}|\psi_n\rangle = \frac{1}{2}\langle\psi_n|\mathbf{r}\cdot\nabla V(\mathbf{r})|\psi_n\rangle. \tag{36}$$

(c) For a homogeneous potential we can use the Euler's theorem:

$$\mathbf{r} \cdot \nabla V(\mathbf{r}) = kV(\mathbf{r}). \tag{37}$$

(To derive the theorem, let us differentiate the homogeneity condition $V(\lambda\mathbf{r}) = \lambda^k V(\mathbf{r})$ with respect to λ:

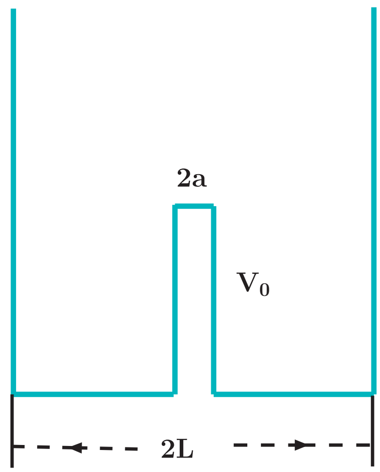
$$x_i \frac{\partial V(\lambda\mathbf{r})}{\partial x_i} = k\lambda^{k-1} V(\mathbf{r}). \tag{38}$$

Setting $\lambda = 1$ we get the statement of Euler's theorem.) Using the theorem together with the result of part 2, we get

$$\langle\psi_n|\frac{\mathbf{p}^2}{2m}|\psi_n\rangle = \frac{k}{2}\langle\psi_n|V(\mathbf{r})|\psi_n\rangle. \tag{39}$$

For the Coulomb potential $V(\mathbf{r}) = \alpha/r$ we have $k = -1$, for a harmonic oscillator $V(\mathbf{r}) = \alpha r^2$ we have $k = 2$. For a bouncing ball we have $V(\mathbf{r}) = mgz$, where z is the vertical coordinate above the Earth's surface. In this case $k = 1$.

5. Square Well



A particle of mass m, is moving in the potential well:

$$V(x) = \begin{cases} V_0, & |x| \leq a, \\ 0, & a < |x| < L, \\ \infty, & |x| \geq L. \end{cases}$$

Find the condition that the parameters of the potential (a, L, and V_0) satisfy when there is a state with the energy $E = V_0$. Solve this condition for V_0 as a function of a and L for the n-th even state (including the ground state for $n = 0$).

Square Well - Solution

Consider the case when the n-th energy level has the energy $E_n = V_0$, and denote

$$k = \frac{1}{\hbar}\sqrt{2mV_0}. \tag{40}$$

The Schrodinger equation in the regions $-L < x < -a$ has a general solution $\psi(x) = A_1 e^{ikx} + B_1 e^{-ikx}$. The boundary condition $\psi(x = -L) = 0$ leads to $\psi(x) = A\sin[k(x+L)]$. Similarly, in the region $a < x < L$ we get $\psi(x) = B\sin[k(x-L)]$. In the middle region $-a < x < a$ we get $\psi(x) = C + Dx$. Thus,

$$\psi(x) = \begin{cases} A\sin[k(x+L)], & \text{for} \quad x \in [-L, -a], \\ C + Dx, & \text{for} \quad x \in [-a, a], \\ B\sin[k(x-L)], & \text{for} \quad x \in [a, L]. \end{cases} \tag{41}$$

Since the potential is an even function of x, all stationary states are characterized by a certain parity. For *even* states $\psi(-x) = \psi(x)$ the coefficients A, B, C, D introduced above are related as

$$A = -B, \qquad\qquad D = 0. \tag{42}$$

Matching the values of the wave function and its derivative at a (or $-a$) we get

$$A\sin[k(L-a)] = C, \qquad\qquad kA\cos[k(L-a)] = 0. \tag{43}$$

The last equation gives the required condition on the parameters of the potential for even states:

$$\cos[k(L-a)] = 0. \tag{44}$$

This can be explicitly solved:

$$\begin{aligned} k(L-a) &= \frac{\pi}{2} + \pi n, & n &= 1, 2, 3, \ldots \\ k &= \frac{(2n+1)\pi}{2(L-a)}, & V_0 &= \frac{(2n+1)^2\pi^2\hbar^2}{8m(L-a)^2}. \end{aligned} \tag{45}$$

In particular, the ground states ($n = 0$) has the energy V_0 when

$$V_0 = \frac{\pi^2\hbar^2}{8m(L-a)^2}. \tag{46}$$

For odd states $\psi(-x) = -\psi(x)$ we have, similarly,

$$A = B, \qquad\qquad C = 0. \tag{47}$$

Matching the values of the wave function and its derivative at a (or $-a$) we get

$$A\sin[k(L-a)] = -Da, \qquad\qquad kA\cos[k(L-a)] = D. \tag{48}$$

The ratio of these two equations gives the required condition on the parameters of the potential for odd states:

$$\tan[k(L-a)] = -ka. \tag{49}$$

6. Fermions in a Gravity Field

(a) A quantum mechanical particle of mass M is placed in the gravitational field of a point mass so that it experiences the potential $U(r) = -\alpha/r$. The energy levels in this potential are given by

$$E_n = -\frac{M\alpha^2}{2\hbar^2 n^2} = -\frac{E_1}{n^2},$$

and are independent of the azimuthal and magnetic quantum numbers l and m. For a given principal quantum number n the azimuthal number can take values $l = 0, 1, 2, \ldots, n-1$. Find the degeneracy d_n of the level E_n.

(b) Consider N identical non-interacting fermions, each of mass M, placed in the same field $U(r) = -\alpha/r$. Assuming that the number of particles is very large, $N \gg 1$, express the Fermi energy E_F and the total energy E_{tot} of the system in terms of E_1 and N.

[Note: The Fermi energy of a system of fermions is the energy of the highest occupied level when the system is in the ground state.]

Fermions in a Gravity Field - Solution

(a) We know that for a given principal quantum number n the azimuthal quantum number l can take n possible values $l = 0, 1, 2, \ldots, n-1$. For each value of l the magnetic quantum number can take $2l+1$ possible values $m = -l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l$. Therefore, the degeneracy of the n-th level (whose energy E_n does not depend on l or m) is

$$d_n = \sum_{l=0}^{n-1} (2l+1) = n^2. \tag{50}$$

(b) The fermions occupy the levels E_n one by one. If the fermions have spin s (can take it to be zero), then the n-th level can accommodate $(2s+1)d_n = (2s+1)n^2$ fermions. Therefore, if the last occupied level has the principal number n_F, the total number of particles is

$$N = (2s+1)\sum_{n=1}^{n_F} d_n = (2s+1)\sum_{n=1}^{n_F} n^2 = \frac{2s+1}{6} n_F(n_F+1)(2n_F+1). \tag{51}$$

Actually, with the assumption $N \gg 1$ we can also assume that $n_F \gg 1$ and keep only the leading term: $N \approx (2s+1)n_F^3/3$. (The same result would be obtained if we replaced the sum over n by an integral in the last equation). Thus, we have $n_F \approx [3N/(2s+1)]^{1/3}$, and the Fermi energy is

$$E_F = -\frac{M\alpha^2}{2\hbar^2 n_F^2} \approx -\frac{M\alpha^2}{2\hbar^2}\left(\frac{2s+1}{3N}\right)^{2/3} = -E_1\left(\frac{2s+1}{3N}\right)^{2/3}. \tag{52}$$

The total energy of all N fermions is found as

$$\begin{aligned} E_{\text{tot}} &= (2s+1)\sum_{n=1}^{n_F} d_n E_n = -\frac{M\alpha^2}{2\hbar^2}(2s+1)\sum_{n=1}^{n_F} 1 = -\frac{M\alpha^2}{2\hbar^2}(2s+1)n_F \\ &\approx -\frac{M\alpha^2}{2\hbar^2}(2s+1)^{2/3}(3N)^{1/3} = -E_1(2s+1)^{2/3}(3N)^{1/3}. \end{aligned} \tag{53}$$

If we consider spinless fermions ($s = 0$), the answers simplify to

$$E_F = -E_1(3N)^{-2/3}, \qquad E_{\text{tot}} = -E_1(3N)^{1/3}. \tag{54}$$

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Ph.D. CANDIDACY EXAMINATION

Day 2

September 15, 2011

(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

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7. Rope on a cylinder

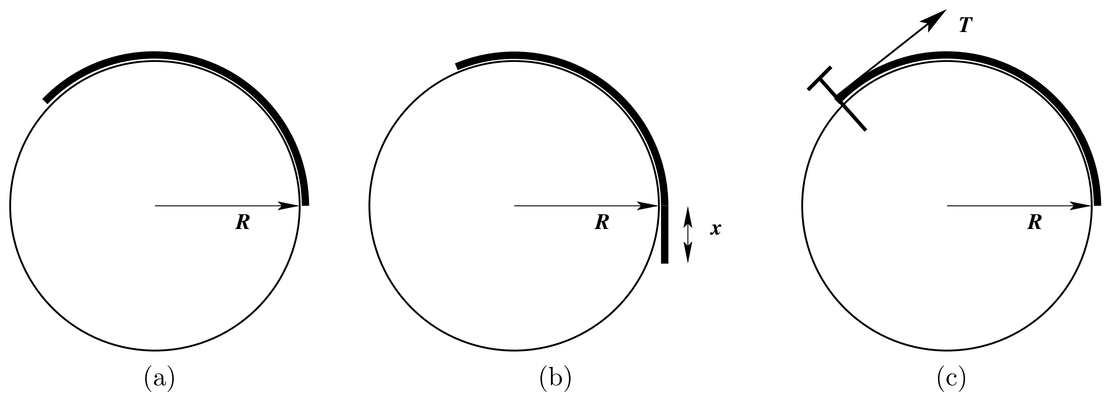


Figure 1: (a) The initial position of the rope. (b) A portion x of the rope has slid off the cylinder. (c) The rope attached to a nail.

A heavy uniform rope of mass m and length l is placed on a cylinder of radius R $(0 < l < \pi R)$ as shown in Fig. 1a.

(a) First assume there is no friction, and the initial velocity of the rope is zero. The rope then starts sliding down along the cylinder. Find its final velocity v when the last portion of the rope slides off the cylinder.

(b) Assume the same setup as in part (a). Use the length x of the portion of the rope that is not touching the cylinder as a generalized coordinate; see Fig. 1b. Write a Lagrangian for this system, find the equations of motion as well as a conserved quantity. Using the conserved quantity, express the time τ it takes for the rope to slide off the cylinder as a definite integral.

(c) To prevent the rope from sliding, its upper end is attached to a nail; see Fig. 1c. Find the force T experienced by the nail from the rope. What is the maximum tension in the rope and at what point is it achieved?

Rope on a cylinder - Solution

(a) Since there is no friction, the total mechanical energy is conserved. Counting potential energy U from the horizontal plane through the axis of the cylinder, we have initially

$$U_1 = \frac{m}{l} g \int_0^{l/R} R \sin\phi \; R d\phi = \frac{mgR^2}{l}\Big(1 - \cos\frac{l}{R}\Big). \tag{1}$$

The kinetic energy at this moment is $T_1 = 0$. At the end, when the rope is not touching the cylinder, its center of mass is located at the height $-l/2$ relative to the origin. Therefore, the final values of the potential and kinetic energies are

$$U_2 = -\frac{1}{2} mgl, \qquad\qquad T_2 = \frac{1}{2} mv^2. \tag{2}$$

Equating the total energies $E = U + T$ at both times we get

$$\frac{1}{2} mv^2 = \frac{1}{2} mgl + \frac{mgR^2}{l}\Big(1 - \cos\frac{l}{R}\Big), \quad v = \left[gl + \frac{2gR^2}{l}\Big(1 - \cos\frac{l}{R}\Big)\right]^{1/2}. \tag{3}$$

(b) The kinetic energy is $T = m\dot{x}^2/2$, the potential energy is

$$U = \frac{m}{l} g \int_0^{(l-x)/R} R \sin\phi \; R d\phi - \frac{mg}{2l} x^2 = \frac{mgR^2}{l}\Big(1 - \cos\frac{l-x}{R}\Big) - \frac{mg}{2l} x^2. \tag{4}$$

The Lagrangian is

$$L = T - U = \frac{1}{2} m\dot{x}^2 + \frac{mg}{2l} x^2 - \frac{mgR^2}{l}\Big(1 - \cos\frac{l-x}{R}\Big). \tag{5}$$

The derivatives

$$\frac{\partial L}{\partial x} = \frac{mg}{l} x + \frac{mgR}{l} \sin\frac{l-x}{R}, \qquad\qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}. \tag{6}$$

The Euler-Lagrange equations of motion are

$$\ddot{x} = \frac{g}{l} x + \frac{gR}{l} \sin\frac{l-x}{R}. \tag{7}$$

t

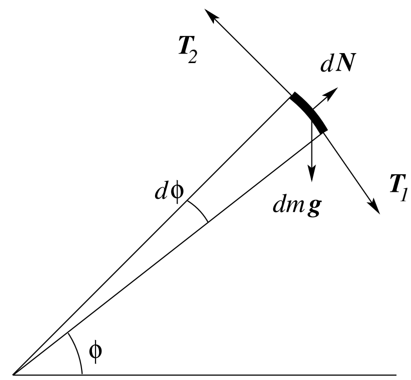


Figure 2: A rope element in equilibrium.

Time translations are a symmetry of this system since the Lagrangian L does not depend explicitly on t. This implies that the total energy

$$E = T + U = \frac{1}{2}m\dot{x}^2 - \frac{mg}{2l}x^2 + \frac{mgR^2}{l}\Big(1 - \cos\frac{l-x}{R}\Big) \tag{8}$$

is conserved. Its initial value is

$$E_0 = \frac{mgR^2}{l}\Big(1 - \cos\frac{l}{R}\Big). \tag{9}$$

Energy conservation gives a first order equation for $x(t)$:

$$\dot{x}^2 = \frac{g}{l}x^2 + \frac{2gR^2}{l}\Big(\cos\frac{l-x}{R} - \cos\frac{l}{R}\Big). \tag{10}$$

Separating variables we can integrate this equation. Let $t = 0$ be the time when the rope starts to slide, we get

$$t = \sqrt{\frac{l}{g}}\int_0^x \frac{dx}{\sqrt{x^2 + 2R^2\Big(\cos\frac{l-x}{R} - \cos\frac{l}{R}\Big)}}. \tag{11}$$

In particular, the time it takes the rope to slide off the cylinder completely is given by

$$\tau = \sqrt{\frac{l}{g}}\int_0^l \frac{dx}{\sqrt{x^2 + 2R^2\Big(\cos\frac{l-x}{R} - \cos\frac{l}{R}\Big)}}. \tag{12}$$

(c) To solve this part of the problem, we need to consider an infinitesimal element of the rope and write the condition of its equilibrium. The tension in the rope is a function of the position, which can be specified by the angle ϕ. Let us look at the element that spans the range of angles $[\phi, \phi + d\phi]$, see Fig. 2. The forces add up to zero:

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + d\boldsymbol{N} + dm\boldsymbol{g} = 0. \tag{13}$$

For this part of the problem we only need the component of this vector equation tangent to the rope:

$$T_2 - T_1 - dmg\cos\phi = 0. \tag{14}$$

Here we have $T_1 = T(\phi)$, $T_2 = T(\phi + d\phi) = T(\phi) + dT$, $dm = (m/l)Rd\phi$. Substituting this leads to the differential equation

$$\frac{dT}{d\phi} = mg\frac{R}{l}\cos\phi. \tag{15}$$

The natural initial condition for this equation is $T(0) = 0$. Then the solution is

$$T(\phi) = mg\frac{R}{l}\sin\phi. \tag{16}$$

In particular, the tension at the end of the rope attached to the nail is

$$T = mg\frac{R}{l}\sin\frac{l}{R}. \tag{17}$$

We can also easily find the maximal tension. If the rope length $l < \pi R/2$, then the maximal tension is where the rope is attached to the nail. However, if the rope is longer that the quarter of the circumference of the cylinder, the maximal tension is achieved at the point $\phi = \pi/2$. Thus

$$T_{\max} = \begin{cases} mg\dfrac{R}{l}\sin\dfrac{l}{R}, & \text{if} \quad l < \pi R/2, \\ mg\dfrac{R}{l}, & \text{if} \quad l > \pi R/2. \end{cases} \tag{18}$$

8. A Physical Particle?

In flat space-time, a particle follows the path (parameterized by σ) given by

$$x^\mu = (x^0, x^1, x^2, x^3) = (\sinh(\sigma), \alpha\sigma, \cosh(\sigma), 0)$$

where x^0 represents the time component, σ runs from 0 to 1, and α is a real number.

(a) Determine the relationship between the proper time τ and σ. [Note: the proper time between two events is the time measured in an inertial frame in which the events occur at the same location in space.]

(b) Determine the components of the four velocity of this particle.

(c) For what values of α might this path describe the motion of an actual (physical) particle?

(d) For $\alpha = 1$, what is the mass of the particle?

A Physical Particle? - Solution

(a) To find τ for this trajectory, recall that

$$d\tau^2 = -ds^2 = -dx{\cdot}dx = d\sigma^2\left(-\frac{dx}{d\sigma}\cdot\frac{dx}{d\sigma}\right) = d\sigma^2\left(\cosh^2\sigma - \alpha^2 - \sinh^2\sigma\right) = d\sigma^2\left(1-\alpha^2\right).$$

Taking the square-root and integrating (and for simplicity setting the constant of integration to 0), we find $\tau = \sigma\sqrt{1-\alpha^2}$, or, for $\alpha \neq \pm 1$, $\sigma = \frac{\tau}{\sqrt{1-\alpha^2}}$.

(b) Assuming $\alpha \neq \pm 1$, we can use the above relation to write down the path x^μ as a function of proper time:

$$x^\mu(\tau) = \left(\sinh\left(\frac{\tau}{\sqrt{1-\alpha^2}}\right), \frac{\alpha\tau}{\sqrt{1-\alpha^2}}, \cosh\left(\frac{\tau}{\sqrt{1-\alpha^2}}\right), 0\right).$$

Thus, the four-velocity is given by

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1-\alpha^2}}\left(\cosh\left(\frac{\tau}{\sqrt{1-\alpha^2}}\right), \alpha, \sinh\left(\frac{\tau}{\sqrt{1-\alpha^2}}\right), 0\right).$$

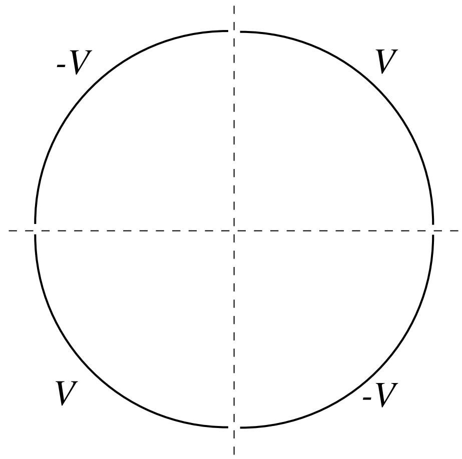
(c) Looking back on the solution to part 1, we see that

$$ds^2 = d\sigma^2(\alpha^2 - 1) \leq 0 \quad \Leftrightarrow \quad |\alpha| \leq 1.$$

In particular, for $\alpha \in (-1, 1)$ the path is time-like, while for $\alpha = \pm 1$ the path is null. Both cases correspond to physical particles (though the latter must be massless).

(d) From the discussion in the last part, it should be clear that when $\alpha = 1$ the proper time along the path vanishes. This can only happen if the particle is massless, so $m = 0$.

9. Infinite Cylinder



A hollow conducting cylinder of radius R and infinite length is divided into four quadrants by infinitely thin insulating pieces. Each quadrant is kept at a constant potential $\pm V$ as shown in the figure. There are no charges inside the cylinder. Find the potential $\phi(r, \theta, z)$ inside the cylinder.

Infinite Cylinder - Solution

Since there are no charges in the cylinder the potential ϕ inside satisfies the Laplace's equation $\nabla^2\phi = 0$. In cylindrical coordinates this takes the form

$$\frac{1}{r}\partial_r\left(r\partial_r\phi\right) + \frac{1}{r^2}\partial_\theta^2\phi + \partial_z^2\phi = 0. \tag{19}$$

Because of the translational symmetry in the z direction the potential ϕ does not depend on z: $\partial_z\phi = 0$. Then the Laplace's equation simplifies to

$$\frac{1}{r}\partial_r\left(r\partial_r\phi\right) + \frac{1}{r^2}\partial_\theta^2\phi = 0. \tag{20}$$

We try the separation of variables $\phi = \rho(r)\psi(\theta)$, then the Laplace's equation reduces to

$$\frac{1}{r}\psi\left(r\rho'\right)' + \frac{1}{r^2}\rho\psi'' = 0. \tag{21}$$

The variables r and θ separate after dividing the equation through by $\rho\psi$:

$$\frac{r}{\rho}\left(r\rho'\right)' = -\frac{\psi''}{\psi}. \tag{22}$$

Each side is a function of its own variable, therefore, they both should be constants. Denoting this constant ν^2 we get for the angular part

$$\psi''(\theta) + \nu^2\psi(\theta) = 0. \tag{23}$$

If $\nu = 0$ then $\psi_0 = a_0 + b_0\theta$, but the periodicity $\psi(\theta) = \psi(\theta + 2\pi)$ forces $b_0 = 0$. The radial part of the equation must satisfy $r^2\rho_0'' + r\rho_0' = 0$ which can be integrated to yield

$$\rho_0(r) = A_0 + B_0 \ln r. \tag{24}$$

If $\nu \neq 0$ then the angular part has the general form

$$\psi_\nu(\theta) = a_\nu \cos(\nu\theta) + b_\nu \sin(\nu\theta). \tag{25}$$

The 2π periodicity then requires that $\nu = 1, 2, \ldots$. This is equivalent to saying that the periodic function $\psi(\theta)$ can be expanded in a Fourier series. For a given value of ν the radial part satisfies

$$r^2\rho_\nu''(r) + r\rho_\nu'(r) - \nu^2\rho_\nu(r) = 0, \tag{26}$$

whose solution is a power-law (notice the convenient matching of the powers of r to the order of the derivative):

$$\rho_\nu(r) = A_\nu r^\nu + B_\nu r^{-\nu}. \tag{27}$$

Thus, the general solution takes the form:

$$\phi(r,\theta) = [A_0 + B_0 \ln r]\, a_0 + \sum_{\nu=1}^{\infty} \left[A_\nu r^\nu + B_\nu r^{-\nu}\right] [a_\nu \cos(\nu\theta) + b_\nu \sin(\nu\theta)]\,. \tag{28}$$

In the situation we consider the potential should remain finite as $r \to 0$. This requires $B_\nu = 0$, and after a redefinition of the coefficients the potential takes the form

$$\phi(r,\theta) = a_0 + \sum_{\nu=1}^{\infty} r^\nu [a_\nu \cos(\nu\theta) + b_\nu \sin(\nu\theta)]\,. \tag{29}$$

Now we can look at consequences of the symmetry of the problem. First of all, the voltages on the cylinder change sign under reflection across the x axis, which changes $\theta \to -\theta$. Therefore the potential $\phi(r,\theta)$ should be an odd function of θ. This forces $a_\nu = 0$ including $a_0 = 0$:

$$\phi(r,\theta) = \sum_{\nu=1}^{\infty} b_\nu r^\nu \sin(\nu\theta). \tag{30}$$

Next, the potentials on the cylinder do not change under $\theta \to \theta + \pi$, and this property is inherited by the potential inside: $\phi(r,\theta+\pi) = \phi(r,\theta)$. This selects ν to be even: $\nu = 2k$, and

$$\phi(r,\theta) = \sum_{k=1}^{\infty} b_{2k} r^{2k} \sin(2k\theta). \tag{31}$$

Furthermore, the potentials on the cylinder change sign under $\theta \to \theta + \pi/2$, implying $\phi(r,\theta+\pi/2) = -\phi(r,\theta)$ and $k = 2l-1$, $l = 1, 2, \ldots$. Thus

$$\phi(r,\theta) = \sum_{l=1}^{\infty} b_{4l-2} r^{4l-2} \sin\left[(4l-2)\theta\right]. \tag{32}$$

The coefficients of the Fourier series are found from the given potentials on the surface of the cylinder:

$$\phi(r=R,\theta) = \begin{cases} V & \text{for} \quad 0 \le \theta < \dfrac{\pi}{2} \quad \text{and} \quad \pi \le \theta < \dfrac{3\pi}{2}, \\ -V & \text{for} \quad \dfrac{\pi}{2} \le \theta < \pi \quad \text{and} \quad \dfrac{3\pi}{2} \le \theta < 2\pi. \end{cases} \tag{33}$$

We find the coefficients b_{4l-2} by the usual inversion of a Fourier series

$$R^{4l-2}b_{4l-2} = \frac{1}{\pi}\int\limits_0^{2\pi} \phi(R,\theta)\sin\big[(4l-2)\theta\big]\,d\theta = \frac{V}{\pi(4l-2)}\Big(-\cos\big[(4l-2)\theta\big]\Big|_0^{\pi/2} + \cos\big[(4l-2)\theta\big]\Big|_{\pi/2}^{\pi}$$
$$-\cos\big[(4l-2)\theta\big]\Big|_{\pi}^{3\pi/2} + \cos\big[(4l-2)\theta\big]\Big|_{3\pi/2}^{2\pi}\Big) = \frac{4V}{\pi(2l-1)}. \tag{34}$$

Substituting these coefficient in the series gives the answer to the problem:

$$\phi(r,\theta) = \frac{4V}{\pi}\sum_{l=1}^{\infty}\frac{1}{2l-1}\left(\frac{r}{R}\right)^{4l-2}\sin\big[(4l-2)\theta\big]. \tag{35}$$

10. Radiating Pendulum

A charged particle with mass m and charge q is attached to a massless pendulum of length l. The pendulum performs small oscillations whose amplitude is decreasing because of radiation emitted by the charge. Assume that the radiation losses, ΔE, during one oscillation are small compared to the mechanical energy E of the pendulum.

(a) Determine how the average energy of the pendulum decreases with time.

(b) What condition on the parameters of this problem would justify $\Delta E \ll E$?

Radiating Pendulum - Solution

(a) According to Larmor's formula, the total power radiated by an accelerated charge q is given by

$$P = A\frac{q^2a^2}{c^3}, \tag{36}$$

where a is the acceleration of the charge, c is the speed of light, and A is a constant that depend on the system of units used: $A = (6\pi\varepsilon_0)^{-1}$ in SI units, $A = 2/3$ in CGS units.

If we neglect the radiation losses, the pendulum oscillates according to the usual law of small oscillations:

$$\theta(t) = \theta_0 \sin\omega t, \tag{37}$$

where θ is the angle between the pendulum and the vertical, θ_0 is its amplitude, and $\omega = \sqrt{g/l}$. The total mechanical energy of the pendulum is conserved in this case, and is equal to

$$E = mgl(1-\cos\theta_0) \approx \frac{1}{2}mgl\theta_0^2, \tag{38}$$

where in the last equality we have assumed $\theta_0 \ll 1$.

The acceleration of the charge attached to the pendulum is

$$a(t) = l\ddot{\theta} = -l\theta_0\omega^2 \sin\omega t = -g\theta_0 \sin\omega t. \tag{39}$$

Larmor's formula gives the radiated power:

$$P(t) = A\frac{q^2g^2\theta_0^2}{c^3}\sin^2\omega t. \tag{40}$$

Integrating over the period $T = 2\pi/\omega$ of oscillations we get the energy loss ΔE during one oscillation:

$$\Delta E = \int_0^T P(t)\ dt = A\frac{q^2g^2\theta_0^2}{c^3}\int_0^T \sin^2\omega t\ dt = A\frac{q^2g^2\theta_0^2}{c^3}\frac{T}{2} == \pi A\frac{q^2g^2\theta_0^2}{c^3}\sqrt{\frac{l}{g}}. \tag{41}$$

Comparison with Eq. (38) gives the following condition for $\Delta E \ll E$:

$$2\pi A\frac{q^2}{mc^3}\sqrt{\frac{g}{l}} \ll 1. \tag{42}$$

When this condition is satisfied, the average mechanical energy $\bar{E}$ of the pendulum decreases slowly, and we can write a differential equation for the rate of change $d\bar{E}/dt$ approximating it by $-\Delta E/T$:

$$\frac{d\bar{E}}{dt} = -A\frac{q^2 g^2 \theta_0^2}{2c^3} = -A\frac{q^2 g}{mc^3 l}\bar{E} = -\frac{\bar{E}}{\tau}, \tag{43}$$

where we have denoted

$$\tau = \frac{1}{A}\frac{mc^3 l}{q^2 g}. \tag{44}$$

It is clear that this quantity is the time constant characterizing the decay of the mechanical energy. Indeed, the solution of the differential equation (43) is

$$\bar{E}(t) = E_0 e^{-t/\tau}. \tag{45}$$

(b) The condition (42) that justifies $\Delta E \ll E$ can be rewritten as $\tau \gg T$ or $\omega\tau \gg 1$.

11. A Half of an Oscillator

(a) Consider a particle of mass m in two different potentials. One potential is the harmonic oscillator, $U(x) = m\omega^2 x^2/2$, and the other is the closely related (half harmonic oscillator) given by

$$\tilde{U}(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & \text{for} \quad x > 0, \\ \infty, & \text{for} \quad x < 0. \end{cases}$$

Let us label the states by $n = 0, 1, 2, \ldots$ with the energy levels and normalized wave functions of $U(x)$ denoted by $(E_n,\, \psi_n(x))$ and those of $\tilde{U}(x)$ by $(\tilde{E}_n,\, \tilde{\psi}_n(x))$. Determine how the $\tilde{E}_n$ and $\tilde{\psi}_n(x)$ relate to the E_n and $\psi_n(x)$ respectively.

(b) The harmonic oscillator potential is initially separated by an impenetrable wall at $x = 0$ into two symmetric half harmonic oscillators:

$$U(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & \text{for} \quad x < 0 \\ \infty, & \text{for} \quad x = 0 \\ \frac{1}{2}m\omega^2 x^2, & \text{for} \quad x > 0. \end{cases}$$

A particle of mass m is initially in the ground state of the $x > 0$ well. Then, at time $t = 0$, the wall separating the wells is suddenly removed leaving the particle in the ordinary harmonic oscillator potential. Determine the probabilities $p_n(t)$ of finding the particle in the n-th state of the harmonic oscillator and the average energy of the particle $E(t)$ at time $t > 0$.

Useful formulas. The normalized wave functions of a harmonic oscillator are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),$$

where $H_n(u)$ are the Hermite polynomials, the first two being $H_0(u) = 1$, $H_1(u) = 2u$.

You will also need the integrals

$$I_n = \int_0^\infty e^{-u^2} u H_n(u) du = \frac{2^{n-2}\sqrt{\pi}}{\Gamma\left(\frac{3-n}{2}\right)}.$$

The first few of these integrals are $I_0 = \frac{1}{2}$, $I_1 = \frac{\sqrt{\pi}}{2}$, and $I_2 = 1$.

A Half of an Oscillator - Solution

(a) The harmonic oscillator potential is symmetric with respect to the change of sign $x \to -x$. Therefore, all the eigenstates $\psi_n(x)$ have a definite *parity* $(-1)^n$, the ground state being symmetric. For odd states $\psi_{2k+1}(x)$ in the domain $x \geq 0$, the Schrodinger equation and the conditions $\psi_{2k+1}(0) = \psi_{2k+1}(\infty) = 0$ are the same as for the states $\tilde{\psi}$ in the potential $\tilde{U}(x)$. Therefore, the spectrum of energies $\tilde{E}_k$ in the potential $\tilde{U}(x)$ is the same as the spectrum of odd states in the potential $U(x)$:

$$\tilde{E}_k = E_{2k+1}, \qquad k = 0, 1, 2, \dots \tag{46}$$

The normalized wave functions $\tilde{\psi}_k(x)$ differ from $\psi_{2k+1}(x)$ by a factor of $\sqrt{2}$, since they must be normalized on the semi-axis $x \geq 0$:

$$\tilde{\psi}_k(x) = \sqrt{2}\psi_{2k+1}(x), \qquad x \geq 0, \qquad k = 0, 1, 2, \dots \tag{47}$$

(b) The ground state in the right potential well is simply

$$\tilde{\psi}_0(x) = \sqrt{2}\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_1\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-u^2/2}u, \qquad x \geq 0. \tag{48}$$

where we have introduced a re-scaled coordinate

$$u = \sqrt{\frac{m\omega}{\hbar}}x. \tag{49}$$

When the impenetrable partition is removed, the state $\tilde{\psi}_0(x)$ is not an eigenstate anymore, and we have to expand it in the basis of the true eigenstates $\psi_n(x)$ of the full harmonic oscillator:

$$\tilde{\psi}_0(x) = \sum_{n=0}^{\infty} a_n \psi_n(x). \tag{50}$$

Since the eignefunctions ψ_n are orthonormal on the real axis, we can find the coefficients a_n by multiplying the last equation by ψ_m and integrating. Since $\tilde{\psi}(x) = 0$ for negative x, we have

$$a_n = \int_0^\infty \tilde{\psi}_0(x)\psi_n(x)dx = \left(\frac{\hbar}{m\omega}\right)^{1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{\sqrt{2^{n-2}n!}} \int_0^\infty e^{-u^2} u H_n(u) du = \frac{I_n}{\sqrt{\pi 2^{n-2} n!}}, \tag{51}$$

where we used the notation I_n for the integral (given in the formulation of the problem).

Each of the eigenstates in the superposition (50) starts to evolve in time after $t = 0$ by acquiring a phase factor related to its energy, as is usual for eigenstates. Thus, the overall state of the particle at time t becomes

$$\tilde{\psi}_0(x,t) = \sum_{n=0}^{\infty} a_n \psi_n(x) e^{-iE_n t/\hbar}, \tag{52}$$

where $E_n = \hbar\omega(n + 1/2)$ are the eigenenergies of the oscillator.

While the state $\tilde{\psi}_0(x,t)$ evolves in time, it is clear that the probabilities p_n of finding the particle in a particular oscillator state $\psi_n(x)$ are time independent:

$$p_n = \left| a_n e^{-iE_n t/\hbar} \right|^2 = a_n^2 = \frac{I_n^2}{\pi 2^{n-2} n!} = \frac{2^{n-2}}{n! \left[\Gamma\left(\frac{3-n}{2}\right) \right]^2}. \tag{53}$$

The same is true for the average energy

$$E = \sum_{n=0}^{\infty} p_n E_n = \hbar\omega \sum_{n=0}^{\infty} \frac{(2n+1) 2^{n-3}}{n! \left[\Gamma\left(\frac{3-n}{2}\right) \right]^2}. \tag{54}$$

Actually, the energy can be found explicitly in the following way. Before the partition is removed, the energy of the particle in the ground state of the right potential well is

$$\tilde{E}_0 = E_1 = \frac{3}{2}\hbar\omega. \tag{55}$$

The sudden (instantaneous) removal of the partition cannot do any work on the particle, so its energy should remain the same after the partition is removed. Thus, the average energy should be

$$E = \frac{3}{2}\hbar\omega. \tag{56}$$

Indeed, the series in Eq. (54) can be summed (for example, using Mathematica), and the sum is exactly 3/2!

12. n-dimensional universe

We live in a universe with 3 spatial dimensions. Instead, let's consider a hypothetical universe with n spatial dimensions.

(a) For a classical monatomic ideal gas in such a universe, determine the ratio of the specific heat capacity at constant pressure to the specific heat capacity at constant volume, $\gamma = C_P/C_V$.

(b) Do the same for a classical diatomic ideal gas.

(b) The energy density of black body radiation in our n-dimensional universe will still be proportional to T^α, where T is the temperature. Determine α as a function of n.

n-dimensional universe - Solution

(a) In the universe with n spatial dimensions, an atom has n translational degrees of freedom. By the equipartition theorem the internal energy of a monatomic gas is $U = \frac{n}{2}NkT$, where N is the number of atoms. Then $C_V = \frac{n}{2}Nk$. Using $C_P = C_V + Nk$, we have

$$\gamma = C_P/C_V = \frac{n+2}{n}. \tag{57}$$

(b) For a diatomic gas we have to find the number of rotational degrees of freedom. If we choose a cartesian coordinate system with n axes, and place a diatomic molecule of the gas along one of the axes, there are $n-1$ remaining axes to rotate the molecule about. (Alternatively, we can say that the position of the molecular axis is specified by a unit n-dimensional vector, or $n-1$ parameters similar to the polar and azimuth angles.) Therefore, the total number of degrees of freedom of the molecule (including n translational degrees of freedom) is $f = 2n-1$. Repeating the previous calculation we get

$$\gamma = C_P/C_V = \frac{f+2}{f} = \frac{2n+1}{2n-1}. \tag{58}$$

(c) Consider black body radiation in a volume V. The number of oscillations with the components of the wave vector $\boldsymbol{k}$ in the element of the reciprocal space $d^n\boldsymbol{k}$ is $Vd^n\boldsymbol{k}/(2\pi)^n$. The number of oscillations with the absolute value of the wave vector in the interval $[k, k+dk]$ is, correspondingly, $V\Omega_n k^{n-1}dk/(2\pi)^n$, where $\Omega_n = n\pi^{n/2}/\Gamma\big(\frac{n}{2}+1\big)$ is the total solid angle subtended by the surface of the unit n-sphere. Introducing the frequency $\omega = ck$, and multiplying by $n-1$ (the number of independent directions of polarization of the radiation), we get the number of quantum states of photons with frequencies in the interval $[\omega, \omega + d\omega]$:

$$\frac{(n-1)V\Omega_n}{(2\pi c)^n}\omega^{n-1}d\omega. \tag{59}$$

Multiplying this by the Planck distribution $n(\omega) = \big(e^{\hbar\omega/k_BT} - 1\big)^{-1}$, by the energy of a photon $\hbar\omega$, and integrating over the frequencies, we get the total energy of the radiation:

$$E = \frac{(n-1)V\Omega_n\hbar}{(2\pi c)^n}\int d\omega \frac{\omega^n}{e^{\hbar\omega/k_BT} - 1}. \tag{60}$$

Introducing the dimensionless variable $x = \hbar\omega/k_BT$, we get for the energy density

$$\frac{E}{V} = \frac{(n-1)V\Omega_n\hbar}{(2\pi c)^n}\Big(\frac{k_BT}{\hbar}\Big)^{n+1}\int d^n x \frac{x^n}{e^x - 1} = (n-1)\Gamma(n+1)\zeta(n+1)\Omega_n\frac{(k_BT)^{n+1}}{(hc)^n} \propto T^{n+1}, \tag{61}$$

where $\Gamma(n)$ and $\zeta(n)$ are the gamma and the Riemann zeta functions. Thus, the power α in the law $E/V \propto T^\alpha$ is given by $\alpha = n+1$.