Autumn 2012

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 2

September 20, 2012

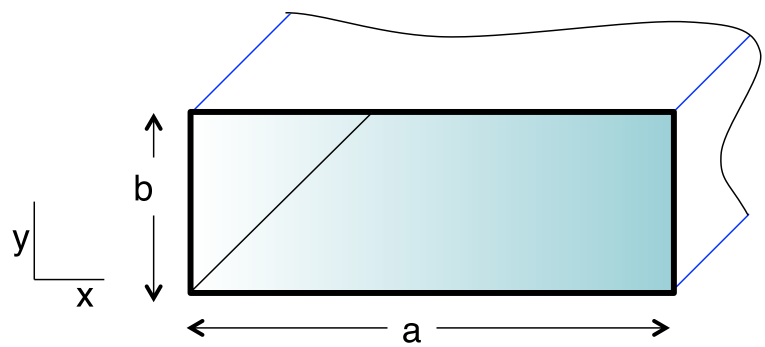
(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

7. Waveguide

Imagine an infinitely long waveguide propagating waves in the z direction. The z direction points out of the page. The waveguide is a hollow metallic pipe. The cross-section of the waveguide, depicted below, is rectangular with width a and height b where $a > b$. The waveguide is a perfect conductor.



(a) What boundary conditions should be imposed on electromagnetic fields propagating in the waveguide?

(b) Find $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ for the lowest frequency propagating mode. The electric field of the lowest mode is only in the y-direction.

(c) Find the phase velocity and group velocity of the lowest propagating mode.

Waveguide Solution

(a) For a perfect conductor, we impose the following boundary condition on $\mathbf{E}$ and $\mathbf{B}$ at each wall

$$\mathbf{n} \cdot \mathbf{B} = 0, \qquad \mathbf{n} \times \mathbf{E} = 0,$$

where $\mathbf{n}$ is the normal vector to the wall. The electric field is normal to the boundary, while the magnetic field is parallel.

(b) The vacuum Maxwell equations describe electromagnetic fields propagating within the waveguide:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

$$\Rightarrow \quad \nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

A wave propagating in the z direction requires the following coordinate dependence:

$$\mathbf{E} = \mathbf{E}_0(x, y)e^{i(kz-\omega t)}, \qquad \mathbf{B} = \mathbf{B}_0(x, y)e^{i(kz-\omega t)}.$$

The frequencies have to be correlated or one cannot solve Maxwell's equations. Let us consider the electric field since you are told that the lowest mode has an electric field only in the y direction. Consider the wave equation for $\mathbf{E}$,

$$\left(\partial_x^2 + \partial_y^2\right) E_{0y} = \left(k^2 - \frac{\omega^2}{c^2}\right) E_{0y}.$$

The boundary condition forces the electric field to vanish at $x = 0, a$. We are free to separate variables for this problem so we can take $E_{0y} = f_1(x)f_2(y)$. We definitely must have x-dependence or we cannot satisfy the boundary conditions with a non-vanishing field. However, any dependence on y increases the value of ω^2. For the lowest mode, we therefore take f_2 to be constant. Up to an overall constant A,

$$E_{0y} = A \sin\frac{\pi x}{a}, \qquad \left(k^2 - \frac{\omega^2}{c^2}\right) = -\left(\frac{\pi}{a}\right)^2, \qquad E_{0x} = E_{0z} = 0.$$

This determines the electric field.

To determine the magnetic field, we see that $\nabla \times \mathbf{E}$ has no component in the y direction so $B_y = 0$ from Maxwell's equations. Evaluating

$$(\nabla \times \mathbf{E})_z = \frac{\partial E_{0y}}{\partial x} = \frac{A\pi}{a} \cos \frac{\pi x}{a} = -\frac{1}{c}(-i\omega) B_{0z} \quad \Rightarrow \quad B_z = -i \frac{Ac\pi}{\omega a} \cos \frac{\pi x}{a} e^{i(kz - \omega t)}.$$

Lastly, we get B_x in a similar way:

$$(\nabla \times \mathbf{E})_x = -\frac{\partial E_{0y}}{\partial z} = -ikA \sin \frac{\pi x}{a} = -\frac{1}{c}(-i\omega) B_{0x} \quad \Rightarrow \quad B_x = -\frac{Ack}{\omega} \sin \frac{\pi x}{a} e^{i(kz - \omega t)}.$$

Note that B_x vanishes at $x = 0, a$ as it should.

(c) The basic dispersion relation gives ω in terms of the wave vector k,

$$\omega = c\sqrt{k^2 + \left(\frac{\pi}{a}\right)^2}.$$

The phase velocity which, by definition, determines how fast the phase propagates is given by

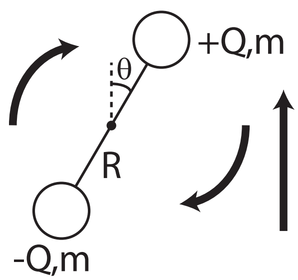
$$v_p = \frac{\omega}{k} = c\sqrt{1 + \left(\frac{\pi}{ka}\right)^2}.$$

The group velocity, which determines how packets travel in a medium, is given by

$$v_g = \frac{\partial \omega}{\partial k} = \frac{c}{\sqrt{1 + \left(\frac{\pi}{ka}\right)^2}}.$$

As $a \to \infty$ or $k \to \infty$, both velocities approach c which is correct for electromagnetic waves in free space.

8. A Polar Molecule in an Electric Field



An idealized polar molecule consists of two point charges of mass m and charge $\pm Q$ separated by a distance $2R$, free to rotate about a fixed axis. Its only degree of freedom is the angle θ. The moment of inertia is $I = 2mR^2$. An electric field E is directed vertically.

(a) With $E = 0$, what are the energy levels?

(b) With $E = 0$, $m = 40$ proton masses and $R = 0.2\,\mathrm{nm}$, at what frequency is the first rotational transition? Please give a numerical value, though it only needs to be accurate to within 10%.

(c) Consider a gas of such molecules at temperature T with $E = 0$. Below what temperature will there be an appreciable population ($> 50\%$) in the rotational ground state?

(d) If a single molecule is placed in a uniform electric field E as depicted in the figure, what is the Hamiltonian in terms of θ?

(e) Assume that the electric field is sufficiently strong so that the molecule only makes small deviations from the E-field axis. Approximate the Hamiltonian as a harmonic oscillator. What is the angular frequency of oscillations, ω, of the dipole about the E-field axis?

(f) Finally, go beyond the harmonic oscillator approximation and treat the next quartic term in the potential as a perturbation. What is the energy shift of the n^{th} energy level? Hint: it may be useful to view the perturbation in terms of annihilation and creation operators with commutator $[\hat{a}, \hat{a}^\dagger] = 1$ and,

$$\hat{\theta} = \left(\frac{\hbar}{2I\omega}\right)^{1/2} \left(\hat{a} + \hat{a}^\dagger\right).$$

A Polar Molecule in an Electric Field Solution

(a) The rotational hamiltonian has only a kinetic term,

$$H = \frac{\hat{L}^2}{2I}.$$

(b) The wavefunctions are of the form $\psi(\theta) = exp(\pm in\theta)$ with n an integer. The angular momentum is thus quantized as $n\hbar$ giving an energy $\hbar^2 n^2/2I$. The first rotational transition will be $E_{01}/h = \hbar/4\pi I$. The inertia of the molecule is approximately $I = 2{\cdot}40\,\mathrm{amu}{\times}(0.2\,\mathrm{nm})^2 \approx 2{\cdot}40{\cdot}(1.66{\times}10^{-27}\,\mathrm{kg}){\times}(2{\times}10^{-10}\,\mathrm{m})^2 \approx 5.4 \times 10^{-45}\,\mathrm{kg{-}m^2}$. The rotational transition is thus $E_{01}/h \approx 1.57\,\mathrm{GHz}$.

(c) Because the spectrum is $\propto n^2$, it is a good approximation that when the ground state has 50% occupation that the third state and beyond have negligible population (suppressed by $exp(-4)$) has negligible population. A quick approximation is that $k_\mathrm{B}T \approx E_{01} \approx 75\,\mathrm{mK}$, there will be significant ground state occupation. The full expression for ground state population including the fact that the excited states are two-fold degenerate is

$$P(n=0) = \left(1 + 2\sum_{n=1}^{\infty} \exp(-E_n/k_\mathrm{B}T)\right)^{-1}$$

The expression can actually be evaluated analytically in terms of elliptic functions but taking just the $n = 0$ and $n = 1$ terms yields only a 4% error. $P(n = 0) \approx 50\%$ when $T = 95\,\mathrm{mK}$.

(d) The potential energy for an electric dipole $\mathbf{d}$ in an electric field is $V = -\mathbf{d}\cdot\mathbf{E}$. The vector $\mathbf{d}$ is oriented from the negative to the positive charge with magnitude $2QR$. The Hamiltonian is therefore,

$$H = -K\frac{\partial^2}{\partial\theta^2} - U\cos(\theta),$$

where $K = \hbar^2/2I$ and $U = 2EQR$.

(e) Expanding the $\cos(\theta)$ gives

$$H = -K\frac{\partial^2}{\partial\theta^2} - U(1 - \theta^2/2 + \theta^4/12 + ...)$$

The constant off-set does not affect the dynamics. By analogy with the linear harmonic oscillator case $U/2 = I\omega^2/2$. Solving for ω gives $\omega = (U/I)^{1/2}$. This can be expressed more generally by eliminating I in favor of K giving,

$$\hbar\omega = \sqrt{2KU}.$$

Thus $\omega = \sqrt{2EQR/I}$.

(f) Using the hint we can express $\theta = (\hbar/2I\omega)^{1/2}(\hat{a} + \hat{a}^\dagger)$. The perturbation is then

$$H_1 = -\frac{U}{12}\theta^4 = -\frac{U}{12}(\hbar/2I\omega)^2\left(\hat{a}^4 + \hat{a}^3\hat{a}^\dagger + ...\right)$$

Of these 24 terms, only the 6 terms with an equal number of $\hat{a}$'s and $\hat{a}^\dagger$'s will contribute to the lowest order energy shift. What's left is

$$-\frac{U}{12}(\hbar/2I\omega)^2\left(\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\right).$$

Using the commutator relation $[\hat{a}, \hat{a}^\dagger] = 1$, the terms in brackets can be rewritten in the form

$$\left(6\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + 6\hat{a}^\dagger\hat{a} + 3\right).$$

Acting on the n^{th} energy level gives a shift,

$$\Delta E_n = \alpha\left(6n^2 + 6n + 3\right), \qquad \alpha = -\frac{U}{12}\left(\frac{\hbar}{2I\omega}\right)^2.$$

We can simplify α using $\omega^2 = U/I$ and $I = \hbar^2/2K$:

$$\alpha = -\frac{K}{24}.$$

It is interesting to note that although the term arises from the non-linear potential, the anharmonicity depends only on the kinetic energy via the moment of inertia.

9. Spy Asteroid

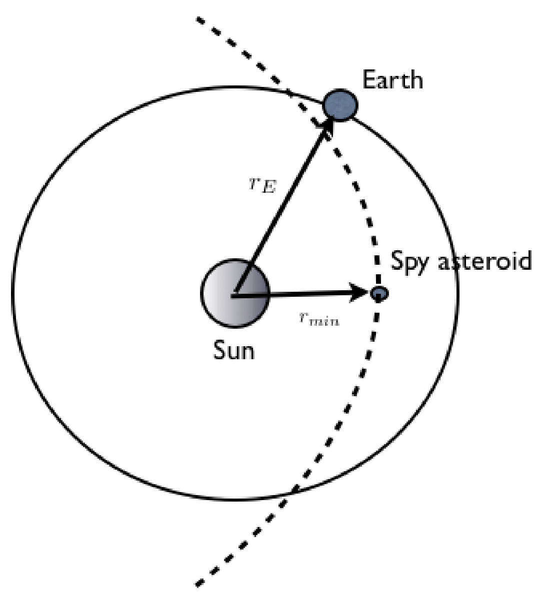
An alien civilization decides they want to spy on us. They hurtle a spy asteroid of mass m in a parabolic trajectory around the sun with mass M. The asteroid is in the same plane of motion as the Earth's orbit. Denote the radius of the Earth's orbit by r_E and assume it is perfectly circular. Ignore any interaction between the Earth and the asteroid. The aliens kindly provide the following possibly useful integral:

$$\int \frac{x dx}{\sqrt{ax-b}} = \frac{2(ax+2b)}{3a^2}\sqrt{ax-b}.$$

(a) What is the total energy of the asteroid?

(b) If the asteroid approaches as close as r_{min} to the sun, how long does it spend inside the orbit of the Earth? You can leave the answer in integral form if you like.

(c) Find the value of r_{min} for which the asteroid spends the most time within the Earth's orbit.



Spy Asteroid Solution

(a) This is another central force Lagrangian. Expressed in terms of the angular momentum l, the Lagrangian takes the form

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - V(r), \qquad V = -\frac{GmM}{r}. \tag{1}$$

For a central force, the Lagrangian is independent of time so energy is conserved. It is also rotationally symmetric so angular momentum is conserved. The asteroid travels out to infinity where it's kinetic and potential energy both vanish so $E = T + V = 0$ for a parabolic orbit.

(b) We want to use the conserved quantities to simplify our analysis. By definition, at the minimum distance r_{min}, all the energy is potential energy. This allows us to relate l to r_{min}

$$\frac{l^2}{2m(r_{min})^2} + V(r_{min}) = 0 \quad \Rightarrow \quad l^2 = 2GMm^2 r_{min}. \tag{2}$$

Now we want a differential equation for the time as a function of r. The conserved total energy $E = T + V$ relates $\dot{r}$ to r:

$$(\dot{r})^2 = \frac{2GM}{r} - \frac{l^2}{m^2r^2}. \tag{3}$$

The time spent inside the Earth's orbit is then given by the integral,

$$t = 2\int_{r_{min}}^{r_E} \frac{rdr}{\sqrt{2GMr - \frac{l^2}{m^2}}}. \tag{4}$$

The integral is quite nice (and given to you by the aliens). Note that

$$\int \frac{xdx}{\sqrt{ax-b}} = \int \frac{2x}{a} d(\sqrt{ax-b}) = \frac{2x\sqrt{ax-b}}{a} - \frac{4(ax-b)^{3/2}}{3a^2} = \frac{2(ax+2b)}{3a^2}\sqrt{ax-b}. \tag{5}$$

In this case, the integral for the time becomes

$$t = \frac{2}{\sqrt{2GM}} \int_{r_{min}}^{r_E} \frac{rdr}{\sqrt{r - r_{min}}} = \frac{2}{\sqrt{2GM}} \frac{2(r_E + 2r_{min})}{3} \sqrt{r_E - r_{min}}. \tag{6}$$

As a basic check, note that when $r_{min} = r_E$, the time inside the Earth's orbit vanishes since the asteroid never gets inside the orbit.

(c) Maximizing t means looking for extrema of the function

$$(a + 2x)\sqrt{a - x}. \tag{7}$$

The extremum is at $x = a/2$ as one can check by differentiating. This corresponds to $r_{min} = r_E/2$.

10. Help!

Imagine you are taking a six hour exam. With two hours left in the exam, your instructor decides to go for a quick ride in a relativistic car. He or she decides that Saturn would be nice to visit this time of year. The car zooms away at velocity v straight to Saturn. On reaching Saturn, the instructor plans to get coffee for one hour before returning at the same speed.

With respect to the rest frame of the Earth, we can model Saturn as 70 light-minutes away. Ignore all planetary motion and gravity for this problem. Assume a space-time signature $(-,+,+,+)$ so that for a four-vector like $x^\mu = (ct, x^1, x^2, x^3)$,

$$|x|^2 = -c^2t^2 + |x^1|^2 + |x^2|^2 + |x^3|^2.$$

(a) A 1/2 hour elapses. You suddenly have an urgent question for your instructor. You need some help! You communicate your question via laser. How fast can your instructor travel so that you can get a response before the exam is over? Assume the instructor replies immediately on receiving the question.

(b) When your message arrives, how much time t' has expired according to your instructor's clock from the moment he or she started the trip? Compare with the time that has expired according to your clock.

(c) Now imagine your instructor takes a much more interesting joy ride to Saturn and back. Let us parametrize the path in space-time by,

$$x^\mu = (70\sigma, 0, 0, 70\sin(\sigma)),$$

where σ is a dimensionless parameter which runs from 0 to π, and the constant 70 is in light-minutes. Is the velocity along the path less than or equal to c? How much time elapses for this trip according to the clock carried by your instructor?

Help! Solution

(a) Assume the instructor moves with velocity v and leaves at $t = 0$. The elapsed time before you send your signal is t_0. First let us think about what happens if the instructor moves at $v = c$. No signal can reach the instructor prior to arriving at Saturn which takes 70 minutes. It would take another 70 minutes plus the time for your signal arrival to get a reply. That's already way too long so $v < c$ and the instructor must be en route to Saturn when your signal arrives.

Your signal intercepts the instructor when,

$$x = vt = c(t - t_0),$$

so

$$t = \frac{t_0}{(1 - v/c)}.$$

At that point, your instructor beams a reply back to you. It takes time x/c to reach you so the total time for this communication to occur is,

$$\Delta t = \frac{x}{c} + \frac{t_0}{(1 - v/c)} - t_0 = (2v/c)\frac{t_0}{(1 - v/c)}.$$

When $v << c$ then this time becomes $2v/c$ which is the classical result. When $v \to c$, the time goes to infinity since your signal can never catch the speedy instructor until they reach Saturn (which is too late). Now $t_0 = 30$ minutes. You have $\Delta t = 90$ minutes left in the exam. Solving for v/c gives $v/c = 3/5$.

For this to be consistent, we can check that the instructor is still en route when your signal arrives. The distance traveled when the signal hits is,

$$x = vt = \frac{3}{5}ct = \frac{3}{5} \times 30 \times \frac{5}{2}c = 45c,$$

which is less than 70 light minutes so the instructor is indeed still traveling.

(b) This part requires a Lorentz transformation. In your frame, the instructor receives the signal at space-time point $(x, t) = (\frac{vt_0}{(1-v/c)}, \frac{t_0}{(1-v/c)})$. If we boost to the instructor's rest-frame with $\gamma = 1/\sqrt{1 - (v/c)^2}$ then

$$(x', t') = \left(\gamma(x - vt), \gamma(t - vx/c^2)\right).$$

So $t' = \frac{\gamma t_0}{(1-v/c)}\left(1 - \frac{v^2}{c^2}\right) = \gamma t_0(1+v/c)$. Let us compare t' with t,

$$\frac{t'}{t} = \gamma(1+v/c)(1-v/c) = \frac{1}{\gamma},$$

so the time is always longer on your clock. As the instructor approaches the speed of light, the elapsed time goes to zero as we should expect for a light-like observer. Clocks tick slower and slower, the faster the observer.

(c) This last part involves a more interesting path through space-time. First we can check that this is a physical path. The velocity along the path is given by

$$v = \frac{dz/d\sigma}{dt/d\sigma} = \frac{70\cos\sigma}{70/c} = c\cos\sigma.$$

So, the path is light-like ($v = c$) at $\sigma = 0$ and π and time-like ($|v| < c$) everywhere else - it is a physical path.

The time measured by an observer following this path is given by the proper time defined as follows,

$$d\tau^2 = -\frac{ds^2}{c^2} = \frac{(70)^2}{c^2}\left(1 - \cos(\sigma)^2\right)d\sigma^2 = \frac{(70)^2}{c^2}\sin(\sigma)^2 d\sigma^2.$$

Conveniently, we can take the square-root and integrate to get,

$$\int_0^\pi d\tau = \frac{70}{c}\int_0^\pi \sin(\sigma)d\sigma = \frac{140}{c}.$$

11. Dark Matter Detection

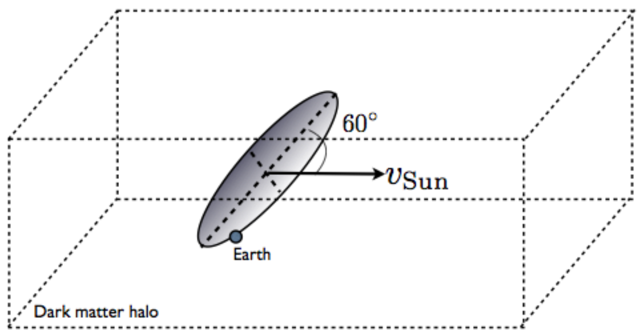
It is possible that our galaxy is embedded in a dark matter “halo”, comprised of nonrelativistic dark matter particles. For definiteness, suppose that dark matter particles have mass

$$m_{\mathrm{DM}} = 100\,\mathrm{GeV}c^{-2}\,,$$

that the dark matter mass density is constant,

$$\rho_0 = 0.3\,\mathrm{GeV}\,c^{-2}\mathrm{cm}^{-3}$$

and neglect the velocity of the gas of matter particles comprising the halo. (The latter assumption is not a good approximation, but simplifies the analysis.)



(a) Suppose that our solar system is moving with speed $v_{\mathrm{Sun}} = 220\,\mathrm{km/s}$ with respect to the dark matter halo. Neglect the velocity of the Earth with respect to the solar system. Compute the flux (number of particles per second, per cm^2) of dark matter particles as seen on Earth.

(b) Suppose that the interaction cross section for a dark matter particle to scatter from an atomic nucleus of mass number A (i.e., the nuclear mass is A times the mass of the proton) is

$$\sigma = A^2 \times 10^{-45}\,\mathrm{cm}^2\,.$$

Ignore the velocity dependence of the cross section. Suppose that a detector is constructed from Germanium, with $A = 73$. Compute the rate of dark matter-nucleus scattering events per year, per kilogram of detector material.

(c) The plane of the Earth’s orbit around the Sun is tilted by an angle $\approx 60^\circ$ relative to the motion of the Sun within the galaxy and the dark matter halo. The Earth’s motion causes an annual variation in the speed of the Earth through the dark matter halo, and hence in the flux of dark matter particles as seen on Earth. Taking into account the motion of the Earth around the Sun, and assuming a circular orbit (radius AU $\sim 1.5 \times 10^{11}$ m), compute the fractional variation in the rate of dark matter-nucleus scattering events.

Dark Matter Detection Solution

(a) The flux is given by

$$\text{flux} = (\text{number density})\times\text{velocity} = \frac{\rho_0}{m_{\rm DM}}\times v_{\rm Sun} = \frac{0.3\times\text{GeVcm}^{-3}}{100\,\text{GeV}}\times 220\,\text{km}s^{-1} = 6.6\times10^4\text{cm}^{-2}s^{-1}$$

(b) The rate per nucleus is given by

$$\text{rate} = (A^2\times10^{-45}\,\text{cm}^2)\times\text{flux} = 3.5\times10^{-37}s^{-1}\text{nucleus}^{-1}\,,$$

Given that there are $360\times24\times60\times60$ seconds per year and $10^3N_{\rm Avagadro}/A$ nuclei per kilogram, the rate per kilogram per year is

$$\text{rate} = (3.5\times10^{-37}s^{-1})\times(360\times24\times60\times60s\,\text{year}^{-1})\times(10^3\times6.0\times10^{23}/73\,\text{kg}^{-1}) \approx 10^{-4}\,\text{kg}^{-1}\text{year}^{-1}$$

(c) The magnitude of Earth's velocity in the Sun reference frame is

$$\frac{2\pi\text{AU}}{\text{year}} = 3.0\times10^4\,\text{m}\,s^{-1}\,,$$

The components of the velocity of the Earth (and Earth-bound detectors) is thus

$$v_{||} = [220\pm(\cos 60^\circ)30]\text{km}\,s^{-1}\,,\quad v_\perp = \sin(60^\circ)30\,\text{km}\,s^{-1}\,.$$

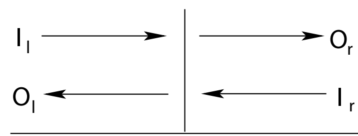
The ratio of maximal to minimal flux (and hence event rate) is thus

$$\frac{v_{\rm max}}{v_{\rm min}} = \frac{\sqrt{(220+15)^2+[(\sqrt{3}/2)30]^2}}{\sqrt{(220-15)^2+[(\sqrt{3}/2)30]^2}} = 1.14\,,$$

so that the fractional variation in event rate is 14 %.

12. Quantum Wire

Coherent disordered conductors are often modeled as collections of randomly placed scattering impurities. We will study the case of a one-dimensional quantum wire.



(a) A particle of mass m scatters from an impurity with scattering matrix S that relates the amplitudes of the incoming (I_l, I_r) and outgoing (O_l, O_r) waves to the left and right of the impurity:

$$\begin{pmatrix} O_l \\ O_r \end{pmatrix} = S \begin{pmatrix} I_l \\ I_r \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} I_l \\ I_r \end{pmatrix}. \tag{8}$$

The matrix S is unitary:

$$S^\dagger S = SS^\dagger = \mathbf{1}.$$

Derive the dependence of $|t'|$ on $|t|$ and $|r'|$ on $|r|$.

(b) The transfer matrix M relates amplitudes on each side of the impurity:

$$\begin{pmatrix} O_r \\ I_r \end{pmatrix} = M \begin{pmatrix} I_l \\ O_l \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} I_l \\ O_l \end{pmatrix}.$$

Given a 2×2 scattering matrix, as in eq. (8), find the corresponding transfer matrix M. Assume $t \neq 0$.

(c) Imagine a section of wire without any impurities. Consider left and right-moving waves with wave-vector k. Find the transfer matrix for free propagation over a distance l.

(d) Let us model an impurity by a delta-function potential,

$$V(x) = \frac{\hbar^2\kappa}{2m}\delta(x).$$

Find the transfer matrix for left and right-moving waves with wave-vector k. What happens to the probability for scattering as $\kappa/k >> 1$ and $\kappa/k << 1$?

Quantum Wire Solution

(a) For an arbitrary scattering matrix we have

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \qquad S^\dagger = \begin{pmatrix} r^* & t^* \\ t'^* & r'^* \end{pmatrix}, \tag{9}$$

and eq. (9) written in components give

$$|t|^2 + |r|^2 = |t'|^2 + |r'|^2 = |t'|^2 + |r|^2 = |t|^2 + |r'|^2 = 1, \tag{10}$$

$$rt^* + t'r'^* = rt'^* + tr'^* = 0. \tag{11}$$

From eq. (10) it follows immediately that $T = |t|^2 = |t'|^2$, $R = |r|^2 = |r'|^2$.

(b) The relation between the amplitudes given by the scattering matrix

$$O_l = rI_l + t'I_r, \qquad O_r = tI_l + r'I_r, \tag{12}$$

can be solved for O_r, I_r in terms of I_l, O_l, which gives the necessary transfer matrix (assuming $t' \neq 0$):

$$m_{11} = t - \frac{rr'}{t'}, \qquad m_{12} = \frac{r'}{t'}, \qquad m_{21} = -\frac{r}{t'}, \qquad m_{22} = \frac{1}{t'}. \tag{13}$$

Using Eq. (11) we can simplify m_{11} to obtain

$$M = \frac{1}{t'} \begin{pmatrix} t'/t^* & r' \\ -r & 1 \end{pmatrix}. \tag{14}$$

(c) Free propagation over a distance l just introduces a phase with no scattering. The corresponding transfer matrix is given by,

$$M_{\text{free}} = \begin{pmatrix} e^{ikl} & 0 \\ 0 & e^{-ikl} \end{pmatrix}. \tag{15}$$

(d) The Schrödinger equation for the delta-function potential states that,

$$-\psi''(x) + \kappa\delta(x)\psi(x) = k^2\psi(x).$$

Delta-function potentials have appeared on past candidacy exams! Integrating this equation once across the impurity located at $x = 0$ forces ψ' to jump by,

$$\psi'(+\epsilon) - \psi'(-\epsilon) = \kappa\psi(0).$$

Integrating this again shows us that ψ is continuous. On either side of the impurity, propagation is free. We take $I_l = A_1 e^{ikx}$ and $O_r = A_2 e^{ikx}$ with $O_l = B_1 e^{-ikx}$, $I_r = B_2 e^{-ikx}$.

Continuity tells us that $A_1 + B_1 = A_2 + B_2$ while the jump condition gives:

$$(ikA_2 - ikB_2) - (ikA_1 - ikB_1) = \kappa(A_1 + B_1).$$

That is 2 conditions on 4 coefficients. For the transfer matrix, we want (O_r, I_r) in terms of (I_l, O_l). So we want (A_2, B_2) in terms of (A_1, B_1). Solving gives two relations:

$$-2ikB_2 = \kappa A_1 + (\kappa - 2ik)B_1, \qquad 2ikA_2 = \kappa B_1 + (\kappa + 2ik)A_1.$$

The transfer matrix is therefore

$$\begin{pmatrix} fa1 - \frac{i\kappa}{2k} & -\frac{i\kappa}{2k} \\ \frac{i\kappa}{2k} & 1 + \frac{i\kappa}{2k} \end{pmatrix}.$$

Note that as $\kappa \to 0$ with no scattering, we recover the identity matrix. When κ/k becomes large, we get total scattering.