Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II

Saturday, April 9, 2011

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student __________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

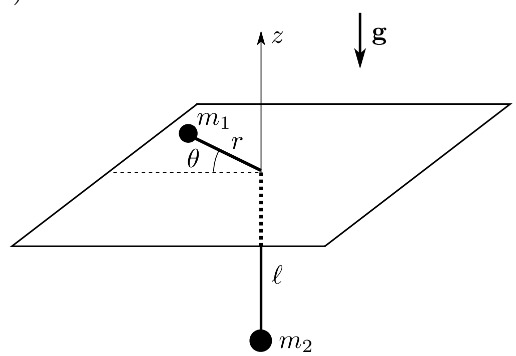
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs 30 min**.

Please, indicate problems you are turning in:

□ **II-1** □ **II-2** □ **II-3** □ **II-4**

II-1. (Classical Mechanics)



Two mass points m_1 and m_2 ($m_1 \neq m_2$) are connected by a massless string of length ℓ passing through a hole in a horizontal table. The string and mass points move without friction with m_1 on the table and m_2 free to move in a vertical line.

(i) What initial velocity must m_1 be given so that m_2 will remain motionless a distance d below the surface of the table?

(ii) If m_2 is slightly displaced in a vertical direction, small oscillations will ensue. Use Lagrange's equations to find the period of these oscillations.

II-2. (Electricity and Magnetism)

Consider a hollow cube of side a. The volume inside the cube is the region

$$0 < x < a\,, \quad 0 < y < a\,, \quad 0 < z < a\,.$$

All of the sides of the cube are metallic and grounded to zero potential. A single point charge of magnitude q is placed in the center of the cube at the point $x = y = z = a/2$.

(i) Solve the differential equation

$$-\epsilon_0 \nabla^2 \Phi \;=\; q\,\delta(x - a/2)\delta(y - a/2)\delta(z - a/2)\,,$$

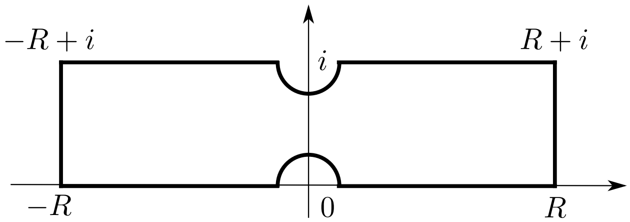
for the electrostatic potential $\Phi(x, y, z)$ everywhere inside the volume of the cube as a double Fourier sine series in the x and y variables. The boundary condition is that the potential vanishes on all of the surfaces of the cube.

(ii) Using the above solution for the potential and Gauss's law, find an expression (also as a double Fourier sine series) for the charge density $\sigma(x, y)$ on the top surface of the cube at $z = a/2$.

(iii) Integrate your result for $\sigma(x, y)$ to find an expression for the total charge q' on the top surface as a double sum. Do not sum the series, but based on symmetry state what the final result for q' must be.

Hint: A useful identity is: $\sinh(\alpha + \beta) \;=\; \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta)$.

II-3. (Mathematical Methods)



Use the contour above with $R \to \infty$ to show that

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)} = \frac{1}{2}\tan\frac{a}{2}, \qquad -\pi < a < \pi .$$

II-4. (Quantum Mechanics)

A spin-half particle with magnetic moment μ is placed in a magnetic field $\vec{B}(t)$ which rotates with frequency ω,

$$\vec{B}(t) = B_z\hat{z} + B_x\hat{x}\cos(\omega t) + B_y\hat{y}\sin(\omega t),$$

where B_z, B_x, and B_y are real constants, with

$$B_z = \frac{\hbar\omega_0}{2\mu}, \qquad B_x = B_y = \frac{\hbar\omega_1}{2\mu}.$$

The Hamiltonian is represented by the 2×2 matrix

$$\mathbf{H}(t) = -\mu\,\vec{\sigma}\cdot\vec{B}(t),$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Write down the equation satisfied by the state ket $|\Psi(t)\rangle$ of the particle.

(ii) Consider the (rotation) operator $\mathbf{R}(\theta\hat{n})$ represented by the matrix

$$\mathbf{R}(\theta\hat{n}) = e^{-i\theta\hat{n}\cdot(\frac{1}{2}\vec{\sigma})}.$$

What is the equation satisfied by $\mathbf{R}(\omega t\hat{z})|\Psi(t)\rangle$, and what state does this ket represent?

(iii) At time $t = 0$ the spin is aligned along the positive z-axis. What is the probability for finding the spin aligned along the negative z-axis at $t > 0$?