Spring 2014

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

March 24, 2014

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Asteroid Collision

At a given time, $t = 0$, it is noted that an asteroid is centered at a position $\mathbf{R}_0$ relative to the center of the earth and moving with velocity $\mathbf{V}_0$ relative to the earth. Treating both bodies as spheres with radii σ_A and σ_E,

(a) find the conditions under which they collide, and

(b) find the time the collision occurs as a function of $\mathbf{R}_0$, $\mathbf{V}_0$, σ_A, and σ_B.

Neglect all gravitational effects.

Asteroid Collision - Solution

(a) The distance of separation between the two bodies at time t is simply

$$r^2(t) = (\mathbf{R}_0 - \mathbf{V}_0 t)^2 \quad . \tag{1}$$

If we define

$$\sigma = \sigma_A + \sigma_E \tag{2}$$

then the condition for a collision is

$$r^2(\tau) = \sigma^2 \tag{3}$$

where τ is the collision time.

(b) We have then

$$\sigma^2 = R_0^2 + V_0^2\tau^2 - 2\mathbf{R}_0 \cdot \mathbf{V}_0 \tau \quad . \tag{4}$$

This is a simple quadratic equation for τ with the two-solutions

$$\tau = \frac{R_0}{V_0}\left[b \pm \sqrt{b^2 + (\sigma/R_0)^2 - 1}\right] \tag{5}$$

where the angular information is carried by

$$b = \hat{\mathbf{R}}_0 \cdot \hat{\mathbf{V}}_0 \quad . \tag{6}$$

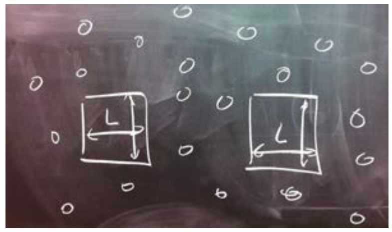
We can choose $(-)$ sign by looking at limiting case $b = 1$ which gives

$$\tau = \frac{R_0 - \sigma}{V_0} \tag{7}$$

Look also at $R_0 \gg \sigma$. One sees intuitively that as R_0 increases b must be progressively closer to one for a collision to take place. One finds

$$b = 1 - \frac{\sigma^2}{2R_0^2} + ... \quad . \tag{8}$$

2. Depletion Interaction

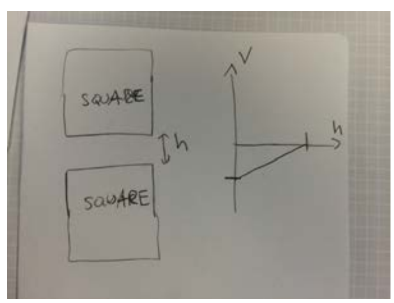


Consider a pair of square particles of perimeter $4L$ surrounded by N disks of radius $r_d \ll L$ in a container of total area V embedded in a two-dimensional fluid at temperature T (see the figure above). The fluid damps the motion of the disks so that they are free to flow, but do not undergo gas-like ballistic motion. The squares are also free to move but not to rotate.

(a) Write down an expression for the free energy of the disks up to a constant, λ^2, that represents the area that an individual disk may occupy. You may assume that $L^2 \ll V$ and that the area V is discretized in units of λ^2.

(b) Derive the corresponding pressure of the ensemble of disks.

(c) Each disk is "hard" in that its center may not get closer than r_d to the surface of a square. The presence of the disks thus gives rise to a short-range interaction potential, a *depletion interaction*, between the square particles of the form sketched in the figure below, where h is the distance between the surfaces of the square particles. Derive an expression for this potential.



(d) This potential allows the squares to bind to each other. Write down an expression for the free energy of the bound squares. You may assume that, if bound, their centers of mass are allowed to explore a fluctuation area V_b and that the quantum of area is again λ^2.

(e) Use this to derive an expression for the change in free energy when the two squares bind.

Depletion Interaction - Solution

(a) Because of the fluid damping, the contribution to the free energy from the kinetic energies of the disks is negligible. Therefore, only the configurartions of the disks matter. The total number of positions available for a single disk is approximately $Z_1 = V/\lambda^2$. Using $Z = Z_1^N/N!$ for N indistinguishable disks we have

$$Z = \frac{1}{N!}\left(\frac{V}{\lambda^2}\right)^N$$

The free energy is given by $F = -k_B T \ln Z$. After using Sterling's approximation, our result becomes

$$F = -k_B T\left[N \ln\left(\frac{V}{\lambda^2}\right) - \ln N\right]$$

(b) The corresponding pressure is given by:

$$P = -\frac{\partial F}{\partial V} = \frac{N}{V} k_B T$$

(c) The squares do not interact until their surfaces get closer than $2r_d$ when disks become 'depleted' from the area between them. They are then attracted by a constant force f given by the magnitude of the pressure times the 'area' L. Using $f = -dU/dr$, we obtain a linear potential U of depth $|U_{min}| = \frac{N}{V} k_B T L 2r_d$.

(d) Before binding, both squares can explore the entire volume V and will each have an entropy of $\ln(V/\lambda^2)$ giving a total entropy of twice this amount. After binding, their center of mass can explore the entire volume, $\sigma_{cm} = \ln(V/\lambda^2)$, but their relative motion can only explore V_b which leads to an entropy $\sigma_b = \ln(V_b/\lambda^2)$. The free energy is $F = E_b - k_B T \sigma_{tot}$, where E_b is the binding energy. Therefore,

$$F = E_b - k_B T \ln\left(\frac{V}{\lambda^2}\frac{V_b}{\lambda^2}\right)$$

(e) The difference between the free energies, bound $-$ unbound, works out to be $\Delta F = E_b + k_B T \ln\left(\frac{V}{V_b}\right)$.

3. Cylindrical Capacitor

A cylindrical capacitor is made of two coaxial conductors. The radius of inner conductor is r_1, and the radius of the outer conductor is r_2. The outer conductor is grounded, and the potential of the inner conductor is V_0.

(a) Consider a point A distance r away from the axis, $r_1 < r < r_2$. Calculate the electric field vector and potential at this point in terms of V_0 and the radii of the cylinders.

(b) Suppose the charge on the inner conductor is Q_1 initially, and there is a small negative charge Q ($|Q| \ll |Q_1|$) at point A. Then the negative charge is released and drifts to r_1. Find the decrease in charge on the inner conductor to first order in Q.

Cylindrical Capacitor - Solution

(a) The potential difference between the inner and outer conductor is V_0, and we have

$$V_0 = \int_{r_1}^{r_2} E dr = \frac{\lambda}{2\pi\epsilon_0} \log\left(\frac{r_2}{r_1}\right). \tag{9}$$

From this, we can solve for line charge density λ. Therefore, the electric field at point A is

$$\vec{E}(r) = \frac{V_0}{\log(r_2/r_1)} \frac{\vec{r}}{r^2}. \tag{10}$$

The correspond potential is

$$V_A(r) = V_0 - \frac{r}{r_1} \vec{E} \cdot d\vec{r} = V_0 \frac{\log(r_2/r)}{\log(r_2/r_1)}. \tag{11}$$

(b) The charge on the inner conductor changes from Q_1 to Q_2, with $\Delta Q = Q_1 - Q_2$. Moving negative charge Q from point A to the inner conductor, the work by the electric field is $Q(V_0 - V_A)$. This is also the change in the energy stored in the capacitor,

$$\frac{Q_1^2}{2C} - \frac{Q_2^2}{2C} = Q(V_0 - V_A) \tag{12}$$

From this, we can solve for ΔQ.

Since Q is very small, we have $\Delta Q \ll Q_1$. Taylor expand the left-hand side of Eq. 12, we have

$$\frac{Q_1}{C} \Delta Q = Q(V_0 - V_A). \tag{13}$$

Therefore,

$$\Delta Q = Q \frac{\log(r/r_1)}{\log(r_2/r_1)}. \tag{14}$$

4. Occupation Number

Consider a quantum mechanical system governed by the Hamiltonian

$$\hat{H} = \sum_i \epsilon_i a_i^\dagger a_i$$

where $a_i^\dagger$ creates a particle with energy ϵ_i and a_i destroys a particle with energy ϵ_i and i labels the quantum single-particle states. The operator $\hat{n}_i = a_i^\dagger a_i$ measures the number of particles in quantum state i. We want to calculate the average number of particles in quantum state i for a system in equilibrium at temperature T and chemical potential μ which is given by

$$\bar{n}_i = \langle \hat{n}_i \rangle = \frac{\mathrm{Tr}[e^{-\beta\hat{K}}\hat{n}_i]}{\mathrm{Tr}[e^{-\beta\hat{K}}]}$$

where we have introduced the operator

$$\hat{K} = \sum_i (\epsilon_i - \mu)\hat{n}_i$$

and $\mathrm{Tr}[\cdots]$ is the quantum mechanical trace which is the sum of the diagonal elements over all states, i.e., $\mathrm{Tr}[\hat{O}] = \sum_i \langle i|\hat{O}|i\rangle$.

The key property of the creation and annihilation operators is the commutator relations for bosons

$$[a_i, a_j^\dagger]_- = a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}$$

together with $[a_i, a_j]_- = 0$ and $[a_i^\dagger, a_j^\dagger]_- = 0$, as well as the anticommutator relations for fermions

$$[a_i, a_j^\dagger]_+ = a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}$$

along with $[a_i, a_j]_+ = 0$ and $[a_i^\dagger, a_j^\dagger]_+ = 0$.

The only additional information we need to evaluate $\bar{n}_i$ is that the quantum mechanical trace obeys cyclic invariance: $\mathrm{Tr}[\hat{A}\hat{B}] = \mathrm{Tr}[\hat{B}\hat{A}]$.

(a) A hint on how to preceed in evaluating $\bar{n}_i$ is to consider the quantity

$$a_i^\dagger(\beta) = e^{-\beta\hat{K}} a_i^\dagger e^{\beta\hat{K}}$$

and take the derivative with respect to β. Use this hint to show that

$$a_i^\dagger(\beta) = e^{-\beta(\epsilon_i - \mu)} a_i^\dagger.$$

(b) Use this result to evaluate $\bar{n}_i$ for both bosons and fermions.

Occupation Number - Solution

(a) Taking the directed derivative one easily finds

$$\frac{\partial}{\partial \beta} a_i^\dagger(\beta) = -e^{-\beta\hat{K}}[\hat{K}, a_i^\dagger]_- e^{\beta\hat{K}} \tag{15}$$

We can evaluate the commutator explicitly. For bosons and fermions we find

$$[\hat{K}, a_i^\dagger]_- = (\epsilon_i - \mu)a_i^\dagger \quad . \tag{16}$$

So

$$\frac{\partial}{\partial \beta} a_i^\dagger(\beta) = -e^{-\beta\hat{K}}(\epsilon_i - \mu)a_i^\dagger e^{\beta\hat{K}} \tag{17}$$

$$= -(\epsilon_i - \mu)a_i^\dagger(\beta) \quad . \tag{18}$$

We then have a simple differential equation with the solution

$$a_i^\dagger(\beta) = e^{-\beta(\epsilon_i - \mu)} a_i^\dagger \quad . \tag{19}$$

(b) Now we can consider

$$Tr e^{-\beta\hat{K}} \hat{a}_i^\dagger \hat{a}_i = Tr \hat{a}_i^\dagger(\beta) e^{-\beta\hat{K}} \hat{a}_i \tag{20}$$

$$= e^{-\beta(\epsilon_i - \mu)} Tr \hat{a}_i^\dagger e^{-\beta\hat{K}} \hat{a}_i \quad . \tag{21}$$

Using the cyclic invariance of the trace, this can be rewritten as

$$Tr e^{-\beta\hat{K}} \hat{a}_i^\dagger \hat{a}_i = e^{-\beta(\epsilon_i - \mu)} Tr e^{-\beta\hat{K}} \hat{a}_i \hat{a}_i^\dagger \quad . \tag{22}$$

However, from the commutation and anticommutation relations

$$a_i a_i^\dagger = 1 + \eta a_i^\dagger a_i \tag{23}$$

where $\eta = +1$ for bosons and $\eta = -1$ for fermions. Then

$$Tr e^{-\beta\hat{K}} \hat{a}_i^\dagger \hat{a}_i = e^{-\beta(\epsilon_i - \mu)} Tr e^{-\beta\hat{K}} (1 + \eta a_i^\dagger a_i) \quad . \tag{24}$$

Dividing by $Z = Tr e^{-\beta\hat{K}}$ gives

$$\bar{n}_i = e^{-\beta(\epsilon_i - \mu)}(1 + \eta \bar{n}_i) \tag{25}$$

So we have finally

$$\bar{n}_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - \eta} \quad . \tag{26}$$

These are the famous results due to Bose-Einstein and Fermi-Dirac.

5. Photon-Electron Interactions

(a) In the photoelectric effect a photon is completely absorbed resulting in the ejection of an electron from a surface. Is it possible for a photon to be completely absorbed by a free electron? Provide a proof of your answer. How does the presence of a heavy nucleus alter the situation?

(b) If a 1 MeV photon is absorbed during the ejection of an electron in the vicinity of a heavy nucleus, what is the velocity of the electron after the interaction? Neglect the initial kinetic energy of the electron, and the recoil energy of the nucleus.

(c) If instead the same photon were to Compton-scatter from this electron, determine the maximum velocity that it would be able to impart to the electron. ($hc = 1240$ MeV-fm).

(d) Compton observed that low-energy photons backscattered from a target sometimes showed no measurable increase in wavelength with respect to their initial value, regardless of the observed 180 degree change in direction of their trajectory, nor did they induce any ionization in the target. Explain how this result can occur.

Photon-Electron Interactions - Solution

(a) In the center-of-mass frame, the total momentum must be zero before and after the interaction. Therefore, the final condition must be an electron at rest. Then, conservation of energy would require $h/\nu + \gamma m_e c^2 = m_e c^2$ which is clearly not possible. A standing heavy nucleus to which the electron is tightly bound contributes momentum and absorbs a negligible amount of recoil energy enabling the process to occur.

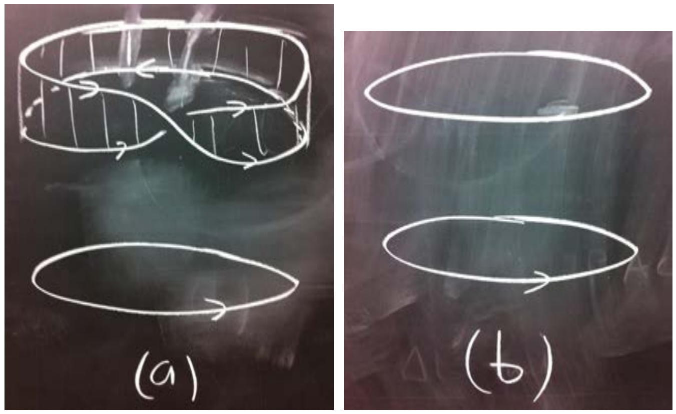
(b) Neglecting the quantities mentioned, the conservation of energy equation reduces to: $E_{\text{photon}} + m_e c^2 = \gamma m_e c^2$. For the given photon energy and the known mass of the electron one can solve for the speed to be $v = 0.94c$.

(c) For Compton scattering we have $\Delta\lambda = \frac{hc}{m_e c^2}(1 - \cos\theta)$. The photon will impart the maximum momentum if it is backscattered ($\theta = 180°$). Inputting constants, we get $\Delta\lambda = 4853$ fm.

The wavelength of the incident 1-MeV photon works out to be $\lambda_i = 1240$ fm. The wavelength of the scattered photon is $\lambda_f = \lambda_i + \Delta\lambda = 6093$ fm. Therefore, the energy of the scattered photon is $E = hc/\lambda_f = 0.2$ MeV. Hence, the electron acquired 0.8 MeV of kinetic energy and the corresponding speed works out to be $v = 0.92c$.

(d) For lower photon energies, the electron binding to the nucleus can be sufficiently strong that the whole atom will recoil, not just the electron. In that case, the mass in the equation for $\Delta\lambda$ is that of the whole atom, making $\Delta\lambda$ negligible. The electron remains bound to the nucleus.

6. Möbius Loops



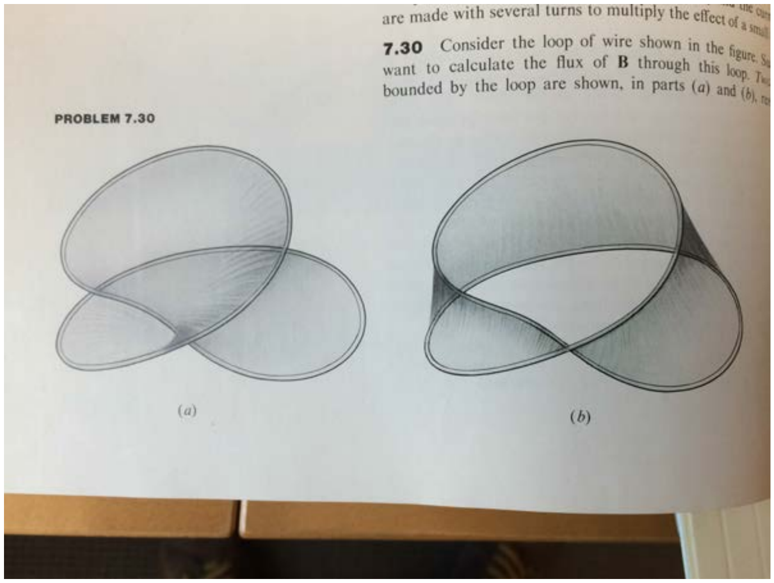
The mutual inductance between two loops is proportional to the magnetic flux:

$$\Phi_{1,2} = \int_{\Omega_2} \mathbf{B}_1 \cdot \mathbf{dA}$$

where $\mathbf{B}_1$ represents the magnetic field generated by a current through loop 1 and Ω_2 is a surface bounded by loop 2.

(a) Consider a loop that traces the edge of a Möbius band as shown at the top of Figure (a). Sketch the corresponding surface Ω that is bound by the loop. Can you think of more than one possible surface? Are all surfaces allowed?

(b) Use this result to determine the mutual inductance between a simple loop and a Möbius loop, in particular: If the mutual inductance between the loops shown in Figure (b) is L, what is the inductance between the Möbius loop at the top of Figure (a) and the simple loop at the bottom of Figure (a)? You can assume the Möbius strip is narrow relative to its diameter.



6. Möbius Loops - Solution

(a) The surface must be orientable for the expression to be well defined. The surface of the möbius band is not orientable but the other surface drawn in the figure above is. This surface is the one that can also be obtained by starting with a loop twice the radius of the band, drawing the trivial disk-like surface then folding it over.

(b) Using the correct surface we see that the inductance will be twice that of the two simple loops, $2L$.

Spring 2014

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 2

March 25, 2014

(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

7. Maxwell's Equations

Consider Maxwell's equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}$$

where ρ and $\mathbf{J}$ are the charge and charge current densities.

(a) Show that ρ and $\mathbf{J}$ satisfy a continuity equation.

(b) Show that Maxwell's equations are invariant under a local gauge transformation.

Maxwell's Equations - Solution

(a) Take the time derivative of the first equation to obtain

$$\dot{\rho} = \frac{\nabla \cdot \dot{\mathbf{E}}}{4\pi} \quad . \tag{1}$$

Take the divergence of the fourth equation and using

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 \tag{2}$$

to obtain

$$\frac{4\pi}{c} \nabla \cdot \mathbf{J} = -\frac{1}{c}\frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \tag{3}$$

Comparing we find

$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0 \tag{4}$$

which is the standard form for a continuity equation.

(b) We can simplify Maxwell's using potentials $\mathbf{A}$ and ϕ where

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

and

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} \quad . \tag{6}$$

Inserting these into Maxwell's equations we obtain two equations for the potentials:

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho \tag{7}$$

and

$$[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\mathbf{A} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi) = -\frac{4\pi}{c}\mathbf{J} \quad . \tag{8}$$

These equations are invariant under the transformation

$$A_i \to A_i' - \nabla_i \Lambda \tag{9}$$

and

$$\phi \to \phi' + \frac{1}{c}\Lambda \tag{10}$$

where Λ is a general fuction of $\mathbf{x}$ and t.

8. Specific Heat

The average energy for a set of noninteracting bosonic quantum excitations, like photons, in thermal equilibrium at temperature T and with zero chemical potential is given by

$$E = V \int \frac{d^3p}{(2\pi\hbar)^3} E_p \frac{1}{e^{\beta E_p} - 1}$$

where V is the volume of the system, $\beta = 1/(k_B T)$ and E_p is the single-particle energy as a function of the momentum p. Suppose the single-particle energy is of the form

$$E_p = \Delta + \epsilon_p$$

where Δ is a positive constant, $\epsilon_p = \epsilon_0 |p|^n$, and ϵ_0 and n are positive.

For this system, determine the specific heat at low-temperatures as a function of temperature up to an overall constant.

Specific Heat - Solution

Since E_p is positive we have the excellant approximation at low temperatures

$$\frac{1}{e^{\beta E_p}-1} = e^{-\beta E_p} + ... \tag{11}$$

and using the equation for the single-particle energy we obtain

$$E = e^{-\beta\Delta} V \int \frac{d^3p}{(2\pi\hbar)^3} (\Delta + \epsilon_p) e^{-\beta\epsilon_p} \quad . \tag{12}$$

Do the angular integration

$$E = e^{-\beta\Delta} V \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty p^2 dp (\Delta + \epsilon_p) e^{-\beta\epsilon_p} \quad . \tag{13}$$

Now change integration variables from p to

$$x = \beta\epsilon_p \tag{14}$$

or

$$p = \left(\frac{(k_B T x)}{\epsilon_0}\right)^{1/n} \quad . \tag{15}$$

Then we have

$$p^2 dp = \left(\frac{(k_B T x)}{\epsilon_0}\right)^{3/n} \frac{1}{nx} \tag{16}$$

and dropping terms higher order in temperature

$$E = \Delta e^{-\beta\Delta} V \frac{4\pi}{(2\pi\hbar)^3} \left(\frac{(k_B T)}{\epsilon_0}\right)^{3/n} \int_0^\infty \frac{x^{3/n}}{nx} dx e^{-x} \quad . \tag{17}$$

This has the form

$$E = K\Delta e^{-\beta\Delta} T^{3/n} \tag{18}$$

where K is a constant independent of temperature.

The specific heat is given by

$$C = \frac{dE}{dT} \tag{19}$$

$$= \frac{K}{k_B} \Delta e^{-\beta\Delta} T^{3/n-2} \tag{20}$$

at low temperatures.

9. Spaceship Signal

A spaceship is flying away from the earth at a constant velocity. It is sending radio signal pulses back to earth. The pulses then reflect from the earth back to the spaceship. It takes a signal pulse 40 seconds to complete the round trip, according to the clock on the spaceship. In addition, when the signal pulse is received by the spaceship, its frequency is half of the value at emission.

(a) When the radio signal is reflected, what is the distance between the earth and the spaceship, in the reference frame of the spaceship?

(b) What is speed of the spaceship relative to the earth?

(c) When the radio signal is received by the spaceship, what is the distance between the spaceship and the earth, in the reference frame of the earth?

Reminder:

Denote the wavelength and the frequency of light as λ and f. Then then wave number $k = 2\pi/\lambda$ and angular frequency $\omega = 2\pi f$ can form a 4-vector. This means $(k, \omega/c)$ in frame O and $(k', \omega'/c)$ in frame O' are related by the usual Lorentz transformation

$$\begin{pmatrix} k' \\ \omega'/c \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} k \\ \omega/c \end{pmatrix},$$

where $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$.

Spaceship Signal - Solution

(a) For this part, it is easiest to just think in the spaceship frame. From this point of view, the trip of the signal to earth took 20 seconds. Therefore, the distance is $20c = 6 \times 10^9$m.

(b) The wave number and frequency in the spaceship frame before the reflection is (k, ω). In the earth frame before reflection, they are (k', ω'). After reflection, in the earth frame, they become $(-k', \omega')$. Transforming $(-k', \omega')$ back to the spaceship frame, we obtain the frequency of the signal when it is received back by the spaceship. Starting from before reflection

$$\begin{pmatrix} k' \\ \omega'/c \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} k \\ \omega/c \end{pmatrix}. \tag{21}$$

We have

$$\begin{aligned} k' &= \gamma(k - \beta\omega/c) = \gamma k(1-\beta) \\ \omega' &= \gamma\omega(1-\beta) \end{aligned} \tag{22}$$

After the reflection, transforming back to the spaceship frame, we have

$$\begin{pmatrix} k_{\mathrm{rec}} \\ \omega_{\mathrm{rec}}/c \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} -k' \\ \omega'/c \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} -\gamma k(1-\beta) \\ \frac{1}{c}\gamma\omega(1-\beta) \end{pmatrix} \tag{23}$$

Therefore,

$$\omega_{\mathrm{rec}} = \frac{1-\beta}{1-\beta^2}\omega(1-\beta) = \frac{1-\beta}{1+\beta}\omega. \tag{24}$$

Therefore, from $\omega_{\mathrm{rec}}/\omega = 1/2$, we can solve $\beta = 1/3$, or $v = 1/3 \cdot c$.

(c) In the spaceship's frame, when the signal is received back, the distance to earth is

$$20 \cdot c + 20 \cdot \frac{1}{3}c = \frac{80}{3}c. \tag{25}$$

Transforming this to the earth frame, the distance is

$$d = \frac{80}{3}c \cdot \gamma = 8.5 \times 10^9\mathrm{m} \tag{26}$$

10. The Elevator Problem

You work in a seven-story physics building serviced by an ideal elevator, i.e., one that continually runs from bottom to top, and down again, in a cyclic fashion. Suppose your office is on the 2nd floor, and you want to take the elevator to visit a colleague on the 6th floor.

(a) Determine the probability for the arriving elevator to be going in the right (upward) direction.

(b) Does the situation change if the building is serviced by two such elevators, operating out of sync? That is, what is the probability that the first elevator to arrive will be going in the desired (upward) direction?

Comment: In this second part we are looking for a reasoned answer more than a formal calculation of the probability – whether or not the probability changes in going from one elevator to two and why.

The Elevator Problem - Solution

(a) Let us define

$$p = \frac{\text{distance from the 2nd floor to the bottom floor}}{\text{distance from the top floor to the bottom floor}}$$

Since you approach the elevator at a random time, the probability that the elevator will be below you is p which is $1/6$ in this case.

(b) **Reasoned Answer**: The situation does change if there are two elevators. Because you are on the second floor, an elevator that is below you, which will definitely be going up, is more likely to arrive first because it is more likely to be closer to you.

More Precise Answer: Consider 3 cases. (1) Both elevators being above you occurs with probability $(1-p)^2$. For this case the first elevator is definitely going down. (2) Both elevators being below you occurs with probabilty p^2. For this case the first elevator is definitely going up. (3) Having one elevator above you and the other below you can occur in two ways, so the probability is $p(1-p)+(1-p)p = 2p(1-p)$.

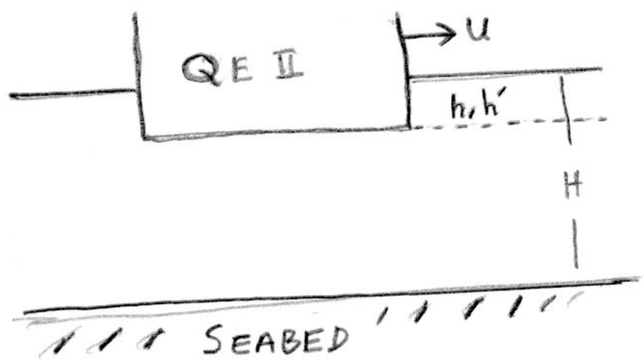
For case (3), if the upper elevator is within a distance from the 2nd floor of twice the distance between the 2nd and bottom floors, which occurs with probability p, it has an equal probability of reaching you first as the other elevator. The probability for this situation is $2p^2$ because it can happen two ways. Otherwise, the lower elevator will arrive first going up. The probability for this latter case is $2p(1-p) - 2p^2 = 2p(1-2p)$.

The probability that the first elevator will arrive going up then is

$$(1-p)^2 \cdot 0 + p^2 \cdot 1 + 2p^2 \cdot \frac{1}{2} + 2p(1-2p) \cdot 1 = 2p(1-p)$$

With $p = 1/6$ we get a probability of $5/18$.

11. Grounding of the Queen Elizabeth II



In 1992 the Queen Elizabeth II struck the seabed on the east coast of the United States. In the area where it was traveling, the ship should have had two meters of clearance above the seabed. What was the Captain's mistake?

To answer this question, consider the simplified model depicted in the figure and described as follows. Assume two dimensional irrotational flow, uniform under the ship. Ignore end effects and let the water be stationary ahead and behind the barge. Let the distance from the surface of the ocean to the bottom of the ship be h when the ship is stationary and h' when the ship is moving with speed U. Let the mass of the ship be M and the density of the water be ρ.

(a) Derive an expression for the height of the ship above the seabed as a function of its forward speed.

(b) Sketch this as a function of the speed of the ship. How does this help explain what might have gone wrong?

Grounding of the Queen Elizabeth II - Solution

(a) By conservation of mass an amount of water $\rho h' A$ must make it from the front of the ship to the back. This induces a flow under the ship with speed

$$v = U\frac{h'}{H - h'} \ .$$

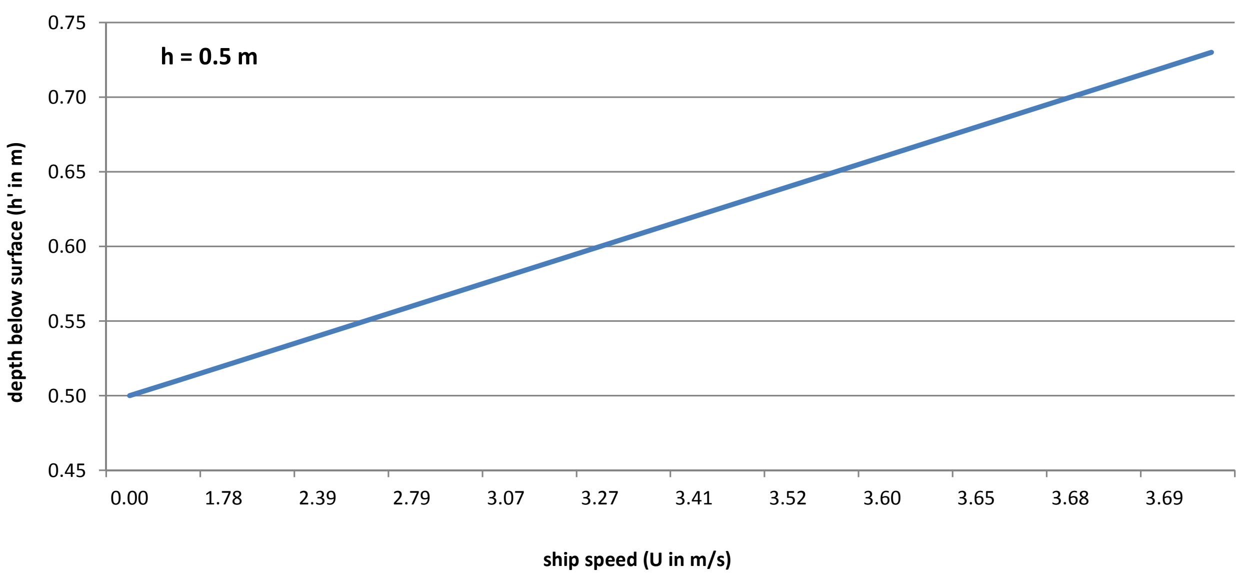
At rest, vertical force balance requires $Mg = PA + \rho g h A$, while Bernoulli's law gives $P = P' + \frac{1}{2}\rho v^2$. During the motion of the ship, vertical force balance requires $Mg = P'A + \rho g h' A$ which can be written

$$Mg = PA - \frac{1}{2}\rho v^2 A + \rho g h' A.$$

Solving this for h' gives $h' = h + \frac{v^2}{2g}$. Substituting for v from the previous result then gives

$$h' = h + \frac{U^2}{2g}\left(\frac{h'}{H - h'}\right)^2 .$$

(b) The plot below shows that the depth initially increases with U. The Captain came in too fast.



12. Coupled Fermions in a Magnetic Field

Consider a system of two spin-$\frac{1}{2}$ fermions in a uniform magnetic field along the z-direction. The Hamlitonian of the system is

$$H = a\sigma_{1z} + b\sigma_{2z} + c_0\vec{\sigma}_1 \cdot \vec{\sigma}_2,$$

where a and b depend on the magnetic field and the intrinsic magnetic moment, and the term proportional to c_0 is the coupling between the two fermions. a, b and c are real constants. We assume $a \neq b$. $\vec{\sigma}$ are Pauli matrices.

(a) Find the exact energy levels of the system.

(b) In the strong magnetic field limit, $c_0 \ll a,\ b$, find the energy levels of the system using perturbation theory to the order of c_0^2.

Coupled Fermions in a Magnetic Field - Solution

(a) We will denote the eigenstates of $\sigma_{1z,2z}$ to be $|\uparrow\rangle_{1,2}$ and $|\downarrow\rangle_{1,2}$. We have

$$\sigma_z|\uparrow\rangle = |\uparrow\rangle, \quad \sigma_z|\downarrow\rangle = -|\downarrow\rangle. \tag{27}$$

It is also useful to consider the eigenstates of total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$, where $\vec{s}_{1,2} = \frac{1}{2}\vec{\sigma}_{1,2}$. We denote this set of eigenstates as $|Lm\rangle$, where $L = 1, 0$. We have

$$\begin{aligned}
|11\rangle &= |\uparrow\rangle_1|\uparrow\rangle_2 \\
|1-1\rangle &= |\downarrow\rangle_1|\downarrow\rangle_2 \\
|10\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2) \\
|00\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)
\end{aligned} \tag{28}$$

From $\vec{S}^2 = \vec{s}_1^2 + \vec{s}_2^2 + 2\vec{s}_1\cdot\vec{s}_2$, we can evaluate

$$\begin{aligned}
\vec{\sigma}_1\cdot\vec{\sigma}_2|1m\rangle &= |1m\rangle \\
\vec{\sigma}_1\cdot\vec{\sigma}_2|00\rangle &= -3|00\rangle
\end{aligned} \tag{29}$$

We also have

$$\sigma_{1z}|10\rangle = |00\rangle,\ \sigma_{2z}|10\rangle = -|00\rangle,\ \sigma_{1z}|00\rangle = |10\rangle,\ \sigma_{2z}|00\rangle = -|10\rangle. \tag{30}$$

In the basis of $\{|11\rangle, |1-1\rangle, |10\rangle, |00\rangle\}$, the Hamiltonian is

$$\begin{pmatrix} c_0+a+b & & & \\ & c_0-a-b & & \\ & & c_0 & a-b \\ & & a-b & -3c_0 \end{pmatrix} \tag{31}$$

Therefore, the energy levels are

$$E = c_0 \pm (a+b), \quad -c_0 \pm \sqrt{4c_0^2 + (a-b)^2} \tag{32}$$

(b) In the limit $a,\ b \gg c_0$, the starting point of perturbative calculation is to identify $H = H_0 + H'$, where $H_0 = a\sigma_{1z} + b\sigma_{2z}$, and $H' = c_0\vec{\sigma}_1 \cdot \vec{\sigma}$. We also start with the eigenstates of H_0

$$|\uparrow\rangle_1|\uparrow\rangle_2, |\downarrow\rangle_1|\downarrow\rangle_2, |\uparrow\rangle_1|\downarrow\rangle_2, |\downarrow\rangle_1|\uparrow\rangle_2 \tag{33}$$

We also note

$$\begin{aligned} &|\uparrow\rangle_1|\uparrow\rangle_2 = |11\rangle, |\downarrow\rangle_1|\downarrow\rangle_2 = |1-1\rangle \\ &|\uparrow\rangle_1|\downarrow\rangle_2 = \frac{1}{\sqrt{2}}\left(|10\rangle + |00\rangle\right), |\downarrow\rangle_1|\uparrow\rangle_2 \frac{1}{\sqrt{2}}\left(|10\rangle - |00\rangle\right). \end{aligned} \tag{34}$$

Then we can use Eq. 29 to evaluate the matrix elements of H'. We obtain

$$H = H_0 + H' = \begin{pmatrix} a+b & & & \\ & -a-b & & \\ & & a-b & \\ & & & -a+b \end{pmatrix} + \begin{pmatrix} c_0 & & & \\ & c_0 & & \\ & & -c_0 & 2c_0 \\ & & 2c_0 & -c_0 \end{pmatrix} \tag{35}$$

To the second order in c_0, we have

$$\begin{aligned} E_1 &= a + b + c_0 \\ E_2 &= -a - b + c_0 \\ E_3 &= a - b - c_0 + \frac{2c_0^2}{a-b} \\ E_4 &= b - a - c_0 + \frac{2c_0^2}{b-a} \end{aligned} \tag{36}$$