Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, April 10, 2010

Do not separate this page from the problem pages. Fill out and turn in at the end of the exam.

Student __________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs 30 min**.

Please, indicate problems you are turning in:

□ **II-1** □ **II-2** □ **II-3** □ **II-4**

II-1. (Mathematical Methods)

Consider a differential equation

$$x(x+1)^2\, y''(x) + \frac{3}{2}x(x+1)\, y'(x) + \frac{3}{2}\, y(x) \;=\; 0\,. \tag{1}$$

(i) Find all singular points of this equation and determine whether they are regular or irregular.

(ii) Calculate the exponents (roots of the indicial equation) at each regular singular point.

(iii) Find a fractional linear transformation of the form

$$z \quad\longrightarrow\quad z' \;=\; \frac{az+b}{cz+d}\,, \qquad a,b,c,d \in \mathbb{C}\,, \tag{2}$$

that reduces (1) to a standard hypergeometric equation.

(iv) Write a solution of the initial equation (1) around $x=\infty$ in terms of the hypergeometric function, $F(\alpha,\beta\,;\gamma\,;z)$, of suitable arguments.

Hint: Recall that the standard hypergeometric equation for a function $f(z)$ has the form

$$z(z-1)\, f''(z) + [(\alpha+\beta+1)\, z - \gamma]\; f'(z) + \alpha\beta\, f(z) \;=\; 0\,. \tag{3}$$

It has regular singular points at $z=0$, 1 and ∞ with exponents $(0,1-\gamma)$, $(0,\gamma-\alpha-\beta)$ and (α,β), respectively. The hypergeometric function $F(\alpha,\beta\,;\gamma\,;z)$ is defined as the analytic solution to this equation around $z=0$.

II-2. (Electricity and Magnetism)

Two concentric spheres have radii a and $b>a$, and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V, while the other hemispheres are maintained at zero potential. Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials.

Hint: The following properties of Legendre polynomials might be useful:

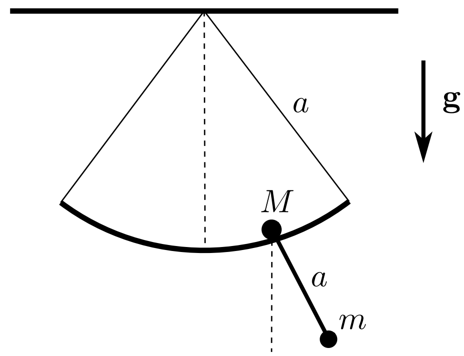
$$P_\ell(-x) \;=\; (-1)^\ell P_\ell(x)\,,$$

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) \;=\; \frac{1}{2\ell+1}\delta_{\ell\ell'}\,,$$

$$\int_{0}^{1} dx\, P_\ell(x) \;=\; \delta_{\ell 0} \qquad \text{for } \ell \text{ even}\,,$$

$$\int_{0}^{1} dx\, P_\ell(x) \;=\; -\left(-\frac{1}{2}\right)^{(\ell+1)/2} \frac{(\ell-2)!!}{\left(\frac{1}{2}(\ell+1)\right)!} \qquad \text{for } \ell \text{ odd}\,.$$

II-3. (Classical Mechanics)



A simple pendulum of mass m and length a is attached to another mass $M = 3m$ which moves without friction along a segment of *smooth* circular weightless structure of radius a. Assume that the motion of m and M is restricted to the same vertical plane.

(i) Find the Lagrangian of the system.

(ii) Write down the corresponding Hamiltonian, if both m and M are subject to small oscillations. Find a *canonical* transformation such that the Hamiltonian becomes the sum of two independent harmonic oscillators. What are the characterstic frequencies of these two harmonic oscillators?

(iii) Find the frequencies of the normal modes of this system under the small oscillation conditions using any other method of your choice. Compare your solution with the results in part (ii).

II-4. (Quantum Mechanics)

An electron moves in one dimension and is confined to the right half-space where it has potential energy

$$V(x) \; = \; -\frac{e^2}{4x}\,, \qquad x \; > \; 0\,,$$

with e the charge of the electron.

(i) What are the boundary conditions on a bound-state wavefunction?

(ii) Consider the function

$$x^\beta e^{-\alpha x^\gamma}\,, \tag{4}$$

where α, β, and γ are real numbers. Do any of these functions satisfy the correct boundary conditions on a bound-state wavefunction? If so, what are possible values of β and γ? Explain.

(iii) For what values of α, β, and γ is (4) a bound-state wavefunction?

(iv) Does the solution represent the ground state? Explain.

(v) What is the expectation value of the position operator, $\widehat{x}$, for this bound-state?