Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, March 24, 2018

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is marked with your Lg-number and the problem number.* Do not write answers to different problems on the same page. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)

A particle of mass m is subjected to the one-dimensional force

$$F(x) = -kx - \frac{\alpha}{x^3},$$

where k and α are real positive constants.

(i) Show that

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + A\frac{p}{x},$$

is a possible Hamiltonian for this system for an appropriately chosen constant A.

(ii) Consider the canonical transformation

$$X(x,p) = \arctan\left(\frac{\lambda x}{p}\right),$$

$$P(x,p) = \frac{1}{2}\left(\frac{p^2}{\lambda} + \lambda x^2\right) + \sqrt{\frac{m}{k}}\frac{Ap}{x},$$

with $\lambda = \sqrt{km}$. Verify that this is a canonical transformation.

(iii) Find the new Hamiltonian, K, associated with the canonical transformation given in part (ii). Solve Hamilton's canonical equations for this Hamiltonian and use your solution to determine the motion of the system in the original coordinates; that is, find $x(t)$. Leave your initial conditions arbitrary.

(iv) Consider the Lagrangian constructed in the usual way by finding the potential energy $V(x)$ associated with the force $F(x)$ and compare it to the Lagrangian constructed from the Hamiltonian in part (i). Does their relationship make sense?

Problem II - 2. (Electricity and Magnetism)

Consider an infinite plane, say the plane $z = 0$, which is **not** a conductor. Assume the potential is specified everywhere on this plane, and vanishes at infinity (Dirichlet boundary condition).

(i) What is the appropriate Green function $G(\vec{x}, \vec{x}')$ for obtaining the potential in the half-space $z \geq 0$? Express $G(\vec{x}, \vec{x}')$ in cylindrical coordinates, i.e. write $\vec{x}$ as a vector whose components along the x-, y-, and z-axes are $\rho \cos \phi$, $\rho \sin \phi$, and z with $0 \leq \rho < \infty$ and $0 \leq \phi \leq 2\pi$, and similarly for $\vec{x}'$.

(ii) On the plane $z = 0$ the potential Φ is held at the constant value V within a circle of radius a centered at the origin, and is held to be zero outside this circle, i.e. $\Phi = V$ in the region $0 \leq \rho < a$, $z = 0$ and $\Phi = 0$ in the region $\rho > a$, $z = 0$. Write down an integral expression (using cylindrical coordinates) for the potential in the region $z > 0$, assuming there is no charge in this region.

(iii) Show that along the axis of the circle (i.e. along the line $\rho = 0$, $z \geq 0$) the potential is

$$\Phi = V \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right).$$

(iv) Assuming that the potential on the plane is maintained at the constant value given in part (iii), how much work must be done on a point charge q to bring it from infinity to the position $z = b > 0$ on the z-axis?

(v) Is this work the same as the change in the total electrostatic energy (excluding the infinite self-energy of the point charge)? Explain.

Problem II-3. (Quantum Mechanics)

Consider a spin-$\frac{1}{2}$ particle moving on a line. The Hamiltonian of the spin-orbit coupling relevant to topological materials is given by

$$\widehat{H} = v\,\sigma^z\,\widehat{p}\,, \tag{3.1}$$

where σ^z is the standard Pauli matrix acting on the spin degrees of freedom (DOF),

$$\sigma^z|\uparrow\rangle = |\uparrow\rangle\,, \qquad \sigma^z|\downarrow\rangle = -|\downarrow\rangle\,,$$

$\widehat{p} = -i\hbar\frac{d}{dx}$ is the linear momentum operator acting on the orbital DOF and v is a non-negative parameter with the dimension of a velocity.

(0) Find the (generalized) eigenvectors and eigenvalues of the Hamiltonian (3.1).

(1) The state vector at $t = 0$ is $|\Psi(0)\rangle = |+\rangle|\psi\rangle$ where

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle + |\downarrow\rangle\big)\,,$$

$$\psi(x) \equiv \langle x|\psi\rangle = \big(2\pi\delta^2\big)^{-1/4}\exp\Big(-\frac{x^2}{4\delta^2}\Big)\,, \qquad \delta > 0\,.$$

Find (as explictly as possible) the time-evolved state-vector

$$|\Psi(t)\rangle = \exp\Big(-\frac{i}{\hbar}\widehat{H}\,t\Big)|\Psi(0)\rangle.$$

Hint: You can use (0) or directly expand the exponential and use properties of σ^z. Also expand $|\psi\rangle$ in $\widehat{p}$-eigenvectors. What's going on physically?

(2) Compute the reduced density matrix of the spin subsystem at time t, that is

$$\rho_S(t) = \mathrm{tr}_O|\Psi(t)\rangle\langle\Psi(t)|\,,$$

where tr_O denotes the *partial* trace over the orbital DOF.

Hint: Depending on the solution, you may or may not need this integral:

$$\int_{-\infty}^{\infty} dp\, e^{-ap^2+bp} = \sqrt{\frac{\pi}{a}}e^{b^2/(4a)}.$$

(3) Argue, as quantitatively as possible, that *entanglement*, $S = -\mathrm{Tr}\,(\rho_S \log \rho_S)$, between the spin and orbital DOF grows from zero at $t = 0$ to the maximal value for $t \to \infty$.

Problem II - 4. (Math Methods)

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\{|i\rangle\}_{i=0}^{\infty}$ a complete orthonormal set (COS) of it. Consider the following operator D over $\mathcal{H}$ whose only non-zero matrix elements are on the diagonal and have values $D_{ii} = N_x x^i$ where the normalization constant is given by $N_x := 1 - x$ and $x \in (0, 1)$ is a fixed parameter (in short: $D = N_x \,\mathrm{diag}(x^i)_{i=0}^{\infty}$).

If X, Y are operators over $\mathcal{H}$ we define

$$\langle X,\, Y\rangle_D := \mathrm{Tr}\left(D X^{\dagger} Y\right) \,. \tag{4.1}$$

(1) Show that Eq. (4.1) above defines a good (non-degenerate) scalar product over the space of *bounded operators* in $\mathcal{H}$, that is those linear maps X for which

$$\|X\| := \sup_{\|v\|=1} \|X(v)\| < \infty \,.$$

(2) Consider now the operators:

a) $X_1 = \mathbf{1}$ (identity),

b) $X_2 = \mathrm{diag}(y^i)_{i=1}^{\infty}, \quad y \in \mathbb{R}$,

c) X_3 whose action over the COS is given by

$$X_3|i\rangle \;= \sqrt{i}|i-1\rangle \,, \qquad i = 0, 1, \ldots, \infty.$$

Discuss whether (and when) the X_α's above ($\alpha = 1, 2, 3$) are (or not) bounded operators.

(3) Find their norms according to Eq. (4.1), that is compute (as explicitly as you can) $\|X_\alpha\|_D := \sqrt{\langle X_\alpha,\, X_\alpha\rangle_D}$, ($\alpha = 1, 2, 3$).