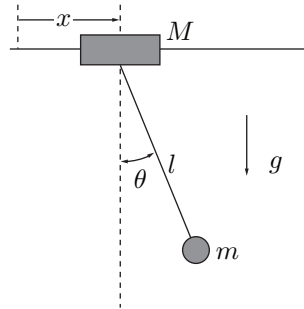# J08M.1 - Pendulum on a Sled

Problem

A plane pendulum consists of a bob of mass m suspended by a massless rigid rod of length l that is hinged to a sled of mass M. The sled slides without friction on a horizontal rail. Gravity acts with the ususal downward acceleration g.

a)   Taking x and θ as generalized coordinates, write the Lagrangian for the system.

b)   Derive the equations of motion for the system.

c)   Find the frequency ω for *small* oscillations of the bob about the vertical.

d)   At time $t = 0$ the bob and the sled, which had previously been at rest, are set in motion by a sharp tap delivered to the bob. The tap imparts a horizontal impulse $\Delta P = F \Delta t$ to the bob. Find expressions for the values of $\dot{\theta}$ and $\dot{x}$ just after the impulse.

J08M.2 - Sphere in a Pipe

Problem

A cylindrical pipe of interior radius R lies horizontally on the ground. A solid sphere of radius $R/2$ is placed inside the pipe. What is the frequency of small oscillations of the sphere? Assume that the axis of rotation of the sphere is always parallel to the symmetry axis of the pipe, and the sphere rolls without slipping.

J08M.3 - Fluid Flow

Problem

When we derive Newton's equations of motion from a Lagrangian or Hamiltonian, the equations are invariant under time reversal, so that if $x(t)$ is a solution, so is $x(-t)$. If we add terms corresponding to damping or viscosity, the invariance is broken, and motions become obviously irreversible. Strangely, a form of reversibility is restored for fluid motion in the limit that viscosities are very large.

Consider a fluid with viscosity η and density ρ, and assume that it is incompressible. The equations of motion are the Navier-Stokes equations,

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right] = \vec{\nabla} p + \eta \nabla^2 \vec{v}$$

$$\vec{\nabla} \cdot \vec{v} = 0 \, ,$$

where $\vec{v}(\vec{x}, t)$ is the velocity of the fluid element at position $\vec{x}$ at time t, and $p(\vec{x}, t)$ is the pressure. To be concrete, imagine that we have a layer of fluid between two (large) parallel plates, a distance d apart.

a) Let one of the plates move at velocity v_0, with the other plate held fixed. Now the natural unit of length is d, the natural unit of velocity is v_0, and the natural unit of pressure is ρv_0^2. Show that, in these natural units, a single term in the Navier-Stokes equations becomes dominant at large viscosity. Since viscosity has units, "large" means large relative to some characteristic scale η_c, which you should determine.

b) In this limit of large viscosity (usually called the "low Reynolds number" limit, $\mathrm{Re} \equiv \eta_c/\eta$), show that if the plate moves for a time T with velocity v_0, and then with velocity $-v_0$ for an equal time T, all elements of the fluid will be returned exactly to their initial locations, so that motion is reversible. You should show this explicitly for the problem of fluid between two plates (by solving the equations), and give a more general argument (which doesn't require solving the equations).

Notes: Recall that fluid immediately adjacent to a wall must move with the velocity of the wall - no slipping. The reversibility of motion means that if we inject a blob of dye into the fluid, then the motion of the wall at v_0 will spread the dye out (you should think about why) but then the motion at $-v_0$ will reassemble the original blob.

J08E.1 - Radiation from an Antenna

Problem

An antenna consists of a circular wire loop of radius R, centered in the x-y plane of a Cartesian coordinate system. The current has the same amplitude, $I = I(t)$, at all locations in the wire at a given time t. There is no net electrical charge on the wire. Assuming that $\dot{I}$, the rate of change of the current, is slow enough that magnetic dipole radiation dominates any higher multipoles, calculate:

a)   the vector potential $\vec{A} = \vec{A}(\vec{r}, t)$ and scalar potential Φ at the location $\vec{r}$ and time t when $r \gg cI/\dot{I}$ (specify your choice of gauge);

b)   the magnetic and electric fields, $\vec{B}$ and $\vec{E}$, at $\vec{r}$ and t;

c)   the energy flux, $S = S(\theta, \phi)$, as a function of the polar angles θ and ϕ;

d)   the total radiated power $P = \int S \sin\theta \, d\theta \, d\phi$.

Retain enough terms of any expansion in powers of $1/r$ to account for radiation. Insofar as possible, express your answers in terms of the magnetic dipole moment, $m = \pi R^2 I/c$, and its time derivatives.

J08E.2 - Rotating Disk in a Magnet

Problem

(Top view)
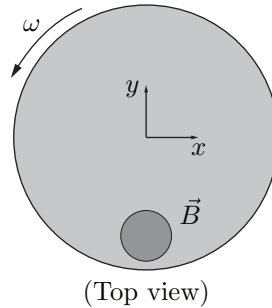
An aluminum disk of radius R, thickness d, conductivity σ, and mass density ρ is mounted on a frictionless vertical axis. It passes between the poles of a magnet near its rim which produces a $\vec{B}$-field perpendicular to the plane of the disk over a small area A of the disk. The initial speed of the disk is $\omega(t = 0) = \omega_0$.

a)   An observer on the disk, moving between the pole pieces of the magnet would feel an electric field. Give the direction and magnitude of this field in terms of R, ω_0, and $\vec{B}$ (assume the angular speed ω is small enough so that $\gamma \sim 1$). This results in a current density.

b)   Calculate the torque due to the Lorentz force produced on this current density by the *vecB*-field of the magnet.

c)   Given the moment of inertia of the disk around its axis $(I = \frac{1}{2}MR^2)$, write out the equation of motion of the disk and calculate the number of revolutions of the disk before it comes to rest.

J08E.3 - Parallel Plate Diode

Problem
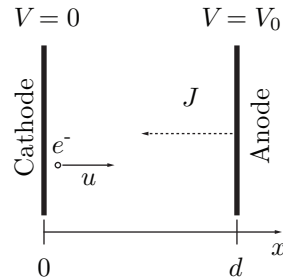
Consider an ideal parallel plate diode in a vacuum tube. A constant potential difference, $V_0 > 0$, is maintained between the cathode and the anode which are separated by a distance d. Electrons are assumed to be released from the cathode at zero potential with negligible velocity, but are accelerated to the anode. The region between the plates is a vacuum except for the electrons that are emitted into it, leading to a finite space charge density, $\rho(x)$, where x is the distance away form the cathode (see figure). Under steady state conditions, ρ is independent of time, and the continuity equation implies that the current density $J = \rho u$ is independent of x.

a)   Use Poisson's equation to find the potential $V(x)$ as a function of x.

b)   Find an explicit expression for the current density J in terms of V_0 (the Child-Langmuir law).

J08Q.1 - Deuteron

Problem

A deuteron is a bound state of a neutron (charge 0, mass 939.5 MeV) and a proton (charge e, mass 938.2 MeV), Scattering measurements determine that the separation of the neutron and proton is about $a = 1.5$ fm and mass measurements determine that the binding energy is $E_b = 2.226$ MeV. Approximate the potential energy as a spherical square well, $V(r) = -V_0$ fro $r < a$ and $V(r) = 0$ for $r > a$. (Recall that $\hbar = 6.5817 \times 10^{-16}$ eV s.)

a)   What is the value of V_0 in MeV?

b)   Can the deuteron have an excited (but still bound!) state with angular momentum $\ell = 0$?

c)   Are there bound states with $\ell > 0$? Explain!

J08Q.2 - Spin in a Magnetic Field

Problem

An electron is subject to a uniform magnetic field $\vec{B}_0 = B_0\hat{z}$ and occupies the spin eigenstate $|\uparrow\rangle$. At time $t = 0$, an additional time-dependent magnetic field $B_1(t) = B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$ is turned on. Calculate the probability of finding the electron with its spin along the negative z-axis at time $t > 0$. Ignore spatial degrees of freedom.

J08Q.3 - Photoelectric Effect

Problem

Compute the differential cross section for the photo-electric effect, i.e., the scattering process by which a photon is absorbed by an atom while kicking an electron out of its orbit. Assume that initially the electron is in the ground state $|\psi_{100}\rangle$ of an H-atom,

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where a_0 denotes the Bohr radius. The incoming photon beam consists of N photons, all in a momentum and polarization eigenstate $|\vec{k}, \hat{\epsilon}\rangle$. The beam and atom are inside a periodic box with volume V. The final state has $N-1$ photons, and you may assume that the electron ends up in a momentum eigenstate $|\vec{k}_f\rangle$.

Hint: use the dipole approximation, where the interaction describing the coupling between the photon field and the electron is given by $(e/m)\vec{A} \cdot \vec{p}$, with

$$\vec{A} = \sqrt{\frac{2\pi\hbar}{V}} \sum_{\vec{k}, \hat{\epsilon}} \frac{1}{\sqrt{c|\vec{k}|}} (a_{\vec{k}, \hat{\epsilon}} + a_{\vec{k}, \hat{\epsilon}}^\dagger) \hat{\epsilon} \,.$$

Here, $a_{\vec{k}, \hat{\epsilon}}$ and $a_{\vec{k}, \hat{\epsilon}}^\dagger$ are the photon creation and annihilation operators, $\hbar\vec{k}$ is the momentum and $\hat{\epsilon}$ the polarization of a photon.

J08T.1 - Graphene

Problem

Graphene is a new material that consists of an essentially two-dimensional array of carbon atoms. The velocity of sound is $c_s \sim 10^4$ m/s, and the area density of the material is ~ 6 atoms/nm^2 (1 nm $= 10^{-9}$ m).

a)   Estimate the Debye temperature Θ_D. Explain clearly how this quantity is defined in relation to experiment, and what physical picture you are using when you make your estimate. By "estimate" we mean to give an approximate numerical answer in Kelvin.

b)   Derive an expression for the contribution of phonons to the specific heat at constant volume, C_V. Find the limiting behaviors at high and low temperature, and use these limits to make a sketch of C_V vs. T. Label the axes with numbers to set the scale of your graph, and be clear about units.

c)   What is different in this problem from the textbook example of a three-dimensional crystal? Derive an expression not for the average energy, but for the mean-square displacement of one atom in the two-dimensional array. Do you see any problems?

J08T.2 - Spin Gas

Problem

Consider a gas of N non-interacting, spin-1/2 fermions, each of mass m. Let the gas initially be in a container of volume V at temperature $T = 0$.

a)   Calculate the total energy of the gas.

b)   Let the gas expand irreversibly into a volume V'. Show that for V' the behavior of the gas becomes classical. What is the final temperature of the gas in this limit?

c)   What was the change in entropy of the gas during its expansion?

J08T.3 - Motion of a Bead

Problem

A spherical particle with radius $r = 1$ μm is placed in water at room temperature ($T = 298$ K). The density of the particle is close to that of the water (1 g/cm^3), and the (dynamic) viscosity of water is $\eta = 0.01$ poise $= 0.01$ g/(cm·s).

a) If the particle has velocity $v = v_0$ at time $t = 0$, and there are no external forces, on what time scale does this initial velocity decay? Once the particle comes to equilibrium with its surroundings, what is its typical speed? For simplicity, assume that the motion is in one dimension, and neglect the force of gravity on the particle.

b) If you observe this system under a microscope with a conventional video camera, you can measure the position of the particle 30 times per second, and if you work hard at the image processing you can obtain a positional accuracy of ~ 1 μm. If you estimate the velocity by taking differences of position from frame to frame, will you see the velocity that you computed in (b)? Why or why not? Would it be worth buying a camera that captures 500 frames per second?

c) Give a quantitative description of the particle's position vs. time that you expect to see in such an experiment. When you sketch the expected results, be sure to indicate (at least approximately) the scales on the axes. Would buying a camera that captures 500 frames per second reveal any new features of this trajectory?