Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 10, 2005
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

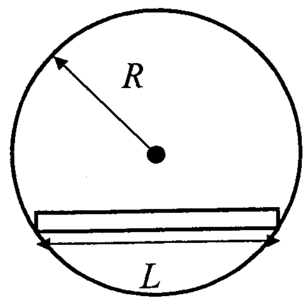
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Section 1 – Question 1

A stick of uniform density, mass, M, and length, L is constrained to move such that its ends rest on the inside of a fixed vertical, circular ring of radius R, as shown below.

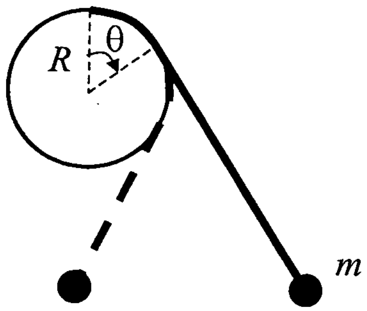


Find the frequency for *small* oscillations about the stick's equilibrium position.

You can neglect friction between the stick and the ring.

Section 1 – Question 2

Consider a pendulum built from a mass m attached to one end of a massless, extensionless string, whose other end is attached to the uppermost point of a fixed, vertical disk of radius R, as shown in the figure below. Assume that the total length of the string is l and that $\pi R < l$.



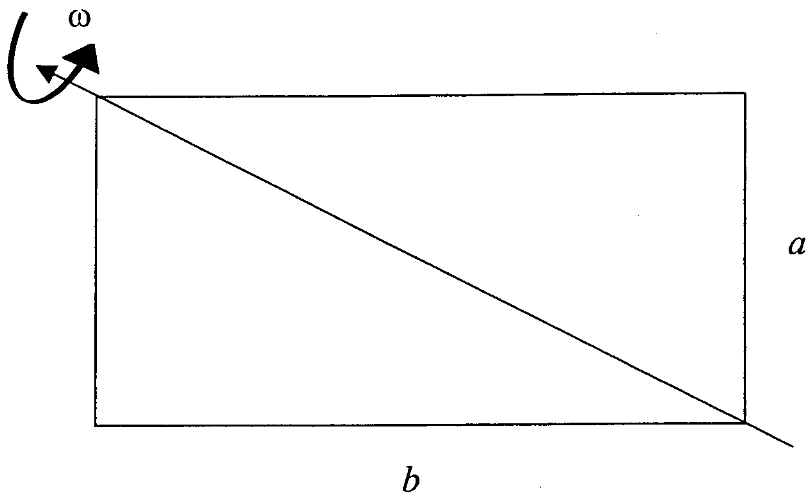
a) Find the equations of motion in terms of the angle, θ, as shown in the figure.

b) What is the equilibrium angle, θ_0?

c) Find the frequency of oscillations about the equilibrium angle.

Section 1 – Question 3

A uniform rectangular object of mass m with sides *a* and *b* ($b > a$) and negligible thickness rotates with constant angular velocity ω about a diagonal through the center. (For this problem, ignore gravity.)



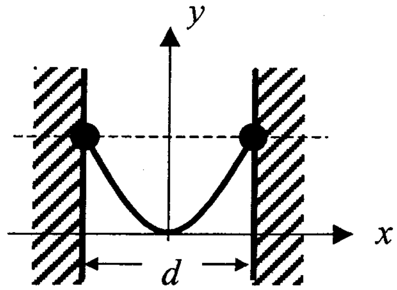
a) What are the principle axes and moments of inertia?

b) What is the angular momentum vector in body coordinate system?

c) What external torque must be applied to keep the object rotating with constant angular velocity around the diagonal?

Section 1 – Question 4

A chain with uniform mass density ρ (per unit length) hangs between two points on two walls as shown below. Assume that these two points are level.



(a) Find the shape of the chain. Apart from an arbitrary additive constant, the function describing the shape should also contain an unknown constant.

(b) Find an equation for this unknown constant in terms of the length l of the chain and the separation d between the walls.

Section 1 – Question 5

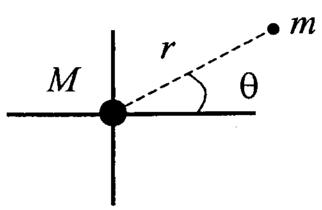
The gravitational potential near a black hole of mass, M and Schwarzschild radius, a can be described by a modified classical potential:

$$U(r) = -\frac{GM}{(r-a)}$$

a) Find an expression for the force acting on a particle of mass, m in this gravitational potential.

b) Expand your answer from part a) to find the lowest order correction to the classical gravitational force when the test particle is at a distance from the black hole large enough to satisfy the condition, $a / r << 1$.

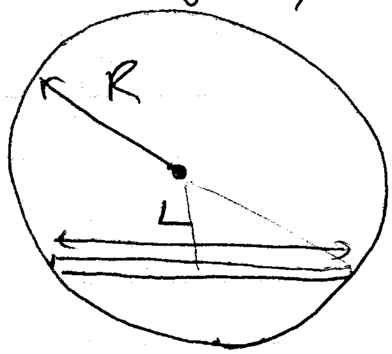
c) Develop a solution for the orbital motion of a test particle near the black hole using the force from part b) – just the classical and first order correction terms. Express your answer in the form $r = r(\theta)$, where θ is the angle between a fixed axis and the radius vector to the particle, as shown below.
You need only find the solutions periodic in θ.



J. Parsons I-1

Oscillation problem

A stick of uniform density, mass M, length L is constrained to move such that its ends rest on the inside of a circular ring of radius R. Find the frequency for small oscillations about its equil^m position.



(You can neglect friction between the stick and ring.)

Answer

about CM $\rightarrow$ $I_{cm} = \frac{M}{L}\int_{-L/2}^{+L/2} x^2\,dx = \frac{M}{L}\left.\frac{x^3}{3}\right|_{-L/2}^{+L/2} = \frac{1}{12}ML^2$

For rotation about center of ring

$$I = I_{cm} + Ml^2 \quad \text{with } l^2 = R^2 - \left(\frac{L}{2}\right)^2$$
$$= \frac{1}{12}ML^2 + M\left(R^2 - \frac{L^2}{4}\right) = M\left(R^2 - \frac{L^2}{6}\right)$$

The $\tau = I\ddot{\theta} = Mgl\sin\theta \simeq Mgl\theta$ for small θ

$\hookrightarrow Mgl\theta \simeq -M\left(R^2 - \frac{L^2}{6}\right)\ddot{\theta}$

$\hookrightarrow$ SHM with $\omega = \sqrt{\frac{gl}{(R^2 - L^2/6)}} = \sqrt{\frac{g\sqrt{R^2 - \frac{L^2}{4}}}{R^2 - L^2/6}}$

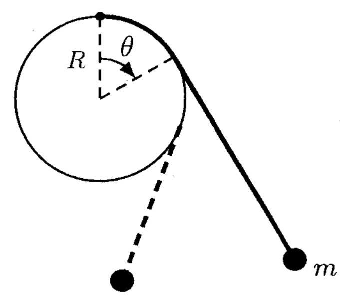
I - 2

DEC 2004

Qualifying exam
Lagrangian mechanics and relativity
Eduardo Pontón

mechanics

1. Consider a pendulum built from a mass m attached to one end of a massless, extensionless string, whose other end is attached to the uppermost point of a vertical disk of radius R, as shown in the figure. Assume that the total length of the string is l and that $\pi R < l$.



   (a) Find the equations of motion in terms of the angle θ as shown in the figure.

   (b) What is the equilibrium angle θ_0? Find the frequency of small oscillations about this position.

Relativity

2. Meson factories produce secondary π^+-meson beams from collisions of high energy protons with nuclear targets. The π^+-meson decays in flight mainly through the channel $\pi^+ \to \mu^+ + \bar{\nu}_\mu$.

   (a) Without approximation derive formulas for the muon and neutrino energies, E_μ and E_ν, in the pion's rest frame, in terms of the pion and muon masses (neglect the neutrino mass).

   (b) The π^+ has spin zero. What is the angular distribution of the μ^+ in the pion's rest frame?

   (c) What is the range of possible energies of the muon, E'_μ, in the laboratory frame? To what physical situations do the maximum and minimum values correspond?

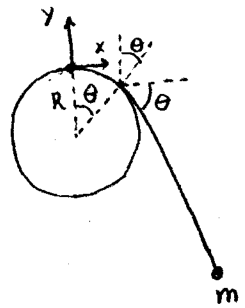
   (d) Making use of your results in parts (b) and (c) obtain the probability distribution of the laboratory μ^+ energy.

Lagrangian mechanics and relativity

Eduardo Pontón

Lagrangian mechanics

1) a)



Since the string extends tangentially to the disk, the geometry is as shown in the figure. Also, the position of m is

$$x = R\sin\theta + (\ell - R\theta)\cos\theta$$

$$y = -R(1-\cos\theta) - (\ell - R\theta)\sin\theta$$

where $(\ell - R\theta)$ is the length of the string that extends from the disk

Therefore,

$$\dot{x} = \cancel{R\dot\theta\cos\theta} + \cancel{(-R\dot\theta)\cos\theta} - (\ell-R\theta)\dot\theta\sin\theta = -(\ell-R\theta)\dot\theta\sin\theta$$

$$\dot{y} = \cancel{-R\dot\theta\sin\theta} - \cancel{(-R\dot\theta)\sin\theta} - (\ell-R\theta)\dot\theta\cos\theta = -(\ell-R\theta)\dot\theta\cos\theta$$

$$\Rightarrow \quad L = \frac{1}{2}m(\ell-R\theta)^2\dot\theta^2 - mg\left[-R(1-\cos\theta) - (\ell-R\theta)\sin\theta\right]$$

$$= \frac{1}{2}m(\ell-R\theta)^2\dot\theta^2 + mgR(1-\cos\theta) + mg(\ell-R\theta)\sin\theta$$

$$\Rightarrow \quad \frac{\partial L}{\partial\dot\theta} = m(\ell-R\theta)^2\dot\theta \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = -2m(\ell-R\theta)R\dot\theta^2 + m(\ell-R\theta)^2\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = m(\ell-R\theta)(-R)\dot\theta^2 + \cancel{mgR\sin\theta} + \cancel{mg(-R)\sin\theta}$$

$$+ mg(\ell-R\theta)\cos\theta$$

$$= -m(\ell-R\theta)\left[R\dot\theta^2 - g\cos\theta\right]$$

Lagrange equation: [cancel out a factor $m(\ell - R\theta)$]

$$(\ell - R\theta)\ddot{\theta} - 2R\dot{\theta}^2 + [R\dot{\theta}^2 - g\cos\theta] = 0$$

$$\Rightarrow \boxed{(\ell - R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0}.$$

b) In equilibrium: $\dot{\theta} = \ddot{\theta} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta_0 = \frac{\pi}{2}}$ (makes sens

Assume $\theta = \frac{\pi}{2} + \varepsilon$, $\varepsilon \ll 1$

$$\Rightarrow \dot{\theta} = \dot{\varepsilon}, \quad \ddot{\theta} = \ddot{\varepsilon}, \quad \cos\theta = -\sin\varepsilon \approx -\varepsilon$$

$$\Rightarrow (\ell - R\tfrac{\pi}{2})\ddot{\varepsilon} + g\varepsilon = 0$$

neglect ε: higher order; also $\dot{\theta}^2 = \dot{\varepsilon}^2$ is higher '

So the frequency of small oscillations is

$$\boxed{\omega = \sqrt{\frac{g}{\ell - R\frac{\pi}{2}}}}$$

I- 3

NOV 8 [illegible]

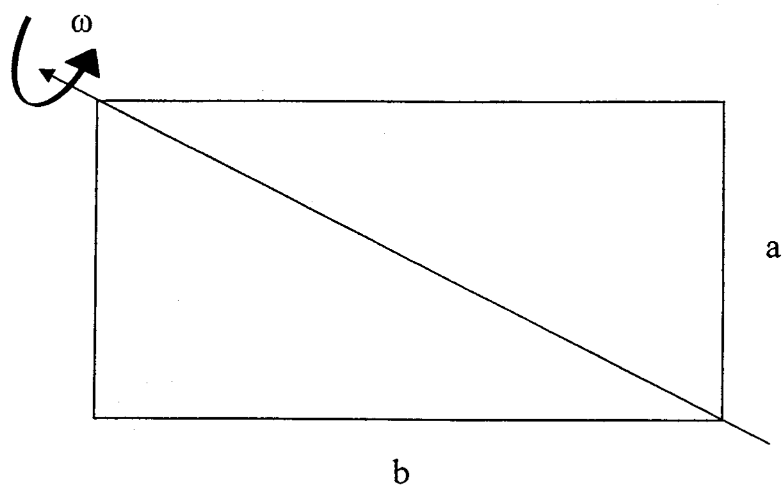
Mechanics Problem M. Shaevitz

A uniform rectangular object of mass m with sides a and b (b > a) and negligible thickness rotates with constant angular velocity ω about a diagonal through the center. (For this problem, ignore gravity.)

a) What are the principle axes and moments of inertia?

b) What is the angular momentum vector in body coordinate system?

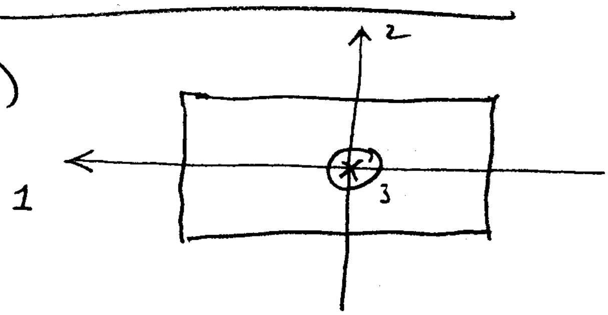
c) What external torque must be applied to keep the object rotating with constant angular velocity around the diagonal?



Mechanics Problem

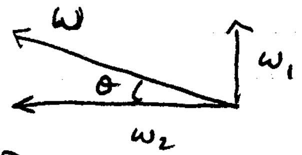
M. Shaevitz
I-3
solution

a)



$$I_1 = \frac{ma^2}{12} \qquad I_2 = \frac{mb^2}{12} \qquad I_3 = I_1 + I_2 = \frac{m(a^2+b^2)}{12}$$

b) $$\omega_1 = \frac{\omega b}{(a^2+b^2)^{1/2}} \qquad \omega_2 = \frac{\omega a}{(a^2+b^2)^{1/2}}$$



$$\omega_3 = 0 \qquad \vec{\omega} = \frac{\omega}{(a^2+b^2)^{1/2}}(b, a, 0)$$

$$\vec{L} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

$$= \left(\frac{ma^2}{12}\right)\frac{\omega b}{(a^2+b^2)^{1/2}}\hat{e}_1 + \left(\frac{mb^2}{12}\right)\left(\frac{\omega a}{(a^2+b^2)^{1/2}}\right)\hat{e}_2 + 0\,\hat{e}_3$$

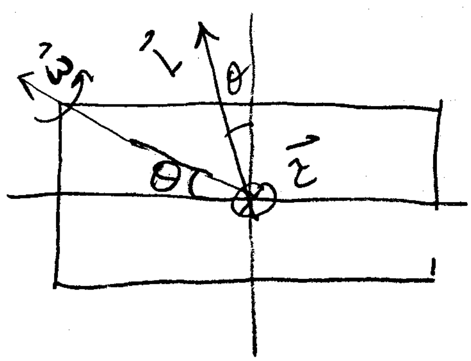
$$\boxed{\vec{L} = \frac{mab\,\omega}{12(a^2+b^2)^{1/2}}(a, b, 0)}$$

c) In body coordinate system $\vec{\omega}$ = constant

$$\vec{\tau} = \cancel{\frac{d\vec{L}}{dt}}^{\,0} + \vec{\omega} \times \vec{L} = \vec{\omega} \times \vec{L}$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & 0 \\ L_1 & L_2 & 0 \end{vmatrix} = (\omega_1 L_2 - \omega_2 L_1)\hat{e}_3$$

$$\boxed{\vec{\tau} = \frac{m\omega^2 ab}{12(a^2+b^2)}(b^2-a^2)\hat{e}_3}$$



Statics

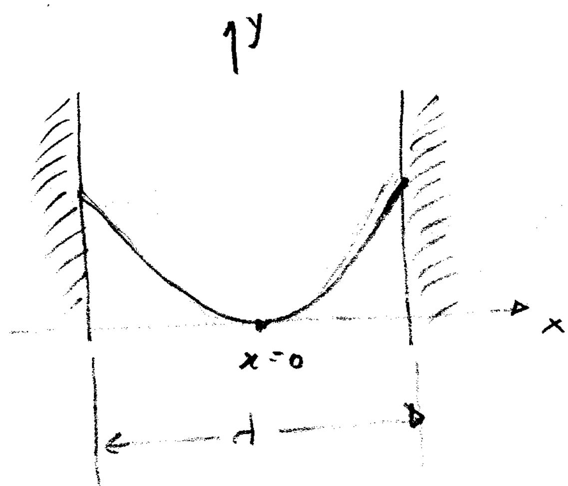
Aprile - I - 4

A chain with uniform mass density ρ / unit length hangs between two given points on two walls. Assume that these two points are are level.

ⓐ Find the shape of the chain.

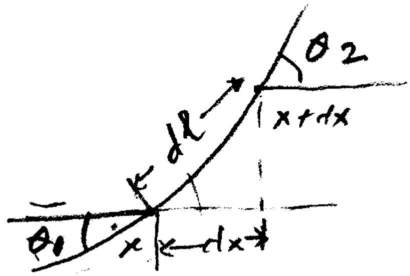
Apart ~~aside~~ from an arbitrary additive constant, the function describing the shape should also contain an unknown constant.

ⓑ Find an equation for this unknown constant in terms of the length l of the chain and the separation d between the walls.



Solution:

(a)



Consider the equilibrium of a section of the chain between x and $x+dx$.

The length of this section is $d\ell = \dfrac{dx}{\cos\theta_1}$

& the weight is $dw = \rho\, d\ell\, g = \rho g \dfrac{dx}{\cos\theta_1}$.

Let $T(x)$ be the tension at x & $T(x+dx)$ that at $x+dx$.

Then: Balance of forces

$$\begin{cases} \text{Horizontal: } & T(x+dx)\cos\theta_2 = T(x)\cos\theta_1 \\ \text{Vertical: } & T(x+dx)\sin\theta_2 = T(x)\sin\theta_1 + \dfrac{\rho g}{\cos\theta_1}dx \end{cases}$$

Squaring & adding,

$$T^2(x+dx) = T^2(x) + 2\rho g\, T(x)\tan\theta_1 + O(dx^2)$$

Recall: $T(x+dx) = T(x) + T'(x)dx + O(dx^2)$

$$\Rightarrow T^2(x) + 2T(x)T'(x)\,dx = T^2(x) + 2\rho g\, T(x)\tan\theta_1 \quad [\text{Dropping } O(dx^2)$$

$$\Rightarrow T'(x) = \rho g \tan\theta_1 .$$

But $\tan\theta_1 = \dfrac{dy}{dx} \Rightarrow T'(x) = \rho g\, y'(x)$

$$\Rightarrow \boxed{T(x) = \rho g\, y(x) + C_1} \quad \ldots\ldots (1)$$

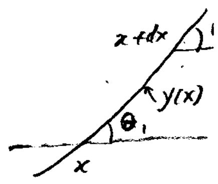
We still have to eliminate T in terms of purely geometric quantities.

③ I-4

again, $y'(x) = \tan\theta_1$ $\Rightarrow \cos\theta_1 = \dfrac{1}{\sqrt{1+\tan^2\theta_1}} = \dfrac{1}{\sqrt{1+(y'(x))^2}}$

& $y'(x+dx) = \tan\theta_2$ $\Rightarrow \cos\theta_2 = \dfrac{1}{\sqrt{1+\tan^2\theta_2}} = \dfrac{1}{\sqrt{1+(y'(x+dx))^2}}$

So, the condition for horizontal equilibrium is:



$$T(x+dx)\,\frac{1}{\sqrt{1+\underbrace{(y'(x+dx))^2}_{\approx\, y'^2(x)+2y'(x)y''(x)\,dx}}} = T(x)\,\frac{1}{\sqrt{1+(y'(x))^2}}$$

$$\Rightarrow \left(T(x)+T'(x)\,dx\right)\frac{1}{\sqrt{1+y'^2(x)}}\left(1+\frac{2y'y''dx}{1+y'^2}\right)^{-1/2}$$

$$= T(x)\,\frac{1}{\sqrt{1+y'^2(x)}}$$

$$\Rightarrow (T+T'dx)\,\frac{1}{\sqrt{1+y'^2}}\left(1-\frac{y'y''dx}{1+y'^2}\right) = T\,\frac{1}{\sqrt{1+y'^2}}$$

$$\Rightarrow \frac{T'}{\sqrt{1+y'^2}} - \frac{y'y''T}{(1+y'^2)^{3/2}} = 0$$

$$\Rightarrow \frac{T'}{T} = \frac{y'y''}{1+y'^2} = \frac{1}{2}\,\frac{d(1+y'^2)}{(1+y'^2)}.$$

$$\Rightarrow 2\ln T = \ln T^2 = \ln(1+y'^2) + \text{const}.$$

$$\Rightarrow \boxed{T^2 = c_2^2(1+y'^2)} \quad \cdots\cdots ②.$$

c_2: constant of integration.

Aprile Statics Solution -3

Eliminate T from ① & ②:

$$(\rho g y + c_1)^2 = c_2^2(1+y'^2).$$

$$\Rightarrow 1+y'^2 = \left(\frac{\rho g}{c_2}y + \frac{c_1}{c_2}\right)^2 = \underbrace{\left(\frac{\rho g}{c_2}\right)^2}_{\alpha^2}\left(y+\underbrace{\frac{c_1}{\rho g}}_{\beta}\right)^2$$

$$\Rightarrow \boxed{1+y'^2 = \alpha^2(y+\beta)^2} \quad \cdots ③$$

We show below that $y(x)+\beta = \frac{1}{\alpha}\cosh\alpha(x+\gamma)$, where α, β, γ are constants, is a solution of ③

Now $1+\sinh^2\theta = \cosh^2\theta$

So, let $\boxed{y+\beta \equiv \frac{1}{\alpha}\cosh\alpha(x+\gamma)}$ constant $\cdots$ ④

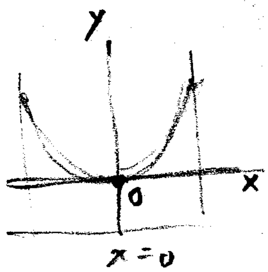
This implies $\Rightarrow y' = \sinh\alpha(x+\gamma)$

$$\Rightarrow 1+y'^2 = \cosh^2\alpha(x+\gamma) = \alpha^2(y+\beta)^2.$$

So $\boxed{y(x)+\beta = \frac{1}{\alpha}\cosh\alpha(x+\gamma) \text{ is indeed a solution of ③}}$ ~~of ③~~

Evaluating the constants:

Now if $y=0$ at $x=0$, (lowest point of chain).



$$0+\beta = \frac{1}{\alpha}\cosh\alpha\gamma.$$

But $\alpha\beta = \frac{\rho g}{c_2}\cdot\frac{c_1}{\rho g} = \frac{c_1}{c_2}$.

$$\Rightarrow \boxed{\cosh\frac{\rho g\gamma}{c_2} = \frac{c_1}{c_2}.} \Rightarrow c_1 = c_2$$

also, at $x=0$, $y'(0)=0$ (tangent is horizontal)

④ $\Rightarrow y'(x) = \sinh\alpha(x+\gamma) \Rightarrow 0 = \sinh\alpha\gamma \Rightarrow \boxed{\gamma = 0} \Rightarrow \boxed{c_1 = c_2}.$

as $\alpha \neq 0$ $\Downarrow$ $\cosh\frac{\rho g \gamma}{c_2}, \gamma=0 \Rightarrow \boxed{\alpha\beta = \frac{c_1}{c_2} = 1}$

$\Rightarrow \boxed{\beta = \frac{1}{\alpha}}$

So, $y(x) + \frac{1}{\alpha} = \frac{1}{\alpha}\cosh\alpha x$.

$$\Rightarrow \boxed{y(x) = \frac{1}{\alpha}(\cosh\alpha x - 1)} \quad \ldots ⑤$$

Shape of the chain (α is the undetermined constant).

Equation for α:

⑥ Length of the chain

$$l = \int_{-d/2}^{d/2} dl = \int_{-d/2}^{d/2} \frac{dx}{\cos\theta_1} = \int_{-d/2}^{d/2} dx \sqrt{1+y'^2}$$

$$= 2\int_0^{d/2} dx \sqrt{1+y'^2}$$

But $y' = \sinh\alpha x$

$\Rightarrow 1 + y'^2 = \cosh^2\alpha x$

$$\Rightarrow \quad l = 2\int_0^{d/2} dx \cosh\alpha x = \frac{2}{\alpha}\int_0^{d/2} dx \frac{d}{dx}\sinh\alpha x$$

$$l = \frac{2}{\alpha}\left[\sinh\frac{\alpha d}{2}\right]$$

$$\Rightarrow \boxed{\sinh\frac{\alpha d}{2} = \frac{\alpha l}{2}} \quad \ldots ⑥$$

This gives α in terms of l & d.

Hailey = gravity/central force

Hailey I, 7, 5
mechanics - Question

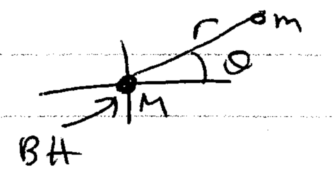
The gravitational potential near a black hole of mass M can be described by a modified classical potential $U(r) = -\frac{GM}{(r-a)}$

where M is the mass of the black hole and a its Schwarzschild radius.

a.) Find an expression for the force acting on a particle of mass m in this gravitational potential.

b.) Expand your answer from part a to find the lowest order correction to the classical gravitational force when the test particle is at a distance from the black hole large enough to satisfy the condition $a/r << 1$.

c.) Develop a solution for the orbital motion of a test particle near the black hole using the force from part b - just the classical and first order correction terms. Express your answer in the form $r = r(\theta)$ where θ is an angle between a fixed axis and the radius vector to the particle.
You need only find the solutions periodic in θ.



Sol'n: gravity Central Force

Hailey
Mechanics - I-V
Solution - p. 1

a.) $U = -\frac{GM}{r-a}$

$$f(r) = -\frac{dU}{dr} m = -\frac{GMm}{(r-a)^2}$$

b.) $f(r) = -\frac{GMm}{r^2(1-\frac{a}{r})^2} \simeq -\frac{GMm}{r^2}\left(1+\frac{2a}{r}\right)$

$$f(r) = \underbrace{-\frac{GMm}{r^2}}_{\text{classical}} - \underbrace{2\frac{GMma}{r^3}}_{\text{correction}} \quad \text{for } \frac{a}{r} \ll 1$$

c.) $m(\ddot{r} - r\dot{\theta}^2) = f(r)$

$m r^2 \dot{\theta} = l$ $\quad l =$ Angular momentum

$\dot{\theta} = \frac{l}{mr^2}$

Make the standard substitution $u = \frac{1}{r}$

$$\dot{r} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = \frac{dr}{du}\frac{du}{d\theta}\dot{\theta} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}$$

$$\dot{\theta} = \frac{l}{m}u^2 \Rightarrow \dot{r} = -\frac{l}{m}\frac{du}{d\theta}$$

$$\frac{d}{dt} = \frac{d}{d\theta}\dot{\theta} = \frac{l u^2}{m}\frac{d}{d\theta} \Rightarrow \ddot{r} = -\frac{l^2}{m^2}u^2\frac{d^2u}{d\theta^2}$$

$$f(r) = - GMmu^2 - 2GMmau^3$$

$$-\frac{l^2}{m}u^2\frac{d^2u}{d\theta^2} - \frac{l^2}{m}u^3 = -GMmu^2 - 2GMma$$

$$d^2u/d\theta^2 + \left(1 - \frac{GMm^2a}{l^2}\right)u = \frac{GMm^2}{l^2}$$

Let $\alpha = \sqrt{1 - \frac{GMm^2}{\ell^2}}$

$$d^2u/d\theta^2 + \alpha^2 u = \frac{GMm^2}{\ell^2}$$

For $\frac{GMm^2}{\ell^2} < 1 \quad \alpha^2 > 0$

$$u = A\cos\alpha\theta + \frac{GMm^2}{\ell^2}$$

$$r = \frac{1}{A\cos\alpha\theta + \frac{GMm^2}{\ell^2}} = \frac{\ell^2/GMm^2}{1 + \varepsilon\cos\alpha\theta}$$

This is the form for $r = r(\phi)$, although in point of fact the orbits are not in the plane, but helical.

This problem can be solved more elegantly using the revolving orbits approach. The effective potential of the original problem can be related to that of an equivalent problem with ~~shifted~~ shifted angular momentum etc.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 10, 2005
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

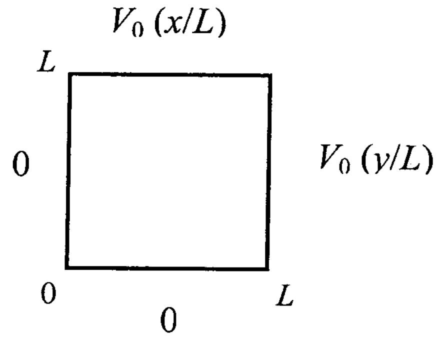
Section 2 – Question 1

A two dimensional box, $0 \le x \le L$, $0 \le y \le L$, is bounded by four conducting plates. The potential boundary conditions are:

$$\phi(0,y) = \phi(x,0) = 0$$

$$\phi(L,y) = V_0 \cdot \frac{y}{L}$$

$$\phi(x,L) = V_0 \cdot \frac{x}{L}$$



a) Determine $\phi(x,y)$ everywhere in the box.

b) Draw qualitatively how equipotential and electric flux lines look like inside the box.

Section 2 – Question 2

A hollow sphere of radius R rotates about one of its diameters with an angular velocity ω. Its surface is electrically charged with a charge density σ.

a) Determine the magnetic moment, $\vec{m}$, of the sphere.

b) Find the magnetic field strength inside and outside the sphere as a function of the distance, r, from its center.

Section 2 – Question 3

The speed of electrons undergoing cyclotron motion can be increased by increasing the magnetic field causing them to move in a circle with time. This is the principle behind a *betatron*. The electrons can be kept in an orbit of constant radius, R, in this process if the magnetic field over the area of the electron's orbit is non-uniform.

Find the relationship between the field at the circumference of the orbit and the average field over the orbit's area that would be required to keep the electron at a constant radius as it is accelerated.

Assume that the electrons start from rest in zero field, and that the apparatus is symmetric about the center of the orbit.

Section 2 – Question 4

Consider a large region of space where the electric field, $\mathbf{E}(x,y,z) = [E_x, E_y, E_z] = [|E|, 0, 0]$, is constant and homogeneous. The effect of gravity can be neglected.

A point particle with mass, m, and positive charge, e, is *accelerated* by the electric field. It moves through the origin at $t_o = 0$. Its velocity vector, $\mathbf{v}$, is known at this point: $\mathbf{v} = [v_o \cos\alpha, 0, v_o \sin\alpha]$, with $v_o << c$ and $\cos\alpha > 0$.

At some later time, t_f, the distance, along the x-axis, between the particle and the origin is L and the particle's speed is still non-relativistic: $v_f << c$.

a) Estimate the total energy emitted, W, in the form of *dipole radiation*, between t_o and t_f, as a function of the variables given above.

b) Describe and interpret the result.

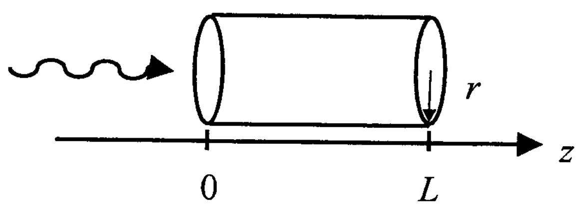
Hint: one can approximate the power radiated by a dipole moment, p, *in non-relativistic situations, by the Larmor formula:*

$$P = \frac{\mu_o \ddot{p}}{6\pi c}$$

Section 2 – Question 5

A plane electromagnetic wave in vacuum is propagating in the positive z-direction. The wave has a frequency ω and its amplitude is being slowly decreased in time. In particular, at $z = 0$ the amplitude is proportional to $(1 - at)$ for time $t = 0$ to $t = 1/a$, with $a/\omega << 1$.

Consider an imaginary cylinder as shown below.



Find the net, average outward energy flow per unit time from the cylinder and show that it equals the rate at which the enclosed energy decreases with time.

2005 Section 2 E+M II - 1

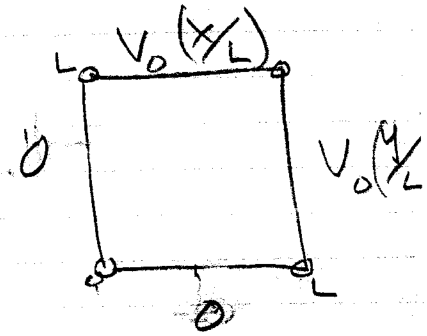
Quals EM Gyulassy

(1) A two dimensional box $0 \leq x \leq L$, $0 \leq y \leq$ is bounded by 4 conducting plates. The potential boundary conditions are

$$\phi(0, y) = \phi(x, 0) = 0$$

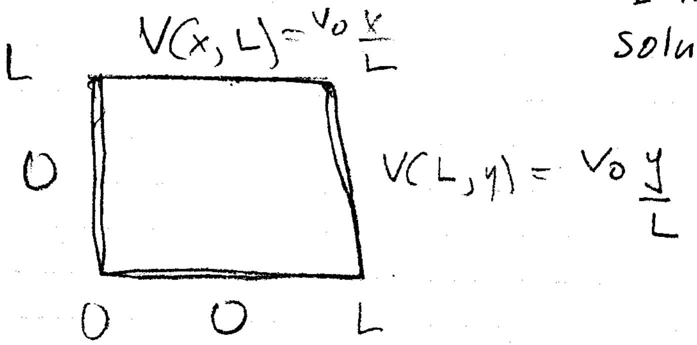
$$\phi(L, y) = V_0 \cdot \frac{y}{L}$$

$$\phi(x, L) = V_0 \cdot \frac{x}{L}$$



a) Determine $\phi(x, y)$ everwhere in the box.

b) Draw qualitatively how equipotential and electric flux lines look like inside the Box.



$\Delta \phi(x\,y) = 0$ $\qquad (\partial_x^2 + \partial_y^2)\,\psi_m(x)\,\phi_m(y) = 0$

take $\partial_y^2 \phi_m(y) = -k_m^2 \phi_m$

to satisfy $\phi(x, 0) = 0$ $\qquad \phi_m = \sin k_m x$

$\Rightarrow (\partial_x^2 - k_m^2)\,\psi_m(x) = 0$

$\psi_m(x) = \mathrm{sh}\, k_m x$ $\qquad$ so that $\psi_m(0) = 0$

$$\phi(x,y) = \sum_m A_m \,\mathrm{sh}(k_m x)\, \sin k_m y$$

if we take $k_m = \frac{m\pi}{L}$ then boundary at $y = L$ vanish

$\underline{\phi(x, L) = 0}$!

but if we take

$k_m = i\,\frac{m\pi}{m}$ $\qquad \underline{(\mathrm{sh}\, ix = i \sin x)}$

then $\phi(L, y) = 0$

thus $\phi = \phi^1 + \phi^2$

where $\phi_1 = \sum_{m=1}^{\infty} A_m \, sh \frac{m\pi x}{L} \sin \frac{m\pi y}{L}$

$\phi_2 = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{L} \, sh \frac{m\pi y}{L}$

$\phi_1(L, y) \neq 0$ but $\phi_2(L, y) = 0$

$\phi_1(x, L) = 0$ $\phi_2(x, L) \neq 0$

Fix A_m via $\int_0^L \sin \frac{m\pi y}{L} \sin \frac{m'\pi y}{L} dy$

$= \frac{L}{2} \delta_{mm'}$

using $\langle \sin^2 \theta \rangle = \frac{1}{2} \Rightarrow \int \left(\sin^2 \frac{m\pi y}{L}\right) dy = L/2$

$sh(m\pi) A_m = \frac{2}{L} \int_0^L dy \left(\sin \frac{m\pi y}{L} \; V_0 \left(\frac{y}{L}\right) \right)$

$= \frac{2V_0}{(m\pi)^2} \int_0^{m\pi} dx (\sin x) \, x$

$f = (-x(\cos x) + \cancel{\sin x})$

$f' = x \sin - \cos + \cos x = x \sin$

$= \frac{2V_0}{(m\pi)^2} (-(m\pi)) \cos m\pi = (-1)^{m+1} \frac{2V_0}{m\pi}$

DEC 9 2004

Gyulassy E+M Solution

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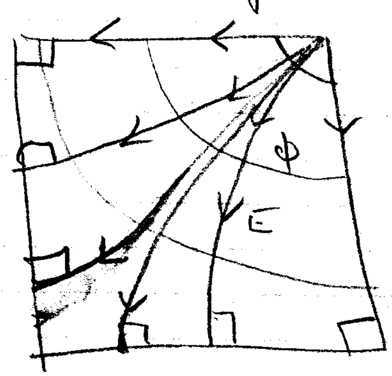
II 1.

$$\boxed{A_m = (-1)^{m+1} \frac{2V_0}{m\pi} \frac{1}{sh(m\pi)}}$$

obviously $B_m = A_m$ by symmetry

$$\boxed{\phi = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{sh(m\pi)} \left(\frac{2V_0}{m\pi}\right) \left\{ sh\frac{m\pi x}{L} \sin\frac{m\pi y}{L} + sh\frac{m\pi y}{L} \sin\frac{m\pi x}{L} \right\}}$$

Draw the equipotential and electric lines qualitatively



Magnetostatics

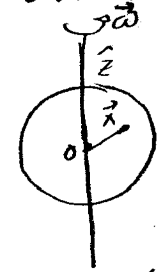
① A sphere of radius R rotates ~~rotates~~ about one of its diameters with an angular velocity ω. Its surface is electrically charged with a density σ.

ⓐ Determine the magnetic moment $\vec{m}$ of the sphere.

ⓑ Find the magnetic field strength inside and outside the sphere.

Solution :

ⓐ The magnetic dipole moment due to a current distribution is given by.



$$m_i = \frac{1}{2c}\int d\vec{x}'\left[\vec{x}'\times \vec{j}(\vec{x}')\right]_i$$

we have to find the $\vec{j}$ due to the rotating sphere.

We can choose the axis of rotation to be the $\hat{z}$ axis.

Since the charge is confined to the surface,

$$\rho(\vec{x}) = \sigma\,\delta(r-R).$$
where $r = |\vec{x}|$.

$$\left\{\left[\begin{aligned} &\text{Check:}\\ &q = \int dv\, \rho(\vec{x})\\ &\quad = \int_0^\infty dr\, r^2 \int d\Omega\; \sigma\,\delta(r-R)\\ &\quad = 4\pi R^2\sigma \end{aligned}\right]\right.$$

The current is

$$\vec{j}(\vec{x}) = \rho(\vec{x})\,\vec{V}$$

$$\&\ \vec{V} = \vec{\omega}\times\vec{x} = \omega\, \hat{z}\times\vec{x} = \omega\, r\sin\theta\left[\cos\phi\,\hat{y} - \sin\phi\,\hat{x}\right]$$
$$= \omega\left[x\hat{y} - y\hat{x}\right]$$

$$\Rightarrow \vec{x}\times\vec{j}(\vec{x}) = \rho(\vec{x})\,\vec{x}\times\vec{V} = \rho(\vec{x})\,\omega r\sin\theta\left[\cos\phi\,\vec{x}\times\hat{y} - \sin\phi\,\vec{x}\times\hat{x}\right]$$
$$= \rho(\vec{x})\,\omega r\sin\theta\left[\cos\phi\left\{x\hat{z} - z\hat{x}\right\} - \sin\phi\left\{-y\hat{z} + z\hat{y}\right\}\right]$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$= \rho(\vec{x})\,\omega\sin\theta\, r\left[\cancel{(-r\cos\theta\cos\phi \mp r\sin\theta\sin^2\phi)\hat{z}}\ r(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)\hat{z} - r\cos\theta\cos\phi\,\hat{x} - r\cos\theta\right.$$

$$= \omega\rho(\vec{x})\, r\sin\theta\left[r\sin\theta\,\hat{z} - r\cos\theta\cos\phi\,\hat{x} - r\cos\theta\sin\phi\,\hat{y}\right]$$

Now $\int_0^{2\pi} d\phi \cos\phi = \int_0^{2\pi} d\phi \sin\phi = 0$.

$$\Rightarrow \int d\vec{x} \left[\vec{x} \times \vec{j}(\vec{x})\right] = \omega \hat{z} \int d\vec{x}\, \rho(\vec{x})\, r^2 \sin^2\theta$$

$$= \omega \sigma \hat{z} \int_0^\infty dr\, r^2 r^2 \int_0^\pi d\theta \sin\theta \sin^2\theta\, \delta(r-R) \int_0^{2\pi} d\phi$$

$$\int_0^\pi d\theta \sin\theta \sin^2\theta = \int_{-1}^{1} d\eta\, (1-\eta^2) \quad \text{where } \eta \equiv \cos\theta$$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

$$\Rightarrow \int d\vec{x} \left[\vec{x} \times \vec{j}(\vec{x})\right] = \omega \sigma \hat{z} R^4 \cdot \frac{4}{3} \cdot 2\pi$$

$$= \frac{8\pi}{3} \omega \sigma R^4 \hat{z}$$

$$\Rightarrow \vec{m} = \frac{4\pi \omega \sigma}{3c} R^4 \hat{z} = \frac{\omega \sigma}{c} V R \hat{z} = \frac{\omega Q}{3c} R^2 \hat{z} = \frac{QR^2}{3c} \vec{\omega}.$$

where V = volume of the sphere.

$$\boxed{\vec{m} = \frac{QR^2}{3c} \vec{\omega}}$$

$$= \frac{4\pi R^4 \sigma}{3c} \vec{\omega}$$

$$\boxed{\vec{m} = \frac{4\pi}{3c} R^4 \sigma\, \vec{\omega}}$$

(b) Since there are no currents either inside or outside the sphere, we can describe the magnetic field inside/outside the sphere in terms of magnetic scalar potentials.

If $\vec{B}_1$ is the field inside the sphere & $\vec{B}_2$ that outside,

$$\left.\begin{aligned} \vec{B}_1 &= -\vec{\nabla}\psi_1 \\ \vec{B}_2 &= -\vec{\nabla}\psi_2 \end{aligned}\right\}.$$

$\vec{B}_1$ & $\vec{B}_2$ satisfy boundary conditions at $r=R$:

$$\left.\begin{aligned} B_{1n} &= B_{2n} \\ \& \quad \hat{n}\times(\vec{B}_2-\vec{B}_1) &= \vec{j}/c \end{aligned}\right\}$$

Now $\vec{j} = \sigma\omega R \sin\theta\,\hat{\phi}$ (from part ⓐ) [Recall: $\hat{\phi} = -\sin\theta\sin\phi\,\hat{x} + \sin\theta\cos\phi\,\hat{y}$]

So $\frac{\sigma\omega}{c}R\sin\theta = \left[\hat{n}\times(\vec{\nabla}\psi_1-\vec{\nabla}\psi_2)\Big|_{r=R}\right]_\phi$

But $\hat{n} = \hat{r}$

$$\Rightarrow \boxed{\frac{\sigma\omega}{c}R\sin\theta = \frac{1}{R}\left(\frac{\partial\psi_1}{\partial\theta} - \frac{\partial\psi_2}{\partial\theta}\right)\Bigg|_R} \quad \text{--- ⓐ}$$

$$\& \quad \boxed{\frac{\partial\psi_1}{\partial r}\Big|_R = \frac{\partial\psi_2}{\partial r}\Big|_R} \quad \text{--- ⓑ}$$

φ_1 & φ_2 satisfy $\nabla^2 \varphi_{1,2} = 0$ (as $\vec{\nabla}\cdot\vec{B} = 0$)

Given the obvious azimuthal symmetry, we can write

$$\varphi_1(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{(Has to be regular at } r = \text{)}$$

$$\& \quad \varphi_2(r,\theta) = \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta) \quad \text{(Has to die out at } r \to \text{)}$$

Now $P_1(\cos\theta) = \cos\theta$ & $\frac{dP_1}{d\theta} = -\sin\theta$.

From (a) $\Rightarrow$ $\underline{A_l, B_l \text{ for } l \neq 1 \text{ are zero.}}$

$$\Rightarrow \frac{\sigma\omega R}{c}\sin\theta = \frac{1}{R}\left[A_1 R(-\sin\theta) - B_1 \frac{1}{R^2}(-\sin\theta)\right]$$

$$\Rightarrow \boxed{A_1 - \frac{1}{R^3}B_1 = -\frac{\sigma\omega R}{c}} \quad \text{.. (c)}$$

from (B), $A_1 \cos\theta + \frac{2B_1}{R^3}\cos\theta = 0$

$$\Rightarrow \boxed{B_1 = -\frac{R^3}{2}A_1} \quad \text{.. (d)}$$

(c),(d) $\Rightarrow$ $\frac{3A_1}{2} = -\frac{\sigma\omega}{c}R$

$$\Rightarrow \boxed{A_1 = -\frac{2}{3c}\sigma\omega R} \quad \text{... (e)}$$

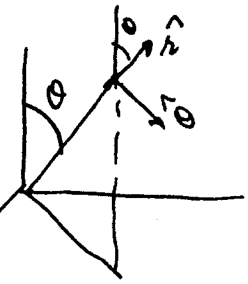
$$\Rightarrow \boxed{B_1 = \frac{\sigma\omega}{3c}R^4} \quad \text{.. (f)}$$

So,
$$\left.\begin{aligned} \varphi_1(r,\theta) &= -\frac{2}{3c}\sigma\omega R\, r\cos\theta \\ \varphi_2(r,\theta) &= \frac{\sigma\omega}{3c}R^4\frac{1}{r^2}\cos\theta \end{aligned}\right\} \quad \text{.. (g)}$$

April Magnetostatics Solution ⑤ II-2

So, $B_r = -\frac{\partial \psi_1}{\partial r} = \frac{2}{3c}\sigma\omega R\cos\theta$

& $B_{1\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\psi_1 = -\frac{2}{3c}\sigma\omega R\sin\theta$.



$\cancel{\vec B_1 = \hat r B_{1r} + \hat\theta B_{1\theta} = (\hat x \sin\theta\cos\phi +}$

$\Rightarrow \vec B_1 = \frac{2}{3c}\sigma\omega R\,(\underbrace{\hat r\cos\theta - \hat\theta\sin\theta}_{=\hat z})$

$$\Rightarrow \boxed{\vec B_1 = \frac{2}{3c}\sigma\omega R\,\hat z = \frac{2\sigma}{3c}R\,\vec\omega = \frac{1}{2\pi R^3}\vec m}$$

$B_{2r} = -\frac{\partial\psi_2}{\partial r} = \frac{2\sigma\omega R^4}{3}\frac{1}{r^3}\cos\theta$.

$B_{2\theta} = -\frac{1}{r}\frac{\partial\psi_2}{\partial\theta} = \frac{\sigma\omega}{3}R^4\frac{1}{r^3}\sin\theta$.

Now $\vec m\cdot\vec r = \frac{4\pi\sigma R^4\omega}{3c}\,\hat z\cdot\vec r = \frac{4\pi\sigma}{3c}R^4\omega r\cos\theta$

Let $\vec\Sigma = \frac{3(\vec m\cdot\vec r)\vec r}{4\pi r^5} - \frac{\vec m}{4\pi r^3} = \frac{\sigma\omega R^4\cos\theta}{c r^3}\hat r - \frac{\sigma\omega R^4}{3c r^3}\hat z$

Then $\Sigma_r = \frac{\sigma\omega R^4}{c r^3}\cos\theta - \frac{\sigma\omega R^4}{3c r^3}\cos\theta = \frac{2}{3}\frac{\sigma\omega R^4}{c r^3}\cos\theta$

[Note $\hat z\cdot\hat r = \cos\theta$] $= \cancel{c}\, B_{2r}$.

& $\Sigma_\theta = -\frac{\sigma\omega R^4}{3c r^3}\underbrace{\hat z\cdot\hat\theta}_{=-\sin\theta} = \frac{\sigma\omega R^4}{3c r^3}\sin\theta = \cancel{c}\, B_{2\theta}$

$$\Rightarrow \boxed{\vec B_2 = c\,\vec\Sigma = \frac{3c(\vec m\cdot\vec r)\vec r}{4\pi r^5} - \frac{c\,\vec m}{4\pi r^3}} .$$

precis the field of a magnetic d dip

Szabi marka
E+M (1)

II - 4

Radiation related problem

Consider a sufficiently large region in vacuum around the origin, where the electric field ($\vec{\mathbf{E}}(x, y, z) = [E_x, E_y, E_z] = [|E|, 0, 0]$) is constant and homogenous (i.e. far beyond the volume of interest). The effect of gravity can be neglected.

A point particle with mass (m) and positive charge (e) is *accelerated* by the electric field. It moves through the origin at $t_\circ$=0. Its velocity vector $\vec{\mathbf{v}}$ is known at this point: $|\vec{\mathbf{v}}| = v_\circ \ll c$ and $v_y = 0$, where c is the speed of the light. The angle between $\widehat{v}$ and $\widehat{x}$ is α; $cos(\alpha) > 0$.

The distance *along the x-axis* between the origin and the point particle will be L at some later time (t_f).

Estimate the total energy emitted (W) in the form of *dipole radiation* between $t_\circ$ and t_f as a function of $\left(\vec{\mathbf{E}},\ v_\circ,\ \alpha,\ L,\ m \right)$. α and L is chosen that the charge never leaves the region of interest where the field is constant and homogenous and that the speed of the particle at t_f is still non-relativistic ($v_f \ll c$). Describe and interpret the result.

(*Hint: Remember, one can approximate the (dipole) radiated power with the Larmor formula (e.g. $P \cong \frac{\mu_\circ \ddot{p}}{6\pi c}$, where p is the dipole moment) for non-relativistic situations.*)

Summary of solution:

One way to estimate this is:

The equation of motion is $m\ddot{\vec{\mathbf{r}}} = e\vec{\mathbf{E}}$. Since $\ddot{\vec{\mathbf{p}}} = e\vec{\mathbf{a}}$ in this case, the radiated power can be approximated as $P \cong \frac{\mu_\circ}{6\pi c}\frac{e^2E^2}{m^2}$ according to the Larmor formula.

The x-projection of the position of the particle at time t_f is $L = \frac{1}{2}\left(\frac{eE}{m}\right)t_f^2 + v_\circ t_f$.

Therefore the total flight time is $t_f = \frac{-mv_\circ cos(\alpha) + \sqrt{m^2v_\circ^2cos^2(\alpha) + 2eEmL}}{eE}$.

Consequently the total energy emitted in the form of dipole radiation between $t_\circ$ and t_f can be estimated as:

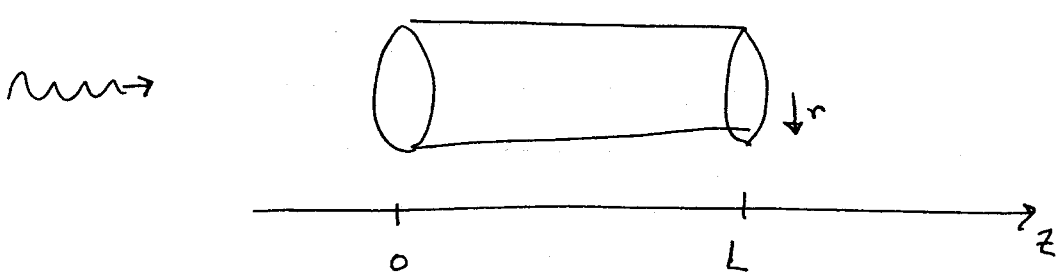
$W \cong \frac{\mu_\circ}{6\pi c}\frac{eEv_\circ}{m}\left(\sqrt{cos^2(\alpha) + \frac{2eEL}{mv_\circ^2}} - cos(\alpha)\right)$

E+M question. December 2004. Robert Mawhinney II-5

DEC 2 [illegible]

A plane electromagnetic wave in vacuum is propagating in the positive z direction. The wave has frequency ω and its amplitude is being slowly decreased in time. In particular, at $z=0$ the amplitude is proportional to $(1 - at)$ for time 0 to time $1/a$ and $a/\omega \ll 1$.

Consider an imaginary cylinder as shown



Find the net average outward energy flow per unit time from the cylinder and show that it equals the rate at which the enclosed energy decreases with time.

Solution - page 1

We need the value of $\vec{E}\times\vec{B}$, the Poynting vector, integrated over the surface.

During the time the amplitude is varying, the electric field at $z=0$ is

$$\vec{E} = \vec{E}_0 (1-at)\, e^{i\omega t}$$

The field at a point z at time t is the same as the field at point 0 at time $t - z/c$, which gives

$$\vec{E} = \vec{E}_0 \left(1 - a\left(t - z/c\right)\right) e^{i\omega\left(t - z/c\right)}$$

$\vec{E}\times\vec{B}$ points in the direction of propagation, so we only need the surface integral over the ends of the cylinder. $B = cE\epsilon_0$ and since a/ω is small averages over an oscillation period yield $1/2$ we have,

$$\int (\vec{E}\times\vec{B})\cdot dA = \frac{1}{2}\epsilon_0 c E_0^2 (\pi r^2)\left[\left(1 - at + aL/c\right)^2 - (1-at)^2\right]$$

contribution from right end   contribution from left end

$$= \frac{1}{2}\epsilon_0 c E_0^2 (\pi r^2)\left[2(1-at)\frac{aL}{c} + \left(\frac{aL}{c}\right)^2\right]$$

Solution - page 2

The energy density is $\frac{\epsilon_0 E^2 + B^2/\mu_0}{2} = \epsilon_0 E^2$ for a plane wave. Averaging over a period gives a factor of ½, yielding an average energy of

$$W = \int_0^L \frac{\epsilon_0 E_0^2}{2}\left(1 - at + \frac{az}{c}\right)^2 dz\,(\pi r^2)$$

$$= \frac{\epsilon_0 E_0^2 \pi r^2}{2} \int_0^L dz\left[(1-at)^2 + 2(1-at)\frac{a}{c}z + \frac{a^2 z^2}{c}\right]$$

$$= \frac{\epsilon_0 E_0^2 \pi r^2}{2}\left[(1-at)^2 L + (1-at)\frac{a}{c}L^2 + \frac{a^2 L^3}{3c}\right]$$

$$\frac{dW}{dt} = \frac{\epsilon_0 E_0^2 \pi r^2}{2}\left[2(1-at)(-aL) - \frac{a^2 L^2}{c}\right] = -\int (\vec{E}\times\vec{B})\cdot d\vec{A}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 12, 2005
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

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Questions should be directed to the proctor.

Good luck!!

Section 3 – Question 1

a) Consider a 1-dimensional delta function potential well, $V(x) = -aV_0\,\delta(x)$, where a and V_0 are positive constants in this problem. A point particle of mass m is bounded in this potential. Show that there is only one bound state in this potential, and find the binding energy and the wavefunction of this bound state.

b) Now, consider two symmetric delta function potential wells, $V(x) = -aV_0[\delta(x+a) + \delta(x-a)]$. Employing only a symmetric argument without solving the Schroedinger equations for this potential, guess the ground state wavefunction and first excited state wavefunction from the wavefunction obtained in a). It is not required to normalize wavefunctions in this problem.

c) Let $\lambda \equiv \frac{2mV_0}{\hbar^2}a^2$. Assume $\lambda \ll 1$, find the energy of ground state in b) up to the correction term to the answer you obtained in a).

Section 3 – Question 2

Two electrons are bound by a spherically symmetric potential, are in the same radial state, and each have total angular momentum $\ell = 1$. Spin-orbit coupling may be neglected.

For parts a) and b), assume that the two electrons are in the spin singlet state.

a) If the total orbital angular momentum, $L_{tot}^2 = (L_1 + L_2)^2$, is measured, what values could be obtained?

b) Give the angular wave functions and degeneracies for all states found in a). You may express the answers in terms of angular harmonics, Y_{lm}, which you do not need to write explicitly.

For parts c) and d) assume that the electrons are in a spin triplet state.

c) Repeat part a).

d) Repeat part b).

Section 3 – Question 3

Muonic atoms are formed when a muon stops in a material and gets "captured" into an atomic state. These muons can then be absorbed by the nucleus through a process that is essentially inverse beta decay (but with a muon being absorbed instead of an electron). We will suppose that a muon is captured in a homogenous material composed of an element with atomic number Z and atomic mass A ($A >>$ 1). Neglect the finite size of the nucleus. You may need the muon mass (rest energy), $m_\mu c^2 \sim 106$ MeV, electron mass, $m_e c^2 \sim 0.511$ MeV, and nucleon mass $m_N c^2 \sim 940$ MeV. You may also find it convenient to use $\hbar c \sim 200$ MeV fm.

a) We can estimate the most likely principle quantum number of the orbital into which the muon gets captured by assuming that the muon ejects the most energetic electron from the atom and occupies a state of comparable energy. Estimate n for the muon capture state using this assumption.

b) The muon will continue to de-excite by ejecting electrons from the atom (thus producing so-called "Auger" electrons) and emitting x-rays. Estimate the energy of the most energetic x-ray that can be emitted by the muon. If the radius of a nucleus with mass number A is, $R \sim (1.2\text{ fm})A^{1/3}$, comment on the validity of neglecting the nuclear size.

c) The absorption of the muon by the nucleus proceeds almost exclusively from S states. Explain why this is so.

d) If we assume that the matrix element and phase space factors are approximately the same for the capture of muons and electrons on nucleons (a crude approximation), estimate the ratio of the probabilities for the muon and electron to be absorbed from an n=1 atomic state, *i.e.* calculate $P(\mu\text{-capt}) / P(\text{e-capt})$.

Section 3 – Question 4

Consider a Hamiltonian, $H = H_0 + V(t)$, where (for V_0 a constant operator)

$V(t) = V_0$ for $0 < t < T$, and

$V(t) = 0$ otherwise.

We label the eigenstates and eigenvalues of H_0 as $|m\rangle$ and E_m, respectively. That is:

$$H_0|m\rangle = E_m|m\rangle.$$

Suppose $|\psi(t)\rangle$ is the state of the system. If $|\psi(t)\rangle = |m\rangle$ for $t < 0$, what is $|\langle n|\psi(t)\rangle|^2$ for $t > T$ when $n \neq m$?

You may work to lowest non-trivial order in V.

Section 3 – Question 5

Consider the Stark effect of an electric field applied to a hydrogen atom in the ground state (1S). In the following, define the x-axis to be the direction of the electric field.

a) First calculate this effect classically. Suppose the atom is composed of a positive charge, e, at the origin ($r = 0$) and a negative charge, $-e$, uniformly distributed throughout a sphere of radius a. Application of the electric field, E, will create a dipole moment by shifting the position of the center of the negatively charged sphere by a distance x from the positive charge. Calculate the magnitude of the electric dipole moment, $\mu = e\,x$, and the polarizability, $\alpha = \mu/E$, of the atom under these conditions.

b) Now calculate the dipole moment and polarizability quantum mechanically, using the true electron wavefunction. Include the effect from the n=2 levels (2s and 2p) only, and forget about effects from n = 3 and higher levels.

Hint: Obtain the energy change using perturbation theory, then equate this energy with the work that the field, E, *does in moving the electron cloud from* r = *0 to* r = x.
You may use the mathematical results given below.

c) Describe how this calculation can be checked experimentally. How can we measure the energy change calculated in part b)?

$$\varphi_{1S} = \frac{1}{\sqrt{4\pi}}\frac{2}{a_0^{3/2}}e^{-r/a_0} \qquad \varphi_{2S} \propto (1 - r/2a_0)e^{-r/2a_0}$$

$$\varphi_{2P_0} \propto re^{-r/2a_0}\cos\theta \qquad \varphi_{2P_\pm} \propto re^{-r/2a_0}\sin\theta\, e^{\pm i\phi}$$

$$x = r\sin\theta\cos\phi \qquad dxdydz = r^2\sin\theta\, drd\theta d\phi$$

$$\iiint x\varphi_{1S}^*\varphi_{2P_0}\,dxdydz = 0 \qquad \iiint x\varphi_{1S}^*\varphi_{2P_\pm}\,dxdydz = \frac{2^7}{3^5}a_0$$

$$E_{1S} = -\frac{e^2}{2a_0} \qquad E_{2S} = E_{2P} = -\frac{e^2}{8a_0} \quad \text{(energies of unperturbed H - atom)}$$

2005 Qualifying Exams

Philip Kim

Quantum Mechanics (potential problem):

(a) Consider a 1-dimensional delta function potential well, $V(x) = -aV_0\,\delta(x)$, where a and V_0 are positive constants in this problem. A point particle of mass m is bounded in this potential. Show that there is only one bound state in this potential, and find the binding energy and the wavefunction of this bound state.

(b) Now, consider two symmetric delta function potential wells, $V(x) = -aV_0\,[\delta(x+a) + \delta(x-a)]$. Employing only a symmetric argument without solving the Schroedinger equations for this potential, guess the ground state wavefunction and first excited state wavefunction from the wavefunction obtained in (a). It is not required to normalize wavefunctions in this problem.

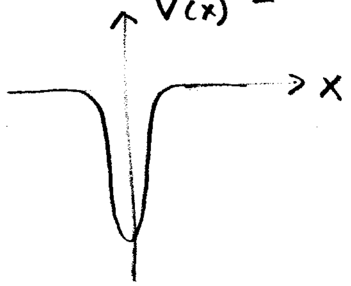
(c) Let $\lambda \equiv \frac{2mV_0}{\hbar^2}a^2$. Assume $\lambda << 1$, find the energy of ground state in (b) up to the correction term to the answer you obtained in (a).

Philip Kim

Q.M.

(a)

$V(x) = -aV_0\,\delta(x)$



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - aV_0\delta(x)\psi(x) = E\psi(x)$$

Set $\psi(x) = e^{-K|x|}$

For $x \neq 0$, $-\frac{\hbar^2}{2m}K^2 = E$

Integrate eq 1, around $x=0$,

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\Big|_{x=0^+} - \frac{d\psi}{dx}\Big|_{x=0^-}\right] - aV_0 = 0$$

or

$$\frac{\hbar^2}{2m}2K = aV_0 \Rightarrow \boxed{K = \frac{ma}{\hbar^2}V_0}$$

unique solution

Therefore, there is only one bound state with the wavefunction

$$\psi_0(x) = \exp\left[-\frac{ma}{\hbar^2}V_0|x|\right]$$

Binding energy : $\frac{\hbar^2}{2m}K^2 = \frac{ma^2}{2\hbar^2}V_0^2$

(b) Now, $V(x) = -aV_0[\delta(x-a) + \delta(x+a)]$

Owing to the symmetry $V(x) = V(-x)$, we can guess

$$\psi_{GND} = \psi_0(x-a) + \psi_0(x+a)$$

$$\psi_{\substack{1st\\ excited}} = \psi_0(x-a) - \psi_0(x+a).$$

where $\psi_0(x) = e^{-\kappa|x|}$

Specifically, up to a common constant free factor,

$$\psi_{GND}(x) = \begin{cases} e^{+\kappa x}, & x<-a \\ \dfrac{e^{-\kappa a}}{\cosh \kappa a} \cosh \kappa x, & |x| \le a \\ e^{-\kappa x}, & a<x \end{cases}$$

$$\psi_{\substack{1st\\ excited}}(x) = \begin{cases} e^{+\kappa x}, & x<-a \\ \dfrac{e^{-\kappa a}}{\sinh \kappa a} \sinh \kappa x, & |x| \le a \\ e^{-\kappa x}, & a<x \end{cases}$$

(C). From the continuity condition at $x=a$,

$$-\frac{\hbar^2}{2m}\left[\left.\frac{d\psi_{GND}}{dx}\right|_{x=a^+} - \left.\frac{d\psi_{GND}}{dx}\right|_{x=a^-} \right] - aV_0 e^{-\kappa a} = 0.$$

$$-\frac{\hbar^2}{2m}\left(-\kappa \cancel{e^{-\kappa a}} - \frac{\kappa \cancel{e^{-\kappa a}}}{\cosh \kappa a} \sinh \kappa a\right) - aV_0 \cancel{e^{-\kappa a}} = 0.$$

or

$$\boxed{\kappa a (1+\tanh \kappa a) = \frac{2m}{\hbar^2} a^2 V_0 \equiv \lambda}$$

since $\lambda \ll 1$, thus $\kappa a \ll 1$. and

$$\lambda \simeq \kappa a\,(1+\kappa a)$$

In order to find the λ^2 term in κa,

let $\kappa a = \lambda + \alpha\lambda^2 \cdots$

Then the above equation becomes

$$\lambda \simeq \lambda(1+\alpha\lambda)(1+\lambda+\alpha\lambda^2)$$

or

$$1 \simeq 1 + (1+\alpha)\lambda + O(\lambda^2).$$

$\Rightarrow$ $\alpha = -1$. or $\boxed{\kappa a \simeq \lambda - \lambda^2}$

Therefore ground state energy

$$E_{GND} = -\frac{\hbar^2}{2m}\kappa^2 \simeq -\frac{\hbar^2}{2m}\frac{1}{a^2}(\lambda-\lambda^2)^2$$

$$\simeq -\frac{\hbar^2\lambda^2}{2ma^2}\left[1-2\lambda\right]$$

Quantum: Millis

- Two electrons are bound by a spherically symmetric potential, are in the same radial state, and each have total angular momentum $\ell = 1$. Spin-orbit coupling may be neglected.

For parts a+b,

- Assume the two electrons are in the spin singlet state.

(a) if the total orbital angular momentum $L_{TOT}^2 = (L_1 + L_2)^2$ ~~is~~ is measured, what values could be obtained.

(b) give the angular wave functions and degeneracies for all states found in (a). You may express the answers in terms of ~~$Y_{\ell m}$~~ angular harmonics $Y_{\ell m}$ which you do not need to write explicitly)

- For parts (c) and (d), assume the two electrons are in a spin triplet state, and repeat parts (a) and (b).

Solution Quantum Millis

For two particles w/ $L=1$, $L_{TOT} = 0, 1, 2$

(a) Spin singlet means symmetric orbital state $\Rightarrow$ L even $\Rightarrow$ $L = 0, 2$.

(b) States: 4 can be written by inspection

$L_{TOT} = 2$ $L_{z,TOT} = 2$: $Y_{1,1}(\theta_1, \varphi_1) Y_{1,1}(\theta_2, \varphi_2)$

$$L_{z\,TOT} = 1 \quad \frac{1}{\sqrt{2}}\left[Y_{1,1}(\theta_1,\varphi_1) Y_{1,0}(\theta_2,\varphi_2) + Y_{1,0}(\theta_1,\varphi_1) Y_{1,1}(\theta_2,\varphi_2)\right]$$

$$L_{z\,TOT} = -1 = \frac{1}{\sqrt{2}}\left[Y_{1,-1}(\theta_1,\varphi_1) Y_{1,0}(\theta_2,\varphi_2) + Y_{1,0}(\theta_1,\varphi_1) Y_{1,-1}(\theta_2,\varphi_2)\right]$$

$$L_{z\,TOT} = -2 = Y_{1,-1}(\theta_1,\varphi_1) Y_{1,-1}(\theta_2,\varphi_2)$$

States of $L_{z,TOT} = 0$:

$$\underbrace{\frac{1}{\sqrt{2}}\left[Y_{1,1}(\theta_1,\varphi_1) Y_{1,-1}(\theta_2,\varphi_2) + Y_{1,-1}(\theta_1,\varphi_1) Y_{1,1}(\theta_2,\varphi_2)\right]}_{|a\rangle} \qquad \underbrace{Y_{1,0}(\theta_1,\varphi_1) Y_{1,0}(\theta_2,\varphi_2)}_{|b\rangle}$$

Raising op. $L=1$: $L_1^+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow L^+_{TOT} \left[| L_{zTOT} = -1 \rangle \right] =$$

$$\left[Y_{1,-1}(\theta_1, \varphi_1) Y_{1,1}(\theta_2, \varphi_2) + Y_{1,1}(\theta_1, \varphi_1) Y_{1,-1}(\theta_2, \varphi_2) \right.$$
$$\left. + 2 Y_{1,0}(\theta_1, \varphi_1) Y_{1,0}(\theta_2, \varphi_2) \right]$$

$\Rightarrow$ Normalize:

State of $L_{TOT} = 2$, $L_{zTOT} = 0$ is

$$\cancel{\frac{1}{\sqrt{6}} \left[Y_{1,1}(\theta_1, \right.} \quad \frac{1}{\sqrt{6}} \left[|a\rangle + 2|b\rangle \right]$$

$\Rightarrow$ State of $L_{TOT} = 0$, is

$$\frac{1}{\sqrt{6}} \left[-2|a\rangle + |b\rangle \right]$$

(c) Triplet $\Rightarrow$ antisym. under exchange $\Rightarrow$ $\underset{TO}{L=1}$

(d) States:

$$\frac{1}{\sqrt{2}}\left[Y_{1,0}(\theta_1,\varphi_1)\,Y_{1,0}(\theta_2\varphi_2) - Y_{1,0}(\theta_2,\varphi_1)\,Y_{1,1}(\theta_2,\varphi\right.$$

$$\frac{1}{\sqrt{2}}\left[Y_{1,1}(\theta_1,\varphi_1)\,Y_{1,-1}(\theta_2,\varphi_2) - Y_{1,-1}(\theta_1,\varphi_1)\,Y_{1,1}(\theta_2\varphi_2\right.$$

$$\frac{1}{\sqrt{2}}\left[Y_{1,0}(\theta_1,\varphi_1)\,Y_{1,+1}[\theta_2,\varphi_2] - Y_{1,-1}[\theta_1,\varphi_1]\,Y_{1,0}(\theta_2\varphi_2)\right.$$

Notes: if you want to be nicer, switch ab $\Leftrightarrow$ cd.

if you want to be more difficult, do it for $L=2$

III - 5

January 10, 2005

Solutions:

Section 3, Question 5, Stark Effect, Tomo Uemura.

a)

Suppose the center of the negatively charged cloud moved by the distance x from the nucleus. The nucleus would be attracted by the negative charge within the radius x of the negatively charged ball. The charge within this area is $-ex^3/a^3$. The attraction force acting upon the nucleus is $e^2 \times (x^3/a^3) \times (1/x^2) = e^2x/a^3$. This force is balanced by the electric field force eE. This leads to $\mu = ex = a^3E$ and $\alpha = \mu/E = a^3$.

b)

Using perturbation: $V = -eEx$. The first order energy change is zero, since $-eE < 1s \mid x \mid 1s >= 0$. Among terms for the second order energy change, we note that $< 1s \mid x \mid 2p_o >= 0$, and $< 1s \mid x \mid 2s >= 0$. As given in the question sheet, $< 1s \mid x \mid 2p_\pm >= (2^7/3^5) \times a_o$. Thus, the second order energy change becomes

$$-\sum[(eE)^2 < 1s \mid x \mid 2p_\pm >]/[E_{2p} - E_{1s}] = -(2^{18}/3^{11})a_o^3E^2 = -1.48a_o^3E^2$$

.

By the electric field E, the electric cloud moves $x = (\alpha/e)E$. The work done by the electric field is

$$-\int_0^x eEdx = -\int_0^x (e^2/\alpha)xdx = -(e^2/2\alpha)x^2 = -(\alpha/2)E^2$$

. This work equals to the second order energy change. Therefore, $\alpha = 2.96a_o^3$.

c)

Measure the atomic spectral line as a function of electric field.

Cole

(with solution)

Section 3 – Question 3

Muonic atoms are formed when a muon stops in a material and gets "captured" into an atomic state. These muons can then be absorbed by the nucleus through a process that is essentially inverse beta decay (but with a muon being absorbed instead of an electron). We will suppose that a muon is captured in a homogenous material composed of an element with atomic number Z and atomic mass A ($A >> 1$). Neglect the finite size of the nucleus. You may need the muon mass (rest energy), $m_\mu c^2 \sim 106$ MeV, electron mass, $m_e c^2 \sim 0.511$ MeV, and nucleon mass $m_N c^2 \sim 940$ MeV. You may also find it convenient to use $\hbar c \sim 200$ MeV fm.

4 a) We can estimate the most likely principle quantum number of the orbital into which the muon gets captured by assuming that the muon ejects the most energetic electron from the atom and occupies a state of comparable energy. Estimate n for the muon capture state using this assumption.

4 b) The muon will continue to de-excite by ejecting electrons from the atom (thus producing so-called "Auger" electrons) and emitting x-rays. Estimate the energy of the most energetic x-ray that can be emitted by the muon. If the radius of a nucleus with mass number A is, $R \sim (1.2\text{ fm})A^{1/3}$, comment on the validity of neglecting the nuclear size.

3 c) The absorption of the muon by the nucleus proceeds almost exclusively from S states. Explain why this is so.

4 d) If we assume that the matrix element and phase space factors are approximately the same for the capture of muons and electrons on nucleons (a crude approximation), estimate the ratio of the probabilities for the muon and electron to be absorbed from an n=1 atomic state, *i.e.* calculate $P(\mu\text{-capt}) / P(\text{e-capt})$.

Qualifying Exam 2005 Solutions

Section 3 Question 3

a) Use the Bohr model: $m v r = n\hbar$

$$v^2/r = \frac{Ze^2}{mr^2} \Rightarrow v^2 m r = Ze^2$$

$$\hookrightarrow v = e^2/n\hbar \quad \text{and} \quad E = \tfrac{1}{2}mv^2 - Ze^2/r = -\tfrac{1}{2}mv^2$$

$$\hookrightarrow E_n = -\tfrac{1}{2}\, m \,\frac{e^4 Z^2}{\hbar^2 n^2} \neq$$

Now, the most energetic electron has $n=1$ and is ~~Unscreed~~ unscreened so we can apply the above without screening. We want $E_\mu = -\tfrac{1}{2} m_e \dfrac{e^4 Z^2}{\hbar^2}$

But $E_\mu = -\tfrac{1}{2}\, m_\mu \dfrac{e^4 Z^2}{\hbar^2 n^2} \rightarrow n^2 = m_\mu/m_e$

$$\hookrightarrow n = \sqrt{m_\mu/m_e}$$

Now $m_\mu/m_e \simeq 200$ So $n \sim 14$ to 15

<u>note:</u> for muon $n < \sqrt{m_\mu/m_e}$ the muon itself is unscreened:

$$r_n = Ze^2/mv^2 = \frac{\hbar^2 n^2}{mZe^2}$$

So if muon has $n = \sqrt{m_\mu/m_e}$, $r = \dfrac{\hbar^2}{m_e Z e^2}$

which is the radius of the $r=1$ electron

$\hookrightarrow$ So for $n < \sqrt{m_\mu/m_e}$ the muon is unscreened

b) The most energetic X rays will be emitted from transitions to the $n=1$ state.

For muon, $E_1 = -\frac{1}{2} M_\mu c^2 \frac{Z^2 e^4}{\hbar^2 c^2}$

Note: $e^2/\hbar c = \alpha = 1/137 \Rightarrow E_1 = -\frac{1}{2} \frac{M_\mu c^2 Z^2}{(137)^2}$

Or you can use E_1 for electrons $= -13.6 \text{ eV } Z^2$
and E_1 for $\mu = \left(\frac{M_\mu}{m_e}\right) E_1$

Either way: $E_1 = -2.8 \text{ KeV } Z^2$

Since Z isn't specified, leave it this way.

Now the energy of an Xray is given by

$$\Delta E = E_1 \left(1/n_1^2 - 1/n_2^2\right)$$ for $n_2 \to n_1$ transition.

We don't know n_2 but, we calculated in part a that $n \sim 14$-15 for capture. While transitions from such large $n \to n=1$ are improbable, they are possible. but $(1/15)^2$ is negligible so we can estimate $E_{max}^{Xray} = -E_1$ $2.8 \text{ KeV } Z^2$

Now $r_n = \frac{\hbar^2 c^2 n^2}{m_\mu c^2 Z e^2} = \frac{\hbar c\, n^2}{m_\mu c^2 Z \alpha}$

for $n=1$ $\quad r_1 = \frac{200 \text{ MeV fm} \times 137}{106 \text{ MeV } Z}$

$\approx \frac{260 \text{ fm}}{Z}$

And nuclear radius $R_A \sim 1.2 \text{ fm } A^{1/3}$

for neglect of finite nuclear size to be valid we want $r_1 \gg R_A$. That will only be true if (approximately)

$200/Z \gg A^{1/3}$ since $Z \sim A$ $\quad 200 \gg A^{4/3}$

or $A \ll \approx 50$

c) ~~Wav~~ The muon is absorbed in the nucleus which is at very small radius compared to the μ orbit (as long as A is not too large). If we take $r \approx 0$, then only the $l=0$ wave functions are non-zero!

d) The wave function for $n=1$ ~~$l=0$~~ state is:

$$\psi(r) = \frac{2}{\left(\sqrt{a_0^\mu}\right)^3} e^{-r/a_0} \rightarrow \psi(0) = \frac{2}{(a_0^\mu)^{3/2}}$$

if we include the angular normalization

$$\psi = \frac{1}{\sqrt{\pi}} \frac{1}{(a_0^{\mu})^{3/2}}$$ (makes no difference).

If all other Contributions to the absorption ~~m~~ probability are the same, the ratio of absorption probabilities is just:

$$\left[\psi^{\mu}(0)/\psi^{e}(0)\right]^2 = \left(\frac{a_0^e}{a_0^{\mu}}\right)^3$$ where a_0 is the Bohr radius of Course

So $$\frac{a_0^e}{a_0^{\mu}} = \cancel{m} \frac{m_{\mu}}{m_e} \approx 200$$

$$\hookrightarrow \quad P(\mu\,\text{capt})/P(e\text{-capt}) \approx (200)^3 = 8 \times 10^6$$

III - 3

No solution provided

2005 Quals
Brian Cole
QM ~~(2)~~

Muonic Atoms

Muonic atoms are formed when a muon stops in a material and gets "captured" into an atomic state. These muons can then be absorbed by the nucleus through a process that is essentially inverse beta decay (but with a muon being absorbed instead of an electron). We will suppose that a muon is captured in a homogenous material composed of an element with atomic number Z and atomic mass A, A >> 1. Neglect the finite size of the nucleus. You may need the muon mass (rest energy), $m_\mu c^2 \approx 106\,\text{MeV}$, electron mass, $m_e c^2 \approx 0.511\ \text{MeV}$, and nucleon mass $m_N c^2 \approx 940\,\text{MeV}$. You may also find it convenient to use $\eta c \approx 200\ \text{eV nm} = 200\ \text{MeV fm}$.

a) We can estimate the most likely principle quantum number of the orbital into which the muon gets captured by assuming that the muon ejects the most energetic electron from the atom and occupies a state of comparable energy. Estimate n for the muon capture state using this assumption.
b) The muon will continue to de-excite by ejecting electrons from the atom (thus producing so-called "Auger" electrons) and emitting x-rays. Estimate the energy of the most energetic x-ray that can be emitted by the muon. If he radius of a nucleus with mass number A is $R \approx (1.2\,\text{fm})A^{1/3}$ comment on the validity of neglecting the nuclear size.
c) The absorption of the muon by the nucleus proceeds almost exclusively from S states. Explain why this is so.
d) If we assume that the matrix element and phase space factors are approximately the same for the capture of muons and electrons on nucleons (a crude approximation), estimate the ratio of the probabilities for the muon and electron to be absorbed from an $n=1$ atomic state (i.e. calculate $P_\mu^{\text{capt}} / P_e^{\text{capt}}$).

QM (Time Dependence)

A. Mueller

Consider a Hamiltonian $H = H_0 + V(t)$ where $V(t) = V_0$ for $0 < t < T$ and $V(t) = 0$ otherwise. V_0 is a constant operator. We label the eigenstates and eigenvalues of H_0 as $|m\rangle$ and E_m, respectively. That is, $H_0|m\rangle = E_m|m\rangle$. Suppose $|\psi(t)\rangle$ is the state of the system. If $|\psi(t)\rangle = |m\rangle$ for $t<0$ what is $|\langle n|\psi(t)\rangle|^2$ for $t > T$ when $m \neq n$. You may work to lowest nontrivial order in V.

Soln:

$$|\psi(t)\rangle = \sum_k C_k(t)\, |k\rangle\, e^{-iE_k t/\hbar}$$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = [H_0 + V(t)]\, |\psi(t)\rangle$$

gives

$$i\hbar \frac{\partial}{\partial t} C_k(t) = \langle k|V(t)|m\rangle\, e^{-i(E_k - E_m)t/\hbar}.$$

This gives

$$C_k(t) = \delta_{km} + \frac{\langle k|V_0|m\rangle}{E_m - E_k}\left(1 - e^{-i(E_k - E_m)t/\hbar}\right)$$

for $0 < t < T$.

$$C_k(T) = \delta_{km} + \frac{\langle k|V_0|m\rangle}{E_m - E_k}\left(1 - e^{-i(E_k - E_m)T/\hbar}\right)$$

$$C_k(t) = C_k(T) \text{ for } t > T$$

Thus, for $t > T$

$$|\psi(t)\rangle \doteq |m\rangle e^{-iE_m t/\hbar} + \sum_k |k\rangle \frac{\langle k|V_0|m\rangle}{E_m - E_k}\left(1 - e^{-i(E_k - E_m)T/\hbar}\right)$$

$$|\langle n|\psi(t)\rangle|^2 = \frac{|\langle n|V_0|m\rangle|^2}{(E_m - E_n)^2} \cdot 4\sin^2\left[\frac{(E_m - E_n)T}{2\hbar}\right]$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 12, 2005
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Section 4 – Question 1

A proton of mass m (0.938 GeV) and energy E collides with a stationary free proton and produces a proton-antiproton pair.

$$p + p \to p + p + \bar{p} + p$$

a) Show that the threshold energy for this process to take place is $7m$ (6.57 GeV).

b) Experimentally, the threshold energy using a copper target is found to be $5.21m$ (4.89 GeV) due to the Fermi motion of the target protons inside the copper nucleus. From this value, what is the maximum Fermi momentum in the copper nucleus? (You can assume that the Fermi momentum is small compared to the proton mass.)

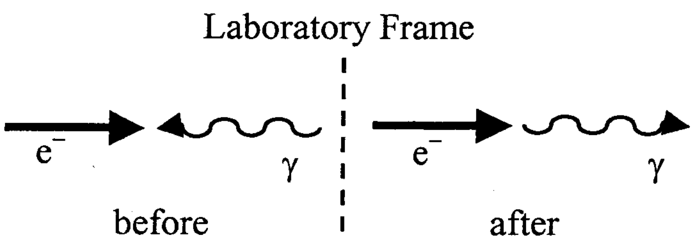
c) From what you know about protons in a copper nucleus (Z=29, A=63), show that this is a reasonable result, *i.e.* derive the value of the Fermi momentum and show that it agrees with the value in part b).

Section 4 – Question 2

A beam of visible photons can be scattered by a relativistic electron beam to produce an intense beam of gamma-rays. Analyze this process assuming,

$E_\gamma = 2$ eV, $KE(e^-) = 6\times10^9$ eV,

and that the collisions are collinear in the lab frame as shown below.



a) In the rest frame of the electron beam, what is the energy of the incoming photon beam?

b) The photons are reflected backwards in the lab frame as shown above. What is the energy of the reflected photons in the lab frame?

Section 4 – Question 3

Meson factories produce secondary π^+-meson beams from collisions of high energy protons with nuclear targets. The π^+-meson decays in flight mainly through the channel $\pi^+ \rightarrow \mu^+ + \upsilon_\mu$.

a) Without approximation, derive formulas for the muon and neutrino energies, E_μ and E_υ, in the pion's rest frame, in terms of the pion and muon masses (neglect the neutrino mass).

b) The π^+ has spin zero. What is the angular distribution of the μ^+ in the pion's rest frame?

c) What is the range of possible energies of the muon, E'_μ, in the laboratory frame? To what physical situations do the maximum and minimum values correspond?

d) Making use of your results in parts b) and c), obtain the probability distribution, $P(E'_\mu)$, of the μ^+ energy in the laboratory frame in terms of the pion and muon masses and the pion's Lorentz factors.

Section 4 – Question 4

Derive Einstein's famous formula, $E = m c^2$ (*i.e.* that mass is a form of energy) from the requirements that:

1. momentum conservation in collisions works independent of the inertial reference frame from which the collision is observed, and
2. velocity transforms between reference frames follow the relativistic (rather than the Galilean) form.

For simplicity, consider two-body collisions in one dimension, but allow the outgoing pair of particles to be different than the incoming pair. You may also work to lowest, non-trivial, order in v/c.

Note: following Einstein, you will have to modify the definition of several classical variables in order for this to make sense. Be sure to specify clearly which variables are getting new, relativistic definitions.

Section 4 – Question 5

In the laboratory frame, a charge, q, moves at a velocity, $v << c$, parallel to a wire carrying a current, I, and zero net charge density. The conduction electrons in the wire move at a velocity, $w << c$.

Show that, for the frame where the charge, q, is at rest, the wire appears to have a net charge density vw/c^2 times the charge density of the conduction electrons in the wire and that this explains the magnetic force on q.

2005

Section 4
Relativity

Relativity Problem

M. Shaevitz

A proton of mass m (0.938 GeV) and energy E collides with a stationary free proton and produces a proton-antiproton pair.

$$p + p \rightarrow p + p + \bar{p} + p$$

a) Show that the threshold energy for this process to take place is 7m (6.57 GeV).

b) Experimentally, the threshold energy using a copper target is found to be 5.21 m (4.89 GeV) due to the Fermi motion of the target protons inside the copper nucleus. From this value, what is the maximum Fermi momentum in the copper nucleus? (You can assume that the Fermi momentum is small compared to the proton mass.)

c) From what you know about protons in a copper nucleus (Z=29,A=63), show that this is a reasonable result, i.e. derive the value of the Fermi momentum and show that it agrees with the value in part b).

Relativity Problem

M. Shaevitz
IV - 1
Solution

a) $(P_B + P_T)^2 = (4P)^2$

$$m^2 + m^2 + 2E_B m = 16m^2$$

$$E_P = \frac{16m^2 - 2m^2}{2m} = 7m$$

b) $P_B = (E_B, 0, 0, P_B) \approx (E_B, 0, 0, E_B)$

$$P_T = (E_T, 0, 0, -x) \approx (m, 0, 0, -x)$$

$$(P_B + P_T)^2 = (4P)^2$$

$$m^2 + m^2 + 2(P_B \cdot P_T) = (4P)^2 = 16m^2$$

$$2m^2 + 2(E_B m + E_B x) = 16m^2$$

$$x = \frac{7m^2 - mE_B}{E_B} = \frac{(7-5.21)m^2}{5.21m} = 0.35m$$

$$= 0.304 \text{ GeV}$$

c) $n_{states} = \dfrac{2\left(\frac{4}{3}\pi p_F^3\right)\left(\frac{4}{3}\pi R^3\right)}{(2\pi\hbar)^3}$

$R = R_0 A^{1/3}$
$R_0 = 1.2$ fm
$\hbar c = 200$ MeV·

$= \text{number of protons} \approx \dfrac{A}{2}$

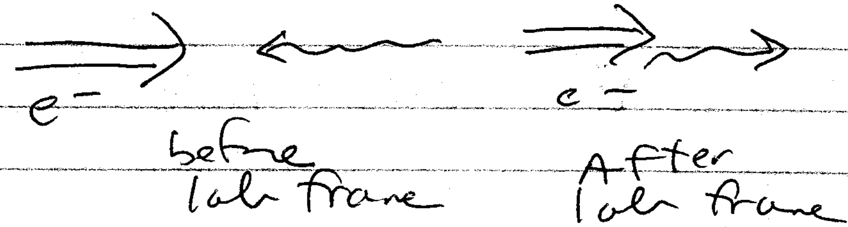
$$\frac{A}{2} = \frac{8}{9}\frac{p_F^3 R_0^3 A}{\pi\hbar^3} \Rightarrow p_F = \frac{\hbar}{R_0}\left(\frac{9\pi}{8}\right)^{1/3}$$

$$\Rightarrow \boxed{p_F = 253 \text{ MeV}}$$

Hailey Relativity: emission of photons
Absorp.

Hailey
Question
Relativity IV-2

A beam of visible photons can be scattered by a relativistic electron beam to produce an intense beam of gamma-rays. Analyze this process assuming $E_\gamma = 2eV$, $KE_{e^-} = 6\times10^9 eV$ and that the collisions are collinear in the lab frame as shown.



a) In the rest frame of the electron beam, what is the energy of the incoming photon beam?

b.) The photons are reflected backwards in the lab frame as shown above. What is the energy of the reflected photons in the lab frame?

IV - 2 Hailey - Relativity (solution)

Soln.

$E_\gamma = 2eV$

$KE = 6\times10^9 eV$

$\gamma \simeq \frac{KE}{mc^2} \simeq 1.2\times10^4$

$\beta \simeq 1$

a.) $E' = \gamma E(1-\beta\cos\theta)$

$E' = \gamma E(1+\beta)$ $\quad \theta = \pi$

$\Rightarrow E' \approx 2\gamma E = 2\cdot(1.2\times10^4)\cdot 2eV$

$E' = 4.8\times10^4 eV$ in e^- frame

b.) $E' = \gamma E(1-\beta\cos\theta)$

$\theta = 0$

$E' = \gamma E(1-\beta) \Rightarrow E = \frac{E'}{\gamma(1-\beta)} = E'\sqrt{\frac{1+\beta}{1-\beta}}$

$E = E'\sqrt{\frac{1+\beta}{1-\beta}\frac{1+\beta}{1+\beta}} = (1+\beta)\gamma E'$

$E \approx 2\gamma E' \approx 1.2\times10^9 eV$

$E \approx 1.2 GeV$

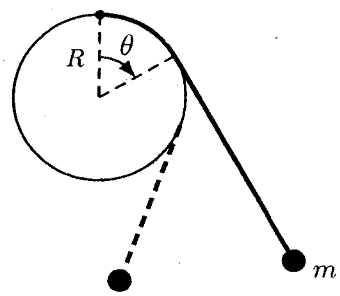
IV - 3

DEC 2004

Qualifying exam
Lagrangian mechanics and relativity
Eduardo Pontón

mechanics

1. Consider a pendulum built from a mass m attached to one end of a massless, extensionless string, whose other end is attached to the uppermost point of a vertical disk of radius R, as shown in the figure. Assume that the total length of the string is l and that $\pi R < l$.



   (a) Find the equations of motion in terms of the angle θ as shown in the figure.

   (b) What is the equilibrium angle θ_0? Find the frequency of small oscillations about this position.

Relativity

2. Meson factories produce secondary π^+-meson beams from collisions of high energy protons with nuclear targets. The π^+-meson decays in flight mainly through the channel $\pi^+ \to \mu^+ + \nu_\mu$.

   (a) Without approximation derive formulas for the muon and neutrino energies, E_μ and E_ν, in the pion's rest frame, in terms of the pion and muon masses (neglect the neutrino mass).

   (b) The π^+ has spin zero. What is the angular distribution of the μ^+ in the pion's rest frame?

   (c) What is the range of possible energies of the muon, E'_μ, in the laboratory frame? To what physical situations do the maximum and minimum values correspond?

   (d) Making use of your results in parts (b) and (c) obtain the probability distribution of the laboratory μ^+ energy.

Relativity : energy and momentum

2) a) Conservation of 4-momentum: $p_\pi = p_\mu + p_\nu$

$$\Rightarrow \quad m_\pi^2 = (p_\mu + p_\nu)^2 = m_\mu^2 + 2\, p_\mu \cdot p_\nu$$

In the rest frame:

$$p_\pi = (m_\pi, \vec{0})$$

$$p_\mu = (E_\mu, \vec{p})$$

$$p_\nu = (E_\nu, -\vec{p}) \qquad \text{with} \quad |\vec{p}| \approx E_\nu$$

$$\Rightarrow \quad p_\mu \cdot p_\nu = E_\mu E_\nu + p^2 = E_\nu (E_\mu + E_\nu)$$

Since $\quad m_\pi = E_\mu + E_\nu$

$$\Rightarrow \quad m_\pi^2 = m_\mu^2 + 2 E_\nu (E_\mu + E_\nu)$$

$$= m_\mu^2 + 2 E_\nu m_\pi$$

$$\Rightarrow \quad \boxed{E_\nu = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi}}$$

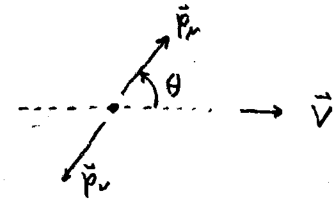
$$\Rightarrow \quad \boxed{E_\mu = m_\pi - E_\nu = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi}}$$

b) The angular distribution is isotropic: there is no preferred direction in the pion's rest frame

c) If the pion has velocity $\vec{v} = v\hat{z}$, we can obtain the answer from the results in the CM frame by boosting by $-\vec{v}$:

$$E'_\mu = \gamma\,(E_\mu + \beta p_z)$$

$$= \gamma\,(E_\mu + \beta p \cos\theta)$$



where θ is the angle between $\vec{v}$ and $\vec{p}_\mu$ in the CM frame

So

$$\gamma\,(E_\mu - \beta p) \leq E'_\mu \leq \gamma\,(E_\mu + \beta p)$$

where $$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

$$p \equiv E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

The highest (lowest) energy is obtained when the muon is emitted parallel (anti-parallel) to the pion's velocity.

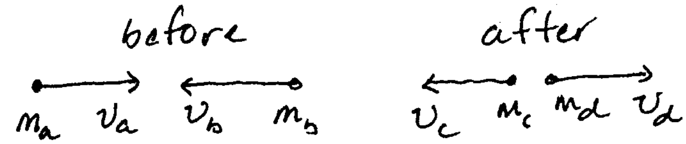
d) From part b), the distribution is uniform in $d\Omega = d\varphi\, d\cos\theta$

$$P(E'_\mu)\,dE'_\mu = P(\varphi,\theta)\,d\Omega = \frac{1}{4\pi}\,d\Omega = \frac{1}{2}\,d\cos\theta$$

$$\Rightarrow P(E'_\mu) = \frac{1}{2}\left(\frac{dE'_\mu}{d\cos\theta}\right)^{-1} = \frac{1}{2}\,\frac{1}{\gamma\beta p}$$

$$= \frac{1}{\gamma\beta}\,\frac{m_\pi}{m_\pi^2 - m_\mu^2} \qquad \text{for } |E'_\mu - \gamma E_\mu| \leq \gamma\beta p$$

- Collision in Frame S:



- Relativistic Velocity Transform to frame S' (moving w/ velocity, u, wrt S)

$$v' = \frac{v+u}{1+\frac{vu}{c^2}} \Rightarrow \beta' = \frac{\beta+\beta_u}{1+\beta\beta_u} \quad (i)$$

- Classically defined momentum ($p = mv$) conservation does not work in both frames with velocities related by (i)

$$m_a v_a + m_b v_b = m_c v_c + m_d v_d \not\Leftrightarrow m_a v_a' + m_b v_b' = m_c v_c' + m_d v_d'$$

- To allow momentum conservation in all frames need to redefine momentum (and mass) relativistically

$$m = \gamma m_o \quad (ii) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$p = mv = \gamma m_o v \quad (iii)$$

- Using this definition, momentum conserv. in frame S'

$$\gamma_a' m_{ao} v_a' + \gamma_b' m_{bo} v_b' = \gamma_c' m_{co} v_c' + \gamma_d' m_{do} v_d' \quad (iv)$$

- Generally

$$\gamma' v' = \left(\frac{1}{1-\beta'^2}\right)^{1/2} \frac{(\beta+\beta_u)c}{1+\beta\beta_u} = \left[\frac{1}{1-\left(\frac{\beta+\beta_u}{1+\beta\beta_u}\right)^2}\right]^{1/2} \frac{v(1+u/v)}{1+\beta\beta_u}$$

$$= \gamma v \left[\frac{(1-\beta^2)(1+\beta\beta_u)^2}{(1+\beta\beta_u)^2 - (\beta+\beta_u)^2}\right]^{1/2} \frac{(1+u/v)}{(1+\beta\beta_u)}$$

$$= \gamma v \left[\frac{1-\beta^2}{1+\beta^2\beta_u^2 - \beta^2 - \beta_u^2}\right] (1+u/v)$$

$$= \gamma v \left\{\frac{1}{1-\beta_u^2}\right\}^{1/2} (1+u/v)$$

- So (iv) becomes

$$\gamma_a m_{ao} v_a \left(\frac{1 + u/v_a}{\sqrt{1-\beta_u^2}} \right) + \gamma_b m_{bo} v_b \left(\frac{1 + u/v_b}{\sqrt{1-\beta_u^2}} \right)$$

$$= \gamma_c m_{co} v_c \left(\frac{1 + u/v_c}{\sqrt{1-\beta_u^2}} \right) + \gamma_d m_{do} v_d \left(\frac{1 + u/v_d}{\sqrt{1-\beta_u^2}} \right)$$

$\Rightarrow$ $\gamma_a m_{ao} v_a + \gamma_b m_{bo} v_b = \gamma_c m_{co} v_c + \gamma_d m_{do} v_d$

if $\gamma_a m_{ao} + \gamma_b m_{bo} = \gamma_c m_{co} + \gamma_d m_{do}$

i.e. relativistic mass is conserved (instead of rest mass)

- Now consider the relativistic mass, expand in terms of v/c

$m = \gamma m_0 \sim m_0 + 1/2\, m_0 (v/c)^2 + \cdots$

$\Rightarrow mc^2 = m_0 c^2 + 1/2\, m_0 v^2 + \cdots$

Kinetic Energy

Energy associated with mass

$\Rightarrow$ $\therefore$ the energy of a particle, comprising its rest energy and kinetic energy is

$E = mc^2$

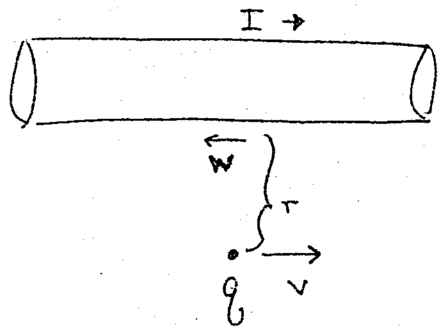
IV-5

Quals Relativity Exam Problems
December 2004
Robert Mawhinney

In the laboratory frame, a charge q moves at a velocity $v \ll c$ parallel to a wire carrying a current I and zero net charge density. The conduction electrons in the wire move at a velocity $w \ll c$.

Show that, for the the frame where the charge q is at rest, the wire appears to have a net charge density vw/c^2 times the charge density of the conduction electrons in the wire and that this explains the magnetic force on q.

Our system, in the lab frame, has current I, produced by electrons drifting with velocity w. The charge q is a distance r from the wire



Let λ_p and λ_n be the linear charge densities for the wire. In the lab frame $\lambda_p = -\lambda_n$, since the wire has zero net charge.

In the frame where q is at rest, the positive charge density is Lorentz contracted to be

$$\lambda_p \left(1 - v^2/c^2\right)^{-1/2}$$

For the negative charge density, we first boost to the frame where it is at rest yielding

$$\lambda_n \left(1 - w^2/c^2\right)^{1/2}$$

and then to the frame where q is at rest. This boost is given by the velocity of q in the rest frame of the negative charges:

$$v' = \frac{w + v}{1 + wv/c^2}$$

giving a linear charge density of

$$\frac{\lambda_n (1 - w^2/c^2)^{1/2}}{\left[1 - \left(\frac{w+v}{1+wv/c^2}\right)^2 / c^2\right]^{1/2}}$$

for the negative charges. Working to lowest order gives

$$\lambda_p\left[1 + \frac{v^2}{2c^2}\right] + \lambda_n\left[1 - \frac{w^2}{2c^2}\right]\left[1 + \frac{(w+v)^2}{2c^2}\right]$$

$$= \lambda_p\left[1 + \frac{v^2}{2c^2}\right] + \lambda_n\left[1 - \frac{w^2}{2c^2} + \frac{w^2}{2c^2} + \frac{wv}{c^2} + \frac{v^2}{2c^2}\right]$$

$$= \lambda_p\left[-\frac{wv}{c^2}\right] = \lambda_n \frac{wv}{c^2}$$

In this frame q sees an electric line charge of

$$\frac{\lambda_n wv}{c^2}$$

and a force of

$$\boxed{F = \frac{-q\lambda_n wv}{2\pi r \epsilon_0 c^2}}$$ in radial direction.

In the lab frame:

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0}{2\pi r}(-\lambda_n \omega)$$

The force is $\boxed{qvB = -\frac{\mu_0 q \lambda_n \omega v}{2\pi r}}$ in the radial direction

Since $c^2 \epsilon_0 \mu_0 = 1$, these results are identical