Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 14, 2008
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

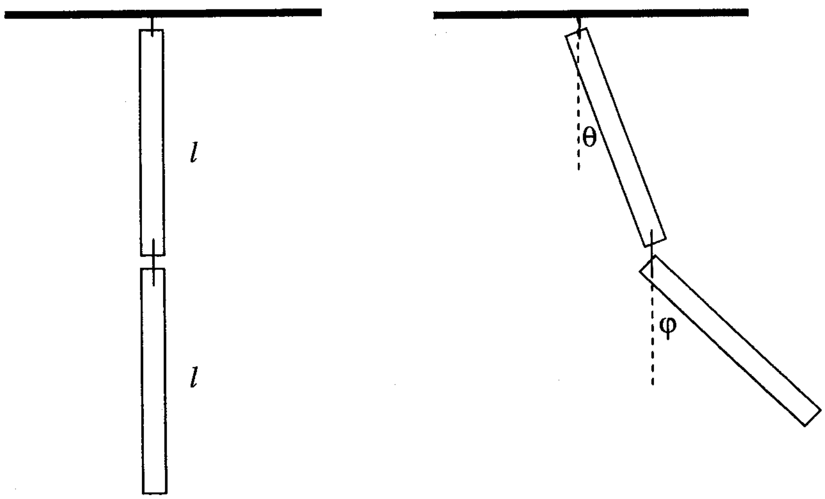
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

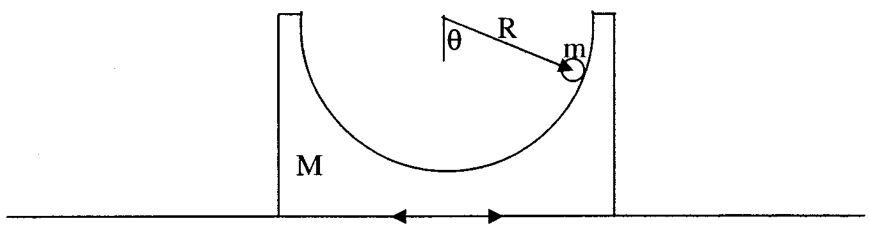
Good luck!!

1. You are mountain climbing on a conical peak described by the equation $z = -\sqrt{x^2 + y^2}$. There is a storm coming and you need to take refuge quickly. What is the equation of the shortest path to the refuge at position $(-1,0,-1)$ if you are currently located at $(1,0,-1)$.

2. Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The system then forms a physical double pendulum. Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. Describe the motion of each of the normal modes.



3. A particle of mass m is constrained to slide without friction on the surface of circular bowl of mass M. The circular bowl has an inner radius R and is free to slide along the horizontal surface without friction. Find the frequency of the normal mode of this system for small oscillations around the equilibrium position at the bottom of the bowl. Describe the motion for this normal mode oscillation.



4. A railroad car can move on a frictionless track. The railroad car has a mass M and is initially at rest. In addition, N people (each of mass m) are initially standing at rest on the car.

(a) Consider the case where all N people run to the end of the railroad car in unison and reach a speed, relative to the car, of v_r. At that point they all jump off at once. Calculate the velocity of the car relative to the ground, after all the people have jumped off.

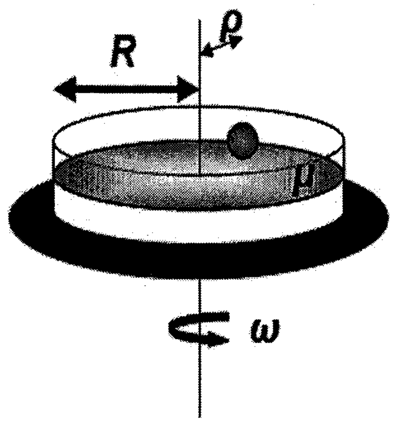
(b) Now consider a different case, in which the people jump off one at a time. In other words the people remain at rest relative to the car, while one of them runs to the end, attains a relative speed of v_r and jumps off. Then the next person starts running, attains a relative speed v_r and jumps off. That continues until all N people have jumped off. Find an expression for the final velocity of the railroad car relative to the ground.

(c) In which case, (a) or (b), does the railroad car attain a greater velocity?

5. We place a cylindrical pan of radius R at the center of a record player. It rotates with a constant angular frequency ω and the pan is half-filled with water.

(a) Compute the equilibrium surface shape of the fluid, which has mass density μ, taking into account the downward acceleration g due to gravity.

(b) A ping-pong ball of mass m and radius b is placed on the rotating fluid at a radial position ρ from the rotation axis. It is given an initial tangent velocity $v(0) = \omega\rho$. Compute all the forces on the ball and discuss qualitatively the subsequent motion of the ball.



Quals 2008 Elena Aprile, Fall 2007

Section 1: Classical Mechanics: Mountain Climbing

You are mountain climbing on a conical peak described by the equation $z = -\sqrt{x^2+y^2}$. There is a storm coming and you need to take refuge quickly. What is the equation of the shortest path to the refuge at position $(-1,0,-1)$ if you are now located at $(1,0,-1)$.

Solution

In cylindrical coordinates the length element is $ds = \sqrt{dr^2 + r^2 d\phi^2 + dz^2}$ and since the mountain is described by the equation $z = -r$ then the length element on the mountain is given by

$$ds = \sqrt{2dr^2 + r^2 d\phi^2}. \tag{1}$$

To find the optimal path we need to minimize the functional

$$\mathcal{P}[r(\phi)] = \int \sqrt{2dr^2 + r^2 d\phi^2} = \int d\phi \sqrt{r^2 + 2\dot{r}^2} = \int d\phi\, \mathcal{L}(r, \dot{r}, \phi)\,. \tag{2}$$

Consequently, the shortest path is the solution of

$$\frac{d}{d\phi}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{3}$$

$$\frac{2\ddot{r}}{\sqrt{r^2+2\dot{r}^2}} - \frac{2\dot{r}\left(\dot{r}r + 2\ddot{r}\dot{r}\right)}{\left(r^2+2\dot{r}^2\right)^{3/2}} - \frac{r}{\sqrt{r^2+2\dot{r}^2}} = 0 \tag{4}$$

$$2\ddot{r}\left(r^2+2\dot{r}^2\right) - 2\dot{r}\left(\dot{r}r + 2\ddot{r}\dot{r}\right) - r\left(r^2+2\dot{r}^2\right) = 0 \tag{5}$$

$$r^2 + 4\dot{r}^2 - 2\ddot{r}r = 0. \tag{6}$$

With $r = 1/u$ then $\dot{r} = -\dot{u}/u^2$ and $\ddot{r} = 2\dot{u}^2/u^3 - \ddot{u}/u^2$ which leads to

$$\frac{1}{u^2} + \frac{4\dot{u}^2}{u^4} - \frac{2}{u}\left(\frac{2\dot{u}^2}{u^3} - \frac{\ddot{u}}{u^2}\right) = 0, \quad \frac{1}{u^2} + \frac{2\ddot{u}}{u^3} = 0, \quad \ddot{u} + \tfrac{1}{2}u = 0. \tag{7}$$

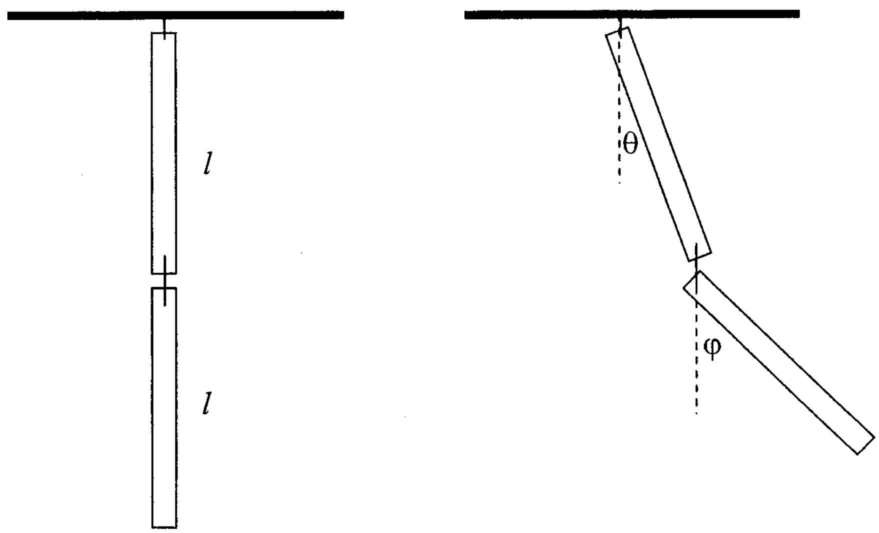
The general solution for the optimal path is thus $u(\phi) = A\cos\frac{\phi}{\sqrt{2}} + B\sin\frac{\phi}{\sqrt{2}}$. From the initial position constraint $(r_0, \phi_0) = (1, 0)$, we obtain $u(0) = A$ which leads to $A = 1$. Similarly, from the final position constraint $(r_1, \phi_1) = (1, \pi)$ we obtain $u(\pi) = \cos\frac{\pi}{\sqrt{2}} + B\sin\frac{\pi}{\sqrt{2}}$ giving $B = \left(1 - \cos\frac{\pi}{\sqrt{2}}\right)/\sin\frac{\pi}{\sqrt{2}}$. Putting everything together we obtain that the shortest path to the refuge is the one described by the equation

$$r(\phi) = \cos\tfrac{\pi}{2\sqrt{2}}\sec\tfrac{\pi - 2\phi}{2\sqrt{2}}. \tag{8}$$

Quals Problem 2

M. Shaevitz
Fall, 2007
Corrected 1/15/07

Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The system then forms a physical double pendulum. Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. Describe the motion of each of the normal modes.



Solution:

Let θ (φ) be the angle of the rod with respect to the vertical for the top (bottom) rod.

$$T = \frac{1}{2}\left(m\left(\frac{l}{2}\dot{\theta}\right)^2 + \frac{1}{12}ml^2\dot{\theta}^2 + m\left(l\dot{\theta} + \frac{l}{2}\dot{\phi}\right)^2 + \frac{1}{12}ml^2\dot{\phi}^2 \right)$$

$$U = mg\frac{l}{2}(1-\cos\theta) + mg\left(\frac{3}{2}l - \left(l\cos\theta + \frac{l}{2}\cos\phi\right)\right) \approx mgl\left(\frac{\theta^2}{4} + \left(\frac{\theta^2}{2} + \frac{\phi^2}{4}\right)\right)$$

$$L = T - U = \frac{4}{6}ml^2\dot{\theta}^2 + \frac{ml^2}{2}\dot{\theta}\dot{\phi} + \frac{1}{6}ml^2\dot{\phi}^2 - \frac{mgl}{4}\left(3\theta^2 + \phi^2\right)$$

Then Lagrange's equations are then given by

$$\frac{1}{2}\left(\frac{8}{3}l\ddot{\theta} + l\ddot{\phi} + \frac{3}{2}g\theta\right) = 0 \quad \frac{1}{2}\left(l\ddot{\theta} + \frac{2}{3}l\ddot{\phi} + \frac{1}{2}g\phi\right) = 0$$

$$\frac{1}{2}\left(\frac{8}{3}\ddot{\theta} + \ddot{\phi} + \frac{3}{2}\omega_0^2\theta\right) = 0 \quad \frac{1}{2}\left(\ddot{\theta} + \frac{2}{3}\ddot{\phi} + \frac{1}{2}\omega_0^2\phi\right) = 0 \text{ where } \omega_0^2 = \frac{g}{l}$$

Assuming small oscillations with $\theta = A\cos\omega t$ and $\phi = B\cos\omega t$ gives

$$\begin{pmatrix} \frac{3}{2}\omega_0^2 - \frac{4}{3}\omega^2 & -\frac{\omega^2}{2} \\ -\frac{\omega^2}{2} & \frac{1}{2}\omega_0^2 - \frac{1}{3}\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

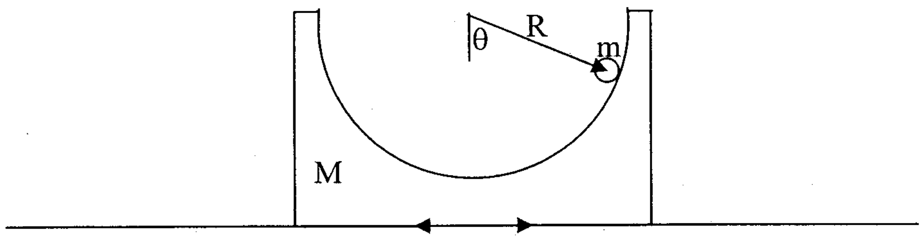
which yields nomal mode frequencies of $\omega^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\omega_0^2 = \begin{cases} 5.27\omega_0^2 \\ 0.73\omega_0^2 \end{cases}$

For the $\omega^2 = \left(3 + \frac{6}{\sqrt{7}}\right)\omega_0^2$ frequency, $B = \left(\frac{-2\sqrt{7}}{3} - \frac{1}{3}\right)A = -2.10A$

For the $\omega^2 = \left(3 - \frac{6}{\sqrt{7}}\right)\omega_0^2$ frequency, $B = \left(\frac{2\sqrt{7}}{3} - \frac{1}{3}\right)A = -1.43A$

Quals Problem 1 "Mass in circ. bowl" M. Shaevitz
Fall, 2007

A particle of mass m is constrained to slide without friction on the surface of circular bowl of mass M. The circular bowl has an inner radius R and is free to slide along the horizontal surface without friction. Find the frequency of the normal mode of this system for small oscillations around the equilibrium position at the bottom of the bowl. Describe the motion for this normal mode oscillation.



Solution:
Let X be the coordinate of the bowl along the horizontal axis, and θ be angular position of the mass, m. The postive X and θ directions are opposite. Then

$$T=\frac{1}{2}M\dot{X}^2+\frac{1}{2}m(\dot{X}-R\dot{\theta})^2 \ , \quad U=mgR(1-\cos\theta)=\frac{1}{2}mgR\theta^2$$

$$L=T-U=\frac{1}{2}M\dot{X}^2+\frac{1}{2}m(\dot{X}-R\dot{\theta})^2-\frac{1}{2}mgR\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=-mR(\ddot{X}-R\ddot{\theta})+mgR\theta \ , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{X}}\right)-\frac{\partial L}{\partial X}=(M+m)\ddot{X}-mR\ddot{\theta}$$

With $\theta=A\cos\omega t$ and $X=B\cos\omega t$, we have

$$\begin{pmatrix} mgR-mR^2\omega^2 & mR\omega^2 \\ mR\omega^2 & -(m+M)\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=0$$

which gives $\omega=\sqrt{\dfrac{g(m+M)}{RM}}$ for the normal mode frequency.

For this mode, the mass m goes one way and bowl goes the other to keep the CM fixed since there are no external forces. $B=\dfrac{mR}{(m+M)}A$

Quals 2008 – Mechanics Problem – Michael Tuts

1. A railroad car can move on a frictionless track. The railroad car has a mass M and is initially at rest. In addition, N people (each on mass m) are initially standing at rest on the car.
A. Consider the case where all N people run to the end of the railroad car in unison and reach a speed, relative to the car, of V_r. At that point they all jump off at once. Calculate the velocity of the car relative to the ground, after all the people have jumped off.
B. Now consider a different case, in which the people jump off one at a time. In other words the people remain at rest relative to the car, while one of them runs to the end, attains a relative speed of V_r and jumps off. Then the next person starts running, attains a relative speed v_r and jumps off. That continues until all N people have jumped off. Find an expression for the final velocity of the railroad car relative to the ground.
C. In which case (A or B) does the railroad car attain a greater velocity?

Answer:

1A. Let v_{car} be the speed of the railroad car relative to the ground. Since there are no external forces on the person-railroad car system, momentum is conserved. So

$$Mv_{car} + Nm(v_{car} - V_r) = 0$$

$$\therefore\ v_{car} = \frac{Nm}{M + Nm} V_r$$

1B. Consider the situation when we have n people on the railroad car, and one is about to jump off. Hence this is the transition from n to n-1 people on the car. Let v_n be the velocity of the railroad car with the n people on it, the total momentum of the railroad car is:

$$p_n = Mv_n + nmv_n$$

After the n^{th} person has jumped off, the total momentum of the railroad car plus the n-1 people and the person that jumped off is

$$p_{n-1} = Mv_{n-1} + m(n-1)v_{n-1} + m(v_{n-1} - V_r)$$

$$p_{n-1} = (M + nm)v_{n-1} - mV_r$$

And since there are no external forces acting between that transition, then

$$p_{n-1} = p_n$$

$$Mv_n + nmv_n = (M + nm)v_{n-1} - mV_r$$

And since $v_N = 0$ because the railroad car and the N people are initially at rest,

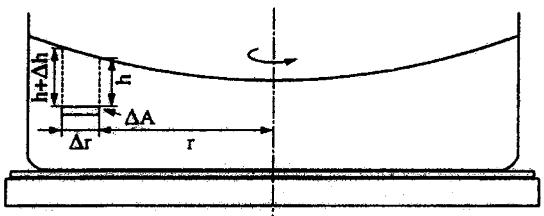
$$v_{final} = \sum_{n=1}^{N} \frac{m}{M + n\,m} V_r$$

1C. Clearly case 1B has the larger final speed.

Sect 1, #5
Mirka

Due to the internal friction of water and the relatively low angular velocity of the turntable the water will be rotating together with constant angular velocity and there are no turbulence related effects. We can also safely neglect vibration and surface waves/ripples.

a. The shape of the surface of the liquid will be a paraboloid, centered on the rotation axis. One can see this by applying 'F = ma' to an infinitesimal volume ($\Delta A\ \Delta h$) of the water:



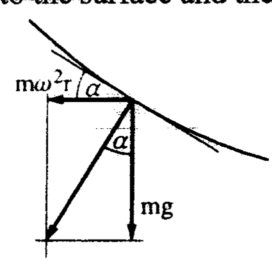
so $$[\varrho g(h+\Delta h)-\varrho g h]\Delta A=\Delta A\cdot\Delta r\cdot\varrho\cdot\omega^2 r$$

which, after some algebra, gives us (to avoid confusion ϱ is not used for the radius here)

$$g\cdot\Delta h=\omega^2 r\cdot\Delta r$$

$$\frac{\Delta h}{\Delta r}=\frac{\omega^2 r}{g}$$

thus the angle between the tangent to the surface and the horizontal:



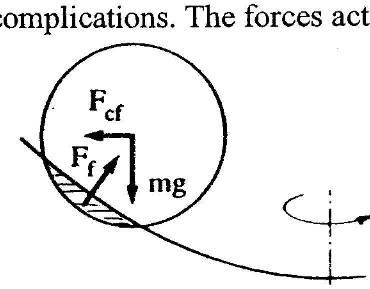
$$\operatorname{tg}\alpha=\frac{\omega^2 r}{g}$$

We can get the shape of the surface through integration:

$$y=\frac{\omega^2}{2g}x^2$$

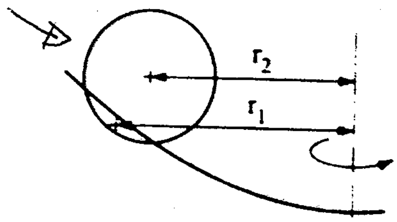
giving us the equation of a paraboloid centered on the rotation axis.

b. Due to the initial conditions and the friction between the water and the ping-pong ball, the ball will rotate together with the water initially. Neglect the air resistance, which in fact would only speed up the process, but would add significant complications. The forces acting on the ping-pong ball:



Decomposing the forces into horizontal and vertical components, we notice that there is a small net horizontal component:

$$F^H = -|H_f^H| + |H_{cf}^H| \approx -m\,\omega^2 r_1 + m\,\omega^2 r_2 < 0 \text{ since } r_2 < r_1$$



due to the paraboloid shaped surface of the liquid. Therefore the ball will slowly spiral into the center and will stay there. (Note, one can also prove that the centrifugal force acting on the ball can be treated as a force concentrated into the center of the ball.)

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 14, 2008
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

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Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A charge Q is uniformly distributed inside a sphere of radius R. The sphere rotates with angular velocity Ω. Find the magnitude of the magnetic field at its poles.

2. A semi-infinite material extends throughout the region $z > 0$ and has a flat boundary at $z = 0$. The region $z < 0$ is vacuum. The non-conducting ($\sigma = 0$) and non-permeable ($\mu = 1$) material has real index of refraction N_+ for circularly polarized waves with positive helicity and N_- for circularly polarized waves with negative helicity, when these waves propagate in the z-direction. $N_+ > 1$ and $N_- > 1$, but $N_+ \neq N_-$. There is no free charge and no free current on the surface of the material or within the volume of the material.

A plane wave linearly polarized in the x-direction and propagating in the z-direction in the vacuum region is incident normally on the material's flat surface. The reflected and transmitted waves also propagate along the z-direction. Calculate the ratio of the reflected intensity to the incident intensity, and describe the polarization of the reflected wave.

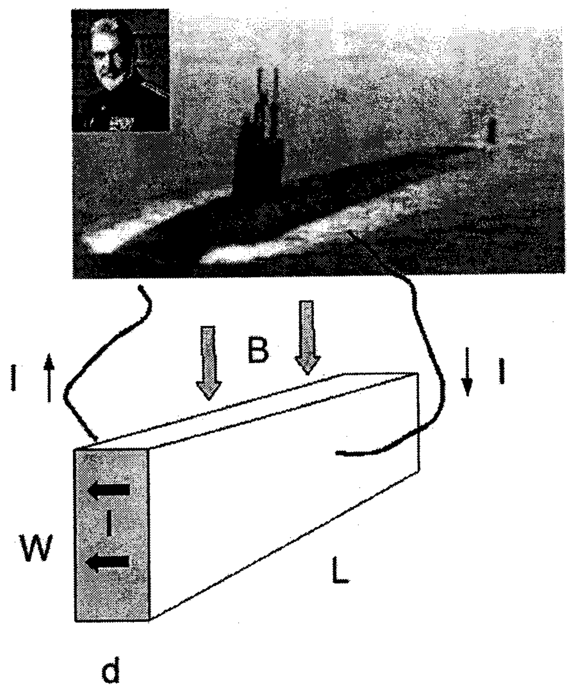
3. A laser beam, constant in time, is shot into space. Show that far from the Earth the transverse dimension of the beam scales as the square root of the wavelength. To be concrete you may consider a beam propagating in the z direction with a Gaussian profile in the transverse directions:

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}\psi(x, y, z)$$

where $\vec{E}_0$ is a polarization vector and

$$\psi(x, y, z) = \exp\left(A(z) - \frac{x^2 + y^2}{2B(z)}\right)$$

4. In a popular film Admiral Connery stole a Russian nuclear submarine with a secret silent magnetohydrodynamic MHD drive. Consider two metal plates with length L = 100 m, width $W = 10$ m separated by a distance $d = 5$ m as shown in the figure. Assume that a $P = 100$ Megawatt nuclear reactor is available to power DC generators on board. DC currents between the plates involve the mobile Na and Cl ions in the sea.



(a) Suppose a voltage difference V is applied to the drive. Given the sea water conductivity $\sigma = 5/(\Omega\text{–m})$ what is the total electric resistance R in Ohms of the drive? What is the current I in Amps if $V = 1$ Volt is applied? What is the theoretical limit of I that the reactor could deliver given its power, P, if there were no other sources of resistance or losses?

(b) Compute the total thrust force, F, in Newtons of a modest MHD drive assuming $I = 1000$ A and assuming the salt ions only feel the earth's magnetic field $B \approx 0.5$ Gauss.

(c) Compute the thrust if an external uniform 10 Tesla B field is also turned on directed toward the Earth over the whole drive. Compare this increased thrust to the conventional propeller thrust $F_{prop} = 10^7$ Newtons that the reactor powers to move it at the maximum cruising speed of 20 Knots. How does the drive thrust depend on dimensions d, L and W?

5. A perpendicularly incident beam of right circularly polarized light is reflected by a stationary mirror. Show that the reflected beam is left circularly polarized.

Problem: Charge Q is uniformly distributed inside a sphere of radius R. The sphere rotates with angular velocity Ω. Find the magnitude of magnetic field at its poles.

Solution:

The rotating charge produces current density $\vec{j} = \rho\vec{v}_{\rm rot}$, where $\rho = 3Q/4\pi R^3$. One can view the rotating sphere as a collection of rings of radii a carrying currents

$$dI = j\,dS = \rho\,\Omega\,a\,dS,$$

where dS is an element of vertical cross section of the sphere. Magnetic field $dB_{\rm pole}$ created by current loop dI is

$$dB_{\rm pole} = \frac{2\pi\,a^2\,dI}{c\,b^3},$$

where b is the distance between the loop and the pole. Integrating over dI (half cross section of the sphere), one finds ($\sin\alpha \equiv a/b$)

$$B_{\rm pole} = \frac{2\pi\,\rho\,\Omega}{c}\int\frac{a^3}{b^3}\,dS = \frac{2\pi\,\rho\,\Omega}{c}\int\!\!\int\sin^3\alpha\;b\,d\alpha\,db = \frac{2\pi\,\rho\,\Omega}{c}\int_0^{2R}db\,b\int_{b/2R}^{1}d\cos\alpha\;\sin^2\alpha = \frac{2}{5}\frac{Q\Omega}{cR}.$$

[The magnetic dipole moment of the sphere is $\mu = (1/5)R^2\Omega\,Q/c$. One can show that the sphere produces dipole magnetic field at all $r \geq R$, so $B_{\rm pole} = 2\mu/R^3$.]

Section 2, E & M, # 2
"Faraday rotator"
Identified by Miklos 12/14/07

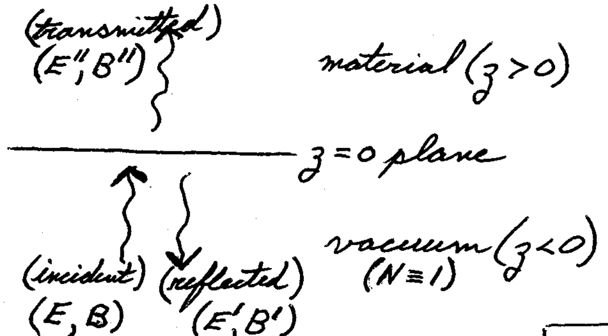
DEC 03 2007

ELECTROMAGNETISM PROBLEM
ALLAN BLAER

A semi-infinite material extends throughout the region $z > 0$ and has a flat boundary at $z = 0$. The region $z < 0$ is vacuum. The non-conducting ($\sigma = 0$) and non-permeable ($\mu = 1$) material has real index of refraction N_+ for circularly polarized waves with positive helicity and real index of refraction N_- for circularly polarized waves with negative helicity, when these waves propagate in the z-direction. $N_+ > 1$ and $N_- > 1$, but $N_+ \neq N_-$. There is no free charge and no free current on the surface of the material or within the volume of the material.

A plane wave linearly polarized in the x-direction and propagating in the z-direction in the vacuum region is incident normally on the material's flat surface. The reflected and transmitted waves also propagate along the z-direction. Calculate the ratio of the reflected intensity to the incident intensity, and describe the polarization of the reflected wave.

Section 2 (E&M) Problem 2 - Allan Blaer



$$\vec{E}_{inc} = E\,\hat{x}\,e^{i(kz-\omega t)}$$

where $\hat{x} = \frac{1}{2}\left[(\hat{x}+i\hat{y}) + (\hat{x}-i\hat{y})\right]$.

This expresses the linear polarization as a superposition of $\oplus$ and $\ominus$ helicity circular polarizations.

Let $\hat{e}_\pm \equiv \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) \Longrightarrow \boxed{\vec{E}_{inc} = \frac{E}{\sqrt{2}}(\hat{e}_+ + \hat{e}_-)e^{i(kz-\omega t)}}$.

Consider the reflection and transmission of each helicity state separately.

In general for plane waves with $\vec{k} = \hat{n}k$ and $N \equiv \frac{kc}{\omega}$:

$$\vec{B} = N\hat{n}\times\vec{E} \quad \left(\text{from } \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right).$$

Therefore, $\vec{E}_\pm = E_0\,\hat{e}_\pm\,e^{i(kz-\omega t)} \Rightarrow \vec{B}_\pm = N\hat{z}\times\vec{E}_\pm = \mp iN\vec{E}_\pm$ (incident, transmitted)

and $\vec{E}_\pm = E_0\,\hat{e}_\pm\,e^{i(-kz-\omega t)} \Rightarrow \vec{B}_\pm = -N\hat{z}\times\vec{E}_\pm = \pm iN\vec{E}_\pm$ (reflected)

At $z=0$, $\vec{E}_{tan}$ is continuous $\Longrightarrow \boxed{E_\pm + E_\pm' = E_\pm''}$.

At $z=0$, $\vec{H}_{tan}$ is continuous $\xRightarrow{(\mu=1)}$ $\vec{B}_{tan}$ is continuous $\Longrightarrow \mp iE_\pm \pm iE_\pm' = \mp iN_\pm E_\pm''$

$$\therefore \boxed{E_\pm - E_\pm' = N_\pm E_\pm''}.$$

Solving simultaneously $\Longrightarrow$ $E_\pm' = \frac{(1-N_\pm)}{(1+N_\pm)}E_\pm$ and $E_\pm'' = \left(\frac{2}{1+N_\pm}\right)E_\pm$

Therefore, $\vec{E}_{inc} = \frac{E}{\sqrt{2}}(\hat{e}_+ + \hat{e}_-)e^{i(kz-\omega t)}$ implies that

$$\boxed{\vec{E}_{reflected} = \frac{E}{\sqrt{2}}\left[\frac{(1-N_+)}{(1+N_+)}\hat{e}_+ + \frac{(1-N_-)}{(1+N_-)}\hat{e}_-\right]e^{i(-kz-\omega t)}}$$

Ratio of time-averaged intensities $= R = \frac{I_{reflected}}{I_{incident}} = \frac{\vec{E}^*_{reflected}\cdot\vec{E}_{reflected}}{\vec{E}^*_{inc}\cdot\vec{E}_{inc}}$

$$\therefore \boxed{R = \frac{\left[\left(\frac{1-N_+}{1+N_+}\right)^2 + \left(\frac{1-N_-}{1+N_-}\right)^2\right]}{2}}$$

(no cross terms between $\oplus$ and $\ominus$ helicities are present.)

Rewriting $\vec{E}_{reflected}$ in terms of $\hat{x}$ and $\hat{y} \Longrightarrow$ elliptical polarization

$$\vec{E}_{reflected} = \frac{E}{2}\left\{\hat{x}\left[\frac{(1-N_+)}{(1+N_+)} + \frac{(1-N_-)}{(1+N_-)}\right] + i\hat{y}\left[\frac{(1-N_+)}{(1+N_+)} - \frac{(1-N_-)}{(1+N_-)}\right]\right\}e^{i(-kz-\omega t)}$$

Section 2, E & M, # 3
"Wave spreading"
Identified by Miklos 12/14/07

Brooijmans

Quals 08, Optics

December 9, 2007

Problem

A laser beam, constant in time, is shot to the moon. Show that the transverse dimension of the beam scales as the square root of the wavelength. As an example, take a beam with a gaussian profile:

$$\psi_0(\vec{r}) = exp\left(A(z) - \frac{r^2}{2B(z)}\right), \tag{1}$$

where ψ_0 is the amplitude of one of the wave components at $z = 0$, z is the direction of motion and $r^2 = x^2 + y^2$.

Solution

The beam propagates according to the wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial^2 t} = 0. \tag{2}$$

While we don't know how the beam is polarized, it's clear that each component, ψ, propagates independently according to the wave equation. In general, we can write

$$\psi(\vec{r}, t) = \psi_0(\vec{r}, t)e^{i(kz-\omega t)}. \tag{3}$$

Here $\omega = kc$. Substituting and neglecting second derivatives of ψ_0 with respect to z (since the beam changes little along the direction of motion compared to the transverse direction), we get

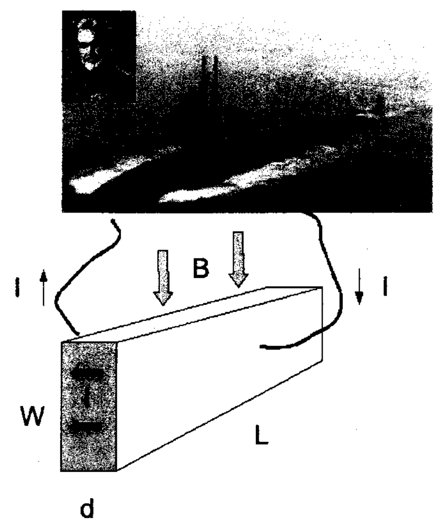
$$\frac{\partial \psi_0}{\partial z} = \frac{i}{2k}\left[\frac{\partial^2 \psi_0}{\partial x^2} + \frac{\partial^2 \psi_0}{\partial y^2}\right], \tag{4}$$

QED. Injecting our example function we find

$$\frac{\partial A}{\partial z} = \frac{-i}{kB}, \frac{\partial B}{\partial z} = \frac{i}{k}. \tag{5}$$

Qualifying exam
EM 2
M. Gyulassy

2 In a popular film Admiral Connery stole a Russian nuclear submarine with a secret silent magnetohydrodynamic MHD drive. Consider two metal plates with length $L = 100$ m, width $W = 10$ m seperated by a distance $d = 5$ m as shown in the figure. Assume that a $P = 100$ Megawatt nuclear reactor is available to power DC generators on board. DC currents between the plates involve the mobile Na and Cl ions in the sea.



a) Suppose a voltage difference V is applied to the drive. Given the sea water conductivity $\sigma = 5/(Ohm - m)$ what is the total electric resistance R in Ohms of the drive? What is the current I in Amps if $V = 1$ Volt is applied? What is the theoretical limit of I that the reactor could deliver given its power, P, if there were no other sources of resistance or losses?

b) Compute the total thrust force, F, in Newtons of a modest MHD drive assuming $I = 1000$ A and assuming the salt ions only feel the earth's magnetic field $B \approx 0.5$ Gauss.

c) Compute the thrust if an external uniform 10 Tesla B field is also turned on directed toward the earth over the whole drive. Compare this increased thrust to the conventional properler thrust $F_{prop} = 10^7$ Newtons that the reactor powers to move it at the maximum cruising speed of 20 Knots. How does the drive thrust depend on dimensions d, L and W?

Solution Gyulassy EM2:

a) Ohms Law $J = \sigma E = \sigma V/d$. Total current is $I = J \times LW = V/R$ so that the drive electric resistance is $R = d/(LW\sigma) = 5m(0.2\Omega - m)/10^3 m^2 = 10^{-3}\Omega$. For $V = 1$ Volt the total current across the drive is then $I = 1,000$ Amps. In theory $P = IV = I^2 R$ can power a lot higher current in the absence of other losses so that $I_{max} = \sqrt{10^8 10^3} \approx 10^6/\sqrt{10}$ Amps.

b) For N ions within the drive volume the current density is $J = Nqv/(dLW) = I/(LW)$, so that $Nqv = Id$. The total magnetic force is then $F = NqvB = IdB = \sigma V(LW)B$ that is independent of d if V is fixed, but indep of L, W if I is fixed.

For the budget MHD drive that uses only the earth magnetic field and a 1000 A rated generator, $F = IdB = (1000A)5m(0.5 \times 10^{-4} Tesla) = 0.25$ Newtons, which is way too small to move the sub.

The total reactor power $P = 10^8$ W can support a max speed of $v_{max} = 20$ Knots $= 10$ m/s under conventional noisy propeller propulsion. The maximum conventional thrust is thus $F_{max} = P/v_{max} = 10^8 W/10 m/s = 10^7$ Newtons.

c) If we used advanced superconductor technology to create a 10 Tesla uniform field over the entire 5000 m^3 volume to make a superduper MHD drive, then we can still can only achieve $F = 0.25N \times 10T/5 \times 10^{-5}T = 5 \times 10^4$ Newtons which is still 200 times smaller than what conventional propeller technology can deliver. With drive segmentation and dimension tuning and using multiple DC generators higher thrusts could be obtained.

Sec 2 #5
Marta

Assuming perpendicular incidence onto a conductive layer mirror the, Fresnel formulae tell us that:

$$E_R = \left(\frac{1-\beta}{1+\beta}\right)E_I = rE_I$$

where the amplitude reflectivity coefficient '*r*' is approaching unity for perfect conductors ($\sigma \to \infty$). They also tell us that upon reflection the reflected field experiences a 180° phase shift relatively to the incident field.

$$E_R = -E_I$$

A right circularly polarized monochromatic plane wave propagating along the *z* axis can be represented as

$$\bar{E}^{RIGHT} = \bar{E}_x + \bar{E}_y \qquad \bar{E}_x = E_0 \cos(kz-\omega t)\hat{x}$$
$$\bar{E}_y = E_0 \sin(kz-\omega t)\hat{y}$$

while the left circularly polarized wave can be represented as

$$\bar{E}^{LEFT} = \bar{E}_x - \bar{E}_y$$

Please note the 180° phase difference between the arguments of the sin and cos terms in contrast to the right circularly polarized wave.

The general validity of the discussion is not compromised by the zero phase angle at t=0 at z=0. We may also choose to place the mirror at z=0 in the x-y plane and choose the wave vector ***k*** of the incident right circularly polarized wave positive (propagating from $-\infty \to z=0$).

Upon reflection, both the ***x*** and the ***y*** component of the ***E*** fields acquire a 180° phase shift, so in itself it would not lead to a differential 180° phase shift between the two components required when changing from right to left circular polarization. Besides this common mode phase shift, the wave vector ***k*** also becomes negative (propagating from $z=0 \to -\infty$). Since the nature of the circularly polarized light is described from the viewpoint of propagation direction, this sign change of ***k*** will change the chirality.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 16, 2008
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

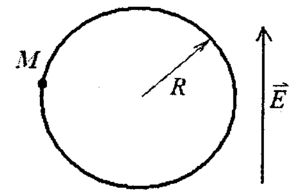
You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A particle of mass M and charge q is constrained to move in a circle of radius R.



(a) If no forces other than those of constraint act on the particle, find its allowed energies and the corresponding eigenstates.

(b) A strong, uniform electric field $\vec{E}$, oriented in the plane of the circle, is applied to the system. Find the first few lowest eigenvalues and corresponding eigenstates. Assume that $qRE >> \hbar^2/MR^2$.

(c) If a uniform magnetic field $\vec{B}$ is applied perpendicular to the plane of the circle, find the resulting eigenvalues and eigenstates. Work this out for both the $\vec{E} = 0$ situation of part (a) as well as the $\vec{E} \neq 0$ case of part (b).

2. Consider two identical, non-interacting spin ½ particles of mass m. The particles are moving in a potential given by

$$V(x_1, x_2) = 0 \qquad \text{for } x_1 \text{ and } x_2 \text{ in the interval } (-a/2, a/2)$$
$$V(x_1, x_2) = \infty \qquad \text{otherwise}$$

This is just the conventional 1-dimensional particle(s) in a box.

(a) For the singlet spin state, explicitly solve the Schrödinger equation to obtain the 2 particle ground state wavefunction. Calculate the energy of the ground state.

(b) For the triplet spin state, explicitly solve the Schrödinger equation to obtain the 2-particle ground state wavefunction. Calculate the energy of the ground state.

Assume that an interaction potential between the particles is turned on. The form of the interaction is $v = v_o b\, \delta(x_1 - x_2)$ where v_o is the interaction strength and b a characteristic length.

(c) Calculate the first order correction to the ground state energy in the triplet and singlet states.

(d) Provide a simple physical explanation for the magnitude of the first order correction for the singlet and triplet states.

3. Consider a single-electron atom within the field of a laser producing an oscillating local electric field of $\mathcal{E}(t) = \mathrm{Re}[\mathcal{E}_0 \exp(-i\omega t)]$ in the z-direction. Treating the laser semi-classically, we can write the perturbation acting on the atom's electron (within the dipole approximation) as $V = -e\,\mathcal{E}(t)\,z$, where $-e$ is the charge of the electron.

We now tune the laser to be exactly resonant with a transition between the ground state $|0\rangle$ and excited state $|1\rangle$ of the atom, i.e., the energy difference between the two states matches the photon energy of the laser: $E_1 - E_0 = \hbar\omega$. In the following, treat the atom as a (non-degenerate) two-level system. *Do not use perturbation theory, but do neglect any weak, rapidly varying terms in the response.*

(a) Find the probability that the atom is in state $|1\rangle$ as a function of time t. Take the atom to be in state $|0\rangle$ at $t = 0$.

(b) For a typical (allowed) optical transition in the atom, estimate the electric field strength needed to achieve the maximum population of state $|1\rangle$ in 10^{-9} s. If this electric field is to be provided by a laser beam with a power of 1 mW, how tightly should the laser be focused?

Hint: Write the solution of the time-dependent Schrödinger equation as

$$|\psi\rangle = C_0(t)\exp(-iE_0 t/\hbar)|0\rangle + C_1(t)\exp(-iE_1 t/\hbar)|1\rangle$$

and find equations for the time-dependent coefficients.

4. Consider a simple harmonic oscillator with frequency ω. A coherent state $|\lambda\rangle$ is defined to be an eigenstate of the lowering operator: $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$. Since the lowering operator isn't Hermitian, λ can be a complex number.

(a) Show that, up to an overall normalization, a coherent state can be expressed as

$$|\lambda\rangle = \exp(\lambda \hat{a}^\dagger)|0\rangle$$

Here $\hat{a}^\dagger$ is the raising operator and $|0\rangle$ is the ground state.

(b) Compute the position-space wavefunction $\psi(x) = \langle x|\lambda\rangle$ for a coherent state. Hint: derive a differential equation for $\psi(x)$.

(c) Compute the normalized expectation values of the position and momentum operators

$$\frac{\langle\lambda|\hat{x}|\lambda\rangle}{\langle\lambda|\lambda\rangle} \qquad \frac{\langle\lambda|\hat{p}|\lambda\rangle}{\langle\lambda|\lambda\rangle}$$

(d) Start with a coherent state $|\lambda_0\rangle$ at time $t = 0$. Show that up to an overall phase, under time evolution this state evolves into a coherent state $|\lambda(t)\rangle$. Express $\lambda(t)$ in terms of λ_0.

Useful facts: $\hat{a} = \sqrt{\dfrac{m\omega}{2\hbar}}\left(\hat{x} + \dfrac{i\hat{p}}{m\omega}\right)$, $\hat{a}^\dagger = \sqrt{\dfrac{m\omega}{2\hbar}}\left(\hat{x} - \dfrac{i\hat{p}}{m\omega}\right)$

5. Consider two identical particles of mass m and spin S = ½ with a spin-dependent interaction between them. Let $r = r_1 - r_2$ be the distance between the two particles and neglect any center of mass motion. The potential is

$$V(r) = \frac{g^2}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

with $\vec{\sigma}$ the usual trio of Pauli matrices.

(a) Prove that the total spin $\vec{S} = \vec{\sigma}_1 + \vec{\sigma}_2$ commutes with the Hamiltonian and find the eigenvalues of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$

(b) Find the energies of all of the bound states.

(c) Find the possible values of the angular momentum for each of the bound states.

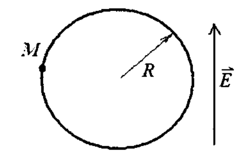
Section 3, QM, # 1
"QM round circle"

N. Christ

November 27, 2007

Quals Problems

1. A particle of mass M and charge q is constrained to move in a circle of radius R.



   (a) If no other forces other than those of constraint act on the particle, find its allowed energies and the corresponding eigenstates.

   (b) A strong, uniform electric field $\vec{E}$, oriented in the plane of the circle, is applied to the system. Find the first few lowest eigenvalues and corresponding eigenstates. Assume that $qRE \gg \hbar^2/MR^2$.

   (c) If a uniform magnetic field $\vec{B}$ is applied perpendicular to the plane of the circle, find the resulting eigenvalues and eigenstates. Work this out for both the $\vec{E} = 0$ situation of part (a) as well as the $\vec{E} \neq 0$ case of part (b). [You should neglect the $\vec{A}^2$ term in the Hamiltonian.]

2. A particle of mass M moves in a three-dimensional harmnic oscillator well. The Hamiltonian for this system is

$$H = \frac{\vec{p}^2}{2M} + \frac{1}{2}k\vec{r}^2$$

   (a) Find the energy and orbital angular momentum of the ground state and the first three excited states.

   (b) If eight identical, non-interacting spin-1/2 particles are placed in such a harmonic potential, find the ground state energy for the eight particle system.

   (c) Assume that these particles each have a magnetic moment $\vec{\mu} = \gamma\vec{s}$ where $\vec{s}$ is the particle's spin. If a magnetic field B is applied, what is the approximate ground state energy of the eight-particle system as a function of B? Plot the magnetization $(-\partial E/\partial B)$ for the ground state as a function of B.

Christ Section 3 QM
#1

Suggested Solutions

1. (a) $\psi(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}$, $m \in Z$ and $E_m = \frac{\hbar^2 m^2}{2MR^2}$.

   (b) For such a large E, the particle will under go simple harmonic motion around the minimum of the potential: $\psi(\theta)_n = h_n(z)e^{-z^2}$ where $z = \sqrt{\frac{M\omega}{\hbar}} R\theta$ and $E_n = \hbar\omega(n + \frac{1}{2})$. Here $\omega = \sqrt{\frac{qE}{MR}}$.

   (c) If $E = 0$, then the eigenstates are not changed but the energies are shifted by the usual magnetic moment–B coupling: $E_m = \frac{\hbar^2 m^2}{2MR^2} - \frac{e\hbar m B}{2Mc}$. For the $\vec{E} \neq 0$ case the effects of the magnetic field can be removed by adding a phase $e^{-i\frac{eBR^2}{2c\hbar}\theta}$ to the wave function and the energies are not affected.

2. (a) The ground state $\psi_0(\vec{r}) = Ne^{-\frac{m\omega}{\hbar}\vec{r}^2}$ and has energy $E_0 = \hbar\omega\frac{3}{2}$ with $\omega = \sqrt{k/m}$. This state has $l = 0$ and $m_l = 0$. The first three excited states all have energy $E_1 = \hbar\omega(1 + \frac{3}{2})$, $l = 1$ and $m_l = \pm 1$ and 0 if written:

   $$(x \pm iy)e^{-\frac{m\omega}{\hbar}\vec{r}^2} \quad \text{and} \quad ze^{-\frac{m\omega}{\hbar}\vec{r}^2}.$$

   (b) The exclusion principle permits us to put two spin-1/2 particles in each of these four lowest states. The resulting energy is then $E_{\text{gnd}} = 2E_0 + 6E_1 = 18\hbar\omega$.

   (c) Since the particle spins are paired, for small B, there is no dependence of E_{gnd} on B. However, when $2\mu B$ becomes greater than the energy separation $\hbar\omega$ between the harmonic oscillator states, it become energetically favorable to move the three $n = 1$ particles with magnetic moments anti-parallel to $\vec{B}$ to one of the six unoccupied single particle states with $E = \hbar\omega(2 + \frac{3}{2})$. Finally when $2\mu B$ becomes greater than the energy separation $2\hbar\omega$ between these six states and the ground state, the anti-aligned ground state particle also moves up to an $E = \hbar\omega(2 + \frac{3}{2})$ state. Thus, the magnetic susceptibility is zero for $0 \leq \mu B \leq \hbar\omega$. It jumps to 6μ for $\hbar\omega \leq \mu B \leq 2\hbar\omega$ and finally jumps to its largest value, 8μ for $2\hbar\omega \leq \mu B$.

C. Haily Quantum solution NOV 1 2007

a.) For the singlet state we need the symmetric spatial wave function which is the product of the single particle states. The single particle states are the solutions to

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$$

For the given boundary conditions this is

$$\psi_n = \sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a} \qquad n = 1, 3, 5 \ldots$$

$$\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \qquad n = 2, 4, 6$$

as can be seen by substitution, normalization and parity.

The symmetric ground state is just

$$\psi_{11}^S(x_1, x_2) = \frac{2}{a}\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a} \qquad \text{Ans } 2$$

This clearly solves the 2-particle Schrodinger

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi_{11}^S(x_1, x_2) = E_{11}\psi_{11}^S(x_1, x_2)$$

and is properly normalized ie

$$\int_{-a/2}^{a/2}\int |\psi_{11}^S(x_1, x_2)|^2 dx_1 dx_2 = 1$$

This immediately gives $E_{11} = \frac{\hbar^2}{m}\left(\frac{\pi}{a}\right)^2$ Ans 2

as the ground state

2008 Quals, Hailey, Section 3, QM, Question 1?

Consider two identical, non-interacting spin ½ particles of mass m. The particles are moving in a potential given by

$V(x_1, x_2) = 0$	for x_1 and x_2 in the interval $(-a/2, a/2)$ and
$V(x_1, x_2) = \infty$	otherwise

This is just the classical 1-dimensional particle(s) in a box.

(a) For the singlet spin state, explicitly solve the Schrödinger equation to obtain the 2-particle ground state wavefunction. Calculate the energy of the ground state.

(b) For the triplet spin state, explicitly solve the Schrödinger equation to obtain the 2-particle ground state wavefunction. Calculate the energy of the ground state.

Assume that an interaction potential between the particles is turned on. The form of the interaction is $v = v_o{*}b\ \delta(x_1 - x_2)$ where v_o is the interaction strength and b a characteristic length.

(c) Calculate the first order correction to the ground state energy in the triplet and singlet states.

(d) Provide a simple physical explanation for the magnitude of the first order correction for the singlet and triplet states.

b.) For the spin triplet we need an anti-symmetric wavefunction

$$\psi^A_{n_1 n_2} = \frac{1}{\sqrt{2}}\left(\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\right)$$

where $n_1 \neq n_2$. The ground state wave function is

$$\psi^A_{12} = \frac{1}{\sqrt{2}}\frac{2}{a}\left(\cos\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a} - \sin\frac{2\pi x_1}{a}\cos\frac{\pi x_2}{a}\right)$$

This clearly solves the 2-body Schrodinger equation.

using $-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi^A_{12} = E_{12}\,\psi^A_{12}$

we immediately obtain $E_{12} = \frac{5}{2m}\left(\frac{\hbar\pi}{a}\right)^2$ Ans 2

as the ground state.

c.) For the singlet state we need $\langle \psi^S_{11} | v | \psi^S_{11} \rangle$

$$E^{(1)} = \frac{4}{a^2}\iint \cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a}\, v_0 b\,\delta(x_1 - x_2)\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a}\,dx_1 dx_2$$

$$E^{(1)} = \frac{4}{a^2} v_0 b \int_{-a/2}^{a/2} \cos^4\frac{\pi x_1}{a}\,dx_1$$

Ans 2 $\quad E^{(1)} = \frac{3}{2} v_0 \frac{b}{a}$ using $\int \cos^n\theta\, d\theta = \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta$

For the triplet state the δ-fcn will yield $x_1 = x_2$ on integration over dx_2, and $\psi_{12}^{A}(x_1, x_1) = 0$

so $E^{(1)} = 0$ Ans for triplet

d.) The Antisymmetry of the spatial wavefunction in the triplet state means the e^-'s can never overlap and thus never feel the interaction potential.. In the singlet state The wavefunction Allows e^- overlap, yielding non-zero $E^{(1)}$.

Sec 3, QM
3
Revised

HEINZ QM PROBLEM - SOL'N

(a) $(H_0 + V)|\psi\rangle = i\hbar \frac{\partial|\psi\rangle}{\partial t}$ with $V = -\frac{e}{2}(e^{+i\omega t} + e^{-i\omega t})\mathcal{E}_0 z$

$|\psi\rangle \equiv c_0(t)|0\rangle + c_1(t)e^{-i\omega t}|1\rangle$

where we set $E_0 = 0$ for convenience and $\hbar\omega = E_1 - E_0 = E_1$

Substituting the expansion into the Schrödinger eq:

$c_0 V|0\rangle + c_1 \cancel{\hbar\omega e^{-i\omega t}}|1\rangle + c_1 e^{-i\omega t} V|1\rangle =$

$i\hbar c_0'|0\rangle + i\hbar c_1' e^{-i\omega t}|1\rangle + \cancel{c_1 \hbar\omega e^{-i\omega t}}|1\rangle$

Now project the equation on states $\langle 0|$ and $\langle 1|$:

$c_1 e^{-i\omega t}\langle 0|V|1\rangle = i\hbar c_0'$

$c_0 \langle 1|V|0\rangle = i\hbar e^{-i\omega t} c_1' \quad (*)$

Here we omit terms $\langle 0|V|0\rangle$ and $\langle 1|V|1\rangle$. They vanish by parity and can also be omitted since they only contribute rapid oscillations. Keeping only terms in (*) oscillating at the same frequency:

$c_1 V_{01} = i\hbar c_0'$; $c_0 V_{10} = i\hbar c_1'$ with $2V_{10} = 2V_{01}^* = \langle 1|-ez\mathcal{E}_0|0\rangle$

combining the above $= \mathcal{E}_0 \langle 1|-ez|0\rangle$

$c_0'' = -\Omega_R^2 c_0$ with $\Omega_R = \frac{|V_{10}|}{\hbar} = \frac{\mathcal{E}_0 p_{10}}{2\hbar}$ $\equiv \mathcal{E}_0 p_{10}$

$\therefore c_0(t) = \cos(\Omega_R t)$ for $c_0(t=0) = 1$

$P_1(t) = 1 - |c_0(t)|^2 = \sin^2(\Omega_R t)$

(b) Maximum population for $\Omega_R \tau = \pi/2 \Rightarrow \mathcal{E}_0 = \frac{\pi\hbar}{e z_{10}\tau}$

$z_{10} \sim a_0 \sim 10^{-10}\,\text{m} \Rightarrow \mathcal{E}_0 \sim 2\times10^4\,\text{V/m}$ for $\tau = 10^{-7}\,\text{s}$

$\therefore I = \langle S\rangle = \frac{\epsilon_0 c \mathcal{E}_0^2}{2} = 600\,\text{kW/m}^2 = \frac{P}{A} \Rightarrow A = 2\times10^{-9}\,\text{m}^2$

Section 3, QM #4

Prof. Kabat

Coherent states

1. $|\lambda\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n (\hat{a}^+)^n |0\rangle$

$$\hat{a}|\lambda\rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n \hat{a} (\hat{a}^+)^n |0\rangle$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n \, n (\hat{a}^+)^{n-1} |0\rangle$$

$$= \lambda |\lambda\rangle$$

2. $\hat{a}\psi = \lambda\psi$

$$\sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{1}{m\omega} \frac{\partial}{\partial x} \right) \psi = \lambda \psi$$

$$\frac{1}{\psi} \frac{d\psi}{dx} = \sqrt{2\hbar m\omega}\, \lambda - m\omega x$$

$$\log \psi = \text{const.} + \sqrt{2\hbar m\omega}\, \lambda x - \frac{1}{2} m\omega x^2$$

$$\psi \sim e^{-\frac{1}{2} m\omega x^2} e^{\sqrt{2\hbar m\omega}\, \lambda x}$$

3. $\langle x \rangle = \dfrac{\langle \lambda | \sqrt{\frac{2\hbar}{m\omega}} \frac{1}{2} (\hat{a} + \hat{a}^+) | \lambda \rangle}{\langle \lambda | \lambda \rangle} = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\, \lambda$

$$\langle p \rangle = \frac{\langle \lambda | \sqrt{2\hbar m\omega}\, \frac{1}{2i} (\hat{a} - \hat{a}^+) | \lambda \rangle}{\langle \lambda | \lambda \rangle} = \sqrt{2\hbar m\omega}\, \mathrm{Im}\, \lambda$$

4. $|\lambda_0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_0^n (\hat{a}^+)^n |0\rangle$

$$|\lambda(t)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_0^n e^{-i\omega(n+\frac{1}{2})t} (\hat{a}^+)^n |0\rangle$$

$$= e^{-i\hbar\omega t/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(e^{-i\omega t} \lambda_0 \right)^n (\hat{a}^+)^n |0\rangle$$

$$\Rightarrow \lambda(t) = \lambda_0 e^{-i\omega t}$$

-3

$1 \cdot 2 \cdot \frac{1}{2} - 3$

Millis 07 Quantum Solution

(a) $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{1}{2}\left((\vec{\sigma}_1 + \vec{\sigma}_2)^2 - \vec{\sigma}_1^2 - \vec{\sigma}_2^2\right) = \frac{1}{2}\left[(\vec{\sigma}_1 + \vec{\sigma}_2)^2 - 3.\right]$ Thus the spin-dependent part of the Hamiltonian is proportional to $(\vec{\sigma}_1 + \vec{\sigma}_2)$ and therefore commutes with it.

(b) The triplet states are not bound (repulsive potential). The singlet state has potential $V(r) = -3g^2/r$. The problem is thus hydrogen with $e^2 \to 3g^2$ and a mass equal to the reduced mass $\mu = m/2$. Thus eigenvalues

$$E_n = \frac{9mg^4}{2n^2}$$

(c) The bound states are fermions (because spin 1/2 and spin singlet, meaning the wave function is odd under interchange of spins; thus even under interchange of particles; thus only EVEN angular momenta are allowed.

$\frac{3}{4} + \frac{3}{4}$

$s(s+1)$

$\frac{1}{2}:$ $\frac{3}{2}$ $\frac{0}{2}$

$-\frac{3}{4}$ $+\frac{1}{4}$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 16, 2008
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code.**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. The Schrödinger equation for helium cannot be solved exactly. However, if we replace the Coulomb force with a spring force, the system can be solved exactly. As an example, consider the Hamiltonian in 3-dimensional space given by:

$$H = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + \frac{1}{2}m\omega^2(r_1^2 + r_2^2) - \frac{\lambda}{4}m\omega^2 |\vec{r}_1 - \vec{r}_2|^2$$

assuming $\lambda < 1$.

(a) Setting first $\lambda = 0$, write the ground state energy of the two (uncoupled) oscillators. Also write the normalized ground state wavefunction (use the notation $\alpha \equiv m\omega/\hbar$).

(b) Based on the above ground state wavefunction, estimate the ground state energy of the system for $\lambda \neq 0$.

(c) What can you say about the sign of the error of your approximation? That is, do you expect the approximate answer to be larger or smaller than the exact result?

(d) By a suitable change of variables, transform the full H into two independent 3-dimensional harmonic oscillators.

(e) What is the exact ground state energy of the system? Does it fulfill your expectation of part (c)?

The following integral may be useful:

$$\int_0^\infty x^{2n} e^{-ax^2} = \begin{cases} \frac{1}{2}\sqrt{\frac{\pi}{a}} & for\ n = 0 \\ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} & for\ n = 1, 2, \ldots \end{cases}$$

2. A long straight wire is carrying a steady current I. A charge q at a distance r is moving with velocity v parallel to the wire. For the special case where the velocity v is the same as that for the charge carriers in the wire, calculate the following quantities in the rest frame of q:

(a) the charge density of the wire;

(b) the electrical force the wire exerts on q. Transform this force to the rest frame of the wire and show that it is precisely the same as the magnetic force exerted on the moving charge in this frame.

Hint: the wire is electrically neutral in its rest frame.

3. The cosmic microwave background radiation has a temperature of 3 K. A cosmic ray proton of sufficient energy can collide with a microwave background photon and form a pion through the process $p + \gamma \rightarrow p + \pi$. Assuming a photon of average energy, what is the minimum proton energy (in electron volts) needed for this process to occur?

(Hint: the mass of a pion is about 1/6 the mass of a proton.)

4. The π^- meson interacts with the deuteron from an s orbital state to form two neutrons.

$$\pi^- + d \rightarrow n + n$$

(a) Determine the allowed L, S, J values for the neutrons.

(b) From this process, deduce the parity of the pion.

(A deuteron has J^P (spin$^{\text{parity}}$) = 1^+, the neutron has $J^P = \frac{1}{2}^+$ and pions have zero spin.)

5. Figure 1 shows C_V, the specific heat at constant volume, of H_2 gas as a function of temperature in units of R, where R is the universal gas constant. We know that $R = k_B N_A$, where $k_B = 1.38\times10^{-23}$ J/K is the Boltzmann constant and $N_A = 6.02\times10^{23}$ is Avogadro's number. From this data for C_V, and the value of the Bohr radius, $a_0 = 0.053$ nm, estimate the value of Planck's constant.

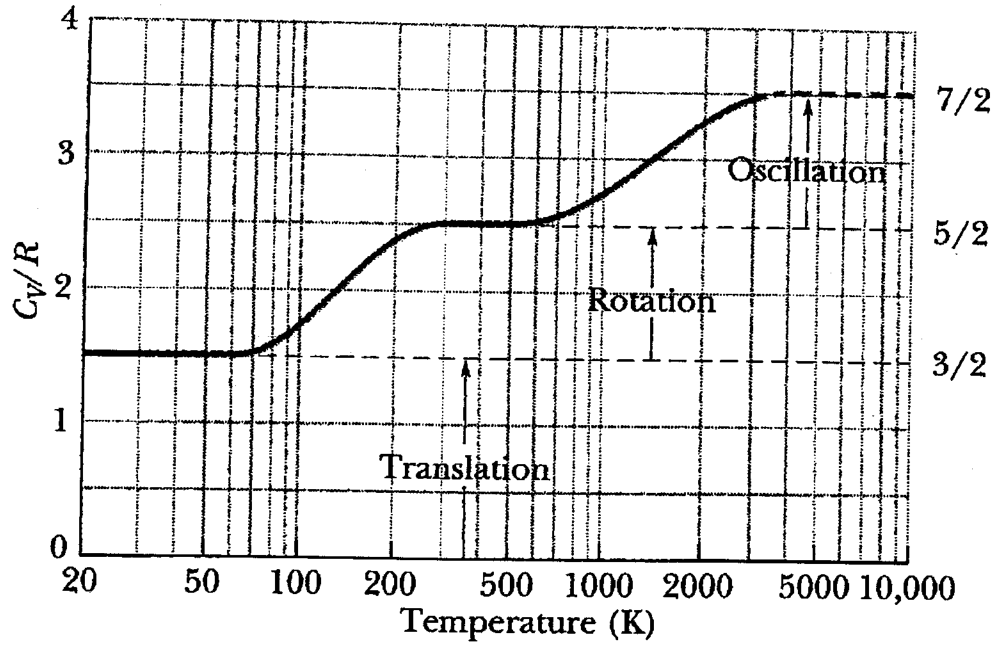


Figure 1.

Qualifying exam
Quantum mechanics
Eduardo Pontón

1. The Schrödinger equation for helium cannot be solved exactly. However, if we replace the Coulomb force with a spring force, the system can be solved exactly. As an example, consider the Hamiltonian in 3-dimensional space given by:

$$H = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + \frac{1}{2}m\omega^2(r_1^2 + r_2^2) - \frac{\lambda}{4}m\omega^2|\vec{r}_1 - \vec{r}_2|^2 \,, \qquad (1)$$

assuming $\lambda < 1$.

   (a) Setting first $\lambda = 0$, write the ground state energy of the two (uncoupled) oscillators. Also write the normalized ground state wavefunction (use the notation $\alpha \equiv m\omega/\hbar$).

   (b) Based on the above ground state wavefunction, estimate the ground state energy of the system for $\lambda \neq 0$.

   (c) What can you say about the sign of the error of your approximation? That is, do you expect the approximate answer to be larger or smaller than the exact result?

   (d) By a suitable change of variables, transform the full H into two independent 3-dimensional harmonic oscillators.

   (e) What is the exact ground state energy of the system? Does it fulfill your expectation of part (c)?

The following integral may be useful:

$$\int_0^\infty x^{2n} e^{-ax^2} = \begin{cases} \frac{1}{2}\sqrt{\frac{\pi}{a}} & \text{for } n = 0 \\ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} & \text{for } n = 1, 2, \ldots \end{cases}$$

Solutions

① a) Ground state wavefunction for $\lambda = 0$:

$$\Psi_0(\vec{r}_1, \vec{r}_2) = \left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha(\vec{r}_1^{\,2} + \vec{r}_2^{\,2})/2} \quad , \quad \alpha = \frac{m\omega}{\hbar}$$

with energy $E_0 = 3\hbar\omega$

b) Use the variational method with trial wavefunction

$$\Psi_\eta \equiv \left(\frac{\eta}{\pi}\right)^{3/2} e^{-\eta(\vec{r}_1^{\,2} + \vec{r}_2^{\,2})/2} \quad , \quad \langle \Psi_\eta | \Psi_\eta \rangle = 1$$

We need

$$\langle \Psi_\eta | \frac{\hat{p}_1^2}{2m} | \Psi_\eta \rangle = \frac{1}{2m} \| \hat{p}_1 | \Psi_\eta \rangle \|^2$$

$$= \frac{\hbar^2}{2m} \left(\frac{\eta}{\pi}\right)^{3/2} \int d^3r_1 \left| \nabla_1 e^{-\eta \vec{r}_1^{\,2}/2} \right|^2$$

$$= \frac{\hbar^2}{2m} \left(\frac{\eta}{\pi}\right)^{3/2} 4\pi \int_0^\infty r_1^2 dr_1 \left(\eta^2 r_1^2 e^{-\eta r_1^2}\right)$$

$$= \frac{\hbar^2}{2m} \left(\frac{\eta}{\pi}\right)^{3/2} 4\pi \eta^2 \frac{3}{2^3 \eta^2} \sqrt{\frac{\pi}{\eta}}$$

$$= \frac{3}{2} \eta \frac{\hbar^2}{2m}$$

$$\langle \Psi_\eta | \tfrac{1}{2} m\omega^2 r_1^2 | \Psi_\eta \rangle = \frac{3}{2} \eta^{-1} \frac{1}{2} m\omega^2$$

$$\langle \Psi_\eta | -\frac{\lambda}{4} m\omega^2 (\vec{r}_1 - \vec{r}_2)^2 | \Psi_\eta \rangle =$$

(angular integration) 0

$$-\frac{\lambda}{4} m\omega^2 \left(\frac{\eta}{\pi}\right)^3 \int d^3r_1\, r_2^2 dr_2\, d\varphi_2\, d\cos\theta_2 \left\{ r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta_2 \right\} e^{-\eta(r_1^2 + r_2^2)}$$

$$= -\frac{\lambda}{4} m\omega^2 \left(\frac{\eta}{\pi}\right)^{3/2} \times 2 \int d^3r\, r^2 e^{-\eta r^2}$$

$$= -\frac{3}{2} \eta^{-1} \frac{\lambda}{2} m\omega^2$$

Then

$$0 = \frac{d}{d\eta}\frac{\langle\psi_\eta|H|\psi_\eta\rangle}{\langle\psi_\eta|\psi_\eta\rangle} = \frac{3}{2}\frac{d}{d\eta}\left\{\frac{\hbar^2}{m}\eta + m\omega^2\eta^{-1} - \frac{\lambda}{2}m\omega^2\eta^{-1}\right\}$$

$$\Rightarrow \quad \eta = \sqrt{1-\frac{\lambda}{2}}\;\frac{m\omega}{\hbar} = \sqrt{1-\frac{\lambda}{2}}\;\alpha$$

and

$$E_0^{\text{estimate}} = \frac{\langle\psi_\eta|H|\psi_\eta\rangle}{\langle\psi_\eta|\psi_\eta\rangle} = 3\sqrt{1-\frac{\lambda}{2}}\;\hbar\omega$$

c) $E_0^{\text{true}} \leqslant E_0^{\text{estimate}}$

d) Do

$$\vec{u} = \frac{1}{\sqrt{2}}(\vec{r}_1 + \vec{r}_2)$$

$$\vec{v} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$$

to obtain

$$H = \left[-\frac{\hbar^2}{2m}\nabla_{\vec{u}}^2 + \frac{1}{2}m\omega^2u^2\right] + \left[-\frac{\hbar^2}{2m}\nabla_{\vec{v}}^2 + \frac{1}{2}m\omega^2(1-\lambda)v^2\right]$$

e)

$$E_0^{\text{true}} = \frac{3}{2}\hbar\omega + \frac{3}{2}\hbar\omega\sqrt{1-\lambda}$$

$$= \frac{3}{2}\hbar\omega\left\{1 + \sqrt{1-\lambda}\right\}$$

The function

$$f(\lambda) = \sqrt{1-\frac{\lambda}{2}} - \frac{1}{2}\left\{1+\sqrt{1-\lambda}\right\}$$

has a minimum at $\lambda = 0$, where $f(0) = 0$, and is otherwise positive (for $\lambda \leqslant 1$) as expected in part c).

Problem: A long straight wire is carrying a steady current I. A charge q at a distance r is moving with velocity v parallel to the wire. For the special case where the velocity v is the same as that for the charge carriers in the wire, calculate the following quantities in the rest frame of q: a) the charge density of the wire and b) the electrical force the wire exerts on q. Transform this force to the rest frame of the wire and show that it is precisely the same as the magnetic force exerted on the moving charge in this frame. **Hint:** the wire is electrically neutral in its rest frame.

Solution: (I'll treat the charge carriers as positive, to avoid an irrelevant minus sign.)

Write $\lambda^0_\pm$ for the charge density of positive and negative charges when at rest in the wire. The presence of the current and the neutrality of the wire requires

$$\frac{\lambda^0_+}{\sqrt{1-\beta^2}} + \lambda^0_- = 0 \quad ; \quad \beta = v/c \quad . \tag{1}$$

In the rest frame of q, $\lambda^0_+/\sqrt{1-\beta^2} \rightarrow \lambda^0_+$ and $\lambda^0_- \rightarrow \lambda^0_-/\sqrt{1-\beta^2}$, so q sees a line charge density

$$\lambda_{NET} = \lambda^0_+ + \frac{\lambda^0_-}{\sqrt{1-\beta^2}} \tag{2}$$

$$= \lambda^0_-(-\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}) \tag{3}$$

$$\Rightarrow E(r) = \frac{\lambda_{NET}}{2\pi\epsilon_0 r} = \frac{\lambda^0_-}{2\pi\epsilon_0 r}(-\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}) \quad , \tag{4}$$

so the electrical force in q's rest frame is $F_q = qE(r)$, directed towards the wire.

To transform back to the wire frame, write

$$F_q = \frac{dp'_\perp}{dt'} = \frac{1}{\sqrt{1-\beta^2}} \frac{dp_\perp}{dt} \tag{5}$$

so that the force F_w in the wire frame is

$$F_w(r) = \sqrt{1-\beta^2}F_q(r) = \sqrt{1-\beta^2}\frac{q\lambda_-^0}{2\pi\epsilon_0 r}(-\sqrt{1-\beta^2}+\frac{1}{\sqrt{1-\beta^2}}) = \frac{q\lambda_-^0}{2\pi\epsilon_0 r}(-1+\beta^2+1) = \frac{q\lambda_-^0 v^2}{2\pi\epsilon_0 c^2 r} \quad . \tag{6}$$

This can be rewritten as

$$F_w(r) = \frac{q\lambda_-^0 v^2}{2\pi\epsilon_0 c^2 r} = qv\mu_0\frac{\lambda_-^0 v}{2\pi r} = qv\frac{\mu_0 I}{2\pi r} = qvB(r) \quad , \tag{7}$$

with a direction (towards the wire) consistent with the right-hand-rule.

Weinberg (Relativistic)

The cosmic microwave background radiation has a temperature of 3 K. A cosmic ray proton of sufficient energy can collide with a microwave background photon and form a pion through the process

$$p + \gamma \rightarrow p + \pi$$

Assuming a photon of average energy, what is the minimum proton energy (in electron volts) needed for this process to occur?

Sec IV #3
Weinberg
DEC 1

GZK Problem - Solution

We want $p + (\text{CMB}\,\gamma) \longrightarrow p + \pi$
$(p_1) \quad (k) \qquad (p_2) \quad q$

$\Rightarrow (p_1 + k)^2 = (p_2 + q)^2 > (m_p + m_\pi)^2$

$\hookrightarrow m_p^2 + 2p_1 \cdot k + 0$

But $2p_1 \cdot k = 2\left(E_p k - \sqrt{E_p^2 - m^2}\, k \cos\theta\right)$
$\approx 2 E_p k (1 - \cos\theta)$

This is maximized by $\cos\theta = -1$

$\Rightarrow \cancel{2}\, 4E_p k > (m_\pi^2 + 2m_p m_\pi) = m_\pi (2m_p + m_\pi)$

$$E_p > \frac{m_\pi (2m_p + m_\pi)}{4k}$$

k = avg E of CMB photons $= 2.7\,T$
$\approx 7 \times 10^{-4}$ eV

$$E_p > \frac{(1.4 \times 10^8\,\text{eV})(2.0 \times 10^9\,\text{eV})}{4\,(7.0 \times 10^{-4}\,\text{eV})} \approx 10^{20}\,\text{eV}$$

Solution

Since quantum mechanics says that angular momentum is quantized, at low temperatures rotational motion of the H_2 molecule is not possible. With only translations, we have $C_V = \frac{3}{2}R$, as for a monatomic gas.

Around 100K, the graph shows that the heat capacity changes and this is due to rotational motion being excited by the average thermal energy. Equipartition then gives

$$E_{\text{rot}} = \frac{L^2}{2I} \sim k_B T \tag{1}$$

Since $L \sim \hbar$, we only need to know I to determine (roughly) Planck's constant.

We can estimate I from the Bohr radius, $a_0 = 0.053\text{nm}$, giving $I \sim 2ma_0^2$. This gives

$$I = 2 \times \frac{2 \times 10^{-3}\,\text{kg}}{6.02 \times 10^{23}} \times (5.3 \times 10^{-11}\,\text{m})^2 = 1.87 \times 10^{-47}\,\text{kg}\,\text{m}^2 \tag{2}$$

If we use $T = 100$ K, we find

$$\begin{aligned} \hbar^2 &\sim 2Ik_BT & (3)\\ &= 2 \times (1.87 \times 10^{-47}\,\text{kg}\,\text{m}^2) \times (1.38 \times 10^{-23}\,\text{J/K}) \times (100\,\text{K}) & (4)\\ &= 5.16 \times 10^{-68}\,\text{J}^2\,\text{s}^2 & (5) \end{aligned}$$

So we estimate

$$\hbar \sim 2 \times 10^{-34}\,\text{J}\,\text{s} \tag{6}$$

Shaevitz

Section 4, Relativity/Appl QM, # 4
"Parity of pin"
Identified by Miklos 12/14/07

~~General~~ XXXX Physics Problem:

The π^- meson interacts with the deuteron from an s orbital state to form two neutrons.

$$\pi^- + d \rightarrow n + n$$

a) Explain why the neutrons necessarily need to be in the state L=1, S=1, and J=1.

b) From this process, deduce the parity of the pion.

(A deuteron has J^P ($\text{spin}^{\text{parity}}$) = 1^+, the neutron has $J^P = 1/2^+$ and pions have zero spin)

Solution:
Neutrons are identical fermions and need to be in an anti-symmetric state

a)

$$0^? + 1^+ \rightarrow \tfrac{1}{2}^+ + \tfrac{1}{2}^+$$

$S = 1 \qquad J = 1$

$L = 0 \qquad L = 0$ and $S = 1$ not allowed since totally symmetric

$L = 1$ and $S = 0$ not allowed since totally symmetric

$L = 1$ and $S = 1$ allowed since symmetric $\otimes$ anti-symmetric

b)

LHS	RHS	
$(?)\,(+)\,(+) = ?$	$(+)\,(+)\,(-) = -$	$\Rightarrow$ Parity (?) of pion = (-)

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 18, 2008
9:00 AM – 11:00 AM

General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5(General Physics) Question 7, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

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Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Consider an ideal gas that consists of N particles. The gas has specific heat C_P, at constant pressure, and specific heat C_V, at constant volume, that are both independent of temperature.

(a) Find the work done in an isothermal change of volume from V_1 to V_2. In the isothermal process the gas is in contact with a thermal bath that is kept at a constant temperature T.

(b) Evaluate the quantity of heat Q absorbed in the process described in (a).

(c) The gas undergoes a change in volume from V_1 to V_2 that takes place at constant pressure (isobaric). The gas is thermally isolated and thus unable to exchange heat with its environment. Find the change in temperature $T_2 - T_1$.

2. In an Ar-ion laser, green light (515 nm) is emitted by a transition of Ar^+ ions in a discharge. The radiative lifetime of the relevant transition is $\tau_{sp} = 10$ ns and its measured linewidth is $\Delta\nu = 3$ GHz. The discharge has an effective temperature of $T = 1000$ K and is at a pressure of 0.1 atmosphere. The laser cavity consists of parallel plane mirrors separated by 1 m.

(a) Estimate the contribution to the linewidth from radiative decay (the natural linewidth).

(b) Estimate the contribution to the linewidth from Doppler broadening.

(c) Estimate the contribution to the linewidth from pressure broadening (collisions).

(d) Estimate how many different longitudinal modes the laser oscillates in.

(e) If the laser is operated in a pulsed mode, estimate the duration of the shortest pulse that it can support.

(f) If the average power of the laser is 10 W, what would be the peak power achievable for the laser operating in a pulsed mode in which a single pulse travels back and forth in the cavity? What would be the peak intensity (irradiance) of such a pulse focused to a diffraction-limited spot?

3. A particle slides without friction through a tunnel connecting two arbitrary points on the Earth's surface. The path of the tunnel is perfectly straight. You can model the Earth as a non-rotating sphere of uniform density.

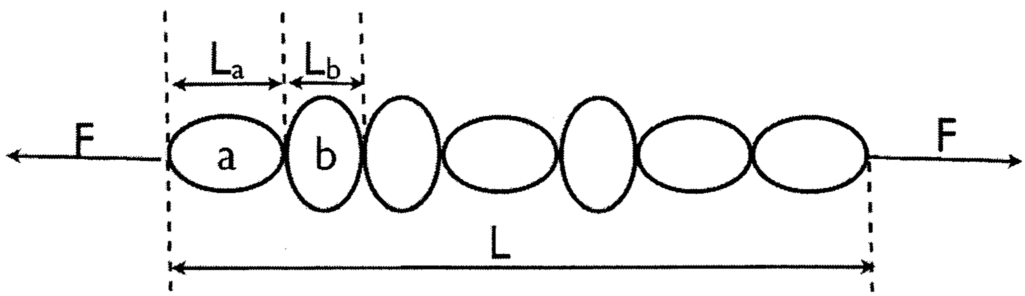
(a) Show that under the influence of gravity the particle undergoes simple harmonic motion in the tunnel. (You can assume the amplitude is small enough that the particle never leaves the tunnel.)

(b) What is the period of oscillation?

(c) Show that the period can be expressed in terms of Newton's constant and the density of the Earth, and is independent of where the endpoints of the tunnel are located.

4. Commencement ceremonies are displayed on huge LED-based television displays. Estimate how close you can get to these displays and still see an image and not the individual LEDs. Please feel free to select reasonable values for the necessary quantities.

5. Consider a one-dimensional chain consisting of N molecules which exist in two configurations, a and b, with corresponding energies E_a and E_b and lengths L_a and L_b. The molecules are in contact but do not otherwise interact (see figure). The chain is subject to a tensile force F and is in equilibrium with a thermal reservoir of temperature T.



(a) Write down the partition function for the system.

(b) Assume that $E_a > E_b$ and $L_a > L_b$. Make a sketch of the average length $< L >$ in the absence of tensile force, $F = 0$, as a function of temperature T. Give the high and low temperature limits and the characteristic temperature at which the changeover between the two limits occurs.

(c) Calculate the average length $< L >$ as a function of F and T.

6. Consider a spherical gas cloud in otherwise empty space. No forces act on the gas besides its own gravity and pressure. The gas molecules have mass m, and an unknown number density $n(r)$ that depends on the distance from the cloud's center (r). Assuming thermal equilibrium, that is, uniform temperature throughout:

(a) Derive the differential equation that $n(r)$ has to obey in order for a stationary configuration to exist.

(b) Solve the equation you derived assuming that $n(r)$ scales like a power of r.

(c) Compute the total mass of the cloud.

(d) What can you conclude about astrophysical gas clouds from your result in part (c)?

Pinczuk

Section 5 - General - I _ Problem #1

(a) $$W = \int_{V_1}^{V_2} P\,dV = Nk_B T \int_{V_1}^{V_2} \frac{dV}{V}$$

$$= Nk_B T \ln \frac{V_2}{V_1} = Nk_B T \ln \frac{P_2}{P_1}$$

because for an ideal gas

$$PV = Nk_B T$$

where $Nk_B = R$ = the ideal gas constant

(b) For an isothermal process there is no change in the internal energy:

$$\Delta U = W + Q = 0$$

$$Q = - W$$

(c)

$$\Delta U = C_V (T_2 - T_1)$$

$$\Delta U = W = P(V_2 - V_1)$$

$$T_2 - T_1 = \frac{P}{C_V} (V_2 - V_1)$$

AP.

Heinz - Problem 2

(a) $\Delta\nu_{sp} = \frac{1}{2\pi\tau_{sp}} = 16\,MHz$

(b) $\Delta\nu_D \approx 2\frac{\bar{v}_z}{c}\nu = \frac{2\bar{v}_z}{\lambda} \approx 2\frac{\sqrt{k_B T/M}}{\lambda} = \frac{2\times 680\,m/s}{511\,nm} = 2.7\,GHz$

(c) $\Delta\nu_{coll} = \frac{f_{coll}}{\pi}$

$f_{coll} \approx \bar{v}\,\pi d^2 n = \sqrt{3}\,\bar{v}_z\,\pi d^2 n$, with $d \approx 1\,\mathring{A}$ atomic diameter

$\approx 110\,MHz$

$n = 6\times10^{23}/210\,\ell$

$= 3\times10^{24}\,m^{-3}$

$\Delta\nu_{coll} \approx 35\,MHz$

(d) $\Delta\nu_{mode} = \frac{c}{2L} = 150\,MHz$

Number of accessible modes $\sim \frac{\Delta\nu}{\Delta\nu_{mode}} = 20$ (2x if you include polarization)

(e) By the time-frequency uncertainty relation

$\tau \geq (4\pi\Delta\nu)^{-1} = 26\,ps$

(f) The longest separation possible between the pulses is one cavity roundtrip of $2m/3\times10^8\,m/s = 6.7\,ns$. Then

$\bar{P} = P_{peak}\frac{25\,ps}{6.7\,ns} \Rightarrow P_{peak} = \bar{P}\frac{6.7\,ns}{25\,ps} = 2.7\,kW$

$I_{peak} = \frac{P_{peak}}{\lambda^2} = 1\times10^{16}\,W/m^2$

(Approximate answers are fine for all questions. Relations above are approximately correct for $\Delta\nu$ as FWHM)

Section 5, #3

Kabat

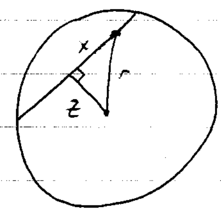
JAN - 3 2008

General I #3 - falling through the earth

$M(r) = \frac{4}{3}\pi r^3 \rho$

$F = \frac{G M(r) m}{r^2} = \frac{4}{3}\pi r \rho G m$

$V(r) = \int_0^r dr\, F = \frac{4}{6}\pi r^2 \rho G m = \frac{4}{6}\pi \rho G m (x^2+y^2+z^2)$



energy construation

$\frac{1}{2}m\dot{x}^2 + \frac{4}{6}\pi \rho G m x^2 = \text{const.}$

harmonic oscillator with $\omega^2 = \frac{4}{3}\pi \rho G$

period $\frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{\rho G}}$ independant of tunnel endpoints

Sec 5, # 4
Marta

Choose the following realistic ranges for the relevant parameters to define the range of the acceptable answers:

Aperture range for human eye (~pupil): $D = 2 - 8 \times 10^{-3}$ m

Wavelength range for visible light: $\lambda = 400 - 700 \times 10^{-9}$ m

Half pitch of LED displays (outdoor and indoor): $h = 3 - 8 \times 10^{-3}$ m

Minimum distance for diffraction limited seeing: $L = ???$ m

From the Rayleigh criterion and some trigonometry (small angles $\sin\alpha \approx \alpha$):

$$L \approx h\,D / 1.22\,\lambda$$

So the lower bound can be as small as:

$$L_{MIN} \approx h_{MIN}\,D_{MIN} / 1.22\,\lambda_{MAX} = 3\text{mm}\ 2\text{mm} / 1.22\ \ 700\text{ nm} = \mathbf{7\ m}$$

So the upper bound can be as large as:

$$L_{MAX} \approx h_{MAX}\,D_{MAX} / 1.22\,\lambda_{MIN} = 8\text{mm}\ 8\text{mm} / 1.22\ \ 400\text{ nm} = \mathbf{131\ m}$$

However, the nominal distance in nice sunlight for decent outdoor displays for peak eye sensitivity:

$$L_{NOM} \approx h\,D / 1.22\,\lambda = 8\text{mm}\ 3\text{mm} / 1.22\ \ 550\text{nm} = \mathbf{35\ m}$$

Please note that often times not the diffraction limit is the limiting factor but the resolution of the retina.

Solution: Millis Statistical mechanics

(a)

$$Z = \left(e^{-\frac{E_a - FL_a}{T}} + e^{-\frac{E_b - FL_b}{T}}\right)^N \tag{1}$$

(b) As $T \to 0$ all molecules will be in lower energy state, so $L = NL_b$. As $T \to \infty$ each molecule has equal probability of being in state a and state b so $L = 0.5N\,(L_a + L_b)$. Characteristic crossover scale is $T \approx E_a - E_b$.

(c)

$$\begin{aligned} < L > &= T\frac{\partial lnZ}{\partial F} \\ &= N\frac{\left(L_a e^{-\frac{E_a - FL_a}{T}} + L_b e^{-\frac{E_b - FL_b}{T}}\right)}{e^{-\frac{E_a - FL_a}{T}} + e^{-\frac{E_b - FL_b}{T}}} \end{aligned} \tag{2}$$

Sec 5, # 6
Nicolis

2008 Problems and Solutions – A. Nicolis

1. Consider a spherical gas cloud in otherwise empty space. No forces act on the gas besides its own gravity and pressure. The gas molecules have mass m, and an unknown number density $n(r)$ that depends on the distance from the cloud's center (r).

Assuming thermal equilibrium, that is uniform temperature throughout:

(a) Derive the differential equation that $n(r)$ has to obey in order for a stationary configuration to exist.

(b) Solve the equation you derived assuming that $n(r)$ scales like a power of r.

(c) Compute the total mass of the cloud.

(d) What does the last result mean?

Solution.

(a) The pressure is $P = n(r)kT$; the gravitational force acting on every molecule is

$$F_g(r) = G\frac{mM(r)}{r^2} = 4\pi G\frac{m^2}{r^2}\int_0^r n(r')r'^2 dr' \,. \tag{1}$$

directed inward. Balancing pressure gradient and gravitational pull we get

$$kT\frac{dn}{dr} = -4\pi G\frac{m^2}{r^2}n(r)\int_0^r n(r')r'^2 dr' \,. \tag{2}$$

We can get rid of the integral on the r.h.s. by deriving with respect to r (after an obvious manipulation).

$$kT\frac{d}{dr}\left(r^2\frac{1}{n(r)}\frac{dn}{dr}\right) = -4\pi G m^2\, n(r) r^2 \,. \tag{3}$$

(b) The ansatz $n(r) = A\, r^\alpha$ yields $n(r) \propto 1/r^2$.

(c)

$$M_{\rm tot} = m\, 4\pi\int_0^\infty n(r) r^2 dr = \infty \tag{4}$$

(d) That a realistic (=finite mass), self-gravitating gas cloud will not reach thermal equilibrium.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 18, 2008
11:10 AM – 1:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6(General Physics) Question 6, etc.).

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted. Questions should be directed to the proctor.

Good luck!!

1. A black hole of mass M can be thought of, schematically, as a spherical region of radius $r_M = 2GM$ (working in $c = 1$ units) where nothing that enters can escape. r_M is known as the Schwarzschild radius. Also, whenever some mass (or energy) M is confined within a sphere of radius r_M, the mass undergoes gravitational collapse and becomes a black hole with the same mass.

Consider a gas of black holes, all with identical masses M, uniformly distributed inside a large cubic box with volume $V = L^3$. Assume that the black holes have negligible velocities.

(a) Show that the black-hole gas can never be so dense as to have the typical distance between two nearby black holes of the order of their Schwarzschild radius.

(b) For a given L and M, estimate the maximum allowed number density of black holes.

[For both questions ignore all numerical factors like 2 and π.]

2. Consider a system of non-interacting particles of spin ½. Each particle has mass m, and is constrained to move with non-relativistic momentum on the very flat surface of superfluid liquid He kept at temperature $T \to 0$. The components of the particle momentum are restricted to the (x,y) plane that defines the surface of the liquid. The interactions between particles and the liquid He substrate are negligible. The area is $A = L^2$, where L is the length of the surface along the x- and y-directions. The number of particles per unit area is n.

(a) Obtain the energy levels E, assuming cyclic boundary conditions. [In cyclic boundary conditions the wavefunctions at position (x,y) are identical to those at positions $(x+L, y)$, $(x, y+L)$, and $(x+L, y+L)$].

(b) Use the results in (a) to obtain the expression for the density of states $g(E)$. Note that $g(E)dE$ gives the number of single-particle energy levels in the range between E and $E+dE$.

(c) Consider the limit of temperature $T = 0$. Obtain the energy difference between the lowest and the highest energy states that are occupied by the particles.

(d) What is the average energy per particle at $T = 0$?

(e) The temperature is raised slightly, so that ΔT is much smaller than the average energy per particle. Describe in words the changes that occur in the system. What is the expected temperature dependence of the change in the total energy of the system?

3. Estimate very roughly, from the heat of vaporization of water ($L \sim 2\times10^3$ Joules/gram), Avogadro's number ($N_A \sim 6\times10^{23}$) and whatever natural constants you might need:

(a) The surface tension of water;

(b) The speed of very small wavelength (λ) ripples on an otherwise flat surface of water;

(c) The lowest frequency (density preserving) oscillation of a drop of water with mass M;

(d) The radius r of capillary tubes inside a tree which bring water from the tree's roots to its leaves 30m higher up.

4. Consider a spherical (radius R) container enclosing very hot (temperature T) fully ionized hydrogen with total mass M.

(a) Estimate roughly the total photon energy contained by the sphere.

(b) About how long will it take for most of the photons of part (a) to escape due to diffusion of photons to the sphere surface? Assume that photons interact with the ionizing hydrogen only by Thomson scattering on free electrons.

(c) Use (a) and (b) to estimate the photon luminosity (L) from the sphere.

(d) Suppose the sphere is held together by gravitational attraction of the hydrogen and kept from collapsing by the kinetic motion of the electrons and protons. Show that

$$L = AM^{\alpha}$$

with A and the exponent α independent of R, T and M. Give A and α in terms of the "other constants".

[Express results in terms of M, R, T and other constants from among electron mass (m_e), hydrogen mass (m_H), Boltzmann constant (k_B), electron charge (e), Planck constant/2π ($\hbar$), speed of light (c), and gravitational constant (G).]

5. Consider a 3-dimensional metal, like Na, where one conduction electron comes from one atom. The crystal structure has one atom per unit cell. The lattice constant of this system is about 3 Angstroms, and the melting temperature is about 700 K. The atomic weight is about 50. (These numbers are given just to qualitatively understand the situation. It may not be necessary to use them in answering the following questions.)

Consider the specific heat of this system.

(a) Show that the specific heat from phonons is proportional to T^3 at low temperatures (Debye model). Estimate the minimum wavelength and hence maximum wavenumber of phonons in the metal.

(b) Show that the specific heat from the conduction electrons is proportional to T at low temperatures. We assume that the effective mass of electrons in this metal is equal to that of a bare electron.

(c) At $T = 300$ K, which one is larger: the specific heat from the lattice or that from the conduction electrons? Describe your reasoning.

(d) When the effective mass of the electrons is 100 times that of the bare electron, how does the specific heat from the electron system change from the value for the bare electron mass?

(e) In the so-called Einstein model, where all phonons are taken to have the same characteristic frequency ω_0, derive the temperature dependence of the specific heat in the limit of low temperature.

6. The flux of solar radiation incident on the earth's atmosphere is 1370 Watts per square meter. About 30% of this is reflected immediately; the remainder is absorbed by the Earth, leading to an effective solar constant of roughly 960 W/m^2. The radiation is peaked in the visual range of frequencies.

(a) Calculate the mean temperature of the earth neglecting the effects of the atmosphere.

(b) Model the atmosphere as a single thin layer which is transparent to the solar radiation, but which totally reabsorbs the (infrared) frequencies emitted by the Earth. Calculate the mean surface temperature in the presence of such a layer.

(c) Assume the density of infrared absorbing gases in the atmosphere doubles. Model this as two thin layers, and calculate the mean surface temperature.

The Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$.

Sec 6, # 1
Nicolis

2. A black hole of mass M can be thought of, schematically, as a spherical region of radius $r_M = 2GM$ (we work in $c = 1$ units) from where nothing that enters can escape. r_M is known as the Schwarzschild radius. Also, whenever some mass (or energy) M is confined within a sphere of radius r_M, the mass undergoes gravitational collapse and becomes a black hole with the same mass.

Consider a gas of black holes, all with identical masses M, uniformly distributed inside a large cubic box with volume $V = L^3$. Assume that the black holes have negligible velocities.

8 *(a)* Show that the black-hole gas can never be so dense as to have the typical distance between two nearby black holes of order of their Schwarzschild radius.

7 *(b)* For given L and M, estimate the maximum allowed number density of black holes.

[For both questions ignore all numerical factors like 2 and π]

Solution.

(a) The problem in making the gas very dense is that when we bring many black holes together they tend to collapse into a larger black hole. Indeed if we have $N \gg 1$ of them, their overall mass is NM, with a corresponding Schwarzschild radius $r_{NM} = Nr_M$. On the other hand, if we imagine that their typical distance is of order of their radius r_M, they all fit in a sphere of radius

$$R \sim N^{1/3} r_M \ll N r_M = r_{NM} \,. \tag{5}$$

Hence, well before they can get so close to one another, they all collapse into a larger black hole.

(b) Call n the black-hole number density. In a sphere of radius R there are $\sim nR^3$ black holes, their total mass is $\sim nR^3 M$, the corresponding Schwarzschild radius is $\sim nR^3 r_M$. For the whole sphere not to collapse into a black hole we want that its radius be larger than its Schwarzschild radius,

$$R \gtrsim nR^3 r_M \,, \tag{6}$$

from which we immediately get $n \lesssim 1/(R^2 r_M)$. The r.h.s. depends on the radius of the sphere we consider. The bound is more stringent for larger R. The absolute bound is thus obtained by taking the largest possible sphere, $R \sim L$,

$$n_{\max} \sim \frac{1}{L^2 r_M} \,. \tag{7}$$

That is, the typical distance among nearby black holes is always parametrically larger than r_M, and at best of order $d_{\min} \sim (L^2 r_M)^{1/3}$.

Pinczuk

Section 6 - General II Problem #2

(a) Quantization of momentum yields

$$p_x = \hbar \frac{2\pi n_x}{L} \; ; \quad p_y = \hbar \frac{2\pi n_y}{L}$$

where n_x and n_y are positive or negative integers.

The energy levels are:

$$E(n_x, n_y) = \frac{\hbar^2}{2m} \left(\frac{2\pi}{L}\right)^2 \left[n_x^2 + n_y^2\right]$$

(b)

$$g(E) = \frac{m}{\pi \hbar^2}.$$ (including spin degeneracy)

independent of energy

(c) The number of states per unit area is:

$$dn = g(E)\, dE$$

$$n = \frac{m}{\pi \hbar^2} \quad E_F$$

$$E_F = \frac{\pi \hbar^2 n}{m}$$

(d) $$\langle E \rangle = \int_0^{E_F} E\, g(E)\, dE$$

$$\langle E \rangle = \frac{m}{2\pi\hbar^2} E_F^2$$

From (c) $$\frac{m}{\pi\hbar^2} = \frac{n}{E_F}$$

$$\langle E \rangle = \frac{1}{2}\frac{n}{E_F} E_F^2 = \frac{1}{2} n E_F$$

$$\langle \varepsilon \rangle = \frac{1}{2} E_F$$

(e) The number of particles that change energy is proportional to ΔT

The typical energy change per particle is also proportional to ΔT

The total change in energy is proportional to $(\Delta T)^2$

$$\Delta \varepsilon \sim (\Delta T)^2$$

Section 6, # 3

General Physics Mal Ruderman

Problem 1

Estimate very roughly, from the heat of vaporization of water ($L \sim 2 \times 10^3$ joules/gram), Avogadro's number ($N_A \sim 6 \times 10^{23}$), and whatever ~~other~~ natural constants you might need

a) the surface tension of water ;

b) the speed of very small wavelength (λ) ripples on an otherwise flat surface of water ;

c) the ~~first~~ lowest frequency (density preserving) oscillation of a drop of water with mass M ;

d) the radius (r) of capillary tubes inside a tree which bring water from the tree's roots to its leaves 30 meters higher up.

Section 6, #4.

General
Problem #2
2.

Mal Ruderman

How bright are typical stars?
Consider a spherical (radius R) container enclosing ~~the~~ very hot (temperature T) fully ionized hydrogen with total mass M.

~~a)~~ [Express results below in terms of M, R, T and other constants from among ~~the~~ electron mass (m_e) hydrogen mass (m_H), Boltzmann constant (k_B), electron charge (e), Planck constant/2π ($\hbar$), speed of light (c), and gravitational constant (G).]

a) Estimate roughly the total ~~contained~~ photon energy contained by the sphere..

b) About ~~How~~ how long will it take for most of the photons of a) to escape if this happens to any photon which diffuses to the sphere surface? Assume that diffusing photons interact with the hydrogen they diffuse through only by Thomson scattering on free electrons

c) Use a) and b) to estimate the photon luminosity (L) from the sphere

General Physics
Problem #2 page 2

a) If the sphere is held together by gravitational attraction of the hydrogen and kept from collapsing by the kinetic motion of the electrons and protons

$$L = A M^{\alpha}$$

with A and the exponent α independent of R, T, and M. Give ~~these~~ A and α in terms of the "other constants".

General Answers   Problem #2   Mal Ruderman

a) $E_{photons} \sim R^3 \, k_B T \left(\frac{k_B T}{\hbar c}\right)^3$

b) $time(t) \sim \frac{R^2}{\lambda_{mfp} c}$

$\lambda = \frac{1}{n_b \sigma_T}$ $n_e = \frac{M}{m_H}$

$\boxed{\sigma_T \sim \left(e^2/mc^2\right)^2}$

$\therefore \; t \sim \frac{M\sigma_T}{c\, m_H R} \sim \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{M}{m_H}\right)\left(\frac{1}{cR}\right)$

c) $L \sim \frac{E_{photons}}{t}$

d) $\frac{GM^2}{R} \sim k_B T \times \frac{M}{m_H}$

$L \sim \frac{G^4 m_H^5}{\hbar^3 c^2 \sigma_T} \, M^3$

$\therefore \; \alpha = 3 \; , \; A = \frac{G^4 m_H^5 \, m_e^2 c^2}{\hbar^3 e^4}$

III. Part 6
Prob. #5.

Tomo Uemura

(a) Phonon $\Rightarrow$ Bose Stat.

Linear Dispersion $\varepsilon = ck$.

$D(k) \sim k^2 dk \quad D(\varepsilon) \sim \varepsilon^2 d\varepsilon$

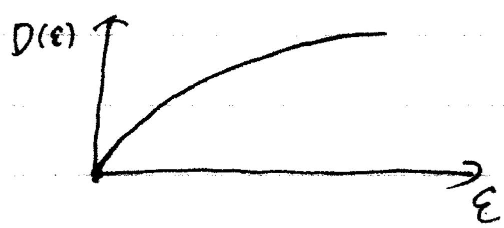
$$n \propto \frac{1}{\exp\left(\frac{\varepsilon}{kT}\right) - 1}$$

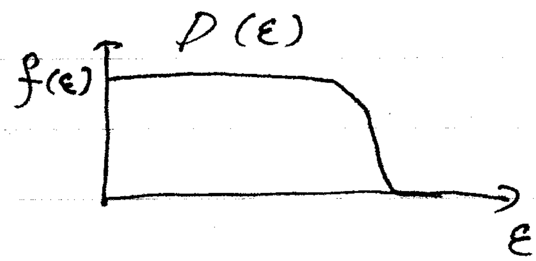
for low-T $\quad n \sim \exp\left(-\frac{\varepsilon}{kT}\right)$

$$\langle E \rangle \sim \int \varepsilon D(\varepsilon)\, d\varepsilon$$
$$\sim \int \varepsilon^3 \exp\left(-\frac{\varepsilon}{kT}\right) d\varepsilon$$
$$\propto T^4$$

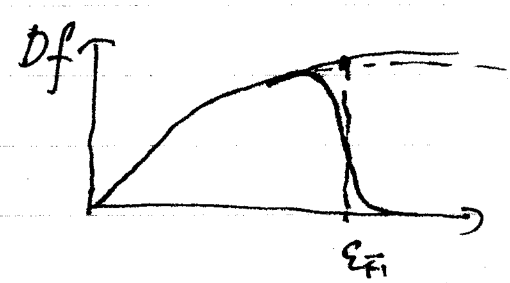
$$C = \frac{d\langle E \rangle}{dT} \sim T^3$$

(b) Electrons : Fermions.





$f(\varepsilon)$: Fermi dist.



Excitable states in width $\sim kT$
each state carries energy $\sim kT$

$\langle \varepsilon \rangle \propto T^2 \qquad C \propto T$

(c). In the high-T limit, both C_{ph} and C_{el} have classical value for a gas.

Crossover to classical region characterized by $\Theta_{Debye} \sim \Theta_{melting} \sim 700°K$ for phonons

Fermi Temp for electrons
↳ order 10^4 K.

At room temp, electron system is still degenerated.

Thus $C_{phonon} \gg C_{electron}$.

(d). $C_{electron}$ or γ Sommerfeld Coeff

$\gamma \propto D(\varepsilon_F) \propto m^*$

∴ 100 times larger, scaling with m^*

(e) $\langle E \rangle \sim n \hbar \omega_0$

at low-T $\langle E \rangle \sim \frac{\hbar\omega}{1 - e^{-\hbar\omega/kT}}$

$\sim \hbar\omega \cdot \exp\left(\frac{\hbar\omega}{kT}\right)$ @ low T

$$C \sim \boxed{\frac{d\langle E \rangle}{dT} \sim \frac{\exp\left(-\frac{\hbar\omega}{kT}\right)}{T^2}}$$

Zajc

Problem: The flux of solar radiation incident on the earth's atmosphere is 1370 Watts per square meter. About 30% of this is reflected immediately; the remainder is absorbed by the earth, leading to an effective solar constant of roughly 960 W/m^2. The radiation is peaked in the visual range of frequencies. a) Calculate the mean temperature of the earth neglecting the effects of the atmosphere. b) Model the atmosphere as a single thin layer which is transparent to the solar radiation, but which totally reabsorbs the (infrared) frequencies emitted by the earth. Calculate the mean surface temperature in the presence of such a layer. c) Assume the density of infrared absorbing gases in the atmosphere doubles. Model this as two thin layers, and calculate the mean surface temperature.

The Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}$W m^{-2} K^{-4}.

Solution: a) The earth intercepts radiation with cross section πR^2 and re-radiates with cross section $4\pi R^2$, so in the absence of atmospheric heat blankets

$$\frac{1}{4}\Phi_{eff} = \sigma T^4 \Rightarrow 240 \text{ W/m}^2 = 5.67 \times 10^{-8} \text{W m}^{-2} \text{ K}^{-4} T^4 \Rightarrow T \approx 255 \text{ K} \quad .$$

b) You can solve this by drawing a picture and counting photons, which is mathematically equivalent to

$$\begin{aligned} \Phi_0 &= \Phi_1 && (1) \\ \Phi_0 + \Phi_1 &= \Phi_S \quad , && (2) \end{aligned}$$

where $\Phi_0 \equiv \Phi_{eff}/4$ (as in part "a"), Φ_S is the flux leaving the earth's surface and Φ_1 is the flux from one side of the thin layer. This immediately gives $\Phi_S = 2\Phi_0$, or

$$T = 2^{1/4} \times 255 \text{ K} \approx 303 \text{ K}$$

(a fairly reasonable estimate).

c) Repeating, with the outermost layer labeled as "2" :

$$\begin{aligned} \Phi_0 &= \Phi_2 && (3) \\ \Phi_S + \Phi_2 &= 2\Phi_1 && (4) \\ \Phi_0 + \Phi_1 &= \Phi_S \quad , && (5) \end{aligned}$$

giving $\Phi_S = 3\Phi_0$, or

$$T = 3^{1/4} \times 255 \text{ K} \approx 336 \text{ K}$$

(!).