Spring 2011

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

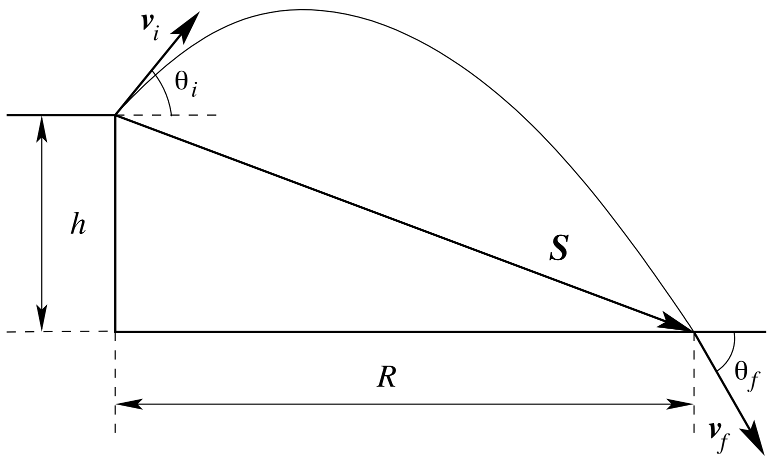
March 23, 2011

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Projectile



A projectile is launched with initial velocity $\boldsymbol{v}_i$ at angle θ_i from height h over level ground in a uniform gravitational field $\boldsymbol{g}$, as shown in the diagram.

(a) If the projectile travels a path of maximal range R_m, what is the relation between the directions of the initial and final velocities? **Hint**: express the vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_f$ in terms of $\boldsymbol{g}$ and the displacement vector $\boldsymbol{S}$. Take a cross product.

(b) Calculate the maximal range R_m.

(c) What is the launch angle θ_m that yields the maximal range?

1. Projectile - Solution

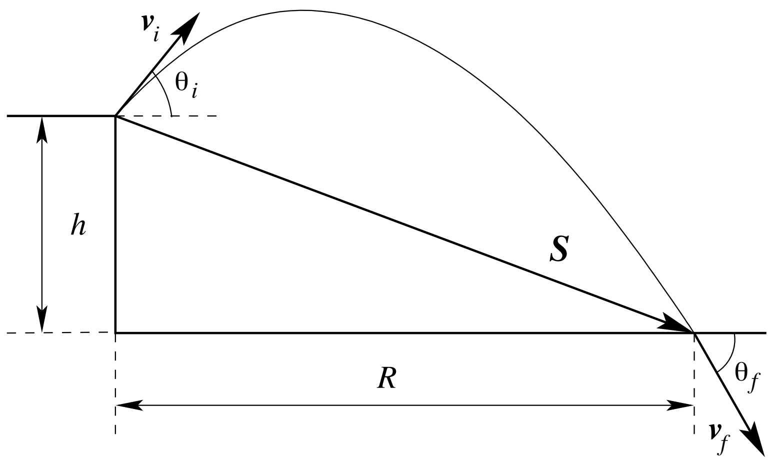


Figure 1:

(a) Kinematics of the motion with a constant acceleration $\boldsymbol{g}$ gives

$$\boldsymbol{S} = \boldsymbol{v}_i t + \frac{1}{2}\boldsymbol{g}t^2, \qquad \boldsymbol{v}_f = \boldsymbol{v}_i + \boldsymbol{g}t. \tag{1}$$

Solve this for $\boldsymbol{v}_i$ and $\boldsymbol{v}_f$:

$$\boldsymbol{v}_i = \frac{\boldsymbol{S}}{t} - \frac{\boldsymbol{g}t}{2}, \qquad \boldsymbol{v}_f = \frac{\boldsymbol{S}}{t} + \frac{\boldsymbol{g}t}{2}. \tag{2}$$

Taking the cross product of these two vectors cancels out the time dependence:

$$\boldsymbol{v}_i \times \boldsymbol{v}_f = \boldsymbol{S} \times \boldsymbol{g}. \tag{3}$$

Equating the magnitudes of these two vector products gives (notice that the range R is exactly the length of the horizontal component of the displacement vector $\boldsymbol{S}$)

$$v_i v_f \sin\beta = gR, \tag{4}$$

where β is the angle between the directions of the initial and final velocities. For a fixed initial speed v_i, the final speed v_f will be uniquely determined by the energy conservation. Therefore, the only variable factor in the left hand side is $\sin\beta$. Its maximal value 1 (achieved at $\beta = \pi/2$) corresponds to the maximal range R_m. Thus, for the motion resulting in the maximal range, the initial and final velocities are perpendicular to each other.

(b) The energy conservation gives

$$\frac{mv_i^2}{2} + mgh = \frac{mv_f^2}{2}, \qquad \Rightarrow \qquad v_f = \sqrt{v_i^2 + 2gh}. \tag{5}$$

Substituting this into Eq. (4) with $\beta = \pi/2$ leads to the answer:

$$R_m = \frac{1}{g} v_i \sqrt{v_i^2 + 2gh}. \tag{6}$$

(c) The horizontal component of the velocity is conserved during the motion, so

$$v_i \cos\theta_i = v_f \cos\theta_f. \tag{7}$$

But $\theta_f = \beta - \theta_i$. For the maximal range this gives $\theta_f = \pi/2 - \theta_m$, and

$$v_i \cos\theta_m = v_f \sin\theta_m, \tag{8}$$

which leads to the answer:

$$\tan\theta_m = \frac{v_i}{v_f} = \frac{v_i}{\sqrt{v_i^2 + 2gh}}. \tag{9}$$

1. Projectile - Brute force solution

In this solution we just find R as a function of v_i, θ_i, h, and g, and minimize with respect to θ_i. The necessary results then appear in the order opposite to the one stated in the problem. To simplify formulas, we replace v_i by v, and θ_i by θ.

First, find the time of flight. Time to go up and the height increment during this time are

$$t_1 = \frac{v}{g}\sin\theta, \qquad h_1 = \frac{v^2}{g}\sin^2\theta. \tag{10}$$

Time to go down is

$$t_2 = \sqrt{2g(h+h_1)} = \frac{v}{g}\sqrt{\sin^2\theta + a}, \qquad a \equiv \frac{2gh}{v^2}. \tag{11}$$

The total time of flight is

$$T = \frac{v}{g}\big(\sin\theta + \sqrt{\sin^2\theta + a}\big). \tag{12}$$

The range R is then

$$\begin{aligned} R = v\cos\theta T &= \frac{v^2}{g}\big(\sin\theta\cos\theta + \sqrt{\sin^2\theta\cos^2\theta + a\cos^2\theta}\big) \\ &= \frac{v^2}{2g}\big(\sin 2\theta + \sqrt{\sin^2 2\theta + 2a(1+\cos 2\theta)}\big), \end{aligned} \tag{13}$$

where in the last equation we have expressed everything in terms of the double angle.

Now we minimize $X = 2gR/v_i^2$ with respect to θ:

$$X'(\theta) = 2\cos 2\theta + 2\frac{\sin 2\theta(\cos 2\theta - a)}{\sqrt{\sin^2 2\theta + 2a(1+\cos 2\theta)}} = 0. \tag{14}$$

Getting rid of the square root and expressing everything in terms of $c \equiv \cos 2\theta$, we get

$$c^2(1+2a-c)(1+c) = (1-c^2)(c-a)^2. \tag{15}$$

Since $c = -1$ is not the solution we need, we cancel out $1+c$. Opening up all the brackets then leads to the solution

$$c_m \equiv \cos 2\theta_m = \frac{a}{a+2} = \frac{gh}{v^2+gh}. \tag{16}$$

Using elementary trigonometry we find from here

$$\cos\theta_m = \sqrt{\frac{a+1}{a+2}}, \qquad \sin\theta_m = \frac{1}{\sqrt{a+2}}, \qquad \tan\theta_m = \frac{1}{\sqrt{a+1}}. \tag{17}$$

Now, substituting Eq. (16) into Eq. (13), we get the maximal range:

$$\begin{aligned} R_m &= \frac{v^2}{2g}\left(\sqrt{1-c_m^2} + \sqrt{1-c_m^2+2a(1+c_m)}\right) \\ &= \frac{v^2}{g}\sqrt{a+1} = \frac{v}{g}\sqrt{v^2+2gh}. \end{aligned} \tag{18}$$

Finally, let us find the angle θ_f when $\theta_i = \theta_m$. Use the same conservation of energy, Eq. (5), and the horizontal component of the velocity, Eq. (7), as before:

$$\cos\theta_f = \frac{v_i}{v_f}\cos\theta_i = \frac{v_i}{\sqrt{v_i^2+2gh}}\cos\theta_i. \tag{19}$$

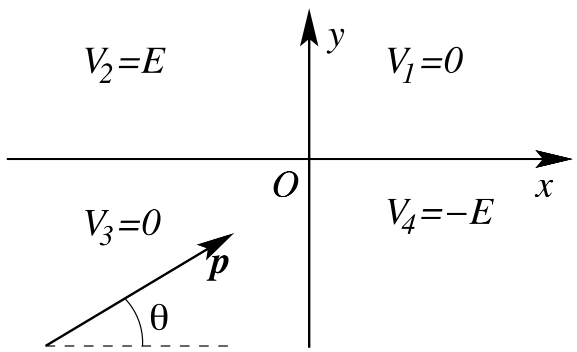
For the motion with the maximal range this is

$$\cos\theta_f = \frac{1}{\sqrt{a+1}}\cos\theta_m = \frac{1}{\sqrt{a+2}} = \sin\theta_m. \tag{20}$$

From this we see that the angle between the directions of the initial and the final velocities is

$$\beta = \theta_i + \theta_f = \pi/2. \tag{21}$$

2. Two-dimensional Scattering



A particle with mass m and energy E is moving in the third quadrant of the $x-y$ plane toward the origin O with momentum $\boldsymbol{p} = \sqrt{2mE}(\cos\theta\hat{x} + \sin\theta\hat{y})$, where $\theta > 0$. The potential in the quadrants of the plane are $V_1 = 0$, $V_2 = E$, $V_3 = 0$, $V_4 = -E$; see the diagram.

(a) Write down the time-independent Schrödinger equation and show that it is separable in x and y coordinates.

(b) Find the scattering solution.

(c) Interpret the obtained solution and determine the probabilities to find the particle in each quadrant when time goes to infinity.

2. Two-dimensional scattering - Solution

(a) In two dimensions the time-independent Schrödinger equation is (in units where $\hbar = 1$)

$$-\frac{1}{2m}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)\psi(x,y)+V(x,y)\psi(x,y)=E\psi(x,y). \tag{22}$$

Notice that the potential $V(x,y)$ can be written as $V(x,y) = V_1(x)+V_2(y)$, where both V_1 and V_2 are step functions:

$$V_1(x)=\begin{cases}0, & \text{for } x<0,\\ -E, & \text{for } x>0,\end{cases} \qquad V_2(y)=\begin{cases}0, & \text{for } y<0,\\ E, & \text{for } y>0.\end{cases} \tag{23}$$

This fact allows us to look for solutions of the Schrödinger equation in the form

$$\psi(x,y)=\psi_1(x)\psi_2(y). \tag{24}$$

Substituting the factorized potential and the ansatz for $\psi(x,y)$ into the equation, we get two one-dimensional problems:

$$\begin{aligned}-\frac{1}{2m}\psi_1''(x)+V_1(x)\psi_1(x)&=E_1\psi_1(x),\\ -\frac{1}{2m}\psi_2''(y)+V_2(y)\psi_2(y)&=E_2\psi_1(y),\end{aligned} \tag{25}$$

where the separation constants E_1, E_2 should add up to E.

(b) The incoming wave with momentum $\boldsymbol{p}$ in the third quadrant (where both $V(x) = V(y) = 0$) is

$$\psi(x,y)=e^{ik_x x}e^{ik_y y}, \tag{26}$$

where

$$k_x=\sqrt{2mE}\cos\theta, \qquad k_y=\sqrt{2mE}\sin\theta. \tag{27}$$

This wave function satisfies the separated equations (25) with

$$E_1=E\cos^2\theta, \qquad E_2=E\sin^2\theta. \tag{28}$$

Using $E_1 = E\cos^2\theta$ the first of the separated equations (25) we get a general solution for ψ_1:

$$\psi_1(x)=\begin{cases}A_1e^{ik_x x}+B_1e^{-ik_x x}, & \text{for } x<0,\\ C_1e^{ik_x' x}+D_1e^{-ik_x' x}, & \text{for } x>0,\end{cases} \tag{29}$$

where

$$k_x' = \sqrt{2m(E+E_1)} = \sqrt{2mE}\sqrt{1+\cos^2\theta}. \tag{30}$$

Since there is no incoming wave in the right hand half-plane, we have $D_1 = 0$. Also, we can set the amplitude of the incoming wave to $A_1 = 1$.

Matching the two expressions for wave function ψ_1 and its derivative at $x = 0$, we get

$$1 + B_1 = C_1, \qquad k_x(1 - B_1) = k_x' C_1. \tag{31}$$

Given the previously determined values of k_x and k_x', the solution is

$$\begin{aligned} B_1 &= \frac{k_x - k_x'}{k_x + k_x'} = -\frac{\sqrt{1+\cos^2\theta} - \cos\theta}{\sqrt{1+\cos^2\theta} + \cos\theta}, \\ C_1 &= \frac{2k_x}{k_x + k_x'} = \frac{2\cos\theta}{\sqrt{1+\cos^2\theta} + \cos\theta}. \end{aligned} \tag{32}$$

Similarly, using $E_2 = E\sin^2\theta$, the general solution of the second of the separated equations (25) is

$$\psi_2(y) = \begin{cases} A_2 e^{ik_y y} + B_2 e^{-ik_y y}, & \text{for } y < 0, \\ C_2 e^{-\kappa_y y} + D_2 e^{\kappa_y y}, & \text{for } y > 0, \end{cases} \tag{33}$$

where

$$\kappa_y = \sqrt{2m(E - E_2)} = \sqrt{2mE}\cos\theta. \tag{34}$$

Discarding the growing solution, we get $D_2 = 0$. Then setting $A_2 = 1$ and matching the two forms of the solution at $y = 0$, we get

$$1 + B_2 = C_2, \qquad ik_y(1 - B_2) = -\kappa_y C_2. \tag{35}$$

The solution is

$$B_2 = -\frac{\kappa_y + ik_y}{\kappa_y - ik_y} = -\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = -e^{-2i\theta}, \tag{36}$$

$$C_2 = 1 - e^{-2i\theta} = 2ie^{-i\theta}\sin\theta. \tag{37}$$

Putting everything together, the scattering solution is $\psi(x, y) = \psi_1(x)\psi_2(y)$, where

$$\psi_1(x) = \begin{cases} e^{ik_x x} + B_1 e^{-ik_x x}, & \text{for } x < 0, \\ C_1 e^{ik_x' x}, & \text{for } x > 0, \end{cases} \tag{38}$$

$$\psi_2(y) = \begin{cases} 2ie^{-i\theta}\sin(k_y y + \theta), & \text{for } y < 0, \\ 2ie^{-i\theta} e^{-\kappa_y y}, & \text{for } y > 0, \end{cases} \tag{39}$$

and the coefficients and momenta involved are given in Eqs. (27, 30, 32, 34).

(c) The factor $\psi_2(y)$ is a standing wave in the lower half-plane, and an exponentially decaying function in the upper half plane. This means that the particle does not propagate into the upper half plane, being reflected off the positive part of the "potential" $V_2(y)$. On the other hand, the factor $\psi_1(x)$ describes a partial reflection and partial transmission in the x direction.

Thus, the incoming wave (26) is totally reflected in the y direction, and partially reflected and partially transmitted in the x direction. The probabilities of finding the particle in four quadrants are, then,

$$p_1 = p_2 = 0, \tag{40}$$

$$p_3 = |B_1|^2 = \frac{1 + 2\cos^2\theta - 2\cos\theta\sqrt{1+\cos^2\theta}}{1 + 2\cos^2\theta + 2\cos\theta\sqrt{1+\cos^2\theta}}, \tag{41}$$

$$p_4 = 1 - |B_1|^2 = \frac{4\cos\theta\sqrt{1+\cos^2\theta}}{1 + 2\cos^2\theta + 2\cos\theta\sqrt{1+\cos^2\theta}}. \tag{42}$$

Notice that p_3 and p_4 are the reflection and transmission coefficient in the one-dimensional problem whose solution is given by $\psi_1(x)$. In this problem the potentials at the positive and negative infinities are different, and the probability fluxes for positive and negative x involve different wave vectors. This is why the reflection coefficient is simply given by $R = |B_1|^2$, but the transmission coefficients is $T = (k'_x/k_x)|C_1|$. It would be a mistake to write $T = |C_1|^2$, as this would violate the conservation of probability.

3. The Hanbury Brown-Twiss Effect

In the 1950's, Hanbury Brown and Twiss showed that one could measure the angular sizes of astrophysical radio sources and stars from correlations of signal intensities, rather than amplitudes, in an array of detectors.

(a) Consider the source a distance L away to consist of a collection of independent point-source radiators of strengths α_i. Simplifying to the situation of two such independent radiators a and b with amplitudes α and β separated by a distance R transverse to the line of sight, calculate the intensity I_1 measured in one of the detectors, averaged over the random phases of the sources.

(b) Calculate the correlation of intensities $\langle I_1 I_2 \rangle$, again averaged over random phases, for two detectors separated by a transverse distance d. Sketch the result as a function of d.

(c) Find the angular size of the source as seen by the detectors by relating the characteristic distance d over which the correlation varies to the angular separation of the sources.

3. Hanbury Brown-Twiss Effect - Solution

(a) Consider sources a and b, and detectors 1 and 2. The wave amplitude at detector 1 from the two sources is

$$A_1 = \frac{1}{L}\Big(\alpha e^{ikr_{1a}+i\phi_a} + \beta e^{ikr_{1b}+\phi_b}\Big)$$

where r_{1a} is the distance from source a to detector 1, etc., and $\phi_{a,b}$ are random phases. The total intensity in detector 1 is thus

$$I_1 = |A_1|^2 = \frac{1}{L^2}\Big(|\alpha|^2 + |\beta|^2 + \alpha^*\beta e^{i(k(r_{1b}-r_{1a})+\phi_b-\phi_a)} + \alpha\beta^* e^{-i(k(r_{1b}-r_{1a})+\phi_b-\phi_a)}\ .$$

(b) Averaging over the random phases ϕ the last two terms average to zero, and one finds as expected that the sources add incoherently:

$$\langle I_1\rangle = \langle I_2\rangle = \frac{1}{L^2}\Big(\langle|\alpha|^2\rangle + \langle|\beta|^2\rangle\Big)\ .$$

On the other hand, multiplying the intensities $I_1 I_2$ before averaging and then averaging over random phases there are additional contributions where the random phases cancel out in the product, resulting in

$$\langle I_1 I_2\rangle = \langle I_1\rangle\langle I_2\rangle + \frac{2}{L^4}\ |\alpha|^2|\beta|^2\cos[k(r_{1a}-r_{2a}-r_{1b}+r_{2b})]\ .$$

The Pythagorean Theorem says that

$$r_{1a} = \sqrt{L^2+(R-d)^2/4} = r_{2b}\ ;\quad r_{2a} = \sqrt{L^2+(R+d)^2/4} = r_{1b}\ .$$

Then for large separation of the sources $L \gg R$, the argument of the cosine is approximately

$$k(r_{1a}-r_{2a}-r_{1b}+r_{2b}) \simeq \frac{2k}{L}\left[\frac{(R-d)^2}{8} - \frac{(R+d)^2}{8}\right] = -\frac{kRd}{L}\ ,$$

so the modulation involves $\cos(kRd/L)$.

(c) One can express $kRd/L = k\ d\ \theta$, where $\theta = R/L$ is the angular size of the source as seen by the observer. Hence one finds θ from the relation $2\pi = (k\ d_{2\pi}\ \theta)$, where $d_{2\pi}$ is the change in separation of detectors 1 and 2 corresponding to one period of the cosine function. Thus $\theta = 2\pi/(k\ d_{2\pi})$.

See also: `http://th-www.if.uj.edu.pl/acta/vol29/pdf/v29p1839.pdf`

4. Scattering Radiation

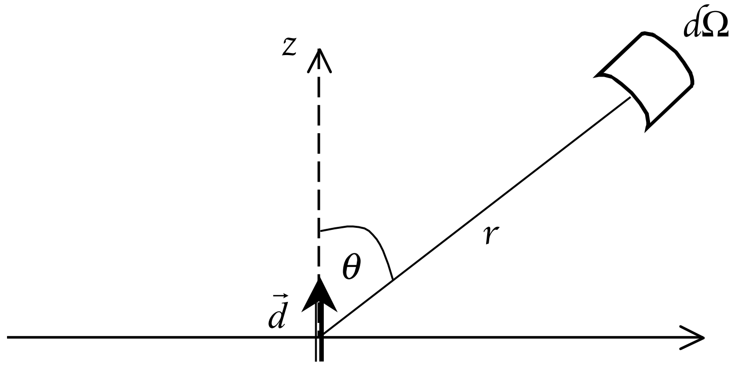
In the dipole approximation, the vector potential due to a localized system of charges and currents varying sinusoidally in time as $\rho(\vec{r}, t) = \rho(\vec{r})e^{-i\omega t}$ and $\vec{J}(\vec{r}, t) = \vec{J}(\vec{r})e^{i\omega t}$ is given by

$$\vec{A}(\vec{r}) = -\frac{ik\vec{d}e^{ikr}}{r}$$

where $k = \omega/c$ and $\vec{d}$ is the dipole vector as shown in the diagram.

(a) Calculate the power radiated into a solid angle $d\Omega$ by $\vec{d}$.

(b) Calculate the effective cross-section for scattering of an electromagnetic wave off a free electron where the effective cross-section is defined as power emitted by the electron into $d\Omega$ divided by the power flux (power per unit area) of the incident wave. Assume the velocity v of the electron is $v \ll c$.



4. Scattering Radiation - Solution

(a)

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left[-\frac{ik\vec{d}e^{ikr}}{r} \right] = \frac{k^2 e^{ikr}}{r} \left[1 - \frac{1}{ikr} \right] \hat{n} \times \vec{d}$$

where $\vec{n}$ = unit vector along r. $\vec{E} = \vec{B} \times \vec{n}$. The Poynting vector gives power/unit area $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}^*$ or energy into $d\Omega$. Keeping only terms $O(1/r)$,

$$dP = \frac{c}{4\pi} r^2 (\frac{k^2}{r})^2 \left(\left[(\vec{n} \times \vec{d}) \times \vec{n} \right] \times (\vec{n} \times \vec{d}) \right) d\Omega = \frac{c}{4\pi} k^4 |\vec{d}|^2 \sin^2 \theta \; d\Omega$$

(b) Consider the free electron responding to the EM wave. Because $v \ll c$, we need only consider $\vec{E}$ force on e$^-$. $E = E_0 \cos \omega t$, therefore

$$m\ddot{r} = eE = eE_0 \cos \omega t \implies \ddot{r} = (eE_0/m) \cos \omega t.$$

So,

$$r(t) = -\frac{eE_0}{m\omega^2} \cos \omega t.$$

The dipole moment at any time is $er = -\frac{e^2 E_0}{m\omega^2} \cos \omega t$. From part (a),

$$dP = \frac{c}{4\pi} k^4 |\vec{d}|^2 \sin^2 \theta \; d\Omega = \frac{ck^4}{4\pi} (\frac{e^2 E_0}{m\omega^2} \cos \omega t)^2 \sin^2 \theta \; d\Omega.$$

For the time averaged power: $(\overline{\cos^2 \omega t} = 1/2)$

$$dP = \frac{ck^4}{c^3 8\pi} \frac{e^4 E_0^2 \sin^2 \theta}{m^2 \omega^4} = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \theta d\Omega$$

or,

$$\frac{d\sigma}{d\Omega} = \frac{\text{energy scattered into } d\Omega}{\text{incident flux}}.$$

Incident flux (time averaged) is $\vec{S} = (c/4\pi) \vec{E} \times \vec{B} = (c/8\pi) E_0^2$. So,

$$\frac{d\sigma}{d\Omega} = \frac{\frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \theta}{c E_0^2 / 8\pi} = \left(e^2/mc^2 \right)^2 \sin^2 \theta.$$

5. Positronium Hyperfine Levels

The spin-dependent terms in the Hamiltonian for positronium (a bound state of an electron e^- and a positron e^+) in its lowest S-wave state involve a hyperfine (spin-spin) interaction between the electron and positron and, in a magnetic field of strength B_0 in the z-direction, an interaction of each magnetic moment with the field, yielding a total effective Hamiltonian

$$H_{\text{eff}} = -A\vec{\mu}_1 \cdot \vec{\mu}_2 - B_0(\vec{\mu}_1 + \vec{\mu}_2) \cdot \hat{\mathbf{z}} \ ;$$

$$\vec{\mu}_1 = -\frac{|e|}{m_e c}\vec{S}_1 \quad ; \quad \vec{\mu}_2 = +\frac{|e|}{m_e c}\vec{S}_2 \quad ,$$

where 1 and 2 refer to the electron and positron, respectively.

(a) Find the energy levels when $B_0 = 0$.

(b) Find the dependence of the levels on B_0 including their asymptotic behavior for large B_0.

5. Positronium Hyperfine Levels - Solution

(a) When $B_0 = 0$, $\langle H_{\text{eff}} \rangle$ depends on the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ since

$$\langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \frac{1}{2}[S(S+1) - S_1(S_1+1) - S_2(S_2+1)] = \left\{ \begin{array}{cc} 1/4 & (S=1) \\ -3/4 & (S=0) \end{array} \right. \quad ; \quad \text{so} \tag{43}$$

$$\langle H_{\text{eff}} \rangle = \left\{ \begin{array}{cc} \alpha & (S=1) \\ -3\alpha & (S=0) \end{array} \right. \quad ; \quad \alpha \equiv \frac{Ae^2\hbar^2}{4m_e^2c^2} \ . \tag{44}$$

The $S = 0$ level is nondegenerate, while the $S = 1$ level is triply degenerate.

(b) When $B_0 \neq 0$, the effective Hamiltonian may be written

$$H_{\text{eff}} = \alpha \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \beta(\sigma_{1z} - \sigma_{2z}) \ , \quad \beta \equiv \frac{B_0|e|\hbar}{2m_e c} \ . \tag{45}$$

Although S is no longer a good quantum number, this Hamiltonian leaves $S_z = S_{1z} + S_{2z}$ invariant. The levels with $S = 1, S_z = 0$ and $S = S_z = 0$ mix with one another due to the $\langle \vec{S}_1 \cdot \vec{S}_2 \rangle$ term. Using

$$\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle = \sigma_{1z}\sigma_{2z} + \frac{1}{2}\left[\sigma_{1-}\sigma_{2+} + \sigma_{1+}\sigma_{2-}\right] \ , \quad \sigma_{\pm} \equiv \sigma_x \pm i\sigma_y \ , \tag{46}$$

we can write the 2×2 mixing matrix for the $S_z = 0$ levels in the basis labeled by $|S, S_z\rangle = |1, 0\rangle \, , |0, 0\rangle$ (setting $\hbar = 1$ in the labels) as

$$H_{\text{eff}} = \left[\begin{array}{cc} \alpha & 2\beta \\ 2\beta & -3\alpha \end{array} \right] \ . \tag{47}$$

where we have used the expectation values

$$\langle 1, 0| \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |1, 0\rangle = 1 \ , \quad \langle 0, 0| \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |0, 0\rangle = -3, \tag{48}$$

$$\langle 1, 0| \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |0, 0\rangle = \langle 0, 0| \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |1, 0\rangle = 0 \ , \tag{49}$$

$$\langle 1, 0| \sigma_{1z} \, |1, 0\rangle = \langle 1, 0| \sigma_{2z} \, |1, 0\rangle = \langle 0, 0| \sigma_{1z} \, |0, 0\rangle = \langle 0, 0| \sigma_{2z} \, |0, 0\rangle = 0 \ , \tag{50}$$

$$\langle 1, 0| \sigma_{1z} \, |0, 0\rangle = \langle 0, 0| \sigma_{1z} \, |1, 0\rangle = 1 \ , \quad \langle 1, 0| \sigma_{2z} \, |0, 0\rangle = \langle 0, 0| \sigma_{2z} \, |1, 0\rangle = -1 \ . \tag{51}$$

The eigenvalues $\lambda_\pm$ of this effective Hamiltonian are the roots of

$$(\alpha - \lambda)(-3\alpha - \lambda) - (2\beta)^2 = 0 \ , \quad \lambda_\pm = -\alpha \pm \sqrt{4\alpha^2 + (2\beta)^2} \ . \tag{52}$$

These eigenvalues reduce to $+\alpha, -3\alpha$ for $B_0 = 0$, and asymptotically approach $-\alpha \pm (2\beta)$ for large B_0. These are the values one would obtain if the electron and positron spins responded independently to the magnetic field.

The eigenvalues for the unmixed states $|1, \pm 1\rangle$ may be calculated using

$$\langle 1, \pm 1 | \vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |1, \pm 1\rangle = 1 \ , \quad \langle 1, \pm 1 | \sigma_{1z} \, |1, \pm 1\rangle = \langle 1, \pm 1 | \sigma_{2z} \, |1, \pm 1\rangle = \pm 1 \ , \tag{53}$$

$$\text{giving} \quad \lambda_{1,\pm 1} = \alpha. \tag{54}$$

In contrast to the eigenvalues $\lambda_\pm$ noted above, these are unaffected by B_0.

6. Short Answer Questions I

(a) The Sun radiates 3.9×10^{26} J/s. Find the equilibrium temperature of an insulated black body facing the Sun at a distance of $d = 2.3 \times 10^8$ km. This is the same distance as Mars is from the Sun. The actual average temperature on the surface of Mars is about 250 K. If this is different from your result, what is the physical mechanism that causes this difference?

(b) In your next breath, approximately how many molecules will you take in that were emitted by Lincoln while giving the Gettysburg Address? The speech was given in 1863 and consisted of 271 words.

(c) What is the mean energy of a diatomic ideal gas of N molecules at temperature T? Consider each molecule as two point particles connected by a spring.

6. Short Answer I - Answers

(a) Dividing the total power emitted by the sun by $4\pi d^2$, the intensity of the sun's radiation at Mars's surface is $J = 600$ W/m^2. If Mars were a perfect black body, then all the radiation it receives is absorbed. The cross-section area for absorption is πR_0^2, where R_0 is Mars's radius. In thermal equilibrium, all the radiation is re-emitted as Mars' own black body radiation via total surface area $4\pi R_0^2$. Thus I, the total energy flux emitted per unit area per unit time by Mars, is simply $J/4$. Using Stefan's Law ($I = \sigma T^4$ where $\sigma = 2\pi^5 k_B^4/15h^3c^2$), we get a temperature of 321 K. The average temperature of Mars being at 250 K is lower than the estimate obtained above. This means Mars reflects a fraction of the solar radiation it receives instead of entirely absorbing all of it. Equivalently, the albedo is nonzero.

(b) Estimate that 10^{23} molecules were emitted in the Gettysburg address. The atmospheric volume is the order of 5000 m (depth) $\times 1,000,000$ m (width) $\times 1,000,000$ m (breadth) $= 10^{16}$ m^3. So, there are around 10^{10} Gettysburg molecules per cubic meter. A breath is, say, 10^{-6} cubic meters. So, we expect around 10 molecules.

(c) $7/2\ NkT$.

Spring 2011

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 2

March 24, 2011

(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

7. Measuring Capacitance

There are many ways to measure the capacitance of a capacitor. In the figure below, a capacitor C and three resistors R_a, R_b, R_c connected in different combinations to make a total resistance R are connected to a 50 Hz alternating voltage source with fixed RMS voltage V_0. The electric power dissipated in R depends on C, R, and the voltage V_0 and frequency f of the voltage source. The table below shows the measurement data. The symbols $R_1 \parallel R_2$ and $R_1 + R_2$ denote R_1 and R_2 connected in parallel and series respectively. (When speaking of AC voltages we shall always mean RMS [root-mean-square] values.) Use the supplied graph paper for plots.

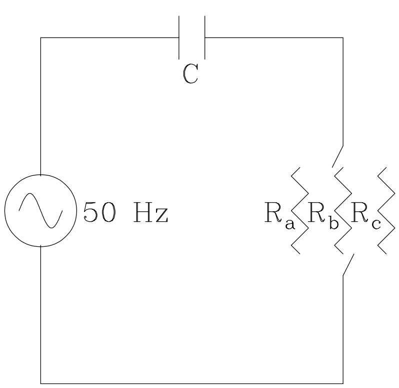
(a) Derive an expression for the average power P dissipated in R.

(b) What is the condition on R for which the dissipated power P is maximum?

(c) Plot P vs. R and from the graph compute the value of the capacitance C.

(d) From (a), convert the dependence into a linear relation between two quantities α and β from whose slope and intercept you can determine the capacitance C.

(e) Construct the plot in (d) and determine C. (The ten highest resistances shown in the Table should be sufficient.)



Resistance R Configuration	Value (Ω)	Voltage across R (V)
R_a	680	9.86
R_b	1500	17.46
R_c	3300	22.81
$R_a + R_b + R_c$	5480	24.40
$R_b + R_c$	4800	23.98
$R_a + R_c$	3980	23.66
$(R_a \parallel R_b) + R_c$	3768	23.43
$R_a + R_b$	2180	20.49
$(R_a \parallel R_c) + R_b$	2064	20.15
$(R_b \parallel R_c) + R_a$	1711	18.63
$(R_a + R_b) \parallel R_c$	1313	16.18
$(R_a + R_c) \parallel R_b$	1089	14.36
$R_b \parallel R_c$	1031	13.78
$(R_b + R_c) \parallel R_a$	596	8.82
$R_a \parallel R_c$	564	8.42
$R_a \parallel R_b$	468	7.28
$R_a \parallel R_b \parallel R_c$	410	6.22

7. Measuring Capacitance - Solution

(a) Denote the reactance of the capacitor by X; its value is $X = 1/(\omega C)$, where $\omega = 2\pi f = 314.2\ \mathrm{s}^{-1}$. The reactance is 90° out of phase with the resistance R, so the power dissipated in R may be written

$$P = \frac{V^2}{R} = \frac{V_0^2 R}{R^2 + X^2}\ . \tag{55}$$

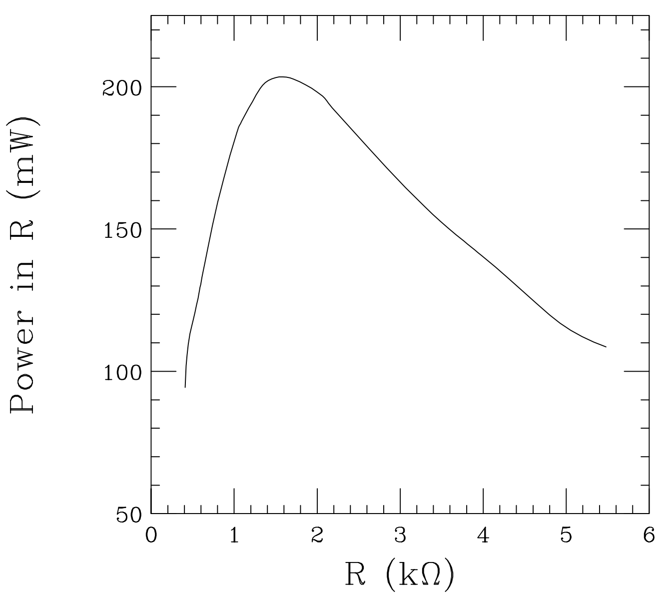
(b) The expression (55) is maximal when $R = X$, as one can see by differentiating with respect to R. (Remember that it is V_0 which is fixed.)

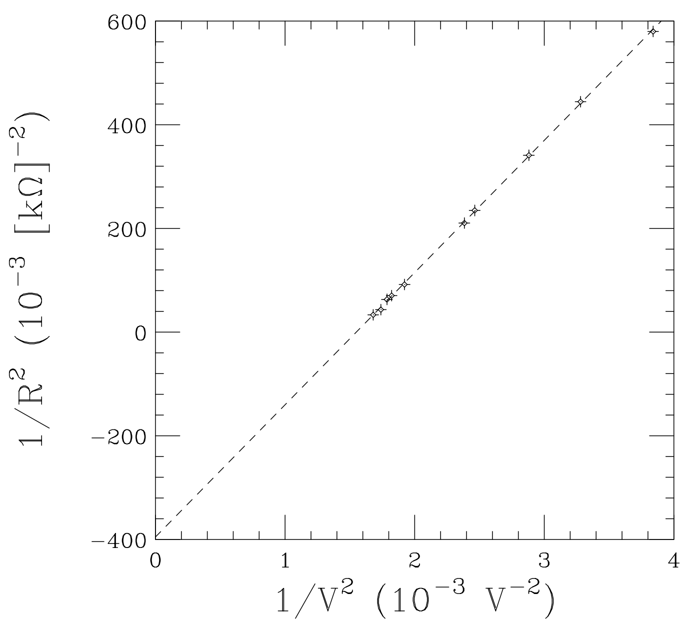
(c) The plot is shown on the left below. P is maximum when $R \simeq 1.6$ kΩ, so $X = 1.6$ kΩ, implying $C = 1/(X\omega) \simeq 2\mu$F.

(d) One can transform Eq. (55) in such a way that $1/R^2$ depends linearly on $1/V^2$:

$$\frac{1}{R^2} = \frac{1}{V^2}\left(\frac{V_0}{X}\right)^2 - \frac{1}{X^2}\ . \tag{56}$$

(e) The plot in (d) is shown below on the right. The dashed line corresponds to an eyeball straight-line fit. Its intercept is about $-0.4(\mathrm{k}\Omega)^{-2}$, implying $X^2 = 2.5(\mathrm{k}\Omega)^2$ or $X = 1.58$ kΩ. As in (c), this implies $C \simeq 2\mu$F. From the slope $(V_0/X)^2$ of the plot, using the value of X just found, one can infer that $V_0 \simeq 25$ V.





8. Short Answer Questions II

(a) Which would extract more water from Lake Michigan in one year, evaporation or the residents of the City of Chicago?

(b) The tidal force produced on Earth by the Moon is about 5 times greater than that produced by the Sun. Use this fact to compare the densities of the Sun and the Moon (without using their known masses and sizes).

(c) You are bicycling along the Lake Michigan path, and stop pedaling. Estimate how far you will coast before your speed is reduced by a factor of two by air friction alone.

(d) The Chicago blizzard of 2011 dumped nearly two feet of snow. Such an event often produces an increase in temperature. Estimate the associated temperature rise if the specific latent heat of fusion of ice is $L = 334\,\mathrm{kJ/kg}$, and the specific heat of air is $c = 1\,\mathrm{J/(g\ K)}$. The average density of a fresh snow is $\rho = 80\,\mathrm{kg/m^3}$.

(e) At what time of day and at what time of year is the Full Moon highest in the Chicago sky? Justify your answer. Assume the orbit of the Moon is in the plane of the ecliptic.

8. Short Answer Questions II - Answers

(a) From drinking, cooking, bathing, flushing, etc., a person consumes on the order of 10 gallons of water per day. There are about 3×10^6 people in Chicago and approximately 3.5×10^2 days in a year. Thus, approximately 10^{10} gallons of water are consumed by Chicagoans per year.

To an order of magnitude about 1 cm of water evaporates from Lake Michigan each day. The lake is about 10^2 km wide by 10^2 km long giving an area on the order of 10^4 km^2. Thus, a volume of 10^8 m^3 = 10^{11} liters $\approx 10^{10}$ gal evaporates per day. Evaporation wins by far.

(b) The tidal force from an object of mass M behaves as $F \propto M/r^3 \approx \rho R^3/r^3$. From total solar eclipses, we know that the Sun and the Moon have about the same apparent size in the sky. So, the ratio (R/r) of their radii to their distances from Earth are about the same. Therefore, the Moon is 5 times more dense than the Sun. [**NOTE**: The original problem mistakenly asked for a comparison of the densities of Earth and the Moon. All students were given credit for this part of the problem.]

(c) The drag force on your body is $F_d \propto A\rho_{air}v^2$. Using the work-energy theorem we get $-A\rho_{air}v^2\Delta x \approx \frac{1}{4}mv^2 - mv^2$. Therefore, to an order of magnitude $\Delta x \sim \frac{m}{A\rho_{air}}$. Taking a mass of 60 kg (which weighs about 130 lbs), a cross sectional area (of your body) of $A \approx 1/2$ m $\times$ $1/3$ m, and the density of air to be $\rho \approx 1$ kg/m^3, we get $\Delta x \approx 360$ m.

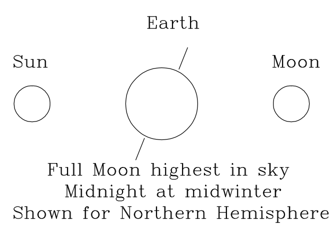
(d) Denote the depth of the snow $h \approx 0.6\,$m, the height of the atmosphere by $H \approx 5\,$km, the are of the snow cover A, the temperature rise ΔT. Assuming that all the latent heat freed out during the formation of the snow was transferred to the air, we have the energy balance:

$$Ah\rho L = AHc\Delta T.$$

The area A cancels, and we have

$$\Delta T = \frac{h\rho L}{Hc} \approx \frac{0.6 \cdot 80 \cdot 3.34 \cdot 10^5}{5 \cdot 10^3 \cdot 10^3} \approx 3.2K.$$

(e) See the following diagram:



9. Spin Cluster

Four $s = 1/2$ spins $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_4$ form a cluster. Each pair of spins interacts via an exchange interaction, so that the Hamiltonian of the system is

$$H = J \sum_{i \neq j = 1}^{4} \boldsymbol{s}_i \cdot \boldsymbol{s}_j .$$

where J is constant.

(a) What is the dimension D of the space of states in this problem?

(b) What are the possible values of the total spin S of the cluster? What are the dimensions d_S of the subspaces corresponding to each value S of the total spin? Check that these dimensions sum up to D.

(c) Express the given Hamiltonian in terms of the operator of the total spin $\boldsymbol{S} = \sum_{i=1}^{4} \boldsymbol{s}_i$.

(d) Find the spectrum of the Hamiltonian and the degeneracies of its energy levels.

(e) What is the total spin S_{grd} in the ground state?

9. Spin cluster - Solution

(a) Each spin has a 2-dimensional space of states. The four spins lead to the space of states of dimension $D = 2^4 = 16$.

(b) Let us break the spins into two pairs. Each pair of spins can form either a singlet $S_{\text{pair}} = 0$ or triplet $S_{\text{pair}} = 1$. Adding the resulting spins of each pair we have the following possibilities:

1. $S_{\text{pair 1}} = 0$ and $S_{\text{pair 2}} = 0$. Then $S = 0$.
2. $S_{\text{pair 1}} = 1$ and $S_{\text{pair 2}} = 0$. Then $S = 1$.
3. $S_{\text{pair 1}} = 0$ and $S_{\text{pair 2}} = 1$. Then $S = 1$.
4. $S_{\text{pair 1}} = 1$ and $S_{\text{pair 2}} = 1$. Then S can be equal to 0, 1, or 2.

We see that the total spin $S = 0$ appears twice, $S = 1$ appears three times, and $S = 2$ appears once. Therefore, the dimensions of the corresponding subspaces are

$$d_0 = 2, \qquad d_1 = 3 \cdot (2 \cdot 1 + 1) = 9, \qquad d_2 = 2 \cdot 2 + 1 = 5. \tag{57}$$

We see that these dimensions sum up to

$$d_0 + d_1 + d_2 = 16 = D. \tag{58}$$

(c) Consider the square of the total spin operator:

$$\boldsymbol{S}^2 = \left(\sum_{i=1}^{4} \boldsymbol{s}_i \right)^2 = \sum_{i=1}^{4} \boldsymbol{s}_i^2 + 2 \sum_{i \neq j = 1}^{4} \boldsymbol{s}_i \cdot \boldsymbol{s}_j. \tag{59}$$

Since each spin has value $s = 1/2$, we have $\boldsymbol{s}_i^2 = s(s+1) = 3/4$, so that

$$\boldsymbol{S}^2 = 3 + 2 \sum_{i \neq j = 1}^{4} \boldsymbol{s}_i \cdot \boldsymbol{s}_j. \tag{60}$$

Comparing this with the Hamiltonian (113) we see that

$$H = \frac{J}{2}\big(\boldsymbol{S}^2 - 3\big). \tag{61}$$

(d) Since the Hamiltonian is now expressed in terms of the total spin only, we can immediately determine its spectrum. The eigenenegries are labeled by the value of the total spin S:

$$E_S = \frac{J}{2}\big(S(S+1) - 3\big), \tag{62}$$

and these energy levels have degeneracies equal to the dimensions d_S. Thus, we have the following energy levels and their degeneracies:

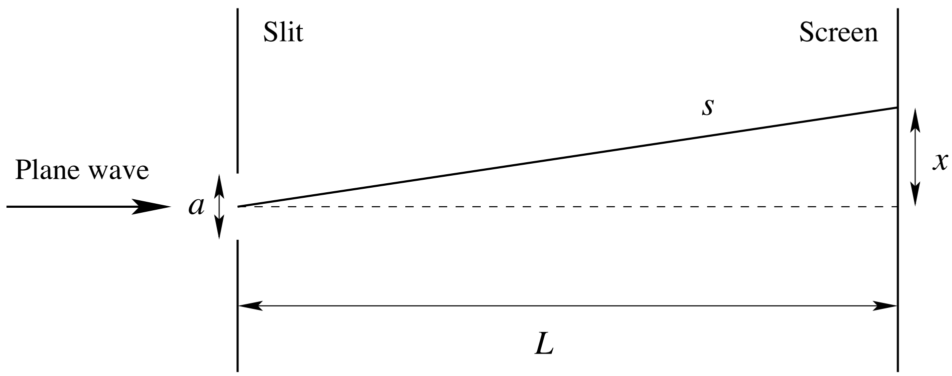
$$\begin{aligned} E_0 &= -\frac{3}{2}J, & d_0 &= 2, \\ E_1 &= -\frac{1}{2}J, & d_1 &= 9, \\ E_2 &= \frac{3}{2}J, & d_2 &= 5. \end{aligned} \tag{63}$$

(e) Which of the energy levels found above corresponds to the ground state depends on the sign of the coupling constant J. For antiferromagnetic coupling $J > 0$ the ground state is a singlet, and the total spin $S_{\text{grd}} = 0$. For ferromagnetic coupling $J < 0$ the lowest energy is E_2, and $S_{\text{grd}} = 2$.

10. Diffraction

(a) For waves of frequency ω in vacuum, write the wave equation for $\boldsymbol{E}(\boldsymbol{r})$ (the time dependent electric field is $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r})e^{i\omega t}$) in cylindrical coordinates (s,θ,z) for the situation where $\boldsymbol{E}$ does not depend on θ or z.

(b) Solve the wave equation in the far field limit $sk \gg 1$ using the substitution $\boldsymbol{E} = \boldsymbol{F}/\sqrt{s}$. You may leave any normalization constants arbitrary.



(c) For the problem of diffraction on a long slit of width a, find the intensity distribution on a flat screen (see the figure), assuming that the distance between the slit and the screen $L \gg x, a, ka^2$ (Fraunhofer diffraction). **Hint**: Replace the vector $\boldsymbol{E}(s)$ by a scalar function $\psi(s)$, and use Huygens' principle to integrate over the slit.

10. Diffraction - Solution

(a) Start with the wave equation:

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}, t) - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t^2} = 0. \tag{64}$$

Assuming that $\boldsymbol{E}(\boldsymbol{r}, t)$ is a monochromatic wave ($\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}) e^{i\omega t}$), the wave equation simplifies to the Helmholz equation

$$(\nabla^2 + k^2) \boldsymbol{E}(\boldsymbol{r}) = 0, \tag{65}$$

where the wave number is $k = \omega / c$.

If the amplitude $\boldsymbol{E}(\boldsymbol{r})$ depends only on the radial distance s in the cylindrical coordinates, the Laplacian acts on this function as

$$\nabla^2 \boldsymbol{E}(s) = s^{-1} (s \boldsymbol{E}'(s))' = \boldsymbol{E}''(s) + \boldsymbol{E}'(s)/s, \tag{66}$$

where prime denotes the derivative with respect to s.

To obtain this it is sufficient to consider the change of the Cartesian coordinates (x, y) to the polar ones (s, θ) in the plane. Then we have

$$x = s \cos\theta, \qquad s = \sqrt{x^2 + y^2}, \tag{67}$$

$$y = s \sin\theta, \qquad \tan\theta = y/x. \tag{68}$$

The partial derivatives transform as

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x} \frac{\partial}{\partial s} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial y} = \frac{\partial s}{\partial y} \frac{\partial}{\partial s} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta}, \tag{69}$$

where

$$\frac{\partial s}{\partial x} = \frac{x}{s} = \cos\theta, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{s^2} = -\frac{\sin\theta}{s}, \tag{70}$$

$$\frac{\partial s}{\partial y} = \frac{y}{s} = \sin\theta, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{s^2} = \frac{\cos\theta}{s}. \tag{71}$$

Acting on a function $f(s)$ that depends on s only, we get

$$\frac{\partial}{\partial x}f(s) = \cos\theta f'(s), \qquad \frac{\partial^2}{\partial x^2}f(s) = \cos^2\theta f''(s) + \frac{\sin^2\theta}{s}f'(s), \tag{72}$$

$$\frac{\partial}{\partial y}f(s) = \sin\theta f'(s), \qquad \frac{\partial^2}{\partial y^2}f(s) = \sin^2\theta f''(s) + \frac{\cos^2\theta}{s}f'(s). \tag{73}$$

Finally, the Laplacian acting on $f(s)$ is

$$\nabla^2 f(s) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(s) = f''(s) + \frac{1}{s}f'(s). \tag{74}$$

This, in cylindrical coordinates the Helmholz equation for the field amplitude that depends on s only, becomes

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} + k^2\right)\boldsymbol{E}(s) = 0. \tag{75}$$

(b) Using the substitution $\boldsymbol{E}(s) = \boldsymbol{F}(s)/\sqrt{s}$, we rewrite the equation (75) as

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} + k^2\right)\frac{\boldsymbol{F}(s)}{\sqrt{s}} = \frac{1}{\sqrt{s}}\left[\boldsymbol{F}''(s) + \left(\frac{1}{4s^2} + k^2\right)\boldsymbol{F}(s)\right] = 0. \tag{76}$$

In the far field ($ks \gg 1$), we can neglect the term $\boldsymbol{F}(s)/s^2$ in the equation, which then becomes much simpler:

$$\boldsymbol{F}''(s) + k^2\boldsymbol{F}(s) = 0. \tag{77}$$

Solutions of this equation correspond to standing ($\cos ks$ and $\sin ks$) or propagating ($e^{\pm iks}$) cylindrical waves.

(c) According to Huygens' principle, the amplitude of the waves reaching the screen form the slit is given by the superposition of the amplitudes of the outgoing cylindrical waves from each portion of the slit at coordinate $x' \in [-a/2, a/2]$. Mathematically this is written as

$$\psi(x) = \int_{-a/2}^{a/2} dx' \frac{e^{iks}}{\sqrt{s}}, \tag{78}$$

where the distance s between a given segment of the slit at x' and the point on the screen at x is

$$s = \sqrt{L^2 + (x - x')^2}. \tag{79}$$

Using the assumptions made in the problem, we have $x, x' \ll L$. The we can simply replace $\sqrt{s}$ by $\sqrt{L}$ in the denominator, and in the exponential we can expand

$$s \approx L - \frac{(x - x')^2}{2L}. \tag{80}$$

Then

$$\psi(x) \approx \frac{1}{\sqrt{L}} e^{ikL} e^{ikx^2/L} \int_{-a/2}^{a/2} dx' e^{-ikxx'/L} e^{-ik(x')^2/L}. \tag{81}$$

The Franuhofer diffraction condition $L \gg ka^2$ allows to drop the last exponential in the above expression, which then gives

$$\psi(x) \approx \frac{2\sqrt{L}}{kx} e^{ikL} e^{ikx^2/L} \sin \frac{kax}{2L}. \tag{82}$$

The intensity then is obtained as

$$I(x) \propto |\psi(x)|^2 = \text{const} \cdot \left(\frac{\sin \alpha x}{\alpha x} \right)^2, \qquad \alpha = \frac{ka}{2L}. \tag{83}$$

11. Anharmonic Oscillator

(a) Consider an anharmonic oscillator with the Hamiltonian (energy)

$$H = (p^2/2m) + bx^{2n},$$

where n is a natural number bigger than 1. This assembly of oscillators is in thermal equilibrium with a heat reservoir at a temperature T high enough that the approximation of classical mechanics is quite good. What is the heat capacity C of the oscillator?

(b) Now consider an anharmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{3}bx^3 + \frac{1}{4}cx^4.$$

Treating the anharmonic terms as perturbations, find (up to the first non-vanishing order in b and c) the heat capacity of the anharmonic oscillator. Do the anharmonic terms increase or decrease heat capacity compared to that of the harmonic oscillator?

The integrals you need for this part of the problem are

$$\int_{-\infty}^{\infty} e^{-ax^2} x^{2k} dx = \frac{\Gamma(k+1/2)}{a^{k+1/2}} = \sqrt{\frac{\pi}{a}} \frac{(2k-1)!!}{(2a)^k}.$$

11. Anharmonic Oscillator - Solution

(a) Before solving this part of the problem, let us evaluate integrals of the form

$$I(2n,a) = \int_{-\infty}^{\infty} e^{-ax^{2n}}\,dx, \qquad n \in \mathbb{N},\ a > 0. \tag{84}$$

We rescale the integration variable: $\xi = a^{1/(2n)}x$, and so

$$I(2n,a) = \frac{1}{a^{1/(2n)}} \int_{-\infty}^{\infty} e^{-\xi^{2n}}\,d\xi = \frac{1}{a^{1/(2n)}} I(2n,1) \tag{85}$$

where $I(2n,1)$ does not depend on the factor a.

We then see that the partition function of the oscillator, in the limit where classical mechanics is applicable, becomes

$$\begin{aligned} Z &= \int \frac{dpdx}{h} e^{-\beta(p^2/2m + bx^{2n})} = \frac{1}{h} I(2,1) I(2n,1) \left(\frac{2m}{\beta}\right)^{1/2} \left(\frac{1}{b\beta}\right)^{1/(2n)} \\ &= \text{const} \cdot \beta^{-\frac{n+1}{2n}}, \end{aligned} \tag{86}$$

where $\beta = 1/(k_B T)$. We never even need to know $I(2n,1)$ to determine the heat capacity as these factors will vanish in the process of finding the heat capacity from the partition function. Then, the energy is given

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{n+1}{2n}\frac{1}{\beta} = \frac{n+1}{2n} k_B T. \tag{87}$$

Hence, the heat capacity of the oscillator is

$$C = \frac{\partial E}{\partial T} = \frac{n+1}{2n} k_B. \tag{88}$$

Alternatively, one can find first the free energy

$$F = -k_B T \ln Z = \text{const} - \frac{n+1}{2n} k_B T \ln(k_B T), \tag{89}$$

then the entropy

$$S = -\frac{\partial F}{\partial T} = \frac{n+1}{2n} k_B + \frac{n+1}{2n} k_B \ln(k_B T), \tag{90}$$

and the energy

$$E = F + TS = \frac{n+1}{2n} k_B T, \tag{91}$$

which gives the same answer (88).

(b) In this part we need to expand the Gibbs weight under the integrals in the partition function in the small parameters b and c.

Writing the energy of the system as

$$H = \frac{p^2}{2m} + \frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{c}{4}x^4, \tag{92}$$

where $a = m\omega^2$, the partition function is

$$\begin{aligned} Z &= \int \frac{dpdx}{2\pi\hbar} e^{-H/k_B T} \\ &= \int_{-\infty}^{\infty} e^{-p^2/(2mk_B T)} \frac{dp}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-ax^2/2k_B T} e^{-bx^3/3k_B T} e^{-cx^4/4k_B T} dx. \end{aligned} \tag{93}$$

Using Eq. (84), the momentum integral gives

$$\int_{-\infty}^{\infty} e^{-p^2/(2mk_B T)} \frac{dp}{2\pi\hbar} = \frac{(2\pi m k_B T)^{1/2}}{2\pi\hbar} = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{1/2}. \tag{94}$$

Considering b and c to be small parameters, we can expand the exponentials in Eq. (93) as

$$e^{-bx^3/3k_B T} \approx 1 - \frac{bx^3}{3k_B T} + \frac{b^2 x^6}{18(k_B T)^2} \tag{95}$$

$$e^{-cx^4/4k_B T} \approx 1 - \frac{cx^4}{4k_B T}. \tag{96}$$

The odd correction (the x^3 term) does not contribute to the partition function. So to first (non-zero) order in each parameter, the partition function is given by

$$Z \approx \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-ax^2/2k_B T} \left(1 + \frac{b^2}{18(k_B T)^2}x^6 - \frac{c}{4k_B T}x^4\right). \tag{97}$$

Integration using Eq. (84) yields the partition function

$$Z \approx \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{1/2}\left(\frac{2\pi k_BT}{a}\right)^{1/2}\left[1+\frac{15b^2}{18(k_BT)^2}\left(\frac{k_BT}{a}\right)^3-\frac{3c}{4k_BT}\left(\frac{k_BT}{a}\right)^2\right]$$
$$=\frac{k_BT}{\hbar}\sqrt{\frac{m}{a}}\left[1+\left(\frac{5b^2}{6a}-\frac{3}{4}c\right)\frac{k_BT}{a^2}\right]=\frac{k_BT}{\hbar\omega}\left[1+\left(\frac{5b^2}{6m\omega^2}-\frac{3}{4}c\right)\frac{k_BT}{m^2\omega^4}\right]. \tag{98}$$

As in the first part, we now find the energy of the oscillator from the formula

$$E=-\frac{\partial \ln Z}{\partial\beta}=k_BT^2\frac{\partial \ln Z}{\partial T}. \tag{99}$$

In doing so, we need to expand $\ln Z$ in powers of b and c again:

$$\ln Z \approx \ln\frac{k_BT}{\hbar\omega}+\left(\frac{5b^2}{6m\omega^2}-\frac{3}{4}c\right)\frac{k_BT}{m^2\omega^4}. \tag{100}$$

This gives

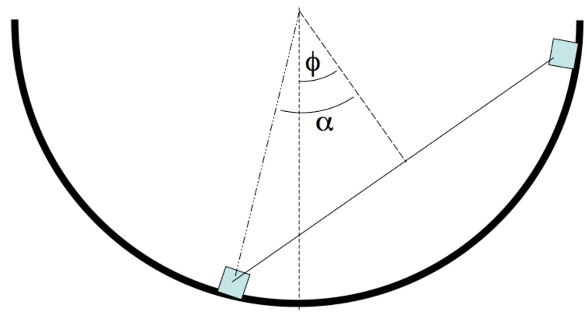
$$E \approx k_BT+\left(\frac{5b^2}{6m\omega^2}-\frac{3}{4}c\right)\frac{k_B^2T^2}{m^2\omega^4}, \tag{101}$$

and, finally, the heat capacity

$$C=\frac{\partial E}{\partial T}\approx k_B+\left(\frac{5b^2}{3m\omega^2}-\frac{3}{2}c\right)\frac{k_B^2T}{m^2\omega^4}. \tag{102}$$

Notice that c and b contributions have different signs, so whether the specific heat is increased or decreased due to anharmonicity depends on the ratio $10b^2/9cm\omega^2$.

12. Half-pipe Twins



Two skateboarders of mass m skate inside a semi-cylindrical half pipe of radius R. They are separated by a massless pole that they hold (see the diagram). Useful co-ordinates are the angular distance ϕ of the midpoint of the rod from the vertical axis, and α, the angle between the first skater and the midpoint; see the diagram. Then both skaters simultaneously start steadily shortening the distance between them in a symmetric way by pulling on the pole, so that $\alpha(t)$ decreases with time. There is no friction, and you can consider the skaters as point masses sliding along a semicircle of radius R.

(a) Find the equation of motion for ϕ in terms of $\alpha(t)$, considered as a given function of time determined by the relative motion of the skaters.

(b) Calculate the Hamiltonian as a function of $\phi, \dot{\phi}$. Is the Hamiltonian equal to the mechanical energy?

(c) Assuming that the angle ϕ stays small ($\phi \ll 1$) during the motion, simplify the equation obtained in part (a) and find the frequency of oscillation of the skaters as a function of $\alpha(t)$ ($\alpha(t)$ is not necessarily small).

(d) Now consider the situation where the skaters start their oscillation with the amplitude $\Phi_0 \ll 1$ and some α_0, and then move toward each other in a symmetric way, and very slowly (compared to the period of oscillation). Eventually they meet at some time T (so that $\alpha(T) = 0$). Find the finite amplitude of oscillation $\Phi(T)$ at time T. **Hint**: use the fact that the adiabatic invariant $I = \oint p dq$, where the integral is taken over the period of oscillation, is approximately conserved.

12. Half-pipe Twins - Solution

(a) The kinetic energy of the skaters is

$$T = \frac{1}{2}m(v_1^2 + v_2^2) = \frac{1}{2}mR^2(\dot{\phi}_1^2 + \dot{\phi}_2^2), \tag{103}$$

and their potential energy is

$$U = mgh_1 + mgh_2 = -mgR(\cos\phi_1 + \cos\phi_2). \tag{104}$$

We denote the angle between skater 1 and the midpoint as α. Then we can write $\phi_1 = \phi - \alpha$ and $\phi_2 = \phi + \alpha$. In terms of ϕ and α, the energies T and U, and the Lagrangian L are given by

$$T = mR^2(\dot{\phi}^2 + \dot{\alpha}^2), \tag{105}$$

$$U = -mgR[\cos(\phi - \alpha) + \cos(\phi + \alpha)] = -2mgR\cos\alpha\cos\phi, \tag{106}$$

$$L = T - U = mR^2(\dot{\phi}^2 + \dot{\alpha}^2) + 2mgR\cos\alpha\cos\phi. \tag{107}$$

Now we find the equation of motion for ϕ in the usual way:

$$\frac{\partial L}{\partial \phi} = -2mgR\cos\alpha\sin\phi, \qquad \frac{\partial L}{\partial \dot{\phi}} = 2mR^2\dot{\phi}, \tag{108}$$

so the Lagrange equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 2mR^2\ddot{\phi} = \frac{\partial L}{\partial \phi} = -2mgR\cos\alpha\sin\phi, \tag{109}$$

or, finally,

$$\ddot{\phi} + \frac{g}{R}\cos\alpha\sin\phi = 0. \tag{110}$$

(b) The Hamiltonian is obtained as a Legendre transform of the Lagrangian with respect to ϕ:

$$H = p_\phi\dot{\phi} - L, \tag{111}$$

where the generalized momentum conjugate to ϕ is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 2mR^2\dot{\phi}. \tag{112}$$

This gives the Hamiltonian:

$$H = mR^2(\dot{\phi}^2 - \dot{\alpha}^2) - 2mgR\cos\alpha\cos\phi. \tag{113}$$

At the same time, the total energy of the system is

$$E = T + U = mR^2(\dot{\phi}^2 + \dot{\alpha}^2) - 2mgR\cos\alpha\cos\phi. \tag{114}$$

We see that $H \neq E$. The two quantities differ in the sign of the term containing $\dot{\alpha}^2$, and this reflects that fact the the problem is not stationary. In other words, both the Lagrangian and the Hamiltonian explicitly depend on time through the function $\alpha(t)$.

(c) For small oscillations $\phi \ll 1$, the equation of motion (110) simplifies to

$$\ddot{\phi} + \frac{g}{R}\big[\cos\alpha(t)\big]\phi = 0, \tag{115}$$

which is the equation for a harmonic oscillator with the time-dependent frequency

$$\omega(t) = \sqrt{\frac{g}{R}\cos\alpha(t)}. \tag{116}$$

(d) Let us find the adiabatic invariant for the harmonic oscillation of the skaters. Splitting the full oscillation into four equal time intervals, we have

$$I = 4\int_0^{\Phi} p_\phi d\phi, \tag{117}$$

where $\Phi = \Phi(t)$ is the amplitude of oscillation at time t. Using the expression (112) for the generalized momentum, we get

$$I = 8mR^2\int_0^{\Phi}\dot{\phi}\,d\phi. \tag{118}$$

The velocity $\dot{\phi}$ can be expressed in terms of ϕ using the equation (114) for the energy. Since the energy can be written in terms of the amplitude Φ as

$$E = mR^2\dot{\alpha}^2 - 2mgR\cos\alpha\cos\Phi, \tag{119}$$

we have from Eq. (114)

$$mR^2\dot{\phi}^2 = 2mgR\cos\alpha[\cos\phi - \cos\Phi]. \tag{120}$$

Using the expression (116), we get

$$\dot{\phi} = \omega(t)\sqrt{2[\cos\phi - \cos\Phi]}. \tag{121}$$

Substituting this into the adiabatic invariant (118) would result in an elliptic integral. However, for small oscillations the velocity simplifies to

$$\dot{\phi} = \omega(t)\sqrt{\Phi^2 - \phi^2}, \tag{122}$$

and the adiabatic invariant is given by an elementary integral

$$I = 8mR^2\omega(t)\int_0^{\Phi}\sqrt{\Phi^2 - \phi^2}\,d\phi = 2\pi mR^2\omega(t)\Phi^2(t). \tag{123}$$

The approximate conservation of the adiabatic invariant means that we can relate the amplitudes of oscillation in the beginning and the end of the relative motion of the skaters:

$$\omega(T)\Phi^2(T) = \omega(0)\Phi^2(0). \tag{124}$$

Substituting the expression (116) for the frequency, and remembering that $\alpha(T) = 0$, we finally obtain

$$\Phi(T) = \Phi_0\left(\frac{\omega(0)}{\omega(T)}\right)^{1/2} = \Phi_0(\cos\alpha_0)^{1/4}. \tag{125}$$