QUALIFYING EXAMINATION, Part 1

1:00 PM – 4:00 PM, Thursday August 28, 2014

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Mathematical Methods

(a) (25 points) Find the radius of convergence R of the series

$$S = \sum_{n=17}^{\infty} \frac{5^n}{n^2+1} x^n \ .$$

Does the series converge *at* $x = R$?

(b) (40 points) Use the contour integration method to evaluate the integral

$$I = \int_0^{\infty} \frac{\cos x}{1+x^2} dx \ .$$

Explain and justify the contour you are using.

(c) (35 points) Show that a three-dimensional vector field $\vec{V}$ that vanishes at infinity is uniquely determined by its curl $\nabla \times \vec{V}$ and divergence $\nabla \cdot \vec{V}$.

Hints: Consider the difference of two vector fields that have the same curl and divergence. You may appeal to the uniqueness properties of the solution to Laplace's equation or use the following identity for a scalar field ϕ

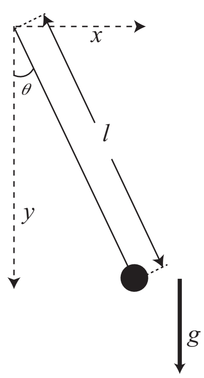
$$\nabla \cdot (\phi \nabla \phi) = |\nabla \phi|^2 + \phi \nabla^2 \phi \ .$$

Problem 2: Classical Mechanics

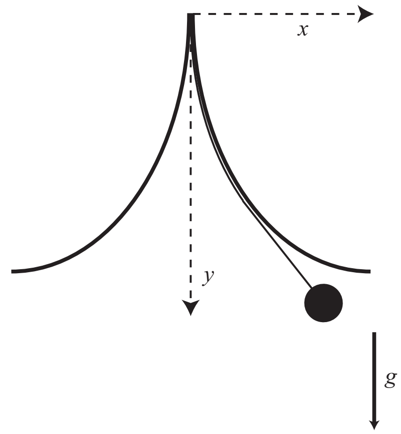
The period of a simple pendulum in a gravitational field g (see left figure) is independent of its amplitude in the limit of small oscillations but for large angles θ the period is a function of the amplitude. The Dutch physicist Huygens constructed a pendulum whose period is independent of its amplitude. He did this by allowing the string of the pendulum to wrap partially around constraining rails that follow the shape of cycloid (see right figure). The x and y cartesian coordinates of the pendulum bob's (see figure for the definition of the x, y coordinates) are then constrained to follow a trajectory that can be parametrized by an angle φ (do not attempt to show this)

$$x = \frac{l}{4}(\varphi + \sin\varphi) \ ; \qquad y = \frac{l}{4}(3 + \cos\varphi) \ ,$$

where l is the length of the string. The equilibrium position of the bob $x = 0, y = l$ is given by $\varphi = 0$.



Simple pendulum



Huygens pendulum

(a) (20 points) Show that in the limit of small φ, the above x, y bob's coordinates (of the Huygens pendulum) have the same values as the x, y coordinates of a simple pendulum with length l and angle $\theta = \varphi/2$.

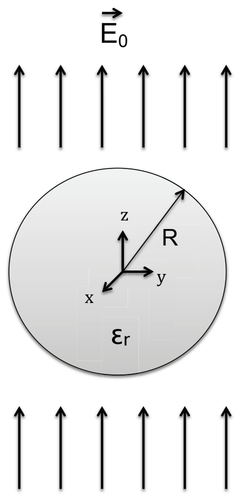
(b) (30 points) Derive the Lagrangian $\mathcal{L}$ of Huygens pendulum, using φ as a generalized coordinate.

(c) (25 points) Rewrite the Lagrangian $\mathcal{L}$ in terms of u and $\dot{u}$, where $u = \sin(\varphi/2)$, and show that it is quadratic in u and $\dot{u}$.

(d) (25 points) Use the Lagrangian in (c) to find the general solution for u as a function of time. Argue that the motion in φ is periodic and determine its period. How does this period compare with the period of small oscillations of the simple pendulum?

Problem 3: Electromagnetism I

A homogeneous dielectric sphere of radius R and dielectric constant ε_r is placed in an external uniform electric field $\vec{E}_0 = E_0\hat{z}$ (see figure).



(a) (25 points) Using spherical coordinates r, θ, φ, where the origin is chosen at the center of the sphere, explain why the electrostatic potential should be independent of the azimuthal angle φ. Write the general form of the potential $V_{\text{in}}(r, \theta)$ inside the sphere using an expansion in Legendre polynomials $P_l(\cos\theta)$.

(b) (25 points) Write the boundary condition for $r \gg R$, and use it to determine the general form of the potential $V_{\text{out}}(r, \theta)$ outside the sphere.

(c) (35 points) Write the boundary conditions at $r = R$, and use them together with your results in (a) and (b) to determine the potential $V_{\text{in}}(r, \theta)$ inside the sphere. What is the electric field inside the sphere ?

(d) (15 points) Sketch the induced bound charge on the surface of the dielectric sphere.

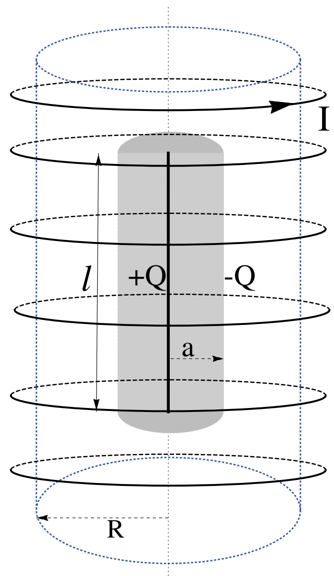
Hint: the solution of Laplace's equation $\nabla^2 V = 0$ in spherical coordinates with azimuthal symmetry around the z axis has the general form

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \, ,$$

where P_l are Legendre polynomials and A_l, B_l are constants. The first two Legendre polynomials are $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$.

Problem 4: Electromagnetism II

Consider an infinitely long solenoid of radius R, n turns per unit length and current I. Inside the solenoid and coaxial to it there is a line charge of length l and total positive charge Q, and a hollow cylinder of same length l, radius a ($a < R$) and total charge negative $-Q$ on its surface (see figure). Assume that the charges are uniformly distributed and that $l \gg a$, so any edge effects from the finite length of the line charge and cylinder can be ignored. The charges on the cylinder are fixed.



(a) (20 points) Find the magnetic field inside the solenoid due to the current I.

(b) (20 points) Find the radial electric field inside and outside the cylinder of radius a due to the charges.

In the following, assume that the current in the solenoid is turned off gradually at a rate of dI/dt. As a result the cylinder begins to rotate. Ignore any field generated by the rotation of the cylinder.

(c) (25 points) Use Faraday's law to determine the induced electric field around the circumference of the cylinder of radius a, expressing it in terms of dI/dt.

(d) (35 points) The induced electric field in part (c) exerts a torque on the charged cylinder. Determine the magnitude of this torque in terms of dI/dt and integrate it over time to find the magnitude of the final angular momentum of the cylinder after the current in the solenoid is reduced to zero. Express this final angular momentum in terms of the initial current I in the solenoid.

Does the cylinder rotate in the same direction as the initial current in the solenoid or in the opposite direction? Explain your answer.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

2×1/2

m_1	m_2	5/2, +5/2
+2	+1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

QUALIFYING EXAMINATION, Part 2

9:00 AM – 12:00 NOON, Friday August 29, 2014

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Quantum Mechanics I

The variational method in quantum mechanics is often accurate for the determination of ground-state energies. It can be especially useful in dealing with systems that are not exactly solvable (most systems) and that cannot be reliably analyzed via perturbation theory. The idea is to guess the form of the normalized ground-state wave function ψ ($\langle\psi|\psi\rangle = 1$) based on physical intuition and symmetry, employing one or more free parameters, and then use $\langle\psi|H|\psi\rangle$ (H is the Hamiltonian of the system) to determine the particular values of the parameters that give the best estimate for the ground-state energy.

(a) (25 points) Show that $\langle\psi|H|\psi\rangle \geq E_0$, where E_0 is the true ground-state energy and ψ is any normalized wave function.

(b) (20 points) As a test of the method, we will estimate the ground-state energy of the hydrogen atom using a Gaussian trial wave function

$$\psi(r, \theta, \phi, \alpha) = \frac{1}{N} e^{-\alpha r^2} ,$$

where r, θ, and ϕ are spherical coordinates, α is a variational parameter, and N is a normalization constant. (The true ground-state wave function is of course not a Gaussian, and the problem can actually be solved exactly.)

What is the orbital angular momentum quantum number l of this Gaussian wave function ? Explain why this value of l is a good choice for the trial wave function.

(c) (25 points) Compute $E(\alpha) \equiv \langle\psi|H|\psi\rangle$ as a function of α. To simplify the calculation, use the result $\langle\psi|T|\psi\rangle = 3\hbar^2\alpha/(2m)$ for the expectation value of the kinetic energy T, and evaluate the expectation value of the potential energy $V = -e^2/r$.

Useful integrals:

$$\int_0^\infty r e^{-br^2} dr = \frac{1}{2b} , \qquad \int_0^\infty r^2 e^{-br^2} dr = \frac{1}{4b}\sqrt{\frac{\pi}{b}} .$$

(d) (30 points) Use your result in (c) to determine the best estimate for the ground-state energy of the hydrogen atom for a Gaussian trial wave function. Express your result in terms of e, m, and $\hbar$ and compare it to the exact ground-state energy $-me^4/2\hbar^2$.

Problem 2: Quantum Mechanics II

Consider a coupled system consisting of a harmonic oscillator with frequency ω_o and a two-level atom with a ground state $|g\rangle$ of energy 0 and an excited state $|e\rangle$ with energy $\hbar\omega_a$. The Hamiltonian of the coupled system (known as the Jaynes-Cummings Hamiltonian) is given by

$$\hat{H}_{JC} = \hat{H}_o + \hat{H}_a + \hat{H}_{int} \ ,$$

where

$$\hat{H}_o = \hbar\omega_o \left(\hat{n} + \frac{1}{2}\right) \ ; \quad \hat{H}_a = \hbar\omega_a |e\rangle \langle e|$$

describe the oscillator and two-level atom Hamiltonians, respectively, and

$$\hat{H}_{int} = \hbar\kappa\hat{a}\hat{\sigma}^+ + \hbar\kappa\hat{a}^\dagger\hat{\sigma}^-$$

describes the interaction between them. Here κ is the coupling strength, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the harmonic oscillator, $\hat{n} = \hat{a}^\dagger\hat{a}$, and

$$\hat{\sigma}^+ = |e\rangle \langle g| \ , \ \ \hat{\sigma}^- = |g\rangle \langle e| \ ,$$

are raising and lowering operators between the ground and excited states of the atom. The n-th excited eigenstate of the oscillator is denoted by $|n\rangle$ and satisfies $\hat{n}|n\rangle = n|n\rangle$ for $n = 0, 1, 2, \ldots$.

(a) (10 points) Show that $|0, g\rangle \equiv |0\rangle |g\rangle$ is an eigenstate of $\hat{H}_{JC}$, and compute its energy.

(b) (25 points) In order to find all eigenstates of $\hat{H}_{JC}$, we introduce another operator

$$\hat{N} = \hat{n} + |e\rangle \langle e| \ ,$$

which is associated with the total number of excitations in the system. Show that $\hat{N}$ is a conserved quantity (i.e., a constant of motion), and thus its eigenvalue is a good quantum number.

(c) (30 points) Find all eigenstates and eigenvalues of $\hat{N}$, and identify the degree of degeneracy associated with each eigenvalue of $\hat{N}$.

(d) (35 points) What is the structure of the matrix describing $\hat{H}_{JC}$ in the basis of eigenstates of $\hat{N}$? Calculate the Hamiltonian matrix within the subspace of each degenerate eigenvalue of $\hat{N}$. Explain briefly how would you proceed from here to find the eigenvalues of $\hat{H}_{JC}$ but do not calculate them.

Useful relations for the oscillator are

$$\hat{a}|n\rangle = \begin{cases} \sqrt{n}\,|n-1\rangle & n > 0 \\ 0 & n = 0 \end{cases} \ ; \qquad \hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \ .$$

Problem 3: Statistical Mechanics I

Consider N non-interacting distinguishable particles in equilibrium at temperature T. Each particle can only occupy two possible states of energies 0 and ϵ (where $\epsilon > 0$).

(a) (20 points) Find the partition function $Z(T)$ of the system.

(b) (25 points) Calculate the heat capacity of the system as a function of temperature. Find its high-temperature limit.

(c) (30 points) Write the free energy F of the system at temperature T and use it to calculate the entropy S as a function of temperature. Determine the high-temperature and zero temperature limits of S and interpret your results in both limits.

(d) (25 points) What is the average number of particles $\langle n_1 \rangle$ and $\langle n_2 \rangle$ in each of the two states? Derive an expression for the entropy S in terms of the relative occupations $\langle n_1 \rangle / N$ and $\langle n_2 \rangle / N$.

Problem 4: Statistical Mechanics II

Consider a gas of non-interacting massless photons in *two dimensions*, confined to a large square area $A = L^2$ with periodic boundary conditions and held at temperature T. Photons in two dimensions only have one possible transverse polarization state.

(a) (15 points) Determine the allowed quantized values of the photon wavenumber $\vec{k}$ and frequency ω. Calculate the density of states $g(\omega)$ of photons with frequency ω.

(b) (25 points) Write an expression for $\ln Z(T)$ as an integral over the photon frequency ω, where $Z(T)$ is the grand-canonical quantum partition function of the photon gas.

(c) (30 points) The total energy density U/A of the photon gas can be written as an integral over the photon frequency ω, i.e., $U/A = \int_0^\infty d\omega u(\omega, T)$. Find the function $u(\omega, T)$ and determine the explicit temperature dependence of the energy density U/A. Your final result for U/A can depend on a dimensionless integral, and there is no need to evaluate it explicitly.

(d) (30 points) Calculate the entropy per photon and show that it is independent of both the temperature T and the area A. Express your final results in terms of dimensionless integrals; there is no need to evaluate them explicitly.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^{\,2}} + \frac{\partial^2\psi}{\partial x_2^{\,2}} + \frac{\partial^2\psi}{\partial x_3^{\,2}}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho} \right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi} \right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2 \left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1
		−1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2
		−3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2
		−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2
		−2
−3/2	−1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3
		−3
−2	−1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2
		−2
−1	−1	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2
		−5/2
−3/2	−1	1