Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 25, 2017

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.


Student _____
Fill in your Sh-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Sh-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to each problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

Problem I-1. (Classicial Mechanics)
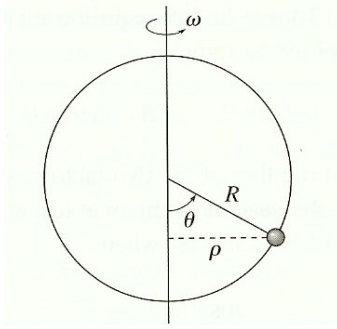
A bead of mass m is threaded on a frictionless circular wire hoop of radius R. The hoop lies in a vertical plane, which is forced to rotate about the hoop's vertical diameter with constant angular velocity $\dot\phi = \omega$, as shown in the figure. The bead's position on the hoop is specified by the angle θ measured up from the vertical.

(i) Write down the Lagrangian for the system in terms of the generalized coordinate θ and find the equation of motion for the bead.

(ii) Find equilibrium positions at which the bead can remain stationary with θ constant.

(iii) Explain the location of those positions using "centrifugal force" $m\omega^2\rho$ (where ρ is the bead's distance from the axis). *Hint:* Draw the force diagram for the bead.

(iv) Discuss the stability of the equilibrium positions.

Problem I-2. (Math Methods)

(i) Solve the ordinary differential equation

$$\frac{dx}{dt} + \lambda x = Be^{i\omega t}, \tag{2.1}$$

where λ and ω are real numbers, B is a complex number, and i is the complex unit, subject to the initial condition $x(0) = x_0$.

(ii) Using your solution in (i), or otherwise, show that, subject to $x(0) = x_0$, the solution of

$$\frac{dx}{dt} + \lambda x = A\sin\omega t,$$

where A is a real number, is given by

$$x(t) = (x_0 + \omega P)e^{-\lambda t} + P\left[\lambda\sin\omega t - \omega\cos\omega t\right],$$

and obtain an expression for P in terms of the quantities in Eq. (2.1).

Problem I-3. (Thermodynamics)

A surface film has a surface tension, σ, given by

$$\sigma(T, A) = -\frac{NkT}{A - b},$$

where k is the Boltzman constant, T is the temperature, and b is a constant with units of area.

(i) Show that the heat capacity at constant area, C_A, is a function of temperature only and is independent of the area of the film.

(ii) Show that the internal energy of the film is a function of temperature alone.

Problem I-4. (Statistical Mechanics)

N diatomic molecules are bound to a metal surface of square symmetry, with each molecule bound to a particular lattice site. Each molecule can either lie flat on the surface, in which case it must be aligned to one of two directions x and y, or it can stand up along the z-direction. There is an energy cost $\epsilon > 0$ associated with a molecule standing up, and zero energy for molecules lying flat along the x- or y-directions.

(a) How many microstates are there for the smallest value of the total energy? How many microstates are there for the largest value of the total energy?

(b) For the macrostate of energy E, calculate the number of microstates $\Omega(E, N)$, and the entropy $S(E, N)$, for $N \gg 1$.

(c) Calculate the heat capacity $C(T)$ and sketch it.

(d) What is the probability that a given molecule is standing up?

(e) What is the largest possible value of the internal energy at any $T > 0$?

Problem I-5. (Electricity and Magnetism)
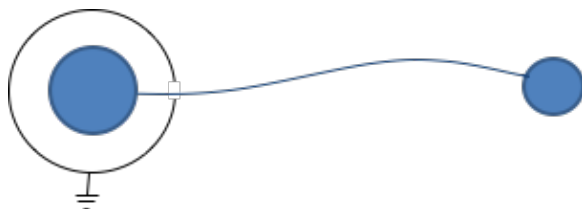
Two solid metal spheres of radii a and b are positioned far away from each other and connected by a long thin wire. A total charge Q is deposited on them. Next, sphere a is surrounded by a thin concentric conducting shell of radius βa, where β is a number greater than 1. This shell is grounded and has a very small hole which allows the wire connecting the spheres to pass without touching it. How much charge flows through the wire as a result, and in what direction?


Problem I-6. (Quantum Mechanics)

Let us consider a three-dimensional anisotropic harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right), \qquad m, \omega_{x,y,z} > 0. \tag{6.1}$$

(i) Using your knowledge of the 1D harmonic oscillator, what are eigenfunctions and eigenvalues of H? (The explicit form of the 1D eigenfunctions is not needed.)

(ii) Assume $\omega_z > \omega_y = \omega_x$ and find the ground state of H and the first excited state(s), the corresponding eigenvalues and degeneracies.

(iii) Set $\omega_z = \omega_y = \omega_x$ (isotropic case). What is $[L_\alpha, H]$ for $\alpha = x, y, z$, where L_α's are the components of the angular momentum? Explain the result.

(iv) Set now $\omega_z = 0$ in (6.1), with $\omega_x = \omega_y > 0$. What are the new eigenvalues and eigenfunctions? What is the main qualitative difference from the former cases?

Problem I-7. (Solid State)
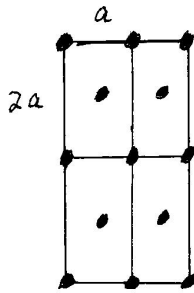
A two-dimensional crystal lattice has a rectangular structure. The rectangle has length a in the x-direction and length $2a$ in the y-direction. Lattice points are located on all four corners of the rectangle. In addition there is another lattice point located in the center of each rectangle, as shown in the picture:

(i) This lattice structure can be described by a primitive Bravais lattice that has only one lattice point per unit cell. Then all lattice points can be written as $\boldsymbol{R} = m\,\boldsymbol{a}_1 + n\,\boldsymbol{a}_2$, where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the primitive lattice vectors and m and n are integers. Find the two vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

(ii) The primitive unit cell is a parallelogram with the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ on two of its sides. Draw a picture of the primitive unit cell.

(iii) Calculate the area of the primitive unit cell.

(iv) The reciprocal lattice is the set of points $\boldsymbol{G} = p\,\boldsymbol{b}_1 + q\,\boldsymbol{b}_2$ where $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are the primitive vectors of the reciprocal lattice and p and q are integers. The vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are determined by the equation $\boldsymbol{b}_i \cdot \boldsymbol{a}_j = 2\pi\delta_{ij}$ so that $\exp(i\,\boldsymbol{G}\cdot\boldsymbol{R}) = 1$ for all points on both lattices. Calculate $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$.

(v) Draw a picture of the reciprocal lattice.

(vi) Calculate the area of the primitive unit cell of the reciprocal lattice.

(vii) The first Brillouin zone of the reciprocal lattice about the point $\boldsymbol{G} = 0$ is the region of reciprocal space that is closer to that point than to any other reciprocal lattice point. Draw a picture of the first Brillouin zone.

Problem I-8. (Relativity)

A photon of frequency ν is moving down the y axis (headed towards $y = -\infty$) of the LAB frame (t, x, y, z). Consider another frame (t', x', y', z') that is moving at a speed v along the x-axis of the LAB frame, headed in the direction of positive x.

(i) What is the 4-momentum of the photon in both the LAB frame and in the moving frame. Give an expression, in terms of v, for the angle, ϕ, that the photon makes to the y' axis in the moving frame. What is the frequency, ν', of the photon in the moving frame?

Suppose that in the moving frame there is a fixed, flat mirror, making an angle θ to the y'-axis (with θ measured in the (t', x', y', z') frame), and tilted back along the negative x'-axis. The mirror reflects the photon.

(ii) What is the 4-momentum of the photon in the (t', x', y', z') after reflection?

(iii) What is the relation between v and θ that would result in the reflected photon traveling along the positive x' axis?

(iv) If the mirror is arranged as in (iii), what is the frequency of the resulting photon in the LAB frame? Express the result in terms of ν and v.

Hint: The Lorentz transformation between the two frames is

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \qquad \frac{v}{c} = \tanh\beta.$$

Problem I-9. (Astrophysics)

A binary-star system with circular orbits has stars of mass 10 and 5 in solar units. *Suddenly*, the mass-10 star undergoes a supernova-II explosion and becomes a neutron star of mass 1, while the ejected mass is lost in space, far away from the binary-star system. The explosion is so sudden that the initial position and velocity of the neutron star are those of the mass-10 star when the explosion occurred.

(i) Show that the resulting system (of masses 5 and 1) becomes unbound.

(ii) Can you imagine a non-generic scenario of the explosion that would allow to keep the stars together?

Hint: To determine the total energies of the system before and after the explosion, use the

Virial Theorem: In a bound system of particles subject only to mutual gravitational interactions, the gravitational potential energy of the entire system $E_{\text{pot}}^{\text{tot}}$ is related to the kinetic energy of the entire system $E_{\text{kin}}^{\text{tot}}$ via

$$E_{\text{pot}}^{\text{tot}} = -2E_{\text{kin}}^{\text{tot}}.$$

Problem I‑10. (Optics)
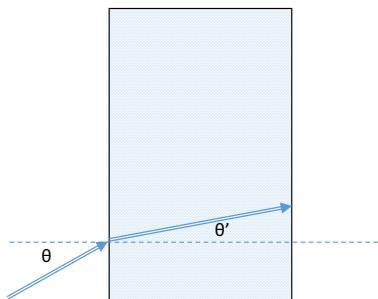
A plane wave of light having wavelength λ in vacuum is incident on a glass slab of thickness d and refractive index n. The entrance and exit surfaces of the slab are perfectly plane and parallel, and are coated to be partially reflecting. The propagation direction of the incident wave makes an angle θ in air with the normal to the left surface, as shown.

For your answers, please use the sheet with the drawing provided in the packet.

(i) On the drawing, **carefully** sketch a few of the transmitted and reflected rays, including rays multiply reflected inside the glass. Label all angles.

(ii) **Derive** a condition on λ, n, d, and either θ or θ' (but not both) so that all the transmitted rays exiting from the glass to the right form an interference maximum. Clearly show all your work, and express your final answer in its simplest form. (Do not use any small-angle approximations.)

Problem I‑11. (Biophysics)

The cyanobacteria *Prochlorococcus* is one of the most numerous organisms on the Earth. It is abundant in the top 100 meters of the world's oceans at a cell density of approximately 10^5 cells/mL.

(a) Estimate the total number of *Prochlorococcus* on the Earth (assuming all oceans have 10^5 cell/mL for the top 100 m of water). (*Hint:* the diameter of the Earth is $12{,}742$ km.)

(b) How does the mass of all the *Prochlorococcus* on Earth compare to the mass of all *Homo sapiens* on Earth?

(c) Given that each *Prochlorococcus* has 8 carboxysomes, and each carboxysome has 250 Rubisco octamers, estimate the number of Rubisco octamers contained in all the *Prochlorococcus* on the Earth.

(d) Given that a Rubisco octamer has the size of a typical protein, how much space in cubic meters (m^3) is taken up by all the Rubisco octamers in all the *Prochlorococcus* cells?

(e) If we assume each Rubisco octamer fixes carbon at a rate of 1 carbon per second, how many molecules of CO_2 are fixed every year by all the *Prochlorococcus* on the Earth? How does this compare with the 10^{14} kg of carbon dioxide fixed through the photosynthetic activity of all organisms on Earth?

(f) Given that it takes approximately 10 photons to fix one atom of carbon, how much energy was needed to fix 10^{14} kg of carbon dioxide?

Department of Physics and Astronomy
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Graduate Screening Examination
Part II
Saturday, March 25, 2017

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ————————————
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Lg-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1   ☐ 2   ☐ 3   ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II‑1. (Classicial Mechanics)

Consider the Lagrangian

$$L = \frac{1}{2}\, m\, e^{2\gamma t}\left(\dot{x}^2 - \omega^2 x^2\right),$$

for the motion of a particle of mass m in one dimension. The constants m, ω and γ are real and positive.

(i) Find the equation of motion. Interpret it by stating what kind of forces the particle is subject to.

(ii) Find the canonical momentum, p, and construct the Hamiltonian, $H(x, p, t)$. Is the Hamiltonian a constant of motion? Is the energy conserved? Explain.

(iii) Find a canonical transformation, $(x, p) \to (X, P)$, under which the transformed Hamiltonian, $K(X, P)$, does not depend explicitly on time. Using your favorite method, verify that the proposed transformation is indeed canonical. Obtain the new Hamiltonian, $K(X, P)$, and the Hamilton equations of motion for $X(t)$ and $P(t)$.

(iv) Assuming that $\gamma < \omega$, find the solution for the motion (i.e., $(X(t), P(t))$) with the initial conditions $X(0) = 0$ and $P(0) = P_0$.

Problem II‑2. (Electricity and Magnetism)

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/(4\pi R^2)$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$, where there is no charge.

(i) Expand the potential in Legendre polynomials and show that the potential inside the spherical surface is, with the origin at the center of the sphere,

$$\Phi = \frac{Q}{8\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]\frac{r^\ell}{R^{\ell+1}}P_\ell(\cos\theta),$$

where here it is to be understood that $P_{-1}(\cos\alpha) = -1$. You may want to use

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}}\, P_\ell(\hat{x} \cdot \hat{x}'),$$

$$\frac{dP_{\ell+1}(\mu)}{d\mu} - \frac{dP_{\ell-1}(\mu)}{d\mu} = (2\ell+1)P_\ell(\mu).$$

(ii) What is the potential outside the surface?

(iii) Find the magnitude and direction of the electric field at the origin.

(iv) Discuss the limiting forms of the potential in part (i) when (a) the spherical cap is very small, and (b) the spherical cap is so large that the area with charge on it is a very small cap at the south pole.

Problem II - 3. (Quantum Mechanics)

Consider a system whose state $|\psi(t)\rangle$ at time t and two observables $\boldsymbol{A}$ and $\boldsymbol{B}$ are given in some basis by the following matrices

$$|\psi(t)\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \boldsymbol{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Compute the eigenvalues a_1, a_2, a_3 of the operator $\boldsymbol{A}$, and similarly for $\boldsymbol{B}$, and exhibit the corresponding (normalized) eigenstates denoted as $|a_1\rangle, |a_2\rangle, |a_3\rangle$, and $|b_1\rangle, |b_2\rangle, |b_3\rangle$.

(b) What is the probability that a measurement of $\boldsymbol{A}$ at time t yields the value -1?

(c) Let us carry out a set of two measurements where $\boldsymbol{B}$ is measured first and then, immediately afterwards, $\boldsymbol{A}$ is measured. Find the probability of obtaining a value of 0 for $\boldsymbol{B}$ and a value of 1 for $\boldsymbol{A}$.

(d) Now we measure $\boldsymbol{A}$ first then, immediately afterwards, $\boldsymbol{B}$. Find the probability of obtaining a value of 1 for $\boldsymbol{A}$ and a value of 0 for $\boldsymbol{B}$.

(e) Compare the results of (c) and (d). Explain what should be expected in such a comparison before doing the computation.

(f) Which among the sets of operators $\{\boldsymbol{A}\}$, $\{\boldsymbol{B}\}$, and $\{\boldsymbol{A}, \boldsymbol{B}\}$ form a complete set of commuting operators (CSCO)?

Problem II-4. (Math Methods)

Part A. The Sokhotski-Plemelj formula is an equality between distributions of the form

$$\frac{1}{x \pm i\epsilon} = a\, P\!\left(\frac{1}{x}\right) + b\, \delta(x)\,, \tag{4.1}$$

where a and b are some constants.

(i) Explain how each term in (4.1) is defined as a distribution acting on test functions.

(ii) Evaluate both sides of (4.1) on the test functions,

$$\varphi_1(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \varphi_2(x) = \begin{cases} x & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases},$$

and use the result to determine the constants a and b.

Hint: The following integrals might be useful

$$\int_{-1}^{1} \frac{1}{x^2 + \epsilon^2}\, dx = \frac{2}{\epsilon} \arctan \frac{1}{\epsilon}\,, \qquad \int_{-1}^{1} \frac{x^2}{x^2 + \epsilon^2}\, dx = 2 - 2\epsilon \arctan \frac{1}{\epsilon}\,.$$

Part B. Let $f(x)$ be a complex valued function of the real argument, x. Suppose that $f(x)$ can be analytically continued to a function, $F(z)$, analytic in the upper-half-plane, including the real line. In other words, $F(z)$ is analytic for $\operatorname{Im} z \ge 0$ and $F(x) = f(x)$ for $x \in \mathbb{R}$. Assume also that

$$\lim_{|z| \to \infty} |F(z)| = 0\,, \qquad 0 \le \arg z \le \pi\,,$$

where the decay is as fast as needed.

(iii) Using the Cauchy integral formula, argue that

$$F(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}\, dx\,, \qquad \operatorname{Im} z_0 > 0\,, \tag{4.2}$$

where the integral is over the real axis and $z_0 = x_0 + i\epsilon$, $\epsilon > 0$, lies above the real axis.

(iv) By considering the limit of (4.2) as $z_0 \to x_0$ becomes real, show that $f(x_0)$ is given by the following principal value integral

$$f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx\,, \qquad x_0 \in \mathbb{R}\,.$$

(v) Suppose that $f(x)$ satisfies

$$f(-x) = \overline{f(x)}\,, \tag{4.3}$$

e.g., $f(x)$ is a Fourier transform of a real function. Show that for such a "symmetric" function $f(x)$ as in (4.3), the real and imaginary parts, $f(x) = u(x) + iv(x)$, satisfy the *disperson relations*

$$u(x_0) = \frac{2}{\pi} P \int_{0}^{\infty} \frac{x v(x)}{x^2 - x_0^2}\, dx\,, \qquad v(x_0) = -\frac{2}{\pi} P \int_{0}^{\infty} \frac{x_0 u(x)}{x^2 - x_0^2}\, dx\,,$$

and hence are not independent.