QUALIFYING EXAMINATION, Part 1

2:00 PM – 5:00 PM, Thursday September 2, 2010

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Mathematical Methods

1. Consider the integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \qquad (0 < a < 1)\ .$$

(a) Evaluate I up to the a^2 term by expanding the integrand to order a^2. (10 points)

(b) Evaluate I by contour integration. (35 points)

Hint: substitute $z = e^{i\theta}$.

(c) Compare the exact result in (b) with the perturbative result in (a). (5 points)

2. Find the eigenvalues and three orthonormal real eigenvectors of the matrix M below. (30 points)

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

3. The function $F(x)$ is defined by

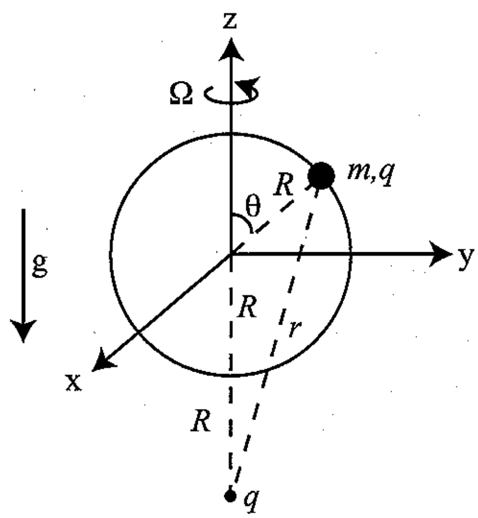
$$F(x) = \left[\frac{1+x}{x^2}\right] \left[\frac{2+2x}{1+2x} - \frac{\ln(1+2x)}{x}\right]\ .$$

Evaluate

$$F(0) = \lim_{x\to 0} F(x)\ . \qquad (20\ \text{points})$$

Problem 2: Classical Mechanics

A charged bead of mass m and electric charge q is threaded on a circular massless and frictionless hoop of radius R. The hoop rotates with angular velocity Ω around a vertical z axis and sits in a vertical gravitational field of acceleration g. A second, stationary charge q is placed along the z axis at a distance R below the bottom of the hoop. Use as generalized coordinate the angle θ between the bead and the vertical z axis (see figure).



(a) Using a cartesian laboratory frame whose origin is at the center of the hoop, write down the transformation equations expressing the cartesian coordinates x, y, z in terms of the generalized coordinate θ. (10 points)

(b) Using your results from part (a) show that the kinetic energy of the bead is

$$T = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \sin^2\theta) \, .$$

Also calculate its potential energy as a function of θ, recalling that the electrostatic contribution has the form

$$V_{\text{elec}} = k\frac{q^2}{r} \quad (k > 0) \, ,$$

where $r = R(5 + 4\cos\theta)^{1/2}$ is the distance between the static charge and the bead. (15 points)

(c) Write down the Lagrangian of the bead and derive the corresponding Euler-Lagrange equation for θ. (25 points)

(d) To simplify the algebra set $\Omega^2 = \frac{4g}{5R}$. Show that there are always two 'trivial' equilibrium positions $\theta_0 = 0, \pi$ for the bead (not necessarily stable) but that, for a certain range of the (dimensionless) ratio

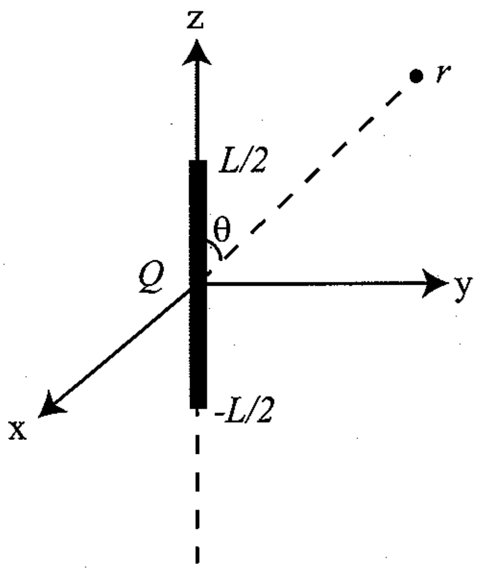
$$\lambda = \frac{kq^2}{mgR^2}$$

there will be an additional non-trivial equilibrium position. Find the corresponding range for λ where this solution exists and express this new equilibrium position in terms of λ. (25 points)

(e) Returning to the exact Lagrangian of part (c), calculate the Hamiltonian of the bead as a function of θ and its conjugate canonical momentum p_θ. Is this Hamiltonian a conserved quantity ? Is it equal to the energy of the bead ? (25 points)

Problem 3: Electromagnetism I

A thin nonconducting rod of length L carries a uniformly distributed charge Q and is oriented along the z-axis (see figure).



(a) Find the electrostatic potential Φ due to the charged rod for any point on the z-axis with $z > L/2$. (35 points)

(b) Find $\Phi(r, \theta)$ for all $|\mathbf{r}| > L/2$, where r and θ are the usual spherical coordinates. Note that since this problem has azimuthal symmetry, Φ is independent of ϕ. (45 points)

Hint: the general solution to Laplace's equation in spherical coordinates in the presence of azimuthal symmetry is

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \,,$$

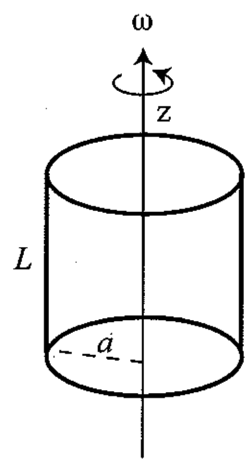
where A_l, B_l are constants and P_l are Legendre polynomials. Also

$$P_l(1) = 1 \,.$$

(c) Check your result in (b) in the limit $r \gg L$. (20 points)

Problem 4: Electromagnetism II

1. A thin cylindrical shell of radius a, height L and constant surface charge density σ rotates along the cylinder axis at a constant frequency ω (see figure).



(a) Calculate the magnetic field $\vec{B}$ along the symmetry axis z of the cylinder as a function of z. Assume the cylinder is centered at z=0 (35 points).

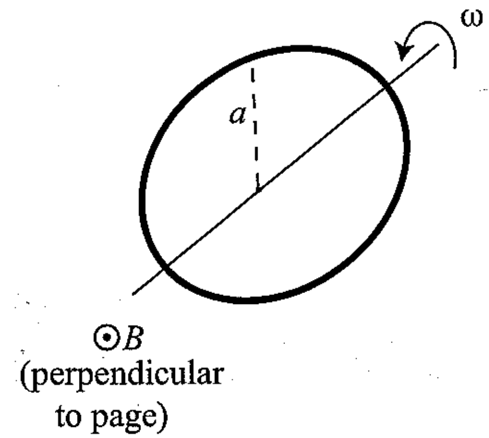
Hint:

$$\int \frac{1}{(a^2+x^2)^{3/2}}dx = \frac{x}{a^2\sqrt{a^2+x^2}} + \text{const.}$$

(b) Consider a circular wire ring carrying a current. What is the single condition on the current and ring radius that will ensure that the same magnetic field is reproduced by the ring as by the rotating cylinder in part (a) at large z ($z >> a, L$)? Note that it is not necessary to take explicitly the limit of the expression you found in (a) (10 points)

(c) Does the rotating cylindrical shell emit electromagnetic radiation? (5 points)

2. A conducting loop of radius a, resistance R, and moment of inertia $\mathcal{I}$ is rotating around an axis in the plane of the loop, initially at an angular frequency ω_0. A uniform static magnetic field B is applied perpendicular to the rotation axis (see figure).



(a) What is the initial kinetic energy of the loop? (5 points)

(b) Calculate the rate of kinetic energy dissipation, assuming it all goes into Joule heating of the loop resistance (30 points).

(c) In the limit that the change in energy per cycle is small, derive the differential equation that describes the time dependence of the angular velocity ω. How long will it take for ω to fall to $1/e$ of its initial value? (15 points)

Hint: In the above limit you can replace the instantaneous rate of energy dissipation by its average value over a cycle.

QUALIFYING EXAMINATION, Part 2

9:00 AM – noon, Friday September 3, 2010

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Quantum Mechanics I

1. A (Stern-Gerlach) magnet with $\vec{B} = B\hat{z}$ ($\hat{z}$ is a unit vector along the z direction) is used to separate a beam of spin 1/2 atoms into spin up atoms in the state $|+\rangle$ such that $\hat{S}_z|+\rangle = \hbar/2|+\rangle$ and spin down atoms in the state $|-\rangle$ such that $\hat{S}_z|-\rangle = -\hbar/2|-\rangle$. Here the operator $\hat{S}_z$ is the z-component of the spin angular momentum $\hat{\vec{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$.

(a) The direction of the magnetic field is suddenly switched from $\hat{z}$ to $\hat{z}'$, where $\hat{z}'$ is a unit vector defined through the polar angle θ and the azimuthal angle ϕ: $\hat{z}' = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. What are the eigenvalues of $\hat{S}_{z'} = \hat{\vec{S}} \cdot \hat{z}'$? Determine the eigenstate $|z', +\rangle$ associated with spin up along $\hat{z}'$ as a linear combination of $|+\rangle$ and $|-\rangle$. (30 points)

Hint: it is convenient to use the spinor eigenvector notation:

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) A measurement of $\hat{S}_z$ is carried out on an atom in the state $|z', +\rangle$. What are the probabilities that this measurement yields the values $\pm\hbar/2$. (20 points)

2. Consider a spinless charged particle in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. The Hamiltonian takes the form

$$H = \tfrac{1}{2m}\left(\hat{\vec{p}} - \tfrac{e}{c}\vec{A}(\vec{r})\right)^2,$$

where e and m are the charge and mass of the particle, respectively, and $\hat{\vec{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ is the momentum conjugate to the particle's position $\vec{r}$. Let $\vec{A} = -By\hat{x}$ corresponding to a constant magnetic field $B\hat{z}$ ($\hat{x}$ and $\hat{z}$ are unit vectors along x and z directions.)

(a) Prove that $\hat{p}_x$ and $\hat{p}_z$ are constants of the motion. (20 points)

(b) It follows from (a) that the corresponding eigenfunction of H is of the form:

$$\psi(x, y, z) = e^{i(xp_x + zp_z)/\hbar}\phi(y)\ ,$$

where p_x and p_z are constants. Derive the eigenvalue equation satisfied by ϕ and determine the quantum energy levels of H. Provide a classical interpretation of the corresponding eigenstates. (30 points)

Problem 2: Quantum Mechanics II

This problem is focused on approximate methods in quantum mechanics.

1. Consider a particle with charge q and mass m in a one-dimensional harmonic oscillator potential described by the Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ (where x is the coordinate and p is the momentum operator). An external electric field b is applied in the positive x direction which adds to H_0 the potential $V(x) = -qbx$.

(a) Use perturbation theory to calculate the first- and second-order shifts in the energy levels. (40 points)

Hint: it is convenient to use creation and annihilation operators, $a^\dagger$ and a, defined by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}\,p\right) \;; \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}\,p\right) .$$

(b) This problem can be solved exactly. Find the exact eigenvalues and compare them with the perturbative result in (a). (15 points)

2. An electron of mass m is moving in a Coulomb potential $V(r) = -\frac{e^2}{r}$. Consider the following (unnormalized) variational wave function in spherical coordinates r, θ, ϕ

$$\Phi(r, \theta, \phi; b) = e^{-br} ,$$

where b is a variational parameter.

(a) Argue that Φ has a good orbital angular momentum l. What is the value of l ? (5 points)

(b) Use Φ to evaluate a variational upper bound to the ground-state energy. (40 points)

Hint:

$$\int_0^\infty z^n e^{-\mu z} dz = \frac{1}{\mu^{n+1}} n!$$

Problem 3: Statistical Mechanics I

Degenerate Fermi gas is of great importance for understanding the properties of white dwarfs, neutron stars and the electron gas in metals. In this problem, we consider the physics of white dwarfs and neutron stars, in which strong gravitational force is balanced by the degeneracy pressure of electrons and neutrons, respectively.

Consider a perfect Fermi gas at zero temperature composed of N particles of mass m and spin-1/2 contained in a cubical box of volume V.

(a) Show that the Fermi energy of the system is given by

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3} ,$$

where $\rho = N/V$ is the number density of particles. (20 points)

(b) Show that the condition for the gas to be non-relativistic is (10 points)

$$\rho \lesssim \left(\frac{mc}{\hbar}\right)^3 .$$

You may ignore constants of order unity.

(c) Show that the total kinetic energy of a non-relativistic Fermi gas is given by

$$E_{\rm nr} = C_1 M^{5/3}/R^2 ,$$

where R is the linear dimension of the gas and M is its total mass. It is not necessary to evaluate the proportionality constant C_1 in this expression. (20 points)

(d) Show that a non-relativistic gas of fixed mass M can always find an equilibrium size R_0 such that the sum of its kinetic and gravitational energies is minimized. Use dimensional arguments to estimate the gravitational energy as a function of M and R. (15 points)

(e) Express the total kinetic energy $E_{\rm rel}$ of a relativistic gas as a function of M and R. It is not necessary to determine the proportionality constant in this expression. Assume the extreme relativistic limit where the rest mass can be set to zero. (20 points)

(f) Show that for a relativistic gas there exists a critical mass M_c above which gravitational collapse occurs. M_c is called the *Chandrasekhar* mass, and it sets the mass of white dwarfs and neutron stars. (15 points)

Problem 4: Statistical Mechanics II

Consider a photon gas enclosed in a volume V and in equilibrium at temperature T.

(a) What is the chemical potential of the gas? Explain why. (10 points)

(b) Calculate the free energy F of the photon gas as a function of T and V from $F = -kT \ln Z$, where Z is its partition function and use it to find its entropy S. You may use the relation $S = -\left(\partial F/\partial T\right)_V$. You do not need to evaluate the proportionality constants (except for their overall sign) in the expressions for F and S. (25 points)

(c) The photon gas undergoes an *adiabatic* quasi-static expansion such that its volume is doubled ($V' = 2V$). Find the final temperature T' of the gas in terms of its initial temperature T. (20 points)

(d) Derive the equation of state

$$pV = \frac{1}{3}E ,$$

where of the p is the pressure and E is the energy of the photon gas. (25 points)
Hint: use the relation $p = kT\left(\frac{\partial \ln Z}{\partial V}\right)_T$. Integration by parts may also be useful.

(e) Show that the equation of state in (d) holds for any non-interacting gas of identical particles of *zero* rest mass, irrespective of their statistics. (20 points)
Hint: while it is possible to follow a method similar to (d) for either bosons or fermions, a shorter derivation starts from

$$p = \sum_r \left(-\frac{\partial \epsilon_r}{\partial V}\right) \langle n_r \rangle ,$$

where ϵ_r is the energy of single-particle state r and $\langle n_r \rangle$ is the average occupation of state r. The dependence of ϵ on V is determined from the dispersion relation (i.e., the relation expressing the energy ϵ versus momentum $\hbar k$) and the quantization condition on $\mathbf{k}$.