Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 24, 2012

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________
Fill in your S-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your S-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve **eight** problems of your choice. Do not turn in more than this number (8) of problems!

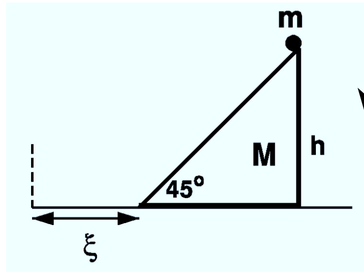
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1	☐ 2	☐ 3	☐ 4	☐ 5
☐ 6	☐ 7	☐ 8	☐ 9	☐ 10

Problems that are not checked above, will not be graded. If you check more than 8 problems, only the lowest 8 scores will count towards your total score.

I - 1. (Classical Mechanics)



A mass m, initially at rest, slides down without friction from the top (i.e., at a height $y = h$) of an inclined plane that is 45° from the horizontal on the left side of a block of mass M, which moves freely on a frictionless horizontal surface as shown. Assume that M is also at rest initially and the motion of the system is limited in a plane. Find the equation of motion in ξ. How much time would it take for the mass m to reach the horizontal floor?

I - 2. (Electricity and Magnetism)

An electric dipole is oscillating at the origin. The corresponding current density is given by

$$\mathbf{J} = \omega\, p\, \delta^{(3)}(\mathbf{r}) \cos(\omega t)\, \widehat{\mathbf{z}}\,,$$

where $\widehat{\mathbf{z}}$ is the unit vector in the z-direction, $\delta^{(3)}(\mathbf{r})$ is the three-dimensional δ-function (which vanishes away from the origin and whose volume integral is unity if the region of integration covers the origin), ω is the angular frequency of oscillation, and p is the magnitude of the oscillating electric dipole moment.

(i) The time-dependent vector potential is given (in a Lorentz gauge) by

$$\mathbf{A}(\mathbf{r}, t) \;=\; \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}\,.$$

Calculate $\mathbf{A}$ for the $\mathbf{J}$ given above.

(ii) The magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$. Calculate the leading contribution to $\mathbf{B}$, which is proportional to $1/r$ for large distances r from the origin (in the radiation zone).

(iii) In the radiation zone for large r, the electric field is given by $\mathbf{E} = -c\,\mathbf{r} \times \mathbf{B}$, so the power P radiated per solid angle Ω is given by

$$\frac{dP}{d\Omega} \;=\; r^2 \, \frac{c}{\mu_0}\, |\mathbf{B}|^2\,.$$

Evaluate the time-averaged $dP/d\Omega$ for the magnetic field in part (iv) as a function of the angle θ with respect to the z-axis.

(iv) Integrate the time-averaged $dP/d\Omega$ over the spherical surface at large r to find the total power P radiated.

I - 3. (Mathematical Methods)

Compute the following integrals:

(i) The integral involving the Dirac δ "function" (distribution):

$$I_\delta \;=\; \int_1^\infty \sin t \; \delta(t^2 - \pi^2/4) \; dt \, .$$

(ii) The contour integral

$$I_{\mathcal{C}} \;=\; \oint_{\mathcal{C}} \frac{z}{\sin z} \, dz \, ,$$

where $\mathcal{C}$ is a circle of radius $3\pi/2$ centered at $z = 0$ and oriented clockwise.

I - 4. (Quantum Mechanics)

Consider the following one-dimensional potential:

$$V(x) \;=\; \begin{cases} +\infty & \text{if } x \leq 0 \\ \frac{1}{2}\, m\omega^2 x^2 & \text{if } x > 0 \end{cases}$$

(i) What is the spectrum of the Hamiltonian $\widehat{H} = \widehat{p}^{\,2}/2m + V(\widehat{x})$?

(ii) What is the eigenfunction of the ground state of this system? Verify by a direct calculation that the expectation value of the momentum operator, $\widehat{p}$, in this state is vanishing.

(iii) Argue that the expectation value of $\widehat{p}$ will vanish in any eigenstate of this Hamiltonian.

I - 5. (Thermodynamics)

The van der Waals equation of state is

$$\left(P + a\frac{N^2}{V^2}\right)(V - Nb) = Nk_BT \, , \tag{1}$$

where k_B is Boltzmann's constant, a and b are constants, and T, V, and N denote the temperature, volume, and number of particles of the gas. The associated internal energy of a van der Waals gas is given by

$$U = \frac{3}{2}Nk_BT - a\frac{N^2}{V} \, . \tag{2}$$

Use (1) and (2) to show that the entropy of a van der Waals gas as a function of temperature and volume, $S = S(T, V)$, is given by

$$S(T, V) = S(T_0, V_0) + c_V \ln\left(\frac{T}{T_0}\right) + Nk_B \ln\left(\frac{V - bN}{V_0 - bN}\right) \, ,$$

and find an expression for c_V, where T_0 and V_0 refer to the temperature and volume in a reference state.

I - 6. (Statistical Physics)

We consider a one dimensional system of classical spins $S_i = \pm 1$ on a chain of length N. We take periodic boundary conditions so that we set formally in all equations $S_{N+1} \equiv S_1$. The energy is made up of a nearest neighbor coupling and a magnetic field, and reads

$$E = -J \sum_{i=1}^{N} S_i S_{i+1} - H \sum_{i=1}^{N} S_i \, .$$

(i) Introduce the two by two matrix whose elements are

$$T_{ss'} = e^{\beta[JSS' + \frac{1}{2}H(S+S')]} \, ,$$

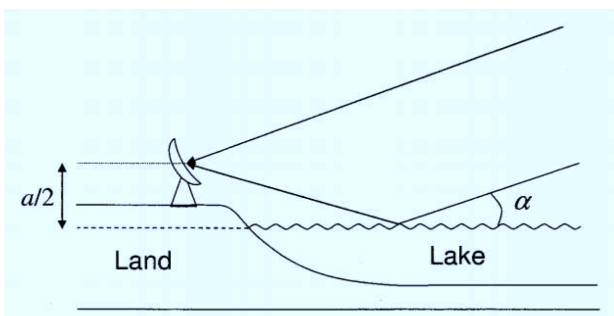
where $S, S' = \pm 1$ label the rows and columns, $\beta = \frac{1}{k_B T}$. Prove that the partition function of the system can be written as

$$Z_N = \mathrm{Tr}\; T^N \, .$$

(ii) Use this to show that the free energy per site in the thermodynamic limit is

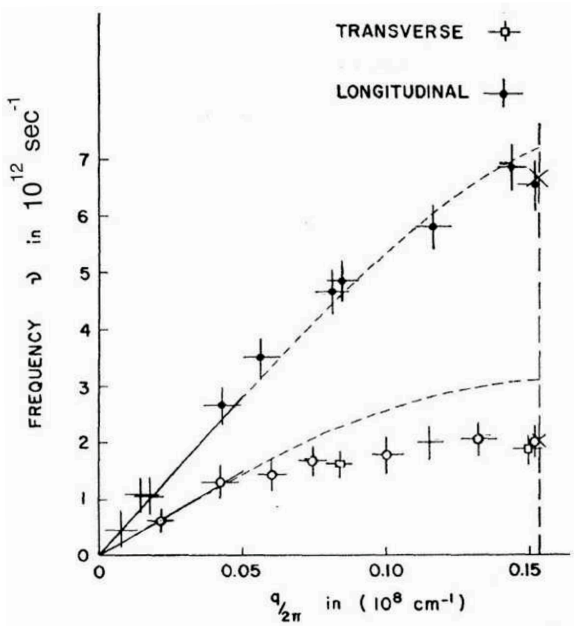
$$f = -\lim_{N\to\infty} \frac{T}{N} \ln Z_N = -k_B T \ln\left[e^{\beta J} \cosh \beta H + \left(e^{2\beta J} \sin^2 \beta H + e^{-2\beta J} \right)^{1/2} \right] \, .$$

I - 7. (Optics)



An antenna at the edge of a lake picks up a signal from a distant radio star (see figure), which is just coming up above the horizon. Write an expression for the phase difference, δ, and for the angular position, α, of the star when the antenna detects its first maximum. Express your answers in terms of α, a and the wavelength λ.

I-8. (Condensed Matter)



The figure shows part of the phonon dispersion relations along the [111]-direction in germanium (Ge, atomic number 32). Here q is the wave vector and ν is the frequency (not the angular frequency). Ge crystallizes in the diamond structure. You may use data from the plot to answer the following questions. All the answers must be explained, concisely but clearly.

(i) An ultrasound generator is placed at the surface of a (111)-oriented Ge crystal. With what speed will the sound waves travel in the material?

(ii) Sketch the molar heat capacity C of pure Ge from 0 K to above room temperature. On the C and the T axes, show the value of the high-temperature heat capacity, and the temperature (in Kelvins) at which the heat capacity transitions to this high-temperature limit.

(iii) Silicon (Si, atomic number 14) is similar to germanium in many ways, and crystallizes in the same structure. Add a drawing to your plot from (ii), showing approximately how you expect the molar heat capacity of Si to behave relative to that of Ge, and explain the key similarities and differences.

I-9. (Relativity)

When the Stanford Linear Accelerator was running, electrons and positrons were accelerated to energies of approximately 40 GeV in a beam pipe of two miles (about 3200 m) long, but only a few centimeters in diameter. To steer an electron through such a narrow path over such a great distance seems impressive. But how long is the accelerator from the point of view of an electron or positron at that energy? (You may use that the mass of an electron is about $0.51\,\mathrm{MeV}/c^2$.)

I-10. (Astrophysics)

Suppose that a star has a radius, $R_\star$, and a surface temperature, $T_\star$, and assume that it may be treated as a black body radiator. A planet, orbiting the star at a distance, d, has an albedo (or reflectivity), a, and a radius R_p. Suppose that the planet attains an equilibrium temperature, T_p.

(i) What is the total amount of power absorbed by the planet from the star's radiation?

(ii) What is the black body energy radiated by the planet into space? Assuming thermal equilibrium, obtain an expression for the equilibrium temperature of a planet in terms of its distance from the star.

(iii) The temperature of the Sun's photosphere is 5800 K and it has a radius of 700,000 km. The distance from the Sun to Venus, Earth and Jupiter is 0.7 AU, 1.0 AU and 5.2 AU where 1 AU = 1.5×10^8 km. For simplicity take $a = 0$ for all the planets and estimate the equilibrium temperatures of these planets.

(iv) The estimates in part (iii) are good for Jupiter, moderately good for Earth and way off for Venus. Venus has an equilibrium temperature of 750 K (which is much higher than what you got in part (iii)). Why do you think there is such a large error?

Hint: Wien's law states that the dominant wavelength of emission from a black body of temperature T is given by:

$$\lambda_{\text{dominant}} \approx 0.0029\,\text{m/T}$$

Think about the wavelengths/types of radiation setting up the equilibrium.

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Graduate Screening Examination
Part II
Saturday, March 24, 2012

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Student ____________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

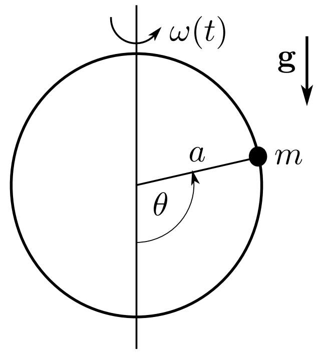
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 h 30 min**.

Please, indicate problems you are turning in:

☐ **II - 1** ☐ **II - 2** ☐ **II - 3** ☐ **II - 4**

II - 1. (Classical Mechanics)



A bead of mass m is constrained to slide without friction on a hoop of radius a. The hoop rotates with angular velocity $\omega(t)$ around a vertical axis which coincides with a diameter of the hoop.

(i) Set up the Lagrangian and obtain equations of motion of the bead.

(ii) Find the Hamiltonian, H, of the bead and calculate the total time derivative dH/dt along trajectories in the phase space.

Assume now that the angular velocity, ω, of the hoop is constant.

(iii) Show that there is a critical angular velocity, ω_0, below which the bottom of the hoop provides a stable equilibrium position for the bead.

(iv) Find the stable equilibrium position for $\omega > \omega_0$ and determine the period of small oscillations around this equilibrium.

(v) For $\omega = 2\omega_0$, describe *qualitatively* the motion of the bead for $t > 0$ if the initial conditions for $t = 0$ are $\theta(0) = 2\pi/3$ and $\dot{\theta}(0) = 0\,\mathrm{rad/s}$.

II - 2. (Mathematical Methods)

Solve

$$\varphi(x) \;=\; 1 + \lambda^2 \int_0^x (x-t)\,\varphi(t)\,dt\,,$$

by each of the following methods:

(i) Reduction to a differential equation (including establishment of boundary conditions).

(ii) The Neumann series.

(iii) The use of Laplace transforms.

II - 3. (Electricity and Magnetism)

A hollow sphere of inner radius R is grounded. Along a straight line passing through the center of the sphere sit three charges. One charge, $-2q$, sits at the center of the sphere. The other two charges are each q, and they sit on opposite sides of the center at a distance $a < R$ from the center. Note that this distribution of charges has no monopole and dipole moments; the lowest nonvanishing moment is quadrupole.

(i) By using the method of images (there are 3 image charges) determine the potential everywhere inside the sphere, for both $r < a$ and $r > a$, in terms of spherical coordinates with the z-axis along the line of charges.

Hint: You may use without proof that the image charge corresponding to q at $z = \pm a$ is $-(R/a)q$ at $z = \pm(R^2/a)$.

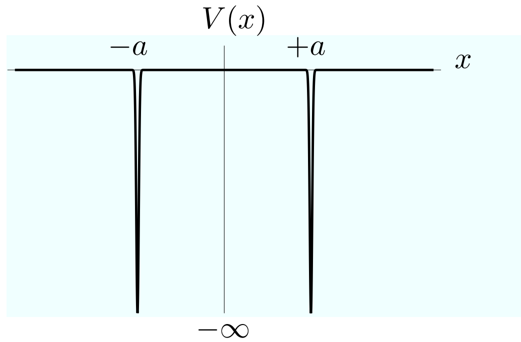
(ii) Show that in the limit $a \to 0$, with $Q \equiv qa^2$ finite, the potential inside the sphere is

$$\Phi(r, \theta, \phi) = \frac{2Q}{r^3}\left(1 - \frac{r^5}{R^5}\right) P_2(\cos\theta)\,.$$

Hint: The following expansion may be useful

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{(r_<)^\ell}{(r_>)^{\ell+1}}\, P_\ell(\cos\theta)\,.$$

II - 4. (Quantum Mechanics)



An electron of mass m is moving from left to right in a one dimensional potential of the form of two delta function wells centered at the origin,

$$V(x) = -\frac{\hbar^2\beta^2}{2m}\,\delta(|x| - a)\,,$$

as in the figure. Assume that the energy, $E = \hbar^2 k^2/(2m)$, is positive and compute the reflection and transmission coefficients. Your answer should include the following parts:

1. Write down the appropriate equations and give the general form of the solution for the three regions: (i) $x < -a$, (ii) $|x| < a$, and (iii) $x > a$. In these expressions identify the parameters that represent the amplitudes for reflection and transmission.
2. Give the equations that connect them into an overall solution.
3. Using the equations you setup in part 2, obtain the solution for the transmission *probability*. If you cannot finish the computation outline the remaining steps.