Spring 2012

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 2

March 21, 2012

(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

7. Two-dimensional Orbit

Consider the following potential in two dimensions with polar coordinates (r, ϕ):

$$V(r) = ar^p + br^q.$$

Imagine a particle with mass m moving in this potential.

(a) Find two conserved quantities for this motion.

(b) Suppose the particle is moving in the potential $V(r)$ along the orbit $r(\phi) = c\phi^2$. Find the values of p and q for which this is possible. Here c is a constant.

(c) Express the values of the two conserved quantities in terms of the parameters of the potential a and b, and the parameter of the orbit c. Find the velocity, v_∞, of the particle at infinity.

Two-dimensional Orbit - Solution

This is a standard problem of a particle motion in a central potential. Here are the typical steps.

(a) The kinetic and potential energies are

$$T = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2}, \qquad V = ar^p + br^q. \tag{1}$$

The Lagrangian

$$L = T - V = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} - ar^p - br^q \tag{2}$$

does not depend explicitly on time, therefore the total energy is conserved:

$$E = T + V = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} + ar^p + br^q. \tag{3}$$

This is the first conserved quantity.

In addition, the coordinate ϕ is cyclic (L does not depend on it), so the corresponding generalized momentum is also conserved. This is the component of the angular momentum of the particle along the axis perpendicular to the plane of the motion:

$$l = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}. \tag{4}$$

This is the second conserved quantity.

(b) When the particle is moving along the trajectory $r(\phi) = c\phi^2$, we can find its radial velocity:

$$\dot{r} = 2c\phi\dot{\phi}. \tag{5}$$

The time derivative of the angle ϕ can be found from the conserved angular momentum (4):

$$\dot{\phi} = \frac{l}{mr^2}. \tag{6}$$

Then the radial velocity becomes

$$\dot{r} = \frac{2c\phi l}{mr^2}. \tag{7}$$

Substituting this and (6) into the energy (3), we obtain

$$E = \frac{2c^2\phi^2 l^2}{mr^4} + \frac{l^2}{2mr^2} + ar^p + br^q = \frac{2cl^2}{mr^3} + \frac{l^2}{2mr^2} + ar^p + br^q. \tag{8}$$

(We have used the equation for the trajectory in the first term.)

The energy is now expressed only in terms of the radial coordinate and constant parameters. Since it is conserved, the r dependence should cancel out. This dictates the possible values of the powers p and q: either

$$p = -2, \qquad q = -3, \tag{9}$$

or vice versa.

(c) Setting now $p = -2$, $q = -3$ in the expression for the energy (8), we also get the following conditions for the parameters that must be satisfied for the energy to be independent of r:

$$\frac{l^2}{2m} = -a, \qquad \frac{2cl^2}{m} = -b. \tag{10}$$

The first one of these gives

$$l = \sqrt{-2ma}, \tag{11}$$

and we also see that the total energy vanishes:

$$E = 0. \tag{12}$$

Since the potential $V(r)$ vanishes at infinity, this implies that the particle starts its motion at infinity with zero velocity:

$$v_\infty = 0. \tag{13}$$

8. Electron on Helium

Electrons can float over the surface of liquid helium because of two effects: (1) attraction of an electron and its image charge, (2) repulsion from the liquid helium surface. Assume that the distance between an electron and the surface of the liquid helium is h, and that the dielectric constant of liquid helium is ϵ.

(a) Find the position and the value of the image charge q' inside the helium (remember that in the case of a dielectric filling the semi-infinite space $z < 0$, one has to introduce different image charges to find potentials and fields for $z > 0$ and $z < 0$). Then find the potential energy of the attraction between the electron and its image charge q' inside the dielectric as a function of h.

(b) Model the repulsion of an electron and the surface of the liquid helium as an infinite potential barrier. Consider the quantum mechanical motion of the electron in the combined potential of the image charge and the surface. Using Heisenberg's uncertainty relation and the quantum virial theorem, estimate the average height $\langle h \rangle$ of the electron above the liquid helium in the ground state. Find the numerical value of $\langle h \rangle$ taking the dielectric constant of liquid helium to be $\epsilon = 1.057$.

Electron on Helium - Solution

(a) By translation symmetry we can place the electron at the point $(0,0,z)$, where the x and y axes are in the plane of the surface of helium, and the z axis is perpendicular to the surface. According to the method of images we should try to find the potential $\phi(\boldsymbol{r})$ for $z>0$ by placing an image charge q' inside the dielectric. Again, using symmetry considerations, the image charge should be placed on the vertical line below the electron, say, at the point $(0,0,-h')$. Denoting the charge of the electron by e, we have for $z>0$

$$\begin{aligned}\phi_+(x,y,z) &= \phi_e(x,y,z)+\phi_{q'}(x,y,z)\\ &= \frac{e}{\sqrt{x^2+y^2+(z-h)^2}}+\frac{q'}{\sqrt{x^2+y^2+(z+h')^2}}, \qquad \text{for } z>0. \end{aligned} \tag{14}$$

Similarly, the potential inside the dielectric is found by placing a charge q'' in the upper half-space at point $(0,0,h'')$, which then gives for $z<0$

$$\phi_-(x,y,z)=\phi_{q''}(x,y,z)=\frac{q''}{\epsilon\sqrt{x^2+y^2+(z-h'')^2}}, \qquad \text{for } z<0. \tag{15}$$

The unknown charges q', q'' and distances h', h'' are found from the following boundary conditions. First, the potential should be continuous across the surface of the dielectric:

$$\phi_+(x,y,0)=\phi_-(x,y,0). \tag{16}$$

This immediately gives that

$$h'=h''=h, \tag{17}$$

$$q''=\epsilon(e+q'). \tag{18}$$

The next boundary condition is the continuity across the surface of the dielectric of the vertical component of the displacement field $\boldsymbol{D}=-\epsilon\nabla\phi$. This gives

$$\left.\frac{\partial\phi_+}{\partial z}\right|_{z=0}=\epsilon\left.\frac{\partial\phi_-}{\partial z}\right|_{z=0}. \tag{19}$$

Using the relations (17, 18), this gives

$$h(q'-e)=-\epsilon h(q'+e), \quad \Rightarrow \quad q'=-\frac{\epsilon-1}{\epsilon+1}e. \tag{20}$$

This is all we need for this problem, but for completeness we can also find the other image charge

$$q'' = \epsilon(e + q') = \frac{2\epsilon}{\epsilon + 1}e. \tag{21}$$

The attraction force acting on the electron from the image charge is

$$F(h) = \frac{eq'}{(2h)^2} = -\frac{\epsilon - 1}{\epsilon + 1}\frac{e^2}{4h^2}. \tag{22}$$

This corresponds to the potential energy

$$V(h) = -\frac{\alpha}{h}, \qquad \alpha = \frac{\epsilon - 1}{\epsilon + 1}\frac{e^2}{4}. \tag{23}$$

(b) The one-dimensional version of the quantum virial theorem states that in a stationary state $|\psi\rangle$ we have the equality

$$\langle\psi|\frac{p^2}{m}|\psi\rangle = \langle\psi|xV'(x)|\psi\rangle, \tag{24}$$

where x is the one-dimensional coordinate, and p is the associated momentum. Applying this to our potential energy $V(h) = -\alpha/h$ gives

$$\langle\psi|\frac{p^2}{m}|\psi\rangle = \langle\psi|\frac{\alpha}{h}|\psi\rangle. \tag{25}$$

Let the spread of the wave function in the ground state is a good estimate for the average $\langle h\rangle$. Then the uncertainty relation gives that the typical value of the momentum in the ground state is of order $p \sim \hbar/\langle h\rangle$. Then, up to a factor of order unity, the virial theorem (25) gives

$$\frac{\hbar^2}{m\langle h\rangle^2} \sim \frac{\alpha}{\langle h\rangle} \qquad \Rightarrow \langle h\rangle \sim \frac{\hbar^2}{m\alpha} = 4\frac{\epsilon + 1}{\epsilon - 1}\frac{\hbar^2}{me^2} = 4\frac{\epsilon + 1}{\epsilon - 1}a_B, \tag{26}$$

where $a_B \approx 0.53$Å is the Bohr radius.

Using the value $\epsilon = 1.057$ gives

$$\langle h\rangle \sim 144a_B \approx 76\text{Å}. \tag{27}$$

The actual analytic solution for the ground state wave function gives

$$\langle h\rangle = \frac{3\hbar^2}{2m\alpha} = 6\frac{\epsilon + 1}{\epsilon - 1}\frac{\hbar^2}{me^2} = 6\frac{\epsilon + 1}{\epsilon - 1}a_B \approx 114\text{Å}. \tag{28}$$

9. Monatomic Gas

Consider one particle, of mass m, in a non-relativistic ideal gas confined to a 3-dimensional region of volume V.

(a) Find the total number of accessible states, $\Gamma(\epsilon)$, with energy less than ϵ for this particle. Also find the density of states $\nu(\epsilon) = d\Gamma/d\epsilon$.

(b) The partition function of this 1-particle system has the form

$$Z_1 = \frac{V}{\lambda_{_T}^3}$$

where $\lambda_{_T}$ is the thermal wavelength. Find an expression for $\lambda_{_T}$ in terms of the gas temperature T.

(c) Now find the partition function, Z, of the entire gas of N such identical particles. Assume Boltzmann statistics. Approximate Z using Stirling's approximation ($N! \approx (N/e)^N$). Using this approximate Z, obtain expressions for the total energy U, the free energy F, the entropy S, and the heat capacity C_V.

(d) Find the results for U, F, and S for a one-dimensional ultra-relativistic ideal gas confined to an interval of length L.

Monatomic Gas - Solution

(a) By applying the usual division of phase-space into cells of volume $(L/h)^3$, the number of allowed states with energy smaller than ϵ is the number of cells in phase-space where the allowed energy is not exceeded:

$$\Gamma(\epsilon) = \frac{V}{h^3}\int_{|\vec{p}|^2/2m<\epsilon} dp_x dp_y dp_z$$

The integral equals the volume of a sphere of radius $\sqrt{2m\epsilon}$, such that

$$\Gamma(\epsilon) = \frac{4\pi V}{3}\left(\frac{\sqrt{2m\epsilon}}{h}\right)^3 = \frac{V}{6\pi^2}\left(\frac{\sqrt{2m\epsilon}}{\hbar}\right)^3$$

and

$$\nu(\epsilon) = \frac{d\Gamma}{d\epsilon} = \frac{V}{4\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3\sqrt{\epsilon}$$

(b) In the Canonical ensemble, the partition function of the single atom is obtained by multiplying the density of states for each energy by the appropriate Boltzmann factor and integrating over the all energies:

$$\begin{aligned} Z_1 &= \int_0^\infty \nu(\epsilon)e^{-\frac{\epsilon}{k_BT}}d\epsilon \\ &= \frac{V}{4\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3\int_0^\infty \sqrt{\epsilon}e^{-\frac{\epsilon}{k_BT}}d\epsilon \\ &= \frac{V}{4\pi^2}\left(\frac{\sqrt{2m}}{\hbar}\right)^3\int_0^\infty 2\alpha^2 e^{-\frac{\alpha^2}{k_BT}}d\alpha \end{aligned}$$

The latter is a Gaussian integral, i.e., the second moment of a Gaussian distribution $\sim (k_BT)^{3/2}$. Comparing to the given form we see

$$Z_1 = \frac{V}{\lambda_T^3} \quad \Rightarrow \quad \lambda_T = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{\frac{1}{2}}$$

It is now convenient to define

$$\beta \equiv \frac{1}{k_BT} \quad , \quad \lambda_T = \left(\frac{2\pi\hbar^2\beta}{m}\right)^{\frac{1}{2}}$$

(c) The partition function of an ideal gas of N noninteracting identical particles can be written down as

$$Z = \frac{{Z_1}^N}{N!} \approx \left(\frac{eZ_1}{N}\right)^N$$

Because of the statistical independence of the particles the overall partition function is $Z \sim Z_1 Z_2 ... Z_{_N} = {Z_1}^N$. However, because the particles are indistinguishable, any permutation of the "particle labels" would result in an identical microstate. Therefore the expression ${Z_1}^N$ counts every microstate $N!$ times. Thus:

$$Z = e^N \left(\frac{r_s}{\lambda_{_T}}\right)^{3N} \quad , \quad r_s = \left(\frac{V}{N}\right)^{\frac{1}{3}}$$

$$F = -\frac{1}{\beta}\ln(Z) = -Nk_{_B}T\left[1 + 3\ln\left(\frac{r_s}{\lambda_{_T}}\right)\right]$$

$$\begin{aligned}
U &= -\left(\frac{\partial \ln(Z)}{\partial\beta}\right)_V = \frac{\partial\left(3N\ln\lambda_{_T}\right)}{\partial\beta} \\
&= \frac{3N}{\lambda_{_T}}\frac{\partial\lambda_{_T}}{\partial\beta} \\
&= \frac{3N}{\lambda_{_T}}\frac{1}{2}\left(\frac{2\pi\hbar^2}{m\beta}\right)^{\frac{1}{2}} = \frac{3}{2}Nk_{_B}T
\end{aligned}$$

$$S = \frac{U-F}{T} = Nk_{_B}\left[\frac{5}{2} + 3\ln\left(\frac{r_s}{\lambda_{_T}}\right)\right]$$

$$C_{_V} = \left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}Nk_{_B}$$

(d) The the dispersion relation of an ultrarelativistic particle is $\epsilon = cp$. Therefore:

$$
\begin{aligned}
\Gamma(\epsilon) &= \frac{L}{h}\int_{-\frac{\epsilon}{c}}^{\frac{\epsilon}{c}} dp = \frac{L\epsilon}{\pi\hbar c} \\
\nu(\epsilon) &= \frac{d\Gamma}{d\epsilon} = \frac{L}{\pi\hbar c} \\
Z_1 &= \int_0^{\infty} \nu(\epsilon) e^{-\beta\epsilon} d\epsilon = \frac{L}{\pi\hbar c\beta} \\
Z &= \frac{Z_1^N}{N!} \approx \left(\frac{eL}{\pi\hbar c\beta N}\right)^N \\
F &= -\frac{1}{\beta}\ln(Z) = -Nk_{_B}T\left[1+\ln\left(\frac{L}{\pi\hbar c\beta N}\right)\right] \\
U &= -\left(\frac{\partial \ln(Z)}{\partial\beta}\right)_V = Nk_{_B}T \\
S &= \frac{U-F}{T} = Nk_{_B}\left[2+\ln\left(\frac{L}{\pi\hbar c\beta N}\right)\right]
\end{aligned}
$$

10. Magnetic Confinement

(a) Imagine passing a very large constant current I through a long wire. The wire vaporizes and forms a neutral plasma with radius R consisting of charge one ions and electrons. For this equilibrium configuration, find a relation between the current I and the temperatures of the electrons and ions. Assume perfect cylindrical symmetry and ignore interactions between the ions and electrons, treating each component as an ideal gas.

(b) Now consider a perfect ion beam of radius R with no electrons. The beam current is I and the velocity of the ions is v. Assume no radial velocity initially. Each ion has charge Q and mass M. Again assume cylindrical symmetry for the charge and current densities, and ignore thermodynamics. Find the net force on an ion living at radius R. Does the ion want to move inward or outward or stay stationary?

(c) Finally, let us consider a more general setup consisting of a neutral plasma moving non-relativistically with velocity $\vec{v}$ in the presence of electric and magnetic fields. The plasma has conductivity σ. Find the current density $\vec{j}$ in terms of $(\vec{v}, \vec{E}, \vec{B})$ for the neutral plasma. This is the general form of Ohm's Law. Ignoring the time variation of $\vec{E}$ (the displacement current), use Maxwell's equations and the Ohm's Law you derived to determine $\frac{\partial \vec{B}}{\partial t}$ in terms of $\vec{v}$ and $\vec{B}$. This is the magnetic induction equation.

Magnetic Confinement - Solution

(a) The magnetic field at radius R is determined by Ampere's Law:

$$B(R) = \frac{\mu_0 I}{2\pi R}.$$

In cgs units, this would read

$$B(R) = \frac{2I}{Rc}.$$

To be in equilibrium, the kinetic pressure and the magnetic pressure must balance at the surface of the wire. The magnetic pressure is

$$P_B = \frac{B^2}{2\mu_0}, \qquad P_B = \frac{B^2}{8\pi}$$

in SI or cgs units, respectively. Let us work in SI units for this part. For an ideal gas, the kinetic pressure is nk_BT. This gives the relation:

$$nk_B\left(T_e + T_i\right) = \frac{\mu_0 I^2}{8\pi^2 R^2}$$

where n is the number of electrons or ions per unit length.

(b) For this ion beam, there is a linear charge density I/v. So an ion at radius R experiences an outward radial electrostatic force

$$\vec{F}_1 = \frac{2IQ}{vR}\hat{r}.$$

Because of the current, there is also a magnetic force determined again by Ampere's Law

$$\vec{F}_2 = Q\frac{\vec{v}}{c} \times \vec{B}.$$

The magnetic field is in the angular direction oriented so that

$$\vec{F}_2 = -\frac{2IQv}{c^2 R}\hat{r}.$$

Combining these results gives

$$F = \frac{2IQ}{Rv}\left(1 - \frac{v^2}{c^2}\right).$$

These formulae are expressed in cgs units. For any massive ion $v < c$, and the electrostatic force dominates over the magnetic force pushing the ion outward.

(c) First Ohm's Law in the rest frame of the plasma is $\vec{j} = \sigma \vec{E}$. Boosting to a frame with velocity $\vec{v}$ gives the relation

$$\vec{j} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right).$$

Maxwell's equations (with the displacement current dropped) state that

$$\nabla \times \vec{B} = \mu_0 \vec{j}, \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

Introduce the magnetic diffusivity $\eta = 1/(\mu_0 \sigma)$ and eliminate $\vec{j}, \vec{E}$ to get the magnetic induction equation,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B},$$

where we use $\nabla \times \left(\nabla \times \vec{B} \right) = \nabla \left(\nabla \cdot \vec{B} \right) - \left(\nabla \cdot \nabla \right) \vec{B}$.

11. Rope on a Cylinder

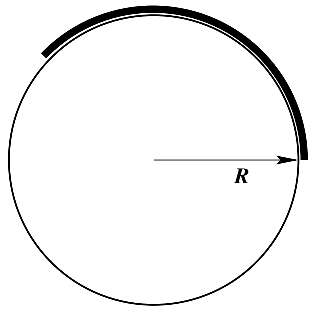

Figure 1: The initial position of the rope.

A heavy uniform rope of mass m and length l is placed on a cylinder of radius R ($0 < l < \pi R$) as shown in Fig. 1. There is a friction between the rope and the cylinder. If the friction coefficient μ exceeds a critical value μ_c, the rope stays in its initial position on the cylinder. If $\mu < \mu_c$, the rope starts sliding down.

Find the critical coefficient of friction μ_c. **Hint**: when $\mu = \mu_c$, the rope is not moving, but the friction force everywhere is related to the normal force by the usual law $F = \mu N$. Assume $\mu = \mu_c$ and derive an equation for the tension in the rope. Solve this equation to obtain a relation which determines μ_c. You do not need to determine μ_c explicitly.

Rope on a Cylinder - Solution

Let us consider an element of the rope in the presence of friction, see Fig. 2. Imagine the friction coefficient being increased gradually from zero to its critical value μ_c. Then for $\mu < \mu_c$ the rope will still slide, and Newton's equation of motion for an element will be

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + d\boldsymbol{N} + dm\boldsymbol{g} + d\boldsymbol{F} = dm\boldsymbol{a}. \tag{29}$$

In addition, when the rope is sliding, the magnitude of the friction force is related to the normal force by $dF = \mu dN$. The bigger the friction coefficient, the smaller the resulting acceleration $\boldsymbol{a}$ will be. Finally, when $\mu = \mu_c$, the acceleration vanishes. Then we can use equation (29) with $\boldsymbol{a} = 0$. The components along and perpendicular to the rope element give

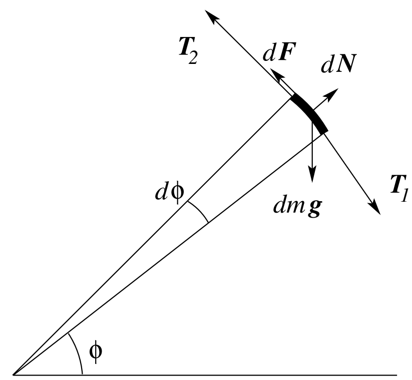

Figure 2: A rope element in equilibrium.

$$T_2 - T_1 + \mu_c dN - dmg\cos\phi = 0, \tag{30}$$

$$Td\phi + dmg\sin\phi = dN. \tag{31}$$

The second equation follows from the fact that the two tension forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are not exactly aligned, the angle between their directions is $\pi - d\phi$, so their vector sum is along the radius of the cylinder, and has the magnitude $Td\phi$ (to linear order in $d\phi$). This leads to the system of equations

$$\frac{dT}{d\phi} + \mu_c\frac{dN}{d\phi} - mg\frac{R}{l}\cos\phi = 0, \qquad \frac{dN}{d\phi} = T + mg\frac{R}{l}\sin\phi. \tag{32}$$

Substituting the second equation into the first leads to an inhomogeneous first order ODE for the tension in the rope:

$$\frac{dT}{d\phi} + \mu_c T + mg\frac{R}{l}(\mu_c\sin\phi - \cos\phi) = 0. \tag{33}$$

A somewhat unusual situation arises since we have *two* boundary conditions for this equation. Indeed, the tension vanishes at both ends of the rope, which gives

$$T(0) = T(l/R) = 0. \tag{34}$$

It is precisely because we have these two conditions that we can both solve the equation and find the critical coefficient of friction μ_c.

The solution of the equation (33) proceeds in a standard manner. First we find a general solution of the homogeneous equation, which gives $T_{\text{homo}}(\phi) = T_0 e^{-\mu_c \phi}$. Then a particular solution of the inhomogeneous equation is found as

$$T_{\text{inhomo}}(\phi) = A \sin\phi + B \cos\phi. \tag{35}$$

The coefficients A and B are found by direct substitution of the solution into the equation, which gives

$$A = \frac{1-\mu_c^2}{1+\mu_c^2} mg \frac{R}{l}, \qquad B = \frac{2\mu_c}{1+\mu_c^2} mg \frac{R}{l}. \tag{36}$$

Thus

$$T(\phi) = T_0 e^{-\mu_c \phi} + \frac{1}{1+\mu_c^2} mg \frac{R}{l} \left[(1-\mu_c^2) \sin\phi + 2\mu_c \cos\phi \right]. \tag{37}$$

The boundary condition $T(0) = 0$ gives

$$T_0 = -\frac{2\mu_c}{1+\mu_c^2} mg \frac{R}{l}. \tag{38}$$

Finally, the second boundary condition at the other free end of the rope gives an equation for the critical coefficient of friction as a function of the angular span of the rope $\lambda = l/R$:

$$2\mu_c e^{-\mu_c \lambda} = \sin\lambda + 2\mu_c \cos\lambda - \mu_c^2 \sin\lambda. \tag{39}$$

By analyzing the left and the right hand sides of this equation it is easy to check that it has a unique solution for any λ in the range $[0, \pi]$.

12. Photon Absorption

Consider an electron trapped in a three-dimensional harmonic-oscillator potential which is anisotropic. The Hamiltonian is given by

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\left(\omega_x^2\hat{x}^2 + \omega_y^2\hat{y}^2 + \omega_z^2\hat{z}^2\right).$$

Assume that this system is exposed to a monochromatic electromagnetic plane wave that is linearly polarized along the z-axis. Further assume that the wavelength of the electromagnetic wave is much larger than the spatial extent of the electron wave functions of interest. If the electron is initially in the ground state of the trap, at what photon energies will one-photon absorption take place?

Photon Absorption - Solution

One-photon absorption takes place only when the photon energy equals $\hbar\omega_z$. This can be shown as follows.

The Hamiltonian of the trapped electron is the sum of three independent one-dimensional harmonic-oscillator Hamiltonians:

$$\hat{H}_0 = \hat{H}_x + \hat{H}_y + \hat{H}_z.$$

Using creation and annihilation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ for each direction ($i = x$, y, and z) we have

$$\hat{H}_i|n_i\rangle = \hbar\omega_i \left(\hat{a}_i^\dagger\hat{a}_i + 1/2\right)|n_i\rangle = \hbar\omega_i \left(n_i + 1/2\right)|n_i\rangle,$$

$$n_i = 0, 1, 2, \ldots, \qquad i = x, y, z.$$

Then it follows that the eigenstates of $\hat{H}_0$ are direct products of the form

$$|n_x\rangle|n_y\rangle|n_z\rangle \equiv |n_x, n_y, n_z\rangle.$$

The eigenenergy associated with $|n_x, n_y, n_z\rangle$ is

$$E_{n_x,n_y,n_z} = \hbar\omega_x\left(n_x + 1/2\right) + \hbar\omega_y\left(n_y + 1/2\right) + \hbar\omega_z\left(n_z + 1/2\right).$$

The energy in the ground state $|0, 0, 0\rangle$ is

$$E_{0,0,0} = \frac{\hbar}{2}\left(\omega_x + \omega_y + \omega_z\right).$$

Since within this model, each of the $|n_x, n_y, n_z\rangle$ has an infinite lifetime, and since the electromagnetic wave is monochromatic, one-photon absorption from the ground state can, in principle, take place only when the photon energy is exactly equal to $E_{n_x,n_y,n_z} - E_{0,0,0}$ (i.e., the linewidth of an absorption resonance is zero). However, this is merely a necessary condition; it is not sufficient.

In order to see which of the $|n_x, n_y, n_z\rangle$ can actually be reached from $|0, 0, 0\rangle$ via one-photon absorption, we need to consider the electromagnetic wave as a perturbation and use perturbation theory to find transition probabilities. Using the fact that the wavelength of the radiation field is long compared to the trap, we can write the perturbing Hamiltonian in the dipole approximation:

$$\hat{H}_1 = Ae^{i\omega t}\boldsymbol{d}\cdot\boldsymbol{\varepsilon}, \tag{40}$$

where $\boldsymbol{d} = e\boldsymbol{r}$ is the electric dipole operator and $\boldsymbol{\varepsilon}$ is the polarization vector of the radiation field.

In perturbation theory, the transition probability from the ground state to the excited state $|n_x, n_y, n_z\rangle$ is proportional to the square of the absolute value of the matrix element $\langle n_x, n_y, n_z|\boldsymbol{d} \cdot \boldsymbol{\varepsilon}|0, 0, 0\rangle$. Since the electromagnetic wave is assumed to be linearly polarized along the z direction, we use that $\hat{z} \propto \hat{a}_z^\dagger + \hat{a}_z$ to obtain

$$\begin{aligned}
\langle n_x, n_y, n_z|\boldsymbol{d} \cdot \boldsymbol{\varepsilon}|0, 0, 0\rangle &\propto \langle n_x, n_y, n_z|\hat{z}|0, 0, 0\rangle \\
&\propto \langle n_x, n_y, n_z|\hat{a}_z^\dagger + \hat{a}_z|0, 0, 0\rangle \\
&= \langle n_x, n_y, n_z|\hat{a}_z^\dagger|0, 0, 0\rangle \\
&= \langle n_x, n_y, n_z|0, 0, 1\rangle \\
&= \delta_{n_x,0}\delta_{n_y,0}\delta_{n_z,1}.
\end{aligned}$$

Therefore, one-photon excitation from the ground state is forbidden for all $|n_x, n_y, n_z\rangle$, except for $|0, 0, 1\rangle$. The corresponding resonance appears at a photon energy of $\hbar\omega_z$.