

Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination

Part I

Saturday, April 9, 2011

Do not separate this page from the problem pages.

Fill out and turn in at the end of the exam.

Student _____
Fill in your S-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your S-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

The total time allowed **3 hrs**.

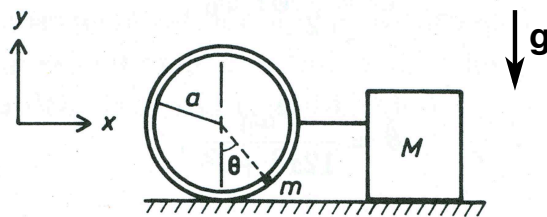
Please, indicate problems you are turning in:

- | | | | | | | | | | |
|--------------------------|---|--------------------------|---|--------------------------|---|--------------------------|---|--------------------------|----|
| <input type="checkbox"/> | 1 | <input type="checkbox"/> | 2 | <input type="checkbox"/> | 3 | <input type="checkbox"/> | 4 | <input type="checkbox"/> | 5 |
| <input type="checkbox"/> | 6 | <input type="checkbox"/> | 7 | <input type="checkbox"/> | 8 | <input type="checkbox"/> | 9 | <input type="checkbox"/> | 10 |

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

Choose 7 out of 10 problems

I-1. (Classical Mechanics)



A massless vertical circular track of radius a is attached through a rigid massless rod to a block of mass M to its right as shown. Both the circular track and the mass M are placed on a frictionless horizontal table. A particle of mass m is confined to move without friction on the circular track.

- Write down the Lagrangian of the system in terms of θ and x . Assume that the motion of the system is confined in a vertical plane.
 - Find the equations of motion.
 - In the limit when $\theta \rightarrow 0$ and for small $\dot{\theta}$, show that the motion of the mass is simple harmonic and find its angular frequency in terms of m , M and a .
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I-2. (Electricity and Magnetism)

A monochromatic wave is incident from the vacuum onto a flat material surface at $z = 0$. The region $z < 0$ is vacuum. For $z > 0$ the material dissipates energy and is characterized by an imaginary dielectric constant $\varepsilon/\varepsilon_0 = i$. The permeability of the material is the same as in the vacuum, $\mu/\mu_0 = 1$. The incoming wave is propagating in the direction normal to the surface, the z direction.

- Calculate the amplitudes of the wave that is transmitted into the material and the wave that is reflected back into the vacuum relative to the amplitude of the incident wave.
- Using the time-averaged Poynting vector $\vec{S} = \vec{E} \times \vec{H}$, calculate the fraction of the incident energy flux perpendicular to the interface that is reflected back and, separately, calculate the fraction that is transmitted into the material. Verify that the two fractions sum to one.

Hint: Maxwell eqs

$$\vec{\nabla} \cdot \vec{D} = \rho, \quad \vec{\nabla} E = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t},$$

where

$$\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H}.$$

Choose 7 out of 10 problems

I-3. (Mathematical Methods)

Consider the differential equation

$$z^2 u''(z) - zu'(z) + u(z) = 0.$$

- (i) Find all singular points of the equation and determine whether they are regular or not.
 - (ii) Use the series expansion or any other method to find the analytic solution around $z = 0$.
What is the radius of convergence of the series?
 - (iii) Find the second solution around $z = 0$.
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I-4. (Quantum Mechanics)

Answer the following questions about a particle moving in one dimension ($-\infty < x < \infty$).

- (i) The wavefunction for a particle with a definite momentum p is

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(i \frac{p}{\hbar} x\right).$$

What are the orthonormality and completeness relations satisfied by the wave functions $\psi_p(x)$?

- (ii) If $\psi(p)$ is the wavefunction of a particle in momentum space, write down an expression for the expectation value of the position operator $\langle x \rangle$.
- (iii) If

$$\psi(x) = \frac{1}{(2\pi\Delta)^{1/4}} \exp\left(-\frac{x^2}{4\Delta}\right),$$

what are the expectation values $\langle p \rangle$ and $\langle p^2 \rangle$, where p is the momentum operator, and what is the uncertainty in momentum?

I-5. (Relativity)

Consider a coordinate system (t', x') that is moving along the positive x -axis with speed v relative to a coordinate system (t, x) . A rocket is moving with a speed u' , measured in the (t', x') coordinate system, along the positive x' axis. Then the speed of the rocket in the (t, x) system is given by:

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

Let a and a' be the accelerations of the rocket measured in the (t, x) and (t', x') systems, respectively.

- (i) Find the relationship between a and a' .
- (ii) Suppose that the rocket undergoes constant *proper acceleration*, α , starting, at $t = 0$ with $x = 0$ and $v = 0$. Using part (i), or otherwise, find the position and velocity of the rocket as a function of t .
- (iii) Show that for small values of t this gives the standard kinematic result.

Hint: Recall that “proper” means “in the instantaneous rest frame.”

Choose 7 out of 10 problems

I-6. (Thermodynamics)

Consider an adiabatically insulated container of volume V , initially empty. After making a tiny puncture, air rushes inside, and because this occurs very quickly we suppose the process is adiabatic.

Assuming air is a perfect gas, denoting by P_0 the atmospheric pressure and T_0 the temperature of the atmosphere, find the volume V_0 of the atmospheric air that enters the container, and the temperature it reaches at the end of the process.

Hints: For an ideal gas, $U = nc_vRT$ where U is the internal energy, c_v the specific heat per mole at constant volume, and $c_p - c_v = R$, $c_p/c_v \equiv \gamma$ a constant depending on the nature of the gas.

I-7. (Statistical Physics)

A system of N bosons in two dimension has energy-momentum relationship

$$E = cp^{3/2},$$

and density $n = N/A$ (A is the area).

- (i) What is Bose-Einstein condensation?
 - (ii) Show that at low temperatures, the system will Bose condense, and that the Bose condensation temperature $T_C \sim n^\alpha$. Find α .
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I-8. (Experimental Physics)

Suppose you had a suspension of small particles with known radius r and material density ρ_m in a liquid of known density ρ_ℓ at temperature T . You also have a microscope that allows you to examine/count particles at different focal planes.

- (i) What is the expected ratio of numbers of particles on two planes at different heights?
- (ii) How can you use this simple set-up, knowing the value of the universal gas constant R , to make a measurement of the Avogadro's constant N_A . Write down an expression for N_A in terms of the measurable and known quantities of this system.

Hint: Boltzmann's constant is given by $k = R/N_A$.

I-9. (Condensed Matter)

Suppose you have a metal A that crystallizes into a simple cubic lattice (the basis is monatomic, the atoms A are monovalent). Now some of the atoms A are replaced by divalent atoms B , without changing the lattice structure or spacing. This gradually increases the electron number density.

- (i) If you approximate the metal's Fermi surface by that of a free electron gas, at what ratio of the electron to atom concentration n_e/n_a (n_e and n_a are the total numbers of electrons and atoms per unit volume, respectively) will it expand to touch the boundary of the first Brillouin zone?
 - (ii) What is the corresponding ratio of the concentrations of B and A atoms, n_B/n_A ?
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Choose 7 out of 10 problems

I-10. (Astrophysics)

Consider the following form of a spherically-symmetric density distribution inside a star:

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right), \quad 0 < r < R,$$

where ρ_c denotes central density and R the radius of the star. Assume that we have zero pressure at the outer boundary ($r = R$).

- (i) Using the differential equation for $M(r)$, the mass enclosed in the sphere of radius r ,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r),$$

find an expression for the central density ρ_c in terms of R and $M(R)$ (the total radius and mass of the star).

- (ii) Using the equation of hydrostatic support

$$\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2},$$

find the pressure $p(r)$ inside the star. Your answer will be of the form

$$p(r) = p_c \times \left(\text{polynomial in } \frac{r}{R}\right),$$

where p_c is the central pressure.
