Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II

Saturday, March 30, 2019

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is marked with your Lg-number and the problem number.* Do not write answers to different problems on the same page. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)

A particle of mass μ moves in one dimension along the q-axis in a potential energy $V(q)$, but is retarded by a damping force $-2\mu\gamma\dot{q}$ proportional to its velocity.

(i) Show that the equation of motion follows from the Lagrangian

$$L = e^{2\gamma t}\left(\frac{1}{2}\mu\dot{q}^2 - V(q)\right).$$

(ii) Show that the Hamiltonian is

$$H = e^{-2\gamma t}\,\frac{p^2}{2\mu} + e^{2\gamma t}\,V(q)\,.$$

(iii) Consider the type-2 generating function

$$F_2(q, P, t) = e^{\gamma t} qP\,.$$

Express dF_2 in terms of the differentials dq, dP, and dt. Recall that if $K(Q, P, t)$ is the transformed Hamiltonian

$$pdq - Hdt = PdQ - Kdt + dF_1\,,$$

and $F_2 = F_1 + QP$. Find $K(Q, P, t)$.

(iv) Now suppose that $V(q)$ is the oscillator potential

$$V(q) = \frac{1}{2}\,\mu\,\omega^2 q^2\,, \qquad \gamma < \omega\,.$$

Show that in this case $K(Q, P, t)$ is a constant of motion.

(v) Write down the Hamilton equations, i.e. first-order coupled differential equations of motion in (P, Q). Eliminate P from the equation of motion that involves $\dot{Q}$ (use the constant of motion) to give a differential equation in Q. Solve this equation, and show that

$$Q(t) = \sqrt{\frac{2K}{\mu(\omega^2 - \gamma^2)}}\,\sin\phi(t)\,,$$

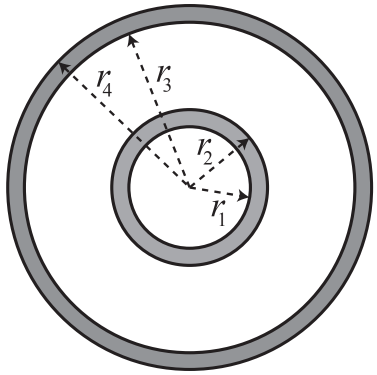
where

$$\phi(t) = \phi(0) + \sqrt{\omega^2 - \gamma^2}\,t\,.$$

You may need $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$.

(vi) Express $q(t)$ as a combination of $\sin(\sqrt{\omega^2 - \gamma^2}\,t)$ and $\cos(\sqrt{\omega^2 - \gamma^2}\,t)$, assuming that K and $\phi(0)$ are known.

Problem II-2. (Electricity and Magnetism)



Two rigid concentric conducting spheres have finite thickness, and sit in empty space, as shown in the diagram. The inner sphere has radii r_1 and r_2, with $r_1 < r_2$, and the outer sphere has radii r_3 and r_4, with $r_3 < r_4$. Charge $Q_{\rm I}$ is put on the inner sphere and charge $Q_{\rm O}$ on the outer sphere.

(i) Find the surface charge density on each of the four surfaces.

(ii) Compute the capacitance of the system when $Q_{\rm O} = -Q_{\rm I} = -Q$.

Now imagine that the space between the spheres is filled with a linear dielectric material with dielectric constant ϵ.

(iii) Find the surface polarization charge density on each of the four surfaces. (Assume again that $Q_{\rm O}$ and $Q_{\rm I}$ are arbitrary).

(iv) Compute the capacitance of the system when $Q_{\rm O} = -Q_{\rm I} = -Q$.

Imagine now that both the outer sphere and the dielectric are compressible. There is zero cost for compressing the conductor, but the dielectric has elastic energy as a function of its radius r_3, modelled by:

$$U = \frac{1}{2}k(r_3 - r_0)^2 ,$$

where $r_0 > r_2$ and k are positive constants.

(v) By sketching the appropriate functions, argue that there is an equilibrium position for the inner surface of the outer sphere that depends on the charge Q. (Let us assume again that $Q_{\rm O} = -Q_{\rm I} = -Q$.)

(vi) Show that the equilibrium position r_3 satisfies:

$$-r_0 k r_3^2 + k r_3^3 + A = 0 , \qquad \text{where} \qquad A = \frac{Q^2}{8\pi\epsilon} .$$

(vii) When r_0 has the value $r_0 = 3(A/4k)^{1/3}$, show that the equilibrium position is at $r_3 = \frac{2}{3}r_0$. (In fact, this is *just* where the equilibrium point starts to appear. There's no physical solution for smaller r_0.)

Problem II-3. (Quantum Mechanics)

A certain system has N possible states. Its state ket belongs to a N-dimensional space spanned by the orthonormal basis $\{|n\rangle, n = 1, 2, ..., N\}$, where the base kets have the periodic boundary condition

$$|n+N\rangle = |n\rangle\,.$$

The system's Hamiltonian is

$$\mathbf{H} = E_0 \sum_{n=1}^{N} |n\rangle\,\langle n| + \lambda \sum_{n=1}^{N} \big(|n\rangle\,\langle n+1| + |n+1\rangle\,\langle n|\big)\,,$$

where λ is a coupling energy. In the zero-coupling (i.e. $\lambda = 0$) limit all of the base kets are eigenkets of the Hamiltonian with the same (degenerate) energy eigenvalue E_0.

(i) Consider the "shift" operator

$$\mathbf{t} \equiv \sum_{n=1}^{N} |n\rangle\,\langle n+1|.$$

Express $\mathbf{H}$ in terms of $\mathbf{t}$, $\mathbf{t}^\dagger$, and the identity operator $\mathbf{1}$.

(ii) Show that $\mathbf{H}$, $\mathbf{t}$, and $\mathbf{t}^\dagger$ mutually commute.

(iii) Represent $\mathbf{t}$ by a $N \times N$ matrix on the basis $\{|n\rangle,\ n = 1, 2, ..N\}$.

(iv) Determine the eigenvalues of $\mathbf{t}$.

(v) Determine the eigenvalues of $\mathbf{H}$.

(vi) Show that the eigenkets of $\mathbf{H}$ are

$$|E_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i(n-1)\theta_m} |n\rangle,$$

where $\theta_m = 2\pi(m/N)$.

Problem II-4. (Math Methods)

The Γ-function is usually defined by the Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt\,. \tag{4.1}$$

(i) Determine for which $z \in \mathbb{C}$ the integral (4.1) is convergent.

(ii) Prove that in the region of $\mathbb{C}$ detemined in (i), $\Gamma(z)$ has the following properties:

- complex conjugation, $\overline{\Gamma(z)} = \Gamma(\overline{z})$.
- recurrence relation, $z\,\Gamma(z) = \Gamma(z+1)$.
- relation to the factorial, $\Gamma(n) = (n-1)!$, $n = 1, 2, \ldots$

Using the recurrence relation in (ii), one can continue $\Gamma(z)$ analytically to the complex plane, where it is a meromorphic function with simple poles at nonpositive integers.

(iii) Show that the residue of $\Gamma(z)$ at $z_0 = -n$ is

$$\mathrm{Res}(\Gamma(z), -n) = \frac{(-1)^n}{n!}\,, \qquad n = 0, 1, 2, \ldots$$

Hint: One method is to use the recurrence relation to express $\Gamma(z-n)$ in terms of $\Gamma(z+n+1)$ and then expand around $z = 0$.

(iv) Sketch the graph of $\Gamma(x)$, $x \in \mathbb{R}$.

(v) Certain function $f(x)$ on $(0, \infty)$ is defined by the inverse Mellin transform given by the following complex integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\, x^{-s}\, ds\,, \qquad x > 0\,,$$

along a contour parallel to the imaginary axis at $\mathrm{Re}\, s = c > 0$. This integral can be evaluated as an infinite series obtained by closing the countour in the left half-plane and then using the Cauchy residue theorem. Find that series and sum it to obtain a closed form formula for $f(x)$.