Department of Physics and Astronomy
University of Southern California

**Graduate Screening Examination**
**Part I**
Saturday, April 9, 2011

| **Do not separate this page from the problem pages.** |
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| Fill out and turn in at the end of the exam. |

Student ________________
Fill in your S-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your S-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

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|---|---|---|---|---|
| ☐ 1 | ☐ 2 | ☐ 3 | ☐ 4 | ☐ 5 |
| ☐ 6 | ☐ 7 | ☐ 8 | ☐ 9 | ☐ 10 |

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

**Choose 7 out of 10 problems**

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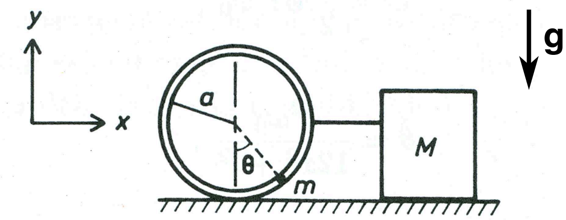
**I-1.** (Classical Mechanics)




A massless vertical circular track of radius $a$ is attached through a rigid massless rod to a block of mass $M$ to its right as shown. Both the circular track and the mass $M$ are placed on a frictionless horizontal table. A particle of mass $m$ is confined to move without friction on the circular track.

(i) Write down the Lagrangian of the system in terms of $\theta$ and $x$. Assume that the motion of the system is confined in a vertical plane.

(ii) Find the equations of motion.

(iii) In the limit when $\theta \to 0$ and for small $\dot{\theta}$, show that the motion of the mass is simple harmonic and find its angular frequency in terms of $m$, $M$ and $a$.

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**I-2.** (Electricity and Magnetism)

A monochromatic wave is incident from the vacuum onto a flat material surface at $z = 0$. The region $z < 0$ is vacuum. For $z > 0$ the material dissipates energy and is characterized by an imaginary dielectric constant $\varepsilon/\varepsilon_0 = i$. The permeability of the material is the same as in the vacuum, $\mu/\mu_0 = 1$. The incoming wave is propagating in the direction normal to the surface, the $z$ direction.

(i) Calculate the amplitudes of the wave that is transmitted into the material and the wave that is reflected back into the vacuum relative to the amplitude of the incident wave.

(ii) Using the time-averaged Poynting vector $\vec{S} = \vec{E} \times \vec{H}$, calculate the fraction of the incident energy flux perpendicular to the interface that is reflected back and, separately, calculate the fraction that is transmitted into the material. Verify that the two fractions sum to one.

*Hint:* Maxwell eqs

$$\vec{\nabla} \cdot \vec{D} \;=\; \rho\,, \qquad \vec{\nabla} E \;=\; -\frac{\partial \vec{B}}{\partial t}\,, \qquad \vec{\nabla} \cdot \vec{B} \;=\; 0\,, \qquad \vec{\nabla} \times \vec{H} \;=\; \vec{j} + \frac{\partial \vec{D}}{\partial t}\,,$$

where

$$\vec{D} \;=\; \varepsilon \vec{E}\,, \qquad \vec{B} \;=\; \mu \vec{H}\,.$$

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**Choose 7 out of 10 problems**

**I-3.** (Mathematical Methods)

Consider the differential equation

$$z^2 u''(z) - z u'(z) + u(z) \;=\; 0\,.$$

(i) Find all singular points of the equation and determine whether they are regular or not.

(ii) Use the series expansion or any other method to find the analytic solution around $z = 0$. What is the radius of convergence of the series?

(iii) Find the second solution around $z = 0$.

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**I-4.** (Quantum Mechanics)

Answer the following questions about a particle moving in one dimension ($-\infty < x < \infty$).

(i) The wavefunction for a particle with a definite momentum $p$ is

$$\psi_p(x) \;=\; \frac{1}{\sqrt{2\pi\hbar}} \exp\left(i\,\frac{p}{\hbar}x\right)\,.$$

What are the orthonormality and completeness relations satisfied by the wave functions $\psi_p(x)$?

(ii) If $\psi(p)$ is the wavefunction of a particle in momentum space, write down an expression for the expectation value of the position operator $\langle x\rangle$.

(iii) If

$$\psi(x) = \frac{1}{(2\pi\Delta)^{1/4}} \exp\left(-\frac{x^2}{4\Delta}\right)\,,$$

what are the expectation values $\langle p\rangle$ and $\langle p^2\rangle$, where $p$ is the momentum operator, and what is the uncertainty in momentum?

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**I-5.** (Relativity)

Consider a coordinate system $(t', x')$ that is moving along the positive $x$-axis with speed $v$ relative to a coordinate system $(t, x)$. A rocket is moving with a speed $u'$, measured in the $(t', x')$ coordinate system, along the positive $x'$ axis. Then the speed of the rocket in the $(t, x)$ system is given by:

$$u \;=\; \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

Let $a$ and $a'$ be the accelerations of the rocket measured in the $(t, x)$ and $(t', x')$ systems, respectively.

(i) Find the relationship between $a$ and $a'$.

(ii) Suppose that the rocket undergoes constant *proper acceleration*, $\alpha$, starting, at $t = 0$ with $x = 0$ and $v = 0$. Using part (i), or otherwise, find the position and velocity of the rocket as a function of $t$.

(iii) Show that for small values of $t$ this gives the standard kinematic result.

*Hint:* Recall that "proper" means "in the instantenous rest frame."

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**Choose 7 out of 10 problems**

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**I-6.** (Thermodynamics)

Consider an adiabatically insulated container of volume $V$, initially empty. After making a tiny puncture, air rushes inside, and because this occurs very quickly we suppose the process is adiabatic.

Assuming air is a perfect gas, denoting by $P_0$ the atmospheric pressure and $T_0$ the temperature of the atmosphere, find the volume $V_0$ of the atmospheric air that enters the container, and the temperature it reaches at the end of the process.

*Hints:* For an ideal gas, $U = nc_vRT$ where $U$ is the internal energy, $c_v$ the specific hear per mole at constant volume, and $c_p - c_v = R$, $c_p/c_v \equiv \gamma$ a constant depending on the nature of the gas.

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**I-7.** (Statistical Physics)

A system of $N$ bosons in two dimension has energy-momentum relationship

$$E = c\,p^{3/2}\,,$$

and density $n = N/A$ ($A$ is the area).

(i) What is Bose-Einstein condensation?

(ii) Show that at low temperatures, the system will Bose condense, and that the Bose condensation temperature $T_C \sim n^{\alpha}$. Find $\alpha$.

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**I-8.** (Experimental Physics)

Suppose you had a suspension of small particles with known radius $r$ and material density $\rho_m$ in a liquid of known density $\rho_\ell$ at temperature $T$. You also have a microscope that allows you to examine/count particles at different focal planes.

(i) What is the expected ratio of numbers of particles on two planes at different heights?

(ii) How can you use this simple set-up, knowing the value of the universal gas constant $R$, to make a measurement of the Avogadro's constant $N_A$. Write down an expression for $N_A$ in terms of the measurable and known quantities of this system.

*Hint:* Boltzmann's constant is given by $k = R/N_A$.

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**I-9.** (Condensed Matter)

Suppose you have a metal $A$ that crystallizes into a simple cubic lattice (the basis is monatomic, the atoms $A$ are monovalent). Now some of the atoms $A$ are replaced by divalent atoms $B$, without changing the lattice structure or spacing. This gradually increases the electron number density.

(i) If you approximate the metal's Fermi surface by that of a free electron gas, at what ratio of the electron to atom concentration $n_e/n_a$ ($n_e$ and $n_a$ are the total numbers of electrons and atoms per unit volume, respectively) will it expand to touch the boundary of the first Brillouin zone?

(ii) What is the corresponding ratio of the concentrations of $B$ and $A$ atoms, $n_B/n_A$?

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**I-10.** (Astrophysics)

Consider the following form of a spherically-symmetric density distribution inside a star:

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right) , \qquad 0 < r < R \, ,$$

where $\rho_c$ denotes central density and $R$ the radius of the star. Assume that we have zero pressure at the outer boundary ($r = R$).

(i) Using the differential equation for $M(r)$, the mass enclosed in the sphere of radius $r$,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \, ,$$

find an expression for the central density $\rho_c$ in terms of $R$ and $M(R)$ (the total radius and mass of the star).

(ii) Using the equation of hydrostatic support

$$\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \, ,$$

find the pressure $p(r)$ inside the star. Your answer will be of the form

$$p(r) = p_c \times \left(\text{polynomial in } \frac{r}{R}\right) \, ,$$

where $p_c$ is the central pressure.

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**Graduate Screening Examination**
**Part II**
Saturday, April 9, 2011

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| **Do not separate this page from the problem pages.**<br>Fill out and turn in at the end of the exam. |

Student ________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

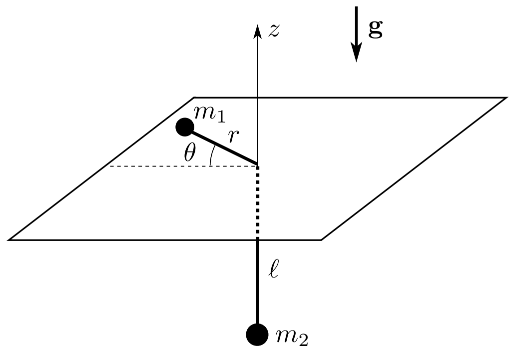
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs 30 min**.

Please, indicate problems you are turning in:

☐ **II-1** ☐ **II-2** ☐ **II-3** ☐ **II-4**

**II-1.** (Classical Mechanics)




Two mass points $m_1$ and $m_2$ ($m_1 \neq m_2$) are connected by a massless string of length $\ell$ passing through a hole in a horizontal table. The string and mass points move without friction with $m_1$ on the table and $m_2$ free to move in a vertical line.

(i) What initial velocity must $m_1$ be given so that $m_2$ will remain motionless a distance $d$ below the surface of the table?

(ii) If $m_2$ is slightly displaced in a vertical direction, small oscillations will ensue. Use Lagrange's equations to find the period of these oscillations.

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**II-2.** (Electricity and Magnetism)

Consider a hollow cube of side $a$. The volume inside the cube is the region

$$0 < x < a\,, \quad 0 < y < a\,, \quad 0 < z < a\,.$$

All of the sides of the cube are metallic and grounded to zero potential. A single point charge of magnitude $q$ is placed in the center of the cube at the point $x = y = z = a/2$.

(i) Solve the differential equation

$$-\epsilon_0 \nabla^2 \Phi \;=\; q\,\delta(x - a/2)\delta(y - a/2)\delta(z - a/2)\,,$$

for the electrostatic potential $\Phi(x, y, z)$ everywhere inside the volume of the cube as a double Fourier sine series in the $x$ and $y$ variables. The boundary condition is that the potential vanishes on all of the surfaces of the cube.

(ii) Using the above solution for the potential and Gauss's law, find an expression (also as a double Fourier sine series) for the charge density $\sigma(x, y)$ on the top surface of the cube at $z = a/2$.

(iii) Integrate your result for $\sigma(x, y)$ to find an expression for the total charge $q'$ on the top surface as a double sum. Do not sum the series, but based on symmetry state what the final result for $q'$ must be.

*Hint:* A useful identity is: $\sinh(\alpha + \beta) \;=\; \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta)$.

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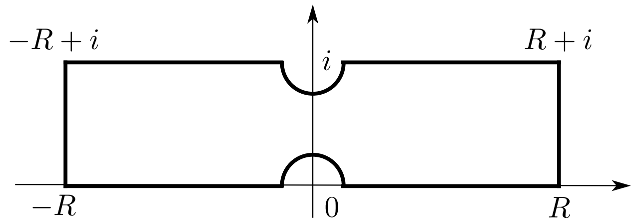
**II-3.** (Mathematical Methods)




Use the contour above with $R \to \infty$ to show that

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)} = \frac{1}{2}\tan\frac{a}{2}\,, \qquad -\pi < a < \pi\,.$$

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**II-4.** (Quantum Mechanics)

A spin-half particle with magnetic moment $\mu$ is placed in a magnetic field $\vec{B}(t)$ which rotates with frequency $\omega$,

$$\vec{B}(t) = B_z\hat{z} + B_x\hat{x}\cos(\omega t) + B_y\hat{y}\sin(\omega t)\,,$$

where $B_z$, $B_x$, and $B_y$ are real constants, with

$$B_z = \frac{\hbar\omega_0}{2\mu}\,, \qquad B_x = B_y = \frac{\hbar\omega_1}{2\mu}\,.$$

The Hamiltonian is represented by the $2\times 2$ matrix

$$\mathbf{H}(t) = -\mu\,\vec{\sigma}\cdot\vec{B}(t)\,,$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Write down the equation satisfied by the state ket $|\Psi(t)\rangle$ of the particle.

(ii) Consider the (rotation) operator $\mathbf{R}(\theta\hat{n})$ represented by the matrix

$$\mathbf{R}(\theta\hat{n}) = e^{-i\theta\hat{n}\cdot(\frac{1}{2}\vec{\sigma})}\,.$$

What is the equation satisfied by $\mathbf{R}(\omega t\hat{z})|\Psi(t)\rangle$, and what state does this ket represent?

(iii) At time $t = 0$ the spin is aligned along the positive $z$-axis. What is the probability for finding the spin aligned along the negative $z$-axis at $t > 0$?