Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, March 24, 2012

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

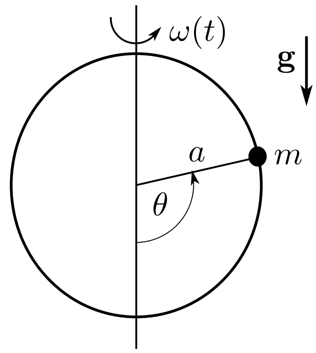
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 h 30 min**.

Please, indicate problems you are turning in:

☐ **II - 1** ☐ **II - 2** ☐ **II - 3** ☐ **II - 4**

II - 1. (Classical Mechanics)



A bead of mass m is constrained to slide without friction on a hoop of radius a. The hoop rotates with angular velocity $\omega(t)$ around a vertical axis which coincides with a diameter of the hoop.

(i) Set up the Lagrangian and obtain equations of motion of the bead.

(ii) Find the Hamiltonian, H, of the bead and calculate the total time derivative dH/dt along trajectories in the phase space.

Assume now that the angular velocity, ω, of the hoop is constant.

(iii) Show that there is a critical angular velocity, ω_0, below which the bottom of the hoop provides a stable equilibrium position for the bead.

(iv) Find the stable equilibrium position for $\omega > \omega_0$ and determine the period of small oscillations around this equilibrium.

(v) For $\omega = 2\omega_0$, describe *qualitatively* the motion of the bead for $t > 0$ if the initial conditions for $t = 0$ are $\theta(0) = 2\pi/3$ and $\dot{\theta}(0) = 0\,\mathrm{rad/s}$.

II - 2. (Mathematical Methods)

Solve

$$\varphi(x) \;=\; 1 + \lambda^2 \int_0^x (x - t)\, \varphi(t)\, dt \,,$$

by each of the following methods:

(i) Reduction to a differential equation (including establishment of boundary conditions).

(ii) The Neumann series.

(iii) The use of Laplace transforms.

II - 3. (Electricity and Magnetism)

A hollow sphere of inner radius R is grounded. Along a straight line passing through the center of the sphere sit three charges. One charge, $-2q$, sits at the center of the sphere. The other two charges are each q, and they sit on opposite sides of the center at a distance $a < R$ from the center. Note that this distribution of charges has no monopole and dipole moments; the lowest nonvanishing moment is quadrupole.

(i) By using the method of images (there are 3 image charges) determine the potential everywhere inside the sphere, for both $r < a$ and $r > a$, in terms of spherical coordinates with the z-axis along the line of charges.

Hint: You may use without proof that the image charge corresponding to q at $z = \pm a$ is $-(R/a)q$ at $z = \pm(R^2/a)$.

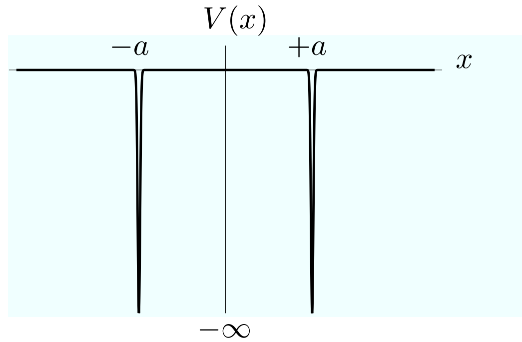
(ii) Show that in the limit $a \to 0$, with $Q \equiv qa^2$ finite, the potential inside the sphere is

$$\Phi(r, \theta, \phi) = \frac{2Q}{r^3}\left(1 - \frac{r^5}{R^5}\right) P_2(\cos\theta)\,.$$

Hint: The following expansion may be useful

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{(r_<)^\ell}{(r_>)^{\ell+1}}\, P_\ell(\cos\theta)\,.$$

II - 4. (Quantum Mechanics)



An electron of mass m is moving from left to right in a one dimensional potential of the form of two delta function wells centered at the origin,

$$V(x) = -\frac{\hbar^2\beta^2}{2m}\,\delta(|x| - a)\,,$$

as in the figure. Assume that the energy, $E = \hbar^2 k^2/(2m)$, is positive and compute the reflection and transmission coefficients. Your answer should include the following parts:

1. Write down the appropriate equations and give the general form of the solution for the three regions: (i) $x < -a$, (ii) $|x| < a$, and (iii) $x > a$. In these expressions identify the parameters that represent the amplitudes for reflection and transmission.
2. Give the equations that connect them into an overall solution.
3. Using the equations you setup in part 2, obtain the solution for the transmission *probability*. If you cannot finish the computation outline the remaining steps.