Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 24, 2018

Do not separate this page from the problem pages. Fill out and turn in at the end of the exam.

Student ______________
Fill in your Sh-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Sh-number and the problem number.* Do not write answers to different problems on the same page and do not write anything on the back of page. Staple *separately* your answers to each problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

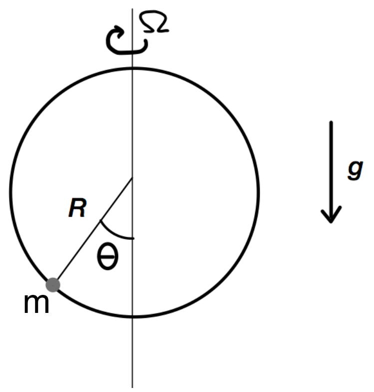
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6
☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

Problem I-1. (Classicial Mechanics)



A bead of mass m slides without friction in a uniform gravitational field on a vertical circular hoop of radius R. The hoop is constrained to rotate at a fixed angular velocity Ω about its vertical diameter. Let θ be the position of the the bead on the hoop measured from the lowest point, as shown in figure.

(i) Write down the Lagrangian $L(\dot{\theta}, \theta)$.

(ii) Determine the equilibrium values of θ.

(iii) Determine if the equilibrium positions are stable or unstable.

Problem I-2. (Electricity and Magnetism)

A point charge q is situated at a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.

Problem I-3. (Math Methods)

(i) Find the Fourier transform, $G_\omega(k)$, of the generalized function

$$P\frac{1}{x-\omega}\,, \qquad \omega \in \mathbb{R}\,,$$

by evaluating the principal value integral

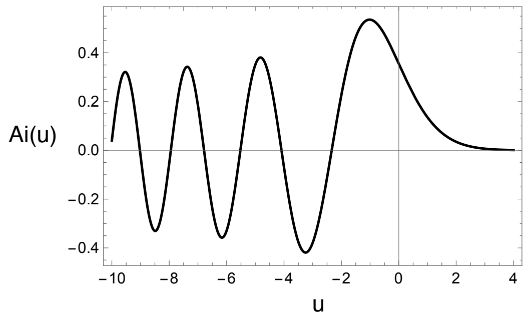
$$G_\omega(k) = P\int_{-\infty}^{\infty} \frac{1}{x-\omega}\, e^{ikx}\, dx\,.$$

(ii) What is the inhomogenous differential equation satisfied by $G_\omega(k)$?

Hint: You can use the Sokhotski-Plemelj identity

$$\frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x)\,.$$

Problem I-4. (Quantum Mechanics)



Consider the one-dimensional motion in the z-direction of a particle of mass m in a uniform gravitational field above an impenetrable plane located at $z = 0$. The potential energy of the particle is infinite if $z < 0$, while the gravitational force on the particle is $F_z = -mg$ for $z > 0$.

One can show that the eigenfunctions $\psi_n(z)$ $(n = 1, 2, \ldots)$ of the Hamiltonian are given in terms of the Airy function,

$$\psi_n(z) = C_n \,\mathrm{Ai}(\alpha_n z + \beta_n)\,, \tag{4.1}$$

where α_n and β_n are constants to be determined and C_n is a normalization constant. The function $\mathrm{Ai}(u)$ is a solution to the Airy equation,

$$\left(\frac{d^2}{du^2} - u\right)\mathrm{Ai}(u) = 0\,,$$

and its plot is shown in the figure.

Using the form of the wave function in (4.1) and the boundary conditions that $\psi_n(z)$ must satisfy, express the energy eigenvalues E_n of the Hamiltonian in terms of the (negative) zeros u_n of $\mathrm{Ai}(u)$.

Problem I-5. (Condensed Matter)

The relation between frequency f and wavelength λ for tension waves on the surface of a liquid of density ρ and surface tension σ is

$$f^2 = \frac{2\pi\sigma}{\rho\lambda^3}\,.$$

Such waves, called "ripplons," constitute bosonic excitations. They exist for example on the surface of liquid helium and contribute to its heat capacity. How does the heat capacity of these ripplons depend on the temperature T at very low temperatures? Specify only the form of the temperature dependence, don't worry about any accompanying coefficients or factors.

Problem I-6. (Statistical Mechanics)

Consider an ideal gas in the grand canonical ensemble.

(a) Show that the grand partition function for an ideal gas of classical particles is given by

$$\mathcal{Q}(T,\mu,V) = \exp\left(e^{\beta\mu}\frac{V}{\lambda^3}\right), \tag{6.1}$$

and give an expression for the thermal (de Broglie) wavelength $\lambda = \lambda(T)$.

Hint:

$$e^x = \sum_{i=0}^{\infty}\frac{1}{i!}x^i\,, \qquad \int_0^{\infty} e^{-x^2/\alpha^2} = \frac{\sqrt{\pi}}{2}\alpha\,.$$

(b) By considering the equilibrium variations of the grand potential $\mathcal{G} = E - TS - \mu N$, or otherwise, find P in terms of T and μ for an ideal gas.

(c) Find an expression for N in terms of (T,μ,V). Combine this result with your result of (b) to obtain the ideal gas equation of state. Comment on whether this result depends on the ensemble considered, and provide a qualitative justification for your answer.

(d) Find the chemical potential of the ideal gas in terms of (T,N,V). Obtain the high- and low-temperature limits of the chemical potential at constant particle density $n = \frac{N}{V}$, and provide an intuitive explanation for the sign of μ.

Problem I-7. (Solid State)

Properties of face-centered cubic (fcc) lattice.

(a) Consider a face-centered cubic lattice with the conventional cube side length of a. Is the lattice a Bravais lattice? Are the "corner" sites and the "face-centered" sites inequivalent or equivalent? Draw a 3D picture of the fcc lattice.

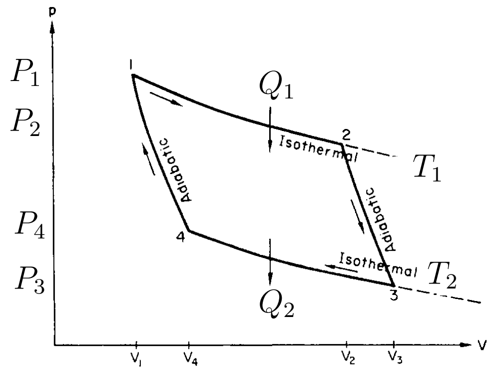
(b) Write down a set of primitive lattice vectors for fcc. Do these primitive lattice vectors "span" the fcc lattice?

(c) What is the volume of a primitive cell of the fcc lattice in terms of a? And what is the density of lattice points (number per unit volume) in terms of a?

(d) Consider a spherical "atom" placed on each lattice site. Assume that these spheres are "close-packed". How many such close-packed spheres are contained in each primitive cell? What is the radius of such a close-packed sphere in terms of a?

(e) What is the volume of one of the close-packed spheres? Based on this and the answer from part (c), calculate the packing fraction for the close-packed spheres in the fcc lattice.

Problem I-8. (Thermodynamics)



A solid is the substance in a Carnot engine working between temperatures $T_1, T_2 < T_1$. Within the relevant interval of temperatures and pressures, the solid obeys the equation of state $V = V_0 + aT - bP$ where V_0, a, b are constants. One also assumes that C_P depends only on T.

(i) What is the physical significance of Carnot cycles?

(ii) Calculate explicitly the heat exchanged Q_1, Q_2 on the two isotherms as a function of temperature and pressures.

(iii) Using the adiabats, find a relation between Q_1 and Q_2 and calculate directly the efficiency of the engine.

(iv) Does the efficiency of this engine depend on the equation of state of the working substance?

Hint: The following relations might be useful:

$$\begin{aligned} TdS &= C_V\, dT + T(\partial P/\partial T)_V\, dV\,, \\ TdS &= C_P\, dT - T(\partial V/\partial T)_P\, dP\,, \\ TdS &= C_V\, (\partial T/\partial P)_V\, dP + C_P(\partial T/\partial V)_P\, dV\,. \end{aligned}$$

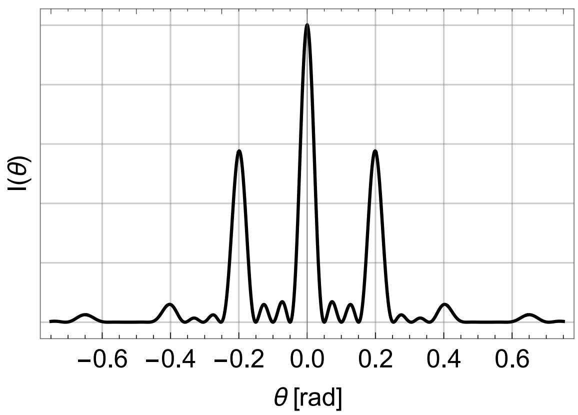
Problem I-9. (Relativity)

A photon of wavelength λ strikes an electron of mass m that is initially at rest. The photon is scattered at an angle θ, and its new wavelength is $\tilde{\lambda}$.

(1) Write down the relativistic equations for momentum and energy conservation. Express your equations in terms of just momenta, the speed of light c and the mass of the electron.

(2) What is the difference in photon wavelengths $\tilde{\lambda} - \lambda$ in terms of m, c and θ?

Problem I-10. (Optics)



The Fraunhoffer diffraction pattern in the figure results from a monochromatic light of wavelength λ incident on a diffraction grating with N slits. The slit separation is d and the width of each slit is w, where both d and w are integer multiples of λ. The angle θ is measured in radians from the central maximum.

(i) Determine the integers d/λ and w/λ.
(ii) How will the pattern change if the slit width, w, is much smaller than λ?
(iii) What is the number of slits, N, for this grating?
(iv) How will the pattern change if the number of slits, N, is large?

Problem I-11. (Experimental Physics)

A *lock-in amplifier* is a piece of experimental equipment that takes in a time-varying voltage signal $V(t)$, multiplies it by a known reference signal $V_{\text{ref}} \cos(\omega_{\text{ref}} t + \phi_{\text{ref}})$ using a hardware component known as a *mixer*, then low-pass filters the product with a filter bandwidth $B \ll \omega_{\text{ref}}$ and displays the resulting *voltage*. In other words, the lock-in displays only that part of the the mixer's output voltage whose frequency lies within a narrow range B around zero.

(i) Use an argument based on Fourier analysis to explain which portion of the input signal the lock-in is sensitive to, and which portion it rejects.
Hint: Use angle addition formulas.
(ii) A *spectrum analyzer*, by comparison, uses a bandpass filter to integrate and display the *total power* within a narrow frequency range around ω_{ref}. What disadvantages and advantages does this offer compared to the lock-in amplifier?
(iii) The lock-in has an effective *integration time* (the time over which the signal needs to be averaged in order to display the result). This implies that the instrument has an equal response time (the minimum time before a new signal will significantly affect the result). Can you deduce this time on the basis of the lock-in's operation parameters given above?

Problem I-12. (Astrophysics)

An estimate of the central conditions of the Sun:

Assuming the structure of Sun to be perfectly spherically symmetric, the following additional simplifications are made:

– The innermost 10% in radius contains exactly 10% of the total mass.

– In the innermost 10% in radius, the density ρ is exactly constant.

(a) What is the value of the density ρ in the center of the Sun according to these approximations? In the calculation, you are allowed to use the following approximate values: solar mass $\mathrm{M} = 2 \times 10^{30}$ kg, solar radius $\mathrm{R} = 7 \times 10^{8}$ m.

Now, assume that these central 10% (in radius as well as in mass) exactly contain the whole nuclear energy generating region of the Sun. In addition assume that in this region, not only ρ but also the nuclear energy production rate ϵ (power per mass) is constant. Using an approximative value for the the solar luminosity of $\mathrm{L} = 4 \times 10^{26}$ W, compute

(b) ϵ in the center of the Sun, and

(c) $\epsilon\rho$, the energy production rate per volume.

Finally, compare the result of (c) with the heat production rate of a typical car engine. To be specific assume that such a car engine produces 100 kW of heat, which is produced inside the engine's cylinders, here assumed to have a total volume of 1000 cm^3.

(d) What is the (heat) energy production rate (per volume) of such an engine? Compare it with the solar value computed in (c)!

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The exam is **closed book**. Use only the paper provided and *make sure that each page is marked with your Lg-number and the problem number.* Do not write answers to different problems on the same page. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)

A particle of mass m is subjected to the one-dimensional force

$$F(x) = -kx - \frac{\alpha}{x^3},$$

where k and α are real positive constants.

(i) Show that

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + A\frac{p}{x},$$

is a possible Hamiltonian for this system for an appropriately chosen constant A.

(ii) Consider the canonical transformation

$$X(x,p) = \arctan\left(\frac{\lambda x}{p}\right),$$

$$P(x,p) = \frac{1}{2}\left(\frac{p^2}{\lambda} + \lambda x^2\right) + \sqrt{\frac{m}{k}}\frac{Ap}{x},$$

with $\lambda = \sqrt{km}$. Verify that this is a canonical transformation.

(iii) Find the new Hamiltonian, K, associated with the canonical transformation given in part (ii). Solve Hamilton's canonical equations for this Hamiltonian and use your solution to determine the motion of the system in the original coordinates; that is, find $x(t)$. Leave your initial conditions arbitrary.

(iv) Consider the Lagrangian constructed in the usual way by finding the potential energy $V(x)$ associated with the force $F(x)$ and compare it to the Lagrangian constructed from the Hamiltonian in part (i). Does their relationship make sense?

Problem II - 2. (Electricity and Magnetism)

Consider an infinite plane, say the plane $z = 0$, which is **not** a conductor. Assume the potential is specified everywhere on this plane, and vanishes at infinity (Dirichlet boundary condition).

(i) What is the appropriate Green function $G(\vec{x}, \vec{x}')$ for obtaining the potential in the half-space $z \geq 0$? Express $G(\vec{x}, \vec{x}')$ in cylindrical coordinates, i.e. write $\vec{x}$ as a vector whose components along the x-, y-, and z-axes are $\rho \cos\phi$, $\rho \sin\phi$, and z with $0 \leq \rho < \infty$ and $0 \leq \phi \leq 2\pi$, and similarly for $\vec{x}'$.

(ii) On the plane $z = 0$ the potential Φ is held at the constant value V within a circle of radius a centered at the origin, and is held to be zero outside this circle, i.e. $\Phi = V$ in the region $0 \leq \rho < a$, $z = 0$ and $\Phi = 0$ in the region $\rho > a$, $z = 0$. Write down an integral expression (using cylindrical coordinates) for the potential in the region $z > 0$, assuming there is no charge in this region.

(iii) Show that along the axis of the circle (i.e. along the line $\rho = 0$, $z \geq 0$) the potential is

$$\Phi = V\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right).$$

(iv) Assuming that the potential on the plane is maintained at the constant value given in part (iii), how much work must be done on a point charge q to bring it from infinity to the position $z = b > 0$ on the z-axis?

(v) Is this work the same as the change in the total electrostatic energy (excluding the infinite self-energy of the point charge)? Explain.

Problem II-3. (Quantum Mechanics)

Consider a spin-$\frac{1}{2}$ particle moving on a line. The Hamiltonian of the spin-orbit coupling relevant to topological materials is given by

$$\widehat{H} = v\,\sigma^z\,\widehat{p}\,, \tag{3.1}$$

where σ^z is the standard Pauli matrix acting on the spin degrees of freedom (DOF),

$$\sigma^z|\uparrow\rangle = |\uparrow\rangle\,, \qquad \sigma^z|\downarrow\rangle = -|\downarrow\rangle\,,$$

$\widehat{p} = -i\hbar\frac{d}{dx}$ is the linear momentum operator acting on the orbital DOF and v is a non-negative parameter with the dimension of a velocity.

(0) Find the (generalized) eigenvectors and eigenvalues of the Hamiltonian (3.1).

(1) The state vector at $t = 0$ is $|\Psi(0)\rangle = |+\rangle|\psi\rangle$ where

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\rangle + |\downarrow\rangle\big)\,,$$

$$\psi(x) \equiv \langle x|\psi\rangle = \big(2\pi\delta^2\big)^{-1/4}\exp\Big(-\frac{x^2}{4\delta^2}\Big)\,, \qquad \delta > 0\,.$$

Find (as explictly as possible) the time-evolved state-vector

$$|\Psi(t)\rangle = \exp\Big(-\frac{i}{\hbar}\widehat{H}\,t\Big)|\Psi(0)\rangle.$$

Hint: You can use (0) or directly expand the exponential and use properties of σ^z. Also expand $|\psi\rangle$ in $\widehat{p}$-eigenvectors. What's going on physically?

(2) Compute the reduced density matrix of the spin subsystem at time t, that is

$$\rho_S(t) = \mathrm{tr}_O|\Psi(t)\rangle\langle\Psi(t)|\,,$$

where tr_O denotes the *partial* trace over the orbital DOF.

Hint: Depending on the solution, you may or may not need this integral:

$$\int_{-\infty}^{\infty} dp\, e^{-ap^2+bp} = \sqrt{\frac{\pi}{a}}e^{b^2/(4a)}.$$

(3) Argue, as quantitatively as possible, that *entanglement*, $S = -\mathrm{Tr}\,(\rho_S \log \rho_S)$, between the spin and orbital DOF grows from zero at $t = 0$ to the maximal value for $t \to \infty$.

Problem II-4. (Math Methods)

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\{|i\rangle\}_{i=0}^{\infty}$ a complete orthonormal set (COS) of it. Consider the following operator D over $\mathcal{H}$ whose only non-zero matrix elements are on the diagonal and have values $D_{ii} = N_x x^i$ where the normalization constant is given by $N_x := 1 - x$ and $x \in (0, 1)$ is a fixed parameter (in short: $D = N_x \,\mathrm{diag}(x^i)_{i=0}^{\infty}$).

If X, Y are operators over $\mathcal{H}$ we define

$$\langle X,\, Y\rangle_D := \mathrm{Tr}\left(DX^{\dagger}Y\right)\,. \tag{4.1}$$

(1) Show that Eq. (4.1) above defines a good (non-degenerate) scalar product over the space of *bounded operators* in $\mathcal{H}$, that is those linear maps X for which

$$\|X\| := \sup_{\|v\|=1} \|X(v)\| < \infty\,.$$

(2) Consider now the operators:

a) $X_1 = \mathbf{1}$ (identity),

b) $X_2 = \mathrm{diag}(y^i)_{i=1}^{\infty}, \quad y \in \mathbb{R}$,

c) X_3 whose action over the COS is given by

$$X_3|i\rangle \;= \sqrt{i}|i-1\rangle\,, \qquad i = 0, 1, \ldots, \infty.$$

Discuss whether (and when) the X_α's above ($\alpha = 1, 2, 3$) are (or not) bounded operators.

(3) Find their norms according to Eq. (4.1), that is compute (as explicitly as you can) $\|X_\alpha\|_D := \sqrt{\langle X_\alpha,\, X_\alpha\rangle_D}$, ($\alpha = 1, 2, 3$).