# M99M.1—Ball Rolling in a Cylinder

## Problem

A solid ball of radius $r$ and mass $m$ is rolling without slipping inside a long hollow vertical cylinder of radius $R > r$ under the influence of gravity. Initially the velocity $v(t = 0)$ is in the horizontal direction and there is no spin perpendicular to the wall about the point of contact. (Though later the ball may develop rotation about the point of contact.)



Side view        Top view at $t = 0$
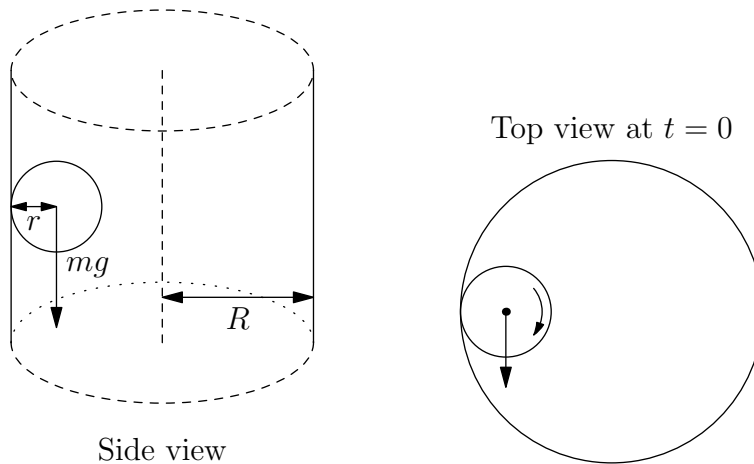
a) Show that there is a rotating coordinate system, with constant angular velocity along the symmetry axis of the cylinder, with respect to which the ball moves only in the vertical direction. What is the angular velocity $\Omega$ of this system? (Hint: you may find it helpful to consider the vertical component of the angular momentum about a conveniently chosen reference point.)

b) Find, and sketch, the trajectory of the ball as a function of time for the given initial conditions. Write down explicit equations of motion for all three components of the angular velocity of the ball.

You may answer the questions in any order, but make sure to write explicitly a complete set of equations of motion, e.g., for the three components of the angular velocity of the ball.

# M99M.2—Two Repulsive Particles

## Problem

There are two particles $A$ and $B$ with a repulsive potential,

$$U(r) = \frac{a}{r^2}$$

between them. Particle $A$ has mass $M$ and particle $B$ has mass $2M$. Suppose particle $A$ is shot at velocity $v$ towards a stationary particle $B$ with impact parameter $b$. Find the deflection angle of particle $A$ after the collision.

Remark: the question can be solved exactly, however an answer incorporating a constant defined by a dimensionless integral is also acceptable (explain your derivation).

# M99M.3—Drum Head

## Problem

A drum can be considered to be a uniform membrane of mass per unit area $\rho$, stretched on a rim of radius $R$. The tension $\tau$, per unit length of the membrane, is so large that it can be considered to be constant even when the membrane is slightly deformed from its equilibrium (flat) shape. (If the membrane is imagined to be cut, then $\tau$ is the force which, applied to unit length on either side of the cut will hold the membrane in its place.)

a)  How far does the center of the membrane sag below the level of the rim when the drum is held horizontal and the acceleration of gravity is $g$?

b)  Find the lowest vibrational frequency $f$ of the membrane. Ignore gravity in this part of the problem.

A reasonable approximate solution will be accepted. However, you may want to know that the first zero of the Bessel function $J_0(x)$, which solves the differential equation $J_0'' + \frac{1}{x}J_0' + J_0 = 0$, is $J_0(0.766\pi) = 0$.

# M99E.1—Radiation from a Rotating Sphere

## Problem

Electromagnetic radiation of wave length $\lambda$ is observed to originate from a system consisting of an electrically charged sphere of radius $R$ placed in a uniform magnetic field $B$ and spinning about its axis with a very large angular velocity $\omega$. The spin axis of the sphere, which is free to move, makes an angle $\alpha$ with the field direction. Assume $R \ll \lambda$.
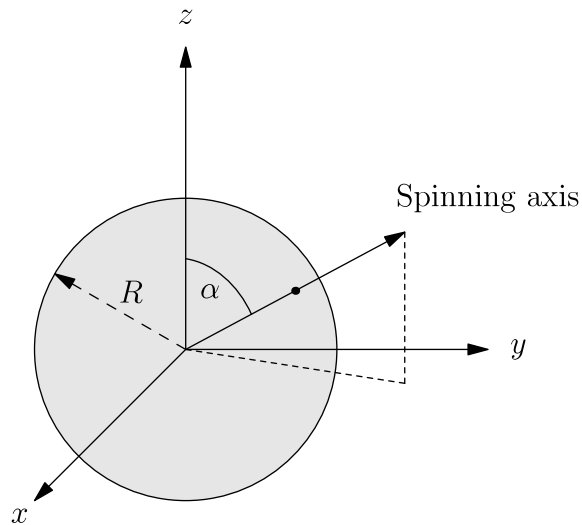


a)    Explain briefly why the system radiates electromagnetic energy.

b)    Find in terms of the given quantities, not all of which may be necessary, the ratio $Q/M$ of the total charge $Q$ to the mass $M$ of the sphere assuming that both charge and mass are uniformly distributed over its volume.

c)    What is the polarization of the radiation field?

# M99E.2—5-Spoke Wheel

## Problem

A 5-spoked wheel (radius $r$), made out of thin wire with resistance per unit length $\lambda$ is freely rotating with angular velocity $\omega(t)$ in a wedge-shaped constant magnetic field $B$ whose field lines are parallel to the axis of the wheel:
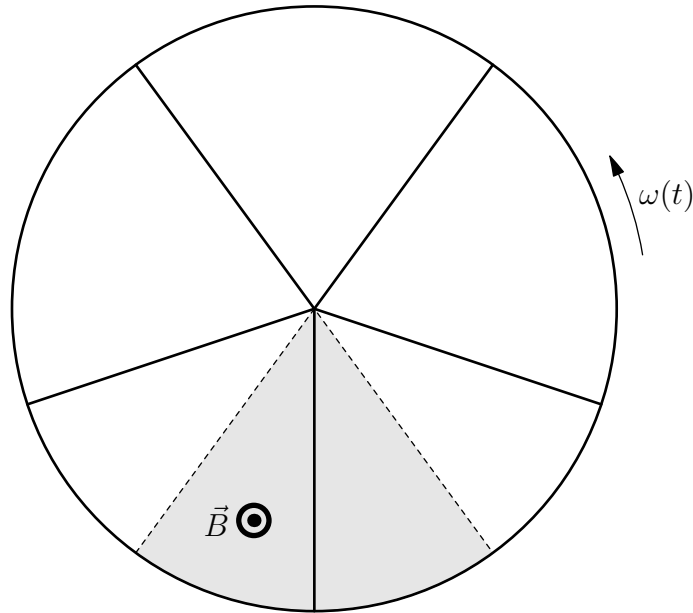


a)   Calculate the resistance of all 10 wire segments.

b)   Find the current in the bottom center spoke.

c)   Determine $\omega(t)$ for an initial angular velocity $\omega(0) = \omega_0$, if the wheel has moment of inertia $I$.

# M99E.3—Dielectric Cylinder

## Problem

A long uniform dielectric solid cylinder with radius $R$ and dielectric constant (electric permittivity) $\varepsilon$ is placed in an external electric field. Far from the cylinder the electric field is uniform with magnitude $E_0$ along a direction perpendicular to the axis of the cyclinder.

Determine the electric field inside and outside the cylinder.

# M99Q.1—Interacting Magnetic Dipoles

## Problem

Two electrons are fixed at positions a distance $L$ apart along the $z$-axis. Their spins interact due to the magnetic dipole interaction:

$$H = \frac{4e^2}{m^2c^2L^3}(\vec{S}_1 \cdot \vec{S}_2 - 3S_{1z}S_{2z}).$$

a)   At time $t = 0$, both spins are aligned along the $z$-axis (so $S_{1z} = S_{2z} = \frac{\hbar}{2}$). At an arbitrary later time $t$, what are the possible results of a measurement of $(S_{1z} + S_{2z})$, and with what probabilities?

b)   Now assume instead that at time $t = 0$, both spins are aligned along the $x$-axis (so $S_{1x} = S_{2x} = \frac{\hbar}{2}$). At an arbitrary later time $t$, what are the possible results of a measurement of $(S_{1x} + S_{2x})$, and with what probabilities?

c)   Compare your answers in parts a) and b) to what would happen for two classical magnetic dipoles.

# M99Q.2—Excited States of Helium

## Problem

This question is about the excited states of the $^4$He atom where one electron has been excited to an $n = 2$ state. (The $^4$He nucleus has spin zero.)

a) How many different such $n = 2$ atomic energy levels are there? (Include all fine-structure effects.) Give the quantum numbers and degeneracy of each level.

b) Which $n = 2$ state has the highest energy? the lowest energy? Give a short explanation of the physics behind your answers, based on symmetries and/or qualitative considerations; no precise calculation is necessary here.

c) For each $n = 2$ state, what is the strongest decay process, and to what atomic state does it decay? Which state has the shortest lifetime, and why? Which state do you think has the longest lifetime, and why?

# M99Q.3—Beam Me Up, Charlie

## Problem

This question relates to a method which was proposed for the "teleportation" of a quantum state by means of classical bits of information (plus a pair of photons in an entangled EPR state).

Alice has a photon – actually a beam of photons but let us focus on one, labeled (1), whose state $|\Psi\rangle$ is an unknown superposition of the two polarization states: vertical polarization, denoted here $|+\rangle$, and horizontal polarization, which we denote $|-\rangle$. Our goal is to enable Bob to reproduce in his lab a photon in exactly this state $|\Psi\rangle$.

To facilitate the "teleportation" of the state to Bob, Charlie sends to each of them one of the photons which are emitted from his EPR (Einstein–Podolsky–Rosen) machine, in the entangled state (of photons labeled (2) and (3)):

$$|\Phi^{(a)}\rangle_{2,3} = \frac{1}{\sqrt{2}}[\ |+,-\rangle_{2,3}\ -\ |-,+\rangle_{2,3}\ ]$$

(where $|+,-\rangle_{2,3} = |+\rangle_2|-\rangle_3$, etc...). He sends photon (2) to Alice and photon (3) to Bob.

Alice has a device which can measure the state of the pair of photons (1) and (2), distinguishing between the four entangled states:

$$|\Phi^{(a)}\rangle_{1,2} = \frac{1}{\sqrt{2}}[\ |+,-\rangle_{1,2}\ -\ |-,+\rangle_{1,2}\ ], \qquad |\Phi^{(b)}\rangle_{1,2} = \frac{1}{\sqrt{2}}[\ |+,-\rangle_{1,2}\ +\ |-,+\rangle_{1,2}\ ]$$

$$|\Phi^{(c)}\rangle_{1,2} = \frac{1}{\sqrt{2}}[\ |+,+\rangle_{1,2}\ -\ |-,-\rangle_{1,2}\ ], \qquad |\Phi^{(d)}\rangle_{1,2} = \frac{1}{\sqrt{2}}[\ |+,+\rangle_{1,2}\ +\ |-,-\rangle_{1,2}\ ]$$

a)   What is the state of the photon (3), which Bob has, if Alice finds her pair of photons to be in the state $|\Phi^{(a)}\rangle_{1,2}$, and what is it if Alice finds her pair of photons to be in the state $|\Phi^{(b)}\rangle_{1,2}$? (It helps to start by writing down the state of all three photons, $|\ldots\rangle_{1,2,3}$.)

b)   Show that by doing the measurement and communicating its outcome (a, b, c or d) to Bob, Alice will provide Bob with enough information to transfrom the state of the photon (3) into the exact state $|\Psi\rangle$. Determine, for each possible outcome of Alice's measurement, which (unitary) operation does Bob need to apply to his photon to make it end up in the polarization state $|\Psi\rangle$? (Assume that he knows how to implement experimentally each of these unitary transformations.)

(Based on: C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).)

# M99T.1—The Otto Cycle

## Problem

Calculate the efficiency of a car engine modeled by the Otto cycle depicted in the diagram. The cycle consists of two isochorous and two adiabatic processes between volumes $V_1$ and $V_2$.
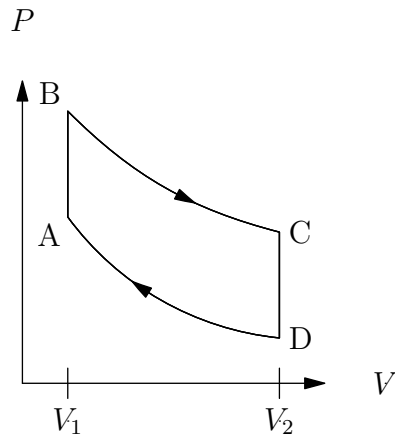


The working medium's equation of state is given by:

$$p = \frac{n}{V}\left(RT + \frac{n\alpha RT}{V}\right) , \quad (-V_1/2 < n\alpha < V_1)$$

(the ideal gas law corrected by the first virial coefficient $RT\alpha$), and the medium's molar heat capacity $C_V$ (at constant volume) remains approximately unchanged through the cycle. Find the efficiency of the depicted cycle in terms of $V_1, V_2, R, C_V$, and $n\alpha$.

# M99T.2—Polymer Chain

## Problem

As a model of rubber, consider a polymer of $N$ molecules of length $a$, connected end to end. One end of the molecular chain is fixed at $x = 0$. Assume the molecules may be oriented only parallel to the $x$-axis, and that the energies of all the configurational states are equal. The chain is kept at fixed temperature $T$ by contact with a heat bath.
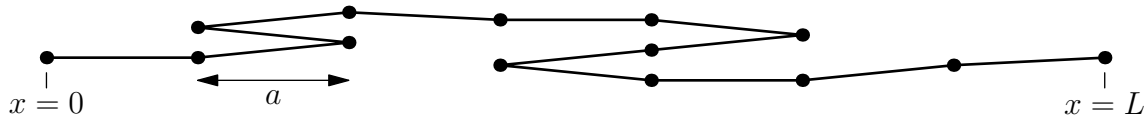


a)  What is the entropy of the chain when one of its ends is at $x = 0$ and the other end is at $x = L$.

b)  What is the tension in the chain when it is stretched to length $L$, with $L \ll L_{max}(= Na)$ ?

c)  For $N$ large, how much work is required to stretch the chain from its rest configuration to length $L \le L_{max}$?

d)  During the stretching process - does the heat bath absorb, or yield, heat? (Explain your answer.)

# M99T.3—Surface Waves

## Problem

Consider waves on a liquid surface where the restoring force is produced by surface tension. Assume there is a single polarization and the dispersion relation is

$$\omega^2 = \frac{\gamma}{\rho} k^3$$

where $\gamma$ is the surface tension of the liquid, $\rho$ is its density, $\omega$ is the frequency of the waves and $k$ is the wavenumber of the waves. (For example, this could be liquid helium, which remains liquid down to arbitrarily low temperatures.) Our goal is to find the contribution of these waves to the low temperature heat capacity of the liquid.

At low temperature $T$ what are the total energy and heat capacity, per unit volume, of these surface waves? Your answer may involve a constant defined by a dimensionless integral. You need not compute its value (denote it $I$). However, you should explain why, and under what conditions, it is OK to set the upper limit to $\infty$.