QUALIFYING EXAMINATION, Part 1

1:00 pm – 5:00 pm, Thursday September 3, 2015

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Mathematical Methods

(a) The formula

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 a^2}\right)$$

is known to work for a particular value of a.

(i) (10 points) Find this value of a by requiring that the right-hand side of the formula reproduces all the zeros of $\sin x$.

(ii) (10 points) Evaluate

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) .$$

(iii) (15 points) Evaluate

$$\prod_{\mathbf{n=2}}^{\infty} \left(1 - \frac{1}{n^2}\right) .$$

(b) (30 points) Starting with Cauchy's theorem

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{z' - z} ,$$

where C is a contour in the complex plane which encloses z and within which $f(z')$ is analytic, derive the formula for $df(z)/dz$ as a contour integral.

Evaluate the integral

$$\oint_{|z|=3.141} \frac{e^{az} dz}{z^2(z-1)} .$$

(c) A classical particle of mass $m = 1$ moves in a potential

$$V(x, y) = x^2 + 2xy - 2y^2. \tag{1}$$

(i) (20 points) Determine the stationary point of the potential (i.e., the equilibrium point of the particle). Is it a maximum, minimum or saddle point?

(ii) (15 points) Determine at least one direction along which the particle may be displaced from the stationary point so that it will execute stable oscillations. What is the corresponding frequency ω for this oscillation?

Problem 2: Classical Mechanics

Consider an N-pendulum, the generalization of the double pendulum (see Fig. 1). Each link consists of a massless string of length l ending in a bob of mass m. The bottommost mass is denoted by $n = 1$ and the topmost mass is $n = N$. The angular displacement of mass n from the vertical is denoted by θ_n and is assumed to be small.

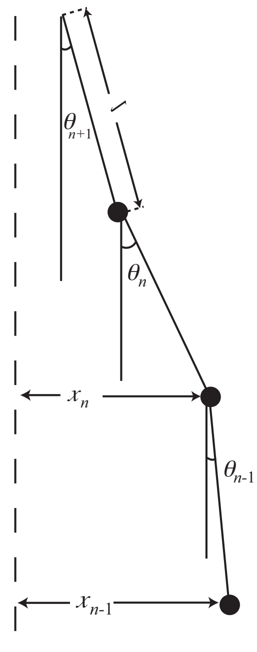

Figure 1: N-pendulum

(a) (10 points) Find an expression for $\sin\theta_n$ in terms of the horizontal displacements x_n and x_{n+1} of the pendulum bobs from their equilibrium positions.

(b) (25 points) Write the (linear) equation of motion for each displacement x_n in terms of other displacements.

Hint: in the limit of small θ_n, you can assume that the tension T_n in the n-th string (just above mass n) is a constant $T_n = nmg$, independent of the displacements.

(c) (25 points) Assuming a harmonic solution $x_n \propto \sin(\omega t + \delta)$ with the same frequency and phase for all x_n, derive a recurrence relation for x_{n+1} in terms of x_{n-1} and x_n.

(d) (40 points) Find the normal frequencies of the double pendulum ($N = 2$) using the recurrence relations in (c).

Hint: The x-displacement is zero at the top hanging point of the pendulum.

Problem 3: Electromagnetism I

An infinitely long hollow (non-conducting) circular cylinder of radius R is fixed at a potential $V = V_0 \sin\varphi$ (see Fig. 2).

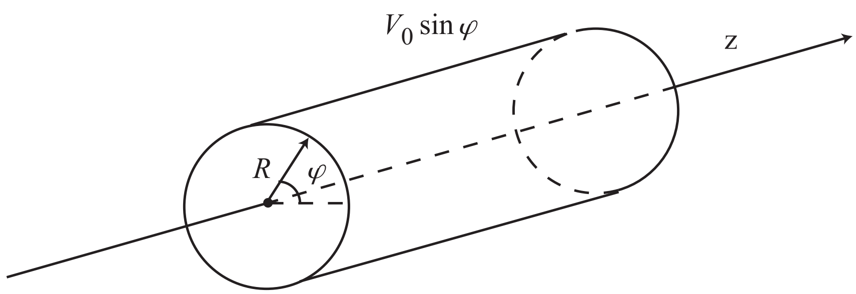

Figure 2: Hollow cylinder with a potential $V = V_0 \sin\varphi$.

(a) (15 points) Using cylindrical coordinates r, φ, z where the z axis is taken along the symmetry axis of the cylinder, argue that the electrostatic potential V is independent of z and must satisfy $V(r, -\varphi) = -V(r, \varphi)$.

(b) (20 points) Find the electrostatic potential $V_{\text{in}}(r, \varphi)$ inside the cylinder ($r \leq R$).

(c) (20 points) Find the electrostatic potential $V_{\text{out}}(r, \varphi)$ outside the cylinder ($r \geq R$).

(d) (25 points) Calculate the surface charge density σ on the cylinder as a function of φ.

(e) (20 points) Find the local capacitance of the device per unit length as a function of the angle φ, i.e., calculate $C = dQ/dV$ where $dQ/d\varphi$ is the charge per unit angle (and per unit length of the cylinder) and $dV/d\varphi$ is the voltage per unit angle.

Hint: the solution of Laplace's equation in cylindrical coordinates which is independent of z has the general form

$$V(r, \varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} a_n r^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n r^{-n} \sin(n\varphi + \beta_n) \ ,$$

where $a_n, b_n, \alpha_n, \beta_n$ are constants.

Problem 4: Electromagnetism II

An infinitely long infinitesimally thin and massless cylindrical non-conducting tube of radius R carries a charge with uniform surface charge density of σ (charge/unit area). The tube rotates around its symmetry axis with a fixed angular velocity ω in free space (see Fig. 3). The formulas below are given in both Gaussian and SI units.

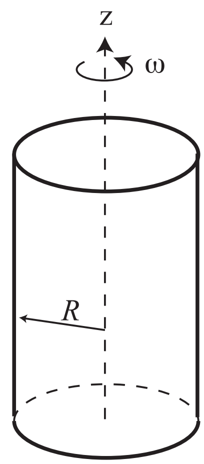

Figure 3: Rotating cylindrical tube.

(a) (10 points) Calculate the electric field $\vec{E}$ inside and outside the cylinder.

(b) (15 points) Calculate the magnetic field $\vec{B}$ inside and outside the cylinder.

Hint: treat the system as a solenoid.

(c) (20 points) Next consider the case where the cylinder's angular velocity ω is not constant, but given by $\omega = \alpha t$, starting with $\omega = 0$ at $t = 0$. What is the magnetic field inside the cylinder? Calculate the electric field (including its direction) just inside the cylinder.

Hint: use Faraday's law of induction to calculate the electric field due to the changing magnetic field.

(d) (20 points) Calculate the Poynting vector

$$\vec{S} = \begin{cases} \frac{c}{4\pi}(\vec{E} \times \vec{B}) & \text{(Gaussian units)} \\ \frac{1}{\mu_0}(\vec{E} \times \vec{B}) & \text{(SI units)} \end{cases}$$

just outside and just inside the cylinder.

(e) (35 points) Integrate the Poynting vector over the inner surface of the cylinder to determine the rate of energy flow per unit length into the cylinder's interior. Compare this to the rate of change of the total field energy within the cylinder per unit length. Are these rates the same? Explain why.

Hint: the energy density u in an electromagnetic field is given by

$$u = \begin{cases} \frac{1}{8\pi}\left(E^2 + B^2\right) & \text{(Gaussian units)} \\ \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) & \text{(SI units)} \end{cases} \quad .$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian
$(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical
(ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical
(r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J ...
		M	M ...
m_1	m_2	Coefficients	
m_1	m_2		
.	.		
.	.		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1		
+1/2	+1/2	1		

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

QUALIFYING EXAMINATION, Part 2

9:00 am – 1:00 pm, Friday September 4, 2015

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators and cell phones may NOT be used.

Problem 1: Quantum Mechanics I

An electron with charge e and mass m_e is placed in uniform magnetic field of strength B pointing in the positive z-direction. The electron spin operator is $\vec{S}$. The Hamiltonian for this system is $H = \omega \hat{S}_z$, where $\omega \equiv |e|B/m_e c$.

At $t = 0$, the electron is in an eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ lies in the x-z plane making an angle φ with the z-axis (see Fig. 1).

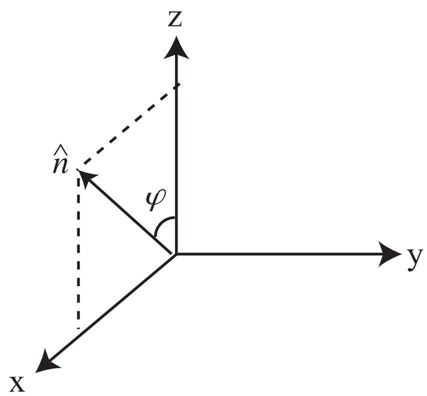

Figure 1: A unit vector $\hat{n}$ in the x-z plane.

(a) (15 points) What is the unitary time-evolution operator $\hat{U}(t) \equiv \hat{U}(t, 0)$ for this system in terms of $\hat{S}_z$ and ω ?

(b) (20 points) Express the initial, normalized state of the system $|t = 0\rangle$ in terms of the normalized spin up/down eigenkets $|+\rangle$ and $|-\rangle$ of $\hat{S}_z$. Do this by acting on the state $|+\rangle$ with the rotation operator of angle φ about the y axis: $|t = 0\rangle = \hat{R}_y(\varphi)|+\rangle = e^{-i\hat{S}_y \varphi/\hbar}|+\rangle$. In the spin 1/2 representation, $\hat{S}_y = (\hbar/2)\sigma_y$, where σ_y is a Pauli matrix.

(c) (30 points) Determine the state $|t\rangle$ of the system at later times t by acting on $|t = 0\rangle$ with the time evolution operator $\hat{U}(t)$. What is the probability that a measurement of $\hat{S}_x$ at time t will yield the eigenvalue $+\hbar/2$?

(d) (20 points) What is the expectation value of $\hat{S}_x$ as a function of time t?

(e) (15 points) What are the answers to parts (c) and (d) for $\varphi = 0$? Check your answers by a direct calculation without using your results in (c) and (d).

The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Problem 2: Quantum Mechanics II

The quantum Zeno effect is a quantum analog of the Zeno paradox. One of its version can be summarized as "a watched pot never boils." Here we will use a two-level system to demonstrate this behavior.

Consider a two-level system with two basis states identified as the "cold" state and the "boiled" state

$$|\psi_{\text{cold}}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\psi_{\text{boiled}}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

We assume that the system undergoes the following unitary time evolution

$$\hat{U}\left(t+\Delta t, t\right) \equiv U\left(\Delta t\right) = \begin{pmatrix} \cos(\nu \Delta t/2) & -\sin(\nu \Delta t/2) \\ \sin(\nu \Delta t/2) & \cos(\nu \Delta t/2) \end{pmatrix} .$$

At an initial time $t = 0$, the pot is prepared in the "cold" state $|t = 0\rangle = |\psi_{\text{cold}}\rangle$. The time evolution can transform this "cold" state into the "boiled" state at a later time.

When we observe whether the pot is "cold" or "boiled" at time t, we effectively perform a measurement of the observable σ_z, with the $+1$ outcome for the "cold" state, and the -1 outcome for the "boiled" state. The pot will be in the corresponding eigenstate of σ_z right after the measurement.

(a) (15 points) Suppose that no measurement is carried out. At what time t_B does the pot enter into the "boiled" state for the first time? In other words, find the smallest positive t_B such that $|t_B\rangle = |\psi_{\text{boiled}}\rangle$.

(b) (30 points) Suppose that the pot is observed twice, first at time $t = t_B/2$ and then at $t = t_B$. Our observations will change the evolution discontinuously. Compute the probabilities associated with the four possible measurement outcomes, and complete the following table.

1st measurement outcome	2nd measurement outcome	Probability
"cold"	"cold"	
"cold"	"boiled"	
"boiled"	"cold"	
"boiled"	"boiled"	

(c) (30 points) Suppose that the pot is observed N times at $t_1 = \frac{t_B}{N}$, $t_2 = \frac{2t_B}{N}, \ldots, t_N = t_B$. Find the probability $P_{Zeno,N}$ that all N measurements yield the "cold" state.

(d) (15 points) Find an approximation of $P_{Zeno,N}$ for large values of N ($N \gg 1$) to the first order in $1/N$.

(e) (10 points) Express the probability in (d) in terms of the time interval $\Delta t = \frac{t_B}{N}$ and find its value in the limit $\Delta t \to 0$, i.e., when the pot is observed continuously.

Problem 3: Statistical Mechanics I

A molecular zipper is comprised of N links, each of which has two states: (i) the zipped-up state, which has energy 0 and degeneracy 1, and (ii) the unzipped state, which has energy ϵ ($\epsilon > 0$) and degeneracy g ($g > 1$). This zipper can only unzip from its left end, and a link cannot unzip unless all links to its left are already unzipped (see Fig. 2). In the following consider the limit $N \to \infty$.

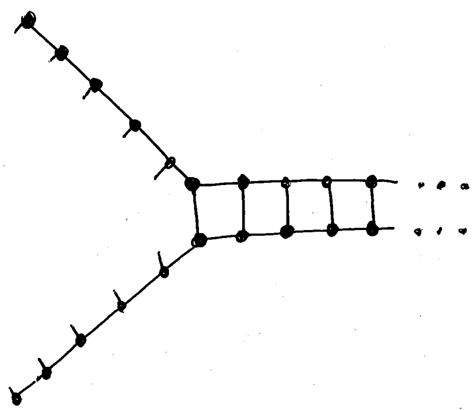

Figure 2: Schematic of the molecular zipper discussed in parts (a)-(c). In this figure there are $m = 5$ unzipped links. The dots indicate additional zipped links.

(a) (30 points) The molecular zipper's partition function Z is finite at temperatures $T < T_M$. Calculate the partition function and determine the maximal temperature T_M for which your calculation is valid.

Hint: For $x < 1$,

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x} .$$

(b) (25 points) Calculate the average number of unzipped links $\langle m \rangle$ for $T < T_M$.

(c) (20 points) For $T < T_M$, what is the probability that one or more links of the zipper are unzipped?

(d) (25 points) A new, improved molecular zipper can unzip from both ends (see Fig. 3). A link cannot unzip unless all the links to its left or all the links to its right are already unzipped. Determine the partition function Z_2 and the mean number of unzipped links $\langle m_2 \rangle$ for this new improved molecular zipper at $T < T_M$. Assume that the number of links is infinite.

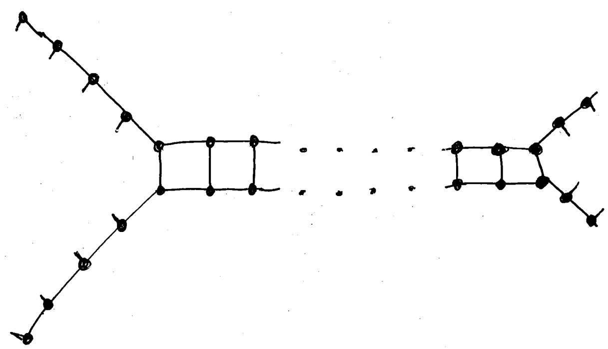

Figure 3: Schematic of the new, improved molecular zipper discussed in part (d). The dots indicate additional zipped links.

Problem 4: Statistical Mechanics II

Consider a gas of N identical non-interacting spin-zero bosons in d dimensions, confined to a large volume $V = L^d$ (you may assume periodic boundary conditions) and held at temperature T. The dispersion relation, expressing the single-particle energy ϵ in terms of the magnitude of its momentum $p = |\vec{p}|$, is given by $\epsilon = ap^s$ where a and s are positive numbers.

(a) (25 points) Find the single-particle density of states $g(\epsilon)$ as a function of energy.

Hint: in d dimensions $d^d\vec{p} = S_d\, p^{d-1}dp$ where S_d is a constant.

(b) (25 points) Write an integral expression for the total number of bosons N in the limit of large volume V when the system is described by a chemical potential μ (assuming there is no Bose-Einstein condensation). Evaluate the integral explicitly in terms of the fugacity $z = e^{\beta\mu}$ and temperature T using the formula given below.

(c) (25 points) To determine whether Bose-Einstein condensation occurs, we have to examine the expression for N in (b) in the limit $\mu \to 0$. Explain why this is the relevant limit and determine the condition satisfied by s and d for which Bose-Einstein condensation occurs. Check that this relation holds for the usual case of $d = 3$ and $s = 2$.

(d) (25 points) When Bose-Einstein condensation occurs, find for $T < T_c$ the fraction $\langle n_0 \rangle / N$ of bosons in the single-particle ground state $\vec{p} = 0$ as a function of T/T_c.

Useful relations:

$$\int_0^\infty dx \frac{x^{r-1}}{z^{-1}e^x - 1} = \Gamma(r) F_r(z) \ ,$$

where $\Gamma(r)$ the gamma function and $F_r(z)$ is the function defined by its series expansion $F_r(z) = \sum_{n=1}^{\infty} z^n/n^r$.

$F_r(1)$ diverges for $r \leq 1$ and converges for $r > 1$.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
⋮ ⋮			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1 m_2	$J=1$, $M=+1$
+1/2 +1/2	1

m_1 m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=1$, $M=-1$
−1/2 −1/2	1

1×1/2

m_1 m_2	$J=3/2$, $M=+3/2$
+1 +1/2	1

m_1 m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	$J=3/2$, $M=-3/2$
−1 −1/2	1

2×1

m_1 m_2	$J=3$, $M=+3$
+2 +1	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	$J=3$, $M=-3$
−2 −1	1

1×1

m_1 m_2	$J=2$, $M=+2$
+1 +1	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	$J=2$, $M=-2$
−1 −1	1

2×1/2

m_1 m_2	$J=5/2$, $M=+5/2$
+2 +1/2	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	$J=5/2$, $M=-5/2$
−2 −1/2	1

3/2×1/2

m_1 m_2	$J=2$, $M=+2$
+3/2 +1/2	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	$J=2$, $M=-2$
−3/2 −1/2	1

3/2×1

m_1 m_2	$J=5/2$, $M=+5/2$
+3/2 +1	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-5/2$
−3/2 −1	1