Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 30, 2019

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your Sh-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Sh-number and the problem number.* Do not write answers to different problems on the same page and do not write anything on the back of page. Staple *separately* your answers to each problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

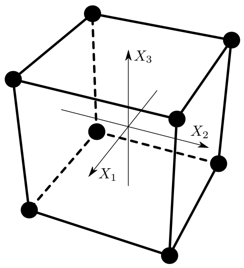
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1	☐ 2	☐ 3	☐ 4	☐ 5	☐ 6
☐ 7	☐ 8	☐ 9	☐ 10	☐ 11	☐ 12

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

Problem I-1. (Classicial Mechanics)



A rigid body consists of 8 point masses, m, located at the corners of a cube and held in place by massless rods of length $2a$. The body frame at the center of mass of the body is oriented such that its axes X_1, X_2 and X_3 pass through the centers of the opposite faces as in the figure.

(i) Calculate the tensor of inertia, I_{ij}, with respect to the center of mass in the body frame and argue that this rigid body is a spherical top.

Consider now a force-free motion of the body. At time $t = 0$, the axes of the body frame coincide with the axes of the fixed frame and the components of the angular momentum of the body with respect to its center of mass are

$$L_1 = \ell_1, \qquad L_2 = \ell_2\,, \qquad L_3 = \ell_3\,,$$

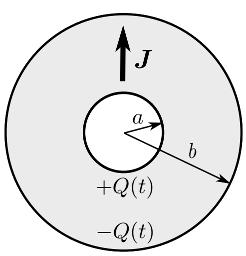
where ℓ_1, ℓ_2 and ℓ_3 are given.

(ii) Find the components of the angular momentum and the angular velocity in the fixed frame at time t .

(iii) What is the kinetic energy of the body with respect to the center of mass at time t?

(iv) Consider the mass at the corner (a, a, a). In the fixed frame, the mass is moving uniformly along a circle. Find the period of rotation and outline the calculation to find the radius of this circle.

Problem I-2. (Electricity and Magnetism)



Imagine two concentric metal spherical shells. The inner one of radius, a, carries a charge, $+Q(t)$, and the outer one of radius, b, an opposite charge, $-Q(t)$. The space between them is filled with Ohmic material of conductivity, σ, so a radial current, $\boldsymbol{J}$, flows.

(i) Using symmetry, compute the electric field, $\boldsymbol{E}$, in the region between the shells.

(ii) Using Ohm's law, find the current density, $\boldsymbol{J}$, in the region between the two shells.

(iii) Charge conservation gives a relationship between the charge $Q(t)$ on the inner shell and the total current I flowing between the two shells. Use this relation to derive a differential equation in time for $Q(t)$.

(iv) Solve this differential equation to determine $Q(t)$ for $t > 0$ starting with the initial value $Q(0) = q$.

Problem I-3. (Quantum Mechanics)

Consider two energy eigenstates of the hydrogen atom, $\psi_{2,1,0}(\boldsymbol{r},t)$ and $\psi_{1,0,0}(\boldsymbol{r},t)$, with the energies $E_{2,1,0}$ and $E_{1,0,0}$, respectively. At time $t=0$,

$$\psi_{2,1,0}(\boldsymbol{r},t=0) = \rho_{2,1}(r)\cos\theta\,, \qquad \psi_{1,0,0}(\boldsymbol{r},t=0) = \rho_{1,0}(r)\,, \tag{3.1}$$

where $\rho_{2,1}(r)$ and $\rho_{1,0}(r)$ are radial wave functions that include all normalization constants. The explicit form of these radial functions is not needed in the following.

(i) What are the wave functions $\psi_{2,1,0}(\boldsymbol{r},t)$ and $\psi_{1,0,0}(\boldsymbol{r},t)$ at time t?

Consider now a superposition of the two states with the wave function of the atom given by

$$\psi(\boldsymbol{r},t) = \alpha\,\psi_{2,1,0}(\boldsymbol{r},t) + \beta\,\psi_{1,0,0}(\boldsymbol{r},t)\,. \tag{3.2}$$

For simplicity assume that both constants α and β are real.

(ii) Show that the dipole moment of the atom in this superposition state is

$$\langle z(t)\rangle = r_0\cos\omega t\,. \tag{3.3}$$

Find an explicit expression for r_0 in terms of α, β and an integral of radial functions. Calculate the frequency ω.

The polar coordinates are: $x = r\sin\theta\cos\phi$, $\quad y = r\sin\theta\sin\phi\,$, $\quad z = r\cos\theta\,.$

Problem I-4. (Math Methods)

Let S be an invertible operator over an n-dimensional complex Hilbert space, $\mathcal{H}$.

(i) Show that the operator $\Delta = S^\dagger S$ is positive definite, that is all its expectation values are positive.

(ii) Show that one can write S as $S = J\Delta^{1/2}$, where J is unitary and unique.

(iii) Assume that S is diagonal, $S \equiv \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{C}$. Find explicitly Δ and J.

Problem I-5. (Condensed Matter)

Consider acoustic waves propagating in a three-dimensional solid of volume V with the average velocity v. In the Debye model their density of states is given by

$$D(\omega) = \frac{3}{2\pi^2}\,V\,\frac{\omega^2}{v^3}\,.$$

(i) The total number of vibrational modes must be limited to $3N$, where N is the number of atoms in the solid. Use this fact to find the Debye frequency.

(ii) Find the zero-point vibrational energy per *one* atom in the solid in terms of the Debye frequency.

(iii) Recall the thermodynamic identity $dU = -pdV + TdS$. At $T=0$ we don't have to worry about the entropy variable and the heat flow term, and the pressure is then simply $p = -(\partial U/\partial V)$. What is the pressure exerted by the zero-point acoustic vibrations in the solid? Express your answer in terms of the Debye frequency and the atom number density.

Problem I-6. (Statistical Mechanics)

We consider a random "polymer" in three dimensions. The polymer is represented by a set of independent vectors $\boldsymbol{r}_i$ all of length a. These vectors encode the orientations of the "monomers." The only energy term is due to a force $\boldsymbol{F}$ exerted along the z-axis, so that, if a monomer i is oriented along the vector $\boldsymbol{r}_i$, the energy for that monomer is $-\boldsymbol{F}\cdot\boldsymbol{r}_i = -Fa\cos\theta_i$.

(i) Argue that the partition function can be written as

$$\mathcal{Z}_N = \frac{1}{(4\pi)^N}\int_0^{2\pi} d\phi_1 \ldots d\phi_N \int_0^{\pi} d\theta_1 \ldots d\theta_N \sin\theta_1 \ldots \sin\theta_N \exp\left(Fa\sum_{i=1}^{N}\cos\theta_i/k_BT\right).$$

where θ_i and ϕ_i are spherical angles.

(ii) Show that

$$\mathcal{Z}_N = \left(\frac{\sinh\beta Fa}{\beta Fa}\right)^N ,$$

where $\beta = 1/k_BT$,.

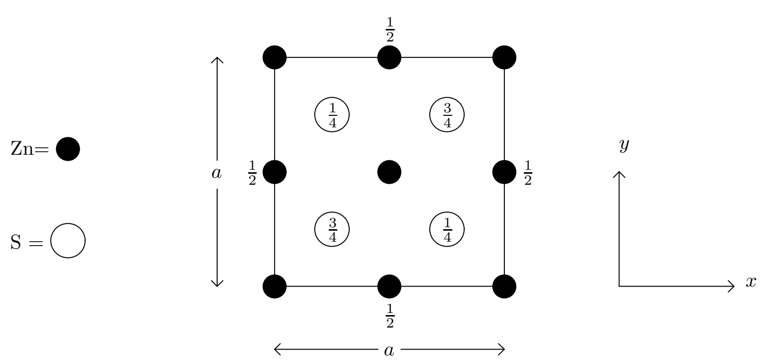
(iii) Show then that the extension of the polymer along the z-axis is

$$\langle z_N\rangle = Na\left[\coth\beta Fa - \frac{1}{\beta Fa}\right].$$

(iv) How does $\langle z_N\rangle$ behave at small F? Compare with a harmonic oscillator: what is the effective spring constant? How does it vary with T? What is the force needed to have $\langle z_N\rangle \approx Na$? What does it mean?

Hint: $\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \ldots$.

Problem I-7. (Solid State)



The diagram above shows a two-dimensional view of a structure of cubic ZnS (zinc blende structure) looking down the z-axis (chosen along a specific crystal direction). The numbers attached to some atoms represent the heights of the atoms above the $z = 0$ plane expressed as a fraction of the cube edge a. Unlabeled atoms are at $z = 0$ and $z = a$.

(i) What is the Bravais lattice type?

(ii) Which are the basis vectors?

(iii) Given that $a = 0.541$ nm, calculate the nearest-neighbor Zn-Zn, Zn-S and S-S distances.

(iv) Cubic SiC is one possible phase of polytypic silicon carbide, which has the same zinc blende structure but a different lattice constant, $a = 0.436$ nm. What are the Si-Si, Si-C and C-C distances?

Problem I - 8. (Thermodynamics)

(i) Derive the Maxwell relation,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V . \tag{8.1}$$

Application: From his electromagnetic theory Maxwell found that the pressure, p, from an isotropic radiation field is proportional to the energy density, $u(T)$,

$$p = \frac{1}{3}\, u(T) , \qquad u(T) \equiv \frac{U(T)}{V} .$$

(ii) Using the first and the second laws of thermodynamics together with the Maxwell identity (8.1) show that u obeys the equation

$$u = \frac{1}{3}\, T\, \frac{du}{dT} - \frac{1}{3}\, u .$$

(iii) Solve this equation and obtain Stefan's law relating u and T.

Problem I - 9. (Relativity)

Alice and Betty are twin sisters. On their 20th birthday Alice decides to board a spaceship and travel to a faraway Galaxy. The speed of the spaceship is $0.8\,c$. Betty remains in Los Angeles. When Betty turns 21 she uses her red laser to send Alice a birthday message.

(i) How old is Betty when Alice receives this message?

(ii) How far is Alice according to Betty from Earth when she receives this message? (Give the answer in light years.)

(iii) How old is Alice when she receives Betty's message?

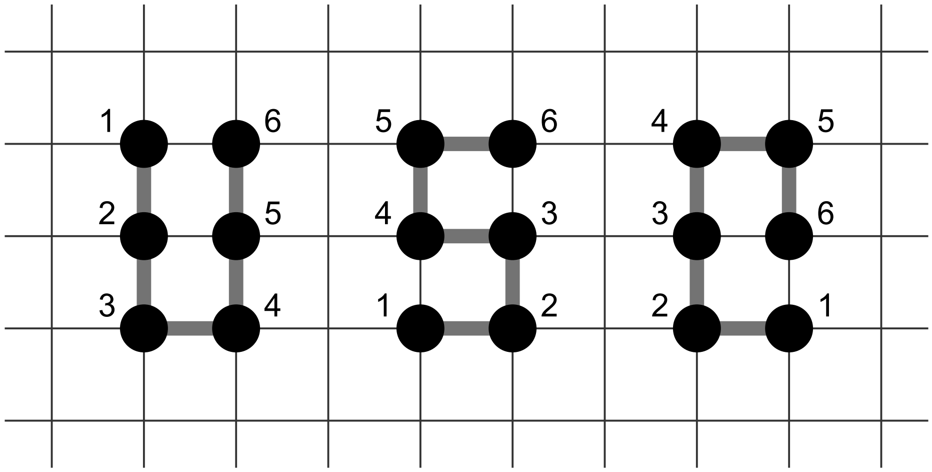
(iv) How far is Alice from Earth, as measured by Alice? (Give the answer in light years.)

Problem I - 10. (Astrophysics)

(i) During the supernova explosion of SN 1987 A, the number of neutrinos per area arriving at the location of Earth was estimated to be 1.3×10^{14} m^{-2}. Assuming the neutrinos had an average energy of 4.2 MeV, estimate the total amount of neutrino energy released during the supernova explosion. In the calculation, you are allowed to use the following approximate values: for the distance d to the neutrino source (located in the Large Magellanic Cloud), d = 50 kpc (with 1 pc = 3.1×10^{16} m), 1 eV = 1.6×10^{-19} J.

(ii) The gravitational binding energy of a spherical object of mass M and radius R is, to a good approximation, $U_g = -3GM^2/(5R)$. Compute the gravitational binding energy of a neutron star of mass $1.4\,\mathrm{M}_\odot$ and radius 10 km. Compare your answer with the amount of the neutrino energy released during the collapse of the iron core of the progenitor star of SN 1987 A. In the calculation, you are allowed to use the following approximate values: $G = 6.7 \times 10^{-11}$ N m^2 kg^{-2}, solar mass $\mathrm{M}_\odot = 2 \times 10^{30}$ kg.

Problem I-11. (Biophysics)



Protein folding is the physical process by which a protein chain acquires its unique compact 3-dimensional structure, which in turn can determine its function. It is also a (very) hard problem: if you knew the amino acid sequence of the protein, how do you predict this native compact structure?

To gain some insight, let's simplify the alphabet of amino acids. Instead of ~ 20 different amino acids, let's just deal with 2; H represents a hydrophobic amino acid and P represents a polar amino acid. As you know, H does not want to neighbor any solvent (surrounding the protein) or a P. A free energy penalty, ε, applies to any such contact. Now, let's predict the structure of such an HP molecule made of 6 amino acids with the following sequence: PHPPHP on a 2×3 lattice. The figure shows the 3 possible compact structures (that are unrelated by lattice rotations, translations or reflections).

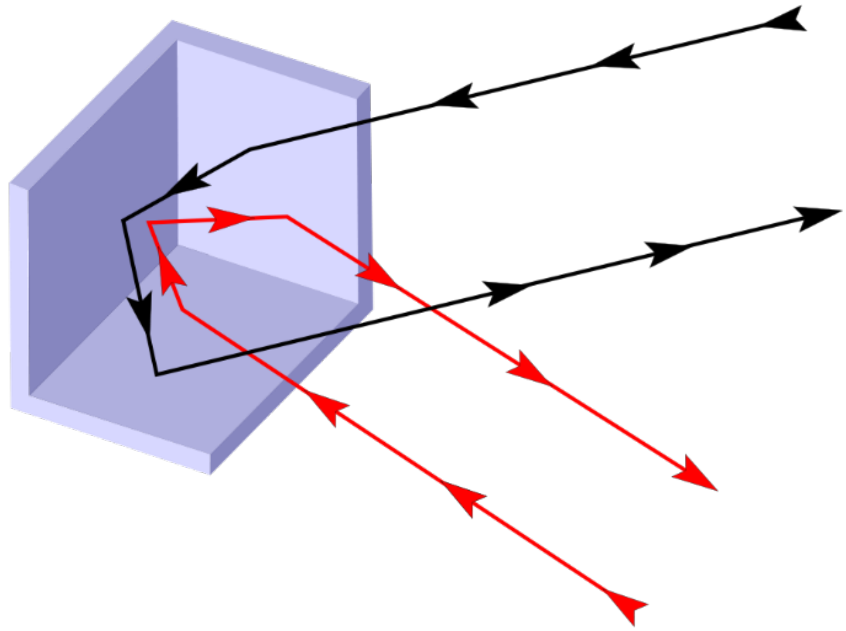
(i) If a sequence has a unique native structure, we call it "protein like." Is PHPPHP protein like? If so, which one is that native structure?

(ii) At a given temperature T, what is the probability of finding this molecule in the native structure?

(iii) Sketch this probability as a function of temperature. Comment on what happens at low temperature and high temperature.

Comment: While simplistic, this kind of approach has been used to design new peptides (from scratch) to accomplish specific functions.

Problem I-12. (Experimental/Optics)



A **corner retroreflector** is a set of 3 square mirrors arranged at right angles to each other (like the inside corner of a cube) as shown in the diagram.

(i) Prove that incoming light is reflected back along the same angle.

(ii) Outside of a certain range of incident angles, the incoming light will not be reflected along the same path. What is that range? You may draw diagrams in 2D if it helps to explain your answer.

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Graduate Screening Examination
Part II
Saturday, March 30, 2019

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Student ________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is marked with your Lg-number and the problem number.* Do not write answers to different problems on the same page. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)

A particle of mass μ moves in one dimension along the q-axis in a potential energy $V(q)$, but is retarded by a damping force $-2\mu\gamma\dot{q}$ proportional to its velocity.

(i) Show that the equation of motion follows from the Lagrangian

$$L = e^{2\gamma t}\left(\frac{1}{2}\mu\dot{q}^2 - V(q)\right).$$

(ii) Show that the Hamiltonian is

$$H = e^{-2\gamma t}\,\frac{p^2}{2\mu} + e^{2\gamma t}\,V(q)\,.$$

(iii) Consider the type-2 generating function

$$F_2(q, P, t) = e^{\gamma t} qP\,.$$

Express dF_2 in terms of the differentials dq, dP, and dt. Recall that if $K(Q, P, t)$ is the transformed Hamiltonian

$$pdq - Hdt = PdQ - Kdt + dF_1\,,$$

and $F_2 = F_1 + QP$. Find $K(Q, P, t)$.

(iv) Now suppose that $V(q)$ is the oscillator potential

$$V(q) = \frac{1}{2}\,\mu\,\omega^2 q^2\,, \qquad \gamma < \omega\,.$$

Show that in this case $K(Q, P, t)$ is a constant of motion.

(v) Write down the Hamilton equations, i.e. first-order coupled differential equations of motion in (P, Q). Eliminate P from the equation of motion that involves $\dot{Q}$ (use the constant of motion) to give a differential equation in Q. Solve this equation, and show that

$$Q(t) = \sqrt{\frac{2K}{\mu(\omega^2 - \gamma^2)}}\,\sin\phi(t)\,,$$

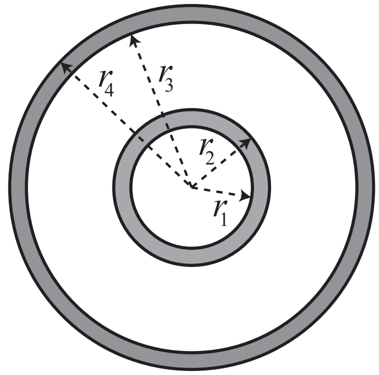
where

$$\phi(t) = \phi(0) + \sqrt{\omega^2 - \gamma^2}\,t\,.$$

You may need $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$.

(vi) Express $q(t)$ as a combination of $\sin(\sqrt{\omega^2 - \gamma^2}\,t)$ and $\cos(\sqrt{\omega^2 - \gamma^2}\,t)$, assuming that K and $\phi(0)$ are known.

Problem II-2. (Electricity and Magnetism)



Two rigid concentric conducting spheres have finite thickness, and sit in empty space, as shown in the diagram. The inner sphere has radii r_1 and r_2, with $r_1 < r_2$, and the outer sphere has radii r_3 and r_4, with $r_3 < r_4$. Charge Q_{I} is put on the inner sphere and charge Q_{O} on the outer sphere.

(i) Find the surface charge density on each of the four surfaces.

(ii) Compute the capacitance of the system when $Q_{\mathrm{O}} = -Q_{\mathrm{I}} = -Q$.

Now imagine that the space between the spheres is filled with a linear dielectric material with dielectric constant ϵ.

(iii) Find the surface polarization charge density on each of the four surfaces. (Assume again that Q_{O} and Q_{I} are arbitrary).

(iv) Compute the capacitance of the system when $Q_{\mathrm{O}} = -Q_{\mathrm{I}} = -Q$.

Imagine now that both the outer sphere and the dielectric are compressible. There is zero cost for compressing the conductor, but the dielectric has elastic energy as a function of its radius r_3, modelled by:

$$U = \frac{1}{2}k(r_3 - r_0)^2 ,$$

where $r_0 > r_2$ and k are positive constants.

(v) By sketching the appropriate functions, argue that there is an equilibrium position for the inner surface of the outer sphere that depends on the charge Q. (Let us assume again that $Q_{\mathrm{O}} = -Q_{\mathrm{I}} = -Q$.)

(vi) Show that the equilibrium position r_3 satisfies:

$$-r_0 k r_3^2 + k r_3^3 + A = 0 , \qquad \text{where} \qquad A = \frac{Q^2}{8\pi\epsilon} .$$

(vii) When r_0 has the value $r_0 = 3(A/4k)^{1/3}$, show that the equilibrium position is at $r_3 = \frac{2}{3}r_0$. (In fact, this is *just* where the equilibrium point starts to appear. There's no physical solution for smaller r_0.)

Problem II-3. (Quantum Mechanics)

A certain system has N possible states. Its state ket belongs to a N-dimensional space spanned by the orthonormal basis $\{|n\rangle, n = 1, 2, ..., N\}$, where the base kets have the periodic boundary condition

$$|n + N\rangle = |n\rangle \, .$$

The system's Hamiltonian is

$$\mathbf{H} = E_0 \sum_{n=1}^{N} |n\rangle \langle n| + \lambda \sum_{n=1}^{N} \big(|n\rangle \langle n+1| + |n+1\rangle \langle n| \big) \, ,$$

where λ is a coupling energy. In the zero-coupling (i.e. $\lambda = 0$) limit all of the base kets are eigenkets of the Hamiltonian with the same (degenerate) energy eigenvalue E_0.

(i) Consider the "shift" operator

$$\mathbf{t} \equiv \sum_{n=1}^{N} |n\rangle \langle n+1|.$$

Express $\mathbf{H}$ in terms of $\mathbf{t}$, $\mathbf{t}^\dagger$, and the identity operator $\mathbf{1}$.

(ii) Show that $\mathbf{H}$, $\mathbf{t}$, and $\mathbf{t}^\dagger$ mutually commute.

(iii) Represent $\mathbf{t}$ by a $N \times N$ matrix on the basis $\{|n\rangle, \; n = 1, 2, ..N\}$.

(iv) Determine the eigenvalues of $\mathbf{t}$.

(v) Determine the eigenvalues of $\mathbf{H}$.

(vi) Show that the eigenkets of $\mathbf{H}$ are

$$|E_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i(n-1)\theta_m} |n\rangle,$$

where $\theta_m = 2\pi(m/N)$.

Problem II-4. (Math Methods)

The Γ-function is usually defined by the Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt\,. \tag{4.1}$$

(i) Determine for which $z \in \mathbb{C}$ the integral (4.1) is convergent.

(ii) Prove that in the region of $\mathbb{C}$ detemined in (i), $\Gamma(z)$ has the following properties:

- complex conjugation, $\overline{\Gamma(z)} = \Gamma(\overline{z})$.
- recurrence relation, $z\,\Gamma(z) = \Gamma(z+1)$.
- relation to the factorial, $\Gamma(n) = (n-1)!$, $n = 1, 2, \ldots$

Using the recurrence relation in (ii), one can continue $\Gamma(z)$ analytically to the complex plane, where it is a meromorphic function with simple poles at nonpositive integers.

(iii) Show that the residue of $\Gamma(z)$ at $z_0 = -n$ is

$$\mathrm{Res}(\Gamma(z), -n) = \frac{(-1)^n}{n!}\,, \qquad n = 0, 1, 2, \ldots$$

Hint: One method is to use the recurrence relation to express $\Gamma(z-n)$ in terms of $\Gamma(z+n+1)$ and then expand around $z = 0$.

(iv) Sketch the graph of $\Gamma(x)$, $x \in \mathbb{R}$.

(v) Certain function $f(x)$ on $(0, \infty)$ is defined by the inverse Mellin transform given by the following complex integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\, x^{-s}\, ds\,, \qquad x > 0\,,$$

along a contour parallel to the imaginary axis at $\mathrm{Re}\, s = c > 0$. This integral can be evaluated as an infinite series obtained by closing the countour in the left half-plane and then using the Cauchy residue theorem. Find that series and sum it to obtain a closed form formula for $f(x)$.