Autumn 2012

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

September 19, 2012

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Conical Capacitor

A capacitor consists of two coaxial metal cones. Their vertices touch but are electrically insulated from each other. A potential difference $\Delta\Phi$ is maintained between the cones with the outer cone (cone 2) at $\Phi = 0$ and the inner cone (cone 1) at $\Phi = V$. The cones are sufficiently large that end effects may be neglected. Assume the equipotential surfaces between the conductors are cones coaxial with the conductors.

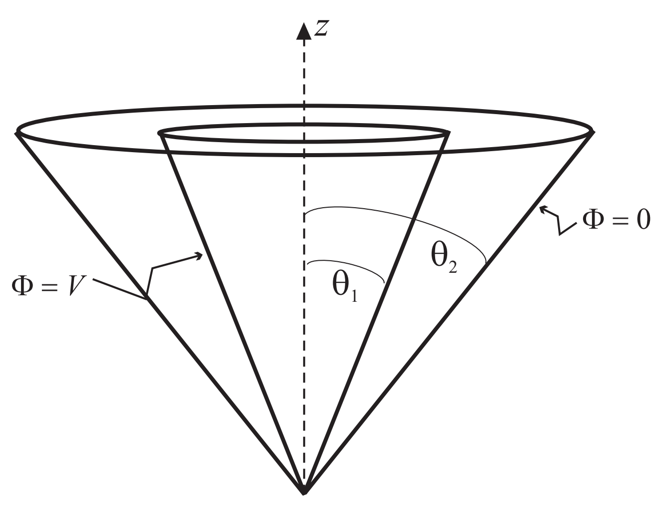
(a) Determine the electric field between the cones in terms of r, θ, and V.

(b) Determine the charge density on each cone.

(c) Explain whether your solution changes if the assumption about the equipotential surfaces is not made.

You may find the following result useful:

$$\int \frac{dx}{\sin ax} = \frac{1}{a} \ln \left[\tan \frac{ax}{2} \right].$$



Conical Capacitor Solution

(a) Because the equipotential surfaces between the conductors are cones the potential is independent of r and ϕ where ϕ is the azimuthal angle in spherical coordinates. Therefore, Laplace's equation in spherical coordinates reduces to

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)=0 \implies \sin\theta\frac{\partial\Phi}{\partial\theta}=c_1$$

where c_1 is constant. This gives

$$\Phi=c_1\int\frac{d\theta}{\sin\theta}=c_1\ln\left(\tan\frac{\theta}{2}\right)+c_2.$$

The constants c_1 and c_2 are found by evaluating Φ at each cone:

$$c_1\ln\left(\tan\frac{\theta_1}{2}\right)+c_2=V \quad \text{and} \quad c_1\ln\left(\tan\frac{\theta_2}{2}\right)+c_2=0$$

Solving these simultaneously for c_1 and c_2 gives

$$c_1=\frac{V}{\ln[\tan\theta_1/2]-\ln[\tan\theta_2/2]} \quad \text{and} \quad c_2=-c_1\ln[\tan\theta_2/2].$$

The potential can now be written as

$$\Phi(\theta)=\frac{V}{A}\ln\left[\frac{\tan\theta/2}{B}\right],$$

where, for convenience, we have taken $A=\ln\left[\frac{\tan\theta_1/2}{\tan\theta_2/2}\right]$ and $B=\tan\theta_2/2$. Note that A is negative.

The electric field is given by $\vec{E}=-\nabla\Phi$. Thus,

$$\vec{E}(r,\theta)=-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}=-\frac{1}{r}\frac{V}{A\sin\theta}\hat{\theta}.$$

(b) The electric field is zero for $\theta<\theta_1$ and $\theta>\theta_2$. Therefore, at the inner cone the surface charge density σ is determined by $E(r,\theta_1)-0=\sigma_1/\epsilon_0$ and at the outer cone by $0-E(r,\theta_2)=\sigma_2/\epsilon_0$. Therefore,

$$\sigma_1=-\frac{\epsilon_0 V}{A\sin\theta_1}\frac{1}{r}$$

and

$$\sigma_2=\frac{\epsilon_0 V}{A\sin\theta_2}\frac{1}{r}.$$

(c) We have found a solution to Laplace's equation for the given boundary conditions that is consistent with the assumption. It is well established that a solution to Laplace's equation under these conditions is unique. Therefore, the solution does not change.

2. Frictional Rope

(a) (b)

Figure 1: (a) The initial position of the rope. (b) A portion x of the rope has slid off the cylinder.

A uniform unstretchable rope of mass m is placed on a fixed cylinder of radius R as shown in Fig. 1a. Its length l is within the range $0 < l < \pi R$. There is dry friction between the rope and the cylinder with a coefficient of friction μ that is insufficient to prevent the rope from sliding.

At a certain moment when the rope remaining on the cylinder spans an angle $\theta = (l - x)/R$, its speed is v and its acceleration is $\dot{v}$; see Fig. 1b. Determine the instantaneous tension along the rope $T(\phi)$ as a function of v, $\dot{v}$, and the angle ϕ.

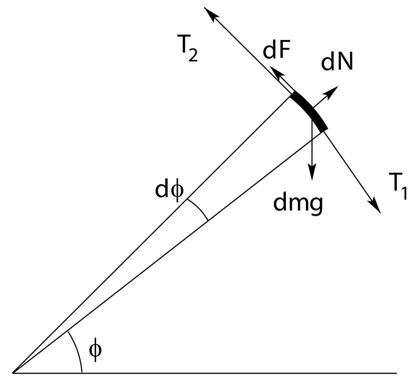


Figure 2: Force diagram for a rope element.

Frictional Rope Solution

Let us consider an element of the rope in the presence of friction; see Fig. 2. The Newton equation of motion for the element is

$$\boldsymbol{T}_1 + \boldsymbol{T}_2 + d\boldsymbol{N} + dm\,\boldsymbol{g} + d\boldsymbol{F} = dm\,\boldsymbol{a}. \tag{1}$$

In addition, when the rope is sliding, the magnitude of the friction force is related to the normal force by $dF = \mu dN$.

The acceleration $\boldsymbol{a}$ can be decomposed into the tangential and the normal components,

$$a_\tau = \dot{v}, \qquad a_n = \frac{v^2}{R}, \tag{2}$$

where v is the instantaneous velocity of the rope. The components along and perpendicular to the rope element give:

$$T_2 - T_1 + \mu dN - dm\,g\cos\phi = -dm\,\dot{v}, \tag{3}$$

$$T d\phi + dm\,g\sin\phi - dN = dm\frac{v^2}{R}. \tag{4}$$

The second equation follows from the fact that the two tension forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are not exactly aligned, the angle between their directions is $\pi - d\phi$, so their vector sum is along the radius of the cylinder, and has the magnitude $Td\phi$ (to linear order in $d\phi$). This leads to the system of equations:

$$\frac{dT}{d\phi}+\mu\frac{dN}{d\phi}-mg\frac{R}{l}\cos\phi+m\frac{R}{l}\dot{v}=0, \qquad \frac{dN}{d\phi}=T+mg\frac{R}{l}\sin\phi-\frac{m}{l}v^2. \tag{5}$$

Substituting the second equation into the first leads to an inhomogeneous first order ODE for the tension in the rope:

$$\frac{dT}{d\phi}+\mu T+mg\frac{R}{l}(\mu\sin\phi-\cos\phi)=\frac{\mu m}{l}v^2-m\frac{R}{l}\dot{v}. \tag{6}$$

At a given instance in time the velocity v and its time derivative $\dot{v}$ are independent of the angle ϕ. Therefore, Eq. (6) is a first order inhomogeneous ODE for $T(\phi)$. The natural boundary condition for it is,

$$T(\theta)=0. \tag{7}$$

The solution of equation (6) proceeds in a standard manner. First we find a general solution of the homogeneous equation, which gives $T_{\text{homo}}(\phi)=T_0e^{-\mu\phi}$. Then a particular solution of the inhomogeneous equation is found as

$$T_{\text{inhomo}}(\phi)=A\sin\phi+B\cos\phi+C. \tag{8}$$

The coefficients A, B, and C are found by direct substitution of the solution into the equation, which gives

$$A=\frac{1-\mu^2}{1+\mu^2}mg\frac{R}{l}, \qquad B=\frac{2\mu}{1+\mu^2}mg\frac{R}{l}, \qquad C=\frac{m}{l}v^2-\frac{mR}{\mu l}\dot{v}. \tag{9}$$

The boundary condition (7) now gives

$$T(\theta)=T_0e^{-\mu\theta}+A\sin\theta+B\cos\theta+C=0, \quad\Rightarrow\quad T_0=-\big(A\sin\theta+B\cos\theta+C\big)e^{\mu\theta}. \tag{10}$$

Finally, the tension in the rope is,

$$\begin{aligned} T(\phi)&=A\sin\phi+B\cos\phi+C-\big(A\sin\theta+B\cos\theta+C\big)e^{\mu(\theta-\phi)},\\ &=e^{-\mu\phi}\big[A(e^{\mu\phi}\sin\phi-e^{\mu\theta}\sin\theta)+B(e^{\mu\phi}\cos\phi-e^{\mu\theta}\cos\theta)+C(e^{\mu\phi}-e^{\mu\theta})\big], \end{aligned} \tag{11}$$

where the constants A, B, and C are given by Eq. (9).

3. Neutrino Oscillations

The Standard Model of particle physics contains three neutrinos, which are now known to have mass. For the present problem, let us consider only two neutrinos. Weak decay processes produce "flavor" eigenstates of neutrinos, which we denote by $|\nu_e\rangle$ and $|\nu_\mu\rangle$. These flavor eigenstates are linear superpositions of mass eigenstates, which we denote by $|\nu_1\rangle$ (mass m_1) and $|\nu_2\rangle$ (mass $m_2 > m_1 > 0$):

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \, , \\ |\nu_\mu\rangle &= -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \, , \end{aligned}$$

where $\sin\theta$ is a fundamental constant. Suppose that at $t = 0$, a neutrino is produced in the $|\nu_e\rangle$ flavor eigenstate with definite three-momentum p. The Hamiltonian of the system is

$$H_0 = E_1|\nu_1\rangle\langle\nu_1| + E_2|\nu_2\rangle\langle\nu_2|,$$

with $E_i = \sqrt{m_i^2c^4 + c^2p^2}$.

(a) Working to first order in the small quantities m_1^2/p^2, m_2^2/p^2, find the probability $P(t)$ that the neutrino remains in the $|\nu_e\rangle$ state if observed at time t.

(b) Cosmic rays impinging on the upper atmosphere produce unstable particles that subsequently decay to final states including neutrinos of known relative flavor abundance. Detectors on Earth can observe neutrinos of a definite flavor. Suppose that a detector is sensitive to neutrinos of energy MeV to TeV. Assuming $\theta \sim 45°$, estimate (order of magnitude) the smallest value of $\Delta m^2 = |m_1^2 - m_2^2|$ that can be probed by the detector. Express your answer numerically in eV^2.

(c) The weak force between electrons and neutrinos results in an effective potential at finite electron density,

$$H = H_0 + V \, , \quad V = \frac{G_F\rho_e}{\sqrt{2}}\left(|\nu_e\rangle\langle\nu_e| - |\nu_\mu\rangle\langle\nu_\mu|\right) \, ,$$

where G_F is the Fermi constant, and ρ_e is the number of electrons per unit volume. Electron neutrinos $|\nu_e\rangle$ are produced in the core of the sun. Suppose that ρ_e varies adiabatically from a large positive value ($G_F\rho_e \gg m_1, m_2$) to zero as the neutrino traverses the sun. Neglect thermal effects. What fraction of the neutrinos will be observed as $|\nu_e\rangle$ on Earth?

Neutrino Oscillations Solution

(a) Expand

$$E_i = pc + \frac{m_i^2 c^4}{2pc},$$

and find

$$H = pc + \frac{m_1^2 c^4}{2pc} |\nu_1\rangle\langle\nu_1| + \frac{m_2^2 c^4}{2pc} |\nu_2\rangle\langle\nu_2|.$$

The probability to find $|\nu_e\rangle$ at time t is

$$\begin{aligned} |\langle\nu_e|e^{\frac{-iHt}{\hbar}}|\nu_e\rangle|^2 &= |\langle\nu_e| \left(\cos\theta e^{\frac{-im_1^2c^4t}{2\hbar pc}} |\nu_1\rangle + \sin\theta e^{\frac{-im_2^2c^4t}{2\hbar pc}} |\nu_2\rangle \right) |^2 \\ &= |\cos^2\theta e^{\frac{-im_1^2c^4t}{2\hbar pc}} + \sin^2\theta e^{\frac{-im_2^2c^4t}{2\hbar pc}}|^2 \\ &= |\cos^2\theta + \sin^2\theta e^{\frac{-i(m_2^2-m_1^2)c^4t}{2\hbar pc}}|^2 \\ &= 1 - \sin^2 2\theta \sin^2\alpha, \end{aligned} \tag{12}$$

where $\alpha = (m_2^2 - m_1^2)c^4 t/(4\hbar pc)$.

(b) From the first problem, a significant probability for oscillation requires

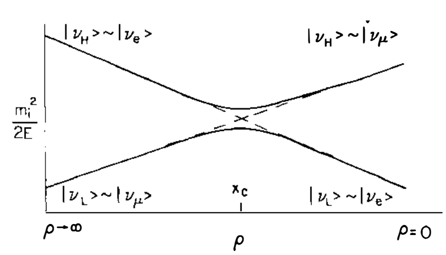
$$\alpha = \Delta m^2 c^4 t/(4\hbar pc) \sim \frac{\Delta m^2 c^2 L}{4E} \sim \frac{1.27 \Delta m^2[\text{eV}^2] L[\text{km}]}{E[\text{GeV}]},$$

be order unity. If the smallest energy is $E \sim \text{MeV} = 10^{-3}\,\text{GeV}$ and the baseline is of order $L \sim R_{\text{Earth}} = 10^4\,\text{km}$, the smallest value is $\Delta m^2 \sim 10^{-7}\,\text{eV}^2$.

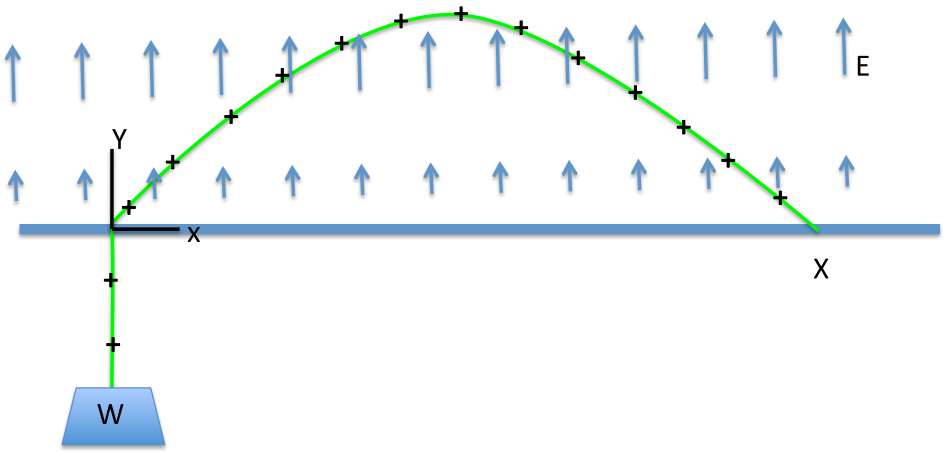
(c) By assumption, the $G_F \rho_e$ term dominates at the core of the sun, so that the $|\nu_e\rangle \approx |\nu_2\rangle$. Assuming adiabaticity, the neutrino emerges still in the higher-mass eigenstate, which in vacuum is

$$|\nu_2\rangle = \sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle.$$

The probability that it is observed as $|\nu_e\rangle$ is thus $\sin^2\theta$. (The value $\sin^2\theta \approx 1/3$ explains why only $\sim 1/3$ of solar neutrinos are observed in detectors sensitive only to $|\nu_e\rangle$.)



4. Charged Thread



A uniformly charged thread with charge-per-unit-length ρ is attached to the x axis at point X. There is a transverse electric field in the y direction of magnitude Ay. The thread passes through a hole at $x = 0$ and is attached to a weight exerting a force W. The region beneath the hole has no electric field. The field causes the thread to deflect in the y direction so that its path is some function $y(x)$. The equilibrium $y(x)$ is the path which minimizes the system's energy $\mathcal{E}$.

(a) Find the potential energy $\mathcal{E}$ as a functional of the path $y(x)$. Note that the length ds of a segment of thread between x and $x + dx$ can be written $ds = \sqrt{1 + y'(x)^2}\, dx$, where $y' \equiv dy/dx$.

(b) Since the integrand of (a) is independent of x, there exists a function f of y and y' that is constant (independent of x) when $y(x)$ minimizes $\mathcal{E}$. Find such a function. You are not asked to determine $y(x)$.

(c) Find the value of this conserved quantity $f(y, y')$, in terms of the maximum deflection y_{max} and the parameters of the problem. Note that y_{max} depends on the attachment point X.

(d) Find the slope y' of the thread at the point where it enters the hole, as a function of y_{max} and the parameters of the problem.

Charged Thread Solution

(a) The electrostatic energy $\Delta\mathcal{E}$ of a charge $\Delta q = \rho\,\Delta s$ at height y is $-\rho\Delta s\int E(y)\,dy = -\rho\Delta s\int Ay\ dy = -\frac{1}{2}\rho\Delta s A y^2$. The gravitational energy of displacing the weight is $Ws = \int W\ ds$. Defining $y' \equiv dy/dx$, using $ds = dx\sqrt{1+y'^2}$ and combining

$$\mathcal{E} \ = \int_0^X dx\sqrt{1+y'^2}\ \left[-\frac{1}{2}(\rho A\ y^2) + W\right],$$

(We view this as an Euler-Lagrange variational problem, where the integrand $\mathcal{L}$ is

$$\mathcal{L} = \sqrt{1+y'^2}\ \left[-\frac{1}{2}(\rho A\ y^2) + W\right]$$

For use below we identify the momentum conjugate to y, denoted p_y:

$$p_y = \frac{\partial\mathcal{L}}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}\left[-\frac{1}{2}\rho A\ y^2 + W\right]$$

(b) In any motion coming from Lagrange equations with no explicit dependence on the independent variable (*eg.* time), the Hamiltonian $\mathcal{H}$ is a constant of the motion: $\mathcal{H} \equiv y' p_y - \mathcal{L}$.

$$\mathcal{H} = \frac{y'^2}{\sqrt{1+y'^2}}\left[-\frac{1}{2}\rho A\ y^2 + W\right] - \sqrt{1+y'^2}\ \left[\frac{1}{2}(\rho A\ y^2) + W\right]$$

$$\mathcal{H} = \left(\frac{y'^2}{\sqrt{1+y'^2}} - \sqrt{1+y'^2}\right)\left[-\frac{1}{2}\rho A\ y^2 + W\right]$$

$$\mathcal{H} = \left(\frac{y'^2 - (1+y'^2)}{\sqrt{1+y'^2}}\right)\left[-\frac{1}{2}\rho A\ y^2 + W\right]$$

$$= -\frac{-\frac{1}{2}(\rho A\ y^2) + W}{\sqrt{1+y'^2}}$$

Of course $-\mathcal{H}$ is also a constant of the motion.

(c) At $y = y_{\max}$, $y' = 0$, so $\mathcal{H} = -\left[-\frac{1}{2}(\rho A\ {y_{\max}}^2) + W\right]$

(d) At the point where the thread enters the hole, $y = 0$. Then the result of b) reads

$$\left[-\frac{1}{2}(\rho A\ {y_{\max}}^2) + W\right] = \frac{-\frac{1}{2}(\rho A\ 0^2) + W}{\sqrt{1+y'^2}}$$

Simplifying,

$$1 - \frac{1}{2}(\rho\frac{A}{W}\ {y_{\max}}^2) = \frac{1}{\sqrt{1+y'(0)^2}}$$

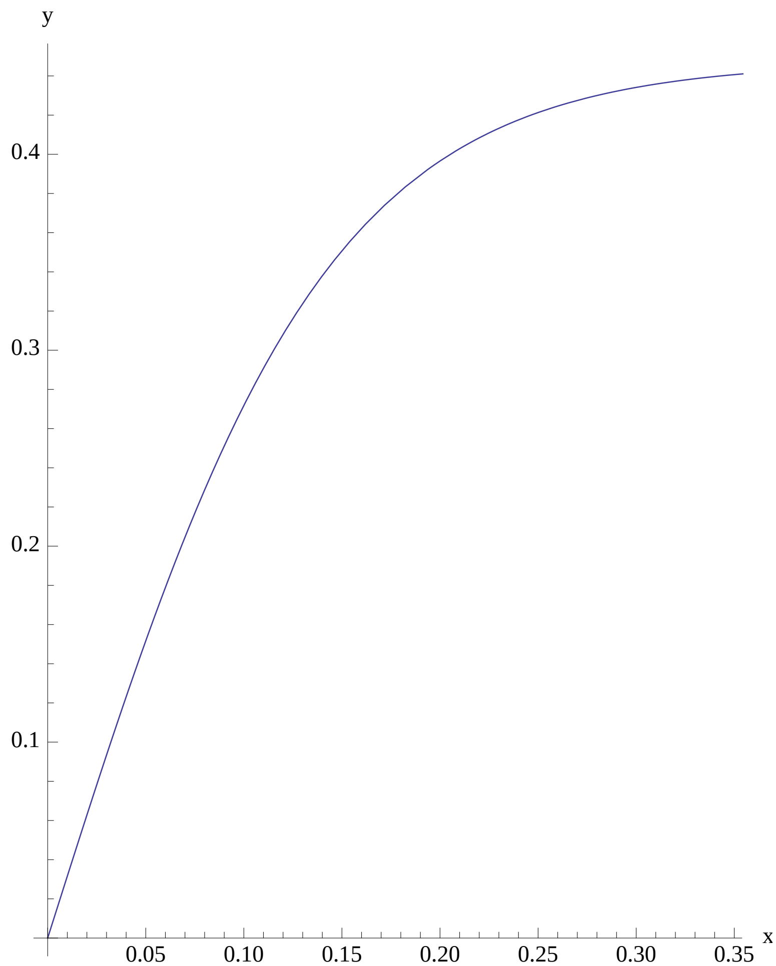
$$\left(\frac{1}{1-\frac{1}{2}(\rho\frac{A}{W}\ y_{\max}{}^2)}\right)^2 = 1 + y'(0)^2$$

$$y'(0) = \sqrt{\left(\frac{1}{1-\frac{1}{2}(\rho\frac{A}{W}\ y_{\max}{}^2)}\right)^2 - 1}$$

As $y_{\max}$ increases from zero, $y'(0)$ increases from zero. For given $\rho A/W$, there is some value of $y_{\max}$ where $y'(0) \to \infty$. Any larger value of $y_{\max}$ gives no solution. (The given W is not strong enough to hold the thread.)

Just to be sure that there is a solution, I integrated the equation to get y(x).

ıape of charged thread for W=4, H = −3.9 with unit rho = A =



5. Bending Waves

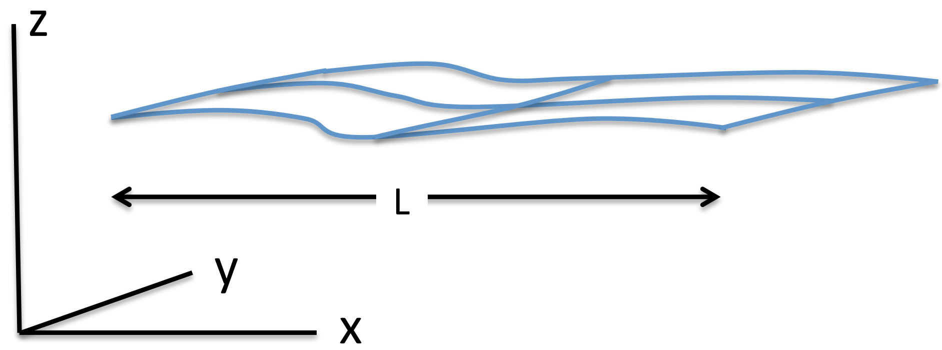


Figure 3: a fluctuating membrane

One form of liquid crystal consists of stacks of fluctuating molecular membranes. A single membrane in the stack can be approximated as a surface $z(x, y)$ as shown in the figure. The surface fluctuates about the horizontal, so that dz/dx and dz/dy are much smaller than 1 everywhere. Deforming the surface away from a flat surface requires bending energy,

$$E = \frac{1}{2} B \int dx\ dy \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)^2,$$

where the "bending stiffness" B is a material-dependent constant. The membrane shape may be expressed as a sum of plane waves: $z(x, y) = \sum_{\vec{q}} A_{\vec{q}} \cos(q_x x)\ \cos(q_y y)$. In a large square membrane of side L, these $\vec{q}$'s have the form $(m, n)\frac{\pi}{L}$, where m and n are positive integers. The membrane is in equilibrium at temperature T.

(a) Find the mean-squared amplitude $\left\langle (A_{\vec{q}})^2 \right\rangle$ as a function of $|q|$, B, T and L.

(b) Now let neighboring membranes confine each membrane. This confinement effect may be modeled as an external potential $U \equiv \frac{1}{2} K \int dx\ dy\ z(x, y)^2$. Find the mean-squared height $\langle z^2 \rangle_K$ as a function of K, B and T.

The following integral might be useful:

$$\int_0^\infty \frac{dx}{c_1 + c_2 x^2} = \frac{\pi}{2\sqrt{c_1\ c_2}}.$$

Bending Waves Solution

Erratum: Problem said that $z(x,y) = \sum_{\vec{q}} A_{\vec{q}} \cos(q_x x)\ \cos(q_y y)$, where $(q_x, q_y) = (m,n)\frac{\pi}{L}$. It should be $z(x,y) = \sum_{\vec{q}} A_{\vec{q}} \sin(q_x x)\ \sin(q_y y)$, where the origin is taken to be at one corner of the square. This has no effect on any of the answers, since cos only entered the calculation via $\langle \cos^2 \rangle = \frac{1}{2}$. These should be replaced by $\left\langle \sin^2 \right\rangle = \frac{1}{2}$. The solution below uses the (cos cos) expansion as stated in the problem.

(a) The energy E is a sum over contributions from the plane waves, and each $\vec{q}$ contribution $E_{\vec{q}}$ is quadratic in its amplitude $A_{\vec{q}}$. We can evaluate the energy

$$E = \sum_{\vec{q}} E_{\vec{q}} = \sum_{\vec{q}} \frac{1}{2} B \int dx\, dy \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]^2 = \sum_{\vec{q}} \frac{1}{2} B \int dx\, dy \left(A_{\vec{q}}\, q^2 \cos(q_x x)\cos(q_y y)\right)^2,$$

$$= \sum_{\vec{q}} \frac{1}{2} B A_{\vec{q}}{}^2 q^4 \int dx\ dy\ (\cos(q_x x)\cos(q_y y))^2 = \sum_{\vec{q}} \frac{1}{8} B A_{\vec{q}}{}^2 q^4 L^2,$$

so that $E_{\vec{q}} = \frac{1}{8} L^2 B q^4 A_{\vec{q}}{}^2$. Then the probability distribution for $A_{\vec{q}}$ is given by the Boltzmann distribution function $f(A_{\vec{q}}) = \text{const} \times \exp(-E_{\vec{q}}/k_B T)$. Each mode of the membrane is a decoupled system since there are no interactions mixing modes. We can therefore compute the partition function for each mode separately,

$$Z_q \equiv \int dA_{\vec{q}}\ \exp(-\beta E_q),$$

where $\beta \equiv 1/(k_B T)$. The expectation value $\langle A_{\vec{q}}{}^2 \rangle$ is given by the integral

$$\left\langle A_{\vec{q}}{}^2 \right\rangle = \frac{1}{Z_q} \int dA_{\vec{q}}\, (A_{\vec{q}})^2 \exp(-\beta E_q) = \frac{4 k_B T}{L^2 B q^4}.$$

(Since E_q is quadratic in $A_{\vec{q}}$ one may also use the equipartition theorem: $\langle E_q \rangle = \frac{1}{2} k_B T$ to obtain this result.)

(b) The energy E_q is now given by

$$E_q = \int dx\ dy\ A_{\vec{q}}^2 \left[\frac{1}{2} K + \frac{1}{2} B\, q^4\right] (\cos(q_x\, x)\cos(q_y\, y))^2 = \frac{1}{8} L^2 [K + B q^4]\ A_{\vec{q}}^2.$$

Again $\langle E_q \rangle = \frac{1}{2} k_B T$ so $\langle A_{\vec{q}}{}^2 \rangle = \frac{1}{2} k_B T / (\frac{1}{8} L^2 [K + B q^4])$. Said differently, this potential does not couple the different modes together so we can treat this case exactly as the prior case. Evaluating,

$$\langle z^2 \rangle = \frac{1}{L^2}\int dx\ dy\ \langle z(x,y)^2 \rangle = \sum_{\vec{q}} \langle A_{\vec{q}}^2 \rangle \ \left(\frac{1}{L^2}\int dx\ dy\ \ \cos^2(q_x x)\cos^2(q_y y) \right) = \sum_{\vec{q}} \langle A_{\vec{q}}^2 \rangle \left(\frac{1}{4} \right)$$

and substituting for $A_{\vec{q}}$,

$$\langle z^2 \rangle = k_B T \sum_{\vec{q}} \frac{1}{L^2[K + Bq^4]}.$$

We are left approximating the sum over $\vec{q}$. Given the good high-q behavior, we can replace the sum $\sum_{m,n}$ by an integral $\int dm\ dn = \int dq_x\ dq_y\ (L/\pi)^2$, giving

$$\langle z^2 \rangle \to k_B T \int_{q_x,q_y>0} d^2q \left(\frac{L}{\pi} \right)^2 \frac{1}{L^2[K + Bq^4]}$$

Switching to angular coordinates gives

$$\langle z^2 \rangle = k_B T\ \frac{1}{4}\ \int (2\pi q\ dq) \frac{1}{\pi^2}\ \frac{1}{K + Bq^4} = k_B T \frac{1}{4\pi} \int_0^\infty dq^2\ \frac{1}{K + Bq^4}$$

Defining $u \equiv q^2$, $\langle z^2 \rangle = \frac{k_B T}{4\pi} \int du/[K + Bu^2]$. Using the given integral formula,

$$\langle z^2 \rangle = \frac{k_B T}{4\pi} \frac{\pi}{2\sqrt{K\ B}} = \frac{k_B T}{8\sqrt{K\ B}}$$

(Using this formula, one may find the pressure p within a stack of sheets at mean separation $\bar{z}$ [W. Helfrich, *Z. Naturforsch.* **33a** 305 (1978)]:

$$p(\bar{z}) = \text{const}\ \frac{(k_B T)^2}{B\bar{z}^3} \quad)$$

6. Electron in a Magnetic Field

Consider an electron moving in the (y, z)-plane in the presence of a uniform magnetic field B in the x-direction: $\vec{B} = B\hat{x}$. In the absence of a $\vec{B}$-field, the electron has momentum (p_y, p_z).

(a) Evaluate the commutator $[\pi_y, \pi_z]$ where,

$$\pi_y = p_y - \frac{eA_y}{c}, \quad \pi_z = p_z - \frac{eA_z}{c},$$

and $\vec{A}$ is the vector potential for $\vec{B}$.

(b) Recall that the Hamiltonian for a charged particle in an electromagnetic field is given by:

$$H = \frac{1}{2m}\left(\pi_y^2 + \pi_z^2\right) + e\phi$$

where ϕ is the electrostatic potential. Derive the complete set of energy levels in the uniform x-directed magnetic field.

Electron in a Magnetic Field Solution

(a) One choice for a vector potential is to take $A_z = By$ and $A_y = 0$. This gives the correct $x-$directed constant magnetic field via $\vec{B} = \nabla \times \vec{A}$. There are many choices of vector potential which are all gauge equivalent i.e., they give the same physical magnetic field. Any two choices are related by:

$$\vec{A} \quad \rightarrow \quad \vec{A} + \nabla f.$$

The commutator can be evaluated straightforwardly in any convenient gauge to give $[\pi_y, \pi_z] = \frac{i\hbar Be}{c}$.

(b) We can take $\phi = 0$ for a constant magnetic field. This would be a free particle in two dimensions if π_y and π_z commuted. However, you have already hopefully seen they do not commute because of the magnetic field. To learn something about the spectrum, let us imagine π_y, π_z are analogues of x and p of the usual simple harmonic oscillator with π_y playing the role of x and π_z playing p. The commutator differs by a factor of $\frac{Be}{c}$ from the usual one. The usual SHO Hamiltonian is

$$H_{\text{SHO}} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

The energy spectrum is $\hbar\omega(n+\frac{1}{2}) \rightarrow \frac{\hbar Be}{mc}(n+\frac{1}{2})$ labeled by the oscillator excitation n.

The preceding discussion is almost but not quite complete. Because we are talking about a system with two degrees of freedom, there must be an additional quantum number labeling eigenstates. For example, if this had been a free particle in two dimensions with $H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m}$ then eigenfunctions would be proportional to $e^{ik_x x + ik_y y}$ labeled by momentum (k_x, k_y).

In our choice of gauge, we note that nothing in H depends on z; therefore $[H, p_z] = 0$. We can simultaneously diagonalize H and p_z by taking eigenfunctions of the form $\psi(z, y) = e^{ikz}\hat{\psi}(y)$. The effective Hamiltonian acting on $\hat{\psi}$ is,

$$\left[H_{\text{eff}} = \frac{1}{2m}\left(p_y^2 + (\hbar k - \frac{eBy}{c})^2\right)\right]\hat{\psi}(y) = E\hat{\psi}(y).$$

This is just a displaced harmonic oscillator no longer centered at $y = 0$. The energy spectrum does not depend on k but eigenfunctions are labeled by their oscillator number along with the quantum number k, which captures a large degeneracy in the spectrum.