J99M.1—Rolling Disk

Problem

A thin uniform disk of radius a and weight Mg rolls without slipping with constant speed in a circle of radius b on a horizontal plane. The plane of the disk is inclined at an angle θ from the vertical. Find the period τ for motion around the circle as a function of θ.

J99M.2—Particle Rolling on a Hyperboloid

Problem

A particle of mass m moves on the surface $x^2 + y^2 - z^2 = R^2$, subject to a uniform force due to gravity g directed along the negative z axis. Deduce the frequency of small oscillations about orbits that lie in a plane normal to the z axis. For what values of z are such orbits unstable?

J99M.3—Rod Orbiting the Sun

Problem

A rod of length L, mass m and uniform mass density is circling in an orbit around the Sun at distance R (between the center of the rod and the Sun). The mass of the Sun is M. Assume that the rod is always pointed in the radial direction. Calculate the tension at the center of the rod.

J99E.1—Charged Particle Trap

Problem

It is well known that a charged particle cannot be held at rest by purely electrostatic fields. In your answers below, you will give a (simple) classical explanation of how a neutral atom of polarizability α can be "trapped" at the focus of a laser beam.

a)   First, ignore magnetic interactions, and deduce that there is a (time-averaged) trapping force dependent on the electric field of the laser.

b)   Atoms have some probability of absorbing photons from the laser beams, thereby being kicked along the direction of the beam. This processes can be modelled classically by supposing that the polarizability of the atom has an imaginary part: $\alpha = \alpha' + i\alpha''$. Deduce the force on an atom along the direction of propagation of a linearly polarized plane electromagnetic wave in terms of α'', the imaginary (absorptive) part of the polarizability.

c)   For an idealized atom with a single natural frequency ω_0, deduce the ratio α'/α'' at the frequency ω for which the real part, α', of the polarizibility is a maximum. For this, you may use a classical model of an atom as an electron on a spring of frequency ω_0, subject to a damping force $-\gamma m\dot{\mathbf{x}}$, where $\gamma \ll \omega_0$ is the reciprocal of the lifetime of the 'excited state'.

In practice, the trapping force a) must be larger than the longitudinal force b). This requires the laser beam to be tightly focused.

J99E.2—Transverse Momentum from an Electromagnetic Wave

Problem

When a charged particle (of mass m and charge e) interacts with a linearly polarized plane wave (with electric field $E_x = E_0 \cos(kz - \omega t)$), the particle's motion includes a transverse oscillation. In the frame in which the particle is at rest on average, the motion is purely transverse if $eE_0/m\omega c \ll 1$, as may be assumed. Hence, the particle has transverse momentum, while the wave carries only longitudinal momentum. How is Newton's 3rd law satisfied in this situation?

Hint: Demonstrate that an appropriate piece of the total field momentum is equal and opposite to the mechanical momentum of the particle. Give an argument based on electric and magnetic fields, rather than, say, one based on the canonical momentum $\mathbf{p} + e\mathbf{A}/c$.

J99E.3—Magnetic Field at a Pulsar

Problem

The x-ray pulsar SGR1806-20 has recently been observed to have a period T of 7.5 s and a relatively large "spindown" rate $\dot{T} = 8 \times 10^{-11}$; C. Kouveliotou *et al.*, Nature **393**, 235 (1998). Calculate the maximum magnetic field at the surface of this pulsar, assuming it to be a standard neutron star of mass $1.4 M_{\odot} = 2.8 \times 10^{30}$ kg and radius 10 km, that the mass density is uniform, that the spindown is due to electromagnetic radiation, and that the angular velocity vector is perpendicular to the magnetic dipole moment of the pulsar.

Give a numerical answer for the surface magnetic field in units of the so-called QED critical field strength $m^2 c^3 / e\hbar = 4.4 \times 10^{13}$ gauss.

J99Q.1—Hydrogen Molecule

Problem

Consider two hydrogen atoms with their nucleii separated by a fixed distance $R > 2r_b$ where r_b is the Bohr radius

a)   In terms of the electronic wavefunctions on the separate atoms write an approximate two electron wavefunction for the ground state, including spin and orbital degrees of freedom.

b)   There are terms in the hamiltonian of the molecule which are absent in two separate atoms. Write an expression for these perturbations. Using perturbation theory find an expression for the lowest order nonvanishing energy difference between the singlet and triplet states. (The result may be left in terms of an integral.)

c)   For large separation between the hydrogen molecules, the perturbation can be treated in a dipole approximation. Find an expression for the effective interaction energy and in particular find its *dependence on the separation R* in the limit $R \gg 2r_b$.

J99Q.2—Spins on a Square

Problem

Four spins interacting antiferromagnetically (only with nearest neighbors) are located at the vertices of a square. The Hamiltonian can be written as:

$$H = \sum_{i,j} \vec{S}_i \cdot \vec{S}_j$$

a)   What are a set of good quantum numbers that can be used to fully classify the eigenstates?

b)   For spin $1/2$: Give the eigenenergies and the degeneracy of each level.

c)   For general S what is the energy and degeneracy of the ground state?

J99Q.3—Electron Transition out of a Square Well

Problem

An electron is in the ground state of a finite but very deep one-dimensional square well,

$$V(x) = \begin{cases} -V_0 & \text{for } 0 < x < a \\ 0 & \text{otherwise,} \end{cases}$$

where V_0 and a are positive constants. A weak electric field $\vec{\varepsilon} = \varepsilon_0 \hat{x} \sin(\omega t)$ is switched on at t=0, and the electron is excited out of the well.

a) What is the momentum p of the electron after the transition?

b) What is the transition rate?

J99T.1—Weirdons

Problem

Suppose a new kind of particle is discovered. This particle is known as the weirdon since it obeys weird statistics in which a given state may contain 0, 1, or 2 particles. Furthermore, weirdons are one dimensional and we will be considering a gas of non-interacting weirdons confined to a straight line of length L. The weirdons are weakly coupled to a thermal reservoir at temperature τ and the weirdon mass is m.

a) Suppose the chemical potential of the weirdons is μ. What is the occupancy of a state with energy ϵ? In addition, give numerical values of the occupancy for $(\mu-\epsilon)/\tau = -\infty$, $(\mu-\epsilon)/\tau = 0$, and $(\mu - \epsilon)/\tau = +\infty$

b) What is the density of states? (That is, the number of states per unit energy as a function of energy?) Remember, the weirdons are one dimensional and are confined to a "box" of length L.

c) Suppose the weirdon gas is cold $(\tau \to 0)$ and contains N weirdons. What is the chemical potential?

d) Under the same conditions as part c), what is the total energy of the weirdon gas? Be sure to eliminate μ from your expression.

e) The low temperature heat capacity of the weirdon gas is proportional to the temperature to some power, $C \propto \tau^\alpha$. What is α?

J99T.2—Maxwell Construction

Problem

Consider a system whose equilibrium states are described by the temperature, T and density ρ.

a)   What is the relation between the free energy per unit volume, $f(T, \rho)$, and the pressure, $P(T, \rho)$?

b)   For thermodynamic stability, the pressure has the property that $P(T, \rho)$ is a monotonically increasing function of ρ for fixed T. What property of $f(T, \rho)$ is equivalent to this?

From a model equation of state, and assuming the system is *homogeneous*, we find the $P(1/\rho)$ curve at constant temperature $T < T_c$, shown in the Figure.
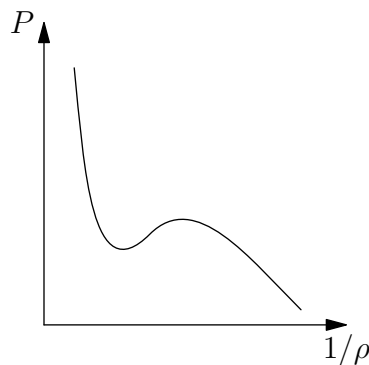
c)   Sketch $f(\rho)$ at temperature $T < T_c$. Assuming the system in equilibrium will phase separate if this reduces the free energy, construct the physical free energy curve, $\bar{f}(\rho)$.

d)   How does your construction of the new free energy translate to a new isotherm in the $(P, 1/\rho)$ plane? Make sure you show *analytically* that the construction for f and the construction for P are equivalent.

J99T.3—Ice Making Machine

Problem

In a large thermally insulated room 100g of water is initially at room temperature, $T_r = 27°$C and at atmospheric pressure. An ice making machine, initially at 27°C, is also in the room. What is the minimum amount of work by the machine needed to transform all the water into ice at 0°C and atmospheric pressure?

Useful constants:

Specific heat of water $c = 4.2$ kJ/kgK

Heat of fusion $\lambda = 330$ kJ/kg