Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, August 22, 2020

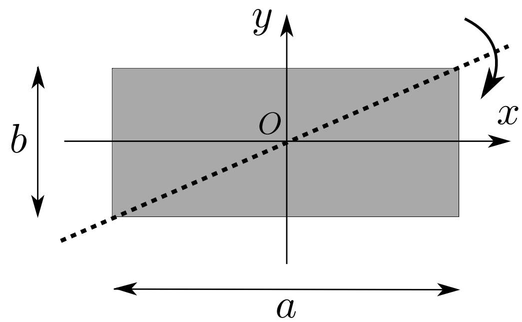
The exam is **closed book/notes**. Simple scientific calculators are allowed. Make sure that each page is signed with your code number (S-#) and the problem number. Do not write answers to different problems on the same page. Scan solutions to each problem as separate PDF files and upload as instructed before.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems! If you submit more than 7 problems, only the lowest 7 scores will count towards your total score.

The total time allowed **3 hrs** till 12:00 p.m.

Problem I-1. (Classical Mechanics)

A flat uniform sheet has mass M and edges of length a and b. The sheet is placed in the xy-plane with center of mass at the origin, O, and edges aligned with the axes.



The inertia tensor with respect to O in the basis $(\widehat{\boldsymbol{x}},\, \widehat{\boldsymbol{y}},\, \widehat{\boldsymbol{z}})$ has components

$$(I_{ij}) \;=\; \begin{bmatrix} \frac{1}{12}Mb^2 & 0 & 0 \\ 0 & \frac{1}{12}Ma^2 & 0 \\ 0 & 0 & \frac{1}{12}M(a^2+b^2) \end{bmatrix} .$$

The sheet rotates with angular speed ω in a right-handed sense about the **diagonal** through the corner at $(\frac{1}{2}a,\, \frac{1}{2}b,\, 0)$. At time $t=0$ the sheet is in the xy-plane as shown.

(i) What is the angular velocity vector $\boldsymbol{\omega}$?

(ii) What is the instantaneous angular momentum $\boldsymbol{L}$ at $t=0$?

(iii) We know the instantaneous rate of change of the angular momentum vector is

$$\frac{d\boldsymbol{L}}{dt} \;=\; \boldsymbol{\omega} \times \boldsymbol{L} \,.$$

Using this relationship, find the required instantaneous torque $\boldsymbol{\tau}$ on the sheet at $t=0$.

Problem I-2. (Quantum Mechanics)

Take a Hamiltonian $\boldsymbol{H}$ with nondegenerate eigenvalues $E_0 < E_1 < E_2 < \ldots$ for the ground state, first excited state, etc. Take an arbitrary normalized wavefunction $|\psi\rangle$ which need not be an eigenfunction of $\boldsymbol{H}$.

(i) Show that $E_0 \le \langle\psi|\boldsymbol{H}|\psi\rangle$.

(ii) Show that if the ground state wavefunction is $|\phi_0\rangle$, and $|\psi\rangle$ is orthogonal to $|\phi_0\rangle$, then $E_1 \le \langle\psi|\boldsymbol{H}|\psi\rangle$.

Problem I-3. (Thermodynamics)

Suppose a Carnot cycle is modified so that its isotherms and adiabats are irreversible. By expressing the heat flows in terms of entropy changes, show that the efficiency of such a cycle is less than the Carnot efficiency.

Problem I-4. (Statistical Mechanics)

A system consists of two identical fermions in contact with a heat bath. (For simplicity assume that the fermions have no spin, but still obey the Fermi-Dirac statistics.) There is an infinite number of 1-particle states that each fermion can occupy. The energies of those states are

$$\epsilon_n = n\,\epsilon\,, \qquad n = 0, 1, 2, \dots\,.$$

A. Show that the canonical partition function of the system is

$$Z = \frac{1}{2}\left[\left(\frac{1}{1-e^{-\beta\epsilon}}\right)^2 - \frac{1}{1-e^{-2\beta\epsilon}}\right], \qquad \beta = \frac{1}{k_B T}\,.$$

B. In the following assume that the temperature is

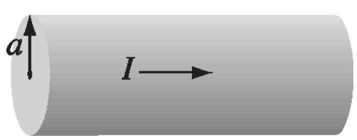
$$kT = \frac{\epsilon}{\ln 2}\,.$$

(i) What is the probability that the system is in its ground state (i.e., the lowest energy state allowed)?

(ii) What is the probability that one of the fermions has zero energy ($n = 0$)?

Hint: If you cannot derive the partition function in A, assume it and proceed with part B.

Problem I-5. (Electricity and Magnetism)



A steady current I flows down a long cylindrical wire of radius a as in the figure. Find the magnetic field, both inside and outside the wire, if:

(i) The current is uniformly distributed over the outside surface of the wire;

(ii) The current is distributed in such a way that its density, J, is proportional to s, the distance from the axis.

Problem I-6. (Math Methods)

Evaluate the complex contour integral

$$\oint_{|z|=4} \frac{1}{z^2 \sinh z}\, dz\,,$$

where the contour is a circle of radius 4, centered at 0 and oriented counterclockwise.

Hint: Recall that $\sinh z = \frac{1}{2}(e^z - e^{-z})$, $z \in \mathbb{C}$.

Problem I-7. (Solid State)

Consider a one-dimensional chain of identical atoms of mass M. The springs are not only between nearest neighbors but between all pairs of atoms. Thus, the elastic energy reads

$$E_{\text{el}} = \frac{1}{2}\sum_n \sum_{m>n} K_m (u_n - u_{n+m})^2 \,,$$

where u_n is the displacement of atom n.

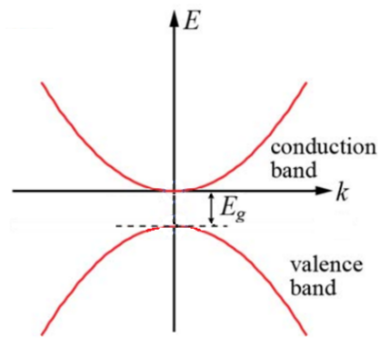
(i) Find the dispersion relation, i.e. the vibrational frequency ω as a function of wave number q. Leave your answer in the form of a series without trying to sum it up.

(ii) Assume $K_m = K_0/m^p$ with $p > 1$ a parameter controlling how rapidly the interaction drops off with distance. Determine the sound velocity for the long-wavelength limit of the dispersion relation for $p > 3$. Leave your answer in the form of a series.

(iii) Consider the long-wavelength limit of the dispersion relation for $1 < p < 3$. Show that one gets anomalous sound, i.e. the frequency is not proportional to the wavenumber.

Hint: Approximate the m-sum by an integral over m from 1 to ∞.

Problem I-8. (Condensed Matter)

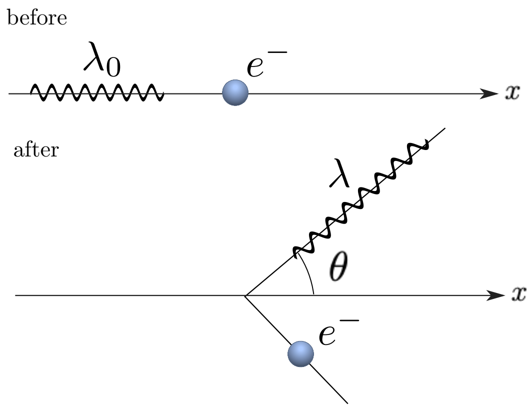


Consider a three-dimensional semiconductor crystal with a parabolic conduction band (i.e., the energy E is a quadratic function of the crystal momentum $\boldsymbol{k}$) with its bottom at $E = 0$.

(i) Show that the density of electronic states in the conduction band has the form $g(E) \propto E^{1/2}$.

(ii) The distribution of electrons in the conduction band is given by $g(E)f(E)$, where $f(E)$ is the Fermi-Dirac function, i.e. the average E-state occupancy for fermions. Assume that our semiconductor is non-degenerate, which is defined as one where the electrons' chemical potential ("Fermi energy") lies within the bandgap at least a few k_BT below the bottom of the conduction band (k_B is the Boltzmann constant). Show that in this case the peak of the electron distribution lies at $E = \frac{1}{2}k_BT$.

Problem I-9. (Special Relativity/Particle Physics)



In Compton scattering in the xy-plane, a beam of photons of wavelength, λ_0, traveling along the x-direction hits a metallic plate and causes electrons to be ejected from the plate along with photons of wavelength, λ, scattered at some angle θ as shown in the figure above. It is observed in experiments that λ is a function of λ_0 and θ given by

$$\lambda = \lambda_0 + \frac{h}{m_e c}(1 - \cos\theta) \,, \tag{9.1}$$

where m_e is the mass of electron, c is the speed of light, and h is the Planck constant. Derive this relation using relativistic kinematics for the conservation of energy and momentum.

Hint: Note that for the initial photons $E_0 = |\vec{p}_0|\, c$ and $\lambda_0\, |\vec{p}_0| = h$ and similarly for the final photons; therefore (9.1) amounts to a relation between the photon's initial and final momenta's magnitudes and the scattering angle for the photons.

Problem I-10. (Astrophysics)

Neutrinos are generated in the core of the Sun, as a byproduct of nuclear fusion processes. The Super-Kamiokande (Super-K) neutrino detector, located under Mount Ikeno near the city of Hida in Japan, detects about 15 solar neutrinos per day. Super-K's target is a tank filled with 20 metric kilotons of water. Assume that the neutrinos only interact with electrons in water molecules, with interaction cross section of $\sigma_e = 10^{-44}$ cm^2, and that the detection efficiency of Super-K is very high (close to unity). Mass density of water is $\rho_{\mathrm{H_2O}}$=1 g/cm^3, the atomic number of oxygen is $Z = 8$, its mass is $16\, m_p$, and proton mass is $m_p = 1.67 \times 10^{-24}$ g.

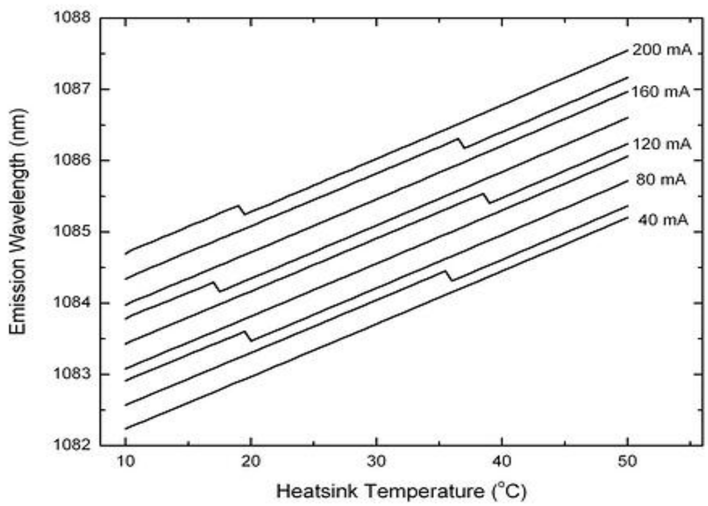
(i) Using the information above, estimate the number flux of neutrinos at the surface of the Earth (how many arrive per second, per cm^2).

Hint: Remember that the effective surface area of the target is *not* the area of the water tank, but rather relates to the cross section and the number density of electrons in it.

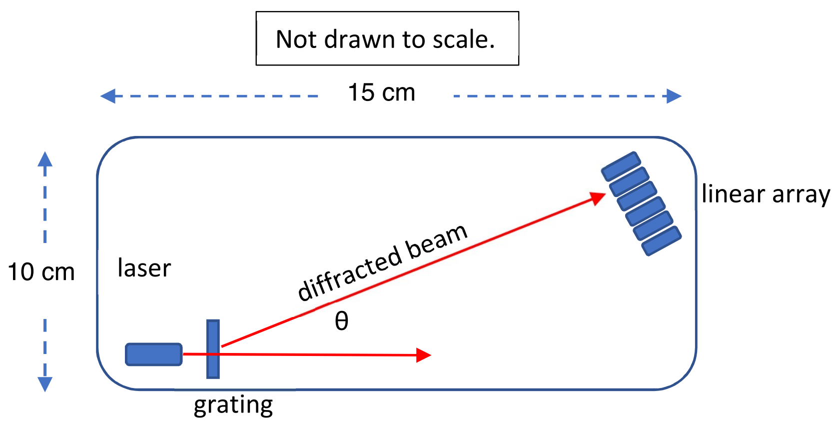
(ii) If you hold out your hand with your palm towards the Sun, how many neutrinos are passing through your hand every second?

Problem I-11. (Experimental Physics)

Apple wants to incorporate a thermometer into their new iPhone. The device is based on the change in the wavelength of a solid-state laser with temperature. Below is a plot of the output wavelength of the laser vs. temperature at various laser currents.



The device sends a laser beam initially parallel to the long dimension of the iPhone. It passes through a diffraction grating having 250 lines per mm. A diffracted beam travels in a straight line along the diagonal of the iPhone and hits a linear detector array whose pixels are spaced 10 microns apart. The iPhone measures 15 cm by 10 cm. Note: Assume that the laser, grating, and array take up negligible space.



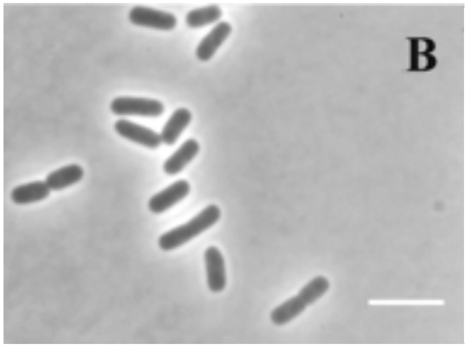
Inside of the iPhone.

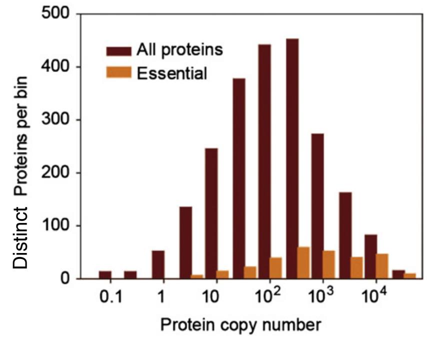
(i) What is the variation of wavelength with temperature $\Delta\lambda/\Delta T$ of this laser?

(ii) What order of the diffraction grating should be used?

(iii) What is the smallest temperature change ΔT that the device can reliably measure?

Problem I-12. (Biophysics)





protein copy number	distinct proteins per bin
0.1	15
0.32	15
1	53
3.2	136
10	246
32	378
100	443
320	452
1000	273
3200	164
10000	84
32000	16

Consider a membrane permeable to solvent but not solute molecules. Take one side of the membrane (region 1) to contain no solute molecules, and the other side (region 2) to contain solute molecules at a density n/V, where n is the number of solute molecules in region 2 and V is the volume of region 2. In analogy to the ideal gas law, the van 't Hoff formula states $\Delta P = (n/V) \cdot k_B T$, where $\Delta P = P_2 - P_1$ is the pressure difference between regions 2 and 1, and $k_B T$ for biological conditions is approximately $4.1\,\mathrm{pN} \cdot \mathrm{nm}$.

Above are two pieces of experimental data. The first is an optical microscopy image of *E. coli* cells, in which the scale bar is 2 microns. This picture could be used to estimate the size of a cell. The second is a histogram showing the copy number, or the total count of molecules, for every distinct protein within a single cell of *E. coli*. For example, the graph shows there are approximately 53 different types of protein for which there is only 1 protein molecule per cell. For reference, the values for the dark bars in the histogram (all proteins) are listed in the chart.

Use these data to answer the following questions. Note, you may have to approximate numerical values for some of the data. Please explain briefly how you arrived at your answers. Avogadro's number is approximately 6.02×10^{23} and 1 cubic micron is the same as 1 femtoliter.

(i) Approximately, how many different types of protein are in an *E. coli* cell?

(ii) Assuming each protein is 100 amino acids in length, how many basepairs of DNA would be approximately needed in the *E. coli* genome to code for all of these proteins?

(iii) What is the total concentration of all proteins inside of an *E. coli* cell in moles/liter?

(iv) What is the pressure difference (in Pa) across the cell membrane due only to the proteins inside the cell?

Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, August 22, 2020

The exam is closed book. Make sure that each page is signed with your code number (S-#) and the problem number. Do not write answers to different problems on the same page. Scan solutions to each problem as separate PDF files and upload as instructed before.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems! If you submit more than 3 problems, only the lowest 3 scores will count towards your total score.

The total time allowed **3 hrs**.

Problem II-1. (Classical Mechanics)

The Lagrangian of a particle of mass m moving in one dimension is given by

$$L = \frac{1}{2}m\dot{x}^2 + A\cos(\omega t)\, x\,,$$

where A and ω are both constants.

(i) Find the Hamiltonian which describes the motion of this particle and solve the Hamilton's equations of motion.

(ii) Set up the Hamilton-Jacobi equation for the Hamilton principal function $S(x, P, t)$. Find a complete solution for $S(x, P, t)$.

(iii) Solve for $x(t)$ and $p(t)$ as functions of time, t, using the complete solution for $S(x, P, t)$ found in (ii).

(iv) Assuming the initial conditions at time $t = 0$, $x(0) = 0$ and $\dot{x}(0) = v_0$, show that the solutions found in (i) and (iii) are indeed the same.

Hint: The following integrals might be useful

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C\,, \qquad \int \cos^2 x\, dx = \frac{x}{2} + \sin 2x + C\,.$$

Problem II-2. (Quantum Mechanics)

Consider a particle which moves in one dimension in a potential $V(x)$ that is invariant under an inversion through the origin, i.e. $V(-x) = V(x)$.

(i) Show that each eigenfunction of the Hamiltonian with nondegenerate eigenvalue is also an eigenfunction of the inversion operator. What are the eigenvalues of the inversion operator?

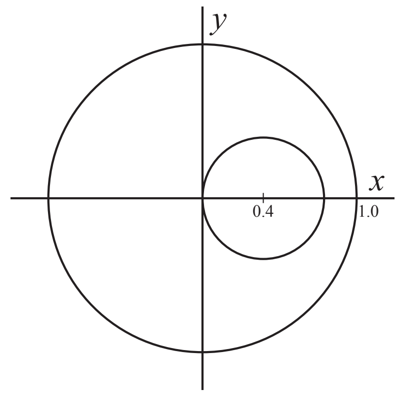
(ii) Consider a state which is represented by the *real unit-norm* wavefunction

$$\Psi(x) = A_1\, \Psi_1(x) + A_2\, \Psi_2(x)\,,$$

where $\Psi_1(x)$ and $\Psi_2(x)$ are *real unit-normalized* eigenfunctions of the Hamiltonian which have different eigenvalues of the inversion operator. What constraints do A_1 and A_2 satisfy?

(iii) Consider $\langle x \rangle$, the expectation value of x in the state Ψ. What are the values of A_1 and A_2 that (a) maximize $\langle x \rangle$, or (b) minimize $\langle x \rangle$?

Problem II-3. (Electricity and Magnetism)



Consider a pair of long conducting cylinders, with circular cross section, one inside the other, running along the z-direction. They are long enough that you can ignore the effects of the ends. Note that they are *not* concentric cylinders! Let the outer cylinder be of radius 1, while the inner one be of radius $2/5$. In the (x, y)-plane the cylinders are placed as in the diagram, where the inner circle passes through the center of the other.

Your task will be to study the potential $\varphi(\boldsymbol{r})$ and the electric field $\boldsymbol{E}(\boldsymbol{r})$ between the two conductors. The outer one has $\varphi = 0$, and the inner one has $\varphi = V$.

Consider the complex coordinate $w = x + iy$. The "complex potential"

$$f(w) = \varphi(x, y) + i\psi(x, y) \ ,$$

is an analytic function that is useful for solving this kind of (effectively two dimensional) problem. Here $\varphi(x, y)$ is the potential and $\psi(x, y)$ is a scalar that can (also) be used to reconstruct the electric field: $E_x = -\partial\psi/\partial y$, $E_y = \partial\psi/\partial x$.

(i) Show using the above information that lines of constant φ (*i.e.* equipotentials) intersect perpendicularly to lines of constant ψ (*i.e.* field lines).

(ii) Consider the map from w to a new complex coordinate $g = u + iv$ given by:

$$g(w) = \frac{2w - 1}{w - 2} \ .$$

Verify (by explicitly mapping at least three points of each circle) that it maps the two non-concentric cylinders to two concentric cylinders, where in the g-plane the outer one has radius 1 and the inner one has radius $1/2$.

It can be shown that solutions for complex potentials in the g-plane map to solutions for complex potentials in the w-plane.

(iii) Solve for the complex potential in the g-problem.

(iv) Hence, map back to the w-problem and write the solution for $\varphi(x, y)$.

(v) Prove that equipotentials in the (x, y)-plane are circles. Write an equation for their radii and the locations of their centers.

(vi) Describe and sketch (on the same diagram) sample equipotentals and electric field lines.

Problem II-4. (Mathematical Methods)

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $(e_i)_{i\in\mathbb{N}}$ an orthonormal complete system over $\mathcal{H}$. Moreover, let

$$\sigma : \mathbb{N} \longrightarrow \mathbb{N} \quad / \quad i \longrightarrow \sigma(i)\,,$$

be a bijection of the natural numbers set, $\mathbb{N}$, onto itself. Let's define

$$A_\sigma : \mathcal{H} \longrightarrow \mathcal{H} \quad / \quad v \longrightarrow \sum_{i\in\mathbb{N}} \langle e_i, v\rangle\, e_{\sigma(i)}\,.$$

(i) Show that A_σ is defined on a dense subset of $\mathcal{H}$.

(ii) Find an explicit formula, like the one above, for the adjoint $A_\sigma^\dagger$.

(iii) Show that A_σ is unitary.

(iv) If $\chi \equiv (\chi_i)_{i\in\mathbb{N}}$ is a sequence in $\mathbb{C}$, when is the new operator

$$A_{\sigma,\chi} : \; v \longrightarrow \sum_{i\in\mathbb{N}} \chi_i\, \langle e_i, v\rangle\, e_{\sigma(i)}\,,$$

still unitary?