Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 12, 2009
1:00 PM - 3:00 PM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1 (Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

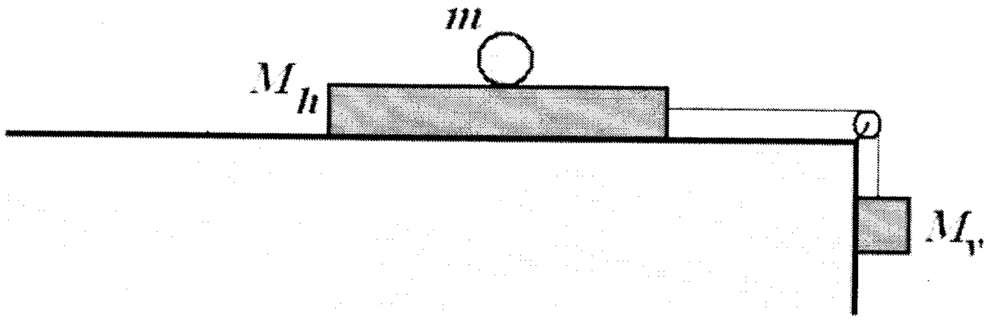
You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A block of mass M_h slides without friction on a horizontal table. It is connected by a massless rope passing over a massless frictionless pulley to a second hanging mass M_v pulled downward by gravity. A sphere of mass m and radius R, initially at rest, rolls without sliding on the top surface of the first block. Find the resulting acceleration of the mass M_v and the center of mass of the sphere.

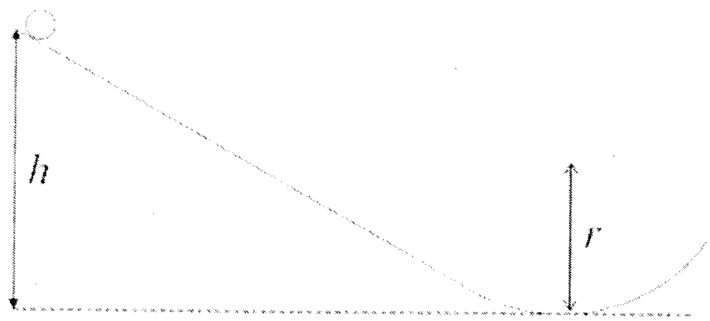


2. A typical disk drive has platters of mass $m = 20$ g, radius $R = 4$ cm, and thickness $t = 1$ mm rotating around a central shaft ("spindle"). New disk drives have platters rotating at frequencies ≥ 10 kHz and at such high frequencies, a small misalignment of the rotation axis and the shaft can cause serious mechanical problems. ***Estimate*** the torque around the center of mass of the platter required to keep a single platter rotating at a frequency $f = 10$ kHz when the shaft is tilted by an angle of 0.1 mrad (0.0001 rad) from the angular velocity vector.

Hint: you may use the fact that the moment of inertia of a disk about an axis in the plane of the disk through the center is one half (½) of the moment of inertia about an axis normal to the disk and through the center.

3. A non-uniform ball of mass M and radius R rolls smoothly from rest down a ramp and onto a circular loop of radius r. The ball is initially at a height h above the bottom of the loop. At the bottom of the loop, the normal force on the ball is twice its weight.

Expressing the rotational inertia of the non-uniform ball in the general form $I = \beta M R^2$, determine an expression for β in terms of h and r.



4. A particle of mass m moves in the 2-dimensional potential

$$V = \frac{1}{2} k \sin^2(\sqrt{x^2 + y^2 - xy})$$

(a) Write the Lagrangian for the system.

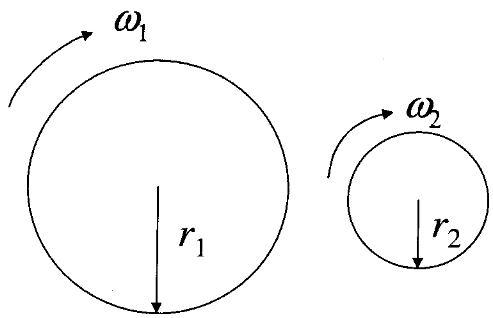
(b) Is the origin a stable equilibrium for the particle?

(c) Write the Lagrangian appropriate for small oscillations about $x = y = 0$.

(d) Calculate the normal frequencies.

(e) Write the general small oscillation solution and sketch the normal modes.

5. Two uniform cylinders spin independently about their axes (the axes are parallel to each other). The first has radius r_1 and mass m_1, the other has radius r_2 and mass m_2. Initially they rotate in the same sense of rotation with angular speeds ω_1 and ω_2 respectively. They are then brought together so that they touch. After the steady state is achieved, what is the final angular velocity of cylinder 1, ω_1'?



Christ

sec 1
Mechanics
#1

Suggested Solutions

1.
- Introduce the tension T in the rope and the force F which the mass M_h exerts to the right on the sphere. Use X, Y and x for the laboratory coordinates of M_h, M_v and m respectively.
- Write down equations for the acceleration of the cm of each mass:

$$\begin{aligned} M_h\ddot{X} &= T - F \\ m\ddot{a} &= F \\ M_v\ddot{Y} &= T - M_v g \end{aligned}$$

- Let θ represent the angular orientation of the sphere (increasing with clockwise motion) and write the equation for the angular acceleration of the sphere and the relation between $\ddot{\theta}$, $\ddot{a}$ and $\ddot{X}$:

$$\begin{aligned} \frac{2}{5}mR^2\ddot{\theta} &= -FR \\ \ddot{a} &= \ddot{X} + R\ddot{\theta} \end{aligned}$$

- These five equations can then be solved for T, F, $\ddot{\theta}$, $\ddot{X}$, $\ddot{Y}$,

2. [redacted]

Cole Sec 1 MECH
Problem # 2

Qualifying exam solutions: Mechanics, Disk Drive Problem

In these solutions I will take the axis of the angular momentum vector as the inertial z axis. Then, the angular momentum should be evaluated in body-centered coordinates using principle axes defined as follows:

- z′ axis normal to disk through the center
- x′ axis in plane of the disk through the center and lying in the plane containing the z and z′ axes
- y′ normal to the others

Then, the total angular momentum in body centered coordinates is

$$\vec{L}_{bc} = I_1\omega'_x\hat{x}' + I_2\omega'_y\hat{y}' + I_3\omega'_z\hat{z}'$$

where ω'_x, ω'_y, ω'_z are the components of the angular velocity vector projected onto the body-centered, principle axes. If we write the angle between the angular velocity vector and the normal to the disk as β, then

$$\omega'_z = \omega_z \cos\beta = \omega\cos\beta \approx \omega$$

and

$$\omega'_x = \omega_z \sin\beta = \omega\sin\beta \approx \omega\beta$$

while $\omega'_y = 0$. Now,

$$I_3 = \frac{1}{2}mR^2$$

and using the hint,

$$I_1 = \frac{1}{4}mR^2$$

So, the body-centered angular momentum is

$$\vec{L}_{bc} = \frac{1}{4}mR^2\omega\left(\beta\hat{x}' + 2\hat{z}'\right)$$

Because the disk is rotating, the angular momentum vector will rotate. We can evaluate the time derivative of the angular momentum vector in inertial coordinates using

$$\dot{\vec{L}}_{in} = \vec{\omega}\times\vec{L}_{bc} + \dot{\vec{L}}_{bc}$$

for which the second term is zero in this problem. Expressing the angular velocity in body centered coordinates as above, we have

$$\dot{\vec{L}}_{in} = -\frac{1}{4}mR^2\omega^2\beta\hat{y}'$$

The torque is equal to the time derivative of the angular momentum, so

$$|\vec{\tau}| = \frac{1}{4}mR^2\omega^2\beta$$

Col Sec 1
Problem # 2
page 2

But this expression needs to be expressed in terms of the rotational frequency, not the angular frequency, so

$$|\vec{\tau}| = m\beta(\pi R f)^2$$

Plugging in numbers, $\pi Rf \approx 1.25\text{x}10^5$ cm s^{-1} = $1.25\text{x}10^3$ m s^{-1}. Then

$|\tau| \approx 0.02$ kg x $1\text{x}10^{-4}$ x $1.6\text{x}10^6$ $m^2s^{-2} \approx 3$ n m.

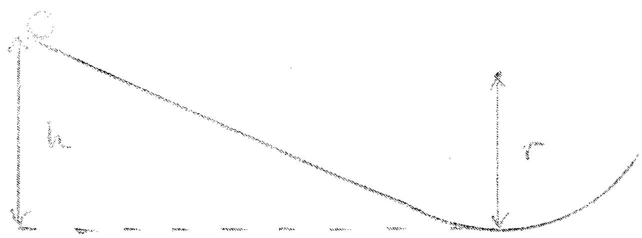
Sec 2
Dodd - Mechanics
Problem # 3

Quals Exam Questions (Dodd)

Mechanics Q

A non-uniform ball of mass M and radius R rolls smoothly from rest down a ramp and onto a circular loop of radius r. The ball is initially at a height h above the bottom of the loop. At the bottom of the loop, the normal force on the ball is twice its weight.

Expressing the rotational inertia of the non-uniform ball in the general form $I = \beta MR^2$, determine an expression for β in terms of h and r.



Solution:

As the ball rolls around the circular loop at the bottom of the track, it experiences centripetal acceleration (v^2/r) which we find from:

$$N - W = \frac{Mv^2}{r}$$

where N is the normal force and W is the weight. We are given than $N = 2Mg$, and so:

$$2Mg - Mg = \frac{Mv^2}{r}$$

i.e. $v^2 = gr$. Relating the angular and translational velocities by $v=r\omega$, we next use the expression for total kinetic energy of a rolling object (no slipping), viz:

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

and apply energy conservation for the ball between its initial position at rest and its position at the bottom of the loop:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Dodd

Sec 1
MECH
Problem # 3
page 2

Substituting for v and for ω (from above), and also for $I = \beta M R^2$, we find after some simple manipulation:

$$h = \frac{1}{2}r + \frac{1}{2}\beta r$$

and re-arranging:

$$\beta = \frac{2h}{r} - 1$$

A. Mueller

Mechanics

A particle of mass m moves in the 2-dimensional potential

$$V = \frac{1}{2} k \sin^2 \sqrt{x^2+y^2-xy}\,.$$

(a) Write the Lagrangian for the system.
(b) Is the origin a stable equilibrium for the particle?
(c) Write the Lagrangian appropriate for small oscillations about $x=y=0$.
(d) Calculate the normal frequencies.
(e) Write the general small oscillation solution and sketch the normal modes.

Solution:

(a) $L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - \frac{1}{2} k \sin^2 \sqrt{x^2+y^2-xy}$

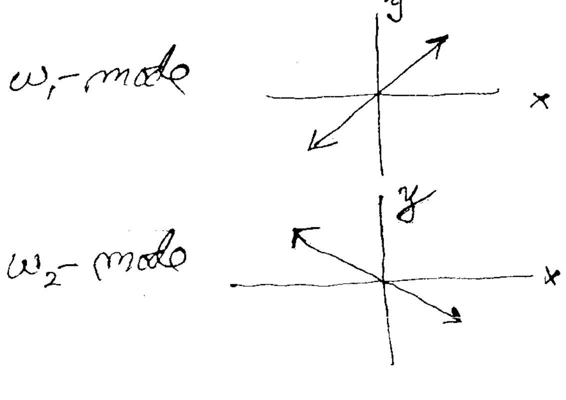
(b) yes

(c) $L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - \frac{1}{2} k (x^2+y^2-xy)$

(d)
$$\begin{aligned} m\ddot{x} &= k(x - y/2) \\ m\ddot{y} &= k(y - x/2) \end{aligned} \Rightarrow \begin{aligned} m(\ddot{x}+\ddot{y}) &= -\frac{k}{2}(x+y) \quad (i) \\ m(\ddot{x}-\ddot{y}) &= -\frac{3k}{2}(x-y) \quad (ii) \end{aligned}$$

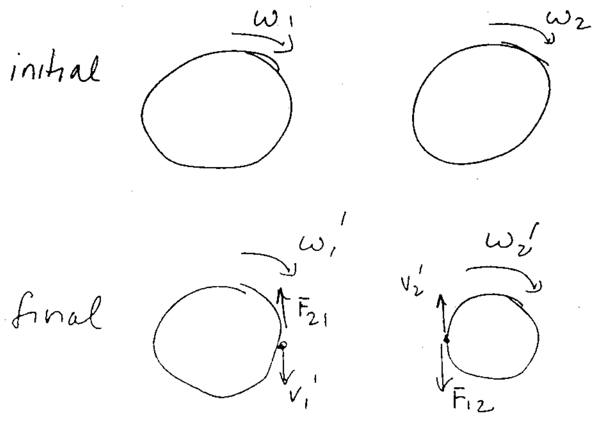
for (i) $\omega_1 = \sqrt{\frac{k}{2m}}$ for (ii) $\omega_2 = \sqrt{\frac{3k}{2m}}$

(e)
$$\begin{pmatrix} x \\ y \end{pmatrix} = A\cos(\omega_1 t + \beta_1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\cos(\omega_2 t + \beta_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



Mechanics Solution - Tuts

Thursday, December 18, 2008
7:07 AM



Since the tangential final speeds must be the same (but opposite directions)

$$\omega_1' r_1 = -\omega_2' r_2 \quad (5)$$

$$\Rightarrow \omega_2' = -\frac{r_1}{r_2}\omega_1'$$

Since the integral of the torque equals the change in angular momentum

$$I_1(\omega_1' - \omega_1) = -\int |r_1||F_{21}|dt \quad (5)$$

and $I_2(\omega_2' - \omega_2) = -\int |r_2||F_{12}|dt$ → but $|F_{21}| = |F_{12}|$ from Newtons 3rd law

$$\therefore \frac{I_1}{r_1}(\omega_1' - \omega_1) = \frac{I_2}{r_2}(\omega_2' - \omega_2)$$

subst in for $\omega_2' = -\frac{r_1}{r_2}\omega_1'$

$$\therefore \frac{I_1}{r_1}\omega_1' + \frac{I_2}{r_2}\cdot\frac{r_1}{r_2}\omega_1' = \frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2$$

$$\omega_1' = \frac{\frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2}{\frac{I_1}{r_1} + I_2\frac{r_1}{r_2}} = \frac{\frac{L_1}{r_1} - \frac{L_2}{r_2}}{\frac{I_1}{r_1} + I_2\frac{r_1}{r_2}}$$

with $I = \frac{1}{2}mr^2$ and some algebra (2)

$$\boxed{\omega_1' = \frac{m_1 r_1 \omega_1 - m_2 r_2 \omega_2}{(m_1 + m_2) r_1}} \quad (3)$$

$$\frac{I_1}{r_1}\omega_1' - \frac{I_1}{r_1}\omega_1 = \frac{I_2}{r_2}(\omega_2' - \omega_2)$$

$$\cancel{\frac{I_1}{r_1}\omega_1' + \frac{I_2}{r_2}\omega_2' = I_1}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 12, 2009
3:10 PM - 5:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2 (Electricity etc.) Question 4, etc.)

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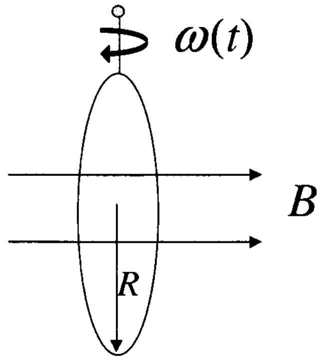
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Calculate the spin frequency decay time, τ, of a thin ring of mass M and radius R that hangs on a string and spins with an angular frequency $\omega(t)$ in a horizontal magnetic field B. The ring has conductivity σ, and a small cross-sectional area $\pi r^2 \ll \pi R^2$.

Assume initially $\omega(0) = \omega_0$ and that the energy lost to Joule heating per period is small compared to the rotation kinetic energy at all times. You can assume the string does not exert any torque. (Hint: use $\langle \sin^2\theta(t) \rangle = ½$ over a period.)



2. An optically active medium can rotate the plane of polarization of light. The susceptibility tensor of such a medium can be expressed as:

$$\overleftrightarrow{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}$$

where $\overleftrightarrow{\chi}$ is related to the polarizability tensor in the usual fashion, $\vec{P} = \varepsilon_0 \overleftrightarrow{\chi} \cdot \vec{E}$. In $\overleftrightarrow{\chi}$, χ_{11}, χ_{12} and χ_{33} are all real. Assume a plane wave propagates in this medium in the z-direction (which is also the 3-direction) with frequency ω. Use Maxwell's equations to establish the following.

(a) That in an optically active medium the propagating EM wave is transverse.

(b) Show the medium admits EM waves with two distinct k-vectors of magnitude k_R, k_L. Find k_R, k_L in terms of ω and the necessary χ_{ij}.

(c) Show that the two k-vectors k_R, k_L correspond to the propagation of right and left circularly polarized EM waves.

(d) Find an expression for the rotary power $\equiv n_R - n_L$ in terms of the χ_{ij}.

3. Part (a) of the figure shows two coils with self-inductances L_1 and L_2. In the relative position shown, their mutual inductance is M. The positive current direction and the positive electromotive force direction in each coil are defined by the arrows in the figure. The equations relating currents and electromotive forces are

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} \pm M \frac{dI_2}{dt} \tag{1}$$

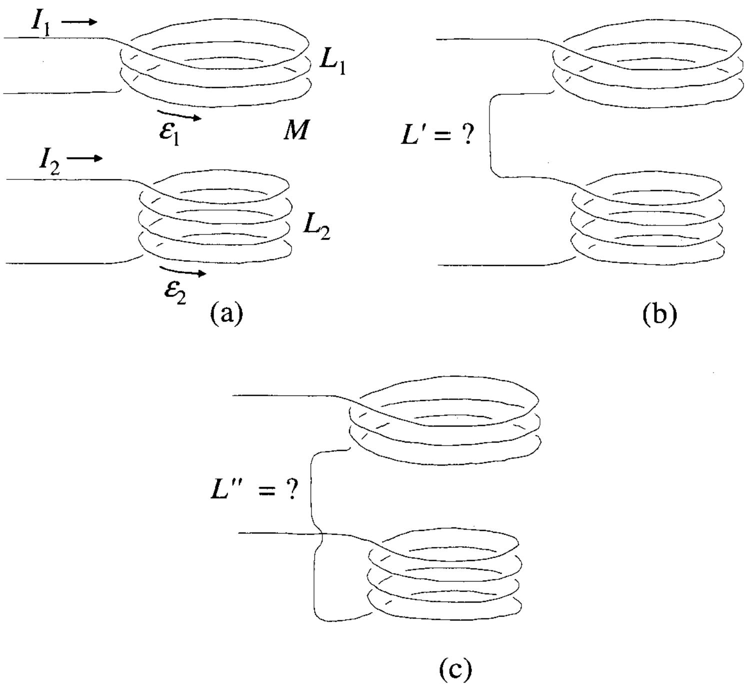
and

$$\mathcal{E}_2 = -L_2 \frac{dI_2}{dt} \pm M \frac{dI_1}{dt} \tag{2}$$

Given that M is always to be taken as a positive constant, how must the signs be chosen in these equations? What if we had chosen, as we might have, the other direction for positive current, and for positive electromotive force, in the lower coil?

Now connect the two coils together as in part (b) of the figure to form a single circuit. What is the inductance L' of this circuit, expressed in terms of L_1, L_2 and M? What is the inductance L'' of the circuit formed by connecting the coils as shown in (c)? Which circuit, (b) or (c), has the greater self-inductance?

Considering that the self-inductance of any circuit must be a positive quantity, see if you can draw a general conclusion, valid for any conceivable pair of coils, concerning the relative magnitude of L_1, L_2, and M.



4. The N-th multipole moment of a charge distribution $\rho(\vec{x})$ is a rank-N tensor $M_{(N)}^{i_1 i_2 \ldots i_N}$ defined as

$$M_{(N)}^{i_1 i_2 \ldots i_N} \equiv \int d^3x \, \rho(\vec{x}) \, x^{i_1} x^{i_2} \ldots x^{i_N} \tag{1}$$

So, for instance, the monopole moment ($N = 0$) is a scalar – the total charge:

$$M_{(0)} = \int d^3x \, \rho(\vec{x}) \equiv Q \tag{2}$$

and the dipole moment ($N = 1$) is the familiar vector

$$M_{(1)}^{i} = \int d^3x \, \rho(\vec{x}) x^i \equiv p^i \tag{3}$$

(a) In equation (1) the position vector $\vec{x}$ is measured with respect to a predetermined, but arbitrary origin. Show that for a given N, the resulting value for $M_{(N)}^{i_1 i_2 \ldots i_N}$ does not depend on the choice of origin *if and only if* the lower order multipoles (those with smaller N's) vanish.

(b) Explain why this ambiguity does not impair the multipole expansion for the electrostatic potential.

5. A thin, non-conducting disk of radius R is spinning around its symmetry axis with angular velocity ω. The disk is uniformly charged with a charge density per unit area of σ.

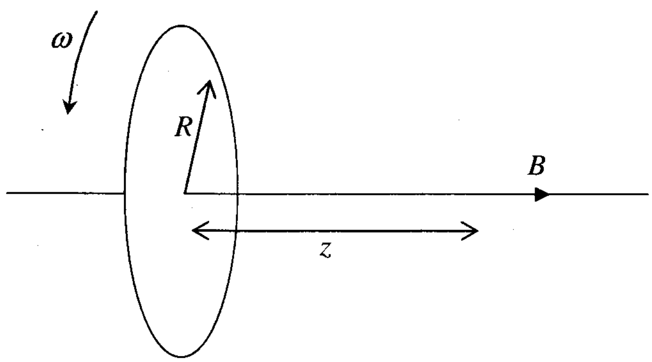
(a) What is the exact expression for the magnetic field along the symmetry axis of the disk as a function of the distance, z, from the disk?

(A useful integral may be $\int \frac{r^3 dr}{(r^2+z^2)^{3/2}} = \left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right))$

(b) For distances far from the disk, the disk looks like a magnetic dipole. What is the effective magnetic dipole moment?

(c) Show that the expressions in part (a) and (b) agree at large distances.

(The American physicist Henry Rowland in 1868 used such a rotating disk to show that the magnetic field due to moving charge distributions is identical with the magnetic field due to an electric current having the same geometry.)



SOLUTIONS E+M

Gyulassy - E+M

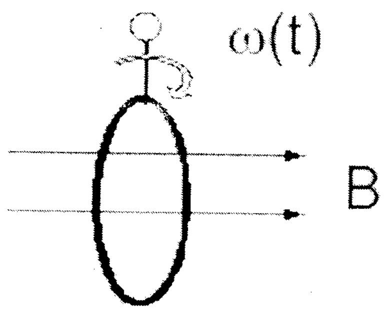
Sec 2 E+M

Problem # 1

Quals 2009: Electromagnetism (M.Gyulassy)

1. Calculate the spin frequency decay time, τ, of a thin ring of mass M and radius R that hangs on a string and spins with an angular frequency $\omega(t)$ in a horizontal magnetic field B. The ring has conductivity, σ and a small area $\pi r^2 \ll \pi R^2$.

   Assume initially $\omega(0) = \omega_0$ and that the energy lost to Joule heating per period is small compared to the rotation kinetic energy at all times. (Hint: use $\langle \sin^2\theta(t)\rangle = \frac{1}{2}$ over a period.)



Solution

The rotational kinetic energy is $KE(t) = \frac{1}{2}I_0\omega^2$ where $I_0 = \frac{1}{2}MR^2$ for the ring. When the ring is at an angle $\theta(t)$ with respect to the horizontal constant B field, there is a magnetic flux, $\Phi(t) = B\pi R^2\cos\theta(t)$ through the loop with $d\theta/dt = \omega(t)$. Faraday's law says that the induced EMF $E(2\pi R) = I(t)\Omega = -1/c\ d\Phi/dt = \pi BR^2\omega(t)/c\ \sin\theta(t)$, where the ring resistance is $\Omega = (2\pi R)/(\sigma\pi r^2)$.

The induced alternating current $I(t)$ dissipates energy according to Joule heating at a rate $P = IV = \Omega I(t)^2 = (d\Phi/dt)^2/\Omega = (\pi BR^2\omega(t)/c\ \sin\theta(t))^2(\sigma\pi r^2)/(2\pi R)$. Over a period, the time averaged power dissipated using $< sin^2 >= \frac{1}{2}$ is $< P >= \frac{1}{2}(\pi BR^2\omega(t)/c)^2(\sigma\pi r^2)/(2\pi R)$.

The rotational kinetic energy decreases according to $dKE/dt = -P$. Using $M = \rho\pi r^2(2\pi R)$ in terms of the mass density ρ, $\frac{1}{2}\,\frac{1}{2}\rho\pi r^2(2\pi R)R^2(2\omega\dot{\omega}) = -(\pi BR^2\omega(t)/c)^2(\sigma\pi r^2)/(2\pi R)$.

Therefore , $\dot{\omega} = -\omega/\tau$ where the spin relaxation time is $\tau = 4\rho c^2/(B^2\sigma)$.

Check dimensions: $[rhoc^2] = En/Vol$, $[B^2] = En/Vol$ whereas $[\sigma] = 1/Time$.

Sec 2 E+M
Problem # 2
Hailey

solution

1.) Maxwell $-\frac{\partial^2 \vec{E}}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$ (1-d)

[get from $\vec{\nabla} \times \vec{E} = -\mu_0 \partial H/\partial t$; $\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}$
if necessary]

put in plane wave solution $e^{i(kz-\omega t)}$
And $\vec{\chi} \cdot \vec{E}$ $\Rightarrow$

$\underline{x}$ $\quad -k^2 E_x + \omega^2/c^2 \, E_x = -\frac{\omega^2}{c^2}(\chi_{11} E_x + i\chi_{12} E_y)$

$\underline{y}$ $\quad -k^2 E_y + \omega^2/c^2 \, E_y = -\omega^2/c^2(-i\chi_{12} E_x + \chi_{11} E_y)$

$\underline{z}$ $\quad \omega^2/c^2 \, E_z = -\omega^2/c^2 \, \chi_{33} E_z$

Last equation $\Rightarrow$ $E_z = 0$ $\Rightarrow$ transverse wave $\underline{\underline{\text{Ans}}}$

2.) The x and y equation have non-trivial solution if

$$\begin{vmatrix} -k^2 + \omega^2/c^2(1+\chi_{11}) & i\frac{\omega^2}{c^2}\chi_{12} \\ -i\,\omega^2/c^2\,\chi_{12} & -k^2 + \frac{\omega^2}{c^2}(1+\chi_{11}) \end{vmatrix} = 0$$

$\Rightarrow$ $k_{R,L} = \omega/c\sqrt{1+\chi_{11} \pm \chi_{12}}$ $\underline{\underline{\text{Ans}}}$

3.) sub k_R, k_L into x or y above and get
$E_x = \pm i E_y$ $\Rightarrow$ circularly polarized E-M $\underline{\underline{\text{Ans}}}$

4.) $n\omega = kc \rightarrow n_R - n_L = \sqrt{1+\chi_{11}+\chi_{12}} - \sqrt{1+\chi_{11}-\chi_{12}}$
$\underline{\underline{\text{Ans}}}$ $n_R - n_L \propto \chi_{12}$ for $\chi_{12} \ll 1+\chi_{11}$

Sec 2 E & M
Problem # 3
McWhinney

Solution

1. Since in part (a) of the figure, both coils have the same conventions for current flow and electromotive force, and since they are aligned so that increasing the magnetic flux through one increases the flux through the other, minus signs must be chosen for the mutual inductance term.

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt} \tag{3}$$

and

$$\mathcal{E}_2 = -L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt} \tag{4}$$

2. If the convention for I_2 and $\mathcal{E}_2$ is reversed, then the mutual inductance enters with a plus sign.

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} \tag{5}$$

and

$$\mathcal{E}_2 = -L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} \tag{6}$$

3. For part (b) of the figure, we have

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_1 + \mathcal{E}_2 & (7) \\ &= -(L_1 + L_2 + M + M)\frac{dI}{dt} & (8) \end{aligned}$$

giving $L' = L_1 + L_2 + 2M$

4. For part (c) of the figure, we have

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_1 + \mathcal{E}_2 & (9) \\ &= -(L_1 - M + L_2 - M)\frac{dI}{dt} & (10) \end{aligned}$$

giving $L'' = L_1 + L_2 - 2M$

5. Circuit (b) has the greater self-inductance

6. To keep the self-inductance of the combined circuit positive, we need $L_1 + L_2 - 2M > 0$, or $M \leq \sqrt{L_1 L_2}$. This follows since $L_1 + L_2 \pm 2\sqrt{L_1 L_2} = (\sqrt{L_1} \pm \sqrt{L_2})^2 \geq 0$.

Nicolis E&M

Sec 2 E+M
Problem #4
Nicolis

2 E&M

The N-th multipole moment of a charge distribution $\rho(\vec{x})$ is a rank-N tensor $M_{(N)}^{i_1 i_2 \cdots i_N}$ defined as

$$M_{(N)}^{i_1 i_2 \cdots i_N} \equiv \int d^3x\, \rho(\vec{x})\, x^{i_1} x^{i_2} \cdots x^{i_N} \,. \tag{19}$$

So, for instance, the monopole moment ($N = 0$) is a scalar—the total charge

$$M_{(0)} = \int d^3x\, \rho(\vec{x}) \equiv Q \,, \tag{20}$$

and the dipole moment ($N = 1$) is the familiar vector

$$M_{(1)}^i = \int d^3x\, \rho(\vec{x})\, x^i \equiv p^i \,. \tag{21}$$

1. In eq. (??) the position vector $\vec{x}$ is measured with respect to a predetermined, but arbitrary origin. Show that for a given N, the resulting value for $M_{(N)}^{i_1 i_2 \cdots i_N}$ does not depend on the choice of origin *if and only if* the lower order multipoles (those with smaller N's) vanish.
2. Explain why this ambiguity does not impair the multipole expansion for the electrostatic potential.

Solution

1. Suppose we choose a different origin O', displaced by $\vec{a}$ with respect to our original choice O. Then our new definition of the multipole moments is

$$M'^{\,i_1 i_2 \cdots i_N}_{(N)} \equiv \int d^3x\, \rho(\vec{x})\, (\vec{x} + \vec{a})^{i_1} (\vec{x} + \vec{a})^{i_2} \cdots (\vec{x} + \vec{a})^{i_N} \,. \tag{22}$$

For instance, for the new dipole moment we have

$$\vec{p}\,' \equiv \int d^3x\, \rho(\vec{x})\, (\vec{x} + \vec{a}) = \vec{p} + Q\, \vec{a} \,, \tag{23}$$

which indeed is equal to the original dipole moment $\vec{p}$ if and only if the monopole moment vanishes. It is clear from eq. (??) that the same conclusion holds for all N's. Indeed, by expanding the products in eq. (??) and collecting terms homogeneous in $\vec{a}$ we get schematically

$$M'^{\,i_1 i_2 \cdots i_N}_{(N)} = M_{(N)}^{i_1 i_2 \cdots i_N} + (M_{(N-1)}\, a)^{i_1 i_2 \cdots i_N} + (M_{(N-2)}\, a\, a)^{i_1 i_2 \cdots i_N} + \cdots + (M_{(0)}\, a \cdots a)^{i_1 i_2 \cdots i_N} \,, \tag{24}$$

where each term denotes a suitable tensor combination (actually, totally symmetric) of the quantities in parentheses. It is clear that the new multipole moment coincides with the original one for a generic displacement $\vec{a}$ if and only if all the lower order multipole moments vanish.

2. The multipole expansion can be thought of as an expansion in powers of (d/r), where d is the typical size of the charge distribution, and r is the distance from the charge distribution to the observation point $\vec{x}_{\rm obs}$. Of course r so defined is ambiguous, at the level of $\delta r \sim d$, for it is not specified which point inside the charge distribution we are computing the distance from. This is exactly equivalent to the ambiguity discussed above for the multipole moments. In other words, by changing our choice of origin, we change both the multipole moments and r, in such a way that the potential at $\vec{x}_{\rm obs}$ computed via the multipole expansion is unaltered. The bottom line is that, as long as we compute r *with respect to the same origin* that we use for computing the multipole moments, the multipole expansion is consistent, and origin-independent.

Sec # 2 E&M
Problem # 5

Quals Problem 1

M. Shaevitz - E+M
Fall, 2008

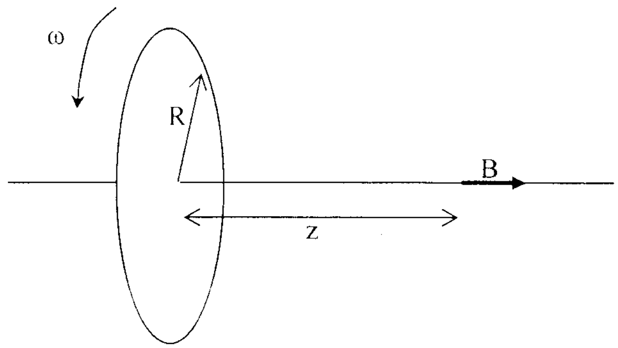
A thin, non-conducting disk of radius R is spinning around its symmetry axis with angular velocity ω. The disk is uniformly charged with a charge density per unit area of σ.

a) What is the exact expression for the magnetic field along the symmetry axis of the disk as a function of the distance, z, from the disk?
(A useful integral may be $\int \frac{r^3 dr}{\left(r^2+z^2\right)^{3/2}} = \left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right)$)

b) For distances far from the disk, the disk looks like a magnetic dipole. What is the effective magnetic dipole moment?

c) Show that the expressions in part a) and b) agree at large distances.

(The American physicist Henry Rowland in 1868 used such a rotating disk to show that the magnetic field due to moving charge distributions is identical with the magnetic field due to an electric current having the same geometry.)



Solution:
The current density of the disk is $di = \frac{\sigma ds dr}{dt} \Rightarrow \frac{di}{dr} = \sigma \frac{ds}{dt} = \sigma \mathrm{v} = \sigma\omega r$

a) The parallel component of the magnetic field of a current loop along the axis can be derived from the Biot-Savart $d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\vec{r}}{r^3}$ formula:

$$dB_{\parallel} = \int_0^{2\pi r} \frac{\mu_0}{4\pi}\frac{di}{\left(r^2+z^2\right)}\frac{r}{\sqrt{r^2+z^2}}dl = \frac{\mu_0 r^2 di}{2\left(r^2+z^2\right)^{3/2}}$$

One can then integrate over the disk to obtain the total field

$$B = \int_0^R \frac{\mu_0 r^2 di}{2\left(r^2+z^2\right)^{3/2}} = \int_0^R \frac{\mu_0 r^2\left(r\omega\sigma dr\right)}{2\left(r^2+z^2\right)^{3/2}}$$

$$= \frac{\mu_0\omega\sigma}{2}\int_0^R \frac{r^3 dr}{\left(r^2+z^2\right)^{3/2}} = \frac{\mu_0\omega\sigma}{2}\left(\frac{r^2+2z^2}{\sqrt{r^2+z^2}}\right)_0^R = \frac{\mu_0\omega\sigma}{2}\left(\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2z\right)$$

b) Calculate the magnetic moment from $i \times Area$

$$\mu = \int_0^R \pi r^2 di = \int_0^R \pi r^2\left(r\omega\sigma\right)dr = \frac{\pi\omega\sigma R^4}{4}$$

c)

Calculating B_{axis} for $z \gg R$ gives $B_{axis} = \frac{\mu_0 \omega \sigma}{2}\left(\left(R^2+2z^2\right)\left(R^2+z^2\right)^{-1/2}-2z\right)$

$$= \frac{\mu_0 \omega \sigma}{2}\left(\left(R^2+2z^2\right)\frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}+\frac{3}{8}\frac{R^4}{z^4}\cdots\right)-2z\right) = \frac{\mu_0 \omega \sigma}{2}\left(\frac{R^4}{4z^3}\right)$$

For a dipole $B_{axis} = \frac{\mu_0}{2\pi}\frac{\mu_{dipole}}{z^3} \Rightarrow \mu_{dipole} = \frac{\pi \omega \sigma R^4}{4}$ which agrees with part b)

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 14, 2009
1:00 PM - 3:00 PM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3 (QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

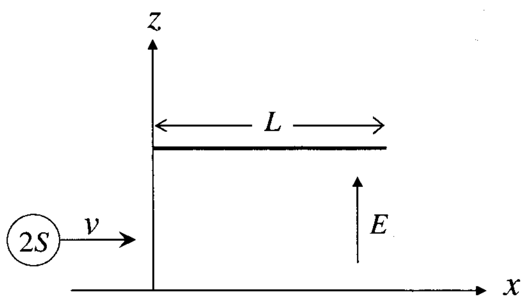
1. An electron beam is prepared by heating a filament, subjecting the emitted electrons to a potential difference V_0, and using a series of electrostatic lenses to guide the electrons on a trajectory parallel to the x-axis. A steel plate with two slits parallel to the y-axis is located at $x = 0$. The distance between the slits is d. The electrons are detected on a screen located at $x = X \gg d$. The screen is free to move along the z-axis. On the screen we observe an interference pattern. In principle, it is possible with this setup to determine through which slit the electron traveled. How? Does this destroy the interference pattern?

2. A beam of excited hydrogen atoms is prepared in the $2S$ state and passed between the plates of a capacitor in which a uniform electric field $\vec{E}$ exists over a distance L (see figure). The hydrogen atoms have velocity v and move parallel to the x axis in the $+x$ direction; the electric field is directed along the z axis. Assume that all $n = 2$ states of hydrogen are degenerate in the absence of the E-field (i.e. neglect hyperfine splittings). In the presence of the E-field, certain of the states will mix. You may neglect coupling to states of $n \neq 2$.

(a) Which of the $n = 2$ states are mixed by the E field to first order in E? Justify your statements by symmetry or other arguments.

(b) Write the Hamiltonian describing the time evolution of the $n = 2$ states for $0 < x < L$.

(c) For an atom which enters the capacitor at time $t = 0$ in the $2S$ state, find the wave function at time $t < L/v$.



(d) If the entering beam contains only atoms in the $2S$ state, find the probability that the emergent beam contains atoms in the $2S$, $2P_x$, $2P_y$ and $2P_z$ states.

Some possibly useful information:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

Hydrogenic wave functions:

$$\psi_{1,0,0} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_B}\right)^{3/2} e^{-\frac{r}{a_B}}$$

$$\psi_{2,0,0} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\left(2 - \frac{r}{a_B}\right) e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,x} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2} \frac{x}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,y} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2} \frac{y}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,z} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2} \frac{z}{a_B} e^{-\frac{r}{2a_B}}$$

3. The energy levels of an isotropic three-dimensional oscillator are easily found to be $E = (N+\frac{3}{2})\hbar\omega$ (where $N = N_x + N_y + N_z$ and N_x, N_y, N_z are the number of quanta in cartesian coordinates). All the degenerate states (fixed N) can also be classified in terms of orbital angular momentum $l = N,\ N-2,\ N-4,\ \ldots,\ 0$ for N even and $l = N,\ N-2,\ N-4,\ \ldots,\ 1$ for N odd.

Now suppose there is a small perturbation which breaks the rotational invariance of the system. In this problem you will determine the manner in which the perturbation breaks the degeneracy, for two forms of the perturbation:

$$H_{\text{pert.1}} = \alpha r^5 Y_{5,0}(\theta, \phi)$$

and

$$H_{\text{pert.2}} = \beta r^6 Y_{6,0}(\theta, \phi)$$

Hint: You do not need to evaluate any integrals to solve this problem.

(a) Show that the perturbation $H_{\text{pert.1}}$, when evaluated to first order in α, does not lead to a splitting of the levels of the degenerate multiplets (labeled by N).

(b) Show that the Hamiltonian including $H_{\text{pert.2}}$ (but not $H_{\text{pert.1}}$) has a spectrum for which m is a good quantum number. Moreover, show that for any level $m \neq 0$ there is a degenerate level with $-m$.

(c) Show that the perturbation due to $H_{\text{pert.2}}$, when evaluated to first order in β, does not lead to a splitting of any of the levels in the $n = 2$ multiplet.

(d) Show that for the $N = 3$ multiplet, $H_{\text{pert.2}}$, when evaluated to first order in β, has the following effects on the spectrum:

i. The $l = 1$ states are not split from each other (all m remain degenerate).

ii. Among the $l = 3$ states, the states with $m = \pm 3, \pm 2, \pm 1, 0$ are split from one another, and also from the $l = 1$ states.

4. Consider a spinless particle in a three-dimensional potential, with the Hamiltonian

$$H = \frac{\boldsymbol{P}^2}{2m} + \frac{k}{2}r^2$$

(a) Find the energy eigenvalues, and determine the degeneracy of the lowest four.

(b) Now suppose instead that five identical particles are in this potential. What is the ground state energy of this system if these particles have
(i) spin $\frac{1}{2}$, (ii) spin 1, or (iii) spin $\frac{3}{2}$? Assume that these particles do not interact with each other.

5. A static uniform electric field $\vec{E} = E\vec{x}$ is applied to an electron in a harmonic potential with natural frequency ω.

(a) Describe the effect of the field on the harmonic oscillator wave functions and on its spectrum.

(b) Calculate the induced dipole moment of the electron, and the polarizability (the ratio of the induced dipole moment to the field strength).

SOLUTIONS
Section 3
QM

Brooijmans - Quantum

Sec 3 QM
Problem # 1
Brooijmans

Quals 08. Quantum

December 18, 2008

Problem

An electron beam is prepared by heating a filament , submitting the emitted electrons to a potential difference V_0, and using a series of electrostatic lenses to guide the electrons on a trajectory parallel to the x-axis. A steel plate with two slits parallel to the y-axis is located at $x = 0$. The distance between the slits is d. The electrons are detected on a screen located at $x = X \gg d$, which is free to move along the z-axis. On the screen we observe an interference pattern. In principle, it is possible with this setup to determine through which slit the electron traveled. How? Does this destroy the interference pattern?

Solution

To be able to distinguish the interference pattern on the screen we need to be able to measure its position with precision better than the distance between the peaks, so

$$\sigma_{pos} < a\frac{\lambda X}{2d} = a\frac{hX}{2p_0 d} \tag{1}$$

where p_0 is the electron's momentum and a is a positive number smaller than 1. Let us take $a = 1/3$.

When an electron hits the screen at a point $z \neq \pm d/2$, its momentum vector is not parallel with the x-axis so it will transfer momentum to the screen which is free to move along the z-axis. Let θ_1 be the angle between the top slit and the point at z:

$$\theta_1 \simeq (z - d/2)/X, \tag{2}$$

then the momentum transfered to the screen if the electron passed through slit 1 is $q_1 \simeq -p_0\theta_1$ (since θ_1 is small). Similarly, $q_2 \simeq -p_0\theta_2$ with $\theta_2 \simeq$

$(z + d/2)/X$. So if we can measure the momentum of the screen along the z-axis well enough to distinguish q_1 from q_2 we can determine through which slit the electron passed.

To do that, our resolution on the momentum needs to be

$$\sigma_{mom} < b(q_1 - q_2) = bp_0 \frac{d}{X} \tag{3}$$

with b a number less than one. Let's again take $1/3$. To see the interference pattern **and** determine which slit the electron went through we need

$$\sigma_{pos}\sigma_{mom} < ab\frac{hX}{2p_0 d}p_0\frac{d}{X} = ab\frac{h}{2}, \tag{4}$$

and we see that this violates the uncertainty principle.

Millis
Sec 3 QM
Problem # 2

Millis Quals 08 Quantum Solution

(a): Label the $n = 2$ states as $2S$, $2P_x$, $2P_y$, $2P_z$. The perturbing potential is $V(x, y, z) = Ez$, so over the scale of the atom is a constant term (shifting all levels) and a term odd under reflection in z. Rotation invariance about z ensures that $2P_x$ and $2P_y$ remain decoupled. The perturbation therefore mixes $2S$ and $2P_z$.

(b) The atom moves on the line $z = z_i$. On this line the potential is

$$V = Ez_0 + E(z - z_0)$$

In the basis $(2S, 2P_z, 2P_x, 2P_y)$ the Hamiltonian is

$$H = \begin{pmatrix} E & M & 0 & 0 \\ M & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}$$

with

$$E = \frac{13.6eV}{4} + Ez_0$$

and

$$M = E\int_0^{2\pi} d\phi \int_{-1}^{1} dcos\theta \int_0^{\infty} r^2 dr \frac{1}{32\pi a_B^3} \frac{z^2}{a_B}\left(2 - \frac{r}{a_B}\right) e^{-\frac{r}{a_B}}$$

adopting dimensionless coordinates we find

$$M = \frac{Ea_B}{24}\int_0^{\infty} u^2 du u^2 (2 - u) e^{-u} = \frac{Ea_B}{24}(2 * 4! - 5!) = -3Ea_B \qquad (1)$$

(c) The eigenfunctions are $\psi_{\pm} = \frac{1}{\sqrt{2}}(|2S> \pm |2P_z>)$ with energy $E \pm M$. The initial state is $|2S> = \frac{1}{\sqrt{2}}(|\psi_+> + |\psi_->)$.Thus at time t we have

$$\psi(t) = e^{iEt}\frac{1}{\sqrt{2}}\left(e^{iMt}|\psi_+> + e^{-iMt}|\psi_->\right) = e^{iEt}\left(cos(Mt)|2S> + sin(Mt)|2P_z>\right)$$

(d) The beam has probability $cos(ML/v)^2$ to be in the $2S$ state and $sin(ML/v)^2$ to be in the $2P_z$ state and no probability to be in any of the other states.

Sec. 3 QM
Problem # 2
Millis

Millis Quals 08 Possibly Useful Information

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

Hydrogenic wave functions

$$\psi_{1,0,0} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_B}\right)^{3/2} e^{-\frac{r}{a_B}}$$

$$\psi_{2,0,0} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\left(2-\frac{r}{a_B}\right) e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,x} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\frac{x}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,y} = \frac{1}{8\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\frac{y}{a_B} e^{-\frac{r}{2a_B}}$$

$$\psi_{2,1,z} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_B}\right)^{3/2}\frac{z}{a_B} e^{-\frac{r}{2a_B}}$$

Pontm QM
SEC 3
Problem # 3

Soln Angular mom.

In general, diagonalize perturbation in each degenerate subspace, but here simpler.

For given $N = N_x + N_y + N_z$:

$l = N, N-2, \ldots, 0$ if N even

$l = N, N-2, \ldots, 1$ if N odd

a) $\langle N, l', m' | Y_{5,0} | N, l, m \rangle = 0$ since $l - l'$ even (can't talk to 5)

or also $|N, l', m'\rangle$ & $|N, l, m\rangle$ have same parity $(-1)^{l'} = (-1)^{l}$

but $Y_{5,0}$ is parity odd.

b) Since $\langle N, l', m' | Y_{6,0} | N, l, m \rangle \propto \delta_{mm'}$ → perturbation commutes with L_z

Can also use time-reversal $Y_{6,0} \xrightarrow{T} (-1)^6 Y_{6,0} = Y_{6,0}$

$\Rightarrow [H_0 + H_{pert.2}, T] = 0$

But $|l, m\rangle \xrightarrow{T} |l, -m\rangle \Rightarrow |m\rangle$ & $|-m\rangle$ degenerate

c) $N = 2$ has $l = 0, 2$.

Wigner - Eckart Th. : $\langle l', m | Y_{k,0} | l, m \rangle \neq 0$ only if $|l-k| \leq l' \leq l+k$

$\Rightarrow$ vanishes for $k = 6$, $l, l' = 0, 2$.

d) For $N = 3 \Rightarrow l = 1, 3$, but $Y_{6,0}$ does not connect $l=1$ with $l=3$

$\Rightarrow$ can treat $l=1$ & $l=3$ seperately

i) $\langle 1, m' | Y_{6,0} | 1, m \rangle = 0 \Rightarrow$ no energy shift from $H_{pert.2}$

ii) Wigner - Eckart :

radial integral : $\neq 0$ (m-independent)

$$\langle 3, m' | Y_{6,0} | 3, m \rangle = A\, \delta_{mm'} \underbrace{\langle 3, m | k=6, q=0, l=3, m \rangle}_{\text{Clebsch - Gordan coeff: } \neq 0, \text{ m-dependent}}$$

$\Rightarrow$ m-dependent energy shift for $l = 3$.

Sec. # 3 QM
Problem # 4
Weinberg

Weinberg 2

a) $H = \frac{\vec{p}^{\,2}}{2m} + \frac{1}{2}kr^2$

$$= \left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right) + \left(\frac{p_y^2}{2m} + \frac{1}{2}ky^2\right) + \left(\frac{p_z^2}{2m} + \frac{1}{2}kz^2\right)$$

Energy levels $= \hbar\omega(n_x + n_y + n_z + 3/2)$, $\omega = \sqrt{\frac{k}{m}}$

$n = 0, 1, 2 \ldots$

Lowest levels $(N = n_x + n_y + n_z)$

$N=0$	$E = 3/2\,\hbar\omega$	deg $= 1$
$N=1$	$= 5/2\,\hbar\omega$	3
$N=2$	$7/2\,\hbar\omega$	$6 = \frac{(3\times 4)}{2}$
$N=3$	$9/2\,\hbar\omega$	$10 = \frac{(3\times 4\times 5)}{3!}$

b) (i) Spin 1/2 $\rightarrow$ two spin states for each $\{n_x, n_y, n_z\}$
$\Rightarrow$ ground state has $N = 0, 0, 1, 1, 1$
$\Rightarrow E = \frac{21}{2}\hbar\omega$

(ii) Spin 1 $\rightarrow$ bosons, all in lowest state
$\Rightarrow E = 15/2\,\hbar\omega$

(iii) Spin 3/2 $\rightarrow$ four spin states for each $\{n_x, n_y, n_z\}$
$\Rightarrow$ ground state has $N = 0, 0, 0, 0, 1$
$\Rightarrow E = 17/2\,\hbar\omega$

Sec 3 QM
Problem # 5
Zelevinsky

MODERN PHYSICS – QUANTUM MECHANICS

Polarizability. SOLUTION.

The Hamiltonian of the harmonic oscillator including the field is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - e\mathcal{E}x = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x-x_0)^2 - \frac{e^2\mathcal{E}^2}{2m\omega^2}. \qquad (1)$$

where

$$x_0 = \frac{e\mathcal{E}}{m\omega^2}. \qquad (2)$$

a) From Eq. (1), it follows that the field shifts the equilibrium position of the oscillator to x_0, and shifts the spectrum as a whole by $(-e^2\mathcal{E}^2/(2m\omega^2))$.

b) The induced dipole moment along $\hat{x}$ is given by the displacement x_0:

$$\langle d\rangle = \langle -ex\rangle = -ex_0 = -\frac{e^2\mathcal{E}}{m\omega^2}, \qquad (3)$$

which corresponds to the polarizability

$$\alpha \equiv \frac{d}{\mathcal{E}} = -\frac{e^2}{m\omega^2}. \qquad (4)$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 14, 2009
3:10 PM - 5:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Four electrons are localized in the four vertices of a tetrahedron. Due to the strong Coulomb repulsion the transition of the electrons between the vertices is forbidden, so the only low energy degrees of freedom are the electron spins, $\widehat{\vec{s}}^{(i)}, i = 1, 2, 3, 4; s^{(i)} = \frac{1}{2}$.

(a) What is the total number of states?

(b) Let the system be described by the exchange Hamiltonian

$$\widehat{H} = K \sum_{i>j} \widehat{\vec{s}}^{(i)} \widehat{\vec{s}}^{(j)} \tag{1}$$

Find the energy levels and their degeneracies.

(c) The exchange Hamiltonian (1) can be generalized to include the interactions of more than two spins. How many of those higher order interactions are allowed by spatial and time reversal symmetries? (Neglect the spin-orbit interaction)

(d) A generalization of Eq. (1) may have the form

$$\begin{aligned} \widehat{H} &= K \sum_{i>j} \widehat{\vec{s}}^{(i)} \widehat{\vec{s}}^{(j)} \\ &+ K' \left(\widehat{\vec{J}}^2\right)^2 \left[\left(\widehat{\vec{s}}^{(1)} \cdot \widehat{\vec{s}}^{(2)}\right) \left(\widehat{\vec{s}}^{(3)} \cdot \widehat{\vec{s}}^{(4)}\right) + \left(\widehat{\vec{s}}^{(1)} \cdot \widehat{\vec{s}}^{(3)}\right) \left(\widehat{\vec{s}}^{(2)} \cdot \widehat{\vec{s}}^{(4)}\right) \right. \\ &\left. + \left(\widehat{\vec{s}}^{(1)} \cdot \widehat{\vec{s}}^{(4)}\right) \left(\widehat{\vec{s}}^{(2)} \cdot \widehat{\vec{s}}^{(3)}\right) \right] \end{aligned} \tag{2}$$

Find the energy levels and their degeneracies.

2. If an atom or nucleus makes a transition from state A to state B emitting a photon of energy E equal to the energy difference between the two states, an identical atom or nucleus in state B can absorb that photon and go to state A. This process, called *resonance fluorescence*, will not proceed, however, if recoil by the emitter (or absorber) reduces the photon energy by an amount greater than the line-width of the transition.

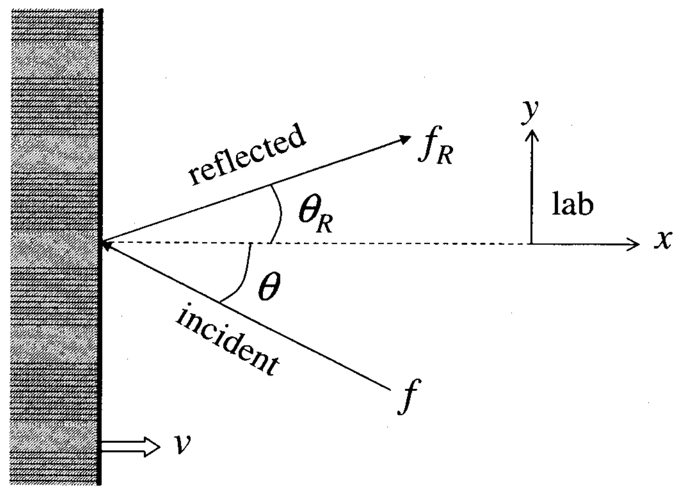
(a) Using typical lifetimes of atomic (10^{-8} s) and nuclear (10^{-10} s) transitions, estimate the typical line-widths for such transitions.

(b) Using typical photon energies in atomic and nuclear transitions show that recoil will not stop atomic resonance fluorescence but will stop nuclear resonance fluorescence.

(c) In 1958 Rudolf Mössbauer discovered the effect that bears his name and for which he won the 1961 Nobel prize. This effect can drastically reduce the recoil energy loss in a nuclear gamma transition. How does it work?

(d) An early application of this effect was made by Robert Pound and Glen Rebka who allowed photons from the nuclear gamma transition of ^{57}Co to fall down the Harvard Tower. Assuming that the transition is a typical nuclear gamma transition and that the tower is a typical tower, estimate the expected energy shift and argue that the Mössbauer effect allows one to detect such a shift. What is so important about this experiment?

3. A monochromatic beam of light is incident on a flat mirror. With respect to the laboratory, the mirror is traveling in at relativistic speed v in the $+x$ direction. (The plane of the mirror is perpendicular to the x-axis.) Also with respect to the laboratory, the incident light beam has frequency f and is traveling at angle θ with respect to the x-axis. Find the frequency f_R and the angle θ_R of the reflected light beam as measured in the laboratory.



4. A capacitor is formed from two large parallel conducting plates of proper area A_p. As seen in a laboratory frame, one plate is moving to the left and the other is moving to the right, both with velocity (magnitude) v. Ignore fringe field effects in answering the following questions. Carry out the following calculations for the situation when the two plates are fully overlapped.

(a) What is the ratio (R) of the capacitance of the moving plate capacitor to a capacitor with the same geometry but with static plates (evaluate in laboratory frame)?

(b) If the potential difference between the plates is V, what is the pressure due to electromagnetic forces on one of the plates as seen in the rest frame of the plate?

(c) If the potential difference between the plates is V, what is the pressure due to electromagnetic forces on one of the plates as seen in the laboratory frame?

5. A photon of wavelength λ Compton scatters off a free electron (mass m) which is initially at rest. The scattered photon has wavelength λ' and scatters at an angle θ (measured from the incident direction). Express your answers in terms of m, λ, θ and h.

(a) Calculate λ'.

(b) Calculate the kinetic energy of the recoil electron.

Section # 4 CEL
Problem # 1
Aleiner

Solution

1. $2^4 = 16$.

2. Introduce the operator of total momentum:

$$\hat{\vec{J}} = \hat{\vec{s}}^{(1)} + \hat{\vec{s}}^{(2)} + \hat{\vec{s}}^{(3)} + \hat{\vec{s}}^{(4)}. \tag{3}$$

   Obviously

$$\hat{\vec{J}}^2 = 3 + 2\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(j)}. \tag{4}$$

   Thus, the Hamiltonian (1) can be rewritten as

$$\hat{H} = \frac{K}{2}\left(\hat{\vec{J}}^2 - 3\right). \tag{5}$$

   Eigenvalues of Eq. (5) are classified according to their total angular momenta $\hat{\vec{J}}^2 = J(J+1)$:

$$\begin{aligned} J = 0;\quad & E = -\frac{3K}{2};\quad 2\times 1 - \text{folded;} \\ J = 1;\quad & E = -\frac{K}{2};\quad 3\times 3 - \text{folded;} \\ J = 2;\quad & E = \frac{3K}{2};\quad 5 - \text{folded.} \end{aligned} \tag{6}$$

3. Terms with odd number of spins are forbidden by the time of the time reversal symmetry. Thus only terms involving either two or four spins are allowed. As all the vertices in tetrahedron are equivalent the only paiwise interaction allowed is given by Eq. (1).

   Furthermore,

$$\hat{s}^{(1)}_\alpha \hat{s}^{(1)}_\beta = \frac{1}{4}\delta_{\alpha\beta} + \frac{i}{2}\epsilon^{\alpha\beta\gamma}\hat{s}^{(1)}_\gamma;\ \alpha, \beta, \gamma = x, y, z. \tag{7}$$

   It yields

$$\begin{aligned} &\left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)}\right)^2 = \frac{3}{16} - \frac{1}{2}(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)}; \\ &\left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)}\right)\left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)}\right) + \left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)}\right)\left(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)}\right) = \frac{1}{2}(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)}; \end{aligned} \tag{8}$$

   This means that only combination where all the spins are different generate something new in the Hamiltonian. As all the vertices in tetrahedron are equivalent, Hamiltonian (2) is the only possibility without involving the spin-orbit interactions.

4. Consider

$$\left(\hat{\vec{J}}^2\right)^2 = \left(3 + 2\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(j)}\right)^2. \tag{9}$$

Using Eq. (8), we find

$$\begin{aligned}
\left(\hat{\vec{J}}^2\right)^2 &= 9 + 12\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(j)} \\
&+ 8\left[(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)})(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)})\right] \\
&+ 4\sum_{i<j}\left(\frac{3}{16} - \frac{1}{2}(\hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(2)})\right) \\
&+ 4 * 2 * \frac{1}{2}\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(2)} \\
&= \frac{27}{2} + 14\sum_{i<j} \hat{\vec{s}}^{(i)} \cdot \hat{\vec{s}}^{(2)} + 8\left[(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)})(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)})\right] \\
&= -\frac{15}{2} + 7\hat{\vec{J}}^2 + 8\left[(\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(2)})(\hat{\vec{s}}^{(3)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(3)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(4)}) + (\hat{\vec{s}}^{(1)} \cdot \hat{\vec{s}}^{(4)})(\hat{\vec{s}}^{(2)} \cdot \hat{\vec{s}}^{(3)})\right]
\end{aligned} \tag{10}$$

Therefore, Hamiltonian (2) can be rewritten as

$$\hat{H} = -\frac{3K}{2} + \frac{15K'}{16} + \left(\frac{K}{2} - \frac{7K'}{8}\right)\hat{\vec{J}}^2 + \frac{K'}{8}(\hat{\vec{J}}^2)^2; \tag{11}$$

Eigenvalues of Eq. (11) are classified according to their total angular momenta $\hat{\vec{J}}^2 = J(J+1)$:

$$\begin{aligned}
J = 0;\quad & E = -\frac{3K}{2} + \frac{15K'}{16};\quad 2 \times 1 - \text{folded}; \\
J = 1;\quad & E = -\frac{K}{2} - \frac{5K'}{16};\quad 3 \times 3 - \text{folded}; \\
J = 2;\quad & E = \frac{3K}{2} + \frac{3K'}{16};\quad 5 - \text{folded}.
\end{aligned} \tag{12}$$

Ponton solution
Appl QM-RFL
SEC 4 Problem # 2

Soln. Mössbauer

Will need approximate conversions, e.g.

$$1\text{ fm} = 10^{-15}\text{m} \leftrightarrow (200\text{ MeV})^{-1}$$ (nuclear scales)

$$1\text{ m} \leftrightarrow (2\times10^{-13}\text{ MeV})^{-1}$$

$$\frac{1\text{m}}{c} = \frac{1}{3}\times10^{-8}\text{ s} \leftrightarrow (2\times10^{-13}\text{ MeV})^{-1}$$

$$1\text{ s} \leftrightarrow (10^{-21}\text{ MeV})^{-1}$$

a) Atomic: $\Gamma_A \sim \tau^{-1} \sim 10^{8}\text{ s}^{-1} \sim 10^{-13}\text{ MeV}$

Nuclear: $\Gamma_N \sim 10^{10}\text{ s} \sim 10^{-11}\text{ MeV}$

b) Nuclear recoil energy $E_R = \frac{p^2}{2M_N} = \frac{E_\gamma^2}{2M_N}$. Use $M_N \sim 50\text{ GeV}$.

Typical atomic photon energy $E_\gamma \sim \text{eV} \Rightarrow E_R^A \sim \frac{(10^{-6}\text{ MeV})^2}{100\times10^3\text{ MeV}} = 10^{-17}\text{ MeV}$

" nuclear " " $E_\gamma \sim \text{MeV} \Rightarrow E_R^N \sim 10^{-5}\text{ MeV}$

(could be 10-100 keV $\Rightarrow E_R \sim 10^{-3} - 10^{-1}\text{ eV}$)

Hence $E_R^A \ll \Gamma_A$ but $\underbrace{E_R^N \gg \Gamma_N}$

$\hookrightarrow \gamma$ non-resonant

c) Mössbauer: nucleus in a lattice $\rightarrow$ whole solid can absorb momentum with negligible E_R since huge recoiling mass

(there is some probability that no phonons are excited, so emission is essentially recoil-free in such cases)

d) Pound & Rebka: measure gravitational (blue) shift for falling photon.

Estimate using effective photon mass: $m_\gamma = \frac{E_\gamma}{c^2}$

$$\Rightarrow \Delta E_\gamma \sim m_\gamma g h \Rightarrow \frac{\Delta E_\gamma}{E_\gamma} \sim \frac{gh}{c^2} \sim \frac{10\cdot10}{(3\times10^8)^2} \sim 10^{-15}$$ (for $h = 10\text{m}$)

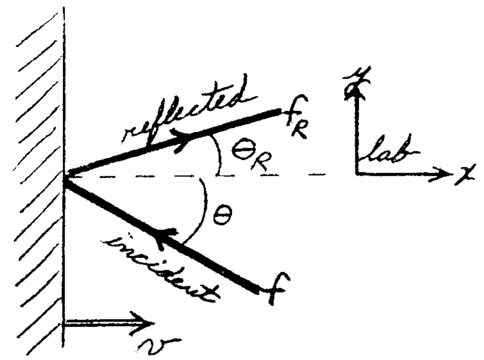
For $E_\gamma \sim 1\text{ MeV} \Rightarrow \Delta E_\gamma \sim 10^{-15}\text{ MeV}$ (without Mössbauer effect, no hope to see this)

Sec. # 4 REL

Qualifying Examination
Relativity Problem

December 2008
Allan Blaer - Relativity

Problem # 3



A monochromatic beam of light is incident on a flat mirror. With respect to the laboratory, the mirror is traveling at relativistic speed v in the +x direction. (The plane of the mirror is perpendicular to the x-axis.) Also with respect to the laboratory, the incident light beam has frequency f and is traveling at angle θ with respect to the x-axis. Find the frequency f_R and the angle θ_R of the reflected light beam as measured in the laboratory.

Solution: Let $\omega = 2\pi f$ and use $c \equiv 1$. Then, $(\omega, \vec{k})$ transforms as a 4-vector under Lorentz transformations, with $|\vec{k}| = \omega$. Let x'-y' axes be fixed with respect to the mirror (moving at $v_x = +v$ with respect to the lab axes x-y). $\left(\gamma \equiv \frac{1}{\sqrt{1-v^2}}\right)$

Incident beam (mirror frame): $\omega' = \gamma(\omega - v k_x)$ and $k_x' = \gamma(k_x - v\omega)$ and $k_y' = k_y$ as measured in the mirror frame of reference.

In the mirror frame, the incident & reflected beams have the same frequency and the angle of incidence equals the angle of reflection.

Reflected beam (mirror frame): $\omega_R' = \omega'$, $k_{Rx}' = -k_x'$, $k_{Ry}' = k_y'$.

We now Lorentz transform the reflected beam's $(\omega_R', \vec{k}_R')$ back to the laboratory frame (which moves at $v_x = -v$ with respect to the mirror):

$$\omega_R = \gamma(\underbrace{\omega_R'}_{\omega'} + v\underbrace{k_{Rx}'}_{-k_x'}) \text{ and } k_{Rx} = \gamma(\underbrace{k_{Rx}'}_{-k_x'} + v\underbrace{\omega_R'}_{\omega'}) \text{ and } k_{Ry} = \underbrace{k_{Ry}'}_{k_y'}.$$

$$\therefore \omega_R = \gamma(\underbrace{\omega'}_{\gamma(\omega - vk_x)} - v\underbrace{k_x'}_{\gamma(k_x - v\omega)}) \text{ and } k_{Rx} = \gamma(-\underbrace{k_x'}_{\gamma(k_x - v\omega)} + v\underbrace{\omega'}_{\gamma(\omega - vk_x)}) \text{ and } k_{Ry} = \underbrace{k_y'}_{k_y}$$

$$\therefore \omega_R = \gamma^2[\omega(1+v^2) - 2v\underbrace{k_x}_{(-\omega\cos\theta)}] \text{ and } \underbrace{k_{Rx}}_{(\omega_R\cos\theta_R)} = \gamma^2[2v\omega - (1+v^2)\underbrace{k_x}_{(-\omega\cos\theta)}] \text{ and } \underbrace{k_{Ry}}_{(\omega_R\sin\theta_R)} = \underbrace{k_y}_{(\omega\sin\theta)}.$$

$$\therefore \boxed{\underbrace{\omega_R}_{2\pi f_R} = \frac{\overbrace{\omega}^{2\pi f}}{(1-v^2)}[1 + 2v\cos\theta + v^2]} \text{ and } \boxed{\tan\theta_R = \frac{(1-v^2)\sin\theta}{[2v + (1+v^2)\cos\theta]}}$$

Cole - Relativity
Sec 4 REL
Problem #4
Cole

Quals problem solutions: Moving plate capacitor

a) An applied potential difference will generate a surface charge density and that surface charge will generate a surface current density. The choice of directions for the plates means that the surface currents are in the same direction. Thus, the magnetic field between the surfaces of the conductor in the laboratory frame is zero but, the magnetic field outside the surface currents, and thus is the conductors is non-zero. Thus, the usual condition for zero electric field in the conductor will not apply. To apply the standard condition on fields at conductor surfaces, we can transform to the rest frame of one of the plates. The electric field in that frame is given by $E' = \gamma E$, since the is zero magnetic field between the surface charges/currents. So, the induced surface charge in the rest frame of one of the plates has magnitude $\sigma' = E'/4\pi = \gamma E/4\pi$. In that frame the surface current is zero, so we can easily transform the charge density back to the laboratory frame, $\sigma = \gamma\sigma' = \gamma^2 E/4\pi$. The total charge stored on the plate is $Q = \sigma A$ where the area is smaller than the proper area due to Lorentz contraction, $A = A_p/\gamma$. So, we have $Q = A_p \gamma E/4\pi = A_p \gamma V/4\pi d$. Thus, the ratio of the capacitance to that of static plates is $R = \gamma$.

If this part is analyzed in the laboratory frame, then the surface charge density is that which produces an electric field in the conductor that exactly cancels the magnetic forces. The magnetic field is $B = 4\pi\varkappa/c$ where $\varkappa = \sigma v$. So, $B = 4\pi\beta\sigma$. The magnetic force on a hypothetical charge moving with the plate is $F = q\beta B = q(4\pi\beta^2\sigma)$. The electric field in the conductor would be $E_c = E - 4\pi\sigma$. We want $qE_c + F = 0$ (signs are correct). Or we want $E - 4\pi\sigma = -4\pi\beta^2\sigma$ or

$$\sigma\left(1-\beta^2\right) = \frac{E}{4\pi} \Rightarrow \sigma = \gamma^2\frac{E}{4\pi}$$

consistent with the above result.

b) The pressure on the plate in the plate rest frame is simply $P' = \sigma' E'/2 = \gamma^2 E^2/8\pi$ or $P = \gamma^2 V^2/8\pi d^2$.

c) The simplest way to obtain the pressure in the laboratory frame is to transform the force from the conductor rest frame in part b. The total force on the plate in the plate rest frame is $F' = P' A_p$. The component of the force normal to the boost direction is reduced by a factor of γ under a Lorentz boost so the total force in the laboratory frame is $F = P' A_p/\gamma$. The pressure in the laboratory frame is therefore, $P = F/A = P'$. So the pressure in the laboratory frame is the same as the result in part b.

Sec 4 REL
Problem # 5 Tuts

Relativity problem solution

From conservation of energy

$$pc + mc^2 = p'c + \sqrt{p_e^2c^2 + m^2c^4}$$

$$(pc + mc^2 - p'c)^2 = p_e^2c^2 + m^2c^4$$

and conservation of momentum

$$\vec{p} - \vec{p}\,' = \vec{p}_e \Rightarrow (\vec{p} - \vec{p}\,')^2 = p_e^2$$

$$\therefore \cancel{p^2} + \cancel{p'^2} + \cancel{m^2c^2} - \cancel{2}pp' - \cancel{2}p'mc + \cancel{2}pmc - \cancel{p^2} - \cancel{p'^2} + \cancel{2}\vec{p}\cdot\vec{p}\,' = \cancel{m^2c^2}$$

$$pp' - \vec{p}\cdot\vec{p}\,' = (p - p')mc$$

$$pp'(1-\cos\theta) = (p-p')mc$$

$$1 - \cos\theta = \left(\frac{1}{p'} - \frac{1}{p}\right)mc$$

where $p = \frac{h}{\lambda}$

$$\therefore \quad \lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$

$$\boxed{\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)}$$

b. from above $\frac{1}{p'c} = \frac{1}{pc} + \frac{1}{mc^2}(1-\cos\theta)$

$$\therefore p'c = \frac{mc^2}{1-\cos\theta + \frac{mc}{p}}$$

and $K = \sqrt{p_e^2c^2 + m^2c^4} - mc^2 = pc - p'c$

but from energy conservation that equals $pc - p'c$ (from 1st equation in a)

$$\therefore \quad K = pc - \frac{mc^2}{1-\cos\theta + \frac{mc}{p}}$$

$$K = \frac{pc(1-\cos\theta + \frac{mc}{p}) - mc^2}{1-\cos\theta + \frac{mc}{p}}$$

$$K = pc(1-\cos\theta)$$

$$K = \frac{pc\,(1-\cos\theta)}{1-\cos\theta+\frac{mc}{p}}$$

$$\boxed{K = \frac{(1-\cos\theta)\,\frac{hc}{\lambda}}{1-\cos\theta+\frac{mc\lambda}{h}}}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 16, 2009
1:00 PM - 3:00 PM

General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5 (General Physics), Question 6, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A star of radius $R_\star = 10^{11}$ cm and mass $M_\star = 10^{33}$ g moves on a parabolic orbit toward a black hole of mass $M = 10^{40}$ g. The angular momentum of the orbit is $L = 10^{23}$ cm^2/s (per unit mass). The black hole can be approximately described by the gravitational potential

$$\Phi = \begin{cases} -\frac{GM}{r-r_g} & r > r_g \\ -\infty & r < r_g \end{cases}$$

where $r_g = 2GM/c^2$, $c = 3\times10^{10}$ cm/s, and $G = 6.67\times10^{-8}$ cm^3/s^{2}g is the gravitational constant.

Does the star's orbit have a pericenter (point of minimum radius r)? Will the star survive the interaction with the black hole?

2. In condensed matter some vibration properties are represented by those of a linear chain of identical atoms of mass M that are joined by springs of force constant K (this is a simplest form of the harmonic approximation). The equilibrium length of each spring is a. The vibration modes are represented by plane waves that have propagation vector $\vec{k}$ and frequency ω. Here $|\vec{k}| = k = 2\pi/\lambda$, where λ is the mode wavelength. Assume that the chain has N atoms. Consider the 'longitudinal' waves that have displacements along the direction of the chain.

(a) Write the classical equation of motion for displacements u along the chain.

(b) Obtain the mode dispersion $\omega(\vec{k})$. Why are modes with k and $-k$ degenerate?

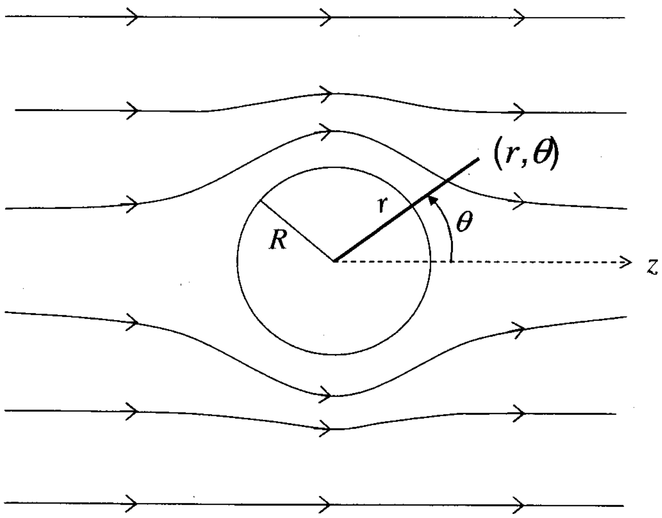
(c) Use cyclic boundary conditions to obtain the density of states of modes. How many modes are in the first Brillouin zone of the linear chain? Explain the result.

(d) Consider the long wavelength limit of $k \to 0$ and $\lambda/a \to \infty$. Show that the equation of motion for the displacements obtained in (a) reduces to a continuum elastic wave equation $\partial^2 u/dt^2 = v^2(\partial^2 u/dx^2)$, where x is the position along the direction of the chain. What is the speed v of the waves?

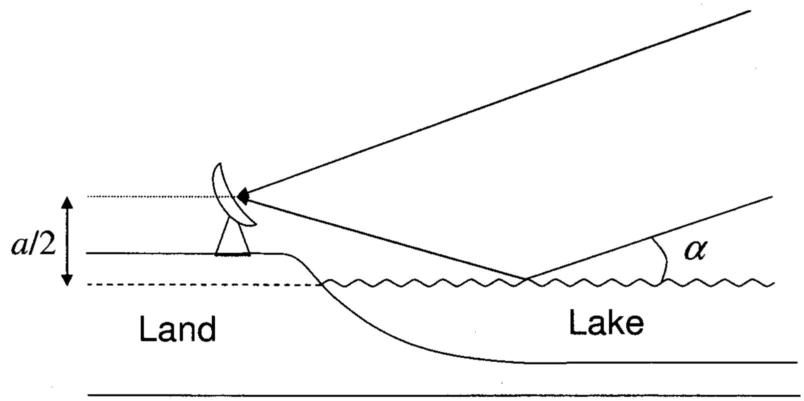
(e) Describe the Debye model for thermal properties due to vibration modes.

(f) Obtain the expression for the thermal energy due to vibrations in the chain within the Debye approximation. What is the low temperature limit of the specific heat due to these vibrations?

3. The steady-state flow of a fluid is specified by the fluid velocity $\vec{v}(\vec{r})$ at each position $\vec{r}$. Consider an incompressible fluid ($\nabla \cdot \vec{v} = 0$) undergoing irrotational flow ($\nabla \times \vec{v} = 0$). An impenetrable solid sphere of radius R is fixed in position while the fluid flows around it. Far from the sphere, the fluid flows uniformly in the z-direction ($\vec{v} = v_0 \hat{z}$ for $r \rightarrow \infty$). For this azimuthally-symmetric problem, find the $\hat{r}$ and $\hat{\theta}$ components of $\vec{v}(r, \theta)$ for all $r > R$.



4. Imagine an antenna at the edge of a lake picking up a signal from a distant radio star (see figure below), which is just coming up above the horizon. Write an expression for the phase difference, δ, and for the angular position of the star when the antenna detects its first maximum. Express answers in terms of α and a.



5. (a) A box is filled with Planckian radiation of temperature $kT \gg m_e c^2$, where m_e is electron mass. A population of $e^{\pm}$ pairs is maintained in equilibrium with radiation through the reaction $e^+ + e^- \leftrightarrow \gamma + \gamma$. Find the number density of positrons in the box.

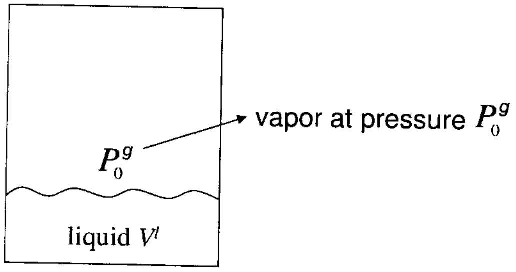
(b) Consider the same problem but now assume that the box is also filled with neutral electron-proton matter at the equilibrium temperature T. The proton number density n_p is given. Find the electron chemical potential μ assuming $\mu \ll kT$.

You can use the following integrals:

$$I_n = \int_0^\infty \frac{x^n dx}{e^x + 1} = \left(1 - \frac{1}{2^n}\right) n! \zeta(n+1)$$

where $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.20$.

6. Consider a liquid and its vapor – both at the same temperature – in equilibrium with each other. Let the equilibrium vapor pressure be P_0^g. An inert, insoluble gas is added to the (closed) container holding the liquid. Find an expression for the new equilibrium vapor pressure of the liquid in terms of V^l, the volume of the liquid, P, the pressure of the inert gas, P_0^g, the vapor pressure without the inert gas, and T, the temperature of the system. Assume the gas phase of the liquid obeys the ideal gas law, and that the liquid is incompressible, i.e. $V^l(P) \approx V^l(P_0^g)$.



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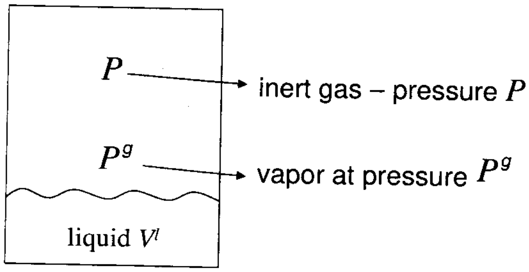


After

Beloborodov . Astro

SOLUTIONS
Section 5
General Part I

Sec # 5 GEN I
Problem # 1
Beloborodov

GENERAL:

A star of radius $R_\star = 10^{11}$ cm and mass $M_\star = 10^{33}$ g moves on a parabolic orbit toward a black hole of mass $M = 10^{40}$ g. The angular momentum of the orbit is $L = 10^{23}$ cm^2/s (per unit mass). The black hole can be approximately described by the gravitational potential

$$\Phi = \begin{cases} -\dfrac{GM}{r - r_g} & r > r_g \\ -\infty & r < r_g \end{cases}$$

where $r_g = 2GM/c^2$, $c = 3 \times 10^{10}$ cm/s, and $G = 6.67 \times 10^{-8}$ cm^3/s^{2}g is the gravitational constant. Does the star's orbit have a pericenter (point of minimum radius r)? Will the star survive the interaction with the black hole?

Solution:

The star's velocity at the pericenter is $v = L/r$, if it exists. Then the pericenter radius is found from

$$\frac{L^2}{2r^2} - \frac{GM}{r - r_g} = 0 \quad \text{(parabolic orbit)}.$$

Denote $x = r/r_g$ and $l = L/cr_g$; then the equation gives

$$x^2 - l^2 x + l^2 = 0 \qquad \Rightarrow \qquad x = \frac{2}{1 - \sqrt{1 - 4/l^2}}.$$

(The root is chosen so that the Keplerian limit $l \gg 1$ gives the correct pericenter radius $r = L^2/2GM$.) The orbit has a pericenter, i.e. does not plunge into the black hole, if $l > 2$. In our case, $l \approx 2.2 > 2$. The pericenter radius is $r \approx 3.5 r_g$; $r_g \approx 1.5 \times 10^{12}$ cm.

The star is on an escaping orbit but it will not survive, because it will be disrupted by tidal forces. The star is stretched in the radial direction with tidal acceleration

$$a_T = \frac{\partial g}{\partial r} R_\star = \frac{2GMR_\star}{(r - r_g)^3} \qquad \text{where} \quad g = -\frac{\partial \Phi}{\partial r} = -\frac{GM}{(r - r_g)^2}.$$

Near the pericenter, this acceleration is much larger than $a_\star = GM_\star/R_\star^2$,

$$\frac{a_T}{a_\star} = \frac{M}{M_\star}\left(\frac{R_\star}{r - r_g}\right)^3 \sim 10^3.$$

This tidal acceleration is created for time $\Delta t \sim r/v$ where v is the star velocity near the pericenter, $v^2/2 = GM/(r - r_g)$. The velocity imparted by the tidal forces, $v_T \sim a_T r/v$, is larger than the escape velocity from the star, $v_\star = (2GM_\star/R_\star)^{1/2}$,

$$\frac{v_T^2}{v_\star^2} \sim \frac{a_T^2 r^2 R_\star}{2GM_\star v^2} = \frac{1}{4}\frac{M}{M_\star}\frac{R_\star^3 r^2}{(r - r_g)^5} = \frac{1}{4}\frac{a_T}{a_\star}\left(\frac{r}{r - r_g}\right)^2 \gg 1.$$

Sec 5 GEN I
Problem # 2
Piriczuk

(a) We have a linear chain of period a

The positions of the atoms are

$$X_n = n a \qquad n = 0, 1, 2 \ldots$$

The displacements are

$$u(X_n) = u(na)$$

The equation of motion is derived from the condition

$$M \ddot{u}(X_n) = - \frac{\partial V}{\partial u(X_n)}$$

$$V = \frac{1}{2} K \sum_n \left[u(X_n) - u(X_{n+1}) \right]^2$$

The equation of motion is

$$M \ddot{u}(X_n) = -K \left[2u(X_n) - u(X_{n-1}) - u(X_{n+1}) \right]$$

Look for solutions that are:

$$u(X_n) = A e^{i(k X_n - \omega t)}$$

(b) $$\omega(k) = \sqrt{\frac{2K}{M}(1-\cos ka)}$$

$$= 2\sqrt{\frac{K}{M}}\,\left|\sin \tfrac{1}{2} ka\right|$$

the modes with k and $-k$ are degenerate because the direction of propagation does not change the mode frequency.

(c) Cyclic boundary conditions mean

$$u(x_n) = u(x_n + Na)$$

This requires that

$$k = \frac{2\pi}{a}\frac{l}{N} \qquad l = 0, 1, 2 \ldots$$

The first Brillouin ~~zone~~ has

$$-\pi/a \le k \le \pi/a$$

It has all the N degrees of freedom for one orientation of the displacement $u(x_n)$.

The density of states in k-space is

$$g(k) = \frac{Na}{2\pi}$$

In frequency space it is

$$g(\omega) = \frac{g(k)}{(d\omega/dk)}$$

The number of modes in the 1st Brillouin zone is

$$\int_{-\pi/a}^{\pi/a} g(k)\, dk = N$$

(d) Write the right handside of the eq. in (a)

$$-K\left[2u(x_n) - u(x_{n-1}) - u(x_{n+1})\right]$$

$$= K\left[\left(u(x_{n+1}) - u(x_n)\right) - \left(u(x_n) - u(x_{n-1})\right)\right]$$

In the limit $\lambda \gg a$

$$u(x_{n+1}) - u(x_n) = \frac{\partial u(x_{n+1})}{\partial x_{n+1}}\, a$$

and

$$u(x_n) - u(x_n) = \frac{\partial u(x_n)}{\partial x_n} a$$

so that the right hand side of the eq. in (a) is

$$K a \left[\frac{\partial u(x_{n+1})}{\partial x_{n+1}} - \frac{\partial u(x_n)}{\partial x_n} \right]$$

$$= K a \left[\frac{\partial^2 u}{\partial x^2} a \right] = K a^2 \frac{\partial^2 u}{\partial x^2}$$

The value of the velocity is

$$v = \sqrt{\frac{K}{M}}\, a$$

(e) The Debye model assumes

(i) A dispersion $\omega(k) = v k$

(ii) A cut-off wave vector k_D

given by the condition

$$\int_{-k_D}^{k_D} g(k)\, dk = N$$

(f) The thermal energy is

$$E_D \sim (k_B T)^2 \int_0^\infty \frac{x}{e^x - 1}$$

The specific heat is

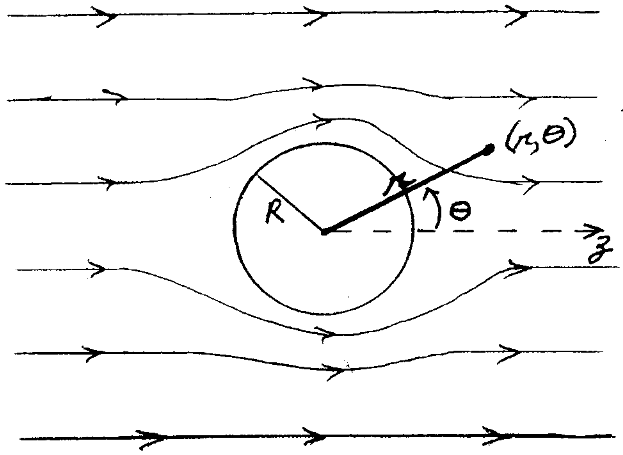
$$C_V \sim T$$

Blaer · General Fluids

Sec 5 GEN I
Problem # 3
Blaer

Qualifying Examination Problem
Fluids

Allan Blaer
December 2008



The steady-state flow of a fluid is specified by the fluid velocity $\vec{v}(\vec{r})$ at each position $\vec{r}$. Consider an incompressible fluid (div $\vec{v} = 0$) undergoing irrotational flow (curl $\vec{v} = 0$). An impenetrable solid sphere of radius R is fixed in position while the fluid flows around it. Far from the sphere, the fluid flows uniformly in the z-direction ($\vec{v} = v_0\hat{z}$ for $r \longrightarrow \infty$). For this azimuthally-symmetric problem, find the $\hat{r}$ and $\hat{\theta}$ components of $\vec{v}(r,\theta)$ for all r>R.

Solution: curl $\vec{v} = 0 \Longrightarrow \vec{v} = -\nabla\Phi$. div $\vec{v} = 0 \Longrightarrow \nabla^2\Phi = 0$.

For azimuthally-symmetric solutions to Laplace's equation,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta) + \sum_{l=0}^{\infty} B_l r^l P_l(\cos\theta).$$

$$r \to \infty \Longrightarrow \vec{v} = v_0\hat{z} \Longrightarrow \Phi = -v_0 z = -v_0 r\cos\theta = -v_0 r P_1(\cos\theta).$$

$\therefore B_{l=1} = -v_0$ and $B_{l>1} = 0$.

$$\therefore \Phi(r,\theta) = -v_0 r P_1(\cos\theta) + B_0 + \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta)$$

At $r = R$, $v_r = 0$ (impenetrable sphere). $\therefore \left(\frac{\partial\Phi}{\partial r}\right)_{r=R} = 0$.

$$\therefore -v_0 P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{-A_l(l+1)}{R^{l+2}} P_l(\cos\theta) = 0 \text{ for all } \theta.$$

$\therefore A_{l=1} = \frac{-v_0 R^3}{2}$ and $A_{l\neq 1} = 0$.

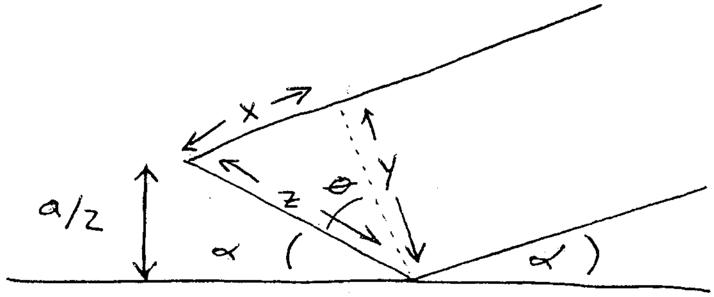
$$\therefore \Phi(r,\theta) = B_0 - v_0\left(r + \frac{R^3}{2r^2}\right)\underbrace{P_1(\cos\theta)}_{\cos\theta}$$

$$v_r = \frac{-\partial\Phi}{\partial r} = +v_0\left(1 - \frac{R^3}{r^3}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -v_0\left(1 + \frac{R^3}{2r^3}\right)\sin\theta$$

OPTICS

Sec 5 GEN I
Problem # 4
Hughes



$$\delta = kz - kx + \pi$$ ← water reflection

$$\sin\alpha = \frac{a/2}{z} \Rightarrow z = \frac{a}{2\sin\alpha}$$

$$\theta = 90 - 2\alpha \qquad \sin\theta = \frac{x}{z}$$

$$x = z\sin\theta = \frac{a}{2\sin\alpha}\sin(90 - 2\alpha)$$

$$= \frac{a\cos 2\alpha}{2\sin\alpha}$$

$$\boxed{\delta = \frac{ka}{2\sin\alpha}(1 - \cos 2\alpha) + \pi}$$

Maximum occurs @ $\delta = 2\pi$

$$\pi = \frac{ka}{\sin\alpha}\left(\frac{1-\cos 2\alpha}{2}\right) = \frac{ka}{\sin\alpha}\sin^2\alpha = ka\sin\alpha = \frac{2\pi a\sin\alpha}{\lambda}$$

First maximum @ $\boxed{\alpha = \sin^{-1}\left(\frac{\lambda}{2a}\right)}$

Beloborodov Stat. Mech

Sec 5 GEN I
Problem # 5
Beloborodov

MODERN:

(a) A box is filled with Planckian radiation of temperature $kT \gg m_e c^2$, where m_e is electron mass. A population of $e^{\pm}$ pairs is maintained in equilibrium with radiation through reaction $e^+ + e^- \leftrightarrow \gamma + \gamma$. Find the number density of positrons in the box.

(b) Consider the same problem but now assume that the box is also filled with neutral electron-proton matter at the equilibrium temperature T. The proton number density n_p is given. Find the electron chemical potential μ assuming $\mu \ll kT$.

You can use the following integrals:

$$I_n = \int_0^\infty \frac{x^n\, dx}{e^x + 1} = \left(1 - \frac{1}{2^n}\right) n!\, \zeta(n+1),$$

where $\zeta(2) = \pi^2/6$, $\zeta(3) \approx 1.20$.

Solution:

(a) The $e^{\pm}$ are relativistic ($kT \gg m_e c^2$) and their energies are $E = cp$ where p is the particle momentum. The occupation number of $e^{\pm}$ is described by Fermi-Dirac function

$$f = \frac{1}{\exp\left(\dfrac{cp - \mu_\pm}{kT}\right) + 1},$$

where μ_+ and μ_- are the chemical potentials of e^+ and e^-. The number densities of e^+ and e^- are

$$n_\pm = \frac{2}{h^3} \int_0^\infty f\, 4\pi p^2 dp. \tag{1}$$

Since $e^{\pm}$ are in equilibrium with photons (which have zero chemical potential), $\mu_+ + \mu_- = 0$. By symmetry, $\mu_+ = \mu_-$ and hence $\mu_+ = \mu_- = 0$. Then equation (1) gives

$$n_+ = n_- = \frac{3\zeta(3)\,\theta^3}{2\pi^2 \lambda\!\!\!^{-}{}^3}, \qquad \text{where} \quad \theta = \frac{kT}{m_e c^2}, \qquad \lambda\!\!\!^{-} = \frac{\hbar}{m_e c}.$$

(b) Electrons and positrons are still in equilibrium with radiation, therefore $\mu_+ = -\mu_- \equiv \mu$. Neutrality requires $n_- - n_+ = n_p \Rightarrow n_+ \neq n_- \Rightarrow \mu \neq 0$. Using $\mu \ll kT$, one can expand the integral in equation (1) in μ/kT, keeping the linear term. Then one finds

$$n_\pm = \frac{1}{\pi^2 \lambda\!\!\!^{-}{}^3}\left[\frac{3}{2}\zeta(3)\,\theta^3 \mp \frac{\pi^2}{6}\,\theta^2\,\frac{\mu}{kT}\right],$$

$$n_- - n_+ = n_p \qquad \Rightarrow \qquad \frac{\mu}{kT} = \frac{3\lambda\!\!\!^{-}{}^3 n_p}{\theta^2}.$$

Sec 5 GEN I
Problem # 6
Hailey

Solution thermo I

The liquid and gas phases are in equilibrium so The Gibb's free energies are equal
And for $dG = -s\,dT + V\,dP + \mu\,dn$
we have for $dT = 0 \quad dn = 0$

$$V^l dp^l = V^g dp^g \quad \text{And} \quad p^g V^g = RT$$

$$V^l dp^l = V^g dp^g = RT\, dp^g / p^g$$

$$\int_0^P \frac{V^l}{RT} dp^l = \int_{p_0^g}^{p^g(P)} dp^g / p^g$$

$$\frac{V^l}{RT} P = \ln \frac{p^g(P)}{p_0^g}$$

$$p^g(P) = P_0^g\, e^{\frac{V^l P}{RT}} \quad \text{Ans}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 16, 2009
3:10 PM - 5:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6 (General Physics), Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code.**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Provide an example of one phenomenon in solid state physics where a "quantum" aspect is important. What is the corresponding observable property which can be estimated using the uncertainty principle? Explain how the observable value can be derived using the uncertainty principle, and describe how you would experimentally measure this observable quantity.

2. You have to measure the index of refraction of a transparent liquid in the hope that it will help you identify the substance. You have plenty of the liquid, a 632nm red laser pointer and a thin walled rectangular glass container ($n_G = 1.5$). Paper, pencil, protractors and rulers are also available.

Describe three different ways to determine (measure) the index of refraction of the liquid. Try to qualitatively compare their accuracy.

3. Bound states composed of a heavy quark/anti-quark pair can be well approximated by non-relativistic quantum mechanics. Investigations of such systems provided early insight into the force between quarks. Consider the currently known spectrum of states for charmonium, made of a c quark and a c anti-quark and denoted by J/ψ, and bottomonium, made of a b and an anti-b and given by Υ.

The figure gives current values for the J/ψ and Υ masses for states labeled by quantum numbers for radial eigenfunctions and orbital angular momentum. The question marks on the figure are for states whose identity is not fully confirmed.

(a) It is known that the S states in the figure have total angular momentum $J = 1$. What is the orbital angular momentum and spin contribution to the total J?

(b) Is a $J = 0$ and $L = 0$ state allowed?

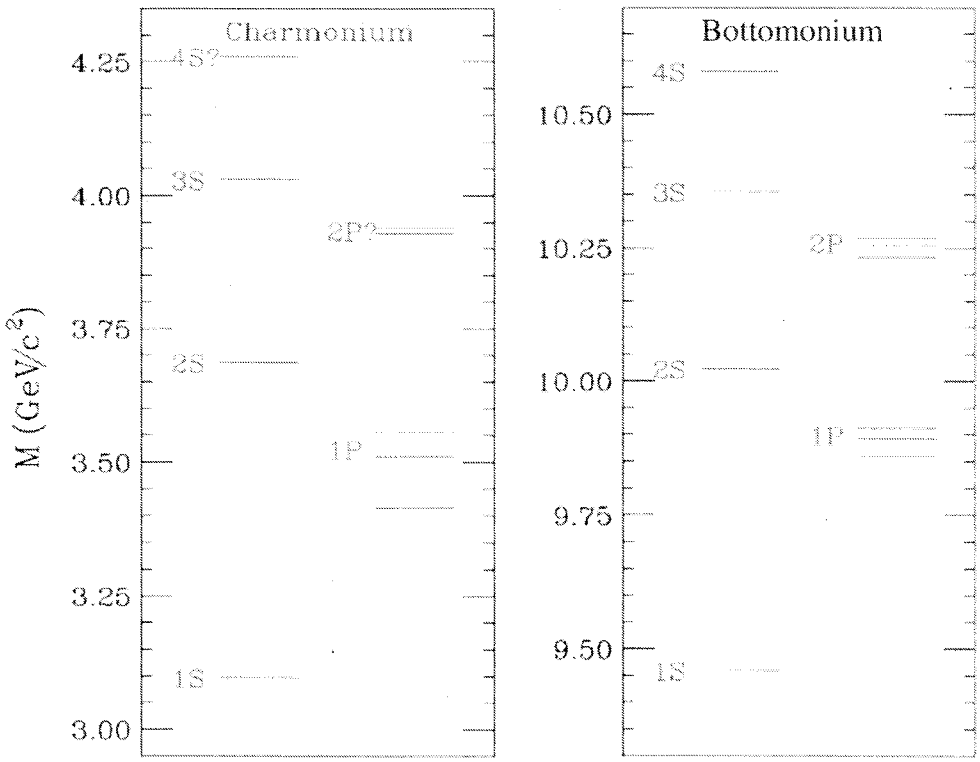
(c) If charmonium and bottomonium bound states are produced in e^+e^- annihilation, will it be easier to produce $J = 0$ states or $J = 1$ states?

(d) What values of J are present in the $1P$ states?

(e) Notice that the splittings between the states are very similar, in spite of the fact that the mass of the Υ is about 2.5 times the mass of the J/ψ. This means that the b quark is much heavier than the c quark, provided the binding energy of the two systems are not wildly different. In the radial part of the Schrödinger equation

$$Hu(r) = \left[-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\vec{L}\cdot\vec{L}}{2mr^2} + V(r)\right]u(r) = Eu(r)$$

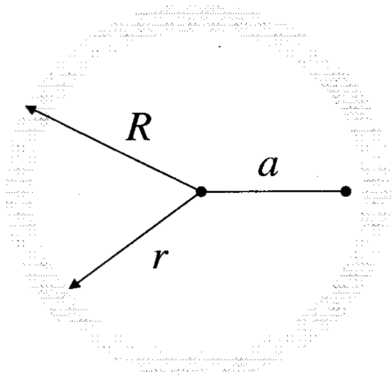
the mass enters and should also generally enter the expression for the eigenvalues. Show that a simple potential of the form $V(r) = \alpha \ln(r/r_0)$ leads to mass independent splittings between the states.


Charmonium
Bottomonium
M (GeV/c^2)
4.25
4.00
3.75
3.50
3.25
3.00
4S?
3S
2S
1S
2P?
1P
10.50
10.25
10.00
9.75
9.50
4S
3S
2S
1S
2P
1P

4. A glass spherical shell of thickness 1 cm and inner radius $r = 6.5\,\text{cm}$ ($n_2 = 1.5$) is filled with a non-absorbing liquid of $n_1 = 1.6$. A point source of light is installed at $a = 6\,\text{cm}$ from the center of the sphere.

What fraction of the light gets out of the sphere?

Hint: Neglect absorption. If you have to use approximations, please explain them precisely. Also, $n_{air} = 1$.



SOLUTIONS Section 6
General Part II

Uemura · quantum exp.
Problem #1

2009 Quals: Problem and some examples of solution: by Tomo Uemura

Provide an example of one phenomenon in solid state physics where a "quantum" aspect is important. What is the corresponding observable property which can be estimated using the uncertainty principle. Explain how the observable value can be derived using the uncertainty principle, and describe how you would experimentally measure this observable quantity.

Examples of possible solutions:

(1) Bose Condensation Temperature and Fermi Temperature:

Bose condensation can be visualized as a phenomenon which occurs when the thermal wave-length (spread of wave function in real space) of a boson becomes comparable to the inter-particle distance (distance to nearest neighbour boson).

At a given temperature T, the kinetic energy of a boson in a 3-dimensional gas is given as

$$ (3/2)*k_{B}T = (1/2)p^{2}/m $$, where m is the mass of a boson.

The thermal wavelength is defined as $ \lambda_{th} = h/p $
Let us assume the particle density of Bose gas as n. Then the interparticle distance is $n^{-1/3}$

This logic is the same as treating the momentum p as Δp, and λ_{th} as Δx, and then applying uncertainty principle (forget about ambiguity of 2π).

These treatments would lead to the condensation temperature T_{BE} as
$ k_{B}T_{BE} = (h^{2}) (n^{2/3})/m $

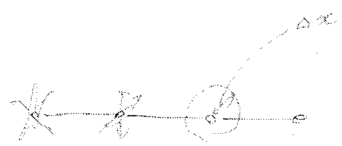
Similar argument can be used to make a crude estimate of Fermi temperature. To satisfy Pauli principle of fermions, with the particle density n, one fermion occupies the volume of $1/n$. If we assign a cube for this volume, the edge of the cube Δx will become $n^{-1/3}$. To confine each electron in such a small region, we need to give a momentum $\Delta p = h / \Delta x$, according to uncertainty principle. Then the kinetic energy of each fermion would become
$ (1/2)(\Delta p)^{2}/m = (h^{2}/2)(n^{2/3})/m \sim k_{B}T_{F}$

This is a very crude way to estimate the Fermi energy $k_{B}T_{F}$. This argument shows that T_{F} and T_{BE} have the comparable magnitudes with n as the particle density, and m as the mass. For non-interacting Bose gas and Fermi gas, better calculation really shows that T_{BE} is very close to T_{F} if the densities and masses of fermions and bosons are equivalent. For BE-BCS crossover, we usually consider two fermions to form a boson with twice the fermion mass, T_{BE} becomes about ¼ T_{F}. This relationship was established by the BE-BCS crossover experiments for cold atoms.

The Bose condensation temperature in superfluid He, for example, can be determined by measuring viscosity using tortional oscillator. Fermi temperature of electron gas is usually determined by two independent measurements of n and m, and/or their combinations. n can be measured by Hall effect, while m can be measured by cyclotron resonance mass. Combination of n and m can be measured by specific heat linear term at low temperatures (Sommerfeld Constant), Pauli susceptibility, Plasma oscillation frequency, etc.

(2) Debye Waller Factor

At finite temperature T, atoms of a crystal lattice are fluctuating around their equilibrium positions due to thermal effect (in some cases like He or other light atoms, quantum fluctuations are also important). Let us express the amplitude of this fluctuation by using the average displacement Δx.



Classically

$$\frac{1}{2} k (\Delta x)^2 \sim k_B T$$

$$\exp\left(-\frac{r^2}{2(\Delta x)^2}\right) \quad \cdots \quad \text{real space}$$

This gives "spread in real space", and thus results in slow decay of Bragg peak intensities with increasing momentum transfer q in X-ray or neutron scattering experiments. This phenomenon can be derived by Fourier transforms of real-space correlation function. However, the essence comes from the uncertainty principle
$\Delta x * \Delta p \sim \hbar$

Usually, the decreasing Bragg peak intensity with increasing momentum transfer q is fitted by a Gaussian decay $I(q) \sim \exp[(-1/2)(q^{2}/(\Delta p)^{2}]$, which corresponds to the Gaussian spread in real space with the probability of displacement
$P(x) = \exp[(-1/2) x^{2}/(\Delta x)^{2}]$

We can connect this phenomenon with thermal excitations of phonons in the following way:

k_{B}T ~ (1/2)C(\Delta x)^{2}, where C is the spring constant of lattice vibration.

The frequency for this vibration is $\omega~ \sqrt(C/M) $ where M is the atomic mass.

The Debye temperature Θ_{D} is proportional to the vibration energy
$ \hbar\omega ~ \Theta_{D}$

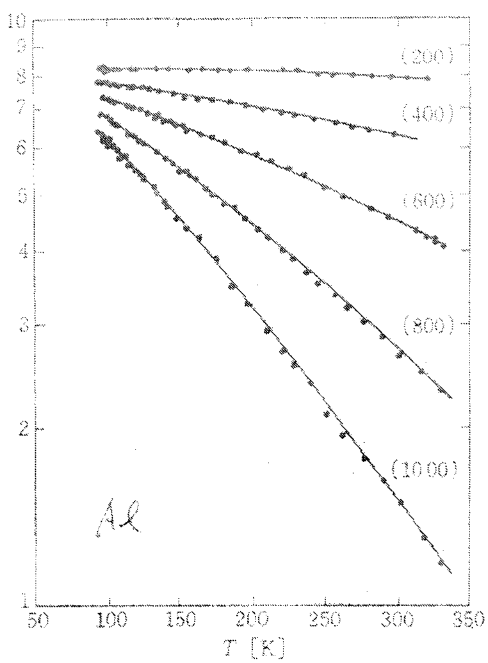
These relationships lead to

$ [1/ (\Delta p)^{2}] ~ (\hbar)^{2}*T / [M*k_{B}*\Theta_{D}^{2}] $,

as found by Debye.

One can verify this relationship by performing measurements of phonon dispersion (to determine Debye temperature) and the momentum-transfer dependence of the Bragg Peak intensity I (q) (to determine Δp) by neutron and/or X-ray scattering, temperature T by a thermometer, and the atomic mass M from chemistry.

I attach an example of T- and q- dependences of the scattering intensity measured in Aluminum by X-rays. The vertical axis represents relative scattering intensity of several Bragg Peaks.



SOLUTIONS Section 6 General Part II

Marka Exp. Techniques Sec 6 GEN II Problem #2 Marka

Problem:

You have to measure the index of refraction of a transparent liquid (n_L) in a hope that it will help you identify the substance. You have plenty of the liquid, a 632nm red laser pointer and a thin walled rectangular glass container (n_G=1.5). Paper, pencil, protractors and rulers are also available.

Describe three different ways to determine (measure) the index of refraction of the liquid. Try to qualitatively compare their accuracy.

Hints of solutions:

This problem is intentionally open ended. It is hoped that many ingenious and precise measurements shall be invented. We mention three possible ways to measure the index of refraction of the liquid (n). (As the wall of the container is very thin its existence is neglected for simplicity. Of course, precise measurements can take that into account.)

1. Fix the laser pointer above the container and ensure a large enough angle of incidence (to the vertical). Mark the point at the bottom of the empty container where the laser hits it ('***A***'). Fill the container with the liquid and measure the distance between ***A*** and the new laser spot ('***B***') at the bottom. From the index of refraction of the air (n_{AIR}=1), the angle of incidence, d_{AB}, and the height of the liquid one can compute n_L.

2. Aim the laser pointer to the surface of the liquid from below (i.e., through the side of the container. Adjust the angle of the laser pointer until the threshold of total internal reflection (i.e., no outgoing beam). From n_{AIR}, and the angle of the laser to the horizontal n_L can be computed.

3. Place a sheet of paper at the bottom of the container (under the liquid). Shine the laser pointer to the middle of the paper from below and observe the light pattern on the paper (opaque screen). You will see a bright point in the middle (the laser) a darker ring around and a larger bright ring around. The inner edge of the bright ring corresponds to the onset of total internal reflection on the liquid-air interface. From n_{AIR}, the thickness of the liquid, and the inner radius of the bright ring one can compute n_L.

Naturally, there are a number of other possibilities. Find some! The methods listed above are quite rudimentary and their accuracy should be comparable (not too good).

Sec 6 GEN II
Problem # 3
Mawhinney

Solutions

1. For an S state, $L = 0$, so the only way to get $J = 1$ is to have the total spin $S = 1$. Since a quark and an anti-quark are not identical particles, they can be in a symmetric spin state without violating the Pauli exclusion principle.

2. Yes. $S = 0$ is allowed, since the quark and anti-quark can have their spins opposite.

3. It is easier to produce $J = 1$ states, since the photon is what couples $e^+ e^-$ to quark/anti-quark states and the photon has spin 1.

4. The 1P states have $L = 1$ and the spins can be $S = 0, 1$, so $J = 0, 1$ and 2 are allowed.

5. Let $\rho \equiv m^{1/2} r$ and write the Schrodinger equation in terms of ρ. Now, the only m dependence in the Schrodinger equation is from

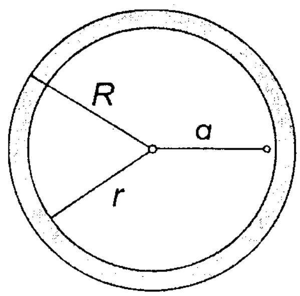
$$V(r) = \alpha \ln(\rho) - \alpha \ln(m^{1/2} r_0) \tag{2}$$

and the second term is a constant. Thus it shifts all the energy eigenvalues, but the splitting between eigenvalues is independent of mass.

Marka - Optics
Sec 6 gen II
Problem # 4

Problem:

A glass spherical shell of thickness 1cm and inner radius of r=6.5cm (n_2=1.5) is filled with a non-absorbing liquid of n_l=1.6. A point source of light is installed at a=6cm from the center of the sphere.

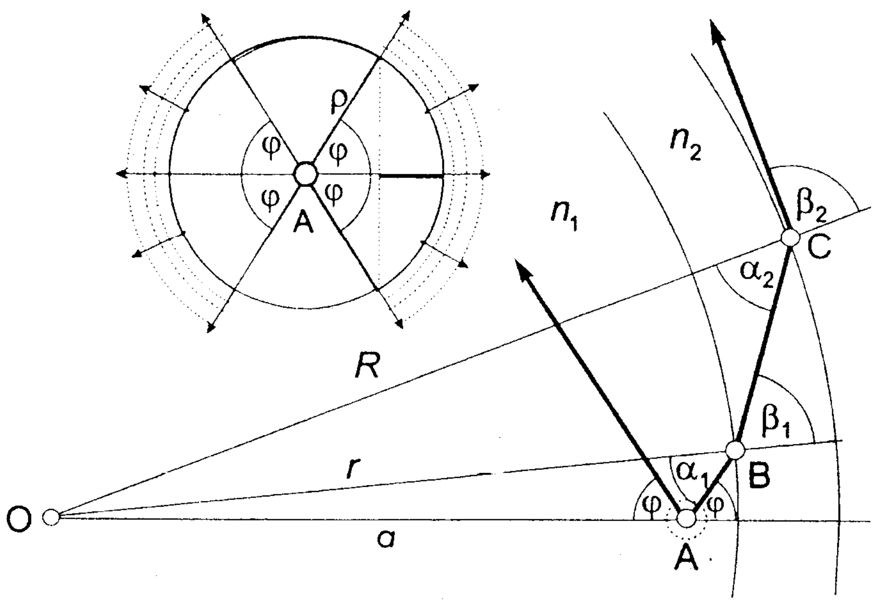


What fraction of the light gets out of the sphere?

(Hint: Neglect absorption. If you have to use approximations, please explain them precisely. Also, n_{AIR}=1.)

Hint of a solution:

Please study the geometry of the system:



We give an approximate solution to the problem. Assuming that the absorption is negligible, we we do not have to track the large number of reflections and long travel inside the materials. Due to the symmetry of the system it is reasonable to state that light-rays where $\beta_2 \leq 90°$ are the ones that will (eventually) leave the system. Therefore one needs to determine the half-opening-angle (φ) of the cone of light that gets out of the system ($\beta_2 = 90°$).

Nicolis - Order of magnitude
Sec 6 GEN II
Problem # 5

Quals 2008-09

1 General: order of magnitude estimates

A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density- and velocity-fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{1}$$

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p\,. \tag{2}$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho)\,. \tag{3}$$

It is well known that the above equations, when *linearized* about the homogeneous, static background configuration

$$\rho = \rho_0\,, \qquad \vec{v} = 0\,, \qquad p = p_0 \equiv p(\rho_0)\,, \tag{4}$$

admit wave-like solutions, which describe sound waves. Sound waves are just small density perturbations

$$\rho(\vec{x},t) = \rho_0 + \delta\rho(\vec{x},t)\,, \qquad \delta\rho(\vec{x},t) = \varepsilon\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.} \tag{5}$$

that propagate at the speed

$$c_s = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_0}}\,. \tag{6}$$

1. Without keeping track of the vector structure and of order one coefficients, estimate from the equations of motion (??, ??) the amplitude of the fluctuations in $\vec{v}$ and p corresponding to the sound wave (??). Express your results in terms of ε, k, and c_s.
2. Estimate the regime of validity of the linear approximation, that is find a condition on the quantities ε, k, and c_s that tells you when you can trust the linear approximation. Do you find your result reasonable, on physical grounds?

Solution

1. The perturbed values for ρ, $\vec{v}$, and p are

$$\rho = \rho_0 + \delta\rho\,, \qquad \vec{v} = \delta\vec{v}\,, \qquad p = p_0 + \delta p\,. \tag{7}$$

However by assumption p is purely a function of ρ, so that at linear order in $\delta\rho$ we have

$$\delta p = \left.\frac{dp}{d\rho}\right|_{\rho=\rho_0}\delta\rho \sim c_s^2\,\varepsilon\,. \tag{8}$$

To estimate $\delta\vec{v}$ we have to linearize the equations of motion in $\delta\rho$, $\delta\vec{v}$, and δp. Using that for wave solution behaving like $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\nabla \sim i\vec{k}\,, \qquad \frac{\partial}{\partial t} \sim -i\omega \sim -ic_s k\,, \tag{9}$$

and neglecting the vector structure and numerical factors, we get

$$c_s k\,\varepsilon + k\rho_0\,\delta v \sim 0 \tag{10}$$

$$c_s k\,\delta v \sim -\frac{1}{\rho_0} k\,\delta p \tag{11}$$

which both yield

$$\delta v \sim c_s \frac{\varepsilon}{\rho_0}\,. \tag{12}$$

2. To determine the regime of validity of the linear approximation, we have to estimate the size of the terms left out by the latter—the non-linear terms in the equations of motion.

   In eq. (??) there is only one such term:

$$\vec{\nabla}\cdot(\delta\rho\,\delta\vec{v}) \sim k\,\varepsilon\, c_s \frac{\varepsilon}{\rho_0}\,, \tag{13}$$

   whereas the linear pieces are of order $c_s k\varepsilon$.

   In eq. (??) we have one source of non-linearities on the l.h.s.:

$$(\delta\vec{v}\cdot\vec{\nabla})\delta\vec{v} \sim k\,c_s^2\,\frac{\varepsilon^2}{\rho_0^2}\,, \tag{14}$$

   and one on the r.h.s.:

$$-\frac{1}{\rho}\vec{\nabla}p = -\frac{1}{(\rho_0+\delta\rho)}\vec{\nabla}p\,, \tag{15}$$

   which when expanded in $\delta\rho$, δp gives us the linear term we already kept in eq. (??) and a quadratic piece of the form

$$\frac{1}{\rho_0}\frac{\delta\rho}{\rho_0}\vec{\nabla}p \sim k\,c_s^2\,\frac{\varepsilon^2}{\rho_0^2}\,, \tag{16}$$

   together with infinitely many higher order terms.

   In all cases, we see that the size of the non-linear terms neglected by the linear approximation, relative to the linear terms is of order ε/ρ_0. Therefore, the linear approximation for sound waves holds as long as

$$\varepsilon \ll \rho_0\,, \tag{17}$$

   whereas it breaks down for

$$\varepsilon \sim \rho_0\,. \tag{18}$$

   On physical grounds, this is very reasonable: it means that density perturbations can be consistently treated as small as long as their typical amplitude is smaller that the average density.

Sec 6 GEN II
Problem #6
Heinz

Solution:

a) Since the block of ice is very large, running the heat engine will just melt part of it to produce liquid water at 0 ^{0}C. The process will continue until the water of the heat source is cooled to 0 ^{0}C. At this point, the system will be in thermal equilibrium and no further work can be extracted.

b) The efficiency of an ideal heat engine running between the water at temperature T and the ice at temperature T_0 is

$$e(T) = 1 - T_0 / T$$

From the definition of efficiency, we have for an infinitesimal process at temp T:

$$e(T) = dW/dQ_H = (dQ_H - dQ_C)/ dQ_H = 1 - dQ_C/ dQ_H = 1 + dQ_C/ (mc\ dT).$$

Here mc is the heat capacity of the water; a minus sign has been introduced since dQ_H is positive as the temperature decreases. Equating the two expressions, we have

$$dQ_C = - mcT_0\, dT/ T$$

Integrating from $T = 373$ K to $T = 273$ K:

$$Q_C = mc\ T_0 \ln (373/273) = \underline{358 \text{ kJ.}}$$

c) The amount of melted ice will be

$$M = Q_C /L = \underline{1.08 \text{ kg}}$$

d) The amount of work done by the engine is

$$W = Q_H - Q_C = |\Delta T|\ mc - Q_C = 420 \text{ kJ} - 358 \text{ kJ} = 62 \text{ kJ}$$

e) The overall efficiency of conversion of heat to work is

$$E = W/ Q_H = 62 \text{ kJ}/420\text{kJ} = \underline{14.8\%}$$

5. A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density- and velocity- fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{1}$$

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p \tag{2}$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho) \tag{3}$$

It is well known that the above equations, when *linearized* about the homogeneous, static background configuration

$$\rho = \rho_0, \qquad \vec{v} = 0, \qquad p = p_0 \equiv p(\rho_0) \tag{4}$$

admit wave-like solutions, which describe sound waves. Sound waves are just small density perturbations

$$\rho(\vec{x},t) = \rho_0 + \delta\rho(\vec{x},t), \quad \delta\rho(\vec{x},t) = \varepsilon e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \text{c.c.} \tag{5}$$

that propagate at the speed

$$c_s = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho=\rho_0}} \tag{6}$$

(a) Without keeping track of the vector structure and of order one coefficients, estimate from the equations of motion (1, 2) the amplitude of the fluctuations in $\vec{v}$ and p corresponding to the sound wave (5). Express your results in terms of ε, k, and c_s.

(b) Estimate the regime of validity of the linear approximation, that is find a condition on the quantities ε, k, and c_s that tells you when you can trust the linear approximation. Do you find your result reasonable, on physical grounds?

6. Consider running an ideal heat engine (at the Carnot efficiency) operating between the following two objects: 1 kg of water at 100 °C and a very large block of ice at 0 °C. We run the engine between this heat source and sink until work can no longer be extracted from the system.

In analyzing this problem, assume that the water and ice are thermally isolated from the external environment and neglect any effect of the changes in volume. Take the specific heat of water to be $c = 4.2$ kJ/kg-K, independent of temperature; the latent heat to melt ice is $L = 330$ kJ/kg.

At the completion of the process:

(a) What is the temperature of the water?

(b) How much heat has been absorbed by the block of ice during the process?

(c) How much ice has been melted?

(d) How much work has been done by the engine?

(e) What is the overall efficiency of the heat engine for the entire process?