Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, April 4, 2009

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Partial credits for each problem are indicated.

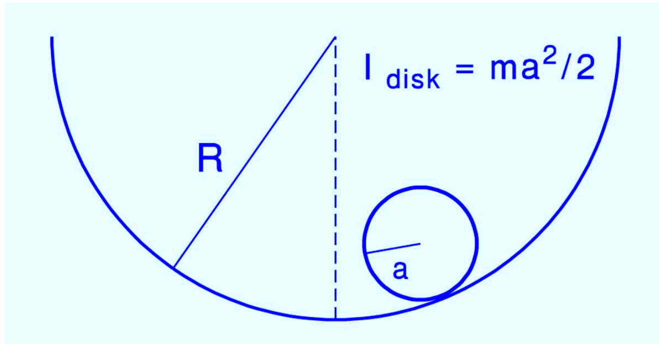
Solve 2 problems of your choice. Do not turn in more than 2 problems.

The total time allowed **2 hrs**.

Please, indicate problems you are turning in

☐ II-1 ☐ II-3 ☐ II-4

II-1. (Classical Mechanics)



A uniform thin disk of mass m and radius a is rolling without sliding inside a circular track of radius R as shown. Assume that the motion is limited to a vertical plane.

(i) (7 pts) Find the Lagrangian of the system and also the Lagrange's equation of motion.

(ii) (3 pts) Find the Hamiltonian which describes the motion of this system.

(iii) (3 pts) Write down the Hamilton-Jacobi equation.

(iv) (7 pts) If the motion of the disk is such that the center of mass moves slightly away from the vertical at the bottom of its track, find the frequency of the small oscillation with the method of *action-angle variable* by first expressing the Hamiltonian in terms of the action-angle variable J.

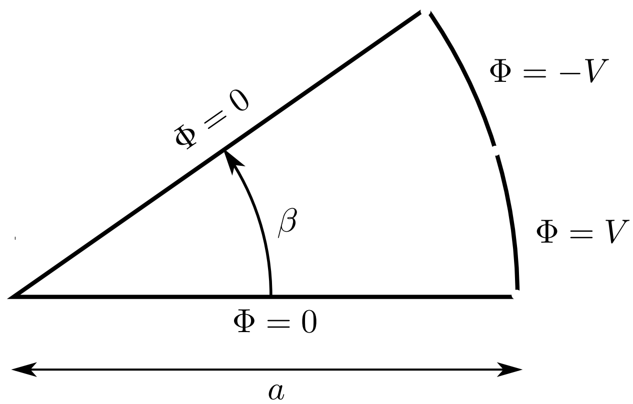
Hint: You may find useful the following integrals:

$$\int \sin^2 x\, dx = \frac{1}{2} - \frac{1}{4}\sin 2x\,, \qquad \int \cos^2 x\, dx = \frac{1}{2} + \frac{1}{4}\sin 2x\,,$$

$$\int (a^2 - x^2)^{1/2} dx = \frac{1}{2}\left[x(a^2-x^2)^{1/2} + a^2 \arcsin\frac{x}{a}\right], \qquad a > 0\,,$$

$$\int \frac{dx}{(a^2-x^2)^{1/2}} = \arcsin\frac{x}{a}\,, \qquad a > 0\,.$$

II-2. (Electricity and Magnetism)



Consider a two-dimensional wedge geometry. The volume enclosed is expressed in polar coordinates as $0 < \rho < a$ and $0 < \theta < \beta$. There is no charge in the volume, but there is a potential on the surface given by (see, the figure):

- $\Phi = 0$ for $\theta = 0$ and $\theta = \beta$;
- on the surface at $\rho = a$, Φ takes two different constant values, $\Phi = V$ for $0 < \theta < \beta/2$ and $\Phi = -V$ for $\beta/2 < \theta < \beta$.

(i) (12 pts) Find a Fourier sine-series solution in the θ variable to the Laplace equation, $\nabla^2\Phi = 0$, everywhere in the volume satisfying the above boundary conditions.

(ii) (8 pts) Then sum the series explicitly.

Hint: Potentially useful formula

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots .$$

II-3. (Quantum Mechanics)

A certain system has the Hamiltonian

$$\boldsymbol{H} \;=\; E_0 \sum_{n=1}^{N} |n\rangle\langle n| + \lambda \sum_{n=1}^{N} \left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right) ,$$

where λ is the coupling energy and E_0 is the (degenerate) energy in the zero-coupling (i.e., $\lambda = 0$) limit. The base kets $|n\rangle$, $n = 1, 2, \ldots, N$, are complete and orthonormal, and satisfy the periodic boundary condition

$$|N + n\rangle \;=\; |n\rangle \, .$$

Define the operator

$$\boldsymbol{b} \;=\; \sum_{n=1}^{N} |n\rangle\langle n+1| \, .$$

(i) (2 pts) Express $\boldsymbol{H}$ in terms of $\boldsymbol{b}$, $\boldsymbol{b}^{\dagger}$, and the identity operator $\mathbf{1}$.

(ii) (2 pts) Show that $\boldsymbol{H}$, $\boldsymbol{b}$, and $\boldsymbol{b}^{\dagger}$ mutually commute.

(iii) (4 pts) Represent $\boldsymbol{b}$ by a $N \times N$ matrix on the basis $\{|n\rangle, n = 1, 2, \ldots, N\}$.

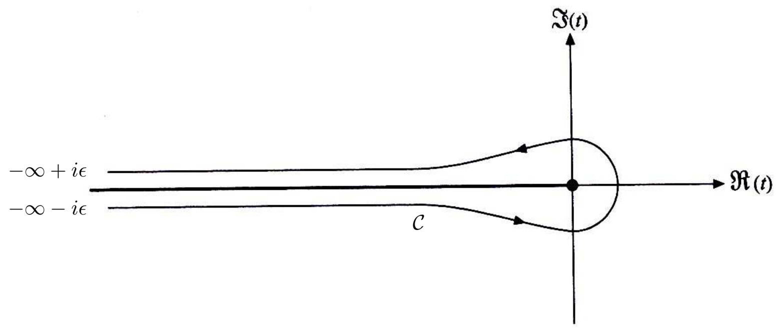
(iv) (6 pts) Determine the eigenvalues of $\boldsymbol{b}$ and $\boldsymbol{b}^{\dagger}$.

(v) (6 pts) Determine the eigenvalues of $\boldsymbol{H}$ and show that the eigenkets are

$$|E_m\rangle \;=\; \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i(n-1)\theta_m} |n\rangle \, ,$$

where $\theta_m \;=\; 2\pi(m/N)$.

II-4. (Mathematical Methods)



The standard definition of the gamma function, $\Gamma(z)$, in terms of the Euler integral is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \,, \qquad \mathrm{Re}\, z > 0 \,. \tag{1}$$

(i) (5 pts) Show that (1) implies that for positive integers,

$$\Gamma(n+1) = n! \,, \qquad n = 0, 1, 2, \ldots .$$

Now, consider a function $G(z)$ defined by the contour integral,

$$G(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{e^t}{t^z} dt \,, \tag{2}$$

where the contour $\mathcal{C}$ is shown in the figure above. The integrand is made single valued by introducing a branch cut in the t-plane along the real axis from $-\infty$ to 0, such that $\mathrm{Arg}\, t = +\pi$ and $\mathrm{Arg}\, t = -\pi$ right above and below the cut, respectively.

(ii) (2 pts) For which complex z's does the above integral converge and hence $G(z)$ is well defined? What is the domain of analyticity of $G(z)$? (Don't worry about rigorous proofs, just give a plausible argument that supports your claims.)

(iii) (4 pts) Evaluate the complex integral (2) for $z = n$, where $n = 0, \pm 1, \pm 2, \ldots$, is a real integer.

(iv) (7 pts) Use (2) to calculate the derivative $G'(z)$ for $z = 0, -1, -2, \ldots$.

(v) (2 pts) Based on the properties of $G(z)$, suggest a relation between $G(z)$ and the gamma function $\Gamma(z)$.