Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 28, 2015

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your Sh-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Sh-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1	☐ 2	☐ 3	☐ 4	☐ 5
☐ 6	☐ 7	☐ 8	☐ 9	☐ 10

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

I-1. (Electricity and Magnetism)

An isolated circular parallel plate capacitor is given a surface charge density K at time $t = 0$. The region between the plates is filled with a slightly conducting dielectric of permittivity ε and conductivity σ through which the capacitor then begins to discharge. Find (neglecting edge effects):

(i) The surface charge density of the capacitor at a time $t > 0$.

Hint: Set up a simple differential equation for $K(t)$.

(ii) The magnitude and direction of the discharge current density.

(iii) The magnitude and direction of the displacement current.

(iv) The magnitude and direction of the magnetic field inside the capacitor caused by the two currents.

I-2. (Thermodynamics)

One can argue using dimensional analysis and the third law of thermodynamics that the entropy and the internal energy of a gas of photons must be of the form

$$S = \frac{4}{3}\alpha V T^3 \qquad \text{and} \qquad U = \alpha V T^4 ,$$

where V is the volume, T is the temperature, and α is a universal constant.

(i) Calculate the pressure, P, of this gas as a function of T and V. Show that the pressure is proportional to the energy density, U/V, of the gas and determine the proportionality constant.

(ii) What is the chemical potential of this gas? Comment on the result.

(iii) Find the Helmholtz free energy, F. Express it in terms of its correct variables.

(iv) Show that during a quasistatic, adiabatic compresion of the gas

$$PV^\gamma = \text{const} .$$

Find the value of the adiabatic constant γ.

I-3. (Stat Mech)

Consider energy fluctuations in the canonical ensemble:

(i) For a gas of particles, state how the thermodynamic energy, E, is related to the average energy of the gas, $\langle \mathcal{E} \rangle$, when the number of particles, N, becomes very large.

(ii) Using the canonical ensemble, find an expression for $\langle (\Delta \mathcal{E})^2 \rangle$ in terms of T and C_V, where $\Delta \mathcal{E} = \mathcal{E} - \langle \mathcal{E} \rangle$.

(iii) Using the result obtained in (ii), find $\langle \left(\frac{\Delta \mathcal{E}}{E}\right)^2 \rangle$ in terms of N for a monatomic ideal gas. How does your result relate to your answer to (i)?

I-4. (Math Methods)

A complex $n \times n$ matrix $\mathbb{A}$ satisfies the relation

$$\mathbb{A} + \mathbb{A}^\dagger = \mathbb{I} ,$$

where $\mathbb{I}$ is the unit matrix.

(i) What can one say about the eigenvalues of $\mathbb{A}$?

(ii) Prove that eigenvectors belonging to different eigenvalues must be orthogonal.

I-5. (Quantum Mechanics)

The Hamiltonian, H, of a two-state system has the following matrix elements in an orthonormal basis $(|\psi_l\rangle, |\psi_r\rangle)$:

$$\langle\psi_l|H|\psi_l\rangle = \langle\psi_r|H|\psi_r\rangle = \epsilon_0 , \qquad \langle\psi_l|H|\psi_r\rangle = \langle\psi_r|H|\psi_l\rangle = -g ,$$

where g is real and positive.

(i) Calculate eigenenergies and the corresponding eigenstates of H.

(ii) In an experiment, the system is prepared in the "left" state, $|\psi_l\rangle$, at time $t = 0$. After a given time, T, it is found in the "right" state, $|\psi_r\rangle$, in 50% of all measurements. What relationship between g and T can you infer from this experiment?

(iii) Give an example of a physical system with such a Hamiltonian. What is the physical meaning of ϵ_0 and g in your example?

I-6. (Condensed Matter)

Consider a one-dimensional crystal with two kinds of atoms, spaced a distance a apart. Atoms of mass M are located at the odd-numbered lattice points $2n-1$, $2n+1$, . . . ; atoms of mass m are located at the even-numbered lattice points $2n$, $2n+2$, ... The equations of motion, assuming nearest neighbor interactions only, are

$$m\ddot{u}_{2n} = \beta(u_{2n+1} + u_{2n-1} - 2u_n) , \qquad M\ddot{u}_{2n+1} = \beta(u_{2n+2} + u_{2n} - 2u_{2n+1}) .$$

Look for solutions of the form

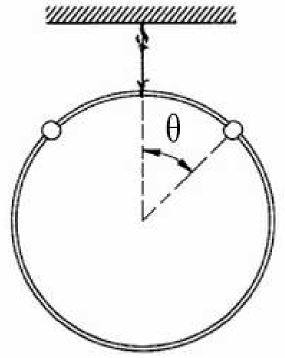
$$u_{2n} = \xi \exp[2nika - i\omega t)] , \qquad u_{2n+1} = \eta \exp[(2n+1)ika - i\omega t] .$$

(i) Solve the equations of motion to find ω as a function of k (the dispersion relation) and also the ratio of amplitudes ξ/η. (You should find two solutions.)

(ii) Find the limiting values of your solutions for ω and ξ/η to the first non-vanishing order near $k = 0$ and near $k = \pi/(2a)$.

(iii) Using these results and assuming that $m > M$, draw a sketch of ω versus k for the range $0 < k < \pi/(2a)$.

I-7. (Classical Mechanics)



A ring of mass M hangs from a thread, and two beads of mass m slide on it without friction. The beads are released simultaneously from the top of the ring and slide away on opposite sides. By examining forces acting on the beads and on the ring:

(i) Show that the ring will start to rise if $m > \frac{3}{2}M$.

(ii) Find the angle, θ, for which this occurs.

I-8. (Optics)

A thick piece of glass with index of refraction n is bent as shown in the diagram. The bend is perfectly cylindrical, with an inner radius of r and an outer radius of R. A narrow laser beam, pointing parallel to the glass before the bend, enters the glass from the left a distance d from the bottom of the glass plate. What constraints on n, r and R ensure that the light beam continues down the glass after the bend and exits out the bottom face of the glass for any entrance position d?

I-9. (Particle Physics)

The Λ and p particles have spin 1/2, the π has spin 0. Suppose that Λ is polarized in the $\hat{z}$ direction and decays at rest,

$$\Lambda \quad \longrightarrow \quad p + \pi \, .$$

Assume that parity is conserved. What is the most general angular distribution of π?
Hint: The parity of Λ and p is $+1$, parity of π is -1.

I-10. (Astrophysics)

We know that galaxies exhibit some random motion relative to the overall Hubble expansion, due to the gravitational pull of their neighbors. This is called "peculiar velocity." Suppose that the root mean square galaxy peculiar velocity is 600 km/s; how far would a galaxy have to be before it could be used to determine the Hubble constant to 10 percent accuracy, supposing that the true value of the Hubble constant is 100 km/s/Mpc?

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Graduate Screening Examination
Part II
Saturday, March 28, 2015

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Fill in your Lg-#

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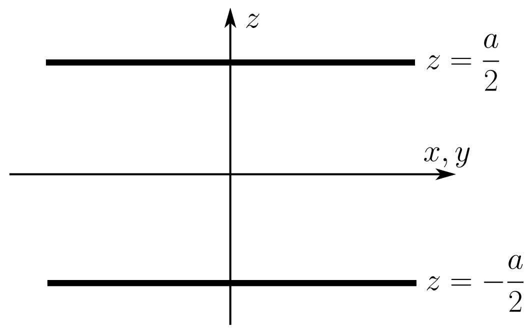
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs 30 min**.

Please, indicate problems you are turning in:

☐ **II-1** ☐ **II-2** ☐ **II-3** ☐ **II-4**

II-1. (E & M)



Consider a point charge, q, placed centrally at the origin ($\vec{r} = 0\,\hat{x} + 0\,\hat{y} + 0\,\hat{z}$) between two parallel infinitely-large grounded conducting plates as shown in the figure. The distance between the two plates is a. Using the method of images answer the following questions:

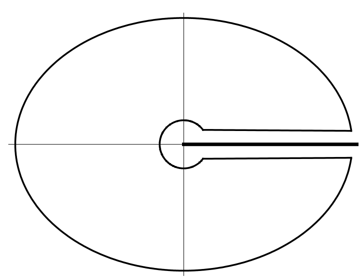
(i) If the bottom plate is removed, what is the electrostatic potential $\phi(x, y, z)$ that satisfies vanishing boundary conditions on the top plate, $\phi(x, y, \frac{a}{2}) = 0$.

(ii) What is then the electric field at the two locations, $\vec{r} = \pm a\hat{z}$?

(iii) If both plates are in place, find the location and charges of the many images and then compute the electrostatic potential $\phi\,(x, y, z)$ that satisfies vanishing boundary conditions on both plates, $\phi(x, y, \pm\frac{a}{2}) = 0$. Give your answer in the form of a sum and argue how this sum will produce the desired vanishing result at the plates.

(iv) In case (iii), what is the induced *total* charge on the top plate? How does it compare to the induced total charge on the bottom plate?

II-2. (Mathematical Methods)



The function

$$f(z) \;=\; \frac{z^{1/2}}{1+z^2}\,\log z\,,$$

is made single valued by introducing a branch cut along the positive real axis. Use the integration(s) along the key-hole contour as above to prove the following result for the real integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,\log x\,dx \;=\; \frac{\pi^2}{2\sqrt{2}}\,.$$

Hint: You may have to do the contour integration more than once.

II-3. (Classical Mechanics)

A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R. (The hoop does not spin, i.e. it stays in the vertical plane perpendicular to the axis of the cylinder.) The only external force is that of gravity. The hoop starts rolling from rest at the top of the cylinder, and while it is rolling there are two constraints, one holonomic, the other semiholonomic. (Actually, the semiholonomic constraint could be integrated to a holonomic one, but do not do this.) Use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.

II-4. (Quantum Mechanics)

Consider a two-level quantum system, e.g., a spin-$\frac{1}{2}$ particle, with the Hamiltonian and the initial state given respectively by

$$H = \frac{\hbar\omega}{2}\sigma_z, \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

where

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \qquad \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \qquad \sigma_y = i\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right),$$

are the standard Pauli matrices.

(i) Find the expression for the time-dependent expectation-values

$$\alpha(t) := \langle\Psi|\sigma_\alpha(t)|\Psi\rangle, \qquad (\alpha = x, y, z)$$

where

$$\sigma_\alpha(t) := e^{i\frac{\omega t}{2}\sigma_z}\sigma_\alpha e^{-i\frac{\omega t}{2}\sigma_z},$$

are the Heisenberg picture evolved operators at time t.

(ii) Compute the variances of $\sigma_x(t)$ and $\sigma_z(t)$ in $|\Psi\rangle$ and discuss the associated Heisenberg Uncertainty Relations.

(iii) Compute the so-called *survival probability*

$$P_{\rm surv}(t) := |\langle\Psi|e^{-i\frac{t}{\hbar}H}|\Psi\rangle|^2$$

What's the first time t^* such that $P_{\rm surv}(t^*) = 0$? What is its physical meaning?

(iv) One way to model the effect of the environment on the two-level system is to assume that the initial state gets randomly *dephased* as follows

$$|\Psi\rangle \longrightarrow |\Psi_\xi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\xi}|1\rangle),$$

where ξ is a *uniformly random angle* in $[0,\, 2\pi]$. Compute the noise averages

$$\frac{1}{2\pi}\int_0^{2\pi} d\xi\, \langle\Psi_\xi|\sigma_\alpha(t)|\Psi_\xi\rangle, \qquad (\alpha = x, z)$$

and comment on the results on physical grounds.