Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II

Saturday, March 25, 2017

Do not separate this page from the problem pages. Fill out and turn in at the end of the exam.

Student ________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Lg-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)

Consider the Lagrangian

$$L = \frac{1}{2}\, m\, e^{2\gamma t}\left(\dot{x}^2 - \omega^2 x^2\right) ,$$

for the motion of a particle of mass m in one dimension. The constants m, ω and γ are real and positive.

(i) Find the equation of motion. Interpret it by stating what kind of forces the particle is subject to.

(ii) Find the canonical momentum, p, and construct the Hamiltonian, $H(x, p, t)$. Is the Hamiltonian a constant of motion? Is the energy conserved? Explain.

(iii) Find a canonical transformation, $(x, p) \to (X, P)$, under which the transformed Hamiltonian, $K(X, P)$, does not depend explicitly on time. Using your favorite method, verify that the proposed transformation is indeed canonical. Obtain the new Hamiltonian, $K(X, P)$, and the Hamilton equations of motion for $X(t)$ and $P(t)$.

(iv) Assuming that $\gamma < \omega$, find the solution for the motion (i.e., $(X(t), P(t))$ with the initial conditions $X(0) = 0$ and $P(0) = P_0$.

Problem II-2. (Electricity and Magnetism)

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/(4\pi R^2)$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$, where there is no charge.

(i) Expand the potential in Legendre polynomials and show that the potential inside the spherical surface is, with the origin at the center of the sphere,

$$\Phi = \frac{Q}{8\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]\frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)\, ,$$

where here it is to be understood that $P_{-1}(\cos\alpha) = -1$. You may want to use

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}}\, P_\ell(\hat{x}\cdot\hat{x}\,')\, ,$$

$$\frac{dP_{\ell+1}(\mu)}{d\mu} - \frac{dP_{\ell-1}(\mu)}{d\mu} = (2\ell+1)P_\ell(\mu)\, .$$

(ii) What is the potential outside the surface?

(iii) Find the magnitude and direction of the electric field at the origin.

(iv) Discuss the limiting forms of the potential in part (i) when (a) the spherical cap is very small, and (b) the spherical cap is so large that the area with charge on it is a very small cap at the south pole.

Problem II-3. (Quantum Mechanics)

Consider a system whose state $|\psi(t)\rangle$ at time t and two observables $\boldsymbol{A}$ and $\boldsymbol{B}$ are given in some basis by the following matrices

$$|\psi(t)\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \boldsymbol{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Compute the eigenvalues a_1, a_2, a_3 of the operator $\boldsymbol{A}$, and similarly for $\boldsymbol{B}$, and exhibit the corresponding (normalized) eigenstates denoted as $|a_1\rangle, |a_2\rangle, |a_3\rangle$, and $|b_1\rangle, |b_2\rangle, |b_3\rangle$.

(b) What is the probability that a measurement of $\boldsymbol{A}$ at time t yields the value -1?

(c) Let us carry out a set of two measurements where $\boldsymbol{B}$ is measured first and then, immediately afterwards, $\boldsymbol{A}$ is measured. Find the probability of obtaining a value of 0 for $\boldsymbol{B}$ and a value of 1 for $\boldsymbol{A}$.

(d) Now we measure $\boldsymbol{A}$ first then, immediately afterwards, $\boldsymbol{B}$. Find the probability of obtaining a value of 1 for $\boldsymbol{A}$ and a value of 0 for $\boldsymbol{B}$.

(e) Compare the results of (c) and (d). Explain what should be expected in such a comparison before doing the computation.

(f) Which among the sets of operators $\{\boldsymbol{A}\}$, $\{\boldsymbol{B}\}$, and $\{\boldsymbol{A}, \boldsymbol{B}\}$ form a complete set of commuting operators (CSCO)?

Problem II-4. (Math Methods)

Part A. The Sokhotski-Plemelj formula is an equality between distributions of the form

$$\frac{1}{x \pm i\epsilon} = a\, P\Big(\frac{1}{x}\Big) + b\,\delta(x)\,, \tag{4.1}$$

where a and b are some constants.

(i) Explain how each term in (4.1) is defined as a distribution acting on test functions.

(ii) Evaluate both sides of (4.1) on the test functions,

$$\varphi_1(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}\,, \qquad \varphi_2(x) = \begin{cases} x & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}\,,$$

and use the result to determine the constants a and b.

Hint: The following integrals might be useful

$$\int_{-1}^{1} \frac{1}{x^2+\epsilon^2}\, dx = \frac{2}{\epsilon} \arctan\frac{1}{\epsilon}\,, \qquad \int_{-1}^{1} \frac{x^2}{x^2+\epsilon^2}\, dx = 2 - 2\epsilon \arctan\frac{1}{\epsilon}\,.$$

Part B. Let $f(x)$ be a complex valued function of the real argument, x. Suppose that $f(x)$ can be analytically continued to a function, $F(z)$, analytic in the upper-half-plane, including the real line. In other words, $F(z)$ is analytic for $\operatorname{Im} z \ge 0$ and $F(x) = f(x)$ for $x \in \mathbb{R}$. Assume also that

$$\lim_{|z|\to\infty} |F(z)| = 0\,, \qquad 0 \le \arg z \le \pi\,,$$

where the decay is as fast as needed.

(iii) Using the Cauchy integral formula, argue that

$$F(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}\, dx\,, \qquad \operatorname{Im} z_0 > 0\,, \tag{4.2}$$

where the integral is over the real axis and $z_0 = x_0 + i\epsilon$, $\epsilon > 0$, lies above the real axis.

(iv) By considering the limit of (4.2) as $z_0 \to x_0$ becomes real, show that $f(x_0)$ is given by the following principal value integral

$$f(x_0) = \frac{1}{\pi i}\, P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx\,, \qquad x_0 \in \mathbb{R}\,.$$

(v) Suppose that $f(x)$ satisfies

$$f(-x) = \overline{f(x)}\,, \tag{4.3}$$

e.g., $f(x)$ is a Fourier transform of a real function. Show that for such a "symmetric" function $f(x)$ as in (4.3), the real and imaginary parts, $f(x) = u(x) + iv(x)$, satisfy the *disperson relations*

$$u(x_0) = \frac{2}{\pi} P \int_0^{\infty} \frac{x v(x)}{x^2 - x_0^2}\, dx\,, \qquad v(x_0) = -\frac{2}{\pi} P \int_0^{\infty} \frac{x_0 u(x)}{x^2 - x_0^2}\, dx\,,$$

and hence are not independent.