Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I

Saturday, April 2, 2016

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________
Fill in your Sh-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Sh-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Write on the front side of each page only. Staple *separately* your answers to each problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

The total time allowed **3 hrs**.

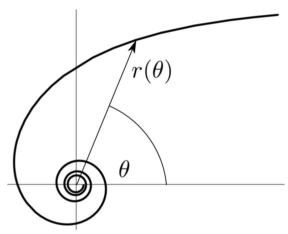
Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

Problem I-1. (Classicial Mechanics)



(i) Determine the central force, $\boldsymbol{F}(\boldsymbol{r}) = F(r)\,\widehat{\boldsymbol{r}}$, that allows a particle of mass m and angular momentum $\ell = mr^2\dot{\theta} > 0$ to move in a spiral orbit given by

$$r(\theta) = \frac{\alpha}{\theta}\,, \qquad \theta > 0\,,$$

where α is a positive constant.

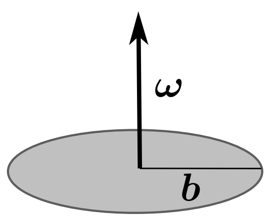
(ii) Show that the potential energy of the particle is of the form

$$U(r) = -\frac{k}{r^n}\,.$$

Determine the constants k and n in terms of α, m and ℓ. Assume that $U(r = \infty) = 0$.

(iii) Find the total energy, E, of the particle in this orbit.

Problem I-2. (Electricity and Magnetism)



A flat insulator in the shape of a circular disc of radius b has a constant surface charge density σ. The disc rotates with angular velocity ω about an axis perpendicular to the disc through its center. Calculate the magnetic field $\boldsymbol{B}$ for all points on the axis of rotation.

Hint: In order to perform the calculation, first divide the disc into a set of rings of radius s and width ds. For each ring, use the Biot-Savart law

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\,,$$

where $d\boldsymbol{\ell}'$ is a vector along the direction of current flow. After integrating over $d\boldsymbol{\ell}'$ for one ring, one should sum over the rings to get the total magnetic field, but in order to save time just write down the integral over the rings that should be evaluated without actually doing the integration explicitly. This remaining integral has the form

$$\int_0^b f(s)ds\,.$$

In other words, find the function $f(s)$.

Problem I-3. (Thermodynamics)

Consider a transformation where one mole of an ideal gas with the adiabatic constant γ is compressed reversibly from P_i to $P_f > P_i$ at constant temperature, then allowed to expand adiabatically and reversibly back to P_i. Imagine doing this transformation N times: calculate the corresponding change of entropy and energy, as well as the final temperature T_f if the initial temperature was T_i.

Problem I-4. (Quantum Mechanics)

Consider a quantum particle of mass m in three dimensions subject to the potential

$$V(\boldsymbol{r}) = \begin{cases} 0 & \text{if} \quad 0 < |\boldsymbol{r}| < R, \\ +\infty & \text{otherwise.} \end{cases}$$

This is the spherical analogue of the infinite barrier potential well.

(i) Which are the conserved quantities for the system and why?

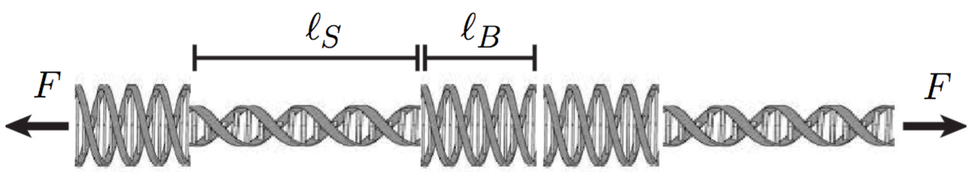
(ii) Find the ground state energy and eigenfunction of the system.

(iii) Using the thermodynamical definition of pressure, find the pressure p that the particle exerts – in its ground state – on the sphere's inner walls.

Hint: The Laplace operator in spherical coordinates can be decomposed into its radial and angular parts

$$\Delta\Psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) - \frac{1}{r^2}\frac{\boldsymbol{L}^2}{\hbar^2}\Psi\,.$$

Problem I-5. (Statistical Physics)



A segment of DNA can exist in two forms, the usual form called B and the stretched form called S. The usual form has length ℓ_B and has an associated energy E_B, and the stretched form has length ℓ_S and has an associated energy E_S. Consider a one-dimensional chain of DNA with N segments, N_B of form B and N_S of form S. The states of segments can be assumed to be independent of each other.

(i) Write down the partition function for the system at temperature T.

(ii) Assume that $E_S > E_B$ and $\ell_S > \ell_B$. Give an expression for the average length of the DNA molecule as a function of temperature at a fixed N.

(iii) What is the expected DNA chain length in the limit of high temperatures?

(iv) A tensile force F is applied. Find the partition function of the DNA chain for a finite force F.

(v) Derive the linear response function,

$$\left.\frac{\partial\langle L\rangle}{\partial F}\right|_{F=0},$$

where $\langle L\rangle$ is the average length of the chain.

Problem I-6. (Condensed Matter)

A donor impurity atom is embedded into a semiconductor with bulk dielectric constant ε and conduction band effective mass m^*.

(i) What are the ground-state radius and binding energy of the Bohr-like orbit of the electron captured by the donor? Please derive the relevant equations (rather than state them by memory) and use them to estimate numerically the values of these parameters for a typical semiconductor.

(ii) Use these estimates to argue why it is legitimate in the present derivation to employ the macroscopic dielectric function and the band effective mass.

(iii) Use your estimates to conclude whether impurity ionization is a significant effect at room temperature.

Problem I-7. (Experimental Physics)

In scanning tunneling spectroscopy, when a conducting tip is brought very close to a sample surface, the tunneling current which flows between the tip and the sample depends on the voltage difference between them.

(i) Describe how this technique can be used to determine the work function of the sample.

(ii) Let the sample be biased by a negative voltage V with respect to the tip. The measurement temperature is low enough so that the Fermi function cuts off sharply at the Fermi surface. Assume that the tip material has a flat density of states $\rho(\varepsilon)$ within the energy range of the Fermi surface that you wish to study, and assume the tunneling matrix element is independent of the energy difference between the tip and sample. Derive an expression for the net tunneling current.

Problem I-8. (Math Methods)

(i) Evaluate the Fresnel integral

$$\int_{-\infty}^{\infty} e^{-ix^2}\, dx\,.$$

Hint: Consider the contour integral of e^{-z^2} around the boundary of the circular sector $0 \le \theta \le \pi/4$, $0 \le r \le R$. You may assume that in the limit $R \to \infty$, the contribution from the circular arc goes to zero.

(ii) Use the result in (i) to show that $f(x) = x^{-1/2}$ is self-reciprocal under both Fourier cosine and sine transforms, that is

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{-1/2} \cos(xt)\, dx = t^{-1/2} \qquad \text{and} \qquad \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{-1/2} \sin(xt)\, dx = t^{-1/2}\,.$$

Assume $t > 0$.

Problem I-9. (Biophysics)

Bacteria and archaea generally rely on passive diffusion for supply of some essential molecules from the environment (e.g. O_2 if they are aerobic). Consider a spherical bacterium of radius R and density ρ, in an environment where the free concentration of O_2 at some distance far away from the cell surface $c(\infty) = c_0$, and where the diffusion coefficient of O_2 is D.

(i) Write down an expression for the number of molecules arriving per unit time (i.e. the rate dn/dt) at the cell surface. To make things a little simpler, assume that all molecules that arrive at the surface are swallowed up by the receptors, in other words: $c(R) = 0$ (*i.e.* the cell surface has machinery that acts as perfect absorbers of O_2). This rate is significant for many reactions in biophysics, from receptor-mediated signaling to polymerization of growing filaments.

(ii) A given biomass of cells has a characteristic burn rate of oxygen b; that is how many molecules of O_2 are consumed per kg per s to satisfy the energy demand of the cells. What condition does this impose on the radius R of the cell? You just found out why bacteria are small (i.e. generally a few micrometers in size).

Problem I-10. (Astrophysics)

The total gravitational energy of a spherically-symmetric astronomical object (star, white dwarf, globular star cluster, spherical galaxy, *etc.*) is

$$E_{\text{grav}} = -\int_0^M \frac{Gm}{r} dm \; . \tag{10.1}$$

Here, M and R are total mass and radius of the object, respectively, and the running variable m (*i.e.*, the mass contained in a spherical subshell of radius r) can be more precisely written as an indexed variable M_r

$$m = M_r = 4\pi \int_0^r s^2 \rho(s) ds \, , \tag{10.2}$$

In astrophysics it is well known that approximately,

$$E_{\text{grav}} = -\alpha \frac{GM^2}{R} \; . \tag{10.3}$$

with α being a constant of order unity.

Your task:

For very special density distributions, α can be computed exactly. Assume a density distribution of the form

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R} \right)^2 \right] \; . \tag{10.4}$$

(i) Determine the central density ρ_c.

(ii) Determine the constant α.

Problem I-11. (Special Relativity)

A rocket ship of proper length L leaves the earth vertically at constant speed $(4/5)c$. A light signal is sent vertically after it from the same launch spot. The light signal arrives at the rocket's tail at $t = 0$ according to both rocket and earth based clocks. When does the signal reach the nose of the rocket according to (a) the rocket clock; (b) the earth clock? *(For simplicity, ignore the motion of the earth.)*

Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, April 2, 2016

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Lg-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Do not write on the back of page. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

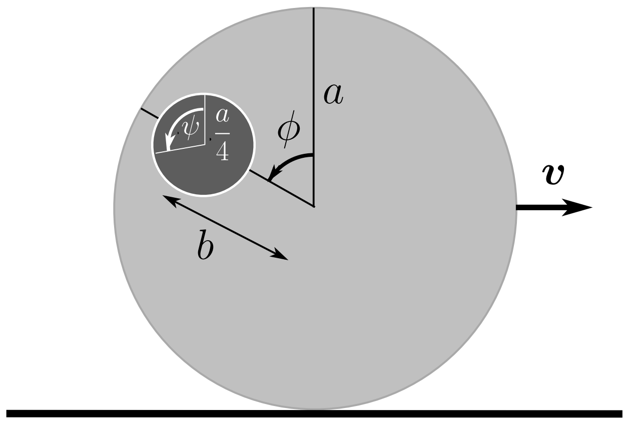
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)



A solid cylinder with uniform mass density, mass M, and radius a can roll without slipping on a rough horizontal surface.

Suppose that a hole of radius $a/4$ is bored through the entire length of the cylinder, parallel to its axis. The line joining the center of the hole to the center of the cylinder is of length $a/4 < b < 3a/4$ and it makes an angle ϕ with the upward vertical. One can show (don't check it) that the moment of inertia of the deformed cylinder about the *instantaneous axis of rotation* is

$$I = \alpha + \beta \cos \phi \, ,$$

where α and β are constants, whose explicit dependence on a, b and M need not be given.

Suppose now that the hole is entirely filled with a solid cylinder of mass M and radius $a/4$, whose length is that of the first cylinder. Assume that there is no friction in the hole so that the smaller cylinder can rotate freely within the hole. This degree of freedom is parametrized by the angle ψ measured from the upward vertical as shown in the figure and is subject to the constraint $f \equiv \dot{\psi} - \dot{\phi} \equiv 0$. The entire system is initially at rest when a horizontal force is applied evenly and perpendicularly to the central axis of the larger cylinder. Consequently, the entire system rolls without slipping in a straight line with speed v. The applied force is adjusted (continually) to ensure that v remains constant; this force in encoded in a further constraint, $g \equiv \dot{\phi} + \omega \equiv 0$ where $\omega = v/a$.

(i) At this stage, what is the kinetic energy of the joint system about the *instantaneous axis of rotation*?

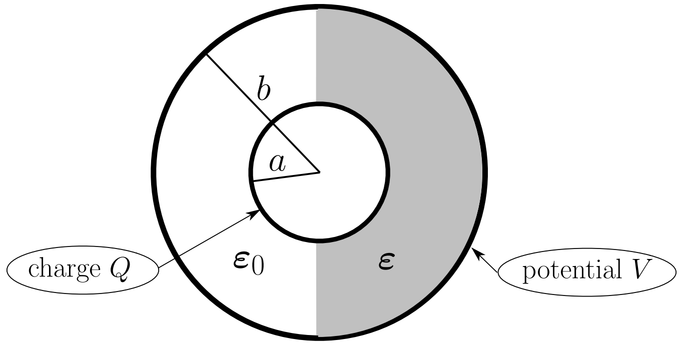
(ii) What is the Lagrangian, L, of the joint system? Include the terms λf and μg in the Lagrangian, where λ and μ are Lagrange multipliers.

(iii) Write down the equations of motion in ϕ and ψ and display the generalized forces Q_ϕ and Q_ψ on the right hand sides of these equations.

(iv) What are the canonical momenta p_ϕ and p_ψ?

(v) Is the Hamiltonian, $H = \dot{\phi} p_\phi + \dot{\psi} p_\psi - L$, a constant of motion? Why or why not?

Problem II-2. (Electricity and Magnetism)



The space between two concentric *conducting* spheres of inner and outer radii a and b, respectively, is half-filled by a hemispherical shell of dielectric with dielectric constant $\varepsilon/\varepsilon_0$. The inner sphere carries charge Q, and the outer sphere is held at a potential V.

(i) Determine the potential everywhere.
(ii) Determine the free charge density everywhere.
(iii) Determine the polarization (bound) charge density everywhere.
(iv) Suppose that the outer sphere is now grounded. How much charge flows to it?

Problem II-3. (Math Methods)

The elements of an $n \times n$ matrix $\mathbb{A} = (A_{ij})$ are given by

$$A_{ij} = \begin{cases} z & \text{for } i = j \\ 1 & \text{for } i \neq j \end{cases}$$

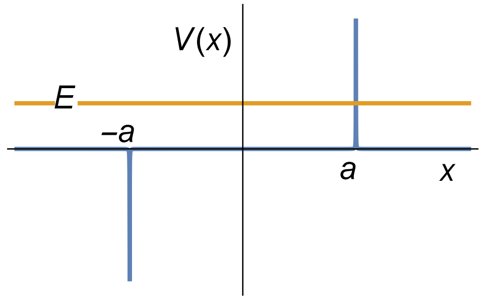
where $z \in \mathbb{C}$.

(i) Show that $\mathbb{A}$ satisfies a matrix equation of the form

$$\mathbb{A}^2 + a\,\mathbb{A} + b\,\mathbb{I} = 0\,,$$

where $\mathbb{I}$ is the $n \times n$ unit matrix and a, b are constants that depend on n and z. Find the constants a and b as functions of n and z.
(ii) For which values of z is the matrix $\mathbb{A}$ normal?
(iii) Argue that $\mathbb{A}$ has at most two distinct eigenvalues. Calculate those eigenvalues and determine their degeneracies.
(iv) Find the determinant of $\mathbb{A}$ as a function of n and z.
(v) For the case $n = 4$ and $z = -1$, find a real orthogonal matrix, $\mathbb{O}$, that diagonalizes matrix $\mathbb{A}$. What is the inverse matrix, $\mathbb{O}^{-1}$?
(vi) What is the freedom in choosing the matrix $\mathbb{O}$ in (v), *i.e.* what is the most general real orthogonal matrix that diagonalizes matrix $\mathbb{A}$?

Problem II-4. (Quantum Mechanics)



Consider a particle of mass m and energy $E = \hbar^2 k^2/2m$ moving in the 1-dimensional non-relativistic potential

$$V(x) = \frac{\hbar^2 \mu^2}{2m}\left[-\delta\left(\frac{x}{a}+1\right)+\delta\left(\frac{x}{a}-1\right)\right],$$

made up of attractive and repulsive delta functions as shown in the figure.

Assuming that the particle initially approaches from the left, compute the transmission (T) and reflection (R) coefficients by following the steps (i)-(vi) below.

(i) Describe in words your strategy for computing T and R and set up the appropriate equations before you solve them. In particular discuss the boundary conditions.

(ii) Give the analytic solutions for the wavefunction $\psi(x)$ in various parts of position space x.

(iii) Impose continuity conditions appropriate for delta function potentials.

(iv) Solve analytically for $T(k)$ and $R(k)$ as a function of $k > 0$.

(v) Discuss the behavior of (T, R) for small and large limits of the parameters k and a. Give a physical reason that anticipates this behavior.

(vi) Determine if there are some special values of energy at which the transmission is 100% and there is no reflection (Ramsauer effect).