Autumn 2013

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

September 17, 2013

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Electrostatics

(a) A positive point charge q is located outside of a spherical conductor of radius a and with positive charge Q. Compute the force between the charge q and the conductor, as a function of the distance r of the charge from the center of the sphere. (By assumption, $r > a$.) Check that your formula reduces to the expected result if $r \gg a$.

(b) Consider a uniformly charged cylinder of radius a and charge density ρ. Take the cylinder to have length $L \gg a$. A cylindrical cavity of radius b and length L is made inside the uniformly charged cylinder (with $a > b$) such that their centers are separated by a distance d. Assume that the cavity exists entirely inside the original cylinder. Find the electric field at an arbitrary point inside the cavity. (You may neglect any effects due to the fact that L is a finite length).

1. Electrostatics - Solution

(a) We treat this problem by superposition. First, consider the case of a grounded sphere (which has no surface charge). Then, the problem as stated is equivalent to a problem of a charge q at a distance r from the center of the sphere and an image charge $q' = -aq/r^2$. Then to account for the charge Q on the sphere, we add a charge $Q - q'$ uniformly on the surface of the spherical conductor. Then, we can equivalently put $Q - q'$ as a point charge at the origin. Considering these three point charges, the force on the charge q is then easily computed by Coulombs law.

$$F = \frac{qq'}{(r-r')^2} + \frac{(Q-q')q}{r^2} = \frac{qQ}{r^2}\left(1 - \frac{q}{Q}\frac{a^3(2r^2-a^2)}{r(r^2-a^2)^2}\right).$$

For $r \gg a$, we have $F = qQ/r^2$, of course.

(b) We solve this problem again by using the principle of superposition. Namely, start with a cylinder of radius a and uniform charge density ρ. By superposition, add to it a cylinder of radius b with uniform charge density $-\rho$. The result is precisely a cylindrical cavity of radius b inside the uniformly charged cylinder of radius a.

Denote the position of the point under consideration $\mathbf{r}$, measured from the center of a uniformly charged cylinder. The electric field is

$$\mathbf{E} = 2\pi\mathbf{r}\rho$$

The superposition in our problem leads to

$$\mathbf{E} = 2\pi\rho\mathbf{r} - 2\pi\rho(\mathbf{r} - \mathbf{d}) = 2\pi\rho\mathbf{d}$$

2. Quantum Dynamics

Consider a quantum mechanical system governed by the Hamiltonian

$$H = \epsilon_a |a\rangle\langle a| + \epsilon_b |b\rangle\langle b|$$

where ϵ_a and ϵ_b are known real numbers and the states $|a\rangle$ and $|b\rangle$ form a complete-orthonormal set. The state of the system at time t, $|\psi(t)\rangle$, satisfies the dynamical equation of motion

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle.$$

Assume that at the initial time $t = t_0 = 0$ the system is in the state

$$|\psi(0)\rangle = a_0|a\rangle + b_0|b\rangle$$

where a_0 and b_0 are real numbers related by normalization of the initial state. Determine the probability that the system is in the initial state $|\psi(0)\rangle$ at time t?

2. Quantum Dynamics - Solution

It is important to note that the states a and b are energy eigenstates

$$H|a\rangle = \epsilon_a|a\rangle$$

$$H|b\rangle = \epsilon_b|b\rangle.$$

The formal solution of the dynamical equation of motion is given by

$$|\psi(t)\rangle = e^{-itH/\hbar}|\psi(0)\rangle$$

$$= e^{-itH/\hbar}\left(a_0|a\rangle + b_0|b\rangle\right)$$

$$= a_0 e^{-it\epsilon_a/\hbar}|a\rangle + b_0 e^{-it\epsilon_b/\hbar}|b\rangle.$$

The probability of interest is given by

$$P(t) = |\langle\psi(0)|\psi(t)\rangle|^2.$$

We first compute

$$\langle\psi(0)|\psi(t)\rangle = a_0^2 e^{-it\epsilon_a/\hbar} + b_0^2 e^{-it\epsilon_b/\hbar}$$

then we have

$$P(t) = \left(a_0^2 e^{-it\epsilon_a/\hbar} + b_0^2 e^{-it\epsilon_b/\hbar}\right)^* \left(a_0^2 e^{-it\epsilon_a/\hbar} + b_0^2 e^{-it\epsilon_b/\hbar}\right)$$

$$= a_0^4 + b_0^4 + 2a_0^2 b_0^2 \cos(t[\epsilon_a - \epsilon_b]/\hbar).$$

Notice that the result is properly normalized if the initial state is properly normalized:

$$a_0^2 + b_0^2 = 1$$

leads to

$$P(0) = 1.$$

3. Pseudopotential

A charged particle cannot be confined in three dimensions by a static electric field. However a carefully chosen dynamical field can achieve such a confinement, this effect is used, for example in ion traps. In this question you will be asked to prove this statement.

(a) By considering a local expansion of a static electrostatic potential $\phi(x, y, z)$, about a minimum (a Taylor series of the potential in Cartesian coordinates about a local minimum, which you may take to be at the origin) prove that a static electric field cannot confine a charged particle in three-dimensions.

(b) Derive the equation of motion for a charged particle (charge e and mass m) moving in potential of the form:

$$V(x,t) = \frac{a}{2}x^2 \cos(\omega t)$$

(c) Take the motion of a particle to consist of a small, rapidly oscillating part $\epsilon(t)$ whose mean value over a period $2\pi/\omega$ is zero and a slower 'average' motion $X(t)$ that changes slowly (compared to $2\pi/\omega$) in time: $x(t) = \epsilon(t) + X(t)$. Use this to separate the equation of motion into two separate equations: one for the average motion $\bar{x}(t) = X(t)$ and one for the rapid oscillations.

(d) From this read off an expression for the effective (time-averaged) potential that smooth motion of the particle sees.

(e) Can you provide a physical argument for the origin of the effective potential?

3. Pseudopotential - Solution

The equation of motion is:

$$m\ddot{x} = -kx\cos(\omega t)$$

Writing $x(t) = X(t) + \epsilon(t)$ we have:

$$m\ddot{X} + m\ddot{\epsilon} = -kX\cos(\omega t) - k\epsilon\cos(\omega t)$$

Which splits into a rapidly oscillating term:

$$m\ddot{\epsilon} = -kX\cos(\omega t)$$

with approximate solution:

$$\epsilon = \frac{k}{m\omega^2}X\cos(\omega t)$$

and a n equation for the average motion:

$$m\ddot{X} = -\frac{k^2}{m\omega^2}X\cos^2(\omega t)$$

which give a period-averaged equation:

$$m\ddot{X} = -\nabla\left(\frac{k^2}{4m\omega^2}X^2\right)$$

From which we can read off

$$V(X) = \frac{k^2}{4m\omega^2}X^2$$

The mass of the particle is the essential ingredient. If the particle is moving away from the minimum, it will overshoot because of its inertia and when the potential flips, feel a greater restoring force than it did a repulsive one on its way out of the minimum. The effect is reversed when the particle is headed towards the minimum, providing an effective confinement.

4. Bose Einstein Condensation

For a simple noninteracting quantum system of Bose particles with mass m we have in d-dimensions the relation in the grand canonical ensemble

$$N = N_{con} + V \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \tag{1}$$

where N is the average number of particles in the system, $\beta = 1/(k_B T)$ where T is the temperature, μ is the chemical potential (which for stability is less than or equal to zero), V is the volume, the single-particle energies are given by

$$\epsilon_k = \frac{(\hbar k)^2}{2m},$$

and N_{con} is the average number of zero-energy particles in a Bose-Einstein condensate. $N_{con} = 0$ if we can find a chemical potential that is negative and satisfies

$$N = +V \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \tag{2}$$

Show that, in two dimensions, we do not have Bose-Einstein condensation for $T > 0$.

4. Bose Einstein Condensation - Solution

In three-dimensions, for fixed $n = N/V$, as one lowers the temperature from high temperatures one can solve Eq.(2) to obtain an increasing negative chemical potential. There is no condensate and $N_{con} = 0$. Eventually, as one lowers T, one finds a temperature $T_0 > 0$ where $\mu(n, T_0) = 0$. For $T < T_0$ the chemical potential remains zero, it can not become positive, and for Eq.(1) to remain valid we require that N_{con} increase as temperature decreases. One has Bose-Einstein Condensation in three dimensions.

For two-dimensions we can find a negative chemical potential for all $T > 0$ and satisfy Eq.(2) with $N_{con} = 0$.

We have in $d = 2$

$$N = V \int \frac{d^2k}{(2\pi)^2} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}.$$

Working in terms of the density and doing the angular integration

$$n = N/V = \int_0^\infty \frac{kdk}{(2\pi)} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}.$$

Change variables to

$$x = \beta\epsilon_k$$

to obtain

$$\begin{aligned} n &= \frac{m}{2\pi\beta\hbar^2} \int_0^\infty dx \frac{1}{e^{(x-\beta\mu)} - 1} \\ &= \frac{m}{2\pi\beta\hbar^2} \int_0^\infty dx \frac{d}{dx} ln\left(1 - e^{-x+\beta\mu}\right) \\ &= -\frac{m}{2\pi\beta\hbar^2} ln\left(1 - e^{\beta\mu}\right). \end{aligned}$$

We can solve this for $\beta\mu$ with the result

$$\beta\mu = ln\left(1 - e^{-2\pi\beta n\hbar^2}\right).$$

We find $\mu < 0$ for $T > 0$ and $N_{con} = 0$ for two dimensions.

5. Specific Heat of Classical Hard Spheres

Compute the specific heat at constant volume for a collection of N identical classical hard spheres each with radius σ. Assume a pair potential $U(r)$, such that $U(r)$ is zero for $r > \sigma$ and infinite for $r < \sigma$. Assume the system is in thermal equilibrium in a volume V at temperature T.

5. Specific Heat of Classical Hard Spheres - Solution
A system of classical hard spheres is governed by a hamiltonian

$$H = K + V_T$$

where the kinetic energy is given by

$$K = \sum_{i=1}^{N} \frac{p_i^2}{2m}$$

with the momentum of particle i given by $\mathbf{p}_i$ with mass m.
The potential energy is given by

$$V_T = \frac{1}{2} \sum_{i \neq j} V(r_i - r_j)$$

with the position of particle i given by $\mathbf{r}_i$. The pair potential is given by

$$V(r) = \infty \text{ for r} < \sigma$$

and zero otherwise. Define the partition function

$$Z[\beta, N, V] = \frac{1}{N!} \int d^3p_1 d^3p_2 \ldots d^3p_N \int d^3r_1 d^3r_2 \ldots d^3r_N e^{-\beta H}$$

where $\beta = 1/k_B T$ and the spatial integrations are over a large volume V. The average energy is given by

$$\langle H \rangle = -\frac{\partial}{\partial \beta} ln Z[\beta, N, V].$$

In this case we can write

$$Z[\beta, N, V] = Z_K[\beta, N, V] Z_{V_T}[\beta, N, V]$$

where

$$Z_K[\beta, N, V] = \int d^3p_1 d^3p_2 \ldots d^3p_N e^{-\beta K}$$

and

$$Z_{V_T}[\beta, N, V] = \frac{1}{N!} \int d^3r_1 d^3r_2 \ldots d^3r_N e^{-\beta V_T}.$$

The key observation is that $Z_{V_T}[\beta, N, V]$ is independent of temperature

$$Z_{V_T}[\beta, N, V] = Z_{V_T}[N, V].$$

This means

$$\langle H \rangle = -\frac{\partial}{\partial \beta} ln Z_K[\beta, N, V].$$

But we recognize this as the ideal gas contribution to the average energy:

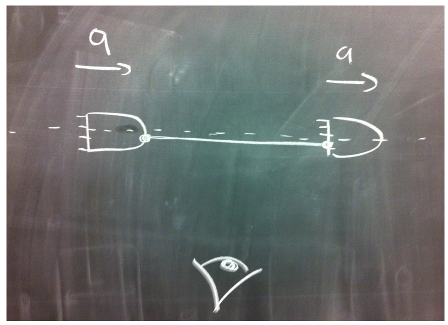
$$E = \langle H \rangle = \frac{3}{2} N k_B T.$$

The specific heat at constant volume is

$$C_V = \frac{dE}{dT} = \frac{3}{2} N k_B.$$

6. Bell's Spaceship Paradox

Consider two spaceships SA and SB, at rest in a common inertial frame of reference. An external observer, equidistant from both ships, sends a signal to both at the same time to turn their engines on and start accelerating in the same direction along the axis that passes through SA and SB, with the same constant acceleration (in each of SA and SB's frames) a.



(a) Derive an expression for the transformation of velocities between two inertial frames of reference moving with relative velocity v. Use this to compute the expression for the transformation of the acceleration of a particle viewed from two inertial frames moving with relative velocity v:

$$a = \gamma(v)^3 A$$

where a and A are the accelerations in the two frames.

(b) Use this to show that the equation:

$$x^2 - c^2t^2 = \frac{c^4}{a^2}$$

defines the the world line of each spaceship in the reference frame of the external observer. Sketch the two world lines. Does the distance between the spaceships remain the same? You may justify your answer either using the world lines or other arguments.

(c) Suppose connected by a fragile massless silk string. Does the string break? Explain your answer. [Note: you may use the result for the world line given above or other considerations.]

6. **Bell's Spaceship Paradox - Solution**

(a) Using coordinates x, t in frame f and X, T in frame F we have:

$$\begin{aligned} x &= \gamma(v)\,(X - vT) \\ t &= \gamma(v)\left(T - \frac{v}{c^2}X\right) \end{aligned}$$

From this we obtain

$$\begin{aligned} dx &= \gamma(v)\,(dX - v dT) \\ dt &= \gamma(v)\left(dT - \frac{v}{c^2}dX\right) \end{aligned}$$

which leads to

$$\frac{dx}{dt} = \frac{\frac{dX}{dT} - v}{1 - \frac{dX}{dT}\frac{v}{c^2}}$$

If we write $u = dx/dt$ and $U = dX/dT$ and take another derivative we get

$$\frac{du}{dt} = \frac{dU}{dT}\frac{1}{\gamma(v)^3}\frac{1}{\left(1 - \frac{Uv}{c^2}\right)^3}$$

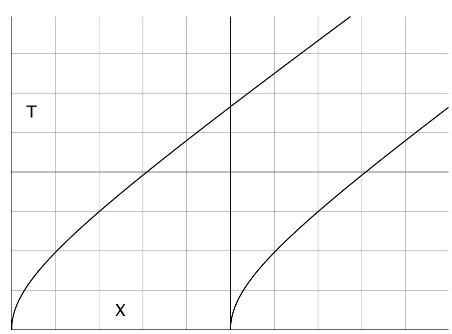
This is the general result.

To specialize to our case, let f be the instantaneous rest frame of SA and F be the frame of the external observer. SA measures its own acceleration to be $a = du/dt =$ const while the external observer measures its acceleration as $A = dU/dT$. Under these conditions, the speed U of the ship as measured in F also corresponds to the instantaneous relative velocity between it and f; that is, $U = v$. The general result then reduces to

$$a = \gamma(v)^3 A$$

(b) For this situation the wordline of each ship is a parabola given by

$$c^2T^2 = (X - X_0)^2 + \frac{2c^2}{a^2}(X - X_0)$$



The distance between the spaceships remains constant in the frame of the observer at rest. This can be seen from the diagram but also from noting that since both ships have exactly the same trajectory, they must remain equidistant.

(c) Considering the string, as it moves relative to the observer with a velocity $v(T)$ along it's axis, it would be observed to contract by a factor $\gamma(T)$. The separation between the spaceships however remains fixed in frame F, therefore the string will feel a tension that will break it.

An alternative explanation can be obtained by considering the point of view of a co-moving observer based on spaceship SA. From that point of view, the string does not undergo Lorentz contraction. However, this observer will see the separation between the spaceships increase. The ships don't start moving simultaneously in that frame. Ship SB accelerates first.

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Ph.D. CANDIDACY EXAMINATION

Day 2

September 18, 2013

(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

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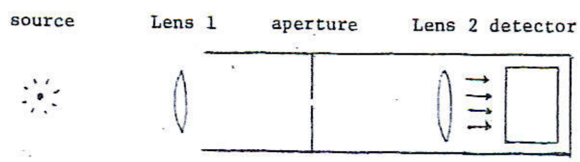
7. Short Answer

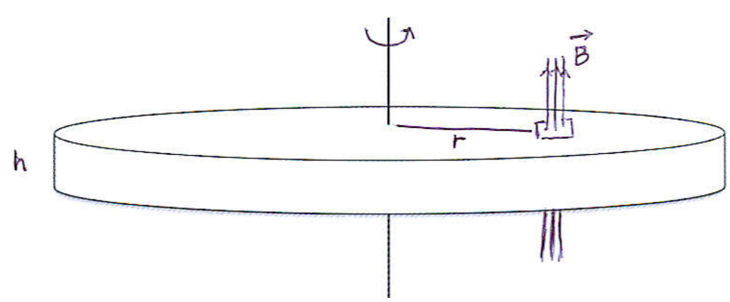
(a) The luminosity of a star represents the radiant power from its surface. Given that the luminosity of the sun is $L_\odot \approx 4 \times 10^{26}$ W, its radius is $R_\odot \approx 7 \times 10^5$ km, and its surface temperature is $T_\odot \approx 5800$ K. Estimate the radius of a star that has a luminosity of $10^4 L_\odot$ and a surface temperature of $0.6 T_\odot$.

(b) You need to design an optical system to focus light from a 1-cm-diameter source into a beam of parallel light rays directed at a detector. To reduce background light you use a 0.5-cm-diameter aperture between two lens and place the second lens in a box.



Your design constraints are that Lens 1 must be 10 cm from the source and be 5 cm in diameter; and the separation between the two lenses must be 20 cm. (1) Determine the needed focal lengths of the two lenses. (2) If the source emits light isotropically, estimate the fraction of the emitted light intensity that will be incident on the detector.

(c) A circular disk has a thickness h of 1 cm, an electrical conductivity of $\sigma = 10^8\ \Omega^{-1} \cdot \mathrm{m}^{-1}$, and is rotating about an axis through its center with an angular speed ω of 10 rad/s as indicated in the figure.



A homogeneous magnetic field of 10 T is applied perpendicular to the disk and is confined to a 0.1-cm $\times$ 0.1-cm area a distance $r = 0.1$ m from the center of the disk. The effect of introducing this magnetic field is to slow the rotation of the disk. Estimate the value of the initial torque that acts to slow the disk's rotation. [Ignore the resistance of the return path.]

7. Short Answer - Solutions

(a) Treating the star as a perfectly spherical blackbody the luminosity is given by Stefan's law $L = 4\pi R^2 \sigma T^4$. Writing the properties of the star in solar units, $R = rR_\odot$ and $T = \tau T_\odot$, we have

$$L = 4\pi\sigma R_\odot^2 T_\odot^4 (r^2\tau^4) = r^2\tau^4 L_\odot.$$

Therefore,

$$r = \left(\frac{L}{\tau^4 L_\odot}\right)^{1/2} \approx 278 \implies R \approx 280 R_\odot.$$

(b) We can use the thin lens equation to get focal length of Lens 1, $f_1 = \frac{d_i d_o}{d_i + d_o}$, where the object distance is given ($d_o = 10$ cm). We want to focus the image on the aperture so the image distance can be related to the object distance from the magnification $m = -h_i/h_o = -d_i/d_o = -0.5$, giving $d_i = 0.5 d_o = 5$ cm. From this we obtain $f_1 = (5 \text{ cm})(10 \text{ cm})/(15 \text{ cm}) = 3.33$ cm. To get parallel rays from Lens 2, its distance from the aperture should equal its focal length. Therefore, $f_2 = 20 \text{ cm} - 5 \text{ cm} = 15$ cm.

We can estimate the fraction, x, of light intensity as the fraction of the area of the lens relative to the area over which the intensity is spread at that distance. The diameter D_1 is given so,

$$x = \frac{\pi D_1^2/4}{4\pi d_o^2} = \frac{D_1^2}{16 d_o^2} = \frac{1}{64}.$$

(c) The part of the disk in the magnetic field can be treated as a wire moving through the field. A motional emf, $|\mathcal{E}| = B\ell v = B\ell r\omega$, will be induced across this region. The resistance is $R = \rho L/A = 1/h\sigma$. The resulting outward induced current is given by Ohm's law $i = |\mathcal{E}|/R = \ell h r \sigma B \omega$. The force on this current-carrying wire is $F = i\ell B$ and the magnitude of the torque is $\tau = rF$. Thus,

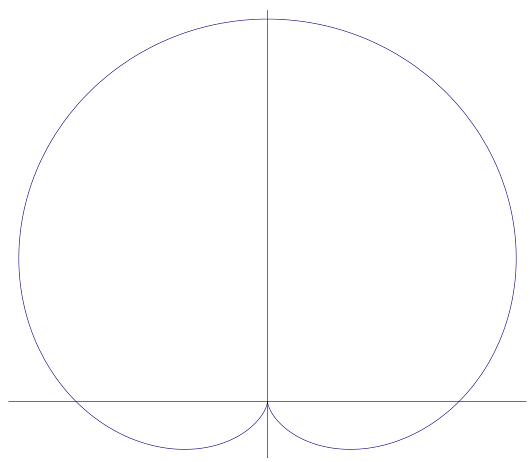
$$\begin{aligned}
\tau &= \ell^2 r^2 B^2 h \sigma \omega \\
&= (10^{-6} \text{ m}^2)(10^{-2} \text{ m}^2)(10^2 \text{ T}^2)(10^{-2} \text{ m})(10^8\ \Omega^{-1} \cdot \text{m}^{-1})(10 \text{ rad/s}) \\
&= 10 \text{ N} \cdot \text{m}
\end{aligned}$$

8. Cardioid

Consider a particle which moves along the cardioid shown below. We can parametrize this cardioid in a plane by the equation

$$r(\phi) = r_0(1 + \sin(\phi))$$

where r and ϕ are the standard radial and angular polar coordinates. Determine the central force, $F(r)$, that results in this motion in terms of conserved quantities.



8. Cardioid - Solution

We can determine the force $F(r)$ by first finding the potential $V(r)$. The total energy of the particle, $E = T + V$, is conserved and can be written as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + V(r)$$

For a central force problem, the angular momentum perpendicular to the plane of motion is conserved according to

$$l = mr^2\dot{\phi} = \text{constant.}$$

Because the orbit, $r(\phi)$, is specified, we can compute the time derivative of r

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = r_0\cos(\phi)\dot{\phi} = \frac{lr_0}{mr^2}\cos(\phi).$$

This allows us to express the kinetic energy in terms of l and ϕ, giving

$$E = \frac{(lr_0)^2}{2mr^4}\cos(\phi)^2 + \frac{l^2}{2mr^2} + V(r).$$

We seek to use this to obtain $V(r)$. Therefore, to eliminate ϕ we use $\cos^2\phi = 1 - \sin^2\phi$ together with the relation $\sin^2\phi = \left(\frac{r}{r_0} - 1\right)^2$ from the given orbit. Doing this, the energy becomes

$$E = \frac{l^2 r_0}{mr^3} + V(r).$$

Therefore, the potential is given by

$$V(r) = E - \frac{l^2 r_0}{mr^3}.$$

Noting that E is constant, the corresponding force is

$$F(r) = -\frac{dV}{dr} = -\frac{3l^2 r_0}{mr^4}.$$

9. Spinor in a Magnetic Field

Consider a positively-charged spin-1/2 particle in an external magnetic field, governed by the Hamiltonian:

$$H = H_0\mathbf{I} - \gamma\mathbf{B}\cdot\mathbf{S},$$

where $\mathbf{I}$ is the identity operator in spin-space, $\mathbf{S}$ is the vector of spin-1/2 spin matrices. γ is a constant (> 0 for a positively-charged particle). H_0 is spin-independent and is independent of the magnetic field $\mathbf{B}$. We assume H_0 possesses exactly one eigenvalue, which is denoted by E.

(a) If the magnetic field is given by $\mathbf{B} = B\hat{z}$ (where $B > 0$), determine the energy eigenstates and eigenvalues of H.

(b) Assume that the magnetic field is given by $\mathbf{B} = B\hat{z}$ for time $t < 0$. The system is initially observed to be in a spin-up state. At $t = 0$, a time-dependent perturbation is added by modifying the magnetic field. The new magnetic field for $t > 0$ is given by:

$$\mathbf{B} = b(\hat{x}\cos\omega t - \hat{y}\sin\omega t) + B\hat{z},$$

where $b > 0$. Using first-order time-dependent perturbation theory, derive an expression for the probability that the system will be found in a spin-down state at some later time $t = T$. For what range of values of ω is this result reliable?

Note: for a system prepared in a state $|\Psi(t=0)\rangle = c_a(0)|a\rangle + c_b(0)|b\rangle$, to the first order in a time-dependent perturbation $H'(t)$,

$$c_b(t) = c_b(0) - \frac{i}{\hbar}\int_0^t \langle b|H'(t')|a\rangle e^{it'(E_b-E_a)/\hbar}dt'$$

9. Spinor in a Magnetic Field - Solution

(a) Eigenstates of H can be chosen to be simultaneous eigenstates of $\mathbf{S}^2$ and S_z.

$$H_0|m\rangle = E|m\rangle, \; S_z|m\rangle = m\hbar|m\rangle, \; m = \pm\frac{1}{2}$$

We have

$$H|m\rangle = (H_0\mathbf{I} - \gamma B S_z)|m\rangle = (E - m\hbar\gamma B)|m\rangle \equiv E_m|m\rangle.$$

(b) Denote the probability amplitude for the spinor to be in state $|m\rangle$ at time t as $c_m(t)$. We have the evolution of state of the spinor $|S(t)\rangle$ as

$$|S(t)\rangle = \sum_m c_m(t)e^{-\frac{i}{\hbar}E_m t}|m\rangle.$$

From first order time dependent perturbation theory, we have

$$c_{-1/2}(t) = c_{-1/2}(0) - \frac{i}{\hbar}\int_0^t \langle -\frac{1}{2}|H^{(1)}(t')|\frac{1}{2}\rangle e^{i(E_{-1/2}-E_{1/2})t'/\hbar}dt',$$

where

$$H^{(1)}(t) = -\gamma b(S_x\cos\omega t - S_y\sin\omega t) \;\rightarrow\; \langle -\frac{1}{2}|H^{(1)}(t)|\frac{1}{2}\rangle = -\frac{1}{2}\gamma b e^{-i\omega t}.$$

Using initial condition $c_{-1/2}(0) = 0$ (spin up initially), we have

$$c_{-1/2}(t) = \frac{1}{2}i\gamma b\int_0^t e^{i(\gamma B-\omega)t'}dt' = \frac{\gamma b}{2(\gamma B - \omega)}\left[e^{i(\gamma B-\omega)t} - 1\right],$$

and the corresponding probability to the order b^2 is

$$P_{-1/2}(T) = |c_{-1/2}(T)|^2 = \frac{\gamma^2 b^2}{(\gamma B - \omega)^2}\sin^2\left(\frac{1}{2}(\gamma B - \omega)T\right).$$

The validity of perturbation theory requires

$$\left|\frac{\gamma b}{(\gamma B - \omega)}\right| \ll 1.$$

In particular, external perturbation frequency ω can not be too close to the resonance frequency γB. And, b needs to be small.

10. Ferromagnets

In ferromagnets a key quantity is the magnetization density $\mathbf{m}$. Assume we have zero external applied magnetic field. Then by definition the equilibrium averaged magnetization density is zero in a high temperature paramagnetic phase and is nonzero in a low temperature ferromagnetic phase.

Let us consider the free-energy density

$$f = \frac{r}{2}\sum_{i=1}^{3} m_i^2 + \frac{u}{4}\left(\sum_{i=1}^{3} m_i^2\right)^2 + \frac{1}{4}\sum_{i=1}^{3} v_i m_i^4$$

where m_i is the ith ($i = x$, y, and z) component of the magnetization density,

$$r = a(T - T_c),$$

where T is temperature, a, u, and T_c are positive, and

$$v_x > v_y > v_z > 0.$$

The parameters a, u, T_c and v_x, v_y, v_z are material parameters determined emperically.

The equilibrium configurations of the system are determined by minimizing the free energy at constant temperature. Determine an equilibrium value of the magnetization density as a function of temperature. What is the physical interpretation of T_c?

10. Ferromagnets - Solution

The first step in minimizing the free energy is to find the zeros of the derivatives of the free energy with respect to the magnetization density:

$$\frac{\partial f}{\partial m_j} = r m_j + u \sum_{i=1}^{3} m_i^2 m_j + v_j m_j^3 = 0.$$

The task is to find any equilibrium value of $\mathbf{m}$. Almost by inspection, we can identify four solutions

$$m_i = 0$$

and

$$m_x = 0$$

$$m_y = 0$$

and

$$r + (u + v_z) m_z^2 = 0,$$

which corresponds to ordering in the z-direction for r negative. There are similar solutions for ordering in the x and y directions. For $T > T_c$ the only physical solution is $m_i = 0$. For $T < T_c$ one has four possible solutions. The equilbrium soltion is the solution that minimizes the free energy. For the paramagnetic phase the free-energy is zero. For the ordered solutions in the i-direction the free energy is given by

$$f = \frac{r}{2} \frac{(-r)}{(u + v_i)} + \frac{(u + v_i)}{4} \left(\frac{-r}{(u + v_i)} \right)^2$$

$$f = -\frac{1}{4} \frac{r^2}{(u + v_i)}$$

and all of these solutions have a lower free energy than the disordered solution. The low temperature equilibrium ordering is along the z direction because v_z is the smallest cubic coupling.

The magnetization density in the ordered phase is $m_x = m_y = 0$ and

$$m_z = \pm \sqrt{\frac{-r}{(u + v_z)}}.$$

The $\pm$ indicates the solution is doubly degenerate.

11. Kinematical Endpoint

Consider a particle A decaying into 3 particles B, C and D. The masses of particle A and B are m_A and m_B, respectively. Particles C and D are massless. Denote the 4-momenta of these particles as p_A, p_B, p_C and p_D. We work in the unit of $c = 1$ in this problem.

The invariant mass of this system is defined as $m_{CD} = \sqrt{(p_C + p_D)^2}$. Write the invariant mass in terms of energies, momenta, and the masses of the particles. What is the maximal value for m_{CD} in terms of m_A and m_B?

11. Kinematical Endpoint - Solution

We begin by writing down all the kinematical constraints. Mass shell conditions are

$$p_A^2 = m_A^2,\ p_B^2 = m_B^2,\ p_C^2 = p_D^2 = 0.$$

Energy momentum conservation requires

$$p_A = p_B + p_C + p_D,$$

which are 4 independent equation for 4 components of p. If we write 4 momentum as $p_i = (E_i, \mathbf{p_i})$, we have

$$E_A = E_B + E_C + E_D,\ \mathbf{p}_A = \mathbf{p}_B + \mathbf{p}_C + \mathbf{p}_D.$$

And mass shell conditions can be written as $E_i = \sqrt{\mathbf{p}_i^2 + m_i^2}$.

Since the invariant mass m_{CD} is Lorentz invariant, we can calculate it in any reference frame. For our purpose, it is useful to consider it in the rest frame of A. We have

$$\begin{aligned}
p_A &= (m_A, 0, 0, 0),\\
p_B &= (E_B, \mathbf{p}_B) = (\sqrt{m_B^2 + \mathbf{p}_B^2}, \mathbf{p}_B),\\
p_C &= (E_C, \mathbf{p}_C),\\
p_D &= (E_D, \mathbf{p}_D).
\end{aligned}$$

Using $E_C + E_D = m_A - E_B$, and $\mathbf{p}_C + \mathbf{p}_D = -\mathbf{p}_B$, we can write

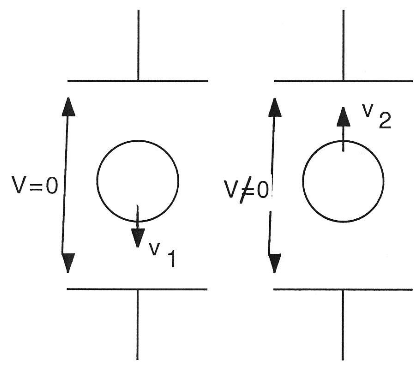
$$m_{CD}^2 = (p_C + p_D)^2 = (m_A - E_B)^2 - |\mathbf{p}_B|^2 = (m_A - \sqrt{m_B^2 + \mathbf{p}_B^2})^2 - |\mathbf{p}_B|^2.$$

We see that $m_{CD}|_{\max}$ obtains when $\mathbf{p}_B = 0$, where $m_{CD}|_{\max} = m_A - m_B$. This is called a kinematical end point. The end point corresponds to the kinematical configuration in which, in the rest frame of A, particle B is also at rest, while C and D fly off back to back with equal momenta. This kinematical end point is very useful in experimental identification of such particle decays.

12. Millikan Oil Drop Experiment

Robert Millikan was able to derive a high-precision value for the charge of the electron from observations of the vertical transit time of an ionized oil droplet moving in air while subjected to an electrostatic potential V.

In this experiment the microscopic droplet was sprayed into the air-filled space between two large metal sheets, after picking up an unknown charge q from a radioactive source. The constant free-fall speed v_1 (when $V = 0$) and constant ascension speed v_2 (when $V \neq 0$) were recorded.



(a) Knowing that the viscous drag force in air is approximately $6\pi r\eta v$, where r is the radius of the droplet, η, the viscous coefficient, is 1.8×10^{-5} in MKS units, and v is the droplet speed. Neglecting buoyancy, find the charge (in multiples of e) of a droplet that was observed to travel with $v_2 = 0.02$ mm/s when $V = 31.5$ volts and $v_1 = 0.002$ mm/s. The distance between plates was 1 cm). No measurement was made of r.

(b) In his infamous droplet #41, Millikan found a charge value of roughly $(2/3)e$. The repercussions of this, and other observations, lasted for some 20 years. Knowing that these anomalous droplets were small and composed of alcohol, use your scientific intuition to come up with a plausible explanation.

12. Millikan Oil Drop Experiment - Solution

(a) In the falling case, force balance implies

$$mg = 6\pi r\eta v_1.$$

When the electrostatic potential V is turned on, and the droplet rises, we have

$$\frac{V}{d}q - mg = 6\pi r\eta v_2.$$

Now, there was no measurement of r and the mass of each droplet is unknown. Here, we want to see some intuition at work. One option is write things in terms of the density of the oil ρ.

Assuming a spherical droplet, $m = 4\pi r^3\rho/3$. Using this in the fall equation and solving for r gives

$$r = \sqrt{\frac{q\eta v_1}{2\rho g}}.$$

Doing likewise in the rise equation and applying the above result once reduces to

$$\frac{V}{d}q = 6\pi r\eta(v_1 + v_2).$$

Applying the result for r a second time and solving for q gives

$$q = \frac{6\pi\eta(v_1 + v_2)}{V/d}\left(\frac{9\eta v_1}{2\rho g}\right)^{1/2}.$$

One way to apply your intuition is to estimate the density of oil. Because oil floats we'll take its density to be slightly less than that of water, $\rho \approx 9\times 10^2$ kg/m^3. Using this estimate together with the other given quantities, we obtain $q \approx 2e$.

(b) For a droplet composed of alcohol, evaporation could lead to a significant reduction in the mass of a droplet during the measurement giving rise to an inaccurate calculation of the charge.