Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 29, 2014

Do not separate this page from the problem pages. Fill out and turn in at the end of the exam.

Student ________________
Fill in your St-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your St-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve **eight** problems of your choice. Do not turn in more than this number (8) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10

Problems that are not checked above, will not be graded. If you check more than 8 problems, only the lowest 8 scores will count towards your total score.

I-1. (Classical Mechanics)

A particle of mass m slides without friction on a straight wire. The wire rotates in the xy-plane so that at time t the position of the mass is

$$x(t) = q(t)\cos\theta(t)\,, \qquad y(t) = q(t)\sin\theta(t)\,,$$

where $\theta(t)$ is a given function of time with $q(t)$ being the single configurational coordinate. In other words, the particle has one degree of freedom and q is the generalized coordinate.

(i) What is the Lagrangian of the particle?
(ii) What is the Hamiltonian of the particle?
(iii) Under which condition for $\theta(t)$ is the Hamiltonian a constant of motion?
(iv) What are the Hamilton equations?
(v) For the particular case $\theta(t) = \sqrt{2}\log t$, find the general solution for $q(t)$ and $p(t)$. *Hint:* Look for solutions of the form at^n.

I-2. (Electricity and Magnetism)

A classical model of the AC conductivity in a material is given by:

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}\,, \qquad \sigma_0 = \frac{ne^2\tau}{m}\,,$$

where n is the number density of charge carriers of charge e and mass m. The quantity ω is the frequency at which the system is being driven by an applied electric field. This model should be familiar.

(i) The parameter τ has dimensions of time. Briefly explain its microscopic interpretation in the model. How would one change τ to make a better conductor?
(ii) On the same axes, give clear sketches of the real and imaginary parts of $\sigma(\omega)$ as a function of real (positive and negative) ω.
(iii) Using Ohm's law and Maxwell's equations, show that, for a harmonic dependence of the form $\mathbf{E}(\omega, \mathbf{x}) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$ with $\nabla \cdot \mathbf{E}(\mathbf{x}) = 0$:

$$\nabla^2\mathbf{E} = -\frac{\omega^2}{c^2}\left(1 + \frac{i\sigma(\omega)}{\omega\epsilon}\right)\mathbf{E}\,,$$

where $1/c^2 = \mu\epsilon$ is the speed of light in the material.
(iv) Hence, give an argument for why (in the limit $\omega\tau \gg 1$) conductors can be reflective (to electromagnetic waves) below a critical frequency $\omega_p = \sqrt{ne^2/m\epsilon}$ and transparent above it.

Hint: The following identities might be useful:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0\,, \qquad \nabla \times (\nabla f) = 0\,, \qquad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}\,.$$

I-3. (Statistical Mechanics)

Consider a monatomic (classical) ideal gas in contact with a catalyst.

(i) The Helmholtz free energy of the gas is given by

$$F = -Nk_BT \ln \frac{Ve}{N\lambda^3}\,, \qquad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}\,,$$

where λ is the thermal wavelength. Show that the chemical potential of the gas particles is related to their temperature and pressure via

$$\mu = k_BT\left[\ln\left(P/T^{5/2}\right) + A_0\right]\,,$$

and determine the constant A_0.

(ii) Assuming that there are $\mathcal{N}$ distinct adsorption sites on the surface of the catalyst, that each adsorption site can be occupied at most by one particle, and that each particle gains an energy ϵ upon adsorption (and has zero energy otherwise), calculate the grand partition function for the particles at the surface, $\mathcal{Q}(T,\mu)$, for a surface chemical potential μ.

(iii) In equilibrium, the gas and surface particles are at the same temperature and chemical potential. The fraction of occupied sites is given by $f(T,P) = \langle N\rangle/\mathcal{N}$, where $\langle N\rangle$ is the average number of occupied sites. Show that $f(T,P) = P/\left[P + P_0(T)\right]$, and find $P_0(T)$.

I-4. (Experimental Physics)

Electron microscopes are capable of achieving much higher resolution than light microscopes. The very basic idea is that the electrons are accelerated to some desired velocity before interacting with the sample and producing a magnified image or diffraction pattern.

(i) In 1-2 sentences maximum, explain why the use of electrons allows higher resolution than, say, the visible photons of a light microscope.

(ii) Write down an expression for the wavelength, λ, of a nonrelativistic electron which is accelerated in an electric potential, U, inside an electron microscope. Your answer should only depend on U and any relevant physical constants/quantities.

(iii) Typically electron microscopes use accelerating potentials larger than 10 kV, even reaching 200 kV in typical transmission electron microscopes (TEM) that achieve atomic resolution. This means very high velocities and relativistic effects must be taken into account. Find the relativistic correction factor to your classical answer from above. Again, your answer should only depend on U and any relevant physical constant/quantities.

I-5. (Quantum Mechanics)

When a photon is emitted by an atom, the atom must recoil to conserve momentum. Consider an atom with mass m.

(i) Calculate the correction $\Delta\lambda$ due to recoil to the wavelength of an emitted photon. Let λ be the wavelength of the photon if recoil is not taken into consideration.

Hint: The correction is very small. Use this fact to obtain an approximate but accurate expression for $\Delta\lambda$.

(ii) Consider a hydrogen atom in which an electron in the nth level returns to the ground level. How does $\Delta\lambda$ depend on n?

I-6. (Thermodynamics)

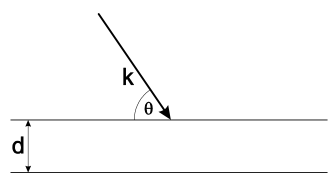
A van der Waals gas undergoes an isothermal expansion to four times its initial volume. What is the change in internal energy? What is the change in entropy?

Hint: The van der Waals equation of state is

$$P = \frac{Nk_BT}{(V - Nb)} - \frac{N^2a}{V^2},$$

where a, b are constants.

I-7. (Solid State/Optics)



Two lattice planes separated by the distance d are shown in the figure above. An X-ray with the wave vector, $\mathbf{k}$, forming the angle, θ, with the planes is scattered by the planes.

(i) Derive the Bragg condition between d, λ and θ.

(ii) Draw the wave vector, $\mathbf{k}'$, of the scattered wave. (Use the special page with the figure.)

(iii) The Laue condition for constructive interference of the scattered waves is

$$\mathbf{k} - \mathbf{k}' = \mathbf{K},$$

where $\mathbf{K}$ is a reciprocal lattice vector. Draw $\mathbf{K}$ based on the figure in (ii) (special page), describe its direction, and show how the Bragg condition in (i) follows from the Laue equation.

I-8. (Particle Physics)

(i) Make two lists: a list of all known quarks and a list of all known leptons. Order each list according to the masses of particles (from the lightest to the heaviest). Include also the electric charges of the particles (in units of electron charge e). Comment on the properties (mass and the electric charge) of their antiparticles.

(ii) What is the quark content of the proton, neutron, positively charged pion and neutral K-meson. What are the baryon numbers of those particles?

(iii) Write down the reaction coresponding to the decay of an antineutron and clearly indicate the decay products including their electric charges, particle or antiparticle properties, and approximate masses.

I-9. (Math Methods)

Let λ_i, $i = 1, 2, 3$, be the eigenvalues of the matrix

$$H = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{pmatrix} .$$

Calculate the sums

$$\sum_{i=1}^{3} \lambda_i \qquad \text{and} \qquad \sum_{i=1}^{3} \lambda_i^2 .$$

I-10. (Astrophysics)

We observe a quasar now at redshift $z = 5$. At what redshift would my descendant see the quasar when the Universe has expanded to three times its current (linear) size?

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Graduate Screening Examination
Part II
Saturday, March 29, 2014

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Student ________________
Fill in your Lg-#

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Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 h 30 min**.

Please, indicate problems you are turning in:

□ **II-1** □ **II-2** □ **II-3** □ **II-4**

II-1. (Classical Mechanics)

A point particle of mass m moves in 3 dimensions in the helical potential

$$V(\rho, \phi, z) = V_0 \rho \cos\left(\phi - \frac{2\pi z}{b}\right).$$

(i) Write down the Lagrangian using generalized coordinates $(q_1, q_2, q_3,) \equiv (\rho, \phi, z)$.

(ii) Find the equations of motion.

(iii) Consider the transformation $q_i \to Q_i(s)$ where s is a continuous parameter and $Q_i(0) = q_i$. Show that if a Lagrangian L is invariant under this transformation, i.e. if $dL/ds = 0$, the quantity

$$\sum_i p_i \frac{dQ_i}{ds}\bigg|_{s=0},$$

is conserved, where p_i is the canonical momentum.

(iv) Show that the Lagrangian of part (i) is invariant for $Q_i = q_i + c_i s$ where the c_i are constants. Find the corresponding constant of motion in terms of the generalized coordinates and velocities.

(v) Is there another constant of motion for the Lagrangian of part (i)? If so, express it in terms of the generalized coordinates and velocities.

II-2. (E & M)

Solve the electrostatic equation

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0},$$

to find the potential Φ inside a cube defined by:

$$0 < x < a, \qquad 0 < y < a, \qquad 0 < z < a.$$

The boundary conditions are that $\Phi = 0$ on all surfaces except the surface at $z = a$, where

$$\Phi(x, y, z = a) = V_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}.$$

The volume charge density is given by

$$\rho(x, y, z) = \sigma(x, z)\delta(y - \frac{a}{4}).$$

It vanishes everywhere inside the cube except on the plane $y = a/4$, where the surface charge density is

$$\sigma(x, z) = \sigma_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{a}.$$

V_0 and σ_0 are constants.

Hint: Useful identity: $\sinh(a + b) = \sinh a \cosh b + \cosh a \sinh b$.

II-3. (Quantum Mechanics)

The creation and annihilation operators, a and $a^\dagger$, of a harmonic oscillator in one dimension are defined in terms of the position operator, x, and the momentum operator, p, as

$$a = \frac{1}{\sqrt{2}}\,(x + i\,p)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}\,(x - i\,p)\,.$$

(i) Using the commutation relations between x and p, evaluate the commutator

$$[a, a^\dagger]\,.$$

The coherent states, $|\alpha\rangle$, are eigenstates of the lowering operator, a,

$$a|\alpha\rangle = \alpha\,|\alpha\rangle\,,$$

where α can be any complex number.

(ii) Calculate the expectation values $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ in the state $|\alpha\rangle$.
Remember that $a^\dagger$ is the hermitian conjugate of a. Do not assume that α is real.

(iii) Show that the state $|\alpha\rangle$ minimizes the uncertainty in position and momentum, that is $\sigma_x\sigma_p = \hbar/2$.
Hint: For any observable A, $\sigma_A^2 = \langle(A - \langle A\rangle)^2\rangle$.

(iv) Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle.$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\,c_0\,.$$

(v) Determine c_0 by normalizing $|\alpha\rangle$.

(vi) Now, put the time dependence:

$$|n\rangle \quad\longrightarrow\quad e^{iE_n t/\hbar}\,|n\rangle\,,$$

and show that $|\alpha(t)\rangle$ remains an eigenstate of a, but the eigenvalue evolves in time. What is that eigenvalue? Interpret the result.

II-4. (Mathematical Methods)

The generating function of the Laguerre polynomials, $L_n(z)$, is

$$g(z,t) \;=\; \frac{1}{1-t}\,\exp\left[-\frac{zt}{1-t}\right],$$

such that

$$L_n(z) \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{g(z,t)}{t^{n+1}}\,dt\,, \qquad n \;=\; 0,1,2,\ldots\,, \tag{$*$}$$

where the closed contour $\mathcal{C}$, oriented counterclockwise, encloses the origin, but not the point $t = 1$.

(i) Calculate the polynomials $L_0(z)$, $L_1(z)$ and $L_2(z)$.

(ii) Using the generating function, show that the polynomials satisfy the recurrence relation

$$(n+1)\,L_{n+1}(z) \;=\; (2n+1-z)\,L_n(z) - n\,L_{n-1}(z)\,.$$

(iii) Consider a Möbius map

$$t \quad\to\quad w = \frac{1}{1-t}\,.$$

What is the image of a circle in the t plane, centered at $t = 0$ and with radius $1/2$, under this map?

(iv) Perform the change of variables

$$t \;=\; \frac{s-z}{s}\,,$$

in the integral $(*)$ to deduce the Rodrigues formula

$$L_n(z) \;=\; \frac{e^z}{n!}\frac{d^n}{dz^n}(z^n e^{-z})\,.$$

Hint: Choose a convenient contour $\mathcal{C}$ in $(*)$, e.g., a circle of small radius, and argue that after the change of variables the new contour in the s-plane will enclose $s = z$, but not $s = 0$. What is the orientation of the contour in the s-plane?

(v) The Laguerre equation satisfied by these polynomials is

$$z\,y'' + (1-z)\,y' + n\,y \;=\; 0\,.$$

Determine singular points of this equation and their type.

Comment: You do not have to check that the equation is satisfied nor to solve it.