Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 28, 2009

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your S-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your S-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

The problems are divided into two groups. Solve

Group A: 4 problems out of 6
Group B: 3 problems out of 7

Do not turn in more than the above number $(4 + 3 = 7)$ of problems.

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

Group A (4 problems):

□ A.1 □ A.2 □ A.3 □ A.4 □ A.5 □ A.6

Group B (3 problems):

□ B.1 □ B.2 □ B.3 □ B.4 □ B.5 □ B.6 □ B.7

Group A. **Choose 4 out of 6 problems**

A.1. (Classical Mechanics)

A superball is bouncing vertically up and down. It has speed v_0 when it hits the ground, and its collisions with the ground are elastic. Suppose that the acceleration, g_0, of the ball due to gravity is *slowly* reduced over a *very long time* by 10% to $0.9\,g_0$. What is the corresponding change in the speed of the ball at the ground?

A.2. (Electricity and Magnetism)

Consider a plane electromagnetic wave incident onto a conducting non-magnetic surface characterized by a dielectric permittivity ε and a conductivity σ. Assume normal incidence for simplicity, so that the wave travels in the x-direction. In this conducting medium the wave equation takes the form

$$\nabla^2 \vec{E} - \mu_0 \left(\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \sigma \frac{\partial \vec{E}}{\partial t} \right) \; = \; 0\,.$$

(i) Write down the frequency dependence of the (complex) wave number, $k(\omega)$.

(ii) Explain in a few clear sentences the meaning and origin of the imaginary part of k.

(iii) In the limit of high conductivity and low frequency, find the penetration distance ("skin depth") for which the amplitude is attenuated by a factor of $1/e$.

(iv) Estimate the skin depth of a good metallic conductor (resistivity $\approx\; 10^{-8}\,\Omega\cdot m$, $\varepsilon \approx \varepsilon_0 = 8.9 \times 10^{-12}\,\mathrm{C^2/N\cdot m^2}$, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$) placed into a household microwave oven ($\lambda \approx 10\,\mathrm{cm}$).

A.3. (Quantum Mechanics)

A spin-one particle is placed in a state represented by the vector

$$|\psi\rangle \;=\; \frac{1}{\sqrt{5}} \begin{pmatrix} 1-i \\ i \\ 1+i \end{pmatrix}\,.$$

The matrix representations of the spin operators in the basis used here are

$$S_x \;=\; \hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \qquad S_y \;=\; \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \qquad S_z \;=\; \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,.$$

(i) What is the probability that a measurement of the spin along the z-axis, S_z, will give the value $+\hbar$?

(ii) What is the probability that a measurement of the spin along the x-axis, S_x, will give the value 0?

A.4. (Thermodynamics)

A small body of constant heat capacity C_P and in equilibrium at temperature T_i is put into contact with a large reservoir, also at equilibrium, at temperature T_f. During the ensuing process, the body is maintained at constant pressure P. Compute the change of the total entropy of the system (body plus reservoir) during the process, and prove that unless $T_f = T_i$ the change is positive, irrespective of the sign of $T_f - T_i$.

A.5. (Statistical Physics)

Consider a system of N distinguishable spins in a magnetic field H. Each spin has a magnetic moment of size μ, and each can point either parallel or antiparallel to the field. Thus, the energy of a particular state is

$$-\sum_{i=1}^{N} n_i \mu H \,, \qquad n_i \;=\; \pm 1 \,,$$

where $n_i\mu$ is the magnetic moment in the direction of the field.

(i) Determine the internal energy of this system as a function of β, H, and N by employing an ensemble characterized by these variables. Here $\beta = 1/k_B T$, where k_B is the Boltzmann's constant and T is the temperature of the system.

(ii) Determine the entropy of the system as a function of β, H, and N.

A.6. (Mathematical Methods)

Let $A = (A_{ij})_{i,j=1,\ldots,n}$ be an arbitrary $n \times n$ complex hermitian matrix, whose eigenvalues are $\lambda_1, \ldots, \lambda_n$. Show that the the diagonal matrix elements, A_{ii}, and the eigenvalues, λ_i, always satisfy

$$\sum_{i=1}^{n} A_{ii} \;=\; \sum_{i=1}^{n} \lambda_i \,,$$

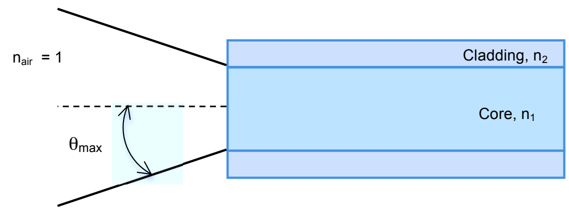
and

$$\sum_{i=1}^{n} |A_{ii}| \leq \sum_{i=1}^{n} |\lambda_i| \;.$$

Hint: Recall that A can be diagonalized by a unitary transformation.

Group B. **Choose 3 out of 7 problems**

B.1. (Optics)



Optical fibers are widely used in fiber-optic communications, which permits data transmission over longer distances. The physics behind reducing light-loss in optical fibers led to the 2009 Nobel Prize. Consider the simple optical fiber configuration shown above, consisting of a core (refractive index n_1) that guides the light, and a cladding (refractive index n_2) surrounding the core. The medium where the light travels before entering the fiber is air, $n_{\rm air} = 1$.

(i) What is the the key physical phenomenon allowing for the light propagation inside the optical fiber? What are the constraints on the ratio n_1/n_2 needed for this to take place?

(ii) The optical fiber shown above will only propagate light that enters the fiber within certain cone, known as the acceptance cone of the fiber. The half-angle of this cone is called the acceptance angle, $\theta_{\rm max}$. Determine the acceptance angle of this fiber. (Simplify formulae in your answer as much as you can.)

B.2. (Solid State)

Graphene is a crystal of carbon made out of sheets with thickness of a single atom. Let the sheet lie in the xy-plane. The crystal structure forms a honeycomb lattice with unit vectors

$$\vec{a}_1 = \frac{a\sqrt{3}}{2}(\sqrt{3},1)\,, \qquad \vec{a}_2 = \frac{a\sqrt{3}}{2}(\sqrt{3},-1)\,.$$

Within the unit cell there are two carbon atoms. One is located at the origin and the other one is located at the position $\vec{b} = a(1,0)$.

(i) Draw a picture of the unit cell and indicate the position of the two basis atoms listed above.

(ii) Calculate the area, A, of the unit cell. *Hint:* You can use the formula for the area of a parallelogram $A = |(\vec{a}_1 \times \vec{a}_2)\cdot\widehat{z}\,|$, where $\widehat{z}$ is the unit vector in the z-direction.

The unit vectors of the reciprocal lattice can be calculated using the formulas

$$\vec{K}_1 = \frac{2\pi}{A}\,\widehat{z}\times\vec{a}_2\,, \qquad \vec{K}_2 = \frac{2\pi}{A}\,\vec{a}_1\times\widehat{z}\,.$$

(iii) Calculate $\vec{K}_1$ and $\vec{K}_2$ explicitly.

(iv) Draw a picture of the first Brillouin zone of the reciprocal lattice.

(v) Calculate the area of the first Brillouin zone. *Hint:* The area of the Brillouin zone is the same as the area of the unit cell of the reciprocal lattice.

B.3. (Experimental Physics)

X-rays are produced when electrons accelerated by a high voltage, V, strike a metal target. The spectrum of wavelengths emitted from an X-ray tube consists of two parts: one continuous spectrum with a cut-off wavelength, λ_c, and a series of peaks.

(i) Explain the underlying physics giving rise to this spectrum.

(ii) What will happen to the spectrum when the accelerating voltage is increased?

(iii) The strongest X-rays are typically the K_α lines, which originate from $n = 2$ (L shell) to $n = 1$ (K shell) transitions. For hydrogen, the energy transition from $n = 2$ to $n = 1$ shell is about 10.2 eV. Estimate the atomic number if the strongest peak from an unknown target occurs at 66 keV.

B.4. (Special Relativity)

(i) Starting with the Lorentz transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v/c^2 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}, \qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}},$$

derive the formula for the relativistic Doppler shift.

(ii) A space-craft is moving, at speed v, directly towards a radar station that is emitting radio waves at an angular frequency, ω. What is the frequency of the reflected radar signal received by the radar station?

B.5. (Particle Physics/Relativity)

Consider the scattering of a photon by an electron at rest in the laboratory. Assume that the energy, E, of the incoming photon is comparable to the rest energy of the electron, mc^2, so that relativistic mechanics must apply. If the photon is scattered at an angle θ relative to the incoming direction, compute its energy, E', as a function of E and θ.

B.6. (Cosmology)

We now observe a galaxy at redshift 2. When the universe expands to 5 times its current (linear) size, at which redshift will our descendants observe that galaxy, assuming they will continue to live on Earth? (If necessary, assume a matter dominated, flat Universe in the calculations.)

B.7. (Astrophysics)

Under the assumption of hydrostatic equilibrium inside a spherical star

$$\frac{dp}{dr} = -\frac{GM_r\rho}{r^2}\,, \qquad \frac{dM_r}{dr} = 4\pi r^2\rho\,,$$

where

$p =$ local pressure,
$\rho =$ local density,
$M_r =$ mass contained in the concentric sub-sphere of radius r,

prove that the following estimate is a rigorous lower bound on the central pressure of the star, valid for any density and pressure distribution of a star (expressed in terms of the total mass M and surface radius R of the star):

$$p_c > \frac{GM^2}{8\pi R^4}\,.$$

Hint: Using the equations of hydrostatic equilibrium, first verify the validity of the following equation

$$\frac{d}{dr}\left(p + \frac{GM_r^2}{8\pi r^4}\right) = -\frac{GM_r^2}{2\pi r^5}\,.$$

Then use the implied monotony of the function

$$p + \frac{GM_r^2}{8\pi r^4}\,,$$

and the fact that $p \approx 0$ at the surface.

Department of Physics and Astronomy
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Graduate Screening Examination
Part II
Saturday, April 10, 2010

Do not separate this page from the problem pages. Fill out and turn in at the end of the exam.

Student ____________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs 30 min**.

Please, indicate problems you are turning in:

□ **II-1** □ **II-2** □ **II-3** □ **II-4**

II-1. (Mathematical Methods)

Consider a differential equation

$$x(x+1)^2\, y''(x) + \frac{3}{2} x(x+1)\, y'(x) + \frac{3}{2}\, y(x) \ = \ 0\,. \tag{1}$$

(i) Find all singular points of this equation and determine whether they are regular or irregular.

(ii) Calculate the exponents (roots of the indicial equation) at each regular singular point.

(iii) Find a fractional linear transformation of the form

$$z \quad \longrightarrow \quad z' \ = \ \frac{az+b}{cz+d}\,, \qquad a,b,c,d \in \mathbb{C}\,, \tag{2}$$

that reduces (1) to a standard hypergeometric equation.

(iv) Write a solution of the initial equation (1) around $x = \infty$ in terms of the hypergeometric function, $F(\alpha, \beta\,;\, \gamma\,;\, z)$, of suitable arguments.

Hint: Recall that the standard hypergeometric equation for a function $f(z)$ has the form

$$z(z-1)\, f''(z) + [(\alpha+\beta+1)\, z - \gamma]\ f'(z) + \alpha\beta\, f(z) \ = \ 0\,. \tag{3}$$

It has regular singular points at $z = 0$, 1 and ∞ with exponents $(0, 1-\gamma)$, $(0, \gamma-\alpha-\beta)$ and (α, β), respectively. The hypergeometric function $F(\alpha, \beta\,;\, \gamma\,;\, z)$ is defined as the analytic solution to this equation around $z = 0$.

II-2. (Electricity and Magnetism)

Two concentric spheres have radii a and $b > a$, and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V, while the other hemispheres are maintained at zero potential. Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials.

Hint: The following properties of Legendre polynomials might be useful:

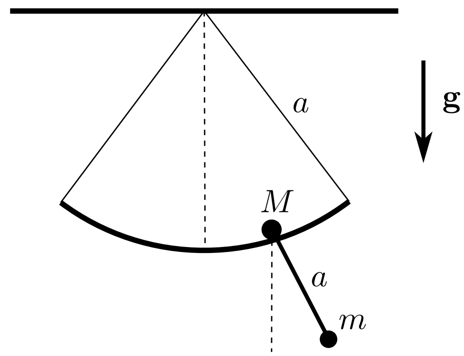
$$P_\ell(-x) \ = \ (-1)^\ell P_\ell(x)\,,$$

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) \ = \ \frac{1}{2\ell+1}\delta_{\ell\ell'}\,,$$

$$\int_{0}^{1} dx\, P_\ell(x) \ = \ \delta_{\ell 0} \qquad \text{for } \ell \text{ even}\,,$$

$$\int_{0}^{1} dx\, P_\ell(x) \ = \ -\left(-\frac{1}{2}\right)^{(\ell+1)/2} \frac{(\ell-2)!!}{\left(\frac{1}{2}(\ell+1)\right)!} \qquad \text{for } \ell \text{ odd}\,.$$

II-3. (Classical Mechanics)



A simple pendulum of mass m and length a is attached to another mass $M = 3m$ which moves without friction along a segment of *smooth* circular weightless structure of radius a. Assume that the motion of m and M is restricted to the same vertical plane.

(i) Find the Lagrangian of the system.

(ii) Write down the corresponding Hamiltonian, if both m and M are subject to small oscillations. Find a *canonical* transformation such that the Hamiltonian becomes the sum of two independent harmonic oscillators. What are the characterstic frequencies of these two harmonic oscillators?

(iii) Find the frequencies of the normal modes of this system under the small oscillation conditions using any other method of your choice. Compare your solution with the results in part (ii).

II-4. (Quantum Mechanics)

An electron moves in one dimension and is confined to the right half-space where it has potential energy

$$V(x) = -\frac{e^2}{4x}, \qquad x > 0,$$

with e the charge of the electron.

(i) What are the boundary conditions on a bound-state wavefunction?

(ii) Consider the function

$$x^\beta e^{-\alpha x^\gamma}, \tag{4}$$

where α, β, and γ are real numbers. Do any of these functions satisfy the correct boundary conditions on a bound-state wavefunction? If so, what are possible values of β and γ? Explain.

(iii) For what values of α, β, and γ is (4) a bound-state wavefunction?

(iv) Does the solution represent the ground state? Explain.

(v) What is the expectation value of the position operator, $\widehat{x}$, for this bound-state?