Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, March 29, 2014

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student __________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Lg-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 h 30 min**.

Please, indicate problems you are turning in:

☐ **II-1** ☐ **II-2** ☐ **II-3** ☐ **II-4**

II-1. (Classical Mechanics)

A point particle of mass m moves in 3 dimensions in the helical potential

$$V(\rho, \phi, z) = V_0 \rho \cos\left(\phi - \frac{2\pi z}{b}\right).$$

(i) Write down the Lagrangian using generalized coordinates $(q_1, q_2, q_3,) \equiv (\rho, \phi, z)$.

(ii) Find the equations of motion.

(iii) Consider the transformation $q_i \to Q_i(s)$ where s is a continuous parameter and $Q_i(0) = q_i$. Show that if a Lagrangian L is invariant under this transformation, i.e. if $dL/ds = 0$, the quantity

$$\sum_i p_i \frac{dQ_i}{ds}\bigg|_{s=0},$$

is conserved, where p_i is the canonical momentum.

(iv) Show that the Lagrangian of part (i) is invariant for $Q_i = q_i + c_i s$ where the c_i are constants. Find the corresponding constant of motion in terms of the generalized coordinates and velocities.

(v) Is there another constant of motion for the Lagrangian of part (i)? If so, express it in terms of the generalized coordinates and velocities.

II-2. (E & M)

Solve the electrostatic equation

$$\nabla^2 \Phi \;=\; -\frac{\rho}{\varepsilon_0},$$

to find the potential Φ inside a cube defined by:

$$0 < x < a\,, \qquad 0 < y < a\,, \qquad 0 < z < a\,.$$

The boundary conditions are that $\Phi = 0$ on all surfaces except the surface at $z = a$, where

$$\Phi(x, y, z = a) \;=\; V_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}\,.$$

The volume charge density is given by

$$\rho(x, y, z) \;=\; \sigma(x, z)\delta(y - \frac{a}{4})\,.$$

It vanishes everywhere inside the cube except on the plane $y = a/4$, where the surface charge density is

$$\sigma(x, z) \;=\; \sigma_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{a}\,.$$

V_0 and σ_0 are constants.

Hint: Useful identity: $\sinh(a + b) \;=\; \sinh a \cosh b + \cosh a \sinh b$.

II-3. (Quantum Mechanics)

The creation and annihilation operators, a and $a^\dagger$, of a harmonic oscillator in one dimension are defined in terms of the position operator, x, and the momentum operator, p, as

$$a = \frac{1}{\sqrt{2}}(x + i\,p)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}(x - i\,p)\,.$$

(i) Using the commutation relations between x and p, evaluate the commutator

$$[a, a^\dagger]\,.$$

The coherent states, $|\alpha\rangle$, are eigenstates of the lowering operator, a,

$$a|\alpha\rangle = \alpha\,|\alpha\rangle\,,$$

where α can be any complex number.

(ii) Calculate the expectation values $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ in the state $|\alpha\rangle$.
Remember that $a^\dagger$ is the hermitian conjugate of a. Do not assume that α is real.

(iii) Show that the state $|\alpha\rangle$ minimizes the uncertainty in position and momentum, that is $\sigma_x\sigma_p = \hbar/2$.
Hint: For any observable A, $\sigma_A^2 = \langle(A - \langle A\rangle)^2\rangle$.

(iv) Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle.$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\,c_0\,.$$

(v) Determine c_0 by normalizing $|\alpha\rangle$.

(vi) Now, put the time dependence:

$$|n\rangle \longrightarrow e^{iE_n t/\hbar}\,|n\rangle\,,$$

and show that $|\alpha(t)\rangle$ remains an eigenstate of a, but the eigenvalue evolves in time. What is that eigenvalue? Interpret the result.

II-4. (Mathematical Methods)

The generating function of the Laguerre polynomials, $L_n(z)$, is

$$g(z,t) \;=\; \frac{1}{1-t}\,\exp\left[-\frac{zt}{1-t}\right],$$

such that

$$L_n(z) \;=\; \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{g(z,t)}{t^{n+1}}\,dt\,, \qquad n \;=\; 0,1,2,\ldots\,, \tag{$*$}$$

where the closed contour $\mathcal{C}$, oriented counterclockwise, encloses the origin, but not the point $t = 1$.

(i) Calculate the polynomials $L_0(z)$, $L_1(z)$ and $L_2(z)$.

(ii) Using the generating function, show that the polynomials satisfy the recurrence relation

$$(n+1)\,L_{n+1}(z) \;=\; (2n+1-z)\,L_n(z) - n\,L_{n-1}(z)\,.$$

(iii) Consider a Möbius map

$$t \quad\to\quad w = \frac{1}{1-t}\,.$$

What is the image of a circle in the t plane, centered at $t = 0$ and with radius $1/2$, under this map?

(iv) Perform the change of variables

$$t \;=\; \frac{s-z}{s}\,,$$

in the integral $(*)$ to deduce the Rodrigues formula

$$L_n(z) \;=\; \frac{e^z}{n!}\frac{d^n}{dz^n}(z^n e^{-z})\,.$$

Hint: Choose a convenient contour $\mathcal{C}$ in $(*)$, e.g., a circle of small radius, and argue that after the change of variables the new contour in the s-plane will enclose $s = z$, but not $s = 0$. What is the orientation of the contour in the s-plane?

(v) The Laguerre equation satisfied by these polynomials is

$$z\,y'' + (1-z)\,y' + n\,y \;=\; 0\,.$$

Determine singular points of this equation and their type.

Comment: You do not have to check that the equation is satisfied nor to solve it.