Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II

Saturday, April 2, 2016

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Lg-number.* Do not write answers to different problems on the same page. Mark each page with the problem number. Do not write on the back of page. Staple *separately* your answers to each problem.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems!

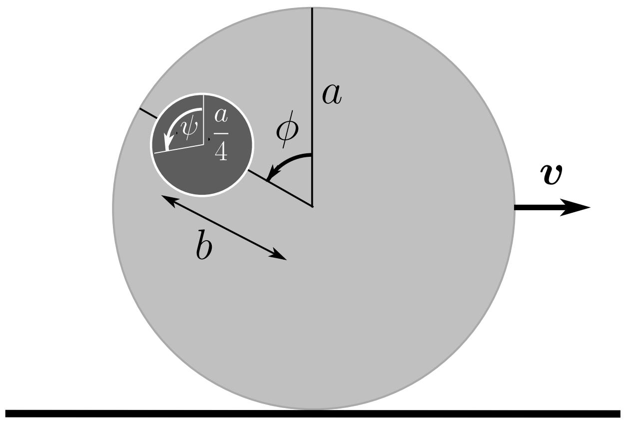
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1 ☐ 2 ☐ 3 ☐ 4

Problems that are not checked above, will not be graded. If you check more than 3 problems, only the lowest 3 scores will count towards your total score.

Problem II-1. (Classicial Mechanics)



A solid cylinder with uniform mass density, mass M, and radius a can roll without slipping on a rough horizontal surface.

Suppose that a hole of radius $a/4$ is bored through the entire length of the cylinder, parallel to its axis. The line joining the center of the hole to the center of the cylinder is of length $a/4 < b < 3a/4$ and it makes an angle ϕ with the upward vertical. One can show (don't check it) that the moment of inertia of the deformed cylinder about the *instantaneous axis of rotation* is

$$I = \alpha + \beta \cos \phi \,,$$

where α and β are constants, whose explicit dependence on a, b and M need not be given.

Suppose now that the hole is entirely filled with a solid cylinder of mass M and radius $a/4$, whose length is that of the first cylinder. Assume that there is no friction in the hole so that the smaller cylinder can rotate freely within the hole. This degree of freedom is parametrized by the angle ψ measured from the upward vertical as shown in the figure and is subject to the constraint $f \equiv \dot{\psi} - \dot{\phi} \equiv 0$. The entire system is initially at rest when a horizontal force is applied evenly and perpendicularly to the central axis of the larger cylinder. Consequently, the entire system rolls without slipping in a straight line with speed v. The applied force is adjusted (continually) to ensure that v remains constant; this force in encoded in a further constraint, $g \equiv \dot{\phi} + \omega \equiv 0$ where $\omega = v/a$.

(i) At this stage, what is the kinetic energy of the joint system about the *instantaneous axis of rotation*?

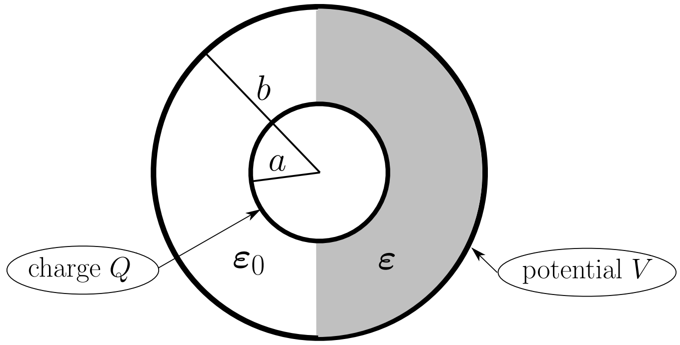
(ii) What is the Lagrangian, L, of the joint system? Include the terms λf and μg in the Lagrangian, where λ and μ are Lagrange multipliers.

(iii) Write down the equations of motion in ϕ and ψ and display the generalized forces Q_ϕ and Q_ψ on the right hand sides of these equations.

(iv) What are the canonical momenta p_ϕ and p_ψ?

(v) Is the Hamiltonian, $H = \dot{\phi} p_\phi + \dot{\psi} p_\psi - L$, a constant of motion? Why or why not?

Problem II-2. (Electricity and Magnetism)



The space between two concentric *conducting* spheres of inner and outer radii a and b, respectively, is half-filled by a hemispherical shell of dielectric with dielectric constant $\varepsilon/\varepsilon_0$. The inner sphere carries charge Q, and the outer sphere is held at a potential V.

(i) Determine the potential everywhere.
(ii) Determine the free charge density everywhere.
(iii) Determine the polarization (bound) charge density everywhere.
(iv) Suppose that the outer sphere is now grounded. How much charge flows to it?

Problem II-3. (Math Methods)

The elements of an $n \times n$ matrix $\mathbb{A} = (A_{ij})$ are given by

$$A_{ij} = \begin{cases} z & \text{for } i = j \\ 1 & \text{for } i \neq j \end{cases}$$

where $z \in \mathbb{C}$.

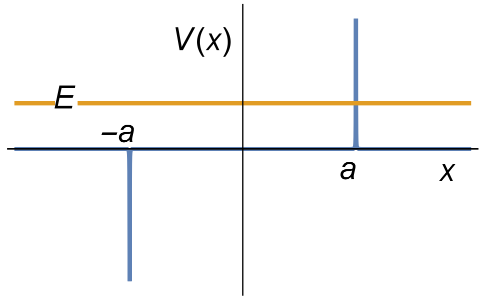
(i) Show that $\mathbb{A}$ satisfies a matrix equation of the form

$$\mathbb{A}^2 + a\,\mathbb{A} + b\,\mathbb{I} = 0\,,$$

where $\mathbb{I}$ is the $n \times n$ unit matrix and a, b are constants that depend on n and z. Find the constants a and b as functions of n and z.

(ii) For which values of z is the matrix $\mathbb{A}$ normal?
(iii) Argue that $\mathbb{A}$ has at most two distinct eigenvalues. Calculate those eigenvalues and determine their degeneracies.
(iv) Find the determinant of $\mathbb{A}$ as a function of n and z.
(v) For the case $n = 4$ and $z = -1$, find a real orthogonal matrix, $\mathbb{O}$, that diagonalizes matrix $\mathbb{A}$. What is the inverse matrix, $\mathbb{O}^{-1}$?
(vi) What is the freedom in choosing the matrix $\mathbb{O}$ in (v), *i.e.* what is the most general real orthogonal matrix that diagonalizes matrix $\mathbb{A}$?

Problem II - 4. (Quantum Mechanics)



Consider a particle of mass m and energy $E = \hbar^2 k^2/2m$ moving in the 1-dimensional non-relativistic potential

$$V(x) = \frac{\hbar^2 \mu^2}{2m} \left[-\delta\left(\frac{x}{a}+1\right) + \delta\left(\frac{x}{a}-1\right) \right] ,$$

made up of attractive and repulsive delta functions as shown in the figure.

Assuming that the particle initially approaches from the left, compute the transmission (T) and reflection (R) coefficients by following the steps (i)-(vi) below.

(i) Describe in words your strategy for computing T and R and set up the appropriate equations before you solve them. In particular discuss the boundary conditions.

(ii) Give the analytic solutions for the wavefunction $\psi(x)$ in various parts of position space x.

(iii) Impose continuity conditions appropriate for delta function potentials.

(iv) Solve analytically for $T(k)$ and $R(k)$ as a function of $k > 0$.

(v) Discuss the behavior of (T, R) for small and large limits of the parameters k and a. Give a physical reason that anticipates this behavior.

(vi) Determine if there are some special values of energy at which the transmission is 100% and there is no reflection (Ramsauer effect).