Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II

Saturday, April 12, 2008

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

If a problem has subparts, each of these will be equally weighted, unless indicated otherwise.

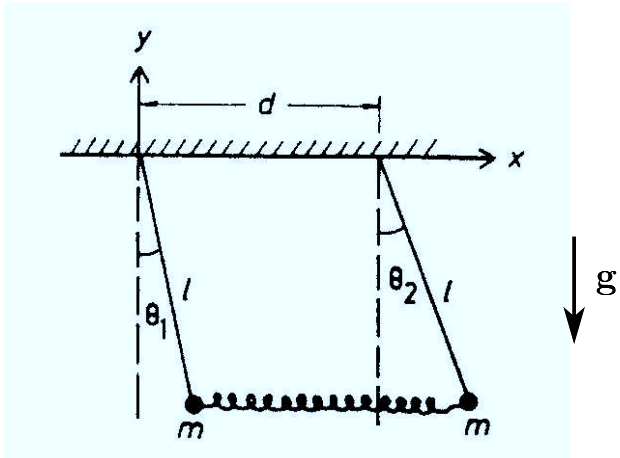
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs**.

Please, indicate problems you are turning in

☐ II-1 ☐ II-2 ☐ II-3 ☐ II-4

II-1. (Classical Mechanics)



Two equal masses are suspended by weightless rigid rods of length ℓ. The masses are also connected by a weightless spring of spring constant k and unstretched length d. The motion is limited to the vertical plane as shown in the figure.

(i) (2 pts) Find the kinetic and the potential energy of the system and write down its Lagrangian.

(ii) (3 pts) Write down the Hamiltonian if the system is subject to small oscillations.

(iii) (3 pts) Find a *canonical transformation* such that the Hamiltonian becomes separable (show the transformation is indeed *canonical*.) What are the characteristic frequencies of this system?

(iv) (2 pts) Find the eigenfrequencies of the normal modes for this system using other method of your choice. Compare your results with the ones from part (iii). Describe qualitatively each normal mode with a simple sketch.

II-2. (Mathematical Methods)

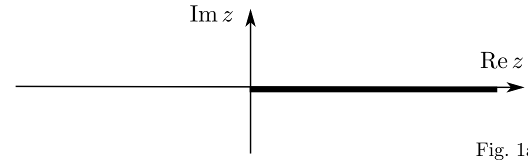


Fig. 1a

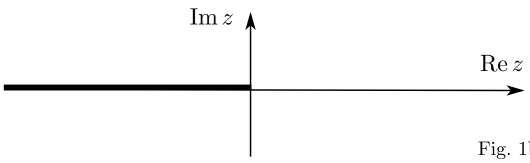


Fig. 1b

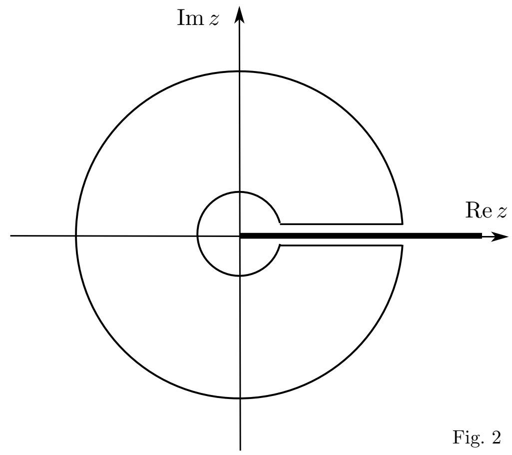


Fig. 2

Recall the definition of the complex mapping

$$z \quad \longrightarrow \quad w \;=\; z^{\alpha} \;\equiv\; e^{\alpha \ln z}\,, \qquad z \in \mathbb{C} \cup \{\infty\}\,,$$

for an arbitrary complex exponent α. Unless α is an integer, the mapping is multivalued. In order to define a single valued function (a single valued branch), one introduces a branch cut, which is used to assign unambiguous phase to each $z \neq 0$. In general, different choices of the branch cut lead to different single valued functions.

(i) (3 pts) Consider two different single valued branches of $w = z^{1/2}$, with the branch cuts slightly below the positive real axis and slightly above the negative real axis as in Figs 1a and 1b, respectively. In each case the phase of $z = 1$ is set to 0. For each of the branches 1a and 1b evaluate the real and imaginary parts of

$$(i)^{1/2}\,, \qquad (-1)^{1/2}\,, \qquad \text{and} \qquad (-i)^{1/2}\,,$$

where $i \;=\; \sqrt{-1}$ is the imaginary unit.

(ii) (7 pts) Consider a definite real integral

$$I(a) \;=\; \int_0^{\infty} \frac{x^{a-1}}{x^2+1}\, dx\,,$$

where a is a real parameter. What is the range of a for which the integral is convergent? Using integration along a contour of the type shown in Fig. 2, where the branch cut is placed along the positive real axis, prove that

$$I(a) \;=\; \frac{\pi}{2} \csc\left(\frac{\pi a}{2}\right)\,.$$

Explain carefully all the steps in your calculation.

II-3. (Electricity and Magnetism)

Two flat conducting plates are parallel and separated by a distance L. One plate occupies the plane $z = 0$, and the other plate occupies the plane $z = L$. The electrostatic potential Φ equals zero in both plates. A single point charge q is placed between the plates at the point $(x = 0, y = 0, z = d)$, where $0 < d < L$.

(i) (7 pts) Find an expression for the electrostatic potential Φ everywhere between the plates $0 < z < L$ as a Fourier integral using rectangular coordinates x, y, z. The potential Φ has the form

$$\Phi(x, y, z) \;=\; \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\ell \, e^{ikx + i\ell y} f(k, \ell, z) \, .$$

(ii) (1.5 pts) Using Φ from part (i) find a Fourier integral expression for the charge density $\sigma(x, y)$ on the metal surface at $z = L$.

(iii) (1.5 pts) Integrate the result from part (ii) to find the total charge

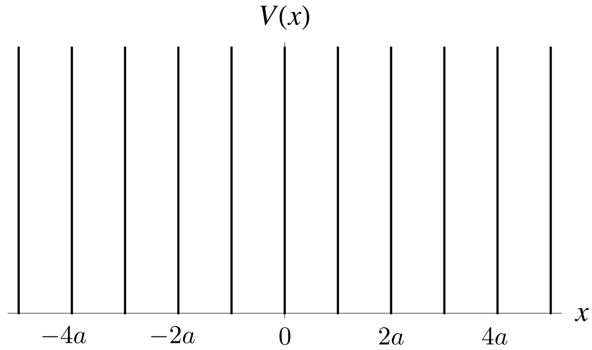
$$Q \;=\; \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \sigma(x, y) \, ,$$

on the metal surface at $z = L$.

Hint:

$$\int_{-\infty}^{\infty} e^{ikx} \;=\; 2\pi \, \delta(x) \, .$$

II-4. (Quantum Mechanics)



The motion of an electron in a one-dimensional crystal can be modelled by an idealized periodic delta function potential

$$V(x) \;=\; \frac{\hbar^2 v}{ma} \sum_{n=-\infty}^{\infty} \delta(x-na)\,,$$

with some period a as shown in the figure. The wave function, $\psi(x)$, of the electron with energy $E = \hbar^2 k^2/(2m)$, $k \neq 0$, can be obtained by gluing together solutions to the Schrödinger equation in the valleys where $V(x) = 0$. In particular, in the n^{th} valley,

$$\psi(x) \;=\; \psi_n(x)\,, \qquad na < x < (n+1)a\,,$$

where

$$\psi_n(x) \;=\; A_n e^{ik(x-na)} + B_n e^{-ik(x-na)}\,,$$

for some complex coefficients A_n, B_n. The wave function, $\psi(x)$, is continuous for all x, but its derivative, $\psi'(x)$, is discontinuous at $x = na$.

(i) (3 pts) Use the Schrödinger equation to show that the discontinuity of $\psi'(x)$ at $x = na$ is given by

$$\lim_{\epsilon\to 0^+} \big(\psi'(na+\epsilon) - \psi'(na-\epsilon)\big) \;=\; \frac{2v}{a}\psi(na)\,.$$

(ii) (3 pts) Write down the boundary conditions that relate (A_n, B_n) and (A_{n+1}, B_{n+1}) for two neighboring valleys. Express your answer in the matrix form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} \;=\; \mathbf{T} \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \qquad \mathbf{T} \;=\; \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}.$$

Compute the parameters α, β in terms of the parameters a and v in the potential and the wave number k. Is there a simple physical reason that $\mathbf{T}$ does not depend on n? Verify that $\det \mathbf{T} = 1$.

[continued on the next page]

(iii) (4 pts) Let λ_+ and λ_- be the eigenvalues of $\mathbf{T}$. Since $\det \mathbf{T} = 1$, we have $\lambda_+ \lambda_- = 1$. In addition, one can show that λ_+ and λ_- must be pure phases, i.e.

$$\lambda_+ \;=\; e^{+i\phi} \qquad \text{and} \qquad \lambda_- \;=\; e^{-i\phi}\,,$$

for some angle ϕ. Use those properties of λ_+ and λ_- (you do not need to prove them) to show that the allowed values of k must satisfy

$$|\cos ka + \frac{v}{ka} \sin ka| < 1\,.$$

Discuss the structure of the resulting spectrum of allowed energies of the electron. How does the spectrum look like in the limit of very high energies? Is the result consistent with your intuition about the energy spectrum in that limit?

Hint: Consider $\mathrm{Tr}\, \mathbf{T}$.