

Department of Physics and Astronomy  
University of Southern California

**Graduate Screening Examination**

**Part I**

Saturday, April 2, 2016

**Do not separate this page from the problem pages.**

Fill out and turn in at the end of the exam.

Student \_\_\_\_\_  
Fill in your Sh-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Sh-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Write on the front side of each page only. Staple *separately* your answers to each problem.

Solve **seven** problems of your choice. Do not turn in more than this number (7) of problems!

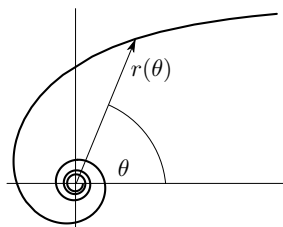
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

- |                            |                            |                            |                             |                             |                            |
|----------------------------|----------------------------|----------------------------|-----------------------------|-----------------------------|----------------------------|
| <input type="checkbox"/> 1 | <input type="checkbox"/> 2 | <input type="checkbox"/> 3 | <input type="checkbox"/> 4  | <input type="checkbox"/> 5  | <input type="checkbox"/> 6 |
| <input type="checkbox"/> 7 | <input type="checkbox"/> 8 | <input type="checkbox"/> 9 | <input type="checkbox"/> 10 | <input type="checkbox"/> 11 |                            |

Problems that are not checked above, will not be graded. If you check more than 7 problems, only the lowest 7 scores will count towards your total score.

**Problem I-1.** (Classical Mechanics)



- (i) Determine the central force,  $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$ , that allows a particle of mass  $m$  and angular momentum  $\ell = mr^2\dot{\theta} > 0$  to move in a spiral orbit given by

$$r(\theta) = \frac{\alpha}{\theta}, \quad \theta > 0,$$

where  $\alpha$  is a positive constant.

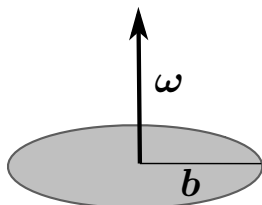
- (ii) Show that the potential energy of the particle is of the form

$$U(r) = -\frac{k}{r^n}.$$

Determine the constants  $k$  and  $n$  in terms of  $\alpha$ ,  $m$  and  $\ell$ . Assume that  $U(r = \infty) = 0$ .

- (iii) Find the total energy,  $E$ , of the particle in this orbit.

**Problem I-2.** (Electricity and Magnetism)



A flat insulator in the shape of a circular disc of radius  $b$  has a constant surface charge density  $\sigma$ . The disc rotates with angular velocity  $\omega$  about an axis perpendicular to the disc through its center. Calculate the magnetic field  $\mathbf{B}$  for all points on the axis of rotation.

*Hint:* In order to perform the calculation, first divide the disc into a set of rings of radius  $s$  and width  $ds$ . For each ring, use the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\ell' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},$$

where  $d\ell'$  is a vector along the direction of current flow. After integrating over  $d\ell'$  for one ring, one should sum over the rings to get the total magnetic field, but in order to save time just write down the integral over the rings that should be evaluated without actually doing the integration explicitly. This remaining integral has the form

$$\int_0^b f(s)ds.$$

In other words, find the function  $f(s)$ .

**Problem I-3.** (Thermodynamics)

Consider a transformation where one mole of an ideal gas with the adiabatic constant  $\gamma$  is compressed reversibly from  $P_i$  to  $P_f > P_i$  at constant temperature, then allowed to expand adiabatically and reversibly back to  $P_i$ . Imagine doing this transformation  $N$  times: calculate the corresponding change of entropy and energy, as well as the final temperature  $T_f$  if the initial temperature was  $T_i$ .

**Problem I-4.** (Quantum Mechanics)

Consider a quantum particle of mass  $m$  in three dimensions subject to the potential

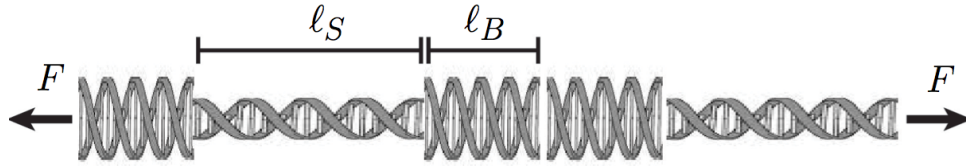
$$V(\mathbf{r}) = \begin{cases} 0 & \text{if } 0 < |\mathbf{r}| < R, \\ +\infty & \text{otherwise.} \end{cases}$$

This is the spherical analogue of the infinite barrier potential well.

- (i) Which are the conserved quantities for the system and why?
- (ii) Find the ground state energy and eigenfunction of the system.
- (iii) Using the thermodynamical definition of pressure, find the pressure  $p$  that the particle exerts – in its ground state – on the sphere's inner walls.

*Hint:* The Laplace operator in spherical coordinates can be decomposed into its radial and angular parts

$$\Delta\Psi = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\Psi) - \frac{1}{r^2} \frac{\mathbf{L}^2}{\hbar^2} \Psi.$$

**Problem I-5.** (Statistical Physics)

A segment of DNA can exist in two forms, the usual form called  $B$  and the stretched form called  $S$ . The usual form has length  $\ell_B$  and has an associated energy  $E_B$ , and the stretched form has length  $\ell_S$  and has an associated energy  $E_S$ . Consider a one-dimensional chain of DNA with  $N$  segments,  $N_B$  of form  $B$  and  $N_S$  of form  $S$ . The states of segments can be assumed to be independent of each other.

- (i) Write down the partition function for the system at temperature  $T$ .
- (ii) Assume that  $E_S > E_B$  and  $\ell_S > \ell_B$ . Give an expression for the average length of the DNA molecule as a function of temperature at a fixed  $N$ .
- (iii) What is the expected DNA chain length in the limit of high temperatures?
- (iv) A tensile force  $F$  is applied. Find the partition function of the DNA chain for a finite force  $F$ .
- (v) Derive the linear response function,

$$\left. \frac{\partial \langle L \rangle}{\partial F} \right|_{F=0},$$

where  $\langle L \rangle$  is the average length of the chain.

**Problem I-6.** (Condensed Matter)

A donor impurity atom is embedded into a semiconductor with bulk dielectric constant  $\varepsilon$  and conduction band effective mass  $m^*$ .

- (i) What are the ground-state radius and binding energy of the Bohr-like orbit of the electron captured by the donor? Please derive the relevant equations (rather than state them by memory) and use them to estimate numerically the values of these parameters for a typical semiconductor.
  - (ii) Use these estimates to argue why it is legitimate in the present derivation to employ the macroscopic dielectric function and the band effective mass.
  - (iii) Use your estimates to conclude whether impurity ionization is a significant effect at room temperature.
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**Problem I-7.** (Experimental Physics)

In scanning tunneling spectroscopy, when a conducting tip is brought very close to a sample surface, the tunneling current which flows between the tip and the sample depends on the voltage difference between them.

- (i) Describe how this technique can be used to determine the work function of the sample.
  - (ii) Let the sample be biased by a negative voltage  $V$  with respect to the tip. The measurement temperature is low enough so that the Fermi function cuts off sharply at the Fermi surface. Assume that the tip material has a flat density of states  $\rho(\varepsilon)$  within the energy range of the Fermi surface that you wish to study, and assume the tunneling matrix element is independent of the energy difference between the tip and sample. Derive an expression for the net tunneling current.
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**Problem I-8.** (Math Methods)

- (i) Evaluate the Fresnel integral

$$\int_{-\infty}^{\infty} e^{-ix^2} dx.$$

*Hint:* Consider the contour integral of  $e^{-z^2}$  around the boundary of the circular sector  $0 \leq \theta \leq \pi/4$ ,  $0 \leq r \leq R$ . You may assume that in the limit  $R \rightarrow \infty$ , the contribution from the circular arc goes to zero.

- (ii) Use the result in (i) to show that  $f(x) = x^{-1/2}$  is self-reciprocal under both Fourier cosine and sine transforms, that is

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{-1/2} \cos(xt) dx = t^{-1/2} \quad \text{and} \quad \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{-1/2} \sin(xt) dx = t^{-1/2}.$$

Assume  $t > 0$ .

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**Problem I-9.** (Biophysics)

Bacteria and archaea generally rely on passive diffusion for supply of some essential molecules from the environment (e.g.  $O_2$  if they are aerobic). Consider a spherical bacterium of radius  $R$  and density  $\rho$ , in an environment where the free concentration of  $O_2$  at some distance far away from the cell surface  $c(\infty) = c_0$ , and where the diffusion coefficient of  $O_2$  is  $D$ .

- (i) Write down an expression for the number of molecules arriving per unit time (i.e. the rate  $dn/dt$ ) at the cell surface. To make things a little simpler, assume that all molecules that arrive at the surface are swallowed up by the receptors, in other words:  $c(R) = 0$  (i.e. the cell surface has machinery that acts as perfect absorbers of  $O_2$ ). This rate is significant for many reactions in biophysics, from receptor-mediated signaling to polymerization of growing filaments.
  - (ii) A given biomass of cells has a characteristic burn rate of oxygen  $b$ ; that is how many molecules of  $O_2$  are consumed per kg per s to satisfy the energy demand of the cells. What condition does this impose on the radius  $R$  of the cell? You just found out why bacteria are small (i.e. generally a few micrometers in size).
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**Problem I-10.** (Astrophysics)

The total gravitational energy of a spherically-symmetric astronomical object (star, white dwarf, globular star cluster, spherical galaxy, *etc.*) is

$$E_{\text{grav}} = - \int_0^M \frac{Gm}{r} dm . \quad (10.1)$$

Here,  $M$  and  $R$  are total mass and radius of the object, respectively, and the running variable  $m$  (i.e., the mass contained in a spherical subshell of radius  $r$ ) can be more precisely written as an indexed variable  $M_r$

$$m = M_r = 4\pi \int_0^r s^2 \rho(s) ds , \quad (10.2)$$

In astrophysics it is well known that approximately,

$$E_{\text{grav}} = -\alpha \frac{GM^2}{R} . \quad (10.3)$$

with  $\alpha$  being a constant of order unity.

**Your task:**

For very special density distributions,  $\alpha$  can be computed exactly. Assume a density distribution of the form

$$\rho(r) = \rho_c \left[ 1 - \left( \frac{r}{R} \right)^2 \right] . \quad (10.4)$$

- (i) Determine the central density  $\rho_c$ .
  - (ii) Determine the constant  $\alpha$ .
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**Problem I-11.** (Special Relativity)

A rocket ship of proper length  $L$  leaves the earth vertically at constant speed  $(4/5)c$ . A light signal is sent vertically after it from the same launch spot. The light signal arrives at the rocket's tail at  $t = 0$  according to both rocket and earth based clocks. When does the signal reach the nose of the rocket according to (a) the rocket clock; (b) the earth clock? *(For simplicity, ignore the motion of the earth.)*

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