M03M.1—Lagrange points and WMAP

Problem

The Earth is ins a circular orbit of angular frequency ω about the Sun. The Sun is so much more massive that the Earth that, for our purposes, it may take to sit at rest at the center of our coordinate system. Lagrange discovered that there exist a certain number of equilibrium points at which an artificial satellite of negligible mass can orbit the Sun with the same frequency ω as the Earth (while maintaining a fixed distance from the Earth and the Sun). Such orbits are ideally suited for space-based observatories of various kinds. We will explore some of the properties of the 'Lagrange points' in this problem.

a) Consdier points on the line that runs from the Sun through the Earth. This line is of course stationary in the reference frame that rotates with the Earth in its orbit. Show that there is one point on this line outside the Earth's orbit where a test particle may sit at equilibrium in the rotating frame. This point is commonly designated as the L2 Lagrange point. (There is a similar Lagrange point, L1, inside the Earth's orbit as well.)

b) Give an approximate expression, correct to leading order in the small quantity $\beta = M_e/M_s$, for the distance from the Earth to the L2 Lagrange point described above. Express your answer in terms of the masses and R, the Earth-Sun distance. The Wilkinson Microwave Anisotropy Probe (WMAP) is stationed at L2: using $\beta \approx 3 \times 10^{-6}$ and $R = 1.5 \times 10^8$ km, find the distance from Earth to WMAP.

c) Determine whether the L2 equilibrium point is stable or unstable against small perturbations in position along the Earth-Sun line.

M03M.2—Rolling Sphere

Problem

The problem of a sphere rolling without slipping directly down an inclined plane is an old chestnut of freshman physics. In this problem, you are asked to examine more general motions of a sphere rolling without slipping on inclined and rotating planes.

The state of motion of a sphere of mass M is specified by giving its position $\vec{R}$ and velocity $d\vec{R}/dt$ with respect to a fixed point and its angular velocity $\vec{\omega}$ with respect to its center of mass. The rolling without slipping constraint is usually expressed as $d\vec{R}/dt = a\vec{\omega} \times \hat{n}$ where a is the sphere's radius and $\hat{n}$ is a unit vector normal to the plane on which the sphere is rolling. There is a constant force $\vec{f}$ which acts in the plane to enforce the constraint and this constraint force has to be taken into account in the equations for acceleration of the center of mass and for the time rate of change of the angular momentum $I\vec{\omega}$ about the center of mass.

a) Suppose that the plane is inclined to the horizontal at an angle θ in the Earth's gravitational field. Show that you can eliminate the constraint force to find an equation for the center of mass alone and show that the center of mass experiences an acceleration of $\frac{5}{7}g\sin\theta$ along the 'downhill' direction of the plane. Hence, the trajectories of this rolling sphere are parabolae.

b) Now consider the case that the pane is not inclined, but is rotating with angular velocity $\vec{\Omega}$ about the vertical axis. The most important modification to the calculation you have just done is that the condition for rolling without slipping changes. Use the changed condition in your previous analysis to show that the center of mass executes circular motion in the horizontal *inertial* frame with frequency $\frac{2}{7}\Omega$! In other words, the rolling sphere executes Larmor orbits, just like a charged particle in a uniform magnetic field!

For reference, note that the moment of inertia of a homogeneous sphere is $\frac{2}{5}Ma^2$.

M03M.3—Helmholtz Resonator

Problem

Consider a cube of size l filled with air. Air in the cube has an equilibrium pressure p_0 and density ρ_0.

a) Write the wave equation for small fluctuations $p_1(\vec{x}, t)$ of the pressure field. State the relation between fluctuations ρ_1 in the density and fluctuations p_1 in the pressure (it involves the velocity of sound c_s). What is the equation that would allow one to infer the local velocity $\vec{v}$ of the air fluid from the gradient of the fluctuating pressure field?

b) State the proper boundary condition which must be satisfied by the pressure field at the walls of the cube and use it to derive an expression for the eigenfrequencies of oscillation of the air in the cube. Explain why the mode which has no spatial dependence ($p = p(t)$) and eigenfrequency zero is not actually in the spectrum. Note that all real modes have frequencies that scale with the size of the cube as $1/L$.

c) Now make a small hole of area S in a wall of the cube and attach an *open* tube of the same area and length l so that air can flow in and out of the cube. Revisit the problem of a mode in which the pressure is independent of position. A pressure excess inside the cube will accelerate the air in the tube, leading to an outflow of air which will then reduce the pressure inside the cube. Write down the coupled equations of motion for the pressure $p(t)$ in the cube and the speed $u(t)$ of airflow in the tube. Solve to find the oscillation frequency of this 'zero mode' and note that it scales as $1/L^{3/2}$ (if l and S are held fixed). This device for making low-frequency sound is called a Helmholtz resonator.
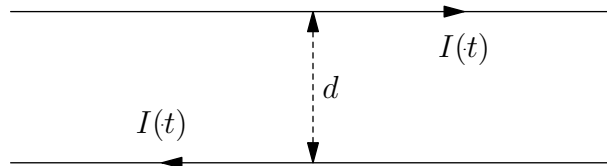
The setup is as indicated in the following figure:
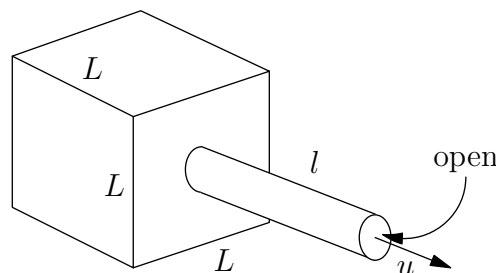
M03E.1—Force Between Current Carrying Wires

Problem

Two infinitely long parallel wires, a distance d apart, carry time-dependent currents $I(t)$ of the same magnitude but opposite direction, as shown in the figure:

We will consider two possible time histories of the current:

a) Suppose that the current switches on suddenly at time $t = 0$ and remains constant thereafter (i.e. that $I(t) = I_0 \theta(t)$). Calculate, as a function of time, the force per unit length $F(t)$ on the wires and sketch your result. It is not really possible to instantaneously turn on a current in a circuit: explain how your answer reflects this fact.

b) Now let's do it in a more realistic fashion by turning on a linearly increasing current at time $t = 0$ which continues to increase until I_0 is reached (i.e. $I(t) = bt$ for $t < 0 < I_0/b$ and $I = I_0$ for later times). As in the previous item, calculate the force per unit length $F(t)$ between the wires and plot your result.

Note the possibly useful indefinite integral:

$$\int \frac{dy}{\sqrt{y^2 - a^2}} = \log(\sqrt{y^2 - a^2} + y)$$

M03E.2—The Method of Images

Problem

The 'method of images' allows us to solve many problems involving point charges and spherical conductors of various kinds. In this question you are asked to derive and then use this method in some representative applications.

a)   Consider a charge Q placed at a distance $R > a$ from the center of a sphere of radius a (for the moment, this is just a geometrical sphere, not a conductor or any other physical object). Show that if a certain charge of opposite sign is placed inside the sphere at the appropriate location then the spherical surface of radius a is a $V = 0$ equipotential surface. The result can be used in the following applications:

b)   A point charge Q is placed at a distance $R > a$ from the center of a conducting sphere of radius a. Find the force exerted on the sphere if the total charge on the sphere is Q.

c)   Consider the distribution of the total charge Q on the surface of the sphere under the conditions just described. Find an equation for the distance R such that surface charge density of opposite sign to Q first appears somewhere on the sphere.

d)   Two perfectly conducting spheres of radius a are placed far apart (their centers are separated by $R \gg 2a$) and kept at the same potential V_0 (this condition could be enforced by connecting the spheres with a fine wire). What is the charge on each sphere, correct to first order in a/R?

e)   What is the charge on the two spheres correct to second order in a/R?

M03E.3—Electromagnetic Wave in a Plasma

Problem

The $\boldsymbol{E}$-vector of a plane electromagnetic wave propagating along the z-axis and having a polarization vector $\vec{e} = (\alpha_x, \alpha_y, 0)$ can be written

$$\vec{E}(\vec{r}, t) = E(\alpha_x \hat{x} + \alpha_y \hat{y}) e^{i(kz - \omega t)}$$

(the polarization vector is taken to be of unit length, $\vec{\alpha} \cdot \vec{\alpha}^* = 1$). In free space, the dispersion relation is $\omega = ck$ and the wave propagates with both phase and group velocity equal to c.

Now let the wave propagate through a dilute plasma containing a density N of free mobile electrons of mass m and charge e (along with a background of compensating positive charge taken to be so massive as to be fixed in place). By solving for the motion of an electron in the electric field of the propagating wave (i.e. ignoring the effect of the wave's B-field on the motion) one can infer a polarization density $\vec{P} \propto \vec{E}$. This in turn allows us to infer a frequency-dependent dielectric constant via

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon(\omega) \vec{E}.$$

The index of refraction of plasma is then given by $n(\omega) = \sqrt{\epsilon/\epsilon_0}$.

a)   Use this line of argument to compute the frequency dependence of the index of refraction $n(\omega)$ of a plasma. Turn your result into a dispersion relation $\omega(k)$ and find the limiting frequency ω_p as the wavelength of the wave goes to infinity. This cutoff frequency, below which waves cannot propagate, is called the plasma frequency.

b)   Now let the plasma be subject to a static magnetic field $B_0 \hat{z}$ in the direction of propagation of the wave. Also assume that the electrons have some kinetic energy so that, in the absence of any other perturbation, they execute circular motion about the static magnetic field lines at the Larmor frequency $\omega_L = |eB_0|/m$. Extend your calculation of the dielectric constant of the plasma to this new case. The response is different for different states of circular polarization so, for definiteness, analyze the case of right circular polarization of the propagating wave $(\vec{\alpha} = (\hat{x} + i\hat{y})/\sqrt{2})$.

c)   Find the lowest frequency at which such a wave can propagate. Express your answer in terms of ω_p and ω_L.

M03Q.1—Flipping Spins With a Magnetic Field

Problem

A particle with spin $1/2$ is at rest in a static magnetic field of strength $B_z = B_0$ oriented along the z-axis. The magnetic moment interaction splits the $S_z = +1/2$ from the $S_z = -1/2$ state. One can manipulate the spin state of the particle by subjecting it to a time-dependent field $\vec{B}_1 = B_1(\cos(\phi(t)), \sin(\phi(t)), 0)$ that rotates in the x-y plane at a variable frequency $\dot{\phi}(t)$. We will discuss different ways of choosing $\phi(t)$ to achieve the goal transforming an initial $S_z = -1/2$ state into an $S_z = +1/2$ state.

The two-component spin wave function ψ of this system evolves under a time-dependent Hamiltonian which can be written as

$$H = \mu B_0 \sigma_z + \mu B_1 (\sigma_x \cos(\phi(t)) + \sigma_y \sin(\phi(t)))$$

where μ is the magnetic moment and $\sigma_{x,y,z}$ are the Pauli matrices (with $\sigma_x^2 = 1$ etc.). For general $\phi(t)$ this is hard to solve, but various special cases and approximations are helpful as we now show.

a)   First show that the 'interaction picture' wave function

$$\hat{\psi} = \exp(-i\phi(t)\frac{\sigma_z}{2})\psi$$

evolves according to a simpler Hamiltonian $H_{rot}(t)$ which becomes time-independent when ϕ is linear in t.

b)   Consider the case that $\dot{\phi} = \omega_1$ for a finite time interval $-T < t < T$ and is zero otherwise (this is not too hard to realize experimentally). H_{rot} is now time-independent and you can solve Schrödinger equation. Find ω_1 and T such that a spin-down state is perfectly converted into a spin-up state by the $-T \to T$ time evolution (this is sometimes called a 'Π pulse'). Note that for this method to work for a collection of spins, they must all be subject to the same field $B_0 \hat{z}$.

c)   Now consider the case of a 'chirped' frequency such that $\dot{\phi}(t) = -\alpha t$ for $-T < t < T$. $H_{rot}(t)$ now varies with time, but if its matrix elements vary slowly (i.e. if α is small), and there is no level crossing, the adiabatic theorem should apply. This means that the system remains in the 'same' eigenstate of the instantaneus Hamiltonian for all time. Make a rough plot of the eigenenergies of the instantaneous Hamiltonian H_{rot} as a function of time for this case. Show that the lowest energy eigenstate evolves from spin down at $t = -T$ to spin up at $t = +T$ if we take $\alpha T \gg \mu B_0, \mu B_1$. Note that this method of spin flipping is insensitive to the value of B_0 an could work for a collection of spins in an inhomogeneous environment.

M03Q.2—Scattering From a Central Potential

Problem

Consider the familiar problem of s-wave ($l = 0$) scattering of a particle of mass m from an attractive square-well potential of depth V_0 and radius r_0: $V = -V_0\theta(r_0 - r)$.

a)   First, let's look at whether the potential has any s-wave bound states. Show that there is a critical potential strength V_{crit} such that for $0 < V_0 < V_{crit}$ there is *no* s-wave bound state. Put another way, show that a bound state (of zero energy) first appears at V_{crit}.

b)   Set up an equation for determining the phase shift $\delta_0(k)$ and show that it implies that $\delta_0 \sim Ak$ as $k \to 0$. Evaluate the coefficient A as a function of V_0.

c)   Show that A vanishes, as it should, as $V_0 \to 0$. More alarmingly, show that A blows up as the potential strength V_0 approaches V_{crit} from below!

d)   Calculate the contribution of the s-wave phase shift to the total cross section in the limit of small k . How does the zero-energy cross section behave as $V_0 \to V_{crit}$? Comment.

M03Q.3—Addition of Angular Momentum

Problem

The total angular momentum of a valence electron in an atom of orbital angular momentum l can be either $J = l + 1/2$ or $J = l - 1/2$. The two total angular momentum states are typically split by spin-orbit interactions, leaving $2J + 1$ degenerate states of the same J, but different J_z. Upon applying a uniform magnetic field $B\hat{z}$, these magnetic substates are split by the interaction between the magnetic moment and the applied magnetic field. The perturbation Hamiltonian is

$$H_{int} = -\frac{e\hbar}{2m_e c}B(L_z + 2S_z) = -\frac{e\hbar}{2m_e c}B(J_z + S_z)$$

where the relative factor of two between L_z and S_z is due to the famous fact that the g-factor of the electron is two. The Zeeman splitting of the degenerate multiplet of total angular momentum J is thus

$$\delta E_{J,J_z} = -\frac{e\hbar B}{2m_e c}\langle J, J_z|(J_z + S_z)|J, J_z\rangle = -\frac{e\hbar B}{2m_e c}(J_z + \langle J, J_z|S_z|J, J_z\rangle).$$

To evaluate this explicitly we need to solve the slightly nontrivial problem of evaluating the matrix elements of S_z.

a)   Consider the special case $l = 1$. Construct the $J = 3/2, 1/2$ states by angular momentum addition and evaluate $\langle J, J_z|S_z|J, J_z\rangle$ for all the states. Show that the energy splittings satisfy $\delta E_{J,J_z} = g_J J_z$ and state the two values of g_J that you have just computed.

b)   Now consider the case of general l. It is possible, but tedious, to directly verify that $\delta E = g_J^l J_z$ for the two possible J multiplets. Assuming that this is so (i.e. that it suffices to compute g_J^l in any magnetic sublevel J_z), find the relevant g_J^l factors for the two multiplets $J = l \pm 1/2$.

c)   The above calculations are particular examples of a general result most easily proved using the Wigner-Eckart theorem. For a general vector operator $\vec{A}$ (one whose commutation relations with $\vec{J}$ are of the form $[J_i, A_j] = i\epsilon_{ijk}A_k$) the theorem says that

$$\langle J, M'|\vec{A}|J, M\rangle = \frac{\langle J|\vec{J} \cdot \vec{A}|J\rangle}{J(J+1)}\langle J, M'|\vec{J}|J, M\rangle.$$

Use this theorem to rederive the g_J^l factor for $J = l+1/2$ multiplet which you computed above.

M03T.1—Thermodynamics of an Elastic String

Problem

An elastic string is found to have the following properties:

- To stretch it to a total length x requires a force $f = \mu x - \alpha T + \beta T x$. Assume that α, β, μ are constants.

- Its heat capacity at constant length x is proportional to temperature: $C_x = A(x)T$.

We can use thermodynamic identities to derive from these facts a variety of other thermal properties. More specifically:

a)   Calculate $\frac{\partial S}{\partial x}\big|_T$.

b)   Show that A has to be independent of x.

c)   Calculate $\frac{\partial S}{\partial T}\big|_x$ and give the general expression for entropy $S(x, T)$ assuming $S(0, 0) = B$, where B is a constant.

d)   Compute the heat capacity at zero tension $C_F = T\frac{\partial S}{\partial T}\big|_{f=0}$.

M03T.2—White Dwarf Star

Problem

Let us model a white dwarf star as a degenerate Fermi gas of electrons, supported against gravitational collapse by the electron degeneracy pressure. For simplicity, we will assume that the star is a sphere of radius R and *uniform* mass density containing N electrons, N protons, and N neutrons for an approximate total mass of $M = 2Nm_p$.

a) First, assume that the electrons are not relativistic. Find their Fermi energy and show that at absolute zero, their total kinetic energy is

$$U_k = \frac{3N(\hbar\pi)^2}{10m_e}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}}$$

where V is the volume of the star. (Note that the total kinetic energy of the nucleons is much smaller than that of the electrons.)

b) The gravitational binding energy of a uniform density sphere is

$$U_{grav} = -\frac{3GM^2}{5R}.$$

FInd the equilibrium radius for the white dwarf. Eliminate N. How does this radius depend on the mass?

c) If instead the electrons are highly relativistic, so that their energy and momentum are related by $\epsilon = cp$, then find the Fermi energy and show that the total kinetic energy is now

$$U_k = \frac{3N\hbar\pi c}{4}\left(\frac{3N}{\pi V}\right)^{\frac{1}{3}}.$$

d) Under what conditions is a highly relativistic degenerate electron star unstable against collapse? This is called the Chandrasehkar limit. A star that violates the limit will collapse into a neutron star or black hole, depending on whether neutron degeneracy pressure can hold up the star.

Constants you may need:

$$G/(\hbar c) = 6.707 \times 10^{-39}\,(\text{GeV}/\text{c}^2)^{-2}$$
$$1\,GeV/c^2 = 1.78 \times 10^{-27}\,\text{kg}$$
$$M_\odot = 1.99 \times 10^{30}\,\text{kg}$$

M03T.3—Particles in a Box

Problem

Consider a single free particle of mass m confined to a volume V. Let $Z_1(m)$ denote the quantum partition function for this system (where the partition sum is taken over the discrete energy levels of a particle of mass m in a box of volume V).

a)   Show that $Z_1(m) \to V/\lambda^3$ with $\lambda = h/\sqrt{2\pi mkT}$ in the classical (or small $\hbar$) limit. Use this result to calculate the classical energy and heat capacity at fixed volume of the single particle system.

b)   Identify the temperature at which this approximation breaks down.

c)   Now consider a system consisting of two identical, non-interacting particles in the same box. Because of the effects of identical particle statistics, the classical expectation for the two-particle partition function $Z_2(m) = Z_1(m)^2$ is not quite correct. Show that the exact free boson and free fermion two-particle partition sums can in fact be expressed in a simple way in terms of the one-particle functions $Z_1(m)$ and $Z_1(m/2)$.

d)   Using the classical approximation $Z_1(m) = V/\lambda^3$ derived in the first part of this problem, calculate the correction to the energy E and the heat capacity C due to Bose or Fermi statistics.