Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 9, 2006
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

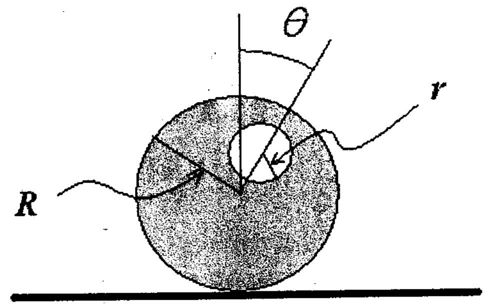
Good luck!!

Problem 1 : Section 1 Classical Mechanics

A cylinder of length L, radius R and mass density ρ rolls on a horizontal surface without slipping. A hole of radius $r<R$ has been drilled through the cylinder parallel to its axis at a distance $R/2$ from its center. Describe the orientation of the cylinder by specifying the angle θ between the vertical direction and a line connecting the centers of the cylinder and the hole. If initially the cylinder is at rest but θ has a small non-zero value,

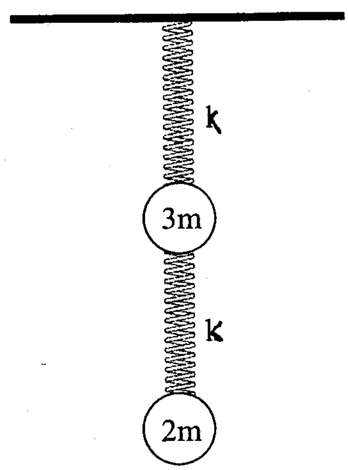
$$\theta\,(t=0) = \theta_0 \ll 1$$

predict the subsequent motion θ (t). Draw a graph of θ (t) indicating the times, if any, where $\theta = 0$.



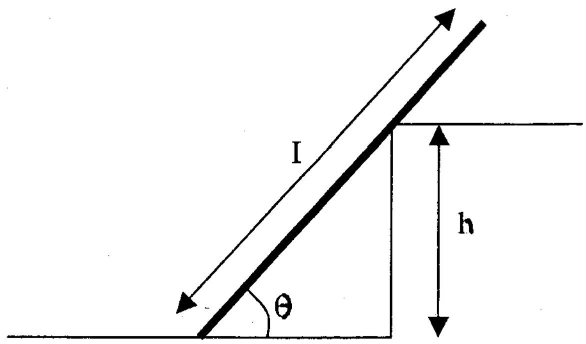
Problem 2 : Section 1 Classical Mechanics

Two massless springs with spring constant k are connected to two masses that hang vertically as shown in the figure. The top one has mass 3m and the bottom one has mass 2m. Find the frequencies of the normal modes of this system for vertical displacements. Describe the motion of each of the normal modes.



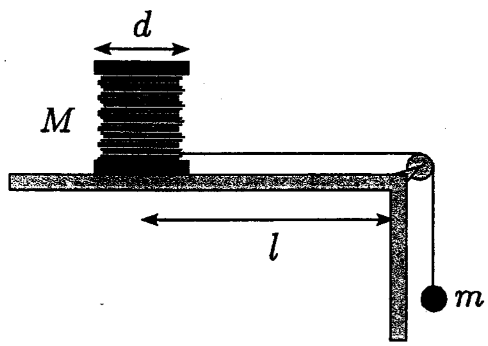
Problem 3 : Section 1 Classical Mechanics

A uniform ladder of weight W and length L is leaning at an angle θ against a structure whose height is $h < L$. The situation is pictured in the figure below. (Note that the normal force at the corner is perpendicular to the ladder.) There is static friction between the ladder and the ground, but negligible friction between the ladder and the vertical structure. Find the coefficient of friction between the ladder and ground that would be necessary to keep the ladder from moving in terms of L, h, and θ.



Problem 4 : Section 1 Classical Mechanics

A solid spool of mass M and diameter d is released from rest a distance l from the edge of a table. The spool is connected via a massless, inextensible string to a hanging mass m. The spool slides and rotates freely. What is the velocity of the mass m when the spool's center of mass reaches the edge of the table?



Problem 5 : Section 1 Classical Mechanics

Consider the motion of the earth around the sun. Let's approximate the orbit as circular. Suppose the sun very slowly loses its mass, from an original mass of M_1 to a mass of M_2. Suppose the initial radius of the orbit is R_1 and the final radius is R_2.
What is R_2 in terms of the other parameters?

N. Christ November 26, 2005

Quals Problems

1. Quantum Mechanics:

   Consider a hydrogen atom in the 1s state. The magnetic interaction of the spin of the proton $\vec{s}_P$ and that of the electron $\vec{s}_e$ is given by the hyperfine Hamiltonian:

$$H_{\mathrm{HF}} = +\frac{8\pi}{3}\frac{g_P g_e}{4 m_P m_e c^2}\,\vec{s}_P \cdot \vec{s}_e\,\delta^3(\vec{r}_e) \tag{1}$$

   where $\vec{r}_e$ is the relative coordinate of the electron, g_e and g_P the g-factors for the electron and proton and m_P and m_e their respective masses.

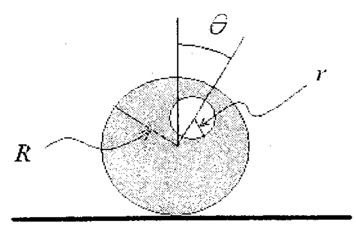
   (a) If the hydrogen atom wave function is $\psi(\vec{r}) = e^{-r/a_0}/\sqrt{\pi a_0^3}$ with $a_0 = \hbar^2/(m_e e^2)$, find the splitting between the $F = 0$ and $F = 1$ hyperfine states. (Here $\hbar\vec{F}$ is the total spin of the electron and proton.) [8 points]

   (b) If a weak magnetic field $\vec{B}$ is applied, determine the shift in the energy, $\delta E(B)$, of the lowest hyperfine state. [10 points]

   (c) Compute the magnetic polarizability, $\alpha_B = -\partial^2\delta E(B)/\partial B^2|_{B=0}$ for this ground state. [2 points]

2. Mechanics:

   A cylinder of length L, radius R and mass density ρ rolls on a horizontal surface without slipping. A hole of radius $r < R$ has been drilled through the cylinder parallel to its axis at a distance $R/2$ from its center. Describe the orientation of the cylinder by specifying the angle θ between the vertical direction and a line connecting the centers of the cylinder and the hole. If initially the cylinder is at rest but θ has a small non-zero value, $\theta(t = 0) = \delta\theta$, describe the subsequent motion. Find the time required for θ to decrease to zero. [20 points]



Sec 1 # 1
Christ p. 2
(solution)

2. Consider rotation about the point of contact, P. Treat the cylinder as a complete cylinder of radius R with mass $M == \rho\pi R^2 L$ and a second of negative mass $-m = -\rho\pi r^2 L$. The first cylinder exerts no torque about P while the second exerts:

$$\tau = -\frac{3R}{2}\rho\pi r^2 Lg\theta \quad [5 \text{ points}] \tag{7}$$

assuming θ to be small.

The moment of inertia about P is that of the cylinder of radius R minus that of r:

$$I = \frac{1}{2}MR^2 + MR^2 - \frac{1}{2}mr^2 - m(3R/2)^2 \quad [5 \text{ points}] \tag{8}$$

where the parallel axis theorem has been used.

Finally we can combine these:

$$I\frac{d^2\theta}{dt^2} = \tau = -\frac{3R}{2}\rho\pi r^2 Lg\theta \quad [5 \text{ points}] \tag{9}$$

which describes simple harmonic motion with period

$$T = \sqrt{2I/3R\rho\pi r^2 Lg} \quad [3 \text{ points}] \tag{10}$$

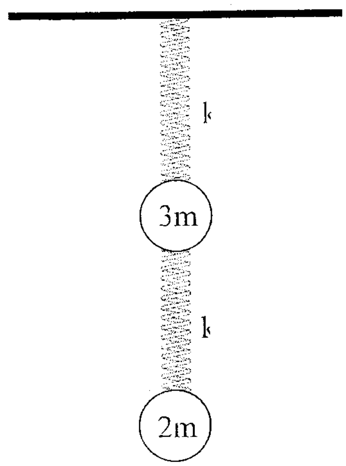
Thus, the cylinder will roll back and forth, executing simple harmonic motion about the equilibrium position $\theta = 0$. It will take $T/4$ time units to first reach $\theta = 0$ [2 points].

SCC 1 # 2

Quals Problem 2 Classical Mechanics

M. Shaevitz
2006
11/28/05

Two massless springs with spring constant k are connected to two masses that hang vertically as shown in the figure. The top one has mass 3m and bottom one has mass 2m. Find the frequencies of the normal modes of this system for vertical displacements. Describe the motion of each of the normal modes.



Solution:

Let $x_1(x_2)$ be the position of the top (bottom) mass with respect to the ceiling.

$$L = \frac{1}{2}3m\dot{x}_1^2 + \frac{1}{2}2m\dot{x}_2^2 + 3mgx_1 + 2mgx_2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k\left(x_2 - x_1\right)^2$$

Then Lagrange's equations are:

$$3m\ddot{x}_1 - 3mg + 2kx_1 - kx_2 = 0$$

$$2m\ddot{x}_2 - 2mg + kx_2 = 0$$

The mg factors can be removed with a change of variables.

Assuming small oscillations with $x_i = A_i \cos\omega t$ gives

$$\begin{pmatrix} 2k - 3m\omega^2 & -k \\ -k & k - 2m\omega^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

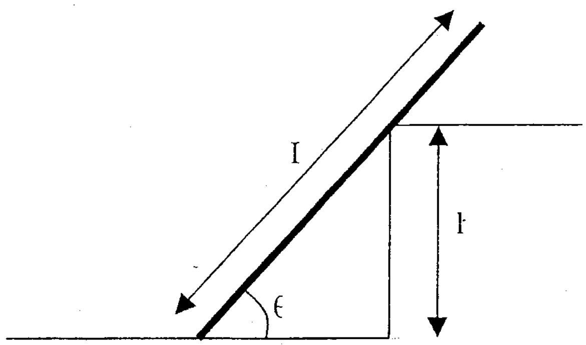
which yields nomal mode frequencies of $\sqrt{k/m}$ and $\sqrt{k/6m}$

For the $\sqrt{k/m}$ frequency the motion has both masses moving in opposite directions with $x_1 = -x_2$ and for the $\sqrt{k/6m}$ frequency the motion has both masses moving in same direction with $x_1 = \frac{3}{2}x_2$.

Quals Problem 3 Classical Mechanics M. Shaevitz 2006 11/28/05

Sec 1 # 3

A uniform ladder of weight W and length L is leaning at an angle θ against a structure whose height is h < L. The situation is pictured in the figure below. (Note that the normal force at the corner is perpendicular to the ladder.) There is static friction between the ladder and the ground, but negligible friction between the ladder and vertical structure. Find the coefficient of friction between the ladder and ground that would be necessary to keep the ladder from moving in terms of L, h, and θ.



Solution:

Let N_1 be the upward normal force of the ground and N_2 be the normal force from the vertical corner.

Vertical Forces: $N_1 + N_2\cos\theta - W = 0$

Horizontal Forces: $-N_2\sin\theta + f = 0$

Torques around ground point: $-W\frac{L}{2}\cos\theta + N_2\frac{h}{\sin\theta} = 0$

Solving these gives:

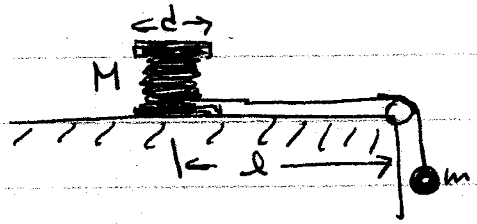
$$N_2 = \frac{WL\sin\theta\cos\theta}{2h} \qquad N_1 = \frac{W(2h - L\sin\theta\cos^2\theta)}{2h} \qquad f = \frac{WL\sin^2\theta\cos\theta}{2h}$$

Then $\mu = f/N_1 = \dfrac{L\sin^2\theta\cos\theta}{(2h - L\sin\theta\cos^2\theta)}$

Amber Miller

Mechanics

Problem - A solid spool of mass M and diameter d is released from rest a distance l from the edge of ~~the~~ a table. The spool is connected via a massless, inextensible string to a hanging mass m. The spool slides and rotates freely. What is the velocity of the mass m when the spools center of mass reaches the edge of the table?



Solution

$$\tau = I\ddot{\theta} = T\frac{d}{2} \qquad T = M\ddot{x} = mg - m\ddot{x} - m\frac{d}{2}\ddot{\theta}$$

$$\ddot{\theta}\left[\frac{1}{2}M\left(\frac{d}{2}\right)^2\right] = T\frac{d}{2} \qquad \cancel{M\ddot{x} = mg - m\ddot{x} - m\frac{d}{2}(\ldots)}$$

$$T = M\frac{d}{4}\ddot{\theta}$$

$$\ddot{\theta} = \frac{4T}{Md} = \frac{4}{Md}(M\ddot{x})$$

$$= \frac{4}{d}\ddot{x}$$

$$M\ddot{x} = mg - m\ddot{x} - m\frac{d}{2}\left(\frac{4}{d}\ddot{x}\right)$$

$$= mg - m\ddot{x} - 2m\ddot{x}$$

$$= mg - 3m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{mg}{M+3m}$$

time to get to edge of table $x = \frac{1}{2}at^2 = l$

$$t = \sqrt{\frac{2l}{a}} = \sqrt{\frac{2l}{\ddot{x}}}$$

$$t = \sqrt{\frac{2l(M+3m)}{mg}}$$

velocity of m $v = at = (\ddot{x} + \frac{d}{2}\ddot{\theta})t$

$$= \left[\frac{mg}{M+3m} + \frac{d}{2}\left(\frac{4}{d}\right)\left(\frac{mg}{M+3m}\right)\right]t$$

$$= \frac{mgt}{M+3m}(1+2)$$

$$= \frac{3mgt}{M+3m} = \frac{3mg}{M+3m}\sqrt{\frac{2l(M+3m)}{mg}}$$

$$= \sqrt{\frac{2l(M+3m)}{mg}\,9\left[\frac{mg}{M+3m}\right]^2}$$

$$\boxed{v = \sqrt{\frac{18lmg}{(M+3m)}}}$$

Classical mechanics sec 1 #5

CORRECTED VERSION

Subject: 2 questions for the Quals committee
From: Lam Hui <lhui@astro.columbia.edu>
Date: Wed, 23 Nov 2005 12:42:20 -0500 (EST)
To: lalla@phys.columbia.edu, lhui@phys.columbia.edu

To the Quals Committee,

Below please find two problems: one mechanics and one general.

Lam

Mechanics -

Problem:

Consider the motion of the earth around the sun. Let's approximate the orbit as circular. Suppose the sun very slowly loses its mass, from an original mass of M_1 to a mass of M_2. Suppose the initial radius of the orbit is R_1 and the eventual radius is R_2. What is R_2 in terms of the other parameters?

Solution:

The angular momentum is an adiabatic invariant. Therefore, $~~M_1~~ R_1 v_1 = ~~M_2~~ R_2 v_2$, with $v_1 = \sqrt{G M_1/R_1}$ and $v_2 = \sqrt{G M_2/R_2}$. Hence, $R_2 = R_1 (M_1/M_2)~~[illegible]~~$ i.e. the orbit expands under mass loss.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 9, 2006
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

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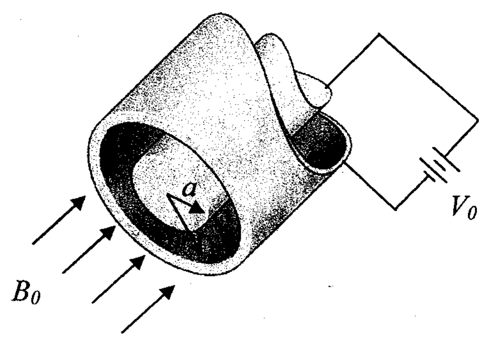
Questions should be directed to the proctor.

Good luck!!

Problem 1 : Section 2 EM

A very long conducting cylindrical rod of radius a and length L is surrounded by a conducting cylindrical shell whose inner radius is b. There is electric potential V_0 is applied between two conductors (the inner conductor is at higher potential) and a uniform magnetic field B_0 is directed along the axis of the cylinder as shown in the figure below.



(a) Find the total net charge on the inner conductor.

(b) Suppose an electron with charge $-e$ and mass m is orbiting around a circular orbit around the inner conductor at a distance R away from its cylindrical axis and well away from the edge of cylinders. Find the velocity v of the electron in this circular orbit.

Problem 2 : Section 2 EM

An oscillating electric dipole moment $\vec{p}(t) = p_0 \cos(wt)\,\hat{z}$ generates radiating electric and magnetic fields. Far away from the dipole, the scalar, $V(\vec{x},t)$, and vector potentials $\vec{A}(\vec{x},t)$, due to this dipole are written as

$$V = -\frac{p_0 w}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[w(t-r/c)] \quad \text{and} \quad \vec{A} = -\frac{\mu_0 p_0 w}{4\pi r}\sin[w(t-r/c)]\,\hat{z}$$

in SI unit where $c^2 = 1/(\mu_0\varepsilon_0)$

(a) Show that the total find power of radiation emitted from this dipole is given by $P = \dfrac{\mu_0 p_0^{\,2} w^4}{12\pi c}$ in SI unit (or $P = \dfrac{p_0^{\,2} w^4}{3c^3}$ in cgs unit).

(Hint: Work in spherical coordinates. This integral might be useful $\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$).

(b) Consider a classical charged simple harmonic oscillator with mass m and charge q is oscillating with angular frequency w. Let A_0 is the oscillation amplitude at $t=0$. Find the time , $T_{1/2}$, when the amplitude of the oscillator reduces in half.

Problem 3 : Section 2 EM

Maxwell's equations yield the following wave equations for a linear, isotropic medium with conductivity σ:

$$\nabla^2 \vec{E} - \epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} - \sigma\mu \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon}\vec{\nabla}\rho_f \tag{1}$$

$$\nabla^2 \vec{H} - \epsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} - \sigma\mu \frac{\partial \vec{H}}{\partial t} = 0 \tag{2}$$

with

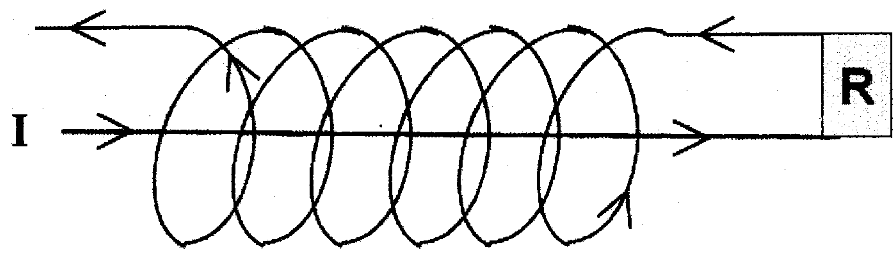
$$\mu \frac{\partial \vec{H}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \tag{3}$$

Consider a plane polarized electromagnetic wave in vacuum, propagating in the positive z direction. It strikes a semi-infinite conducting slab, whose boundary is at $z = 0$. Determine the ratio of the amplitude for the reflected wave to that of the incident wave for the case where the conducting slab is a good conductor ($\sigma >> \omega\epsilon$).

Problem 4 : Section 2 EM

Steady current I flows in the circuit below. The solenoid is long with length L >> radius a, and number of turns n=N/L >> 1/a. The resistance R is given but the resistivity of the wire elsewhere can be neglected. The straight wire inside the solenoid is coaxial with the solenoid.

Find the net flux of electromagnetic energy through the cross section area π a^2 , of the solenoid (far from its edges)

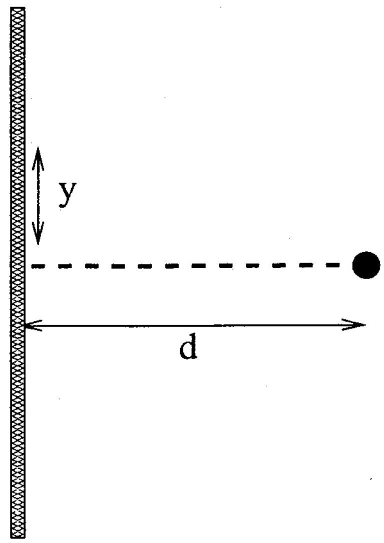


Problem 5 : Section 2 EM

A very long wire of radius a is suspended a distance d above an infinite conducting plane. In the case that $d >> a$, find approximate expressions for

a. The capacitance per unit length of the wire, conducting plane system.

b. The surface charge density on the conducting plane as a function of y, the distance along the plane lateral to the wire.



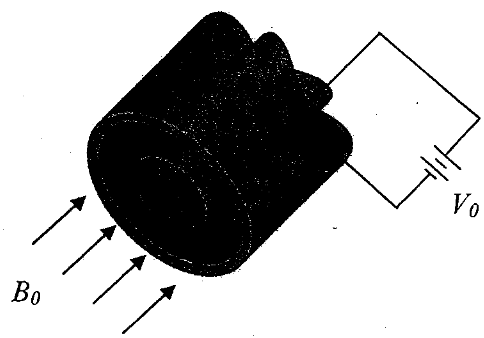
Sccz #1

DEC 5 2005

Philip Kim 2006 Qual

E&M I:

A very long conducting cylindrical rod of radius a and length L is surrounded by a conducting cylindrical shell whose inner radius is b. There is electric potential V_0 is applied between two conductors (the inner conductor is at higher potential) and a uniform magnetic field B_0 is directed along the axis of the cylinder as shown in the figure below.



(a) Find the total net charge on the inner conductor.

(b) Suppose an electron with charge $-e$ and mass m is orbiting around a circular orbit around the inner conductor at a distance R away from its cylindrical axis and well away from the edge of cylinders. Find the velocity v of the electron in this circular orbit.

DEC 5 2005

Sec 2 #1

Kim-solution

Kim

E&M I Sol

(a) The potential between the conductor

$$V(r) = -\frac{V_0 \ln(\frac{r}{a})}{\ln(\frac{b}{a})} \Rightarrow \text{Electric field} \quad \vec{E} = \frac{V_0}{\ln(\frac{b}{a})}\frac{1}{r}\hat{r}$$

At the surface of the inner conductor,

$$\sigma = \hat{r}\cdot\vec{E}\Big|_{r=a} = \frac{V_0}{\ln(\frac{b}{a})}\frac{1}{a}$$

Thus the total net charge on the inner conductor

$$Q = L\cdot 2\pi a\,\sigma = \frac{2\pi V_0 L}{\ln(b/a)}$$

(b) For a circular motion of radius R, considering electrostatic & Lorentz force

$$\frac{mv^2}{R} = e\frac{V_0}{\ln(b/a)}\frac{1}{R} + evB_0$$

$$\Rightarrow \quad v^2 - \left(\frac{eB_0}{m}R\right)v - \frac{eV_0}{m\ln(b/a)} = 0$$

or

$$\boxed{v = \omega_L R \pm \sqrt{(\omega_L R)^2 + \frac{eV_0}{m\ln(b/a)}} \quad \text{where} \quad \omega_L = \frac{eB_0}{2m}}$$

DEC 5 2005 Sec 2 #2

Philip Kim 2006 Qual

E&M II:

An oscillating dipole moment $\vec{p}(t) = p_0 \cos(wt)\,\hat{z}$ generates radiating electric and magnetic field. At far away from the dipole, the vector potential due to this dipole is written as

$\vec{A} = -\frac{\mu_0 p_0 w}{4\pi r}\sin[w(t - r/c)]\hat{z}$ in SI unit (or $\vec{A} = -\frac{p_0 w}{cr}\sin[w(t - r/c)]\hat{z}$ in cgs unit).

(a) Show that the total find power of radiation emitted from this dipole is given by

$P = \frac{\mu_0 p_0^2 w^4}{12\pi c}$ in SI unit (or $P = \frac{p_0^2 w^4}{3c^3}$ in cgs unit).

(This integral might be useful $\int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$).

(b) Consider a classical charged simple harmonic oscillator with mass m and charge q is oscillating with angular frequency w. Let A_0 is the oscillation amplitude at t =0. Find the time where the amplitude of the oscillator reduces in half.

Sec 2 #2

(2)

(b) Let $X(t) = A_0 \cos\omega t$ is the position of the charge

Then the charge density is described by

$$
\begin{aligned}
\rho(x) &= q\,\delta(X(t)) = q\,\delta(A_0 \cos\omega t) \\
&= +q\,\delta(0) - q\delta(0) + q\,\delta(A_0 \cos\omega t) \\
&= +q\,\delta(0) + P_0 \cos\omega t
\end{aligned}
$$

where $P_0 = q A_0$

Since the static charge at $X=0$ does not radiate, thus energy of the SHO is reduced by dipole radiation.

Energy of the SHO

$$E = \frac{m}{2}\omega^2 A_0^2$$

From the result of (a),

$$\frac{dE}{dt} = -P = -\frac{\mu_0 \omega^4}{12\pi c}(qA_0)^2 \qquad \text{Here } \frac{dE}{dt} = m\omega^2 A_0 \frac{dA_0}{dt}.$$

or

$$m\omega^2 A_0 \frac{dA_0}{dt} = -\frac{\mu_0 \omega^4}{12\pi c} q^2 A_0^2$$

$$\Rightarrow \quad \frac{dA_0}{dt} = -\frac{A_0}{\tau} \qquad \text{where} \quad \boxed{\tau = \frac{12\pi c\, m}{\mu_0 \omega^2 q^2}}$$

or

$$A_0(t) = A_0(0)\, e^{-t/\tau}$$

The amplitude reduces in half

when

$$t = \tau \ln 2.$$

Kim

E&M II sol

(1)

(a) $$\vec{A} = -\frac{\mu_0 P_0 \omega}{4\pi r}\sin[\omega(t-\tfrac{r}{c})]\,(\hat{z}) = \cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$= -\frac{\mu_0 P_0 \omega}{4\pi}\left\{\frac{\omega}{c}\frac{1}{r}\sin\theta\cos[\omega(t-\tfrac{r}{c})] + \underbrace{\frac{\sin\theta}{r^2}\sin[\omega(t-\tfrac{r}{c})]}_{\sim\frac{1}{r^2}\text{ not propagating, ignore!}}\right\}\hat{\phi}$$

$$\simeq -\frac{\mu_0 P_0 \omega}{4\pi}\frac{\omega}{cr}\sin\theta\cos[\omega(t-\tfrac{r}{c})]\hat{\phi}$$

Since $\vec{E}$ field is orthogonal to $\vec{B}$ & $\hat{r}$, and $|\vec{B}|/|\vec{E}| = \frac{1}{c}$,

Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}|\vec{B}|^2\,\hat{r}$

$$\langle\vec{S}\rangle = \frac{1}{\mu_0 c}\left(\frac{\mu_0 P_0 \omega^2}{4\pi c r}\right)^2\sin^2\theta\cdot\frac{1}{2}\hat{r} = \frac{\mu_0 P_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}$$

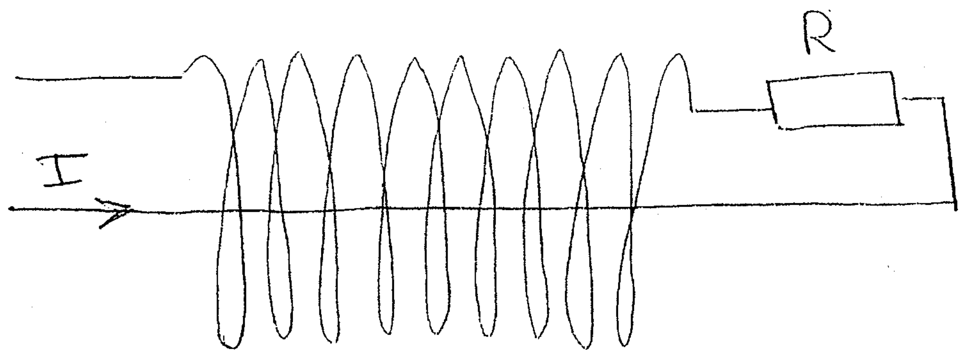
time average

$$P = \int_0^{2\pi}\!\!\int_0^{\pi}\langle\vec{S}\rangle\cdot\hat{r}\;r^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{\mu_0 P_0^2\omega^4}{32\pi^2 c}\cdot 2\pi\cdot\underbrace{\int_0^{\pi}\sin^3\theta\,d\theta}_{=\,4/3} = \frac{\mu_0 P_0^2\omega^4}{12\pi c}$$

Sec 2 #4

Belobordov
11/28/05 E&M



Problem 1.
Steady current I flows in the cicruit shown in the figure. The solenoid is long (length $L \gg$ radius a) and has number of turns $n = N/L \gg a^{-1}$. The resistance R is given; neglect resistivity of the wire everywhere else in the circuit. The straight wire inside the solenoid is coaxial with the solenoid. Find the net flux of electromagnetic energy through the πa^2 cross section of the solenoid (far from its edges).

Solution: Poynting flux inside the solenoid is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}.$$

We'll use cylindrical coordinates r, ϕ, z with the z-axis along the axis of the solenoid. First find electric field $\mathbf{E}$. It is created bceause there is potential drop IR between the solenoid and the wire on its axis. By symmetry, $E_\phi = E_z = 0$, and the non-zero component E_r depends on r only. E_r may be found from $\nabla\cdot\mathbf{E} = 0$ between the wire and the solenoid, which gives

$$\frac{1}{r}\frac{d}{dr}(rE_r) = 0, \qquad E_r = \frac{C}{r}.$$

C is found from the known potential drop. Denote the radius of the wire by b, then

$$IR = \int_b^a E_r\,dr = C\ln\frac{a}{b}, \qquad C = \frac{IR}{\ln(a/b)}.$$

The Poynting flux is then given by,

$$\mathbf{S} = \frac{c}{4\pi}E_r\mathbf{e}_r\times(B_\phi\mathbf{e}_\phi + B_z\mathbf{e}_z) = \frac{c}{4\pi}E_r\,(B_\phi\mathbf{e}_z - B_z\mathbf{e}_\phi),$$

where $\mathbf{e}_r$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$ are unit vectors tangent to the coordinates lines and we have used $\mathbf{e}_r\times\mathbf{e}_\phi = \mathbf{e}_z$ and $\mathbf{e}_r\times\mathbf{e}_z = -\mathbf{e}_\phi$. The net flux of electromagnetic energy through the solenoid is

$$\mathbf{F} = \int_b^a dr\int_0^{2\pi} d\phi\,\mathbf{S} = \int_b^a \frac{c}{4\pi}E_r B_\phi\mathbf{e}_z 2\pi r\,dr \tag{1}$$

(the second term with $B_z\mathbf{e}_\phi$ vanishes after integration by symmetry). It remains to find $B_\phi(r)$ and calculate the integral (1).

The solenoid itself creates a uniform B_z and does not contribute to B_ϕ. The axial wire creates B_ϕ which is found by integrating Maxwell equation $\nabla\times\mathbf{B} = (4\pi/c)\mathbf{j}$ over the cross section of the wire and then applying the Stokes' theorem,

$$2\pi r B_\phi = \frac{4\pi}{c}I, \qquad B_\phi = \frac{2I}{cr}.$$

Substituting the known $E_r(r)$ and $B_\phi(r)$ into equation (1) and performing the integration, one finds

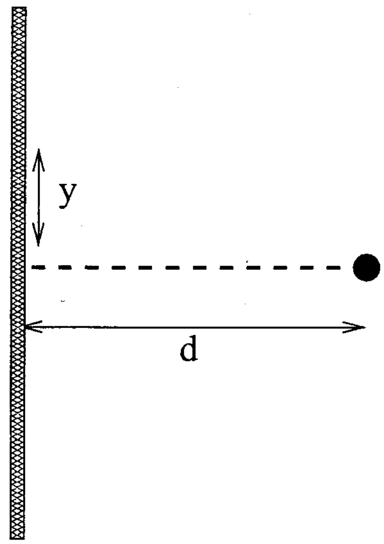
$$F = I^2R.$$

Brian Cole E&M

A very long wire of radius a is suspended a distance d above an infinite conducting plane. In the case that $d >> a$, find approximate expressions for

a The capacitance per unit length of the wire, conducting plane system.

b The surface charge density on the conducting plane as a function of y, the distance along the plane lateral to the wire.



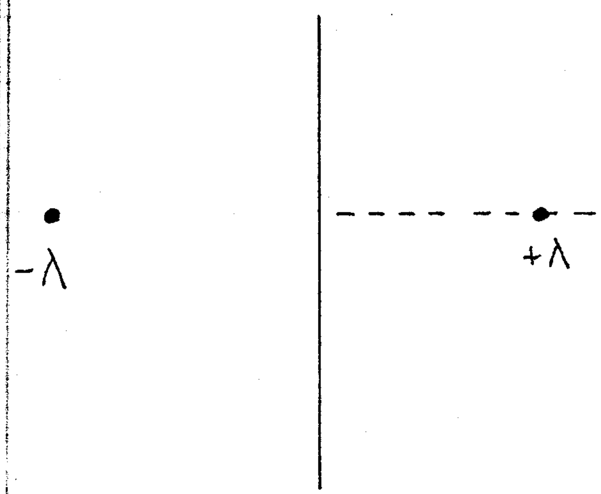
Sec 2 #5

Brian Cole
2006 Qualifying Exam

Classical Physics, E&M
Problem 5 Solutions

a) Because $a \ll d$, we can treat the wire as if it is a carrier of charge of linear density λ.

Use the method of images to account for the induced charge on the surface of the conducting sheet, so imagine linear charge density $-\lambda$ a distance d past the conducting sheet.



Then, if we choose the electrostatic potential, φ, to be zero on the sheet, along the line passing through the charges,

$$\Delta\varphi = -\int_0^{d-a} dx\, E(x)$$

$$E(x) = \frac{-\lambda}{2\pi\varepsilon_0}\left(\frac{1}{-x+d} + \frac{1}{x+d}\right)$$

$$\text{So } \Delta\varphi = \frac{\lambda}{2\pi\varepsilon_0}\Big(\ln(x+d) - \ln(d-x)\Big)\Big|_0^{d-a}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\left(\frac{2d-a}{d}\right) + \ln\left(\frac{d}{a}\right)\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{2d-a}{a}\right)$$

The capacitance per unit length is the charge per unit length$/|\Delta\varphi|$

$$\frac{C}{L} = \lambda/|\Delta\varphi| = \frac{2\pi\varepsilon_0}{\ln\left(\frac{2d-a}{a}\right)}$$

b) The magnitude of the electric field from the wire at $+d$ at the surace of the plane is

$$|E_+(y)| = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\sqrt{d^2+y^2}}$$

The component $\perp$ to the plane is the above multiplied by $d/\sqrt{d^2+y^2}$.

The components $\parallel$ to the plane from the wire and its image cancel of course & the $\perp$ component is doubled:

$$|E| = \frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2} \rightarrow E(y) = -\frac{\lambda}{\pi\varepsilon_0}\frac{d}{d^2+y^2}\hat{x}$$

Then, the charge density is $\sigma = E\varepsilon_0$

So $\sigma(y) = \frac{-\lambda d}{\pi(d^2+y^2)}$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 11, 2006
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1: Section 3 Quantum Mechanics

a) Use only the uncertainty principle to estimate the binding energy E_B of Hydrogen in terms of $m_e, e, \hbar, c$. (Evaluate the answer in terms of Electron Volts to at least 1 digit accuracy, using $m_e c^2 = 5 \times 10^5$ eV and the known value of the fine structure constant, $\alpha = e^2/\hbar c$)

b) In a far off galaxy, long ago, mystery matter changed the Coulomb potential to

$$V(r) = \frac{e^2}{r}\left(\frac{d}{r}\right)^{\epsilon}$$

where d is a new length scale and $|\epsilon| \ll 1$. Assuming that none of the other physical parameters changed, extend part (a) to show that to first order in ϵ, the Bohr radius, r_B, changed to $f\, r_B$ where $f \approx 1 - \epsilon\{1 + \log(d/r_B)\}$.

(Hint: For tiny ϵ, the approximations $1/(1+\epsilon) \approx 1 - \epsilon$ and $x^{\epsilon} \approx 1 + \epsilon \log x$ may be useful)

Problem 2 : Section 3 Quantum Mechanics

Consider *two* identical and non-interacting particles. The particles are in the same spin state, but occupy distinct spatial states a and b, with $\psi_a(x)$ and $\psi_b(x)$ denoting the relevant single-particle 1-D spatial wavefunctions. In the problem below, consider both the case of fermions and bosons.

(a) Write an expression for the normalized two-particle wavefunctions $\psi_F(x_1, x_2)$ and $\psi_B(x_1, x_2)$ for fermions and bosons, respectively. Write the corresponding energy eigenvalues E_F and E_B in terms of the single-particle energies ε_a and ε_b.

(b) Show that the expectation values $\langle x_1^2\rangle$ and $\langle x_2^2\rangle$ for the two-particle system satisfy the following relation for both fermions and bosons:

$$\langle x_1^2\rangle = \langle x_2^2\rangle = \tfrac{1}{2}\left(\langle x^2\rangle_a + \langle x^2\rangle_b\right),$$

where $\langle f(x)\rangle_a \equiv \int_{-\infty}^{+\infty} [\psi_a(x)]^* \, f(x) \, \psi_a(x)\, dx$ (and likewise for b) is the expectation value in the single-particle state.

(c) Define the average separation between the particles as $\langle (x_1 - x_2)^2\rangle$. Show that the average separation for fermions is always greater than or equal to that for bosons. (For simplicity, you may assume that the single-particle wavefunctions satisfy $\langle x\rangle_a = \langle x\rangle_b$.)

Problem 3 : Section 3 Quantum Mechanics

A particle of mass m moves in a 1-dimensional square well potential

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & -a < x < a \end{cases}$$

1. A plane wave with momentum $\hbar k$ hits the potential well from the left. For certain values of k the wave is perfectly transmitted by the potential. That is, the reflection coefficient vanishes and the transmission coefficient is equal to unity. Determine the values of $E = \hbar^2 k^2/2m$ for which this occurs.

2. The cross section for scattering low energy electrons off xenon atoms exhibits a dip at an electron energy of around 0.7 eV. Suppose the xenon atom can be modeled as a 1-D square well potential. Given that the size of the atom is around 1 Angstrom, estimate the depth of the potential V_0.

Useful facts: $\hbar c \approx 2 \times 10^{-5}\,\mathrm{eV \cdot cm}$ and $m_e c^2 \approx 500\,\mathrm{keV}$.

Problem 4 : Section 3 Quantum Mechanics

Consider a particle of mass m moving in the following one dimensional potential:

$$V(x) = \begin{cases} \infty & \text{for } \; 0 < a < x < \infty \\ V_0\,\delta(x) & \text{for } \; -\infty < x < a \end{cases} \tag{1}$$

where $V_0 a$ is a constant and $a > 0$. Assume that there is a wave, $\exp(+ikx)$ incident on the potential. Write the complete solution in the $x < 0$ region as $u(x) = \exp(+ikx) + R\exp(-ikx)$.

a) Determine $R(k)$ and evaluate its magnitude, $|R|$.

b) Using $R(k)$ determine a transcendental equation for possible bound state energies for $V_0 > 0$ and $V_0 < 0$. (hint: Study the small and large k limits of the equation to set a constraint on $V_0 a$).

c) Sketch qualitatively the functional form of the modulus of the bound state wavefunction(s) in this potential.

Problem 5: Section 3 QM

A particle of charge $-e$ and mass m undergoes simple harmonic motion (spring constant k) in one-dimension. The particle is subject to an electric field of constant value $E = E_o$ along the x-direction. Treating the electrostatic potential as a weak perturbation, determine the ground state energy and the first excited state energy to 2^{nd} order. You may either apply perturbation theory or derive the exact solution to this problem.

Brooijmans

... 1 ' 2008

Problem 1: Sec 3 Quantum Mechanics

8

1. a) Use only the uncertainty principle to estimate the binding energy E_B of Hydrogen in terms of $m_e, e, \hbar, c$. (Evaluate the answer in terms of Electron Volts to at least 1 digit accuracy, using $m_e c^2 = 5 \times 10^5$ eV and the known value of the fine structure constant, $\alpha = e^2/\hbar c$)

2. b) In a far off galaxy, mystery matter changes the Coulomb potential to

$$V(r) = \frac{e^2}{r}\left(\frac{d}{r}\right)^{\epsilon}$$

7

where d is a new length scale and $|\epsilon| \ll 1$. Assuming that m_e does not change, show using uncertainty principle that to first order in ϵ, the Bohr radius, r_B, changes to $f \times r_B$ where $f \approx 1 - \epsilon\{1 + \log(d/r_B)\}$.
(Hint: $1/(1+\epsilon) \approx 1 - \epsilon$ and $x^\epsilon \approx 1 + \epsilon \log x$ may be useful)

1 Solution

1. a) $E = p^2/2m - e^2/r > \hbar^2/2mr^2 - e^2/r = E(r)$

   $dE/dr = 0$ gives $r_B = \hbar^2/me^2 = \lambda_e/\alpha$ where $\lambda_e = \hbar/m_e c$ and $\alpha = e^2/\hbar c = 1/137$.
   $E(R_B) = (\hbar^2/2m)m^2e^4/\hbar^2 - e^2me^2/\hbar^2 = -\frac{1}{2}\alpha^2 mc^2 = -1/2(1/137)^2(5 \times 10^5)eV = -5/4 \times 10$ eV, which is reasonably close to the well known Rydberg 13.6 eV.

2. b) Change $E(r, \epsilon) = \hbar^2/2mr^2 - (e^2/r)(d/r)^\epsilon$. Minimize to get $r^{1-\epsilon} = (\hbar^2/me^2(1+\epsilon)d^\epsilon) = r_B/((1+\epsilon)d^\epsilon)$.

   Use $r_B = r_B^{1-\epsilon} r_B^\epsilon$ to write $r/r_B = [(r_B/d)^\epsilon/(1+\epsilon)]^{1/(1-\epsilon)}$.
   Expand $1/(1 \pm \epsilon) \approx 1 \mp \epsilon$. Keep first order only. Use $x^\epsilon \approx 1 + \epsilon \log x$.
   Therefore the new minimum is at $r \approx r_B(1-\epsilon)\underbrace{(r_B/d)^\epsilon}_{1+\epsilon\log(r_B/d)} = r_B(1 - \epsilon(1 + \log(d/r_B)) =$ $+\epsilon$ $-\epsilon$
   $r_B(1 - \epsilon \log(\cancel{e}d/r_B))$.
   For $\epsilon > 0$ the Bohr radius shrinks if $d > r_B/e$.
   Not needed for full credit but for fun: The kinetic $T \approx T_B[1 + 2\epsilon(1 + \log(d/r_B))]$. The pot $V \approx V_B(1 + \epsilon \log(ed/r_B))(d/r_B)^\epsilon \approx V_B[1 + \epsilon(1 + 2\log(d/r_B))]$. Recall $V_B = -2T_B = 2E_B$,

   $E = -E_B[1 + 2\epsilon(1 + \log(d/r_B))] + 2E_B[1 + \epsilon(1 + 2\log(d/r_B))] = E_B + \epsilon E_B[1 + 2\log(d/r_B)]$

Sec 3 # 2

QUANTUM MECHANICS PROBLEM (HEINZ)
12/2/05

Consider *two* identical and non-interacting particles. The particles are in the same spin state, but occupy distinct spatial states a and b, with $\psi_a(x)$ and $\psi_b(x)$ denoting the relevant single-particle 1-D spatial wavefunctions. In the below, consider both the case of fermions and bosons.

(a) Write an expression for the normalized two-particle wavefunctions $\psi_F(x_1, x_2)$ and $\psi_B(x_1, x_2)$ for fermions and bosons, respectively. Write the corresponding energy eigenvalues E_F and E_B in terms of the single-particle energies ε_a and ε_b.

(b) Show that the expectation values $\langle x_1^2 \rangle$ and $\langle x_2^2 \rangle$ for the two-particle system satisfy the following relation for both fermions and bosons:

$$\langle x_1^2 \rangle = \langle x_2^2 \rangle = \tfrac{1}{2} (\langle x^2 \rangle_a + \langle x^2 \rangle_b),$$

where $\langle f(x) \rangle_a \equiv \int_{-\infty}^{+\infty} [\psi_a(x)]^* f(x)\, \psi_a(x)\, dx$ (and likewise for b) is the expectation value in the single-particle state.

(c) Define the average separation between the particles as $\langle (x_1 - x_2)^2 \rangle$. Show that the average separation for fermions is always greater than or equal to that for bosons. (For simplicity, you may assume that the single-particle wavefunctions satisfy $\langle x \rangle_a = \langle x \rangle_b$.)

HEINZ QM PROBLEM (SOLUTION)

(a) Since the spin of the particles is the same, the spin wavefunction must be symmetric. Thus, the spatial wavefunction must be antisymmetric for fermions and symmetric for bosons.

$$\psi_F(x_1,x_2) = [\psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1)]/\sqrt{2}$$
$$\psi_B(x_1,x_2) = [\quad " \quad + \quad " \quad]/\sqrt{2}$$

$E_F = E_B = \epsilon_a + \epsilon_b$ since the particles are non-interacting

(b) $$\langle x_1^2\rangle = \tfrac{1}{2}\iint dx_1 dx_2 \,|\psi_a(x_1)\psi_b(x_2) \mp \psi_a(x_2)\psi_b(x_1)|^2 x_1^2$$
$\langle x_1^2\rangle = \frac{1}{2}[\langle x^2\rangle_a + \langle x^2\rangle_b]$ using the orthonormality of ψ's
$= \langle x_2^2\rangle$, analogously.

(c) $$\langle (x_1-x_2)^2\rangle = \langle x_1^2 + x_2^2\rangle - 2\langle x_1 x_2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x_1 x_2\rangle$$

The last term is the one that differs for F and B.

$$-2\langle x_1 x_2\rangle = -\iint dx_1 x_2 \,|\psi_a(x_1)\psi_b(x_2) \mp \psi_a(x_2)\psi_b(x_1)|^2 x_1 x_2$$

The direct terms are $\propto \langle x_1\rangle$ or $\langle x_2\rangle$ and vanish.
The cross terms are

$$= \pm\iint dx_1 dx_2 [\psi_a(x_1)\psi_b(x_2)\psi_a^*(x_2)\psi_b^*(x_1)\, x_1 x_2 + cc$$
$$= \pm\left(\int dx_1\, \psi_a(x_1)\, x_1\, \psi_b^*(x_1)\, x_1\right)\left(\int dx_2\, \psi_b(x_2)\, x_2\, \psi_a^*(x_2)\, x_2\right) + cc$$
$$\cong \pm[\langle x\rangle_{ab}\langle x\rangle_{ba} + c.c.]$$
$$= \pm 2|\langle x\rangle_{ab}|^2, \text{ for F, B respectively}$$

∴ Fermions are further apart than bosons. ✓

Sec 3 #3

Dan Kabat
11/15/05

Quantum

A particle of mass m moves in a 1-dimensional square well potential

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & -a < x < a \end{cases}$$

1. A plane wave with momentum $\hbar k$ hits the potential well from the left. For certain values of k the wave is perfectly transmitted by the potential. That is, the reflection coefficient vanishes and the transmission coefficient is equal to unity. Determine the values of $E = \hbar^2 k^2/2m$ for which this occurs.

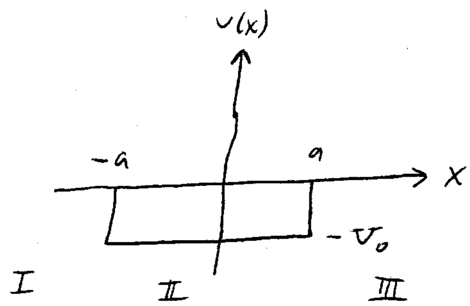
2. The cross section for scattering low energy electrons off xenon atoms exhibits a dip at an electron energy of around 0.7 eV. Suppose the xenon atom can be modeled as a 1-D square well potential. Given that the size of the atom is around 1 Angstrom, estimate the depth of the potential V_0.

Useful facts: $\hbar c \approx 2 \times 10^{-5}\,\text{eV} \cdot \text{cm}$ and $m_e c^2 \approx 500\,\text{keV}$.

Dan Kabat 11/15/05

Sec 3 #3

Quantum problem solution



$\psi_I = e^{ikx}$ $\qquad E = \frac{\hbar^2 k^2}{2m}$

$\psi_{II} = Ae^{ik_2x} + Be^{-ik_2x}$ $\qquad k_2 = \frac{1}{\hbar}\sqrt{2m(E+V_0)}$

$\psi_{III} = Ce^{ikx}$

match at $x = -a$ $\Rightarrow$ $e^{-ika} = Ae^{-ik_2a} + Be^{ik_2a}$

$ike^{-ika} = ik_2(Ae^{-ik_2a} - Be^{ik_2a})$

match at $x = +a$ $\Rightarrow$ $Ce^{ika} = Ae^{ik_2a} + Be^{-ik_2a}$

$ikCe^{ika} = ik_2(Ae^{ik_2a} - Be^{-ik_2a})$

Four equations, three unknowns. Get a solution iff

$C = \pm e^{-2ika}$ and $e^{ik_2a} = \pm 1$ or $\pm i$

$\Rightarrow k_2 = \frac{n\pi}{2a}$ $\quad n = 0, 1, 2, \ldots$

$E = \frac{n^2\pi^2\hbar^2}{8ma^2} - V_0$ (need $E > 0$ for scattering state)

The xenon dip presumably corresponds to $n=1$, so

$V_0 = \frac{\pi^2\hbar^2}{8ma^2} - E$

$= \frac{\pi^2 (2\times10^{-5}\,eV\cdot cm)^2}{8 \times 500\,keV \times (10^{-8}\,cm)^2} - 0.7\,eV$

$\approx 9\,eV$

Quantum Mechanics Quals 2006

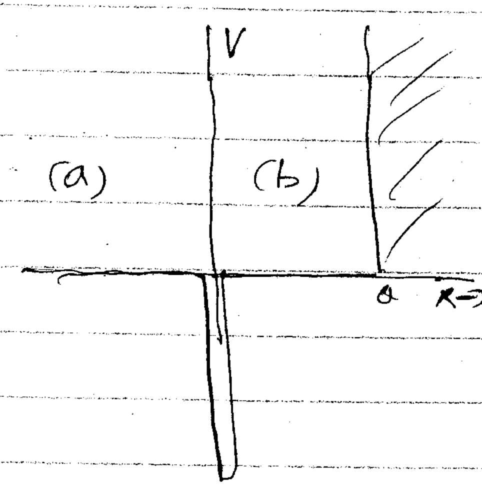
A. Mueller
NOV 21 2005
Sec 3 # 4

Consider a particle of mass m moving in a potential $V(x)$ where $V(x)=\infty$ for $x>a$ and $V(x)=V_0\,\delta(x)$ for $-\infty<x<a$ with V_0 a constant. Further suppose there is a wave e^{ikx} incident on the potential. Write $u(x)=e^{ikx}+Re^{-ikx}$ to describe the wavefunction of the particle for $x<0$.

~~(i) Evaluate R. What is |R|.~~

(ii) From R determine the possible bound state energies for $V_0>0$ and for $V_0<0$.

Solution.



Region (a): $u=e^{ikx}+Re^{-ikx}$

Region (b): $u=A\sin k(x-a)$

$u(0+)=u(0-)\Rightarrow 1+R=-A\sin ka$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2}+V(u)=Eu$$

$$\Rightarrow -\frac{\hbar^2}{2m}\underbrace{(u'(0+)-u'(0-))}_{Ak\cos ka-ik(1-R)}+V_0u(0)=0$$

$$\underbrace{Ak\cos ka}_{-\frac{k(1+R)}{\tan ka}}-ik(1-R)=\frac{2m}{\hbar^2}V_0(1+R)$$

gives

$$R=-\left(\frac{k/\tan ka+2mV_0/\hbar^2+ik}{k/\tan ka+2mV_0/\hbar^2-ik}\right)\qquad |R|=1$$

For bound state, $E=-\hbar K^2/2m$ with $k=iK$ and $R(k=iK)=\infty$.

get $K=-mV_0/\hbar^2\,(1-e^{-2Ka})$

$\Rightarrow$ bound state only if $V_0<0$ and $\frac{2mV_0a}{\hbar^2}>0$.

Sec 3 #5

Chuck Hailey's 2006 Quals problem (typed by Elena)
12/5/05

Quantum problem:

A particle of charge –e and mass m undergoes simple harmonic motion (spring constant k) in one-dimension. The particle is subject to an electric field of constant value $E = E_0$ along the x-direction. Treating the electrostatic potential as a weak perturbation, determine the ground state energy and the first excited state energy to 2^{nd} order. If you do not want to apply perturbation theory feel free to seek an exact solution to the problem.

Solution: This is most easily done with operators. sec3 #5

Hailey QM

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Kx^2 - eE_0x = \mathcal{E}\psi$$ ignoring the

perturbation, $q=\sqrt{\alpha}\,x$ $\alpha = \frac{m\omega}{\hbar}$ brings the SHO to the form $(p^2+q^2)\psi = \frac{2\mathcal{E}}{\hbar\omega}\psi$ $p = -i\frac{\partial}{\partial q}$

And the canonical transformation $q = \frac{1}{\sqrt{2}}(a^\dagger + a)$ Allows us to use operators. The perturbation is

$$V_p = -eE_0x = -\frac{eE_0q}{\sqrt{\alpha}} = -bq \qquad b \equiv \frac{eE_0}{\sqrt{\alpha}}$$

1st order shifts: $\delta E_0 \propto \langle 0|V_p|0\rangle \propto \langle 0|q|0\rangle = 0$ by parity

$\delta E_1 \propto \langle 1|V_p|1\rangle \propto \langle 1|q|1\rangle = 0$ by parity

2) order: $$\delta E_0^{(2)} = \sum_{n\neq 0}\frac{\langle 0|V_p|n\rangle\langle n|V_p|0\rangle}{E_0-E_n}$$

$$\delta E_1^{(2)} = \sum_{n\neq 1}\frac{\langle 1|V_p|n\rangle\langle n|V_p|1\rangle}{E_1-E_n}$$

We need $\langle 0|q|n\rangle$ And $\langle 1|q|n\rangle$

$$\langle 0|q|n\rangle = \langle 0|\frac{(a^\dagger+a)}{\sqrt{2}}|n\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\,\delta_{0,n+1} + \sqrt{n}\,\delta_{0,n-1}\right)$$

only $\langle 0|q|1\rangle = \frac{1}{\sqrt{2}}$ is non-vanishing

$$\langle 1|q|n\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\,\delta_{1,n+1} + \sqrt{n}\,\delta_{1,n-1}\right)$$

$\langle 1|q|0\rangle = \frac{1}{\sqrt{2}}$ $\qquad \langle 1|q|2\rangle = 1$

$$\delta E_0^{(2)} = b^2\left(\frac{(1/\sqrt{2})^2}{-\hbar\omega}\right) = -\frac{1}{2}\frac{e^2E_0^2}{m\omega^2} = -\frac{1}{2}\frac{e^2E_0^2}{K}$$

$$\delta E_1^{(2)} = b^2\left(\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{\hbar\omega} + \frac{1^2}{-\hbar\omega}\right) = -\frac{1}{2}\frac{e^2E_0^2}{m\omega^2} = -\frac{1}{2}\frac{e^2E_0^2}{K}$$

SE 3 #5

So the shifts are the same and

$$E_0 \simeq \frac{1}{2}\hbar\omega - \frac{1}{2}\frac{e^2E_0^2}{m\omega^2}\ ; \quad E_1 = \frac{3}{2}\hbar\omega - \frac{1}{2}\frac{e^2E_0^2}{m\omega^2}$$

You can do with reg. polynomial $\psi_n(q)$ but this would be tougher. It's easier to solve exactly

Exact soln: using $q = \sqrt{\alpha}\,x$ we can write the Hamiltonian as $\left[p^2 + q^2 - \frac{2eE_0 q}{\hbar\omega\sqrt{\alpha}}\right]\psi = \frac{2\varepsilon}{\hbar\omega}\psi$

call $b = \frac{2eE_0}{\hbar\omega\sqrt{\alpha}}$ $\quad -\frac{d^2}{dq^2}\psi + (q^2 - bq)\psi = \frac{2\varepsilon}{\hbar\omega}\psi$

complete the square $\quad -\psi'' + \left(q - \frac{b}{2}\right)^2 - \frac{b^2}{4} = \frac{2\varepsilon}{\hbar\omega}\psi$

And rescaling $q \to \tilde{q} = q - b/2$

$$-\psi'' + \tilde{q}^2\psi = \left(\frac{2\varepsilon}{\hbar\omega} + \frac{b^2}{4}\right)\psi$$

This is just the SHO with new eigenvalues

$$\frac{2\varepsilon}{\hbar\omega} + \frac{b^2}{4} = 2n+1 \qquad \varepsilon_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{b^2}{4\cdot 2}\hbar\omega$$

$$\varepsilon_n = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{e^2E_0^2}{2m\omega^2}$$ same as before

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 11, 2006
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

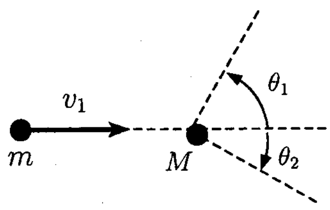
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

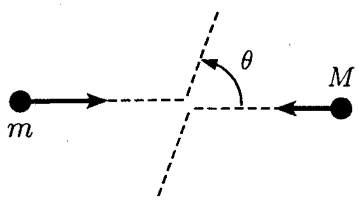
Questions should be directed to the proctor.

Good luck!!

Problem 1: Section 4 Applied QM and Relativity:

A particle of rest mass m and moving with velocity v_1 collides elastically with a stationary particle of rest mass M. Find the recoil and scattering angles in terms of the corresponding angles in the zero momentum system. Verify that your answer has the correct non-relativistic form.





Problem 2: Section 4 Applied QM and Relativity

Quantum Mechanics:
Consider a hydrogen atom in the 1s state. The magnetic interaction of the spin of the proton $\vec{s}_P$ and that of the electron $\vec{s}_e$ is given by the hyperfine Hamiltonian:

$$H_{\rm HF} = +\frac{8\pi}{3}\frac{g_P g_e}{4m_P m_e c^2}\,\vec{s}_P\cdot\vec{s}_e\,\delta^3(\vec{r}_e) \tag{1}$$

where $\vec{r}_e$ is the relative coordinate of the electron, g_e and g_P the g-factors for the electron and proton and m_P and m_e their respective masses.

(a) If the hydrogen atom wave function is $\psi(\vec{r}) = e^{-r/a_0}/\sqrt{\pi a_0^3}$ with $a_0 = \hbar^2/(m_e e^2)$, find the splitting between the $F = 0$ and $F = 1$ hyperfine states. (Here $\hbar\vec{F}$ is the total spin of the electron and proton.)

(b) If a weak magnetic field $\vec{B}$ is applied, determine the shift in the energy, $\delta E(B)$, of the lowest hyperfine state.

(c) Compute the magnetic polarizability, $\alpha_B = -\partial^2\delta E(B)/\partial B^2|_{B=0}$ for this ground state.

Problem 3 : Section 4 Appl. QM and Rel

Two dimensional electron systems can be created on the surface of semiconductors. The electrons are trapped in a potential well and their motion perpendicular to this surface is quantized. The electron system has no degree of freedom perpendicular to the surface (no freedom in the z-direction) but can move freely in the plane (x,y directions).
As an approximation to the well in which the electrons are trapped, we will use a **triangular** potential $V(z) = \mathcal{E}_0 z$ for $z>0$ and V=Infinity for $z<0$. Take $\mathcal{E}_0 = \mathbf{10^5}$ eV/cm.

Part A) Write down the Schrödinger equation for the motion in the z-direction in such a potential well and solve for the wavefunction $\psi_E(z)$, using the Airy function shown in Fig. 1. The Airy function obeys $\frac{d^2}{dw^2}Ai(w) = w\,Ai(w)$ in terms of a variable w and has zeros at approximate values $w_i = -\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$. Discuss the relationship between these values and the energy E in the Schrödinger equation.

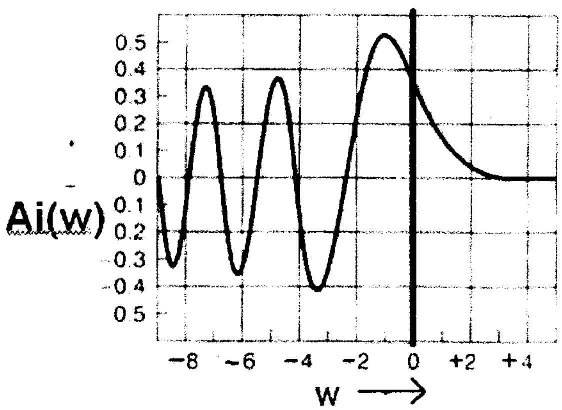


Fig. 1 Airy function

Part B) Find the energy eigenstates $E=E_i$, by inspection of Fig. 1 and by applying the correct boundary conditions. Determine the lowest 3 bound state energies, using the free electron mass, m_e.

Problem 4: Section 4 Applied QM and Relativity

Consider a metal with conduction electron density $\rho \equiv N/V = 5 \times 10^{22}$ per cm^3. Neglect all interactions. The mass of an electron is $m_e c^2 = 500$ keV, and we assume that the effective mass of electrons in this metal is the same as this "bare electron mass".

a) Describe the ground state of this system

b) What is the characteristic temperature T_c in eV for this metal above which most of the electrons are excited out from the ground state.

c) How do you expect T_c to scale in different metals if ρ and the effective mass m vary? determine the powers a, b with which $T_c \propto \rho^a m^b$

d) Assume next that all electrons combine into N/2 bound pairs with a very large binding energy between two electrons composing the pair. The spin of each pair is zero. We then neglect interaction amoung different pairs. In this simplified situation, describe how the ground state in part (a) would change. Estimate using characteristic quantal and thermo kinetic length scales, the characteristic temperature T_c for this new type of paired electron system.

Problem 5: Section 4 Applied QM and Relativity

The detection of neutrinos from Supernova SN 1987A can be used to put an upper limit on the neutrino mass, m_ν. Show that for two neutrino events with different energies E_1 and E_2, the arrival time difference on Earth can be expressed by a definite function

$$\Delta t = \Delta t(m_\nu, E_1, E_2, L) \tag{1}$$

that depends on the velocity of light c as well as the variables shown.

Calculate an upper limit using typical values $E_1 = 10\,\mathrm{MeV}$, $E_2 = 20\,\mathrm{MeV}$ and the fact that the neutrino pulse from SN 1987A lasted less than 10 s and SN 1987A is $L = 170\,000$ light years away. How does this compare with the current limit (3 eV) from tritium beta decay ?

Amber Miller Sec 4
#1

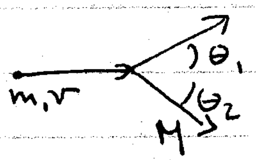
NOV 30 2005

Relativity

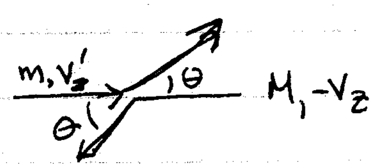
Problem - A particle of rest mass m and velocity v collides elastically with a stationary particle of rest mass M. Find the recoil and scattering angles in terms of the corresponding angles in the zero momentum system. Verify that your answer has the correct non-relativistic form

Solution

Lab frame



zero-p frame



Velocity of zero p system $V_z = \dfrac{\gamma m V}{\gamma m + M}$

$$V' = \frac{V - V_z}{1 - V V_z} = \frac{V - \gamma m V / \gamma m + M}{1 - \dfrac{V \gamma m V}{\gamma m + M}} = \frac{V(\gamma m + M) - \gamma m V}{\gamma m + M - \gamma m V^2}$$

$$= \frac{MV}{M + \gamma m(1 - V^2)}$$

$$\gamma_z = \frac{1}{\sqrt{1 - V_z^2}}$$

$$V' = \frac{MV}{M + m\sqrt{1 - V^2}}$$

$$\tan\theta_1 = \frac{\sin\theta}{\gamma_z(\cos\theta + V_z/V_1)} \xrightarrow{\text{non-rel}} \frac{\sin\theta}{\cos\theta + m/M}$$

$$\tan\theta_2 = \frac{\sin(\theta + \pi)}{\gamma_z[\cos(\theta + \pi) + V_z/V_z]} = \frac{-\sin\theta}{\gamma_z(1 - \cos\theta)} \longrightarrow \frac{-\sin\theta}{1 - \cos\theta}$$

4 # 2

N. Christ November 26, 2005

Quals Problems

1. ~~Quantum Mechanics:~~ Applied QM + Relativity

   Consider a hydrogen atom in the 1s state. The magnetic interaction of the spin of the proton $\vec{s}_P$ and that of the electron $\vec{s}_e$ is given by the hyperfine Hamiltonian:

$$H_{\mathrm{HF}} = +\frac{8\pi}{3}\frac{g_P g_e}{4 m_P m_e c^2}\,\vec{s}_P \cdot \vec{s}_e\,\delta^3(\vec{r}_e) \qquad (1)$$

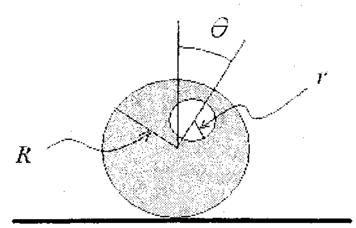
   where $\vec{r}_e$ is the relative coordinate of the electron, g_e and g_P the g-factors for the electron and proton and m_P and m_e their respective masses.

   (a) If the hydrogen atom wave function is $\psi(\vec{r}) = e^{-r/a_0}/\sqrt{\pi a_0^3}$ with $a_0 = \hbar^2/(m_e e^2)$, find the splitting between the $F = 0$ and $F = 1$ hyperfine states. (Here $\hbar\vec{F}$ is the total spin of the electron and proton.) [8 points]

   (b) If a weak magnetic field $\vec{B}$ is applied, determine the shift in the energy, $\delta E(B)$, of the lowest hyperfine state. [10 points]

   (c) Compute the magnetic polarizability, $\alpha_B = -\partial^2\delta E(B)/\partial B^2|_{B=0}$ for this ground state. [2 points]

2. Mechanics:



   A cylinder of length L, radius R and mass density ρ rolls on a horizontal surface without slipping. A hole of radius $r < R$ has been drilled through the cylinder parallel to its axis at a distance $R/2$ from its center. Describe the orientation of the cylinder by specifying the angle θ between the vertical direction and a line connecting the centers of the cylinder and the hole. If initially the cylinder is at rest but θ has a small non-zero value, $\theta(t = 0) = \delta\theta$, describe the subsequent motion. Find the time required for θ to decrease to zero. [20 points]

Suggested Solutions

1. (a) Write the product

$$\vec{s}_P \cdot \vec{s}_e = \frac{1}{2}\Big\{(\vec{s}_P + \vec{s}_e)^2 - \vec{s}_P^{\,2} - \vec{s}_e^{\,2})\Big\} = \frac{\hbar^2}{2}\Big\{f(f+1) - 3/2\Big\} \quad (2)$$

where $f = 1$ or 0 for the $F = 1$ and $F = 0$ states. [4 points]
Then simply substitute into the lowest order perturbation theory formula $E_n = \langle n|V|n\rangle$ to determine the ground state as $F = 0$ with hyperfine energy:

$$E_f = \frac{g_P g_e \hbar^2 e^2}{3 m_e m_P c^2 a_0^3}\Big\{f(f+1) - \frac{3}{2}\Big\} \quad [4 \text{ points}] \qquad (3)$$

(b) For small external field the most important effect will be the mixing of the $f = 0$ and 1 states and the interaction which will do this is

$$\begin{aligned} H_B &= \frac{e}{2c}\Big\{-\frac{g_P}{m_P}\vec{s}_P \cdot \vec{B} + \frac{g_e}{m_e}\vec{s}_e \cdot \vec{B}\Big\} \\ &\approx \frac{g_e e}{2 m_e c}\vec{s}_e \cdot \vec{B}. \quad [2 \text{ points}] \end{aligned} \qquad (4)$$

We can then use second order perturbation theory to find the shift $\delta E(B)$ in the energy of the $f = 0$ state caused by this term:

$$\begin{aligned} \delta E(B) &= \left(\frac{g_e e B}{2 m_e c}\right)^2 \frac{|\langle f = 1, m_f = 0|s_z|f = 0\rangle|^2}{E_0 - E_1} \quad [4 \text{ points}] \\ &= \frac{3}{16}\frac{g_e}{g_P}\frac{m_P}{m_e}B^2 a_0^3 \quad [4 \text{ points}] \end{aligned} \qquad (5)$$

(c) Differentiating with respect to B then gives:

$$\alpha_B = +\frac{3}{8}\frac{g_e}{g_P}\frac{m_P}{m_e}a_0^3 \quad [2 \text{ points}] \qquad (6)$$

2. Consider rotation about the point of contact, P. Treat the cylinder as a complete cylinder of radius R with mass $M == \rho\pi R^2 L$ and a second of negative mass $-m = -\rho\pi r^2 L$. The first cylinder exerts no torque about P while the second exerts:

$$\tau = -\frac{3R}{2}\rho\pi r^2 Lg\theta \quad [5 \text{ points}] \tag{7}$$

assuming θ to be small.

The moment of inertia about P is that of the cylinder of radius R minus that of r:

$$I = \frac{1}{2}MR^2 + MR^2 - \frac{1}{2}mr^2 - m(3R/2)^2 \quad [5 \text{ points}] \tag{8}$$

where the parallel axis theorem has been used.

Finally we can combine these:

$$I\frac{d^2\theta}{dt^2} = \tau = -\frac{3R}{2}\rho\pi r^2 Lg\theta \quad [5 \text{ points}] \tag{9}$$

which describes simple harmonic motion with period

$$T = \sqrt{2I/3R\rho\pi r^2 Lg} \quad [3 \text{ points}] \tag{10}$$

Thus, the cylinder will roll back and forth, executing simple harmonic motion about the equilibrium position $\theta = 0$. It will take $T/4$ time units to first reach $\theta = 0$ [2 points].

Qualifier Question Physics 2005, Stormer, Appl. Quantum Mechanics
11/23/05

Two-Dimensional Electron Systems.

Two dimensional electron systems can be created on the surface of semiconductors. The electrons are trapped in a potential well and their motion perpendicular to this surface is quantized. At low electron densities and at Helium temperatures all electrons can be confined to the lowest bound state while the next state is several kT higher in energy. Under such conditions, the electron system has no degree of freedom perpendicular to the surface (z-direction) but can move freely in the plane (x,y directions). It represents a two-dimensional electron system (2DES), which has shown many interesting physical phenomena. This problem establishes some of the energetics of such systems.

A typical implementation of a 2DES is a Silicon Metal-Oxide-Field-Effect-Transistor (Si MOSFET). It consists of a thick, Si single crystal with a layer of oxide at its surface, followed by a thin layer of metal (see Fig. 1) The oxide acts as an insulator and, assuming that there are already a few electrons at the Si/oxide interface, the whole structure resembles a capacitor. E_F is the Fermi level in the Si.

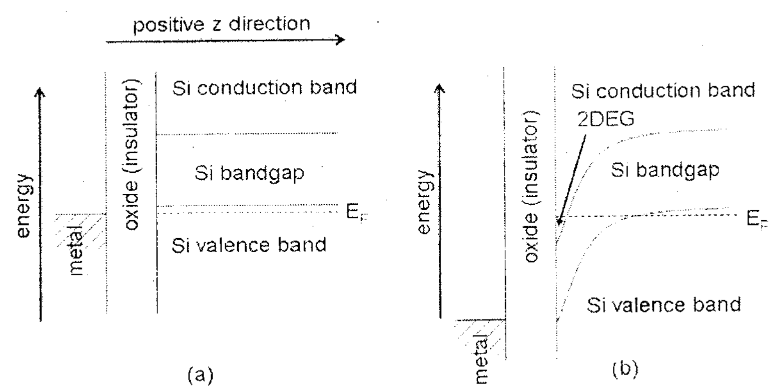


Fig 1. Energetics in a Si MOSFET, before (a) and after (b) biasing

A) Assume the oxide to be d=80nm thick, having a dielectric constant of $\varepsilon_{ox} = 4.5$. Apply 8V bias between the 2DES in the Si and the metal. Calculate the electric field, E_{Si}, within the Si ($\varepsilon_{Si} = 11.8$) right at the interface to the oxide. Neglect any contribution from the 2DES charge density, since quantum mechanically, right at the interface the charge must have dropped to zero.

(2 points)

Sec 4. Prob 3 Solution : Stormer

(8 points)

$$-\frac{\hbar^2}{2m}\psi''(z)+(eE_{Si}z-\varepsilon)\psi(z)=0,$$ **replace** $y=z-\frac{\varepsilon}{eE_{Si}}$, $z_0=\frac{\varepsilon}{eE_{Si}}$ **hence**

$$-\frac{\hbar^2}{2m}\psi''(y)+eE_{Si}y\psi(y)=0,$$

hence $\psi''(y)=\frac{2m}{\hbar^2}eE_{Si}y\psi(y)=\beta^3 y\psi(y)$, **with** $\beta^3=\frac{2m}{\hbar^2}eE_{Si}$

replacing $x=\beta y$ **we get** $\psi''(x)=x\psi(x)$**, which is solved by the Airy function of Fig.2.**

with $x=\beta y=\beta(z-\frac{\varepsilon}{eE_{Si}})$ **we find** $x_0=-\beta\frac{\varepsilon}{eE_{Si}}$ **or** $\varepsilon=-\frac{eE_{Si}x_0}{\beta}$

(7 points)

For stationary states the wave function has to have a node at the Si/oxide interface. This requires x_0 to coincide with one of the zeros of the Airy function. Use

$$x_i=-\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$$ **from above to arrive at** $\varepsilon_i=\frac{eE_{Si}}{\beta}\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$.

x_1=2.32, x_2=4.08, x_3=5.52, $\frac{eE_{Si}}{\beta}=\left(\frac{\hbar^2}{2m_e}\right)^{1/3}(eE_{Si})^{2/3}=14meV$

therefore ε_1=31meV, ε_2=57meV, ε_3= 77meV for a free electron mass, m_e.

Stormer
Sec 4 #3 ?

E-field within oxide: E_{ox} =8V/80nm/ε_{ox}=2.22 x 10^7 V/m. Continuity of D at Si/oxide interface yields: $E_{Si}\,\varepsilon_{Si} = E_{ox}\,\varepsilon_{ox}$ or E_{Si}=8.46 x 10^6 V/m.

B) As an approximation to the well in which the electrons are trapped (see Fig. 2(b)), we will use a triangular potential well made from the oxide (infinitely high barrier) and the linearly varying potential due to the E_{Si}, calculated in A). Write down the Schrödinger equation for the motion in the z-direction in such a well and solve it, using the Airy function shown in Fig. 2, with the properties $Ai''(x) = xAi(x)$ and zeros at approximate position $x_i = -\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$ .Note the relationship between position and energy for a ready solution.

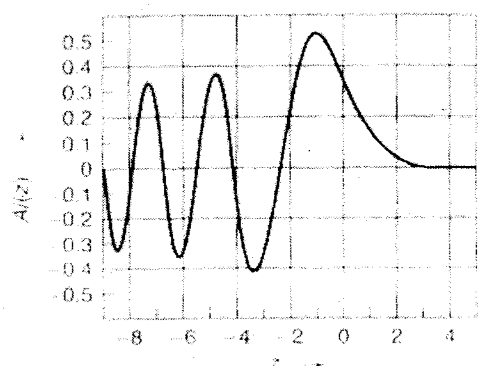


Fig. 2 Airy function

(4 points)

$$-\frac{\hbar^2}{2m}\psi''(z) + (eE_{Si}z - \varepsilon)\psi(z) = 0, \text{ replace } y = z - \frac{\varepsilon}{eE_{Si}}, \; z_0 = \frac{\varepsilon}{eE_{Si}} \text{ hence}$$

$$-\frac{\hbar^2}{2m}\psi''(y) + eE_{Si}y\psi(y) = 0,$$

hence $\psi''(y) = \frac{2m}{\hbar^2}eE_{Si}y\psi(y) = \beta^3 y\psi(y)$, **with** $\beta^3 = \frac{2m}{\hbar^2}eE_{Si}$

replacing $x = \beta y$ **we get** $\psi''(x) = x\psi(y)$, **which is solved by the Airy function of Fig.2.**

with $x = \beta y = \beta(z - \frac{\varepsilon}{eE_{Si}})$ **we find** $x_0 = -\beta\frac{\varepsilon}{eE_{Si}}$ **or** $\varepsilon = -\frac{eE_{Si}x_0}{\beta}$

C) Find the energy eigenstates ε_i , by inspection of Fig. 2 and by applying the correct boundary conditions. Determine the lowest 3 bound state energies, using the free electron mass, m_e.

(3 points)

Sec 4 # 3 ?
Stormer

For stationary states the wave function has to have a node at the Si/oxide interface. This requires x_0 to coincide with one of the zeros of the Airy function. Use

$$x_i = -\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$$ **from above to arrive at** $$\varepsilon_i = \frac{eE_{Si}}{\beta}\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}.$$

x_1=2.32, x_2=4.08, x_3=5.52, $\frac{eE_{Si}}{\beta} = \left(\frac{\hbar^2}{2m_e}\right)^{1/3}(eE_{Si})^{2/3} = 14meV$

therefore ε_1=31meV, ε_2=57meV, ε_3= 77meV for a free electron mass, m_e.

D) While you used the free electron mass to arrive at the previous result, the mass in silicon deviates from the free electron mass and is not isotropic. In Si the energy dispersion around the conduction band minimum, appropriate for the above considerations, reads $\varepsilon(\vec{k}) = \frac{\hbar^2}{2}\sum_{\mu\nu} k_\nu (M^{-1})_{\mu\nu} k_\nu$ with k_μ, k_ν being k-vectors, and

M being the mass tensor $M = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} m_e$ with standard x,y,z notation. What is the effect of this on the previous calculation? By what factor need the eigenstates be scaled?

(2 points)

Only the z-mass m_z=0.9m_e is relevant since the electron motion studied is parallel to z. All energies need to be scaled (increased) by a factor $0.9^{-1/3}$=1.04.

E) To further the description of the 2DES lets assume the capacitor model holds exactly and at V=0 there are negligible carriers in the 2DES, but they are starting to accumulate at V_0=1V (threshould). What is the carrier density at the 8V bias we applied?

(2 point)

Q=ne=C'(V-V_0)=($\varepsilon_{ox}\varepsilon_0$/d)(V-$V_0$), where C' is the capacitance per unit area and n the carrier density per unit area. Hence n=($\varepsilon_{ox}\varepsilon_0$/ed)(V-$V_0$)= 4.5 x 5.5x$10^7$(eV/m)/e80x$10^{-9}$(m) x 7V=2.1x$10^{16}$m^{-2}.

F) Using $D(\varepsilon) = 2m/\pi\hbar^2$ for the density of states of this 2DES, calculate the energy to which the 2D states are filled up. Make sure to use the correct mass deduced from the mass tensor in D). Does this filling reach the second energy level calculated in C) or do all electrons "fit" into the lowest energy level?

(2 points)

Sec 4 # 5: Stormer

We need to use the x,y mass (transverse mass) $0.2m_e$. With this $D(\varepsilon)=1.67\times10^{18}$ $(eVm^2)^{-1}$. Dividing $n=2.1\times10^{16}m^{-2}$ by $D(\varepsilon)$ yields 12.6meV, which is less than the ε_1 to ε_2 spacing. Therefore only the lowest subband is filled.

Sec 4 # 4

Subject: Re: Quals problem 1: applied quantum Tomo Uemura
From: "Yasutomo J. Uemura" <tomo@lorentz.phys.columbia.edu>
Date: Wed, 30 Nov 2005 12:59:19 -0500 (EST)
To: Lalla Grimes <lalla@phys.columbia.edu>

Dear Quals committee:

If you adopt this problem, do NOT say Fermi Energy nor BE condensation in the problem. These are what students are to supposed to find out.

--- Tomo

On Wed, 30 Nov 2005, Yasutomo J. Uemura wrote:

Possible Quals Problem:

(1) Fermi energy, Bose-Einstein condensation

1-a. We consider a system of spin=1/2 neutral (chargeless) particle without interaction among each other. We consider a 3-dimensional system.
We have n such partciles, and each having the mass m.
a-1. Describe the ground state of this system
a-2. The characteristic temperature T_{a} of this system is proportional to the power of n and m as $T_{a} \propto n^{xa}m^{ya}$ obtain the power xa and ya. (hand-waving argument is enough)
a-3. Obtain the exact form of T_{a}.

1-b. We now consider a system where two of these particles are very strongly coupled to form a composite particle of spin = 0. The number of particle is now n/2, while the mass of the new composite particle is 2m. There is no interaction among the different composite particles.
b-1. Describe the ground state of this new system,
b-2. The characteristic temperature $T_{b} of this system is proportional to the power of n and m as $T_{b} \propto n^{xb}m^{yb}$.
Obtain the power xb and yb. Hand-waving argument is sufficient.
b-3. When we compare T_{a} and T_{b}, which is higher ? Describe the reasonings.

to the committee: if you think this is too easy, then we can add

b-4. Obtain the exact value of T_{b}. --- this is not easy,

Sincerely yours,

Tomo Uemura

(1) 1&2

Tomo Uemura

Quals. Prob. 4. Section 4. Relativity SOLUTION JAN 1 7 2008

Fermi Energy, BEC.

a). Fermions will occupy states up to the Fermi Energy in the ground state.

b). Fermi Temperature T_F

Periodic boundary condition $L^3 = V$.

$k = 2\pi n / L$

One state of k per every $\left(\frac{2\pi}{L}\right)^3$

$$N = \frac{4}{3}\pi k_F^3 \times \underset{\uparrow \text{ spin } \downarrow,\uparrow}{2} \times \frac{V}{8\pi^3}$$

$$k_F^3 = 3\pi^2 N/V$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(3\pi^2 N/V\right)^{\frac{2}{3}}$$

$$= k_B T_F$$

For $N/V = 5 \times 10^{22}$, $m_e c^2 = 511$ keV

$$T_F = 5.05 \text{ eV} \simeq 59{,}000 \text{ °K}$$

c) $$T_F \propto \left(\frac{N}{V}\right)^{2/3} \cdot (m_e)^{-1}$$

d) making bosons $N/2 \equiv n_b$ with mass $2m_e \equiv m_b$

ground state : Bose Einstein Condensation.

BEC occurs when thermal wavelength λ becomes comparable to interboson distance

$$\frac{3}{2} k_B T_B = \frac{\hbar^2 k^2}{2 m_b} \qquad k = \frac{2\pi}{\lambda} \cong 2\pi \left(\frac{n_b}{V}\right)^{1/3}$$

$$k_B T_B \sim \frac{1}{3} \cdot \frac{h^2}{2m_e} \left(\frac{n_e}{V}\right)^{2/3} \cdot \left(\frac{1}{2}\right)^{2/3}$$

$$\sim \frac{\hbar^2}{2m_e} \cdot \left(\frac{N}{V}\right)^{2/3} \cdot (8.27)$$

$\longrightarrow$ slightly smaller than $k_B T_F$

Do it rigorously

$$k_B T_{BEC} = (2.612)^{-\frac{2}{3}} \left(\frac{N_b}{V}\right)^{\frac{2}{3}} \cdot \left(\frac{\hbar^2}{m_b}\right) \cdot 2\pi$$

$$= (2.612)^{-\frac{2}{3}} \cdot 2\pi \cdot (1.56)^{-1} \cdot \left(\frac{N_e}{V}\right)^{2/3} \times \left(\frac{\hbar^2}{2m_e}\right)$$

$$= \left(\frac{\hbar^2}{2m_e}\right) \cdot (2.1) \left(\frac{N_e}{V}\right)^{2/3} \sim \frac{1}{4} k_B T_F$$

Westerhoff
11/28/05
General

sec 5 #6

Problem 1

The acceleration due to gravity on the surface of Mercury is $3.5\,\mathrm{m\,s^{-2}}$. The radius of Mercury is $2.4 \times 10^6\,\mathrm{m}$. Suppose that the atmosphere of Mercury were pure H_2 gas.

(a) What would the temperature be so that the rms speed of the H_2 molecules matched the escape speed ? Qualitatively, what is the effect on the temperature of the remaining gas ?

(b) Would there be a similar effect if the actual temperature was less than the result in (a) ?

(c) If Mercury's atmosphere had two or more components, what would happen to the composition as a function of time ?

Problem 2 (10 points)

sec 4 #5

The detection of neutrinos from Supernova SN 1987A can be used to put an upper limit on the neutrino mass. Show that for two neutrino events with different energies E_1 and E_2, the arrival time difference on Earth is given by

$$\Delta t \simeq \left(\frac{Lm^2c^4}{2c}\right)\left(\frac{1}{E_1^2} - \frac{1}{E_2^2}\right) \quad ,$$

where L is the distance to the supernova, and m is the neutrino mass. Calculate an upper limit using typical values $E_1 = 10\,\mathrm{MeV}$, $E_2 = 20\,\mathrm{MeV}$ and the fact that the neutrino pulse from SN 1987A lasted less than 10 s and SN 1987A is 170 000 light years away. Can this limit compete with current limits from tritium beta decay ?

Sec 4
5

Problem 2

For event with energy ϵ, $t_\epsilon = t_{SN} + \frac{L}{\beta}$

Get β using $\epsilon = \frac{m}{\sqrt{1-\beta^2}}$ $\Rightarrow$ $\beta^2 = 1 - \frac{m^2}{\epsilon^2}$,

$$\frac{1}{\beta} \simeq 1 + \frac{m^2}{2\epsilon^2}$$

So

$$t_\epsilon = t_{SN} + L\left(1 + \frac{m^2}{2\epsilon^2}\right)$$

and

$$\Delta t = t_1 - t_2 = \frac{L m^2}{2}\left(\frac{1}{\epsilon_1^2} - \frac{1}{\epsilon_2^2}\right)$$

For the upper limit, $m^2 = \frac{2\Delta t}{L} \frac{1}{1/\epsilon_1^2 - 1/\epsilon_2^2}$,

which gives for $\Delta t < 10\ s$

$$m < 22\ eV,.$$

Currently, tritium beta decay gives 3 eV (PDG 2004).

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 13, 2006
9:00 AM – 11:00 AM

General Physics (Part I)
<u>Section 5.</u>

Two hours are permitted for the completion of this section of the examination. Choose <u>4 problems</u> out of the 6 included in this section. (You will <u>not</u> earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5(General Physics) Question 7, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1 (Section 5 General)

An ideal gas of N atoms in a volume V is in thermal equilibrium with temperature T_i but is assumed to be isolated from other systems. At $t = 0$ all the atoms with kinetic energy larger than $\frac{1}{2}Mv^2 > \alpha\, k_B\, T_i > 0$ are allowed to escape the volume. After that the remaining atoms are assumed to come slowly to a new thermal equilibrium at temperature $T_f(\alpha)$ due to some unspecified weak interactions.

a) Find the dependence of $\mathrm{T}_f(\alpha)$ on α.

b) Find the asymptotic behavior of $T_f(\alpha)$ for very small $\alpha \ll 1$ and very large $\alpha \gg 1$.

Problem 2 (Section 5 General)

The specific heat, C_V, of a system is found to be independent of its volume, i.e., $(\partial C_V/\partial V)_T = 0$.

a) Write down the most general form of the Free Energy $F(T, V)$ that is compatible with this condition.

b) Write down the most general equation of state (i.e. P(V,T), with P the pressure) that is compatible with this condition.

Problem 3 (Section 5 General)

Compute the equilibrium ratio of the number of neutrons to protons in a neutron star. You can assume the electrons, neutrons and protons are all relativistic and degenerate inside the neutron star.

Problem 4 (Section 5 General)

Excitons in semiconductors are bound electron-hole pairs, typically generated after electrons and holes have been created by absorption of light and just before they recombine to emit again a photon. Assume that the conduction band and valence band of the semiconductor follow the dispersion relation

$$E = \pm\sqrt{\left((v\hbar k)^2 + \Delta^2\right)}$$

with $v=c/200$; c being the speed of light and $\Delta = 1\,\mathrm{eV}$.

A) Determine the band edge electron mass, m_e, and the band edge hole mass, m_h.

B) Calculate the binding energy of an exciton in this material, assuming a dielectric constant of $\varepsilon = 16$ for the semiconductor.

C) At what temperatures would we observe excitons.

D) If you wanted to create *free* (non bound) electrons and holes in the semiconductor what is the minimal photon energy required in this example?

E) After formation of exciton and recombination of the electron with the hole, what is the resulting photon energy?

Problem 5 (Section 5 General)

An incompressible liquid is kept under pressure ($P_\circ$) by a movable piston.
A gas bubble of radius $R_\circ$ is trapped within the liquid (see Figure 1).

(a.) What is the pressure (P) of the trapped gas?

(b.) Very roughly, what is the liquid's surface tension (σ) in terms of some of the other parameters given below?

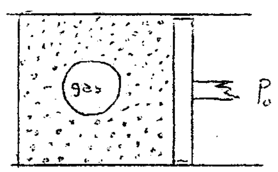


Figure 1: Incompressible liquid, gas bubble and piston.

Please use the following notation:

$R_\circ$ Bubble radius

L Heat of vaporization per unit mass of liquid

σ Surface tension of liquid

ρ_L Density of the liquid

ρ_g Density of the gas in part

$\mathbf{m}_g$ Mass of a gas molecule

$\mathbf{m}_L$ Mass of a liquid molecule

Problem 6 (Section 5 General)

The acceleration due to gravity on the surface of Mercury is $3.5\,\mathrm{m\,s^{-2}}$. The radius of Mercury is $2.4 \times 10^6\,\mathrm{m}$. Suppose that the atmosphere of Mercury were pure H_2 gas.

(a) What should the temperature be for the rms speed of the H_2 molecules to match the escape speed? Qualitatively, what is the effect on the temperature of the remaining gas in this situation?

(b) Would there be a similar effect if the actual temperature was less than the result in (a) ?

(c) If Mercury's atmosphere had two or more components, what would happen to the composition as a function of time ?

Statistical mechanics

IgorAleiner

NOV 22 2005 Sec 5

#1

Evaporating cooling

Ideal gas of atoms with established temperature T_i is kept isolated. At $t=0$, all the atoms with the kinetec energy larger than yT_i are removed from the system ($y>0$, is the numerical coefficient. After that the system is isolated but weak interaction leads to the establishing new temperature at $t\to\infty$; $T(t\to\infty)=T_f$.

Find $T_f(y)$ for an arbitrary y (10pts).

Find the asymtotic behavior of $T_f(y)$ at $y\to 0$, and $y\to\infty$ (5pts)

Solution

$$\mathcal{E}_i = \frac{3}{2}N_i T_i$$

$$\mathcal{E}_f = \frac{3}{2}N_f T_f;$$

$$\frac{N_f}{N_i} = \frac{\int_0^y dx\sqrt{x}\,e^{-x}}{\int_0^\infty dx\sqrt{x}\,e^{-x}};$$

$$\frac{\mathcal{E}_f}{\mathcal{E}_i} = \frac{\int_0^y dx\, x^{3/2}e^{-x}}{\int_0^\infty dx\, x^{3/2}e^{-x}};$$

$$\frac{T_f}{T_i} = \frac{\mathcal{E}_f}{\mathcal{E}_i}\frac{N_i}{N_f} = \frac{2}{5}\,\frac{\int_0^y dx\, x^{3/2}e^{-x}}{\int_0^y dx\, x^{1/2}e^{-x}} = \quad (10pts)$$

$$= \begin{cases} 6/25\, y; & y\to 0; \\ 1-\frac{2}{5}y^{3/2}e^{-y}; & y\to\infty; \end{cases} \quad (5pts).$$

Thermal physics

Igor Aleiner

NOV 2 2 2005 Sec 5 #2

The specific heat of some system does not depend on its volume, $\left(\frac{\partial C_V}{\partial V}\right)_T = 0$;

1) Write down the most general form of the Free energy $F(T,V)$ compatible with this condition. (8pts);

2) Write down the most general equation of state compatible with $\left(\frac{\partial C_V}{\partial V}\right)_T = 0$;

Solution:

1) $$\left.\begin{aligned} C_V &= -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \\ \left(\frac{\partial C_V}{\partial V}\right)_T &= 0; \end{aligned}\right\} \Rightarrow F = F_1(T) + F_2(V) + T F_3(V)$$

where F_1, F_2, F_3 are the arbitrary functions constrained by

$$\frac{\partial^2 F}{\partial V^2} > 0, \text{ i.e. } \boxed{\frac{\partial^2 F_2}{\partial V^2} + T\frac{\partial^2 F_3}{\partial V^2} > 0}$$

2); $$P = -\left(\frac{\partial F}{\partial V}\right)_T,$$

linear T dependence:

$$P = P_1(V) + T P_2(V);$$

$$\left(\frac{\partial P}{\partial V}\right)_T < 0$$

Subject: 2 questions for the Quals committee
From: Lam Hui <lhui@astro.columbia.edu>
Date: Wed, 23 Nov 2005 12:42:20 -0500 (EST)
To: lalla@phys.columbia.edu, lhui@phys.columbia.edu

To the Quals Committee,

Below please find two problems: one mechanics and one general.

Lam

Mechanics -

Problem:

Consider the motion of the earth around the sun. Let's approximate the orbit as circular. Suppose the sun very slowly loses its mass, from an original mass of M_1 to a mass of M_2. Suppose the initial radius of the orbit is R_1 and the eventual radius is R_2. What is R_2 in terms of the other parameters?

Solution:

The angular momentum is an adiabatic invariant. Therefore,
$M_1 R_1 v_1 = M_2 R_2 v_2$, with
$v_1 = \sqrt{G M_1/R_1}$ and $v_2 = \sqrt{G M_2/R_2}$.
Hence, $R_2 = R_1 (M_1/M_2)^3$ i.e. the orbit expands under mass loss.

Sec 5 #3

General -

Problem:

Compute the equilibrium ratio of the number of neutrons to protons in a neutron star. You can assume the electrons, neutrons and protons are all relativistic and degenerate inside the neutron star.

Solution:
The relevant process is a neutron turning into a proton, an electron and an anti-neutrino, and vice versa.
Chemical equilibrium demands the equality of chemical potentials:
$\mu_n = \mu_p + \mu_e$. The chemical potential of anti-neutrinos is zero because they can escape the neutron star.
The chemical potential of each specie is given simply by the Fermi energy, which equals the Fermi momentum in the relativistic regime, which is proportional to the density n to the one-third power.
Therefore, $n_n^{1/3} = n_p^{1/3} + n_e^{1/3}$.
Setting $n_p = n_e$ for charge neutrality then tells us
$n_n/n_p = 8$.

Ses 5 # 4

Qualifier Question Physics 2005, Stormer, General Physics
11/23/05

Excitons in semiconductors are bound electron-hole pairs, typically generated after electrons and holes have been created by absorption of light and just before they recombine to emit again a photon.

Assume that the conduction band and valence band of the semiconductor follow the dispersion relation

$$E = \pm\sqrt{\left((v\hbar k)^2 + \Delta^2\right)}$$

with v=c/200; c being the speed of light and $\Delta = 1\text{eV}$.

A) Determine the band edge electron mass, m_e, and the band edge hole mass, m_h.

(3 points)

$m_{e,h} = \hbar^2(\partial^2 E/\partial k^2)^{-1}$ **yields**

$m_{e,h}/m_0 = \Delta/v^2 = (200)^2 \Delta/m_0c^2 \approx 40000x(1eV)/0.5MeV = 0.08$ **where m_o is free electron mass.**

B) Calculate the binding energy of an exciton in this material, assuming a dielectric constant of $\varepsilon = 16$ for the semiconductor.

(2 points)

Both masses are identical. Therefore like positronium: $E = \dfrac{m_0 e^4}{4\hbar^2} = 1/2Ry = 6.8eV$.

However, mass m_e=m_h=0.08 m_o and E-field is screened by ε. Hence E gets multiplied by $m_{e,h}/(m_o\varepsilon^2)$=0.08/256=3.1 x 10^{-4}. E_{ex}=2.1meV

C) At what temperatures would we observe excitons.

(1 point)

At a temperature smaller than ~2.1meV or ~25K. Otherwise the excitons cannot form; e and h are not bound.

D) If you wanted to create *free* (non bound) electrons and holes in the semiconductor what is the minimal photon energy requires in this example?

(1 point)

The bandgap energy of Δ=1eV.

stormer sect 5
#4

E) After formation of exciton and recombination of the electron with the hole, what is the resulting photon energy?

(1 point)

$E=\Delta-E_{ex}\sim 0.998eV$

F) The same semiconductor also contains impurities; donors as well as acceptors. Calculate the binding energy of carriers to these impurities and specify which carrier binds to which kind of impurity.

(2 points)

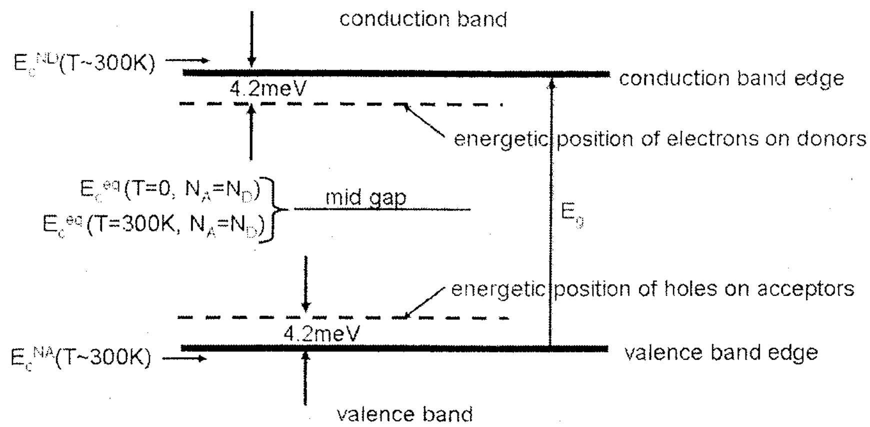
The equations are identical to the exciton case. However, now one of the charges is very heavy (acceptor, donor). Therefore the situation is equivalent to hydrogen and the binding energy is double as big as in the exciton case. $E_{D,A}$= 4.2meV. Electrons bind to donors. Holes bind to acceptors.

G) Make a graph of the energies versus some direction, x, in the semiconductor. Make this graph big, since the following questions require indicating several energetic positions within it. Start by indicating the position of the conduction band, the conduction band edge, the valence band, the valence band edge, the energy gap and its value, the energetic position of carriers bound to donors and carriers bound to acceptors, together with the energies calculated in B).

(2 points)

see graph

Stormer Sec 5 #4



H) Where would you locate the exciton in this graph? Explain in words.

(1 point)

The exciton cannot really be located in this graph, since it is a two-particle system. Sometimes it is indicated as a vertical arrow within the band gap of lengths $E=\Delta-E_{ex}$.

I) If the densities of donors, N_D, and acceptors, N_A, are the same, where would you locate the chemical potential, E_c^{eq} (T=0), at zero temperature and at E_c^{eq} (T~300K)?. Indicate both positions in the graph C).

(1 point)

At mid gap.

J) Also draw the approximate position of E_c^{ND}(T~300K) for N_D>>N_A and E_c^{NA}(T~300K) for N_D<<N_A.

(1 points)

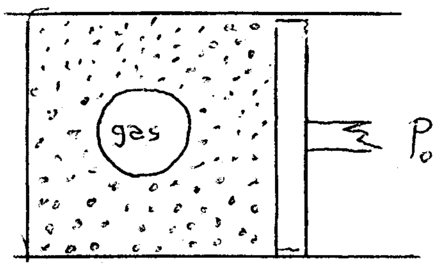
E_c^{ND}(T~300K) in conduction band, close to conduction band edge.
E_c^{NA}(T~300K) in valence band, close to valence band edge.

NOV 28 2005 Sec 5 #5

M. Ruderman

General A

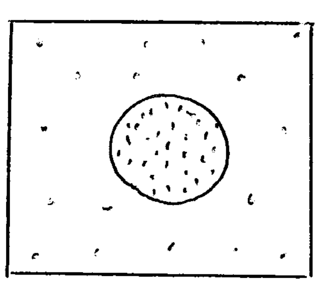
An (incompressible) liquid is kept under pressure (P_o) by a movable piston. Within this liquid a gas is trapped inside a bubble of radius R_o.



a) What is the pressure (P) of the trapped gas?

b) Very roughly, what is the liquid's surface tension (σ) in terms of some of the other parameters given below?

c) Consider a droplet of this same liquid (radius R) in equilibrium with its own vapor. For $R \to \infty$, and the same temperature the equilibrium vapor pressure is P_1. What is the vapor pressure when R is finite? (Assume, if needed, $\sigma/R \ll P_1$)



$R_o \equiv$ bubble radius

$L \equiv$ heat of vaporization per unit mass of liquid

$\sigma \equiv$ surface tension of the liquid

$\rho_L \equiv$ density of the liquid

$\rho_g \equiv$ density of the gas in a)

$m_g \equiv$ mass of gas molecule

$m_L \equiv$ mass of liquid molecule

$\rho_v \equiv$ density of vapor when $R \to \infty$ in c).

Greek rho

ρ_g

m_g

Rudermon General A

NOV 2 8 2005
Sec 5 #5

Answers

a) $\boxed{P = P_0 + \frac{2\sigma}{R}}$

$$-4\pi R^2 \underbrace{\Delta R}_{\Delta \text{Vol}} (P - P_0) + \underbrace{8\pi R \Delta R}_{\Delta \text{Area}} \sigma = 0$$

or
very, very roughly, use dimensional analysis

$$P - P_0 \sim \frac{\sigma}{R}$$

b) Dimensional analysis using only the liquid's parameters

$$\boxed{\sigma \sim (L m_L)\left(\frac{\rho_L}{m_L}\right)^{2/3} \times f}$$

f = fraction of order unity

or

$$\sigma \sim \frac{\text{binding energy deficiency of a surface molecule}}{\text{surface area for each surface molecule}}$$

$$\sim \frac{f L m_L}{(m_L/\rho_L)^{2/3}}$$

$f < 1$,
typically $\sim \frac{1}{6}$
for cubic lattice

c) chemical potential equality is maintained so

$\mu_L = \mu_V \qquad \Delta\mu_L = \Delta\mu_L$

$$(v\Delta p - s\Delta T)_L = (v\Delta p - s\Delta T)_V$$

$\Delta T = 0$
$\Delta P_L \sim \frac{2\sigma}{R}$

$$\boxed{\therefore \Delta P_{\text{vapor}} \cong \frac{2\sigma}{R} \frac{\rho_{\text{vap}}}{\rho_{\text{Liq}}}}$$

Westerhoff
11/28/05
General

Problem 1

sec 5 #6

The acceleration due to gravity on the surface of Mercury is $3.5\,\mathrm{m\,s^{-2}}$. The radius of Mercury is $2.4 \times 10^6\,\mathrm{m}$. Suppose that the atmosphere of Mercury were pure H_2 gas.

(a) What would the temperature be so that the rms speed of the H_2 molecules matched the escape speed ? Qualitatively, what is the effect on the temperature of the remaining gas ?

(b) Would there be a similar effect if the actual temperature was less than the result in (a) ?

(c) If Mercury's atmosphere had two or more components, what would happen to the composition as a function of time ?

Problem 2 (10 points)

sec 4 #5

The detection of neutrinos from Supernova SN 1987A can be used to put an upper limit on the neutrino mass. Show that for two neutrino events with different energies E_1 and E_2, the arrival time difference on Earth is given by

$$\Delta t \simeq \left(\frac{Lm^2c^4}{2c}\right)\left(\frac{1}{E_1^2} - \frac{1}{E_2^2}\right) \quad ,$$

where L is the distance to the supernova, and m is the neutrino mass. Calculate an upper limit using typical values $E_1 = 10\,\mathrm{MeV}$, $E_2 = 20\,\mathrm{MeV}$ and the fact that the neutrino pulse from SN 1987A lasted less than 10 s and SN 1987A is 170 000 light years away. Can this limit compete with current limits from tritium beta decay ?

Solutions

Westerhoff 11/28/05
General # 1

Sec 5 #4

Problem 1:

(a) escape speed $\frac{1}{2} m v_{esc}^2 - G\frac{Mm}{R} = 0$

$$\Rightarrow \quad v_{esc} = \sqrt{2gR} \qquad g = 3.5\,\frac{m}{s^2}$$

average kinetic energy of H_2 is $\frac{1}{2} m v_{rms}^2 = \frac{3}{2} kT$,
so for $v_{rms} = v_{esc}$

$$\frac{1}{2} m\, 2gR = \frac{3}{2} kT \quad \Rightarrow \quad T = \frac{1}{3}\,\frac{2mgR}{k}$$

$$m = 2g/\text{mol} \cdot \frac{\text{mol}}{6.02 \cdot 10^{23}}$$

So

$$T = \frac{1}{3}\,\frac{2 \cdot (3.5\,\text{m s}^{-2})(2g/\text{mol})}{6.02 \cdot 10^{23}\,/\text{mol}\;\; 1.38 \cdot 10^{-23}\,\text{J/K}}\; 2.4 \cdot 10^6\,\text{m}$$

$$= \underline{\underline{1348\ K}}$$

As faster molecules escape, v_{rms} and T <u>decrease</u>.

(b) Yes, since the speed distribution has the speed of some H_2 molecules greater than v_{rms}, but the fraction is less, so H_2 molecules escape more slowly.

(c) The lighter component escapes more rapidly.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 13, 2006
11:10 AM – 1:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6(General Physics) Question 6, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

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Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted. Questions should be directed to the proctor.

Good luck!!

Problem 1: Section 6 General Physics

Energetic photons are attenuated in "free space" due to collisions with cosmic microwave background (CMB) photons (and infrared photons that we will ignore) that result in e^+e^- pair production. In answering the following questions take the CMB temperature to be given by $k_B T = 2.5 \times 10^{-4}$ eV, the electron mass to be $m_e \approx 0.5$ MeV and the product $\hbar c = 200$ eV nm.

a) **[3 pts]** Estimate the minimum energy, $E_{\rm min}$, at which a photon propagating through space will produce an e^+e^- pair assuming that the CMB photons have energy $k_B T$.

b) **[4 pts]** Find a symbolic expression for $k_{\rm min}$, the minimum CMB photon momentum that can produce e^+e^- pairs when colliding with a propagating photon of energy E as a function of θ, the angle between the momentum vectors of the propagating photon and the CMB photon.

c) **[4 pts]** At photon energies above 10^{15} eV the photon-photon scattering is dominated by e^+e^- pair production at threshold. Suppose you are given the pair production cross-section at threshold, σ. Show symbolically how you would calculate the mean free path of a photon with enery E propagating through the universe assuming that it interacts with the full spectrum of CMB photons and only by pair production at threshold. You may leave your results in terms of an unevaluated integral.

d) **[4 pts]** Suppose $E = E_{\rm min}/6$. Obtain an *order-of-magnitude estimate* for the mean free path of photons with this energy using $e^{-6} \approx 1/400$ and $\sigma \approx 1 \times 10^{-25}\text{cm}^2$. You may find it convenient to express your result in terms of parsecs, 1 parsec $\approx 3 \times 10^{16}$ m. **Beware, even when simplified, the integral in part c) cannot be completely evaluated analytically. You must find a way to approximate the integral.**

Problem 2: Section 6 General Physics

A capacitor with plate separation d is placed in an ideal gas of molecules at temperature T. The molecules have polarizability α. Find the ratio of gas pressures inside and outside the capacitor as a function of voltage V applied to it.

Problem 3: Section 6 General Physics

A total of N non-relativistic electrons are confined to a box of volume V. Suppose the electrons are in their ground state, meaning that they form a degenerate Fermi gas. Aside from the Pauli principle you can neglect interactions between electrons.

a) **[3 pts]** Compute the energy of the gas as a function of N and V.

b) **[3 pts]** Compute the pressure P exerted by the gas on the walls of the box, and evaluate the bulk modulus $B = -V\frac{\partial P}{\partial V}$.

Beside the electrons, suppose the box contains a lattice of atoms (which we have ignored up to this point). A longitudinal sound wave propagates through the box in the x direction. The wave can be characterized by a displacement field $\chi(x,t)$. This just means the atoms that are at position x in equilibrium (in the absence of a sound wave) have been displaced to position $x + \chi(x,t)$.

c) **[3 pts]** Show that the mass density in the box is $\rho = \rho_0/(1 + \frac{\partial \chi}{\partial x})$, where ρ_0 is the equilibrium mass density.

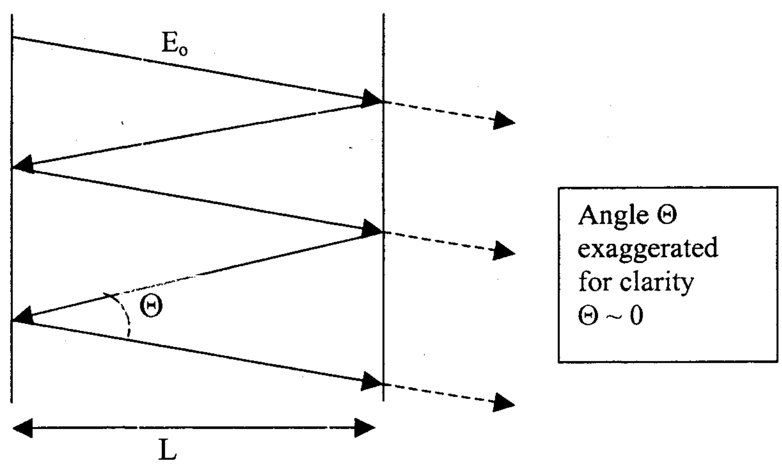
d) **[3 pts]** For small displacements show that $\chi(x,t)$ obeys the wave equation. Show that the speed of sound is $v_s = \sqrt{B/\rho_0}$.

d) **[3 pts]** The density of conduction electrons in copper is $n_e = 8.5 \times 10^{22}$ cm^3, and the mass density of copper is $\rho_0 = 9$ g/cm^3. Estimate the bulk modulus and speed of sound in copper.

Useful facts: $\hbar c \approx 2 \times 10^{-5}$ eV $\cdot$ cm, $m_e c^2 = 500$ keV, 1 eV/c$^2 \approx 2 \times 10^{-33}$ g.

Problem 4: Section 6 General Physics

Consider a cavity with mirrors of amplitude reflectivity r and reflection coefficient $r^2 = R \sim 0.99$ and a separation between the mirrors L. The laser beam continuously reflects between the mirror ends. The high reflectivity permits the beam in opposite directions to be considered of approximately equal amplitudes so that standing waves are generated which constitute the longitudinal modes of the cavity.



a) **[3 pts]** What is the phase change during one round trip of the laser cavity? Express your answer in terms of L and ν, the radiation frequency.
b) **[4 pts]** For what mode frequencies ν can standing waves be maintained in the cavity? Express your answer in terms of L. What is the separation Δν between the mode frequencies?
c) **[4 pts]** Find an expression for the intensity of radiation transmitted out the far end of the laser cavity in terms of L, ν and R.
d) **[4 pts]** As the frequency changes from the standing wave value, the intensity of radiation out of the end of the cavity decreases. Use your results from (a) and (c) to determine the shift in frequency over which the intensity drops to half its maximum value.

Problem 5: Section 6 General Physics

An interface between two materials A and B may be characterized by a surface *tension* σ_{AB} which is the energy cost per unit area of the interface.

Consider a spherical particle of mass density ρ_P and radius R placed in a cylindrical container of radius d filled with liquid to a height h. The mass density of the liquid is $\rho_L < \rho_P$.

Assume that the surface tension of the particle-liquid interface is σ_{PL}, that the surface tension of the liquid-air interface is σ_{LA} and that the surface tension of the particle-air interface vanishes: $\sigma_{PA} = 0$.

[15 pts] Find the height z of the particle above the bottom of the container. You may assume the container radius d is much greater than the particle radius R.

Problem 6: Section 6 General Physics

Consider a radioactive source which emits a positron in every decay. In ordinary matter, the positron is stopped within the source holder, and annihilates with an electron, which has a rest mass of 511 keV.

(a) **[5 pts]** What particle(s) will be emitted, and with what energy, in the annihilation process? How many particles are emitted? If there are multiple particles, is there any correlation in the directions of their emission? Let us define this particle(s) as particle **a**.

(b) **[5 pts]** In the case of a decay of ^{22}Na, the positron decay event leaves the system in an excited state of ^{22}Ne, which then decays with the emission of a gamma ray of 1.27 MeV. We denote this particle as particle **b**.

We want to distinguish between the particles **a** and **b**. What kind of particle detector shall we use for this purpose? Why can we distinguish between these particles?

(c) **[5 pts]** Suppose we have two detectors: detector A which detects particle **a**, selectively, and detector B which detects particle **b**, selectively. The time resolution of these detectors is Δt, which is much longer than the time interval of the successive decay events **a** and **b**. Suppose we have a system to measure the single rate of **a**-decay and the single rate of **b**-decay, by using the detectors A and B. We also can measure the rate of successive decays by taking the time coincidence of the A and B counter signals. Let us define the rates of these as R_a, R_b and R_{ab}. Show that we can determine the strength of the original radioactive source (i.e., N decay events per second) by using this information. Describe why and how we can do this.

Beloborodov
11/28/05
General

Sec 6 #2

Problem 2. A capacitor with plate separation d is placed in ideal gas of molecules with temperature T. The molecules have polarizability α. Find the ratio of gas pressures inside and outside the capacitor as a function of voltage V applied to it.

Solution: Electric field inside the capacitor is $E = V/a$. The dipole moment of a molecule inside the capacitor is $d = \alpha E$ and its potential energy in the electric field is $U = -Ed$. The ratio of gas densities inside and outside the capacitor is given by the Boltzmann factor,

$$\frac{n}{n_0} = \exp\left(-\frac{U}{kT}\right).$$

The temperature is everywhere the same and hence the ratio of gas pressures inside and outside the capacitor is

$$\frac{p}{p_0} = \frac{n}{n_0} = \exp\left(\frac{\alpha V^2}{kTa^2}\right).$$

Sec 6 #3

Dan Kabat
11/15/05

General

A total of N non-relativistic electrons are confined to a box of volume V. Suppose the electrons are in their ground state, meaning that they form a degenerate Fermi gas. Aside from the Pauli principle you can neglect interactions between electrons.

1. Compute the energy of the gas as a function of N and V.

2. Compute the pressure P exerted by the gas on the walls of the box, and evaluate the bulk modulus $B = -V\frac{\partial P}{\partial V}$.

Beside the electrons, suppose the box contains a lattice of atoms (which we've ignored up to this point). A longitudinal sound wave propagates through the box in the x direction. The wave can be characterized by a displacement field $\chi(x,t)$. This just means the atoms that are at position x in equilibrium (in the absence of a sound wave) have been displaced to position $x + \chi(x,t)$.

3. Show that the mass density in the box is $\rho = \rho_0/(1 + \frac{\partial \chi}{\partial x})$ where ρ_0 is the equilibrium mass density.

4. For small displacements show that $\chi(t,x)$ obeys the wave equation. Show that the speed of sound is $v_s = \sqrt{B/\rho_0}$.

5. The density of conduction electrons in copper is $n_e = 8.5 \times 10^{22}/\text{cm}^3$, and the mass density of copper is $\rho_0 = 9\,\text{g/cm}^3$. Estimate the bulk modulus and speed of sound in copper.

Useful facts: $\hbar c \approx 2 \times 10^{-5}\,\text{eV}\cdot\text{cm}$, $m_e c^2 \approx 500\,\text{keV}$, $1\,\text{eV}/c^2 \approx 2 \times 10^{-33}\,\text{g}$.

Sec 6 #3

General problem solution

Box of volume L^3, $\psi_{\vec n} = e^{i 2\pi \vec n \cdot \vec x / L}$

$$E_{\vec n} = \frac{\hbar^2}{2m} \frac{4\pi^2 |\vec n|^2}{L^2}$$

$$E = 2 \int d^3 n \, E_{\vec n} \qquad (\times 2 \text{ for two spins})$$

$$= 2 \frac{\hbar^2}{2m} \frac{4\pi^2}{L^2} \int_0^{n_{max}} 4\pi n^2 dn \cdot n^2$$

$$= 2 \frac{\hbar^2}{2m} \frac{4\pi^2}{L^2} \frac{4\pi}{5} n_{max}^5$$

$$N = 2 \int_0^{n_{max}} d^3 n = 2 \cdot \frac{4}{3} \pi n_{max}^3$$

$$\Rightarrow E = \frac{16\pi^3}{5} \frac{\hbar^2}{mL^2} \left(\frac{3N}{8\pi}\right)^{5/3} = \frac{16\pi^3}{5} \frac{\hbar^2}{m} \frac{1}{V^{2/3}} \left(\frac{3N}{8\pi}\right)^{5/3}$$

$$P = -\frac{\partial E}{\partial V} = \frac{32\pi^3}{15} \frac{\hbar^2}{m} \frac{1}{V^{5/3}} \left(\frac{3N}{8\pi}\right)^{5/3}$$

$$B = -V \frac{\partial P}{\partial V} = \frac{5}{3} \underline{P}$$

density ?

$$dm = \rho_0 L^2 dx = \rho L^2 \left(dx + \frac{\partial \chi}{\partial x} dx\right) \qquad \text{mass conservation}$$

$$\Rightarrow \rho = \frac{\rho_0}{1 + \frac{\partial \chi}{\partial x}}$$

wave equation ?

Newton: $dF = \rho L^2 dx \, \ddot\chi = -\frac{\partial P}{\partial x} dx \, L^2$

$$\rho \ddot\chi = -\frac{\partial P}{\partial x} = -\frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial x} = \frac{V}{\rho} \frac{\partial P}{\partial V} \frac{\partial \rho}{\partial x} \qquad \text{since } \frac{d\rho}{\rho} = -\frac{dV}{V}$$

$$\rho \ddot\chi = -\frac{B}{\rho} \frac{\partial \rho}{\partial x}$$

$$\frac{\rho_0}{1 + \frac{\partial \chi}{\partial x}} \ddot\chi = -\frac{B}{\rho_0} \left(1 + \frac{\partial \chi}{\partial x}\right) \frac{\partial}{\partial x} \frac{\rho_0}{1 + \frac{\partial \chi}{\partial x}}$$

$$\text{linearize} \Rightarrow \ddot\chi = + \frac{B}{\rho_0} \frac{\partial^2 \chi}{\partial x^2}$$

sound speed $v_s = \sqrt{B/\rho_0}$

Sec 6 #3

For copper $B = \frac{32\pi^3}{15} \frac{\hbar^2 c^2}{m_e c^2} \left(\frac{3 n_e}{8\pi}\right)^{5/3}$

$\approx 2.5 \times 10^{23}$ eV/cm^3

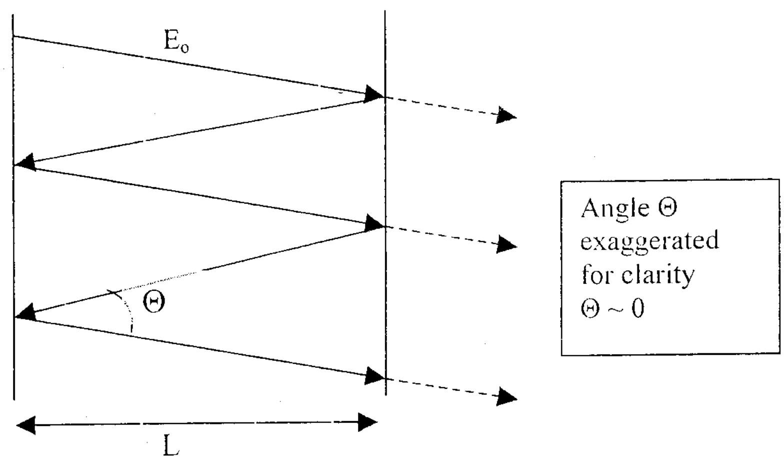
$v_s = \sqrt{\frac{B}{\rho_0}} = \left(\frac{2.5 \times 10^{23} \text{ eV}}{9 \text{ g}}\right)^{1/2} \approx 2100$ m/s

Sec 6
#4

Chuck Hailey's 2006 Quals problem (typed by Elena)
12/5/05

General Problem:

Consider a laser cavity with mirrors of amplitude reflectivity r and reflection coefficient $r^2 = R \sim 0.99$ and a length between the mirrors L. The laser beam continuously reflects between the mirror ends. The high reflectivity permits the beam in opposite directions to be considered of approximately equal amplitudes so that standing waves are generated which constitute the longitudinal modes of the cavity.



a) What is the phase change during one round trip of the laser cavity? Express your answer in terms of L and ν, the radiation frequency.
b) For what mode frequencies ν can standing waves be maintained in the cavity? Express your answer in terms of L. What is the separation $\Delta\nu$ between the mode frequencies?
c) Find an expression for the intensity of radiation transmitted out the far end of the laser cavity in terms of L, ν and R.
d) As the frequency changes from the standing wave value, the intensity of radiation out of the end of the cavity decreases. Use your results from (a) and (c) to determine the shift in frequency over which the intensity drops to half its maximum value.

QM Problem solution:

Hailey
Sec 6
4

General problem solution:

a) $\phi = -2KL + 2\alpha$ $\quad \alpha$ = phase change at mirror

$\phi = -\frac{4\pi L \nu}{c} + 2\alpha$

b.) $2\pi m = \frac{4\pi L \nu}{c} - 2\alpha$

$\nu_m = \frac{mc}{2L} + \frac{\alpha c}{2\pi L}$

c.) $E_t = E_0 t\,(1 + r^2 e^{-i\phi} + r^4 e^{-2i\phi} + \dots)$ $\quad t$ = Amplitude transmission

$E_t = \frac{E_0 t}{1 - Re^{-i\phi}}$ $\quad R \equiv r^2$

$\left|E_t / E_0 t\right|^2 = \frac{1}{(1 - Re^{-i\phi})(1 - Re^{i\phi})}$

$(1 - Re^{-i\phi})(1 - Re^{i\phi}) = 1 + R^2 - 2R\cos\phi = 1 + R^2 - 2R(1 - 2\sin^2\frac{\phi}{2})$

$= (1-R)^2 + 4R\sin^2\phi/2$

Absorbing $(1-R)^{-2}$ into I_{max}

$I_t = \frac{I_{max}}{1 + \frac{4R\sin^2\phi/2}{(1-R)^2}}$ $\quad \phi$ = from part (a)

d) $I_t \to \frac{I_{max}}{2}$ when $\frac{4R\sin^2\delta\phi/2}{(1-R)^2} = 1$

where $\delta\phi$ is shift from peak value $2\pi m$

since $\frac{4R}{(1-R)^2} \gg 1$ $\quad \delta\phi \ll 1$ $\quad 4R\sin^2\frac{\delta\phi}{2} \approx \frac{4R}{4}\delta\phi^2$

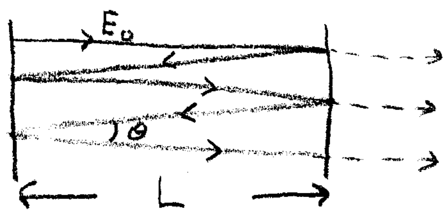
$\delta\phi = \frac{1-R}{\sqrt{R}} = \frac{4\pi L}{c}\delta\nu$ $\quad \delta\nu = \frac{c}{4\pi L}\frac{1-R}{\sqrt{R}}$ $\quad \frac{\text{Ans}}{2}$

Sec 6 #4

NOV 22 2005

General: Hriley

Consider a laser cavity with mirrors of amplitude reflectivity r and reflection coefficient $r^2 = R \sim 0.99$ and a length between the mirrors L. The laser beam continuously reflects between the mirror ends. The high reflectivity permits the beams in opposing directions to be considered of approximately equal amplitude so that standing waves are generated which constitute the longitudinal modes of the cavity.



Angle θ exaggerated for clarity; $\theta \approx 0$

a) What is the phase change during one round trip of the laser cavity? Express your answer in terms of L and ν, the radiation frequency.
b) For what mode frequencies ν can standing waves be maintained in the cavity? Express your answer in terms of L. What is the separation Δν between the mode frequencies?
c) Find an expression for the intensity of radiation transmitted out the far end of the laser cavity in terms of L, ν and R.
d) As the frequency changes from the standing wave value, the intensity of radiation out the end of the cavity decreases. Use your results from (a) and (c) to determine the shift in frequency over which the intensity drops to half its maximum value.

Solution general: Hailey

NOV 22 2005
Sec 6 #4

a.) $\phi = -2KL + 2\alpha$ $\quad \alpha$ = phase change at mirror

$$\phi = -\frac{4\pi L \nu}{c} + 2\alpha$$

b.) $$2\pi m = \frac{4\pi L \nu}{c} - 2\alpha$$

$$\nu_m = \frac{mc}{2L} + \frac{\alpha c}{2\pi L}$$

c.) $$E_t = E_o t\left(1 + r^2 e^{-i\phi} + r^4 e^{-2i\phi} + \ldots\right)$$ t = Amplitude transmission

$$E_t = \frac{E_o t}{1 - Re^{-i\phi}} \qquad R \equiv r^2$$

$$\left|\frac{E_t}{E_o t}\right|^2 = \frac{1}{(1-Re^{-i\phi})(1-Re^{i\phi})}$$

$$(1-Re^{-i\phi})(1-Re^{i\phi}) = 1 + R^2 - 2R\cos\phi = 1 + R^2 - 2R(1 - 2\sin^2\tfrac{\phi}{2})$$
$$= (1-R)^2 + 4R\sin^2\phi/2$$

$$I_t = \frac{I_{max}}{1 + \frac{4R\sin^2\phi/2}{(1-R)^2}}$$ Absorbing $(1-R)^{-2}$ into I_{max}

ϕ = from part (a)

d.) $I_t \to \frac{I_{max}}{2}$ when $\frac{4R\sin^2\delta\phi/2}{(1-R)^2} = 1$

where $\delta\phi$ is shift from peak value $2\pi m$

since $\frac{4R}{(1-R)^2} \gg 1$ $\quad \delta\phi \ll 1$ $\quad 4R\sin^2\frac{\delta\phi}{2} \approx \frac{4R}{4}\delta\phi^2$

$$\delta\phi = \frac{1-R}{\sqrt{R}} = \frac{4\pi L}{c}\delta\nu \qquad \delta\nu = \frac{c}{4\pi L}\frac{1-R}{\sqrt{R}} \quad \frac{\text{ans}}{2}$$

NOV 2 1 2005

Sec 6 #5

Millis General 06 Quals Problem

An interface between two materials A, B may be characterized by a *surface tension* σ_{AB} which is the energy cost per unit area of the interface.

Consider a spherical particle of mass density ρ_P and radius R placed in a cylindrical container of radius d filled with liquid to a height h. The mass density of the liquid is $\rho_L < \rho_P$.

Assume that the surface tension of the particle-liquid interface is σ_{PL}, of the liquid-air interface is σ_{LA} while the surface tension of the particle-air interface vanishes: $\sigma_{PA} = 0$.

Please find the height z of the particle above the bottom of the container.

You may assume the container radius d is much greater than the particle radius R.

NOV 2 1 2005

SEC 6
5

Millis General 06 Quals Problem with Solution

An interface between two materials A, B may be characterized by a *surface tension* σ_{AB} which is the energy cost per unit area of the interface.

Consider a spherical particle of radius R placed in a cylindrical container of radius d filled with liquid to a height h.

Assume that the surface tension of the particle-liquid interface is σ_{PL}, of the liquid-air interface is σ_{LA} while the surface tension of the particle-air interface vanishes: $\sigma_{PA} = 0$.

Please find the position of the center of mass of the particle with respect to the surface of the liquid. You may assume the container radius d is much greater than the particle radius R, neglect gravity and assume that the surface tension of the liquid-air interface is positive.

Solution

There are three cases: particle on surface, particle at bottom of container, particle partly submerged. Choose zero of energy to be state in which particle is on top of liquid. Large d limit means may neglect energy cost of displaced liquid.

- Particle on top of liquid: energy 0.
- Particle submerged: energy of interface $E = 4\pi R^2 \sigma_{PL}$.
- Particle partly submerged. Center of mass moved down a distance L. Area submerged is $2\pi LR$. Liquid-air interface lost is $\pi L(2R - L)$. Total energy is

$$E = 2\pi LR\sigma_{PL} - \sigma_{LA}\pi L(2R - L)$$

This energy is minimized at $L^* = \left(1 - \frac{\sigma_{PL}}{\sigma_{LA}}\right) R$; minimum energy is $E^* = -\frac{1}{4}\sigma_{LA}L^{*2} = -\frac{1}{4}\frac{(\sigma_{LA}-\sigma_{PL})^2}{\sigma_{LA}}R^2$.

Therefore if $\sigma_{PL} > \sigma_{LA}$ the particle is expelled from the liquid while if $\sigma_{PL} < 0$ the particle is fully submerged. For intermediate values, the particle is partly immersed.

Tomo Uemura

General II

Problem 6. Section 6. JAN 17 2006

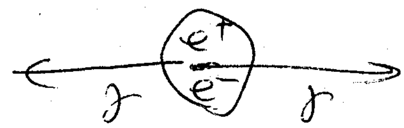
solution

a)

Positron Annihilation

$e^- + e^+ \rightarrow 2\gamma$ 511 keV each

Opposite direction



b) Both are γ-rays 511 keV, 1.27 MeV

To distinguish energy,
use NaI scintillator + Photo Multiplyer,
or Ge detector

Energy deposition proportional to pulse height

c) Originally N decays per sec

$R_a = 2 \cdot N \cdot (\text{detection efficiency } \varepsilon_a) \cdot (\text{solid angle } \Omega_a / 4\pi)$

$R_b = N \cdot \varepsilon_b \cdot \Omega_b / 4\pi$

$R_{ab} = N \cdot \varepsilon_b \cdot \Omega_b / 4\pi \cdot 2 \cdot \varepsilon_a \cdot \Omega_a / 4\pi$

$\therefore \quad N = R_a \cdot R_b \,/\, R_{ab}$