Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I

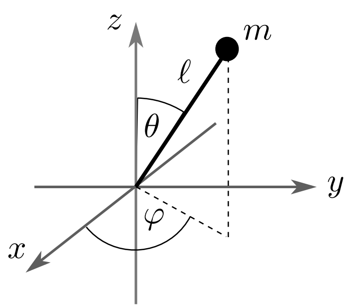
Saturday, August 28, 2021

The exam is **closed book/notes**. Simple scientific calculators are allowed. Make sure that each page is signed with your code number (S-#) and the problem number. Do not write answers to different problems on the same page. Scan solutions to each problem as separate PDF files and upload as instructed before.

Solve **six** problems of your choice. Do not turn in more than this number (6) of problems! If you submit more than 6 problems, only the lowest 6 scores will count towards your total score.

The total time allowed **3 hrs** till 12:00 p.m.

Problem I-1. (Classical Mechanics)



A point mass m is attached to the end of a rigid rod of length ℓ and negligible mass. The rod is pivoted at the coordinate origin (i.e. attached and free to rotate in all directions). Suppose this system is described using spherical coordinates as shown. The zero of gravitational potential energy is chosen at $z = 0$. The mass's velocity is

$$\boldsymbol{v} = v_\theta \,\widehat{\boldsymbol{\theta}} + v_\varphi \,\widehat{\boldsymbol{\varphi}}\,, \qquad v_\theta = \ell\,\dot{\theta}\,, \qquad v_\varphi = (\ell \sin\theta)\,\dot{\varphi}\,.$$

(i) What is the Lagrangian for this particle?

(ii) What are the conjugate momenta p_θ and p_φ in terms of coordinates and velocities?

(iii) Write down the Hamiltonian and the four Hamilton equations.

(iv) Your equations should indicate that p_φ is a conserved quantity. Set $p_\varphi = P = \text{const.}$ Show that a possible motion of the system is

$$\theta(t) = \theta_0 = \text{const}\,, \qquad \varphi(t) = \varphi_0 + \frac{P}{m\ell^2 \sin^2\theta_0}\,, \qquad p_\theta(t) = 0\,,$$

and find the equation that determines θ_0 in terms of P, m, ℓ and g.

Problem I-2. (Statistical Physics)

One of the more surprising aspects of polymers is that the polymer elasticity can follow Hooke's law. Consider a 3D polymer chain of N subunits of length a. Assume that the internal energy of the polymer is independent of the polymer's conformation and that, for a given end-to-end distance r, the number of polymer microstates Ω satisfies

$$\Omega \propto \exp\left(-\frac{3r^2}{2Na^2}\right)\,.$$

Show that the tension τ, with $\tau > 0$ when the polymer is pulling towards decreasing r, satisfies Hooke's law, $\tau = kr$, and find k.

Problem I-3. (Electricity and Magnetism)

Consider two concentric, spherical shells, with radii R_1 and $R_2 > R_1$. They are insulators, are rigid, and fixed in place, with equal and opposite charge uniformly distributed over their surfaces. The net charge on the outer sphere is q.

(i) What is the electric field everywhere in space?

(ii) A uniform magnetic field $\vec{B}(t) = B(t)\hat{z}$ is slowly turned on, beginning at time $t = 0$. It is fully turned on, and no longer changing, by time T. What is the electric field at any time t during the interval $0 < t < T$, as determined by Faraday's Law?

(iii) What is the net torque, relative to the origin, exerted on the two shells by the electric field at any time t during the interval $0 < t < T$?

Problem I-4. (Quantum Mechanics)

Consider a two level system (aka qubit) whose Hamiltonian has eigenvalues E_0 and E_1 (say $E_1 > E_0$) with eigenvectors $|0\rangle$ and $|1\rangle$, respectively.

(i) Write the most general quantum state $|\psi\rangle$ that, upon energy measurement, gives outcomes E_0 and E_1 with equal probability.

(ii) For these states, what is the maximum possible value of the expectation value of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ?$$

What is the minimum?

(iii) Suppose $|\psi(0)\rangle$ is a state of type (i) for which the expectation value of σ_x is minimum. How long does it take to for the time-evolved state to achieve the maximum?

(iv) Suppose now you have started from the maximum value of the expectation value of σ_x. How long does it take to achieve the minimum?

Problem I-5. (Math Methods)

An $n \times n$ complex matrix $\boldsymbol{T}$ is called normal when it commutes with its hermitian conjugate,

$$[\boldsymbol{T}, \boldsymbol{T}^\dagger] = 0\,,$$

with respect to the standard scalar product in $\mathbb{C}^n$,

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{j=1}^{n} \overline{u}_j v_j\,.$$

(i) Show that any hermitian matrix is normal and any unitary matrix is normal. Give an example of a normal matrix that is neither hermitian nor unitary.

(ii) Show the following properties of any normal matrix:

(a) $||\boldsymbol{T}\boldsymbol{v}|| = ||\boldsymbol{T}^\dagger \boldsymbol{v}||$ for any $\boldsymbol{v} \in \mathbb{C}^n$.
Hint: $||\boldsymbol{v}||^2 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle$.

(b) An eigenvector of $\boldsymbol{T}$ is also an eigenvector of $\boldsymbol{T}^\dagger$ with the complex conjugate eigenvalue.
Hint: $\boldsymbol{T}\boldsymbol{v} = \lambda\, \boldsymbol{v}$ if and only if $||(\boldsymbol{T} - \lambda\, \mathbf{1})\, \boldsymbol{v}|| = 0$.

(c) Eigenvectors of $\boldsymbol{T}$ corresponding to distinct eigenvalues are orthogonal.

(iii) Using (a)-(c) argue that $\boldsymbol{T}$ is unitary diagonalizable.

Problem I-6. (Thermodynamics)

A system of n particles is described by the thermodynamic potential

$$U(S,V) = \frac{aS^3}{nV}\,,$$

where a is a constant and S, U, and V denote entropy, internal energy and volume.

(i) Compute pressure, p, as a function of T, V and n.

(ii) Compute $C_p - C_V$ as a function of T, V and n.

Hint: You may use that

$$C_p - C_V = T\Big(\frac{\partial V}{\partial T}\Big)_{p,\,n}\Big(\frac{\partial P}{\partial T}\Big)_{V,\,n}\,.$$

Problem I-7. (Condensed Matter)

Consider a 1D chain of atoms of two species, A and B, with all atoms separated by the same distance a (so the spacing between unit cells is $2a$). Model the atoms as point masses of mass $m_A \neq m_B$, connected by identical springs of spring constant k. Assume that atoms only move longitudinally (along the axis of the chain).

Show that there are two normal modes of oscillation by starting to derive the general dispersion relation $\omega(k)$ in the limit where $ka \ll 1$. You don't have to solve it, just show that it has two branches.

Hint: Write the equations of motion, then look for solutions of the form

$$x_{A,s} = u\,e^{iska}e^{-i\omega t}\,, \qquad x_{B,s} = u\,e^{iska}e^{-i\omega t}\,,$$

where s is the index of the site. Wait until you have a single equation for ω and k before taking any approximations.

Problem I-8. (Solid State)

You have a two-dimensional lattice with the primitive vectors

$$\mathbf{a}_1 = a\,(1,0) \qquad \mathbf{a}_2 = a\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\,.$$

The primitive vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of the two dimensional reciprocal lattice are

$$\mathbf{b}_1 = b\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)\,, \quad \mathbf{b}_2 = b\,(0,1)\,, \qquad \text{where} \qquad b = \frac{4\pi}{\sqrt{3}a}\,.$$

(i) Sketch the reciprocal lattice.

(ii) Sketch the first Brillouin zone (BZ). Denote the corners of the BZ with K_1, $\mathrm{K}_2, \ldots$

(iii) Calculate the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

Hint: In three dimensions, one has $\mathbf{b}_1 = 2\pi\dfrac{\mathbf{a}_2\times\mathbf{a}_3}{(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)}$, etc.

Problem I-9. (Relativity)

As seen from a Lab frame, (t, x, y), an observer, $\mathcal{O}$, starts from the origin $(x = 0, y = 0)$ at time $t = 0$ with constant speed v, moving along the positive x-axis. At $t = 0$ a particle of mass m is moving parallel to the y-axis, at speed u in the negative y-direction. It is on a collision course with $\mathcal{O}$ and will hit $\mathcal{O}$ at time t_c in the Lab frame. This means the particle is observed at an "event" $(t\,, x\,, y) = (0\,, v\,t_c\,, ut_c)$ in the Lab frame.

Comment: A major part of this problem is to compute the time the particle takes to hit $\mathcal{O}$, as measured by $\mathcal{O}$. The obvious guess of $t_c\sqrt{1-v^2}$ is wrong and will get zero points. The components of this question guide you to the right answer.

(i) What is the proper 4-velocity of the particle in the Lab frame?

(ii) What is the Lorentz transformation that takes the Lab frame to the frame of the observer, $\mathcal{O}$.

(iii) Transform the 4-vector position of the particle at $t = 0$ and $t = t_c$ to $\mathcal{O}$'s frame. What is the displacement in space and time of the particle as seen by $\mathcal{O}$ between $t = 0$ and $t = t_c$. What is the time elapsed as measured by $\mathcal{O}$ between the initial position of the particle at $t = 0$ and the collision with $\mathcal{O}$.

(iv) What is the proper 4-velocity of the particle in $\mathcal{O}$'s frame? You should check to see that this is parallel to your answer in part (iii).

(v) What is the energy of the particle as measured by $\mathcal{O}$. Expand this for small u, v and confirm that it matches the Newtonian result.

Problem I-10. (Astrophysics/Cosmology)

Calculate the size of the particle horizon at matter-radiation equality. Assume that the universe is flat, and its content is that of the Benchmark Model. ΛCDM values of density parameters are as follows: $\Omega_{m,0} = 0.31$ and $\Omega_{r,0} = 9 \times 10^{-5}$.

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Graduate Screening Examination
Part II
Saturday, August 28, 2021

The exam is closed book. Make sure that each page is signed with your code number (S-#) and the problem number. Do not write answers to different problems on the same page. Scan solutions to each problem as separate PDF files and upload as instructed before.

Solve **three** problems of your choice. Do not turn in more than this number (3) of problems! If you submit more than 3 problems, only the lowest 3 scores will count towards your total score.

The total time allowed **3 hrs**.

Problem II-1. (Classical Mechanics)

A particle of mass m and charge q moves with velocity $\boldsymbol{v}$ in a uniform, static magnetic field, $\boldsymbol{B} = B\,\widehat{\boldsymbol{z}}$, and hence is acted on by the Lorentz force

$$\boldsymbol{F} = \frac{q}{c}\,\boldsymbol{v} \times \boldsymbol{B}\,, \tag{1.1}$$

in appropriate units.

(i) Show that the Newton's equations of motion of the particle follow from the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x})\,. \tag{1.2}$$

(ii) Argue that spatial translations in any direction are symmetries of this mechanical system and use the Noether theorem to find the corresponding constant of motion.

(iii) Similarly, show that the rotations about the z-axis are continuous symmetries and find the corresponding constant of motion.

(iv) Find the infinitesimal canonical transformations in the particle's phase space generated by the constants of motion found in (ii).

Problem II-2. (Statistical Physics)

A container with volume V is filled with a fluid at temperature T composed of identical particles with short-range repulsive interactions. The particles can enter and leave the container, at (fixed) chemical potential μ. We use a lattice model to describe this system, in which we divide the container into N cells, each with a volume $v = V/N$ comparable to the particle volume. We assume that unoccupied cells and cells occupied by a single particle have zero energy, that a cell occupied by two particles has energy $\epsilon > 0$, and that a cell cannot contain more than two particles. We denote the number of particles in the container by M.

(i) Construct the canonical partition function of the system, $Z(T, V, M)$, in terms of summations and explain the form of the summations and each term in $Z(T, V, M)$. Hence, show that the grand partition function of this system, $\mathcal{Q}(T, V, \mu)$, can be written as

$$\mathcal{Q}(T, V, \mu) = \left(1 + z + e^{-\beta\epsilon} z^2\right)^N \,, \tag{2.1}$$

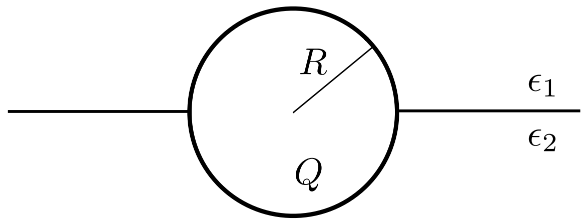
where $\beta = 1/k_B T$ and the fugacity $z = e^{\beta\mu}$.

Hint: $\displaystyle\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^n = (1 + x)^N$.

(ii) Use Eq. (2.1) to find, at thermodynamic equilibrium, the average number of particles per cell, c, and the pressure P in terms of T and μ.

(iii) Using your results from part (b), find P in terms of T and c in the limit of small c. Provide a physical interpretation of your result.

Problem II-3. (Electricity and Magnetism)



A conducting sphere of radius R and charge Q sits at the origin of the xyz-coordinate system. The space outside the sphere is filled by two different linear homogeneous dielectrics: above the xy-plane, the dielectric constant is ϵ_1, below the xy-plane the dielectric constant is ϵ_2, see the figure.

(i) Find the potential and the electric field everywhere outside the conductor, using the solution of Laplace's equation in spherical coordinates and azimuthal symmetry. Exploiting the fact that a conductor is an equipotential object, you should find that only the $\ell = 0$ term contributes to the series of Legendre polynomials of order ℓ of argument $\cos\theta$, where θ is the polar angle in spherical coordinates. To determine the coefficient of the $\ell = 0$ term, use Gauss's law for the electric displacement $\boldsymbol{D}$.

(ii) Find the distribution of free charge, using Gauss's law for the electric displacement.

(iii) Find the distribution of polarization charge at the surface of the conductor. Pay attention to the direction of the unit vector $\widehat{\boldsymbol{n}}$ in the definition of surface polarization charge, which should go outward from the polarized region.

(iv) Explain why there is no polarization charge on the xy-plane, at the interface between the two dielectric materials.

Problem II - 4. (Quantum Mechanics)

A particle of mass m is confined to stay between two concentric spheres of radii a and b ($a < b$), respectively, but otherwise can move freely within this 3-dimensional volume. Then the potential energy $V(r)$ is zero in the space between the spheres and is infinite elsewhere.

(i) *Briefly* describe (without proof) the steps you would take to reduce the 3-dimensional Schrödinger equation for the particle to an effective one-dimensional equation along the radial direction. Give the radial equation for any angular momentum ℓ in all regions of space (a proof is not required, just state the result). Sketch a plot of the effective potential.

(ii) State what the available energy range is, and describe the classical motion of the particle in words in this energy range. From this, determine what the boundary conditions should be for the radial wavefunction for any angular momentum ℓ.

(iii) Using the uncertainty principle, give a rough estimate of the radial momentum, and then estimate the minimum energy for any ℓ.

(iv) Solve the radial Schrödinger equation for the case of zero angular momentum, $\ell = 0$. Discuss how you select the right solution from the two you should have found.

(v) Using the solution above, compute explicitly the quantized energy eigenvalues, E_n, for all excited states with $\ell = 0$. Give E_n as a function of m, a, b and a quantum number n, and list the possible values of n. How does the ground state energy compare to the estimate in part (iii)?

Hint: If you cannot complete some parts due to lack of time, try to outline the remaining steps to receive some partial credit.