Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 11, 2010
1:00 PM - 3:00 PM

Classical Mechanics
Section 1.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2; Section 1 (Classical Mechanics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

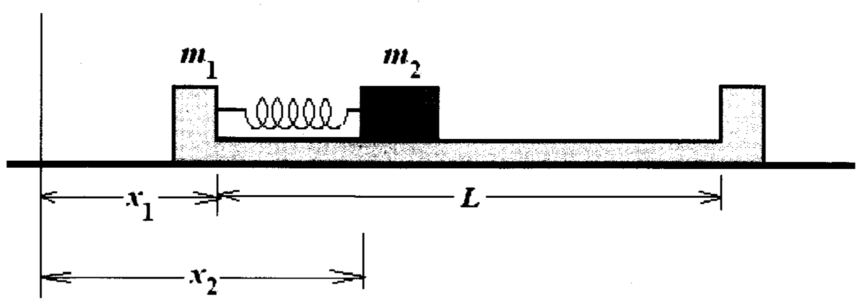
You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Classical Mechanics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A block of mass m_2 slides inside a cavity of length L inside a second block of mass m_1 which rests on a horizontal table. The masses m_1 and m_2 are connected by a massless spring with spring constant k and equilibrium length $l \ll L$. Initially both blocks are at rest and located at $x_1 = 0$ and $x_2 = l - \Delta l$ where Δl specifies the initial compression of the spring.



(a) If the mass m_1 slides without friction on the table and m_2 slides without friction on the second block, find $x_1(t)$ and $x_2(t)$ as a function of time.

(b) If the mass m_1 exerts a frictional force on m_2 proportional to their relative velocity, $F_{1 \text{ on } 2} = -\sigma(\dot{x}_2 - \dot{x}_1)$, again determine the resulting motion of the two masses.

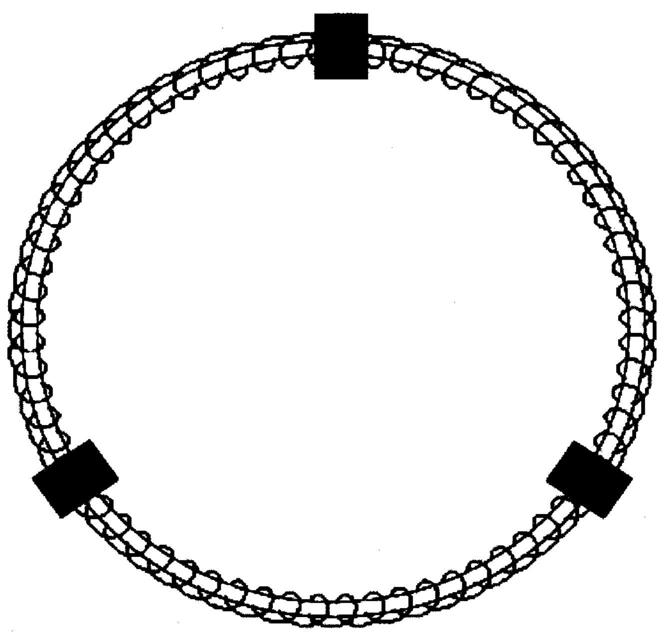
(c) If m_2 slides on m_1 without friction but m_1 experiences a similar frictional force from the table, $F_1 = -\sigma \dot{x}_1$, find the resulting complex frequencies to first order in σ assuming σ to be small. What do those frequencies imply about the resulting motion?

2. Consider the general problem of N beads of mass m, that slide frictionlessly around a fixed horizontal hoop. The beads are attached to, and spaced by, identical massless springs whose natural length is much smaller than their equilibrium length. For any given N, the spring constant is chosen such that in equilibrium the springs are under tension T. Answer the following:

   (a) Suppose $N = 2$. For $t < 0$ bead #1 is held fixed at a reference position, $\theta = 0$, and bead #2 is held fixed at $\theta = \pi + \Delta$ where $\Delta \ll \pi$. At $t = 0$ the beads are released. Find the subsequent motion of the two beads, i.e. $\theta_1(t)$ and $\theta_2(t)$.

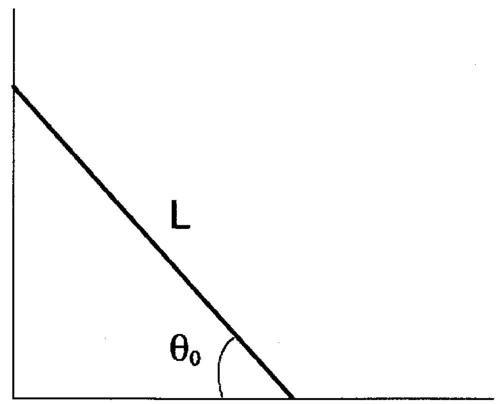
   (b) Suppose N is very large. The mass density of the beads on the hoop is μ. Estimate the two lowest frequencies for the normal modes of the system.

   (c) Suppose $N = 3$. Find the frequencies and corresponding eigenvectors of the normal modes of the system.



$N = 3$

3. A uniform ladder of mass M and length L is placed with one end against a frictionless wall and the other end on a frictionless floor. The ladder initially makes an angle θ_0 with the floor, as shown below.



The ladder is released, and slides under the influence of gravity.

(a) Write the Lagrangian for the sliding ladder as a function of θ (the angle of the ladder with respect to the floor).

(b) At what angle θ does the ladder lose contact with the wall?

(Note: The moment of inertia of a uniform rod of mass M and length L rotating about an axis through its center of mass is $I = \frac{1}{12}ML^2$)

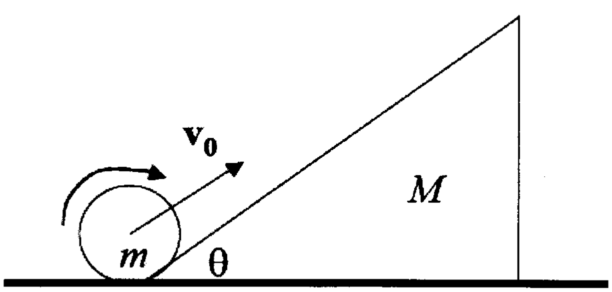
4. A cylinder of radius R and mass m rolls up an inclined plane of angle θ without slipping. The inclined plane has mass M and is free to slide along the horizontal surface without friction.

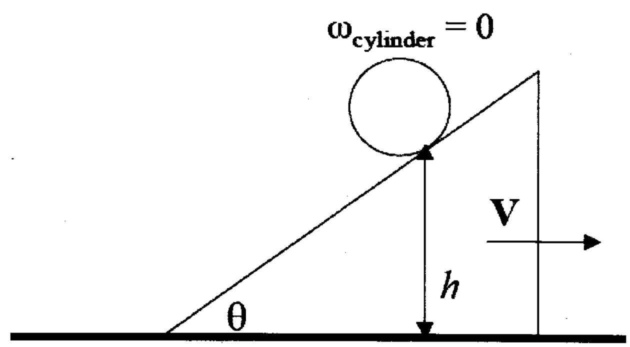
The cylinder has an initial velocity $\vec{v}_0$ up the inclined plane, and the inclined plane is initially at rest with respect to the horizontal surface.

(a) How high does the cylinder rise before it stops rotating and then starts to roll back down the inclined plane (h in the diagram)?

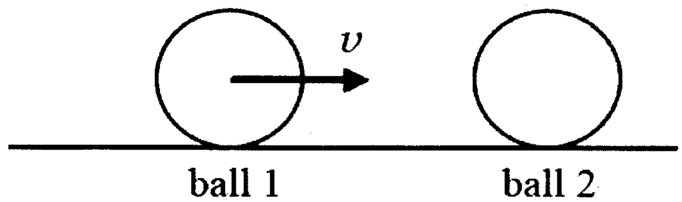
(b) At this point, what is the horizontal velocity of the cylinder and inclined plane ($\vec{V}$ in the diagram)?

(Give your answers in terms of I, R, m, M, θ, g, and v_0.)





5. Consider two identical billiard balls (spheres), each of mass M and radius R. One is stationary (ball 2) and the other rolls on a horizontal surface without slipping, with a horizontal speed v (ball 1), as shown.



Assume that all the frictional forces are small enough so as to be negligible over the time of the collision, and that the collision is completely elastic.

(a) Calculate the moment of inertia of one of the billiard balls about its center.

(b) What is the final velocity of each ball a long time after the collision? *i.e.* when each ball is rolling without slipping once more.

(c) What fraction of the initial energy is transformed into heat?

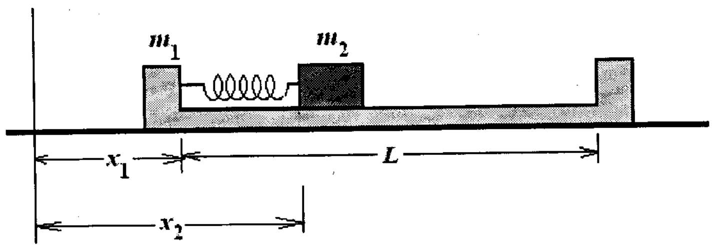
Christ
Sec. 1 MECH
1

N. Christ

November 30, 2008

Quals Problems

1. A block of mass m_2 slides inside a cavity of length L in a second block of mass m_1 which rests on a horizontal table. The masses m_1 and m_2 are connected by a massless spring with spring constant k and equilibrium length $l \ll L$. Initially both blocks are at rest and located at $x_1 = 0$ and $x_2 = l - \Delta l$ where Δl specifies the initial compression of the spring.



(a) If the mass m_1 slides without friction on the table and m_2 slides without friction on the second block find $x_1(t)$ and $x_2(t)$ as a function of time.

(b) If the m_1 exerts a frictional force on m_2 proportional to their relative velocity, $F_{1 \text{ on } 2} = -\sigma(\dot{x}_2 - \dot{x}_1)$, again determine the resulting motion.

(c) If m_2 slides on m_1 without friction but m_1 experiences a similar frictional force from the table, $F_1 = -\sigma\dot{x}_1$, find the resulting complex frequencies to first order in σ assuming σ to be small. What do those frequencies imply about the resulting motion?

Suggested Solution

1. (a) Start with equations of x_1 and x_2:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l) \end{aligned}$$

The sum of these equations describe the free particle motion of the center of mass variable $x_{\text{cm}} = (m_1x_1 + m_2x_2)/(m_1 + m_2)$:

$$\ddot{x}_{\text{cm}} = 0$$

If the first equation is multiplied by m_2 and subtracted from the second multiplied by m_1, we find a simple harmonic equation for the variable $y = x_2 - x_1 - l$:

$$m_1m_2\ddot{y} = -(m_1 + m_2)ky \tag{1}$$

Thus, if we define $\omega_0 = \sqrt{k/\mu}$ with $\mu = m_1m_2/(m_1 + m_2)$ we have the solution:

$$\begin{aligned} y(t) &= -\Delta l\cos(\omega_0 t) \\ x_{\text{cm}} &= \frac{m_2(l - \Delta l)}{m_1 + m_2} \end{aligned}$$

(b) The extra friction force does not change the structure of the equations:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) - \sigma(\dot{x}_1 - \dot{x}_2) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1 - l) - \sigma(\dot{x}_2 - \dot{x}_1) \end{aligned}$$

so they are solved the same variables x_{cm} and y:

$$\begin{aligned} y(t) &= e^{-\gamma t/2}\Delta l\left\{-\cos(\omega t) + \frac{\gamma}{2\omega}\sin(\omega t)\right\} \\ x_{\text{cm}} &= \frac{m_2(l - \Delta l)}{m_1 + m_2}, \end{aligned}$$

where $\gamma = \sigma/\mu$ and $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$.

(c) The equations become less familiar if friction is introduced between the table and m_1:

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) - \sigma\dot{x}_1 \\
m_2\ddot{x}_2 &= -k(x_2 - x_1 - l).
\end{aligned}
$$

Now the center of mass motion will couple with the oscillating variables and the four frequencies present in this system of two coupled second order equations can be found by solving:

$$
\begin{aligned}
0 &= \det\begin{pmatrix} -m_1\omega^2 + k + i\sigma\omega & -k \\ -k & -m_2\omega^2 + k \end{pmatrix} \\
&= m_1 m_2 \omega^4 - (m_1 + m_2)k\omega^2 + i\sigma\omega(k - m_2\omega^2).
\end{aligned}
$$

If $\sigma = 0$, these have the double root $\omega = 0$ and the two roots $\omega = \pm\omega_0$ corresponding to the $x_{\mathrm{cm}}(0) + \dot{x}_{\mathrm{cm}}(0)t$ cm mass and oscillatory motion above. These zeroth-order results can then be substituted in the above equation to find the frequencies to first order in σ:

$$
\begin{aligned}
\omega &= \pm\omega_0 + i\sigma\frac{m_2}{m_1(m_1 + m_2)} \\
\omega &= 0, \quad \omega = +i\frac{\sigma}{m_1 + m_2}.
\end{aligned}
$$

The $\omega = 0$ root corresponds to equilibrium with an arbitrary cm location, while $i\sigma/(m_1{+}m_2)$ describes non-oscillatory behavior with non-zero cm velocity, decreasing exponentially to zero. Finally $\pm\omega_0 + i\sigma m_2/(m_1[m_1 + m_2])$ corresponds to oscillatory motion damped by the motion of m_1.

Cole Section 1
mechanics
Problm #2

N Beads on a hoop – solutions

The displacement of bead with index i from its equilibrium position will be written ξ_i. Since the net force on a bead is zero with all of the beads at their equilibrium positions we can write the equations of motion purely in terms of the displacements ξ.

a. The spring constant k can be expressed in terms of the tension using $T = 2\pi Rk$ or $k = T/2\pi R$. The equations of motion for the two beads can be written

$$\begin{aligned} mR\,\ddot{\xi}_1 &= kR\,(\xi_2 - \xi_1) - kR\,(\xi_1 - \xi_2) = 2kR\,(\xi_2 - \xi_1) \\ mR\,\ddot{\xi}_2 &= kR\,(\xi_1 - \xi_2) - kR\,(\xi_2 - \xi_1) = 2kR\,(\xi_1 - \xi_2) \end{aligned}$$

If we add and subtract the two equations of motion we obtain,

$$\begin{aligned} mR\left(\ddot{\xi}_1 + \ddot{\xi}_2\right) &= 0. \\ mR\left(\ddot{\xi}_2 - \ddot{\xi}_1\right) &= 4kR(\xi_2 - \xi_1) \end{aligned}$$

If we define $\xi_s = \xi_1 + \xi_2$ and $\xi_d = \xi_2 - \xi_1$ and simplify we obtain the two equations

$$\begin{aligned} \ddot{\xi}_2 &= 0. \\ \ddot{\xi}_d &= 4\left(\frac{k}{m}\right)\xi_d \end{aligned}$$

The first equation which describes the first normal mode of the system and has the solution $\xi_s = \xi_0 + \omega_s\,t$, describes the simultaneous motion of the beads around the circle at constant separation. The second equation which describes the second n normal mode corresponds to simple harmonic oscillation of the ξ_d coordinate with frequency $\omega_d = 2\sqrt{k/m}$. We can write the general solution of that equation, $\xi_d = A\cos\omega_d\,t + B\sin\omega_d\,t$. We can obtain the constants ξ_0, ω_s, A, and B from the initial conditions. we have

$$\begin{aligned} \xi_s(t=0) &= \xi_0 = \xi_1(t=0) + \xi_2(t=0) = \Delta/2 \\ \xi_d(t=0) &= A = \xi_2(t=0) - \xi_1(t=0) = \Delta/2 \\ \dot{\xi}_s(t=0) &= \omega_s = \dot{\xi}_1(t=0) + \dot{\xi}_2(t=0) = 0 \\ \dot{\xi}_d(t=0) &= B\omega_d = \dot{\xi}_2(t=0) - \dot{\xi}_1(t=0) = 0 \end{aligned}$$

Or more succinctly, $\xi_0 = \Delta$, $A = \Delta$, $\omega_s = 0$, $B = 0$. Now we express ξ_1 and ξ_2 in terms of ξ_s and ξ_d,

$$\xi_1 = \frac{1}{2}\left(\xi_s - \xi_d\right), \xi_2 = \frac{1}{2}\left(\xi_s + \xi_d\right)$$

with the results for $\xi_1(t)$ and $\xi_2(t)$,

$$\begin{aligned} \xi_1(t) &= \frac{\Delta}{2}\left[1 - \cos\left(\omega_d\,t\right)\right] \\ \xi_2(t) &= \frac{\Delta}{2}\left[1 + \cos\left(\omega_d\,t\right)\right] \end{aligned}$$

b. In the large-N limit the system can be thought of as effectively continuous with a wave equation for the angle-dependent displacement from equilibrium, $\xi(\theta, t)$,

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{v^2}{R^2}\frac{\partial^2 \xi}{\partial \theta^2}$$

The phase velocity can be found purely through dimensional analysis – the only combination of constants in the problem that have the correct dimensions for v^2 is $v^2 = T/\mu$ (similar to waves on a string). We can write the general standing wave solution

$$\xi(\theta, t) = A \sin(kR\theta - \alpha)\cos(\omega t - \phi)$$

where α and ϕ are spatial and temporal phase angles respectively. As usual $\omega/k = v$. Now, waves that propagate on the hoop must satisfy the periodicity condition $\xi(\theta + 2\pi, t) = \xi(\theta, t)$. Thus, we are restricted to solutions where $2\pi kR = n2\pi$ or $kR = n$ where n is an integer. Thus yields values for k, $k = 1/R, 2/R, \ldots$. However, as with the $N = 2$ case in part a, there is a solution corresponding to no oscillation where the beads simply move around the loop at constant angular velocity. The solution corresponds to $\omega = 0$. So, strictly speaking the two lowest frequencies of motion of the systemn have $\omega = 0$ and $\omega = v/R$.

c. The equations of motion take the form

$$\ddot{\xi}_i = -\frac{k}{m}\left(2\xi_i - \xi_{i-1} - \xi_{i+1}\xi\right) \equiv \omega_0^2\left(2\xi_i - \xi_{i-1} - \xi_{i+1}\xi\right)$$

with i cyclic: $i = 0 \rightarrow i = 3$and $i = 4 \rightarrow i = 1$. hewre we have defined with $\omega_0 \equiv \sqrt{k/m}$. If we assume the existence of normal mode solutions to the motion of the form
$U = A\cos(\omega\, t - \phi)$ with $\xi_i = C_i U$ and substitute into the equations of motion we obtain an eigenvalue equation

$$\begin{bmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2 & \omega^2 - 2\omega_0^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \omega_0^2 \begin{bmatrix} r^2 - 2 & 1 & 1 \\ 1 & r^2 - 2 & r^2 \\ 1 & 1 & r^2 - 2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0$$

$r = \omega/\omega_0$. Applying the usual requirement on the determinant (zero)

$$Det\begin{bmatrix} r^2 - 2 & 1 & 1 \\ 1 & r^2 - 2 & 1 \\ 1 & 1 & r^2 - 2 \end{bmatrix} = 0$$

we obtain the characteristic equation

$$\left(r^2 - 2\right)\left(\left(r^2 - 2\right)^2 - 1\right) - \left(r^2 - 2 - 1\right) + \left(1 - \left(r^2 - 2\right)\right) = 0$$

Simplifying, we can write the characteristics equation

$$\left(r^2 - 2\right)^3 - 3\left(r^2 - 2\right) + 2 = 0$$

Expanding out all the terms and cancelling where appropriate we obtain

$$r^6 - 6r^4 + 9r^2 = r^2\left(r^2 - 3\right)^2 = 0$$

with the solutions $r^2 = 0$ and (degenerate) $r^2 = 3$ (taking only the positive root for solutions to normal mode motion. The $r^2 = 0$ solution corresponds to no oscillation. The resulting equation(s) for the (unnormalized) eigenvector taking $C_1 = 1$ are

$$C_2 + C_3 = 2, -2C_2 + C_3 = -1$$

which give as solutions, $C_2 = 1$ and $C_3 = 1$ for a normalized eigenvector,

$$C = \sqrt{\frac{1}{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Clearly this solution corresponds to the simultaneous motion of the beads around the hoop. Now consider the degenerate solution $r^2 = 3$ which means $\omega = \sqrt{3}\omega_0$. We obtain a redundant equation for the eigenvectors, $C_1 + C_2 + C_3 = 0$. The redundancy (due to the degeneracy which, in turn results from the symmetry of the problem under cyclic permutation of the indices) means that we have freedom in choosing the remaining two eigenvectors as long as they are orthogonal. One valid choice based on intuition about how normal modes work is to have one bead fixed and the others to oscillate with opposite phase, namely

$$C = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

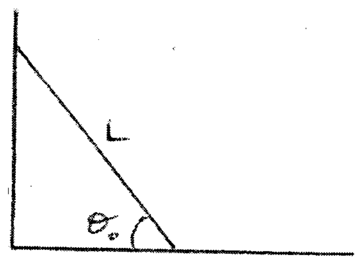
Giving a final eigenvector

$$C = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Dodd
Sec. 1 MECH
3

2010 Quals Question: Mechanics (Dodd)

A uniform ladder of mass M and length L is placed with one end against a frictionless wall and the other end on a frictionless floor. The ladder initially makes an angle θ_0 with the floor, as shown below.



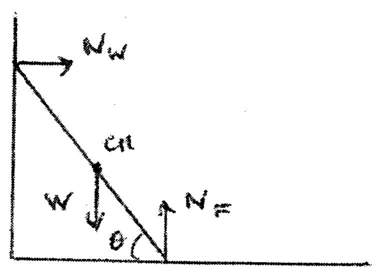
The ladder is released, and slides, under the influence of gravity.

a). Write the Lagrangian for the sliding ladder as a function of θ (the angle of the ladder with respect to the floor).

b). At what angle θ does the ladder lose contact with the wall?

(The moment of inertia of a uniform rod of mass M and length L rotating about an axis through its center of mass is $I = \frac{1}{12}ML^2$.)

Solution:



a). Denoting the center of mass coordinates of the ladder by (x_{CM}, y_{CM}), then the Lagrangian is:

$$L = T - V$$

where:

$$T = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_{CM}\dot{\theta}^2$$

with $I_{CM} = \frac{1}{12}ML^2$, and $(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) = \left(\frac{L}{2}\right)^2 \dot{\theta}^2$, so:

$$T = \frac{1}{2}M\left(\frac{L}{2}\right)^2 \dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\dot{\theta}^2 = \frac{1}{6}ML^2\dot{\theta}^2$$

and:

$$V = Mg\left(\frac{L}{2}\right)sin\theta$$

so the Lagrangian can be written:

$$\underline{L = \frac{1}{6}ML^2\dot{\theta}^2 - \frac{1}{2}MgLsin\theta}$$

b). The equation of motion, via the Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

is:

$$\frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}MgLcos\theta = 0$$

i.e.

$$\frac{1}{3}L\ddot{\theta} + \frac{1}{2}gcos\theta = 0$$

As the ladder slides, energy E is conserved, where:

$$E = T + V = \frac{1}{6}ML^2\dot{\theta}^2 + \frac{1}{2}MgLsin\theta$$

and we know the total energy from the initial condition (with no velocity), viz.:

$$E_0 = \frac{1}{2}MgLsin\theta_0$$

giving:

$$\frac{1}{6}ML^2\dot{\theta}^2 = \frac{1}{2}MgL(sin\theta_0 - sin\theta)$$

i.e.

$$\frac{1}{3}L\dot{\theta}^2 = g(sin\theta_0 - sin\theta)$$

Writing the center of mass coordinates in terms of L and θ: $x_{CM} = \left(\frac{L}{2}\right)cos\theta$, and $y_{CM} = \left(\frac{L}{2}\right)sin\theta$, and looking at horizontal forces, gives:

$$N_W = M\ddot{x}_{CM} = M\left(\frac{L}{2}\right)\left(-\dot{\theta}^2 cos\theta - \ddot{\theta} sin\theta\right)$$

At the point at which the ladder breaks contact with the wall, $N_W = 0$, and so:

$$-\dot{\theta}^2 cos\theta - \ddot{\theta} sin\theta = 0$$

i.e.

$$\ddot{\theta} = -\dot{\theta}^2 cot\theta$$

Substituting in the equation of motion gives:

$$\frac{1}{3}L\left(-\dot{\theta}^2 cot\theta\right) + \frac{1}{2}gcos\theta = 0$$

i.e.

$$\frac{1}{3}L\dot{\theta}^2 = \frac{1}{2}gsin\theta$$

Lastly, substitute into the energy conservation equation, to give:

$$\frac{1}{2}gsin\theta = g(sin\theta_0 - sin\theta)$$

i.e.

$$sin\theta = \frac{2}{3}sin\theta_0$$

and:

$$\underline{\theta = sin^{-1}\left(\frac{2}{3}sin\theta_0\right)}$$

Mech. Shaevitz
sec. 1 MECH

Quals Problem 1 – Mechanics

M. Shaevitz
Fall, 2009

\#4

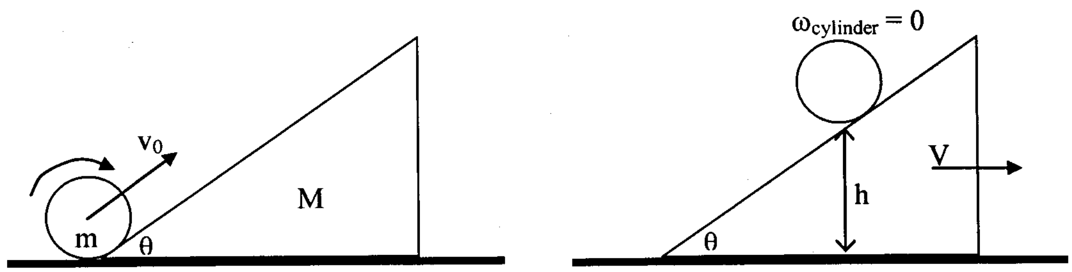
A cylinder of radius **R** and mass **m** rolls up an inclined plane of angle **θ** without slipping. The inclined plane has mass **M** and is free to slide along the horizontal surface without friction.

The cylinder has an initial velocity up the incline of $\mathbf{v_0}$ and the inclined plane is initially not moving with respect to the horizontal surface.

a) How high does the cylinder rise before it stops rotating and then starts to roll back down the inclined plane (**h** in the diagram)?

b) At this point, what is the horizontal velocity of the cylinder and inclined plane (**V** in the diagram)?

(Give your answers in terms of R, m, M, θ, g, and v_0.)



Freshman Physics Solution:

Since No external forces in the x-direction, the horizontal momentum is conserved. (The friction between cylinder and ramp is internal.)

Cons Mom: $m\dot{x}_0 \cos\theta = (m+M)\dot{X}_{TOP}$

Cons. Energy: $\frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}\left(\frac{I}{R^2}\right)\dot{x}_0^2 = \frac{1}{2}(m+M)\dot{X}_{TOP}^2 + mgh$

$$\dot{X}_{TOP} = \frac{m\cos\theta}{(m+M)}\dot{x}_0$$

$$\left(m + \frac{I}{R^2}\right)\dot{x}_0^2 = (m+M)\frac{m^2\cos^2\theta}{(m+M)^2}\dot{x}_0^2 + 2mgh$$

a) $$h = \left(m + \frac{I}{R^2} - \frac{m^2\cos^2\theta}{(m+M)}\right)\frac{1}{(2mg)}\dot{x}_0^2$$

b) $$\dot{X}_{TOP} = \frac{m\cos\theta}{(m+M)}\dot{x}_0$$

Lagrangian Solution: Use x = distance up incline + X position of incline

$$v_{cylinder} = (\dot{x}\cos\theta + \dot{X},\ \dot{x}\sin\theta)$$

$$v_{plane} = \dot{X} \qquad \dot{\theta} = \dot{x}/R$$

$$T = \frac{1}{2}m\left[(\dot{x}\cos\theta + \dot{X})^2 + \dot{x}^2\sin^2\theta\right] + \frac{1}{2}M\dot{X}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{R}\right)^2$$

$$= \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 + \frac{1}{2}(m+M)\dot{X}^2 + m\dot{x}\dot{X}\cos\theta$$

$$U = mgx\sin\theta$$

$$\mathcal{L} = T - \mathcal{U} = \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 + \frac{1}{2}(m+M)\dot{X}^2 + m\dot{x}\dot{X}\cos\theta - mgx\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 = \left(m + \frac{I}{R^2}\right)\ddot{x} + m\ddot{X}\cos\theta + mg\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) - \frac{\partial \mathcal{L}}{\partial X} = 0 = (m+M)\ddot{X} + m\ddot{x}\cos\theta = 0$$

$$\ddot{X} = -\frac{\ddot{x}m\cos\theta}{m+M}$$

$$\left(m + \frac{I}{R^2}\right)\ddot{x} - \frac{m^2\cos^2\theta}{(m+M)}\ddot{x} = -mg\sin\theta$$

$$\ddot{x}\left(m + \frac{I}{R^2} - \left(\frac{m^2}{m+M}\right)\cos^2\theta\right) = -mg\sin\theta$$

Constant acceleration preamblem with

$$M_{eff} = \left(m + \frac{I}{R^2} - \left(\frac{m^2}{m+M}\right)\cos^2\theta\right) \Rightarrow a = \frac{-mg\sin\theta}{M_{eff}}$$

a) At top $\dot{x}_{TOP} = 0$, $x = \frac{h}{\sin\theta}$ Initial $x = 0$ $\dot{x} = \dot{x}_0$

$X = 0$ $\dot{X} = 0$

$$\dot{x}_{TOP}^2 = \dot{x}_0^2 + 2a x_{TOP} = \dot{x}_0^2 - \frac{2mg\sin\theta}{M_{eff}} x_{TOP}$$

$$\Rightarrow h = x_{TOP}\sin\theta = \frac{(M_{eff})\dot{x}_0^2}{2mg}$$ (same as before)

b) From $(m+M)\ddot{X} = -\ddot{x}m\cos\theta$

$$\int_0^{\dot{X}_{TOP}} (m+M)\, d\dot{X} = \int_{+\dot{x}_0}^{0} -m\cos\theta\, d\dot{x} \Rightarrow (m+M)\dot{X}_{TOP} = m\dot{x}_0\cos\theta$$

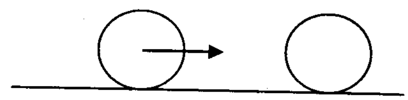
Again $\dot{X}_{TOP} = \left(\frac{m}{m+M}\right)\cos\theta\, \dot{x}_0$

Mechanics Tuts
Sec 1 MECH
#5

Quals – Mechanics Question – Tuts – 11/25/09

QUESTION

Consider two identical billiard balls (spheres), each of mass M and radius R. One is stationary (ball 2) and the other rolls on a horizontal surface without slipping with a horizontal speed v (ball 1), as shown. Assume that all the frictional forces are small enough so as to be negligible over the time of the collision, and that the collision is completely elastic.



A. Calculate the moment of inertia of one of the billiard balls about its center.
B. What is the final velocity of each ball a long time after the collision? *i.e.* when each ball is rolling without slipping once more.
C. What fraction of the initial energy is transformed into heat?

SOLUTION

Part A

$$I = \int r^2 dm$$

$$I = \int x^2 2\pi\rho x\, 2\sqrt{R^2 - x^2}\, dx$$

$$= 4\pi\rho \int x^3 \sqrt{R^2 - x^2} dx$$

$$= 4\pi\rho R^5 \int \left(\frac{x}{R}\right)^3 \sqrt{\left(1 - \left(\frac{x}{R}\right)^2\right)}\, dx$$

$$= 4\pi\rho R^5 \int \left(\frac{x}{R}\right)^2 \sqrt{1 - \left(\frac{x}{R}\right)^2} \frac{1}{2} d\left(\frac{x}{R}\right)^2$$

$$let\ y = \left(\frac{x}{R}\right)^2$$

$$= 2\pi\rho R^5 \int y \sqrt{1 - y} dy$$

$$integrate\ by\ parts\ u = y, dv = \sqrt{1 - y}\ hence\ du = dy\ and\ v = -\frac{2}{3}(1 - y)^{\frac{3}{2}}$$

$$I = 2\pi\rho R^5 [-\frac{2}{3}(1 - y)^{\frac{3}{2}} y - \int (-\frac{2}{3}(1 - y)^{\frac{3}{2}} dy)$$

$$= 2\pi\rho\backslash R^5 \left[-\frac{2}{3}\left(\frac{x}{R}\right)^2 \left(1 - \left(\frac{x}{R}\right)^2\right)^{\frac{3}{2}} - \frac{4}{15}\left(1 - \left(\frac{x}{R}\right)^2\right)^{\frac{5}{2}}\right] \text{from x=0 to R}$$

$$\therefore I = \left(\frac{2}{5}\right) MR^2$$

Part B

Just before the collision

$v_1 = v$

$v_2 = 0$

$\omega_1 = \frac{v_1}{R} = \frac{v}{R}$

$\omega_2 = 0$

Just after the collision (and since friction is negligible during collision and it is elastic)

$v_1' = 0$

$v_2' = v$

And the angular momenta about the center of each ball are conserved, hence

$\omega_1' = \omega_1 = \frac{v}{R}$

And $\omega_2' = \omega_2 = 0$

Now, if we look "a long time later" (where the balls are rolling without slipping), then for each ball we can use angular momentum conservation about a fixed point on the surface where the balls rolls then for ball 1

$$L_1 = MRv_1' + I\omega_1' = MRv_1'' + I\omega_1''$$

$$= v_1''\left(MR + \frac{I}{R}\right)$$

Or replacing in for L

$$I\omega_1' = \frac{Iv}{R} = v_1''\left(MR + \frac{I}{R}\right)$$

$$v_1'' = \frac{vI}{I + MR^2} = \frac{v\left(\frac{2}{5}\right)MR^2}{\left(\frac{7}{5}\right)MR^2}$$

$$v_1'' = \left(\frac{2}{7}\right)v$$

Similarly for ball 2 we arrive at

$$v_2'' = \left(\frac{5}{7}\right)v$$

Part C

$$KE_{initial} = \left(\frac{1}{2}\right)Mv_1^2 + \frac{1}{2}I\omega_1^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}\frac{2}{5}MR^2\left(\frac{v}{R}\right)^2 = \frac{1}{2}Mv^2\frac{7}{5}$$

$$KE_{final} = \frac{1}{2}\frac{7}{5}M\left(\left(\frac{2}{7}v\right)^2 + \left(\frac{5}{7}v\right)^2\right) = \frac{1}{2}\times\frac{7}{5}Mv^2\frac{29}{49}$$

So the energy lost to friction is

$$KE_{initial} - KE_{final} = \frac{1}{2}\times\frac{7}{5}\,Mv^2\frac{20}{49}$$

So the fraction converted to heat is

$$\frac{KE_{initial} - KE_{final}}{KE_{initial}} = \frac{20}{49}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 11, 2010
3:10 PM - 5:10 PM

Electromagnetism
Section 2.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electromagnetism), Question 2; Section 2 (Electromagnetism), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Electromagnetism. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Consider a rigid, ideally conducting sphere of radius R, with total charge equal to zero. The sphere rotates with angular velocity $\vec{\Omega}$; $\Omega R \ll c$. Suppose a dipole magnetic field threads the sphere. The dipole is centered on the center of the sphere. The dipole moment $\vec{\mu}$ is given; it is aligned with $\vec{\Omega}$.

   (a) What voltage is induced between the equator and the poles of the sphere?

   (b) Find the charge density $\rho(r, \theta)$ established inside the sphere. Here r and θ are spherical coordinates: r is the distance from the center and θ is the polar angle measured from the rotational axis.

   (c) Find the electric field outside the sphere.

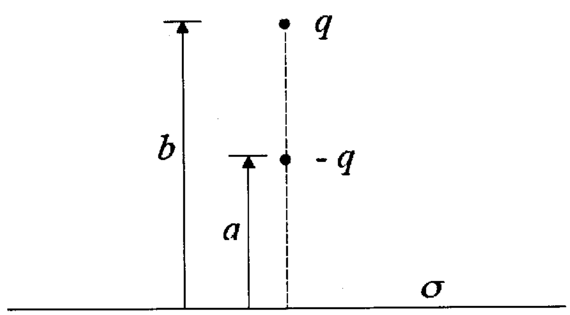
   Hint: any axisymmetric solution of $\nabla^2\Phi = 0$ that vanishes at infinity has the following form in spherical coordinates r, θ, ϕ

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos\theta), \qquad [P_0 = 1,\ P_1 = \cos\theta,\ P_2 = \frac{3\cos^2\theta - 1}{2},\ ...]$$

2. A positive point charge q is fixed 1 cm above a horizontal, grounded conducting x-y plane. An equal negative charge $-q$ can be moved along the perpendicular dropped from q to the plane.

   (a) Where should $-q$ be placed for the total force on it to be zero?

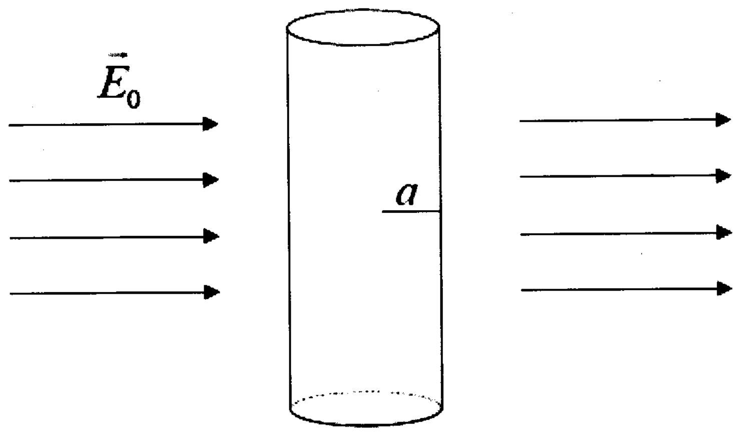
   (b) Taking the distance between q and the plane equal to b, and the distance from $-q$ to the plane equal to be a, what is the surface charge density, $\sigma(x, y)$, on the conductor? Express your answer in terms of a, b, q, x and y.



3. Consider an infinitely long, grounded conducting cylinder, of radius a, which is introduced into a uniform electric field $\vec{E}_0$. The axis of the cylinder is perpendicular to $\vec{E}_0$.

   (a) Find an expression for the external potential after insertion of the cylinder.

   (b) Find an expression for the surface charge induced on the cylinder.



4. A perpendicularly incident beam of right circularly polarized light is reflected by an ideal stationary mirror. Show that the reflected beam is left circularly polarized.

5. A magnetic monopole is a hypothetical particle that is a source for a Coulomb magnetic field

$$\vec{\mathbf{B}} = \frac{g\hat{\mathbf{r}}}{r^2}$$

(a) Consider a particle with mass m and electric charge q that is moving in the magnetic field of a static magnetic monopole. Show that the usual expression for angular momentum, $\vec{\mathbf{r}} \times (m\vec{\mathbf{v}})$ is not conserved, but that there is a conserved angular momentum of the form

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times (m\vec{\mathbf{v}}) + \vec{\mathbf{f}}.$$

Determine $\vec{\mathbf{f}}$.

(b) An electric charge q and a magnetic charge g with fixed positions are located a distance D apart. The combined fields of these charges have a nonzero angular momentum.

i. Show that the magnitude of this angular momentum does not depend on the distance D.

ii. Determine the magnitude and direction of the angular momentum.

The integral

$$\int_0^\infty dy \frac{y}{(y^2 - 2ay + 1)^{3/2}} = \frac{1}{1-a}$$

may be useful.

E+M Beloborodov
Sec. 2 E+M
#1

E&M:

Consider a rigid, ideally conducting sphere of radius R with total charge equal to zero. The sphere rotates with angular velocity $\vec{\Omega}$; $\Omega R \ll c$. Suppose a dipole magnetic field threads the sphere. The dipole is centered on the center of the sphere. The dipole moment $\vec{\mu}$ is given; it is aligned with $\vec{\Omega}$.

(a) What voltage is induced between the equator and the poles of the sphere?

(b) Find the charge density $\rho(r,\theta)$ established inside the sphere. Here r and θ are spherical coordinates: r is the distance from the center and θ is the polar angle measured from the rotational axis.

(c) Find the electric field outside the sphere. Hint: any axisymmetric solution of $\nabla^2\Phi = 0$ that vanishes at infinity has the following form in spherical coordinates r, θ, ϕ

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos\theta), \qquad [P_0 = 1,\ P_1 = \cos\theta,\ P_2 = \frac{3\cos^2\theta - 1}{2},\ ...] \tag{1}$$

Solution:

(a) Electric field in the frame co-rotating with the sphere vanishes inside the ideal conductor: $\vec{E}' = 0$. In the static lab frame, electric field is induced by rotation $\vec{v}_{\rm rot} = \vec{\Omega} \times \vec{r}$:

$$\vec{E} = \vec{E}' - \frac{\vec{v}_{\rm rot} \times \vec{B}}{c} = -\frac{\Omega r \sin\theta}{c}\, \vec{e}_\phi \times \vec{B}, \tag{2}$$

where $\vec{e}_\phi$ is the unit vector in the ϕ-direction of the spherical coordinate system r, θ, ϕ. The dipole magnetic field is given by

$$\vec{B} = \vec{B}' = \frac{3(\vec{\mu}\cdot\vec{e}_r)\vec{e}_r - \vec{\mu}}{r^3} = \frac{\mu}{r^3}\left(2\cos\theta\, \vec{e}_r + \sin\theta\, \vec{e}_\theta\right), \tag{3}$$

where $\vec{e}_r$ and $\vec{e}_\theta$ are the unit vectors in the r and θ directions. Substitution of (2) to (1) gives

$$\vec{E} = \frac{\mu\Omega}{cr^2}\sin\theta\left(\sin\theta\, \vec{e}_r - 2\cos\theta\, \vec{e}_\theta\right), \qquad r < R.$$

Since $\nabla \times \vec{E} = -c^{-1}\partial\vec{B}/\partial t = 0$, the electric field is potential, $\vec{E} = -\nabla\Phi$.

$$\Phi(R,\theta) - \Phi(R,0) = -\int_0^\theta E_\theta\, R\, d\theta = \frac{\mu\Omega}{cR^2}\int_0^\theta \sin 2\theta\, d\theta = \frac{\mu\Omega}{cR^2}\sin^2\theta. \tag{4}$$

The potential difference between the equator and the poles is $\Phi(R,\pi/2) - \Phi(R,0) = \mu\Omega/cR^2$.

(b)

$$\rho = \frac{\nabla\cdot\vec{E}}{4\pi} = \frac{1}{4\pi r^2\sin\theta}\left[\partial_r(r^2\sin\theta E_r) + \partial_\theta(r\sin\theta E_\theta)\right] = -\frac{\mu\Omega}{2\pi c r^3}\left(2\cos^2\theta - \sin^2\theta\right).$$

(c) Potential Φ satisfies Laplace equation $\nabla^2\Phi = 0$ at $r > R$ and has the form (1). The boundary condition for Φ at $r = R$ is given by eq. (4). Since $\sin^2\theta = (2/3)(P_0 - P_2)$, the boundary condition expanded in Legendre polynomials reads

$$\Phi(R,\theta) = \left[\Phi(R,0) + \frac{2\mu\Omega}{3cR}\right]P_0 - \frac{2\mu\Omega}{3cR}P_2 \qquad \Rightarrow \quad a_0 = \Phi(R,0) + \frac{2\mu\Omega}{3cR}, \quad a_2 = -\frac{2\mu\Omega}{3cR}.$$

The boundary condition selects the two non-zero a_n ($n = 0, 2$) in eq. (1). However, since the total charge of the sphere is zero, the monopole contribution must vanish, $a_0 = 0$ [it implies $\Phi(R,0) = -2\mu\Omega/3cR$]. Thus, one finds at $r > R$

$$\Phi(r,\theta) = -\frac{2\mu\Omega R^2}{3cr^3}P_2(\cos\theta), \qquad E_r = -\frac{\partial\Phi}{\partial r} = \frac{\mu\Omega R^2}{cr^4}(3\cos\theta - 1), \qquad E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\mu\Omega R^2}{cr^4}\sin 2\theta.$$

EM Brodjman

Sec 2 E+M

2

Quals 10, EM

December 2, 2009

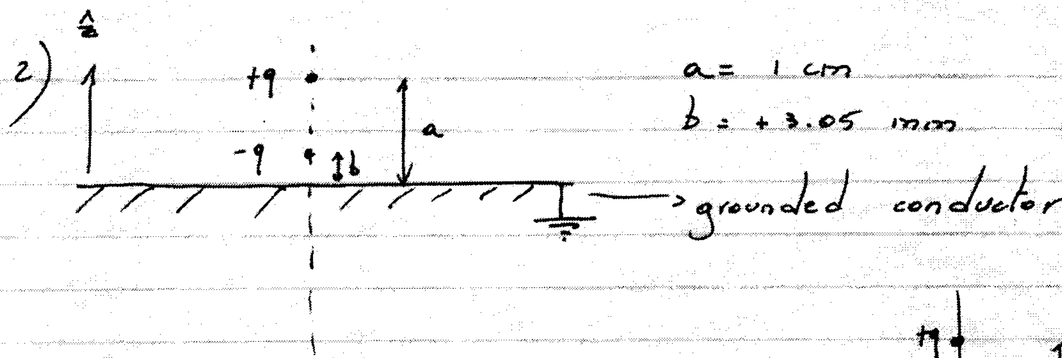
Problem

A positive point charge q is fixed 1 cm above a horizontal, grounded conducting plane. An equal negative charge $-q$ can be moved along the perpendicular dropped from q to the plane. Where should $-q$ be placed for the total force on it to be zero? Taking the distance between q and the plane equal to b, and the distance from $-q$ to the plane equal to a, what is the surface charge density on the conductor?

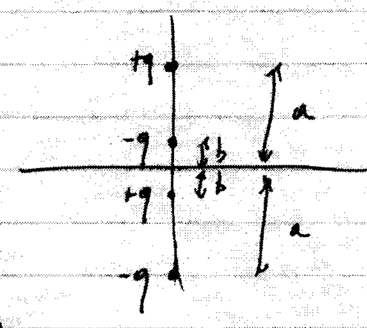
2)

$a = 1$ cm

$b = +3.05$ mm



This is an image problem: take



The force on the charge at b is then

$$F_z = \frac{q^2}{4\pi\varepsilon_0}\left[\underbrace{\frac{1}{(a-b)^2}}_{\text{due to } +q \text{ at } b} - \underbrace{\frac{1}{(2b)^2}}_{\substack{\text{due to} \\ +q \text{ at } -b}} + \underbrace{\frac{1}{(a+b)^2}}_{\substack{\text{due to } -q \\ \text{at } -a}}\right]$$

$$F_z = 0 \quad \text{if} \quad \frac{1}{(a-b)^2} - \frac{1}{(2b)^2} + \frac{1}{(a+b)^2} = 0$$

Both a and b are > 0, and we will assume $a \neq b$

$$\Rightarrow \quad (a+b)^2(2b)^2 - (a-b)^2(a+b)^2 + (a-b)^2(2b)^2 = 0$$

$$\Leftrightarrow \quad 4a^2b^2 + 4ab^3 + 4b^4 - a^4 + 2a^3b - a^2b^2 - 2a^3b + 4a^2b^2 - 2ab^3 - a^2b^2 + 2ab^3 - b^4 = 0$$

$$\Leftrightarrow \quad 10\,a^2b^2 + 7b^4 - a^4 = 0$$

$$\Leftrightarrow \quad 7b^4 + 10\,a^2b^2 - a^4 = 0$$

$$\underline{\text{Sol.}} \qquad x = \frac{-10a^2 \pm \sqrt{100a^4 + 28a^4}}{14}$$

$$a = 1 \quad \Rightarrow \quad x = \frac{-10 \pm \sqrt{128}}{14} = \frac{-10 \pm 11.3}{14}$$

The charge $-q$ is above the plane, (and b is a real number) so we take the positive solution:

$$x = \frac{1.3}{14} \Rightarrow b = \sqrt{\frac{1.3}{14}} = 0.305$$

$\Rightarrow b = 3.05$ mm implies the force on $-q = 0$

What is the surface charge density on the conductor?

• For a conductor, $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$ at the surface

$$\rightsquigarrow \sigma = \varepsilon_0 E_{\hat{n}} = -\varepsilon_0 \frac{\partial V}{\partial n}\Big|_{surface}$$

$\rightarrow$ in this case, $\sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$

$$\bullet\ V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\Delta r_i|}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-b)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{q}{\sqrt{x^2+y^2+(z+a)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+b)^2}}\right]$$

$$\rightsquigarrow \frac{\partial V}{\partial z} = -\frac{1}{2}\frac{2q}{4\pi\varepsilon_0}\left[\frac{(z-b)}{\left(x^2+y^2+(z-b)^2\right)^{3/2}} - \frac{(z-a)}{\left(x^2+y^2+(z-a)^2\right)^{3/2}} + \frac{(z+a)}{\left(x^2+y^2+(z+a)^2\right)^{3/2}} - \frac{(z+b)}{\left(x^2+y^2+(z+b)^2\right)^{3/2}}\right]$$

$$\Rightarrow \sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0} = \frac{q}{4\pi}\left[\frac{2a}{(x^2+y^2+a^2)^{3/2}} - \frac{2b}{(x^2+y^2+b^2)^{3/2}}\right]$$

EM Hailey

Sec. 2 E+M

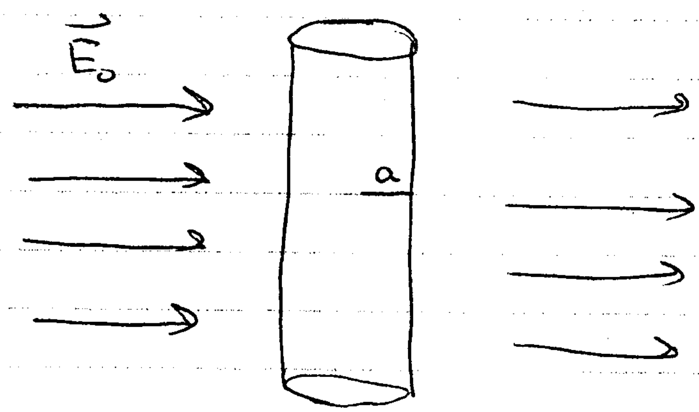
\# 3

Problem 2: E-M = Hailey (Alternate)

Consider an infinitely long, grounded conducting cylinder which is introduced into a uniform electric field $\vec{E}_0$. The axis of the cylinder is perpendicular to $\vec{E}_0$.

a) Find an expression for the external potential after insertion of the cylinder.

b) Find an expression for the surface charge induced on the cylinder



E-M : Harley

Solution 2: This is just the 2-d Laplace equation in cylindrical coordinates for an infinitely long cylinder since there is no z-dependence.

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Separating $\phi(r\,\theta) = R(r)\,Q(\theta) \Rightarrow$

$$r/_R \frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{Q}\frac{d^2Q}{d\theta^2} = 0$$

Let the separation constant be n^2

$$d^2Q/_{d\theta^2} + n^2 Q = 0 \;; \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

$$Q(\theta) \sim e^{\pm in\theta} \;; \quad r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 R = 0$$

By simple substitution the radial equation has solutions $R(r) \sim r^{\pm n}$ so

$$\phi(\rho\theta) \sim \sum_n \left(a_n r^n + b_n r^{-n}\right) e^{\pm in\theta}$$

since $a_n = 0$ to prevent ϕ blowing up as $r \to \infty$ and noting the form of the uniform field

$$\phi_{ext} = -E_0 r\cos\theta + \sum_n b_n r^{-n}\, e^{\pm in\theta}$$

$$\phi_{ext} = -E_0 r\cos\theta + \frac{b_1}{r}\cos\theta$$

Only the $n=1$ term survives so that we can match the boundary condition $\phi(a\ \theta) = 0$

$$\phi_{ext}(a\theta) = 0 = -E_0 a \cos\theta + \frac{b_1}{a}\cos\theta$$

$$b_1 = E_0 a^2$$

$$\phi_{ext} = -E_0 r\cos\theta + \frac{E_0 a^2}{r}\cos\theta$$

a.) $\phi_{ext} = -E_0 r\cos\theta\left(1 - a^2/r^2\right)$ Ans

b.) The induced charge is just the normal component of the E-field ie $E_n = 4\pi\sigma$ from Gauss' Law

$$\sigma = 1/4\pi \; -\frac{\partial \phi_{ext}}{\partial r}\Big|_{r=a}$$

$$\sigma = \frac{E_0 \cos\theta}{4\pi}\left(1 + \frac{a^2}{r^2}\right)\Big|_{r=a}$$

$$\sigma = \frac{E_0\cos\theta}{2\pi}$$ Ans

Marka
Sec. 2 E+M
4

#3: A perpendicularly incident beam of right circularly polarized light is reflected by a stationary mirror. Show that the reflected beam is left circularly polarized.

FROM THE FRESNEL EQUATIONS:

$$\vec{E}_R = \frac{1-\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}\vec{E}_I \underset{\varepsilon\to\infty}{\approx} -\vec{E}_I$$

RIGHT CIRCULAR POLARIZED:

$$E_y = A\cos(\omega t - kx)$$

$$E_z = A\sin(\omega t - kx)$$

LEFT CIRCULAR POLARIZED

$$E_y = A\cos(\omega t - kx)$$

$$E_z = -A\sin(\omega t - kx)$$

WHILE THE REFLECTION INTRODUCES THE SAME PHASE SHIFT FOR y & z components (RELATIVE PHASE SHIFT DOES NOT CHANGE), THE DIRECTION OF PROPAGATION IS REVERSED. THEREFORE THE "TWIST" OF THE POLARIZATION RELATIVELY TO THE DIRECTION OF PROPAGATION FLIPS.

Weinberg
Sec 2 E+M
#5

E&M problem

A magnetic monopole is a hypothetical particle that is a source for a Coulomb magnetic field

$$\mathbf{B} = \frac{g\hat{\mathbf{r}}}{r^2} \tag{1}$$

a) Consider a particle with mass m and electric charge q that is moving in the magnetic field of a static magnetic monopole. Show that the usual expression for angular momentum, $\mathbf{r} \times (m\mathbf{v})$ is not conserved, but that there is a conserved angular momentum of the form

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) + \mathbf{f} \tag{2}$$

Determine $\mathbf{f}$.

b) An electric charge q and a magnetic charge g with fixed positions are located a distance D apart. The combined fields of these charges have a nonzero angular momentum. (i) Show that the magnitude of this angular momentum does not depend on the distance D. (ii) Determine the magnitude and direction of the angular momentum. The integral

$$\int_0^\infty dy \, \frac{y}{(y^2 - 2ay + 1)^{3/2}} = \frac{1}{1-a} \tag{3}$$

may be useful.

a) $\frac{d}{dt}\left[\vec{r}\times(m\vec{v})\right] = \vec{v}\times(m\vec{v}) + \vec{r}\times(m\dot{\vec{v}})$

$= 0 + \vec{r}\times(q\vec{v}\times\vec{B})$

$= \vec{r}\times(q\vec{v}\times\frac{\hat{r}}{r^2})\,g$

$= qg\,\frac{1}{r}\,\hat{r}\times(\vec{v}\times\hat{r})$

$= qg\,\frac{1}{r}\,(\vec{v} - \hat{r}\cdot\vec{v}\,\hat{r})$

$= qg\,\frac{d}{dt}(\hat{r})$

$\Rightarrow$ $L - qg\hat{r}$ is conserved

b) Use the fact that the linear momentum density is

$$\vec{p} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2}\frac{1}{\mu_0}\vec{E}\times\vec{B} = \epsilon_0 \vec{E}\times\vec{B}$$

$$\Rightarrow \vec{L} = \epsilon_0 \int d^3r \; \vec{r}\times(\vec{E}\times\vec{B})$$

Let the magnetic charge be at the origin
" " electric " " " $\vec{D} = (0,0,D)$

By symmetry, $\vec{L}$ is parallel or anti-parallel to $\vec{D}$

$$\Rightarrow (\text{Mag of } \vec{L}) = \frac{1}{D}\vec{D}\cdot\vec{L} \equiv L$$

$$\vec{B} = g\frac{\hat{r}}{r^2}, \quad E = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{D}}{|\vec{r}-\vec{D}|^3}$$

$$L = g\left(\frac{q}{4\pi\epsilon_0}\right)\epsilon_0 \int d^3r \; F$$

$$F = \frac{1}{D}\vec{D}\cdot\vec{r}\times\left[(\vec{r}-\vec{D})\times\hat{r}\right]\frac{1}{|\vec{r}-\vec{D}|^3}\frac{1}{r^2}$$

$$= \frac{-1}{D}\frac{1}{r}\frac{1}{|\vec{r}-\vec{D}|^3}\,\vec{D}\cdot\hat{r}\times(D\times\hat{r})$$

$$= \vec{D}\cdot(\vec{D}-\hat{r}\,\hat{r}\cdot\vec{D})$$

$$= D^2(1-\cos^2\theta)$$

with θ = angle between $\vec{D}$ & $\hat{r}$

$$\Rightarrow F = -\frac{D}{r}(1-\cos^2\theta)\left[r^2\sin^2\theta + (r\cos\theta - D)^2\right]^{-3/2}$$

$$L = -\frac{gq}{4\pi} D\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^{\infty} dr \; r(1-\cos^2\theta)\left[r^2 - 2rD\cos\theta + D^2\right]^{-3/2}$$

Let $r = Dy$

$$\Rightarrow L = -\frac{gq}{4\pi}(2\pi)\int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)\int_0^{\infty} dy \frac{y}{(y^2-2y\cos\theta+1)^{3/2}}$$

$\Rightarrow$ Independent of D

Using the integral given with the problem,

$$L = -\frac{qg}{2}\int_{-1}^{1} d\cos\theta\,(1-\cos^2\theta)\left(\frac{1}{1-\cos\theta}\right)$$

$$= -\frac{qg}{2}\int_{-1}^{1} d(\cos\theta)\,(1+\cos\theta)$$

$$= -qg$$

$\rightarrow \vec{L}$ is antiparallel to $\vec{D}$, points from electric to magnetic charge

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 13, 2010
1:00 PM - 3:00 PM

Quantum Mechanics
Section 3.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (Quantum Mechanics), Question 2; Section 3 (Quantum Mechanics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Quantum Mechanics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Two observers in different inertial frames will need different wave functions to describe the same physical system. To make things simple, consider how it works in the non-relativistic case. The first observer uses coordinates $(\vec{x}, t)$ and a wave function $\psi(\vec{x}, t)$, while the second uses $(\vec{x}\,', t)$ and $\psi(\vec{x}\,', t)$. Of course, $\vec{x}\,' = \vec{x} - \vec{v}t$ where $\vec{v}$ is a constant velocity. The wave functions for the two observers are said to be related as follows:

$$\tilde{\psi}(\vec{x}\,', t) = \psi(\vec{x}, t) \exp\left(\frac{-i}{\hbar}\left[m\vec{v} \cdot \vec{x} - \frac{mv^2}{2}t\right]\right)$$

Despite its innocuous look (it's just a phase!) this transformation has interesting effects.

(a) Let us first verify that it makes sense. Suppose $\psi(\vec{x}, t)$ is the wave function of a free particle of momentum $\vec{p} = (p_x, p_y, p_z)$. Show that $\tilde{\psi}(\vec{x}\,', t)$ indeed describes a free particle with a proper momentum.

(b) Now let us put this to work. Suppose we have a hydrogen atom, which at $t < 0$ was at rest with the electron in the ground *1s* state described by the wave function

$$\psi(\vec{x}) = \psi_{1,0}(\vec{x}) \equiv \frac{1}{\sqrt{\pi a_B^3}} \exp\left(\frac{-r}{a_B}\right); \quad r = |\vec{x}|$$

where a_B is the Bohr radius.

Suppose at $t = 0$ the proton suddenly starts to move (e.g., due to a collision with a neutron) in the z-direction with the velocity v. Let the change in the velocity be so abrupt that the electron wave function remains the same. What is the probability at $t > 0$ to find the moving hydrogen atom with the electron in the ground state?

(c) What is the probability to find the electron in the state with $n = 2, l = 1, m = 1$?

2. (a) Prove that the expectation value of the Hamiltonian $E[\phi]$ is stationary in the neighborhood of a discrete eigenstate i.e., if $H\psi_n = E_n\psi_n$ and $\psi = \psi_n + \delta\psi$, then $\delta\langle\psi|H|\psi\rangle = 0$. Show also that $E[\phi] \geq E_0$, where $E_0 \leq E_n$ is the ground state energy.

   (b) Apply the above to estimate the quantum ground state energy of a simple harmonic oscillator using a trial wave function of the form $\psi(x) = \exp(-x^2/a^2)$. Determine a, and compare $E[\psi]$ to the exact E_0 ground state energy.

(Useful integrals are $\int_{-\infty}^{\infty} dx\, e^{-b^2x^2} = \sqrt{\pi}/b$ and its derivative with respect to b.)

3. Consider an electron of charge e and mass m confined on a ring of radius R. In cylindrical coordinates the Hamiltonian of this confined system can be described by

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d}{d\phi}\right)^2.$$

where ϕ is the azimuthal angle.

(a) Find the energy eigenvalues and normalized eigenfunctions of this system.

(b) Now we consider a magnetic field $\vec{\mathbf{B}} = B\hat{z}$ applied along the z-direction. Employing the "symmetric" gauge, the corresponding vector potential on the ring can be expressed by

$$\vec{\mathbf{A}} = \frac{BR}{2}\hat{\phi},$$

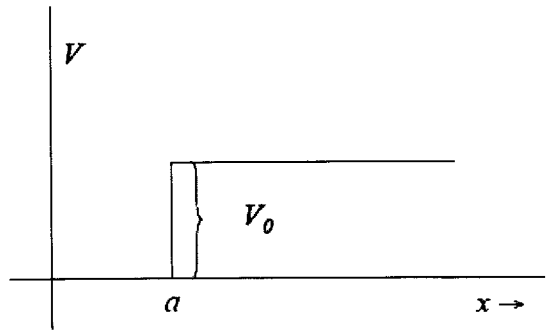
where $\hat{\phi}$ is the unit vector along the azimuthal angle ϕ. In the magnetic field, the Hamiltonian is given by $H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{\mathbf{A}}\right)^2$. Find the energy eigenvalue of an electron confined to this ring in the presence of a fixed magnetic field.

(c) Find the smallest magnetic field for which one can find the non-degenerate ground and doubly degenerate excited states.

4. Consider a particle of mass m in a one-dimensional potential $V(x)$ where

$$
\begin{aligned}
V(x) &= \infty & x &< 0 \\
V(x) &= 0 & 0 < x &< a \\
V(x) &= V_0 & x &> a
\end{aligned}
$$

with $V_0 > 0$.

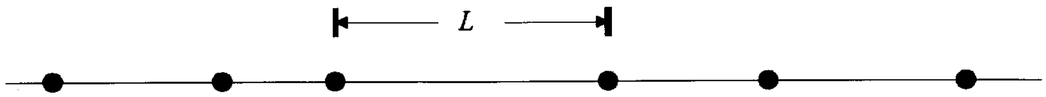


(a) If $E = \frac{\hbar^2 k^2}{2m}$ is a bound state energy and $V_0 - E = \frac{\hbar^2 \alpha^2}{2m}$, give the equation determining possible values of E.

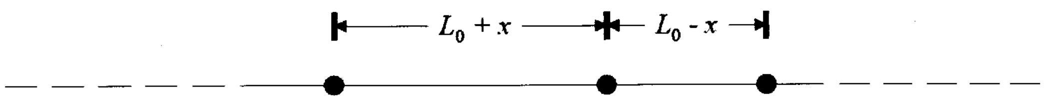
(b) Give the condition on V_0 and a for at least one bound state to exist.

(c) What are the energy levels when $V_0 = \infty$?

5. Consider a quantum system with an infinite set of particles in one dimension as shown in the figure. Particles cannot cross neighbors. We are interested in the probability distribution of spacings L between a particle and its neighbor to the left, given that the average spacing between particles is L_0.



In the simplest approximation to this many-body problem, a single particle moves between two fixed neighbors separated by $2L_0$. Let $x = L - L_0$ denote the deviation from the midpoint.



(a) Find the probability distribution $P(L)$ in the ground state, in the above approximation, and assuming there are no interparticle interactions (other than contact interactions).

(b) Now suppose there are strong repulsive potentials between pairs of neighboring particles of the form AL^{-n}. Considering only the nearest neighbors, write the potential energy for the middle particle for $x \ll L_0$. Write the result explicitly in terms of A, n and L_0.

(c) Still assuming that $x \ll L_0$ and that the neighboring particles are fixed, write the Schrödinger equation for the middle particle, and argue that the problem can be mapped into a familiar one. Based on this analogy, what is the form of the distribution $P(L)$? How does its *width* scale with L_0?

(d) In the limit of strong repulsion, as in *b*) and *c*) above, explain how you can measure the power n and the amplitude A that characterize the potential.

Altshuler
sec. 3 QM
1

Two observers in different inertial frames will need different wave functions to describe the same physical system. To make things simple consider how it works in the non-relativistic case. The first observer uses coordinates $(\vec{x},t)$ and a wave function $\psi(\vec{x},t)$, while the second uses $(\vec{x}',t)$ and $\tilde{\psi}(\vec{x}',t)$. Of course, $\vec{x}' = \vec{x} - \vec{v}t$ where $\vec{v}$ is a constant velocity. The wave functions for the two observers are said to be related as follows:

$$\tilde{\psi}(\vec{x}',t) = \psi(\vec{x},t)\exp\left(\frac{-i}{\hbar}\left[m\vec{v}\vec{x} - \frac{mv^2}{2}t\right]\right)$$

Despite its innocuous look (it's just a phase!) this transformation has interesting effects.

a) Let us first verify that it makes sense. Suppose $\psi(\vec{x},t)$ is the wave function of a free particle of momentum $\vec{p} = (p_x, p_y, p_z)$. Show that $\tilde{\psi}(\vec{x}',t)$ indeed describes a free particle with a proper momentum.

Solution:
Wave function of the free particle of momentum $\vec{p} = (p_x, p_y, p_z)$ can be written as

$$\psi(\vec{x},t) = C\exp\left(\frac{i}{\hbar}\left[\vec{p}\vec{x} - \frac{p^2}{2m}t\right]\right) = C\exp\left(\frac{i}{\hbar}\left[p_x x + p_y y + p_z z - \frac{p_x^{\ 2} + p_y^{\ 2} + p_z^{\ 2}}{2m}\right]\right)$$

where C is the normalization constant.

$$\tilde{\psi}(\vec{x}',t) = C\exp\left(\frac{i}{\hbar}\left[\vec{p}\vec{x} - \frac{p^2}{2m}t - m\vec{v}\vec{x} + \frac{mv^2}{2}t\right]\right)$$

Now we can substitute $\vec{x} = \vec{x}' + \vec{v}t$

$$\tilde{\psi}(\vec{x}',t) = C\exp\left(\frac{i}{\hbar}\left[\vec{p}(\vec{x}' - \vec{v}t) - \frac{p^2}{2m}t + m\vec{v}(\vec{x}' + \vec{v}t) - \frac{mv^2}{2}t\right]\right) =$$

$$= C\exp\left(\frac{i}{\hbar}\left[(\vec{p} - mv)\vec{x}' + t\left(\vec{p}\vec{v} - \frac{p^2}{2m} + mv^2 - \frac{mv^2}{2}\right)\right]\right)$$

$$= C\exp\left(\frac{i}{\hbar}\left[(\vec{p} - m\vec{v})\vec{x}' - t\frac{1}{2m}(p^2 - 2m\vec{p}\vec{v} + m^2v^2)\right]\right) = \exp\left(\frac{-i}{\hbar}\left[(\vec{p} - m\vec{v}) - \frac{1}{2m}(p - m\vec{v})^2\right]\right)$$

One can see that the wave function in a new frame can be written as

$$\tilde{\psi}(\vec{x}',t) = C\exp\left(\frac{i}{\hbar}\left[\vec{p}'\vec{x}' - \frac{(p')^2}{2m}t\right]\right)$$

where $\vec{p}' \equiv \vec{p} - m\vec{v}$ .is the momentum in the new frame. It is indeed the wave function of the free particle in the new frame.

b) Now let us put this to work. Suppose we have a hydrogen atom, which at $t<0$ was at rest with the electron in the ground *1s* state described by the wave function

$$\psi(\vec{x}) = \psi_{1,0}(\vec{x}) \equiv \frac{1}{\sqrt{\pi a_B^3}}\exp\left(\frac{-r}{a_B}\right); \quad r = |\vec{x}|$$

Suppose at $t=0$ the proton suddenly starts to move (e.g., due to a collision with a neutron) in the *z*-direction with the velocity *v*. Let the change in the velocity be so abrupt that the electronic wave function remained the same. What is the probability to find at $t>0$ the moving hydrogen atom with the electron in the ground state?

Solution:

As we learned the wave function of the electron at the new rest frame of the proton is

$$\tilde{\psi}(\vec{x}',t) = \exp\left(\frac{-i}{\hbar}\left[m\vec{v}\vec{x} - \frac{mv^2}{2}t\right]\right)\psi_{1,0}(\vec{x},t) = \frac{1}{\sqrt{\pi a_B^3}}\exp\left[\frac{-r}{a_B} - \frac{i}{\hbar}\left(mvz - \frac{mv^2}{2}t\right)\right]$$

Therefore at *t=0+* the electron wave function would be

$$\psi(\vec{x}') = \frac{1}{\sqrt{\pi a_B^3}}\exp\left[\frac{-r}{a_B} - \frac{i}{\hbar}mvz\right]$$

The ground state of the electron in the moving atom is described by the wave function

$$\psi_{1,0}(\vec{x}') \equiv \frac{1}{\sqrt{\pi a_B^3}}\exp\left(\frac{-r'}{a_B}\right)$$

Note that $\vec{x} = \vec{x}'$ at *t=0*. The probability P that the electron remains in the ground state is

$$P = \left|\int \psi(\vec{x})\psi_{1,0}(\vec{x})d\vec{x}'\right|^2 = \frac{1}{(\pi a_B^3)^2}\left|\int \exp\left[-\frac{2r}{a_B} - \frac{i}{\hbar}mvz\right]d\vec{x}'\right|^2$$

Now we can use polar coordinates: momentum $z = r\cos\theta$; $\quad d\vec{x} = r^2 dr d(\cos\theta)d\varphi$

$$P = \frac{1}{(\pi a_B^3)^2}\left|\int \exp\left[-r\left(\frac{2}{a_B} - \frac{i}{\hbar}mv\cos\theta\right)\right]r^2 dr d(\cos\theta)d\varphi\right|^2$$

Integrals over φ and over θ can be evaluated straightforwardly. The result is

$$P = \frac{4\hbar^2}{a_B^6 m^2 v^2}\left(\operatorname{Im}\int_0^\infty \exp\left[-r\left(\frac{2}{a_B} - \frac{imvr}{\hbar}\right)\right]rdr\right)^2 = \frac{4\hbar^2}{a_B^6 m^2 v^2}\left(\operatorname{Im}\left[\frac{1}{(2/a_B - imvr/\hbar)^2}\right]\right)^2$$

Integral $\int \exp(-cr)rdr$ can be evaluated by parts: $\int_0^\infty \exp(-cr)rdr = 1/c^2$. Therefore

$$P = \frac{4\hbar^2}{a_B^6 m^2 v^2}\left(\operatorname{Im}\int_0^\infty \exp\left[-r\left(\frac{2}{a_B} - \frac{imvr}{\hbar}\right)\right]rdr\right)^2 = \frac{4\hbar^2}{a_B^6 m^2 v^2}\left(\operatorname{Im}\left[\frac{1}{(2/a_B - imvr/\hbar)^2}\right]\right)^2$$

c) What is the probability to find the electron in the state with $n=2, l=1, m=1$?

Solution

This probability vanishes after the integration over the polar angle because $\psi_{2,1,1} \propto e^{i\varphi}$

Gyulassy

Sec. 3 QM

2

3 QM Quals 2010:

a) [10] Prove that the expectation value of the Hamiltonian $E[\phi]$ is stationary in the neighborhood of a discrete eigenstate, i.e., if $H\psi_n = E_n\psi_n$ and $\psi = \psi_n + \delta\psi$, then $\delta\langle\psi|H|\psi\rangle = 0$. Show also that $E[\phi] \geq E_0$ where $E_0 \leq E_n$ is the ground state energy.

b) [10] Apply the above to estimate the quantum ground state energy of a simple harmonic oscillator using a trial wavefunction of the form $\psi(x) = \exp\left(-x^2/a^2\right)$. Determine a, and compare $E[\psi]$ to the exact E_0 ground state energy. (Useful integrals are $\int_{-\infty}^{\infty} dx e^{-b^2x^2} = \sqrt{\pi}/b$ and its derivative with respect to b)

Quals2010 QM sec3 prob2a

QM MG 1

a) $E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ mean energy in state $|\psi\rangle$

For ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$, $E[\psi_0] = E_0$

Try variation $\psi = \psi_0 + \delta\psi$

$$E[\psi_0+\delta\psi] - E[\psi_0] = \frac{\langle\delta\psi|H|\psi_0\rangle + \langle\psi_0|H|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle} - \langle\psi_0|H|\psi_0\rangle\left(\frac{\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle^2}\right)$$

to first order

Let us normalize ground $\langle\psi_0|\psi_0\rangle = 1$

$$= E_0(\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle) - E_0(\quad " \quad " \quad) = 0$$

Thus $\underline{E[\psi_0+\delta\psi] = E[\psi_0]} + \underset{\text{to first order}}{\underline{\underline{0}}} + \binom{2^{nd}\text{ order}}{\text{in }\delta\psi}$

To show that $E[\psi] \geq E[\psi_0]$

$|\psi\rangle = |\psi+\delta\psi\rangle = \sum_n z_n|\psi_n\rangle$ we can expand in complete orthonormal

where $H|\psi_n\rangle = E_n|\psi_n\rangle$, $E_n \geq E_0$, $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle\psi|H|\psi\rangle = \sum_n\sum_m z_n^* z_m E_m \langle\psi_n|\psi_m\rangle$$

$$= \sum_n |z_n|^2 E_n \geq \left(\sum_n |z_n^2|\right) E_0 = E_0\langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Quals2010 QM sec3 prob2b

MG 2

b) $\psi = e^{-x^2/a^2}$, Norm $= \mathcal{N} = \int_{-\infty}^{\infty} |\psi|^2 dx = a\sqrt{\frac{\pi}{2}}$ (from hint)

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \hat{KE} + \hat{V}$$

Use $\langle\psi|\hat{KE}|\psi\rangle = -\hbar^2/2m \int (-)\left(\frac{d\psi}{dx}\right)^2 dx$ (int. by parts)

$$= \frac{\hbar^2}{2m}\int\left(\frac{2x}{a^2}\right)^2|\psi|^2 = \left(\frac{2\hbar^2}{ma^4}\right)\int x^2|\psi|^2 = (\;"\;)\underbrace{\left(\frac{-d}{d(2/a^2)}\langle\psi|\psi\rangle\right)}_{\langle\psi|x^2|\psi\rangle}$$

$$= \frac{2\hbar^2}{ma^4}\left(\frac{a^3}{4}\frac{d}{da}\right)a\sqrt{\pi/2} = \frac{\hbar^2}{2ma^2}\langle\psi|\psi\rangle$$

$$\Rightarrow \langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2ma^2} + \frac{1}{8}m\omega^2 a^2$$

<u>Minimize w.r.t. a^2</u> $\quad \frac{d\langle E\rangle}{da^2} = 0 = -\frac{\hbar^2}{2ma^4} + \frac{1}{8}m\omega^2$

$$\Rightarrow \underline{a^2 = \sqrt{\frac{4\hbar^2}{m^2\omega^2}} = \frac{2\hbar}{m\omega}} \qquad \Rightarrow \underline{\langle KE\rangle = \langle V\rangle}$$

$$\langle E\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{1}{8}m\omega^2\frac{2\hbar}{m\omega} = \frac{1}{2}\hbar\omega$$

This is the exact quantum oscillator ground state energy

By part (a) if we tried any other guess for ψ we would obtain a larger energy

Quals2010 QM sec3 prob2a

QM

MG 1

a) $E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ mean energy in state $|\psi\rangle$

For ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$, $E[\psi_0] = E_0$

Try variation $\psi = \psi_0 + \delta\psi$

$$E[\psi_0+\delta\psi] - E[\psi_0] = \frac{\langle\delta\psi|H|\psi_0\rangle + \langle\psi_0|H|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle} - \langle\psi_0|H|\psi_0\rangle\left(\frac{\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle^2}\right)$$

to first order

let us normalize ground $\langle\psi_0|\psi_0\rangle = 1$

$$= E_0\left(\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle\right)$$
$$- E_0\left(\quad " \qquad " \quad \right)$$
$$= 0$$

Thus $E[\psi_0+\delta\psi] = E[\psi_0] + \underset{\text{to first order}}{\underline{\underline{0}}} + \left(\begin{matrix}2^{nd}\text{ order}\\ \text{in } \delta\psi\end{matrix}\right)$

To show that $E[\psi] \geq E[\psi_0]$

we can expand in complete orthonormal

$$|\psi\rangle = |\psi+\delta\psi\rangle = \sum_n z_n|\psi_n\rangle$$

where $H|\psi_n\rangle = E_n|\psi_n\rangle$, $E_n \geq E_0$, $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle\psi|H|\psi\rangle = \sum_n\sum_m z_n^* z_m E_m \langle\psi_n|\psi_m\rangle$$
$$= \sum_n |z_n|^2 E_n \geq \left(\sum_n |z_n^2|\right) E_0 = E_0\langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Quals2010 QM sec3 prob2b

MG 2

b) $\psi = e^{-x^2/a^2}$, Norm $= \mathcal{N} = \int_{-\infty}^{\infty} |\psi|^2 dx = a\sqrt{\frac{\pi}{2}}$ (from hint)

$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \hat{KE} + \hat{V}$

Use $\langle\psi|\hat{KE}|\psi\rangle = -\hbar^2/2m \int (-)\left(\frac{d\psi}{dx}\right)^2 dx$ (Int. by parts)

$= \frac{\hbar^2}{2m}\int\left(\frac{2x}{a^2}\right)^2|\psi|^2 = \left(\frac{2\hbar^2}{ma^4}\right)\int x^2|\psi|^2 = \left(\,''\,\right)\underbrace{\left(\frac{-d}{d(2/a^2)}\langle\psi|\psi\rangle\right)}_{\langle\psi|x^2|\psi\rangle}$

$= \frac{2\hbar^2}{ma^4}\left(\frac{a^3}{4}\frac{d}{da}\right) a\sqrt{\pi/2} = \frac{\hbar^2}{2ma^2}\langle\psi|\psi\rangle$

$\Rightarrow \langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2ma^2} + \frac{1}{8}m\omega^2 a^2$

Minimize w.r.t. a^2 $\qquad \frac{d\langle E\rangle}{da^2} = 0 = -\frac{\hbar^2}{2ma^4} + \frac{1}{8}m\omega^2$

$\Rightarrow \underline{a^2 = \sqrt{\frac{4\hbar^2}{m^2\omega^2}} = \frac{2\hbar}{m\omega}}$ $\qquad \Rightarrow \underline{\langle KE\rangle = \langle V\rangle}$

$\langle E\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{1}{8}m\omega^2\frac{2\hbar}{m\omega} = \frac{1}{2}\hbar\omega$

This is the exact quantum oscillator ground state energy

By part (a) if we tried any other guess for ψ we would obtain a larger energy

Quals2010 QM sec3 prob2a

CQM MG 1

a) $E[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$ mean energy in state $|\psi\rangle$

For ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$, $E[\psi_0] = E_0$

Try variation $\psi = \psi_0 + \delta\psi$

$$E[\psi_0+\delta\psi] - E[\psi_0] = \frac{\langle\delta\psi|H|\psi_0\rangle + \langle\psi_0|H|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle}$$

to first order

$$- \langle\psi_0|H|\psi_0\rangle\left(\frac{\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle}{\langle\psi_0|\psi_0\rangle^2}\right)$$

Let us normalize ground $\langle\psi_0|\psi_0\rangle = 1$

$$= E_0(\langle\delta\psi|\psi_0\rangle + \langle\psi_0|\delta\psi\rangle)$$
$$- E_0(\quad " \quad\quad " \quad)$$
$$= 0$$

Thus $\underline{E[\psi_0+\delta\psi] = E[\psi_0]} + \underset{\text{to first order}}{\underline{\underline{0}}} + \left(\begin{matrix}\text{2nd order}\\ \text{in } \delta\psi\end{matrix}\right)$

To show that $E[\psi] \geq E[\psi_0]$

we can expand in complete orthonormal

$$|\psi\rangle = |\psi+\delta\psi\rangle = \sum_n z_n |\psi_n\rangle$$

where $H|\psi_n\rangle = E_n|\psi_n\rangle$, $E_n \geq E_0$, $\langle\psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle\psi|H|\psi\rangle = \sum_n\sum_m z_n^* z_m E_m \langle\psi_n|\psi_m\rangle$$
$$= \sum_n |z_n|^2 E_n \geq \left(\sum_n |z_n^2|\right) E_0 = E_0\langle\psi|\psi\rangle$$

$$\Rightarrow \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Quals2010 QM sec3 prob2b

MG 2

b) $\psi = e^{-x^2/a^2}$, Norm $= \mathcal{N} = \int_{-\infty}^{\infty} |\psi|^2 dx = a\sqrt{\frac{\pi}{2}}$ (from hint)

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \hat{KE} + \hat{V}$$

Use $\langle\psi|\hat{KE}|\psi\rangle = -\hbar^2/2m \int (-)\left(\frac{d\psi}{dx}\right)^2 dx$ (int. by parts)

$$= \frac{\hbar^2}{2m}\int\left(\frac{2x}{a^2}\right)^2|\psi|^2 = \left(\frac{2\hbar^2}{ma^4}\right)\int x^2|\psi|^2 = (\ ''\ )\underbrace{\left(\frac{-d}{d(2/a^2)}\langle\psi|\psi\rangle\right)}_{\langle\psi|x^2|\psi\rangle}$$

$$= \frac{2\hbar^2}{ma^4}\left(\frac{a^3}{4}\frac{d}{da}\right) a\sqrt{\pi/2} = \frac{\hbar^2}{2ma^2}\langle\psi|\psi\rangle$$

$$\Rightarrow \langle E\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2ma^2} + \frac{1}{8}m\omega^2 a^2$$

<u>Minimize w.r.t. a^2</u> $\quad \frac{d\langle E\rangle}{da^2} = 0 = -\frac{\hbar^2}{2ma^4} + \frac{1}{8}m\omega^2$

$$\Rightarrow \underline{a^2 = \sqrt{\frac{4\hbar^2}{m^2\omega^2}} = \frac{2\hbar}{m\omega}} \qquad \Rightarrow \underline{\langle KE\rangle = \langle V\rangle}$$

$$\langle E\rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\omega}{2\hbar}\right) + \frac{1}{8}m\omega^2\frac{2\hbar}{m\omega} = \frac{1}{2}\hbar\omega$$

This is the exact quantum oscillator ground state energy

By part (a) if we tried any other guess for ψ we would obtain a larger energy

QM Kim

Sec. 3 QM

#3

2009 Qual Problems – Philip Kim

QM

Electron confined in a ring with magnetic field.

Consider an electron of charge e and mass m confined in a ring of radius R. In a cylindrical coordinate the Hamiltonian of this confined system can be described by

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2 = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d}{d\varphi}\right)^2.$$

where φ is the azimuthal angle.

(a) Find the energy eigen values and normalized eigen wavefunctions of this system.

(b) Now we consider a magnetic field $\vec{B} = B\hat{z}$ applied to z-direction. Employing the symmetry gauge, the corresponding vector potential on the ring can be expressed by

$$\vec{A} = \frac{BR}{2}\hat{\varphi},$$

where $\hat{\varphi}$ is the unit vector along the azimuthal angle φ. In the magnetic field, the Hamitonian is given by $H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\vec{A}\right)^2$. Find the energy eigen value of a confined electron in this ring in the presence of a fixed magnetic field.

(c) Find the smallest magnetic field at which one can find the non-degenerate ground and doubly degenerate excited states?

There are some changes in solution. New solution attached.

Corrected solution.

QM

Electron confined in a ring

(a) Ignoring spin,

$$H = -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{d}{d\varphi}\right)^2$$

Try $\psi(\varphi) = A e^{-i\lambda\varphi} \Rightarrow E_\lambda = \frac{\hbar^2}{2m}\left(\frac{\lambda}{R}\right)^2$

Applying periodic boundary condition,

$$\psi(\varphi+2\pi) = \psi(\varphi) \Rightarrow \lambda = 0, \pm1, \pm2, \cdots$$

Normalization

$$1 = \int_0^{2\pi} d\varphi\, |\psi|^2 = 2\pi A^2 \Rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \psi_n(\varphi) = \frac{1}{\sqrt{2\pi}} e^{-in\varphi}, \quad E_n = \frac{\hbar^2}{2m}\left(\frac{n}{R}\right)^2, \quad n = 0, \pm1, \pm2,$$

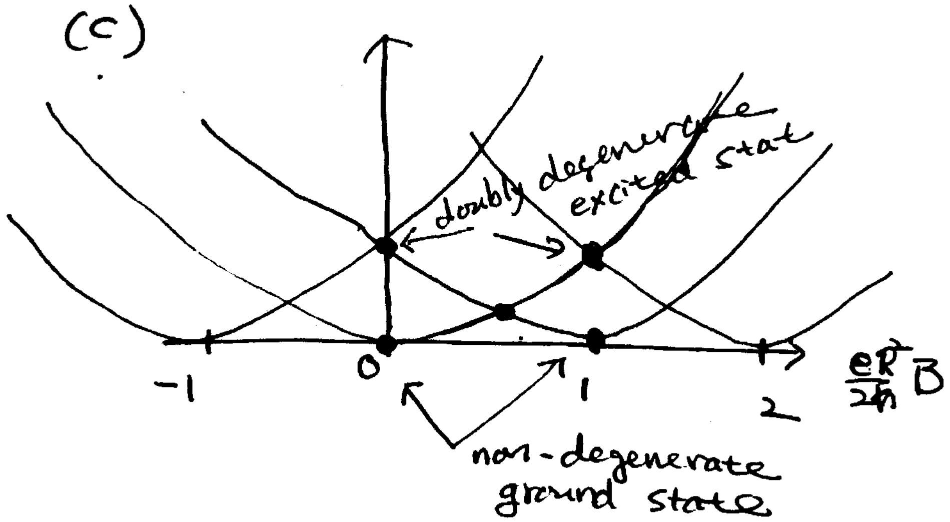
(b) $\vec{A} = \frac{BR}{2}\hat{\varphi}$

$$\tilde{H} = \frac{1}{2m}\left[\frac{\hbar}{i}\frac{1}{R}\frac{\partial}{\partial\varphi} - \frac{eBR}{2}\right]^2 = \frac{\hbar^2}{2mR^2}\left[\frac{1}{i}\frac{\partial}{\partial\varphi} - \frac{eBR^2}{2\hbar}\right]^2$$

$\psi_n(\varphi)$ in (a) becomes eigenfunction.

$$E_n = \frac{\hbar^2}{2mR^2}\left[n - \frac{eBR^2}{2\hbar}\right]^2$$

(c)



$B_{min} = 0$ (for the smallest).

or

$B_{min} = \frac{2\hbar}{eR^2}$ (for the smallest non-zero field)

Columbia Physics Department
2010 QUALIFYING EXAMS

All questions are to be scored on a scale of
<u>0</u> to <u>15</u>
(0 = failing, 15 = highest possible score)

Please write the numerical score in <u>red</u> ink directly on the cover of the exam booklet.

Please be sure to read the problem <u>as it appears in the exam.</u> Some problems have been edited. Make sure that you are grading what the students were asked.

Please return the graded exam booklets to Lalla or to Rasma in 704 Pupin, ideally within 24 hours, or as soon as possible.

Thanks!

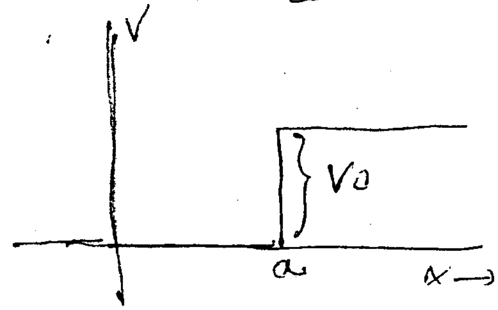
Quantum Mechanics

QM Mueller

Sec 3 QM

4

Consider a particle of mass m in a one-dimensional potential $V(x)$ where



$V(x) = \infty \quad x<0$

$V(x) = 0 \quad 0<x<a$

$V(x) = V_0 \quad x>a$

with $V_0 > 0$.

(i) If $E = \frac{\hbar^2 k^2}{2m}$ is a bound state energy and $V_0 - E = \frac{\hbar^2 K^2}{2m}$, give the equation determining possible values of E.

(ii) Give the condition on V_0 and a for at least one bound state to exist.

(iii) What are the energy levels when $V_0 = \infty$

Solution:

(i) $0<x<a \quad u(x) = A \sin kx$

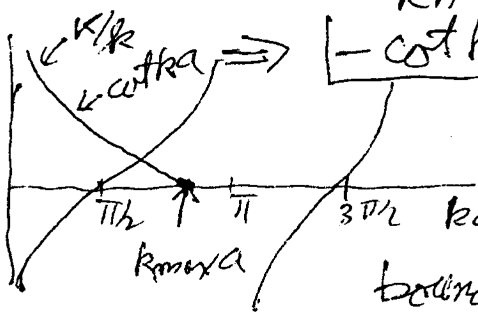
$a<x<\infty \quad u(x) = B e^{-Kx}$

match at $x=a$: $A \sin ka = B e^{-Ka}$

$kA \cos ka = -BK e^{-Ka}$

$\Rightarrow \boxed{-\cot ka = K/k}$

(ii)



$k_{max} = \sqrt{2mV_0}/\hbar$

$k_{max} a = \sqrt{2mV_0}\, a/\hbar$

bound state if $\boxed{\sqrt{2mV_0}\, a/\hbar > \pi/2}$

(iii) $V_0 = \infty \Rightarrow K = \infty \Rightarrow \cot ka = -\infty \Rightarrow ka = n\pi$

$\boxed{E_n = \left(\frac{n\pi\hbar}{a}\right)^2 \frac{1}{2m}} \quad n = 1, 2, 3, \cdots$

QM Ponton

Sec 3 QM

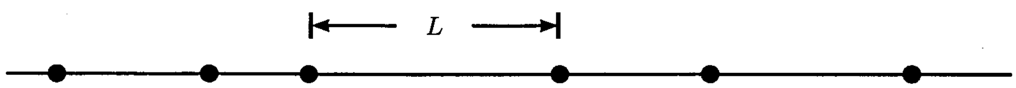
#5

(with corrected solution)

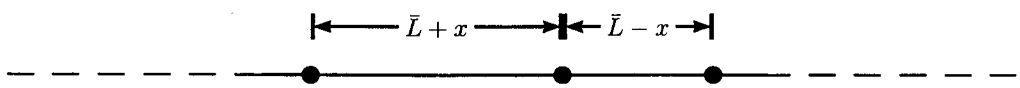
Qualifying exam 2010
Eduardo Pontón

1. **Quantum mechanics**

Consider a quantum system with an infinite set of particles in one dimension as shown in the figure. Particles cannot cross neighbors. We are interested in the probability distribution of spacings L between a particle and its neighbor to the left, given that the average spacing between particles is $\bar{L}$.



In the simplest approximation to this many-body problem, a single particle moves between two fixed neighbors separated by $2\bar{L}$. Let $x = L - \bar{L}$ denote the deviation from the midpoint.



(a) Find the probability distribution $P(L)$ in the ground state, in the above approximation, and assuming there are no interparticle interactions (other than contact interactions).

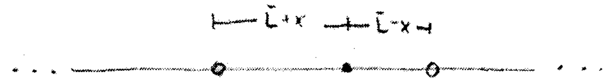
(b) Now suppose there are strong repulsive potentials between pairs of neighboring particles of the form AL^{-n}. Considering only the nearest neighbors, write the potential energy for the middle particle for $x \ll \bar{L}$. Write the result explicitly in terms of A, n and $\bar{L}$.

(c) Still assuming that $x \ll \bar{L}$ and that the neighboring particles are fixed, write the Schrödinger equation for the middle particle, and argue that the problem can be mapped into a familiar one. Based on this analogy, what is the form of the distribution $P(L)$? How does its *width* scale with $\bar{L}$?

(d) In the limit of strong repulsion, as in *b*) and *c*) above, explain how you can measure the power n and the amplitude A that characterize the potential.

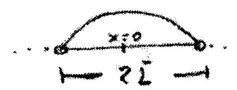
over →

Soln



a) Free particle with boundary conditions $\psi(-\bar{L}) = \psi(\bar{L}) = 0$

Ground state : $\psi_0(x) = N \sin \frac{\pi(\bar{L}+x)}{2\bar{L}}$



$$E_0 = \frac{\hbar^2}{2m} \cdot \left(\frac{\pi}{2\bar{L}}\right)^2 = \frac{\pi^2 \hbar^2}{2m\bar{L}^2}$$

$$N^{-2} = \int_{-\bar{L}}^{\bar{L}} dx \, \sin^2 \frac{\pi(\bar{L}-x)}{2\bar{L}} = \bar{L}$$

For spacing from particle to the left : $L = \bar{L} + x$

$$P(L) = |\psi_0(x)|^2 = \frac{1}{\bar{L}} \sin^2 \frac{\pi L}{2\bar{L}}$$

b) $$V(x) = A(\bar{L}+x)^{-n} + A(\bar{L}-x)^{-n}$$

$$= \frac{2A}{\bar{L}^n} \left\{ 1 + \tfrac{1}{2} n(n+1) \left(\frac{x}{\bar{L}}\right)^2 + O(x^3/\bar{L}^3) \right\}$$

$$= \frac{2A}{\bar{L}^n} + Cx^2 + \dots$$

$$C = \frac{n(n+1)A}{\bar{L}^{n+2}} \equiv \tfrac{1}{2} m\omega^2$$

c) Natural length scale (from m & ω) : $a = \left(\sqrt{\frac{m\omega}{\hbar}}\right)^{-1}$

Ground state for harmonic osc. $\propto e^{-\frac{x^2}{2a^2}}$ (Gaussian)

$\rightarrow$ width $\sim a \sim \omega^{-1/2} \sim \left(\frac{1}{\bar{L}^{\frac{n}{4}+\frac{1}{2}}}\right)^{-1}$

d) Measure width of $P(L)$ for different $\bar{L}$ (densities) to determine C as a function of $\bar{L}$ and fit for A and n.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 13, 2010
3:10 PM - 5:10 PM

Applied QM and Special Relativity
Section 4.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Applied QM and Special Relativity), Question 2; Section 4 (Applied QM and Special Relativity), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Applied QM and Special Relativity. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. The hole spectrum of $GaAs$ at $k = 0$ is four-fold degenerate at $k = 0$ (Γ point of the Brillouin zone). In the vicinity of this point the spectrum is described by the Luttinger Hamiltonian

$$\hat{H} = Ak^2\hat{\mathbf{I}} + B(\vec{\mathbf{k}} \cdot \hat{\vec{\mathbf{J}}}\,)^2$$

where $\hat{\mathbf{J}}_{x,y,z}$ are matrices of angular momentum $J = 3/2$ and $\hat{\mathbf{I}}$ is the unit matrix.

(a) Find the eigenvalues $\epsilon(k)$ of the Luttinger Hamiltonian.

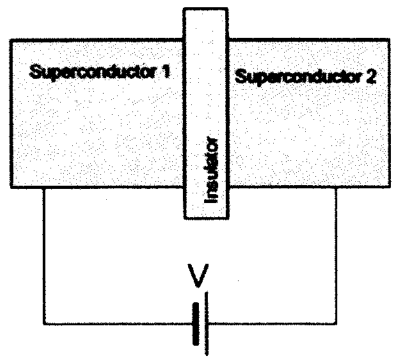
(b) The Luttinger Hamiltonian is spherically symmetric but the crystal has a cubic symmetry. Generalize the Luttinger Hamiltonian so it would have a cubic symmetry.

(c) If the crystal is deformed, the degeneracy at the Γ point can be partially lifted. What is the minimal possible degeneracy of the spectrum at the Γ point?

2. Two identical superconductors are separated by a thin insulator and connected to a battery whose DC voltage is given by V, as shown in the figure below. Let ψ_1 be the wave function of the condensed superconducting electron pairs on one side of the superconductor and ψ_2 be the wave function on the other side. The two wave functions are related to each other by the time dependent Schrödinger equation in the following way:

$$i\hbar\frac{\partial}{\partial t}\psi_1 = eV\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial}{\partial t}\psi_2 = -eV\psi_2 + K\psi_1$$



Here, the constant K is a characteristic of junctions, related to the tunneling process of the electron pairs across the insulator, and V is the voltage applied by the battery.

In this problem we express each wave function in terms of its corresponding condensation density and the phase of the wave function: $\psi_1 = \sqrt{n_1}e^{i\theta_1}$ and $\psi_2 = \sqrt{n_2}e^{i\theta_2}$, where n_1 and n_2 are the densities, and θ_1 and θ_2 are the phases of the condensate wave functions of superconductor 1 and 2, respectively.

(a) Assuming n_1 and n_2 are real, show that the current density of this junction is given by

$$J = \frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t} = J_0 \sin\delta$$

where $\delta = \theta_2 - \theta_1$. Find the expression for J_0 in terms of K, n_1, and n_2.

(b) Assume that initially the condensation densities are equal and large, and that the tunneling probability is small so that $n_1(t) \approx n_2(t)$. Show that the current density J derived in part (a) oscillates periodically over time. Find the frequency of the oscillation in terms of the applied DC voltage V.

3. A spinless particle of charge $-e$ and mass m is constrained to move in the x-y plane. There is a constant magnetic field $\vec{\mathbf{B}}$ along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $\mathrm{A}_x = -\mathrm{B}y$.

   (a) Write the expression for the Hamiltonian of one particle.

   (b) To find the solutions of the Schrödinger equation for the stationary states, consider wavefunctions

   $$\psi(x, y) = f(x)\phi(y)$$

   where

   $$f(x) = \exp\left[(i/\hbar)p_x x\right]$$

   and p_x is the x-component of momentum.

   Write the Schrödinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field $\vec{\mathbf{B}}$. What are the quantum numbers that correspond to a Landau level?

   (c) Assume that the area of the plane is given by the product of two lengths $L_x L_y$, that are along the x- and y-directions. Also assume that the function $f(x)$ satisfies the 'obvious' boundary condition

   $$f(x = 0) = f(x = L_x).$$

   Find the degeneracy of a Landau level as a function of the magnetic field for $L_x = L_y = L$.

4. A perpendicularly incident monochromatic plane wave is reflected from a mirror moving with a constant velocity along the line of propagation of the wave. Using Maxwell's equations, determine the angular frequency of the reflected wave as seen by a stationary observer.

5. In colliding beam detectors, K^0_{short} mesons can be detected through their decay to two charged pions

$$K^0_{\text{short}} \rightarrow \pi^+\pi^-$$

Cylindrical gas trackers composed of many wires in an argon gas volume located inside a solenoidal magnet can detect the ionization trail left by the pions and measure their vector momenta.

The lifetime of the K^0_{short} is 0.89×10^{-10} s and the mass is 498 MeV. (The mass of the charged pion is 140 MeV.)

For the following questions, assume that the energy of the K^0_{short} in the laboratory frame of the detector is 60 GeV.

(a) What is the minimum opening angle in the lab frame of the two pions from the K^0_{short} decay?

(b) How far, on average, does the K^0_{short} go before decaying into two pions?

(c) How far, on average, would the K^0_{short} go before interacting with an argon atom in the gas if the cross section for K+p or K+n interactions is about 20 millibarns (1barn = 10^{-28} m^2)? (The density of argon gas is 1.8×10^{-3} g/cm^3.)

(d) The K^0_{long} has a lifetime of 5.17×10^{-8} s and a substantial fraction (38.7%) decay as

$$K^0_{\text{long}} \rightarrow \pi^{\pm}e^{\pm}\nu_e$$

From this information, what branching fraction would you predict for the decay

$$K^0_{\text{short}} \rightarrow \pi^{\pm}e^{\pm}\nu_e$$

AQM Aleiner
Sec. 4
Rel + Applied QM
1

The hole spectrum of $GaAs$ at $k = 0$ is four-fold degenerate at $k = 0$ (Γ point of the Brillouin zone). In the vicinity of this point the spectrum is described by the Luttinger Hamiltonian

$$\hat{H} = Ak^2\hat{I} + B(\vec{k} \cdot \hat{\vec{J}})^2$$

where $\hat{J}_{x,y,z}$ are the matrices of angular momentum $J = 3/2$ and $\hat{I}$ is the unit matrix.

1. Find the eigenvalues $\epsilon(k)$ of the Luttinger Hamiltonian.
2. The Luttinger Hamiltonian is spherically symmetric and the crystal has a cubic symmetry. Generalize the Luttinger Hamiltonian so it would have a cubic symmetry as well.
3. If the crystal is deformed, the degeneracy at Γ point can be partially lifted. What is the minimal possible degeneracy of the spectrum in Γ point?

Solution:

1. Choose direction of k as z-axis. Then

$$\epsilon(k; J_z = \pm 1/2) = k^2(A + B/4); \quad \text{light holes}$$

and

$$\epsilon(k; J_z = \pm 3/2) = k^2(A + 9B/4); \quad \text{heavy holes.}$$

2.

$$\hat{H} = Ak^2\hat{I} + B(\vec{k} \cdot \hat{\vec{J}})^2 + C(k_x^2\hat{J}_x^2 + k_y^2\hat{J}_y^2 + k_z^2\hat{J}_z^2)$$

3. As the electron has spin 1/2 and the time reversal symmetry is not broken the minimal degeneracy in Γ-point is two because of the Kramers theorem.

AQM Kim

Sec. 4
Relativity + Applied QM
#2

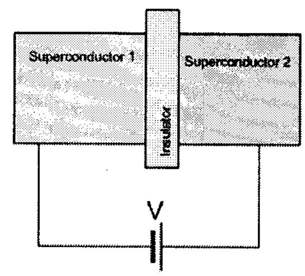
Applied QM

DC Josephson superconductor tunneling

Two identical supercondutors are separated by a thin insulator and connected to a battery whose DC voltage is given by V as shown in the figure below. Let ψ_1 be the wave function of the condensed superconducting electron pairs in one side of super conductor and ψ_2 be the wave function of the other side. The two wave functions are related to each other by the time-dependent Schrödinger equation in the following way:

$$i\hbar\frac{\partial}{\partial t}\psi_1 = eV\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial}{\partial t}\psi_2 = -eV\psi_2 + K\psi_1$$



Here, the constant K is a characteristic of junctions related to the tunneling process of the electron pairs across the insulator and V is voltage applied by the battery outside.

In this problem we express each wav function in its corresponding condensation density and the phase of wave function: $\psi_1 = \sqrt{n_1}e^{i\theta_1}$ and $\psi_2 = \sqrt{n_2}e^{i\theta_2}$ where n_1 and n_2 are the density of condensate and θ_1 and θ_2 are the phase of the condensate wave functions of superconductor 1 and 2, respectively.

(a) Considering n_1 and n_2 are real, show that the current density of this junction defined is given by

$$J = \frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t} = J_0 \sin\delta$$

where $\delta = \theta_2 - \theta_1$. Find the expression of J_0 in terms of K and n_1 and n_2.

(b) We assume that initially the condensation densities are equal and large, and further assume that the tunneling probability is small so that$_2 n_1 \approx n$ for all time.S how that the current density J derived above oscillates periodically over time. Find the frequency of the oscillation in terms of the applied DC voltage V.

Applied QM

DC Josephson tunneling

(a) $\psi_1 = \sqrt{n_1}\, e^{i\theta_1} \Rightarrow \frac{\partial}{\partial t}\psi_1 = \frac{1}{2}\frac{\dot{n}_1}{\sqrt{n_1}}e^{i\theta_1} + i\dot{\theta}_1\sqrt{n_1}e^{i\theta_1}$

$\psi_2 = \sqrt{n_2}\, e^{i\theta_2} \Rightarrow \frac{\partial}{\partial t}\psi_2 = \frac{1}{2}\frac{\dot{n}_2}{\sqrt{n_2}}e^{i\theta_2} + i\dot{\theta}_2\sqrt{n_2}e^{i\theta_2}$

From the coupled eqn.

$$\frac{i\hbar}{2}\frac{\dot{n}_1}{\sqrt{n_1}}e^{i\theta_1} - \hbar\dot{\theta}_1\sqrt{n_1}e^{i\theta_1} = \frac{2eV}{2}\sqrt{n_1}e^{i\theta_1} + K\sqrt{n_2}e^{i\theta_2}$$

or

$$\frac{i\hbar}{2}\frac{\dot{n}_1}{n_1} - \hbar\dot{\theta}_1 = eV + K\sqrt{\frac{n_2}{n_1}}e^{i\delta} \quad —①$$

Like wise we have

$$i\hbar\frac{\dot{n}_2}{n_1} - \hbar\dot{\theta}_2 = -eV + K\sqrt{\frac{n_1}{n_2}}e^{-i\delta} \quad —②$$

Considering n_1, n_2, θ_1 & θ_2 are all real function, from the imaginary part of ① & ② we have

$$\frac{\hbar}{2}\frac{\dot{n}_1}{n_1} = K\sqrt{\frac{n_2}{n_1}}\sin\delta, \quad \frac{\hbar}{2}\frac{\dot{n}_2}{n_2} = -K\sqrt{\frac{n_1}{n_2}}\sin\delta$$

$$\Rightarrow \boxed{J = \dot{n}_1 = -\dot{n}_2 = \underbrace{\frac{2}{\hbar}K\sqrt{n_1 n_2}}_{= J_0}\sin\delta}$$

(b) From the real part of ① & ②, & $n_1 \simeq n_2$

$$-\hbar\dot{\theta}_1 = eV + K\sqrt{\frac{n_2}{n_1}}\cos\delta \simeq eV + K\cos\delta$$

$$\Rightarrow \frac{d\delta}{dt} = \dot{\theta}_2 - \dot{\theta}_1 = \frac{2e}{\hbar} V$$

or $$\boxed{\delta(t) = \delta_0 + \frac{2eV}{\hbar} t}$$

From (a)

$$J(t) = \frac{2}{\hbar} K \sqrt{n_1 n_2} \sin\left[\delta_0 + \underbrace{\frac{2eV}{\hbar}}_{=\omega} t\right]$$

$$\Rightarrow \boxed{\hbar\omega = 2eV}$$

//

Pinczuk
Sec. 4 Rel + Appl. QM
#3

General-Section 4: applied quantum mechanics

A spin-less particle of charge –e and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

(a) write the expression for the Hamiltonian of one particle.

(b) to find the solutions of the Schroedinger equation for the stationary states consider wavefunctions

$\psi(x,y)= f(x)\phi(y)$

where

$f(x)= \exp[(i/\hbar)p_x x]$

and p_x is the x-component of momentum.

Write the Schroedinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field B.
What are the quantum numbers that correspond to a Landau level?

(c) Assume that the area of the plane is given by the product of two lengths L_xL_y, that are along the x- and y-directions.
Also assume that the function f(x) satisfies the 'obvious' boundary condition

$f(x=0)=f(x=L_x)$

Find the degeneracy of a Landau level as function of magnetic field for $L_x=L_y=L$.

Solution

(a) $$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

$\vec{A} = (A_x, 0, 0) \qquad A_x = -By$

$$H = \frac{1}{2m}\left(p_x + \frac{e}{c}By\right)^2 + \frac{1}{2m}p_y^2$$

(b) $$\frac{d^2\phi(y)}{dy^2} + \frac{2m}{\hbar^2}\left[E_n - \frac{1}{2}m\omega_c^2(y-y_0)^2\right]\phi(y) = 0$$

$$\omega_c = \frac{eB}{mc} \qquad |y_0| = \frac{c\,p_x}{eB}$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega_c$$

(c) From the boundary condition

$e^{(i/\hbar)p_x L} = 1$; which implies

$$p_x = m\,2\pi\frac{\hbar}{L}, \quad m = \text{integer}$$

$$y_0 = 2\pi\frac{c\hbar}{eBL}m \le L$$

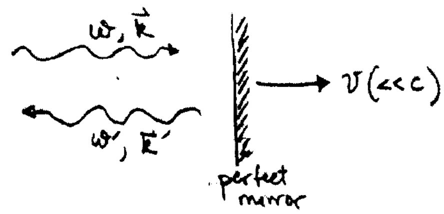
The number of states is equal to m for the maximum value $y_0 = L$

$$N = (hc/eB)A \quad ; A = L^2$$

Marka
Sec. 4 Rel + App QM
4

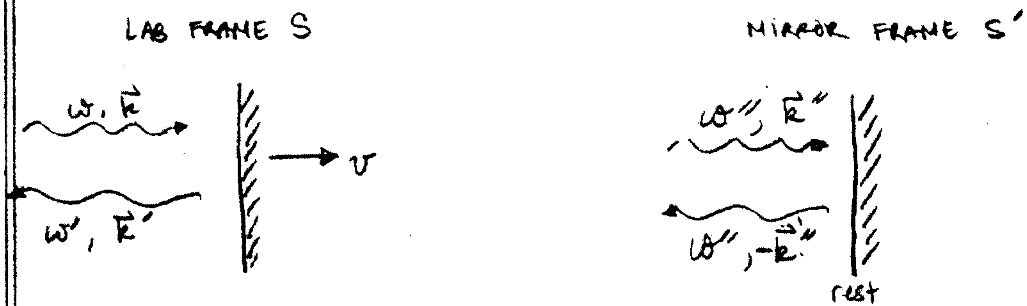
#5: A perpendicularly incident monochromatic plane wave is reflected from a mirror moving with a constant velocity along the line of the propagation of the wave. Using Maxwell's equations, determine the angular frequency of the reflected wave as seen by a stationary observer.

Solution from Gabe-Perez Giz:



This problem has a relativistic solution and can be done with Lorentz transformations. The hint, however, suggests a solution based on the principles of applying BCs at the interface b/w media. First, I'll do the relativistic solution with 4-vectors (it was not expected that you would do the problem this way), and then I'll redo the problem by applying boundary conditions to the waves. Remarkably (or perhaps not remarkably), we will get the same exact answer, and thus the same 1st order approximation (or n^{th} order approximation, for that matter), as well.

SOLUTION W/ LORENTZ TRANSFORMATIONS



In the rest frame of the mirror, we know what happens b/c it is a standard reflection problem of the sort we've seen many times: the incident and reflected waves have the same frequency (which I've called ω'') and their wavevectors differ only in direction (opposite directions, but $|\vec{k}''| = |-\vec{k}''| = \omega''/c$).

To translate this into the frequencies in the lab frame, it's useful to know that ω and $\vec{k}$ form a 4-vector:

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$$ ... so the 4-vector can be written compactly as $(\omega/c, \vec{k})$.

<u>NOTE:</u> You can somewhat understand <u>WHY</u> ω/c & $\vec{k}$ make a 4-vector by thinking of individual photons, for which $\begin{pmatrix} \hbar\omega/c \\ \hbar\vec{k} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$ = 4-momentum of the photon, which is a 4-vector. So $\begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} \equiv k^{\mu} = \frac{1}{\hbar} p^{\mu}$ and is itself a 4-vector.

Strategy: write down $k^{\mu}_{incident}$ in the lab frame. Get $k^{\mu'}_{incident}$ in S' via Lorentz transformation; get $k^{\mu'}_{ref}$ in S' by inspection; get

k^{μ}_{ref} in LAB frame by inverse Lorentz transformation. Then read off the frequency component of k^{μ}_{ref}.

$$k^{\mu}_{inc} = \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ |\vec{k}| \end{pmatrix}$$ if the waves and mirror all move in the $\hat{z}$ direction.

By the way, $|\vec{k}| = \omega/c$ for electromagnetic waves propagating in vacuum, so we write k^{μ}_{inc} more compactly as

$$k^{\mu}_{inc} = \omega/c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, do a Lorentz transformation to the components of this 4 vector in the mirror frame:

$$\underbrace{k^{\mu'}_{inc}}_{\text{mirror frame}} = \omega/c \begin{pmatrix} \gamma[1 - \frac{v}{c}(1)] \\ 0 \\ 0 \\ \gamma[1 - \frac{v}{c}(1)] \end{pmatrix} = \omega/c\, \gamma \begin{pmatrix} 1 - v/c \\ 0 \\ 0 \\ 1 - v/c \end{pmatrix}$$

By arguments made on the prior page, we now know $k^{\mu'}_{ref}$ by inspection: same ω, opposite $\vec{k}$ ...

$$k^{\mu'}_{ref} = \frac{\omega}{c}\gamma \begin{pmatrix} 1 - \frac{v}{c} \\ 0 \\ 0 \\ \frac{v}{c} - 1 \end{pmatrix}$$

Now, do an <u>inverse</u> Lorentz transformation to get the components of k^{μ}_{ref} in the LAB frame:

$$k^{\mu}_{ref} = \frac{\omega}{c}\gamma \begin{pmatrix} \gamma[(1-\frac{v}{c}) + \frac{v}{c}(\frac{v}{c}-1)] \\ 0 \\ 0 \\ \gamma[(\frac{v}{c}-1) + \frac{v}{c}(1-\frac{v}{c})] \end{pmatrix} = \frac{\omega}{c}\gamma^2 \begin{pmatrix} (1-\frac{v}{c})^2 \\ 0 \\ 0 \\ -(1-\frac{v}{c})^2 \end{pmatrix}$$

The timelike component of k^μ_{ref} IS the frequency of the reflected wave in the lab frame, so we get

$$\frac{\omega'}{c} = \frac{\omega}{c} \underbrace{\gamma^2}_{\frac{1}{1-(v/c)^2}} \left(1-\frac{v}{c}\right)^2 = \frac{\omega}{c} \frac{\left(1-\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2} = \frac{\omega}{c} \frac{(1-v/c)(1-v/c)}{(1-v/c)(1+v/c)}$$

$$= \frac{\omega}{c} \frac{1-v/c}{1+v/c} \Rightarrow \boxed{\omega' = \omega \frac{1-v/c}{1+v/c}}$$

We're told $v \ll c$, so we want to expand to lowest order in v/c. Use a binomial expansion: $(1+x)^n \simeq 1 + nx$ to 1st order. Thus,

$$\left(1+\frac{v}{c}\right)^{-1} \simeq 1 - \frac{v}{c} \quad \text{to 1st order}, \quad \text{so}$$

$$\omega' \simeq \omega\left(1-\frac{v}{c}\right)\left(1-\frac{v}{c}\right) = \omega\left(1 - 2\frac{v}{c} + \underbrace{\frac{v^2}{c^2}}_{\substack{\text{neglect b/c} \\ \text{we are only} \\ \text{keeping lowest order in } v/c}}\right) = \boxed{\omega\left(1-2\frac{v}{c}\right)}$$

SOLUTION w/ PLANE WAVES and BC's

Forget relativity... let's just say you have an incident and a reflected wave (but no transmitted wave, since the mirror is, say, a perfect conductor) of <u>different</u> <u>frequencies</u> at the mirror surface. Since the waves are traveling in the z-direction and normally incident on the mirror, then the waves are polarized parallel to the surface of the mirror. For simplicity, let's say the incident wave has a single polarization (and ergo, from HW3, the reflected wave has the same polarization), and called the direction of polarization the $\hat{x}$ direction for ease.

Let's match BC's at the surface of the mirror:

$$\tilde{\vec{E}}_I = \tilde{E}_{0_I}\, e^{i(kz-\omega t)}\,\hat{x} \qquad \tilde{\vec{E}}_T = 0$$
$$\tilde{\vec{E}}_R = \tilde{E}_{0_R}\, e^{i(-k'z-\omega' t)}\,\hat{x}$$

<u>NOTE</u> that the waves have different ω's and thus must have different k's (since $\frac{\omega}{k} = \frac{\omega'}{k'} = c$). Also, <u>NOTE</u> that I've put a $-k'$ in the reflected wave to capture the fact that it moves in the negative z-direction.

There is no component of $\tilde{\vec{E}}$ normal to the surface of the boundary b/c $\tilde{\vec{E}}$ is polarized $\parallel$ boundary. Also, the BC's on $\tilde{\vec{B}}$ will give no new information beyond what the $\tilde{\vec{E}}$ BC's give since all the nonzero waves are in the same medium (vacuum)

$$\Rightarrow \text{BC}: \ \tilde{E}_{0_I}\, e^{i(kz-\omega t)} + \tilde{E}_{0_R}\, e^{i(-k'z-\omega' t)} = 0$$

At the boundary and for all time.

Say the mirror passes the $z=0$ plane at $t=0$. Then the boundary is at $\underline{z = vt}$. Insert that above and also write $k = \omega/c$ and $k' = \omega'/c$:

$$\Rightarrow \tilde{E}_{0_I}\, e^{i(\omega \frac{v}{c} t - \omega t)} + \tilde{E}_{0_R}\, e^{i(-\omega' \frac{v}{c} t - \omega' t)} = 0$$

Now, this needs to hold <u>FOR ALL</u> t. Following the logic we've used repeatedly in class and HW for applying BC's, we see that this can only happen (since $\tilde{E}_{0_I}$ and $\tilde{E}_{0_R}$ do not depend on t) if the arguments of the exponentials are <u>identical</u>:

$$\frac{\omega v}{c}\cancel{t} - \omega \cancel{t} = -\frac{\omega' v}{c}\cancel{t} - \omega' \cancel{t}$$

$$\omega\left(\frac{v}{c} - 1\right) = -\omega'\left(\frac{v}{c}+1\right)$$

$$\Rightarrow \omega' = \omega \frac{1 - v/c}{1 + v/c} \simeq \omega\left(1 - 2\frac{v}{c}\right), \text{ exactly as before!}$$

Pretty awesome way to do this problem, no?

Relativity - Shaevitz
Sec 4 Eel & Appl QM

Quals Problem 2 – Relativity

M. Shaevitz
Fall, 2009

#5

In colliding beam detectors, K^0_{short} mesons can be detected through their decay to two charged pions

$$K^0_{short} \to \pi^+\pi^-$$

Cylindrical gas trackers composed of many wires in an argon gas volume located inside a solenoidal magnet can detect the ionization trail left by the pions and measure their vector momenta.

The lifetime of the K^0_{short} is 0.89×10^{-10} s and the mass is 498 MeV. (The mass of a charged pion is 140 MeV.)

For the following questions, assume that the energy of the K^0_{short} in the laboratory frame of the detector is 60 GeV.

a) What is the minimum opening angle in the lab frame of the two pions from the K^0_{short} decay?

b) How far on average does the K^0_{short} go before decaying into the pions?

c) How far on average would a K^0_{short} go before interacting with an argon atom in the gas if the cross section for K+p or K+n interactions is about 20 millibarns? (The density of argon gas is 1.8×10^{-3} g/cm^3.)

d) The K^0_{long} has a lifetime of 5.17×10^{-8} s and a substantial (38.7 %) decay fraction to

$$K^0_{long} \to \pi^\pm e^\mp \nu_e$$

From this information, what branching fraction would you predict for the

$$K^0_{short} \to \pi^\pm e^\mp \nu_e$$

Solution:

a) Minimum opening angle when $\theta_{cm} = 90°$

$$\gamma_{K^0} = \frac{E_{K^0}}{M_{K^0}} = \frac{60\,GeV}{0.498\,GeV} = 120.5 \qquad \beta_{K^0} \approx 1$$

In K^0 rest frame for $\theta_{cm} = 90°$ $\Rightarrow$ $E^{\pi} = \frac{m_{K^0}}{2}$ $\quad P_{\perp}^{\pi} = \left(\left(\frac{m_{K^0}}{2}\right)^2 - m_{\pi}^2\right)^{1/2}$

Boost to lab

$$P_{\perp}^{LAB} = P_{\perp}^{cm} = \left(\left(\frac{m_{K^0}}{2}\right)^2 - m_{\pi}^2\right)^{1/2} = 0.206\,GeV$$

$$P_{\parallel}^{LAB} = \gamma\left(E_{cm}^{\pi} + P_{\parallel\,cm}^{\pi}\right) = \frac{E_{K^0}}{m_{K^0}}\left(\frac{m_{K^0}}{2}\right) = \frac{E_{K^0}}{2}$$

$$= 30\,GeV$$

$$\tan\theta = \frac{0.206\,GeV}{30\,GeV} = 0.00687 \Rightarrow \theta = 0.00687 = 6.9\,mr$$

$$\theta_{opening} = 2\theta = 13.7\,mr$$

b) $d = \gamma c \tau = (120.5)(3\times10^{8}\,m/s)(0.89\times10^{-10}\,s)$

$$= 3.21\,m$$

c) $$L_{int} = \frac{1}{\rho\sigma N_{avg}} = \frac{1}{(1.8\times10^{-3}\,g/cm^3)(2\times10^{-26}\,cm^2)(6.02\times10^{23}/g)}$$

$$= 461\,m$$

d) $BR_{Ke3}^{Kshort} = \frac{\Gamma_{Ke3}}{\Gamma_{TOTAL}^{Kshort}}$ $\qquad BR_{Ke3}^{KLONG} = \frac{\Gamma_{Ke3}}{\Gamma_{TOTAL}^{KLONG}} \Rightarrow \Gamma_{Ke3} \propto \frac{BR_{Ke3}^{KLONG}}{\tau_{KLONG}}$

$$\Gamma_{TOTAL}^{Kshort} \propto \frac{1}{\tau_{Kshort}}$$

$$BR_{Ke3}^{Kshort} = \frac{BR_{Ke3}^{KLONG}\cdot\tau_{Kshort}}{\tau_{KLONG}} = \frac{(0.387)(0.89\times10^{-10}\,s)}{(5.17\times10^{-8}\,s)} = 6.7\times10^{-4}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 15, 2010
1:00 PM - 3:00 PM

General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 6 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 or 6 problems, the highest scoring problem grade(s) will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5 (General Physics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\ 1/2 \times 11"$ paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

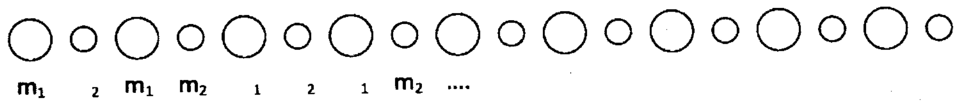
1. The Crab pulsar has a rotational period $p = 33$ msec and a period derivative of $\dot{p} = 4 \times 10^{-13}$.

   (a) Making appropriate estimates of the mass M and radius R for the Crab pulsar, estimate the current luminosity of the Crab pulsar.

   (b) The Crab pulsar emits via magnetic dipole radiation, $L_{\mathrm{m}} = \mathrm{k}\omega^4$. Assuming that the Crab pulsar was born with an initial period $p_i \ll p_{\mathrm{now}}$ ($\omega_i \gg \omega_{\mathrm{now}}$). Use the current values of p and $\dot{p}$ to estimate the age of the Crab pulsar.

   (c) Make a dimensional estimate of the luminosity of the Crab pulsar, due to magnetic dipole radiation, in terms of the magnetic field B, and other pulsar parameters. Use this estimate of the luminosity, along with the L determined in (a), to estimate the magnetic field of the Crab pulsar.

2. Imagine a one-dimensional chain of N atoms (lattice spacing 'a') where alternate atoms have different masses as pictured below:



Assume that the two masses are nearly equal:

$$m_1 = m(1+\Delta) \qquad\qquad m_2 = m(1-\Delta)$$

where $\Delta \ll 1$.

Solve for the normal modes (phonons) of the chain by the following steps:

(a) First solve for the case $\Delta = 0$ (equal masses) as follows. The Hamiltonian of the system is given by

$$H = \sum_n \frac{p_n^2}{2m} + \frac{1}{2} m\omega_0^2 \sum_n (x_n - x_{n+1})^2$$

Here p_n is the momentum of the n^{th} atom and x_n is its displacement from its equilibrium position $X_n = na$. The potential energy is thus determined by the relative position of the nearest neighbors. This can be solved by changing variables to Fourier space:

$$x_n = \sum_k x_k e^{ikna} \quad \text{and} \quad p_n = \sum_k p_k e^{-ikna}$$

with $k = 0, \pm\frac{\pi}{Na}, \pm\frac{2\pi}{Na}, \cdots, \pm\frac{\pi}{a}$.

Show that in Fourier space the Hamiltonian reduces to

$$H = \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_k^2 \sum_k x_k x_{-k}$$

Find ω_k, the dispersion relation between the energy ω and momentum k. Sketch your result. What is this kind of phonon called?

(b) Now solve for the case of unequal masses by expanding the Hamiltonian to first order in Δ. The zeroth order in Δ results in the phonon mode you have found in part (a). What is the first order Hamiltonian?

(c) Solve the first order Hamiltonian you found in (b) in exactly the same way as you did in (a). What is the new dispersion relation? Sketch the results for the two modes. What is the new phonon mode called?

3. In many experiments, the surface of the sample or detector being used has to be placed in a vacuum environment to avoid contamination from air molecules.

   (a) Estimate the pressure in a vacuum chamber (in atmospheres) where one air molecule hits every surface atom of the walls of the chamber every second. Assume that air is composed of only nitrogen molecules (molecular weight 28) that travel at 500 m/s. Assume also that a typical atom on the wall of the chamber has a size of 1 Angstrom (1 atm $= 1.013 \times 10^5$ Pa).

   (b) Such low pressures are reached by the use of vacuum pumps. A vacuum pump operates by displacing a certain volume C per second from the chamber which is then exhausted externally (imagine a chamber where the volume of the chamber is continuously increased by C per second, resulting in a continuous drop in pressure). How long will it take a vacuum pump with a displacement of 1 liter per second to reduce the pressure in a 100 liter chamber from atmosphere to the pressure required in (a)? Assume that temperature is held constant throughout.

4. A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density and velocity fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$
$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p.$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho).$$

(a) Linearize the equations of motion above, for small fluctuations $\delta\rho$ and $\delta\vec{v}$, about the homogeneous, static background configuration

$$\rho = \rho_0\,, \qquad \vec{v} = 0\,.$$

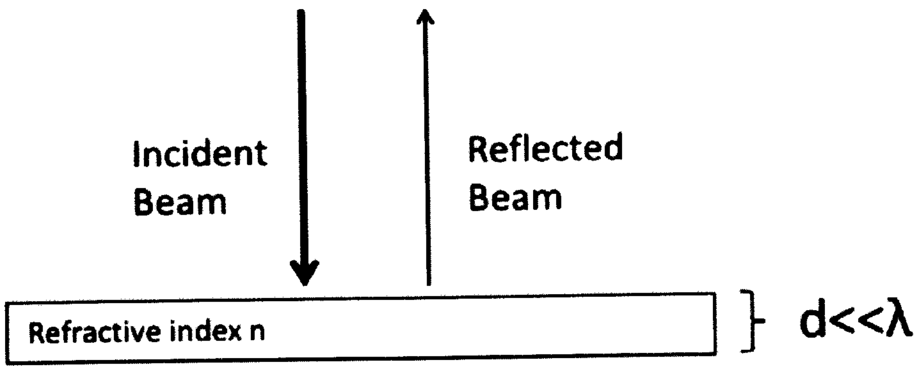
(b) Consider plane wave-like configurations for $\delta\rho$ and $\delta\vec{v}$:

$$\delta\rho(\vec{x},t) = \delta\rho_*(t)e^{i\vec{k}\cdot\vec{x}} + \text{c.c}\,, \qquad \delta\vec{v}(\vec{x},t) = \delta\vec{v}_*(t)e^{i\vec{k}\cdot\vec{x}} + \text{c.c}$$

Solve the linear equations you derived in part (a) for $\delta\rho_*(t)$ and $\delta\vec{v}_*(t)$.
Hint: decompose $\delta\vec{v}$ into transverse and longitudinal parts.

(c) What do these solutions describe physically?

5. We wish to detect the presence of a thin membrane suspended in vacuum by reflection of a light beam impinging at normal incidence. Model the material as a thin slab of homogeneous, transparent material with a refractive index n and a thickness d.



(a) Find an *explicit* expression for the reflectance R of the slab in the limit of $d \ll \lambda$, where λ is the vacuum wavelength of light.

(b) Estimate the minimum effective thickness of a membrane that could, in principle, be detected in this fashion. Assume typical parameters for a dielectric material, that we have available a 1 μW visible laser, and that we are able to detect $10^9/s$ photons of reflected light. Use the relation derived above or, if unavailable, a suitable approximate expression.

6. Consider the rotational degree of freedom of a dilute gas of diatomic CO molecules at temperature T. Suppose that the moment of inertia of each molecule is I.

   (a) Write an explicit expression for the (quantum) partition function Z_{rot} for the rotational degree of freedom of one molecule. Although you may not be able to reduce it to closed form, make sure that all quantities in Z_{rot} are defined so that it could be evaluated numerically.

   (b) Write a general expression for the heat capacity per molecule associated with rotational motion in terms of Z_{rot}.

   (c) Obtain an analytic expression for the asymptotic behavior of the rotational contribution to the heat capacity per molecule in the limit of low temperature.

   (d) For CO molecules, approximately how low does the temperature have to be so that the relation derived in part (c) is applicable. Use suitable estimates of the relevant physical parameters.

Astro Hailey
Sect 5
General I

Hailey : general : astrophysics

#1

The Crab pulsar has a rotational period $P = 33$ msec And a period derivative of $\dot{P} = 4 \times 10^{-13}$ s/s

a.) MAking appropriate estimates of M And R for the Crab pulsar, estimate the current luminosity of the Crab pulsar.

b.) The Crab pulsar emits via magnetic dipole radiation, $L_m \propto K\omega^4$. Assuming that the Crab pulsar was born with an initial period $P_i << P_{now}$ ($\omega_i >> \omega_{now}$), use the current values of P And $\dot{P}$ to estimate the age of the Crab pulsar.

c.) MAke a dimensional estimate of the luminosity of the Crab pulsar ~~in terms~~ due to magnetic dipole radiation in terms of B And other pulsar parameters. Use your estimate of L from (a) along with the L determined in (c) to estimate the magnetic field of the Crab pulsar.

Solution : Hailey : Astrophysics : general

a.) $\dot{E} = \frac{d}{dt}(\frac{1}{2} I \omega^2) = I\omega\dot{\omega}$

$\omega = 2\pi/P \; ; \; \dot{\omega} = \frac{2\pi}{P^2}\dot{P}$

$\omega \simeq 200 \, \frac{rad}{s} \quad \dot{\omega} \simeq 2 \times 10^{-9} \, rad/s$

This is a neutron star so $M \simeq 1 M_\odot \sim 2 \times 10^{33} g$

$R \simeq 10 \, Km \simeq 10^6 \, cm$

$I \sim (2/5) M R^2 \simeq 1.4 \times 10^{45} \, g\text{-}cm^2$

$\dot{E} \simeq 6 \times 10^{38} \, ergs/s$ Ans

b.) $L = I\omega\dot{\omega} = -K\omega^4$

$$\int_{\omega_i}^{\omega} \frac{d\omega}{\omega^3} = \int - \frac{dt \, K}{I}$$

$$-Kt/I = -\left(\frac{1}{\omega^2} - \frac{1}{\omega_i^2}\right)\frac{1}{2} \qquad \omega_i >> \omega$$

$$\omega = \sqrt{\frac{I}{K}} \, t^{-1/2} \; ; \; \dot{\omega} = \sqrt{\frac{I}{K}} \, \frac{1}{2} t^{-3/2}$$

$$\dot{\omega}/\omega = \frac{1}{2t} \qquad t = \frac{1}{2(\dot{\omega}/\omega)} \simeq 1600 \, yrs$$

Ans

c.) Magnetic dipole radiation $L_m \sim \frac{(\ddot{M})^2}{c^3}$

$M_0 \sim \frac{IA}{c} \qquad \int B \, dl \sim BR \approx \frac{4\pi}{c} I$

$M_0 \sim \frac{B}{4\pi} 4\pi R^2 R \qquad \frac{(\ddot{M})^2}{c^3} \simeq \frac{\omega^4 (BR^3)^2}{c^3} \rightarrow$

$$L_m \simeq \frac{B^2 R^6 \omega^4}{c^3}$$

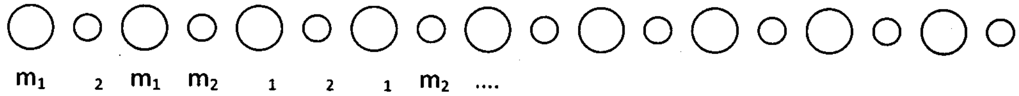
From (a) $L \simeq 6 \times 10^{38} \sim \frac{B^2 R^6}{c^3} \omega^4$

Solving $B \simeq 2 \times 10^{12}$ gauss Ans 2

CM - Pasupathy
Sec. 5 general I
2
condensed matter

Condensed Matter

Imagine a one-dimensional chain of atoms where alternate atoms have different masses as pictured below:



Assume that the two masses are nearly equal:

$$m_1 = m(1+\Delta)$$

$$m_2 = m(1-\Delta)$$

where $\Delta \ll 1$.

Solve for the normal modes (phonons) of the chain by the following steps:

1. First solve for the case $\Delta = 0$ (equal masses) as follows. The Hamiltonian of the system is given by

$$H = \sum_n \frac{p_i^2}{2m} + \frac{1}{2} m\omega_0^2 \sum_n (x_n - x_{n+1})^2$$

here p_n and x_n are the momentum and position of the n^{th} atom in the chain. The potential energy is thus determined by the relative position of the nearest neighbors. This can be solved by changing variables to Fourier space

$$x_n = \sum_k x_k e^{ikn}$$

$$p_n = \sum_k p_k e^{-ikn}$$

Show that in Fourier space the Hamiltonian reduces to

$$H = \sum_n \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_k^2 \sum_n x_k x_{-k}$$

Find ω_k, the relationship (dispersion) between the energy ω and momentum k. Sketch your result. What is this kind of phonon called?

2. Now solve for the case of unequal masses by expanding the Hamiltonian to the first power of Δ. The zeroth order in Δ results in the phonon mode you have found in (1). What is the first order Hamiltonian?

3. Solve this first order Hamiltonian exactly as in (1). What is the new dispersion relation? Sketch the results for the two modes. What is the new phonon mode called?

Solution

$$H = \sum_n \frac{p_i^2}{2m} + \frac{1}{2} m\omega_0^2 \sum_n (x_n - x_{n+1})^2$$

use

$$x_n = \sum_k x_k e^{ikn}$$

$$p_n = \sum_k p_k e^{-ikn}$$

$$H = \sum_{n,k,k'} \frac{p_k e^{-ikn} p_{k'} e^{-ik'n}}{2m} + \frac{1}{2} m\omega_0^2 \sum_n \left(\sum_k x_k e^{ikn} - \sum_k x_k e^{ik(n+1)} \right)^2$$

Use the fact that

$$\sum_n e^{-ikn} e^{-ik'n} = \delta(k + k')$$

to simplify:

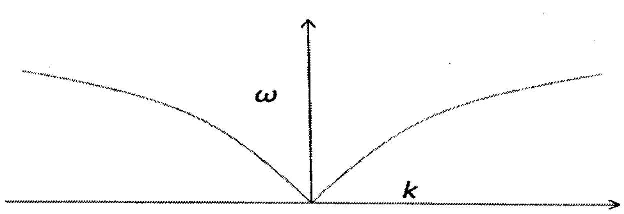
$$H = \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_0^2 \sum_{n,k,k'} 2\left(x_k e^{ikn} x_{k'} e^{ik'n} - x_k e^{ikn} x_{k'} e^{ik'(n+1)}\right)$$

$$= \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_0^2 \sum_{n,k,k'} 2x_k x_{-k}(1 - \cos(k))$$

$$= \sum_k \frac{p_k p_{-k}}{2m} + \frac{1}{2} m\omega_k^2 \sum_k x_k x_{-k}$$

where

$$\omega_k = \omega_0 \sqrt{(1 - \cos(k))} = \omega_0 \left| \sin\left(\frac{k}{2}\right) \right|$$



ACOUSTIC MODE

(2)

$$H = \sum_{odd\ n} \frac{p_i^2}{2m(1+\Delta)} + \frac{1}{2}m(1+\Delta)\omega_0^2 \sum_{odd\ n}(x_n - x_{n+1})^2$$
$$+ \sum_{even\ n} \frac{p_i^2}{2m(1-\Delta)} + \frac{1}{2}m(1-\Delta)\omega_0^2 \sum_{even\ n}(x_n - x_{n+1})^2$$

$$= \sum_n \frac{p_i^2}{2m}(1+(-1)^n\Delta) + \frac{1}{2}m\omega_0^2 \sum_{odd\ n}(x_n - x_{n+1})^2(1+(-1)^n\Delta)$$

$$= H_0 + H_\Delta$$

Here H_0 is the zeroth order Hamiltonian and

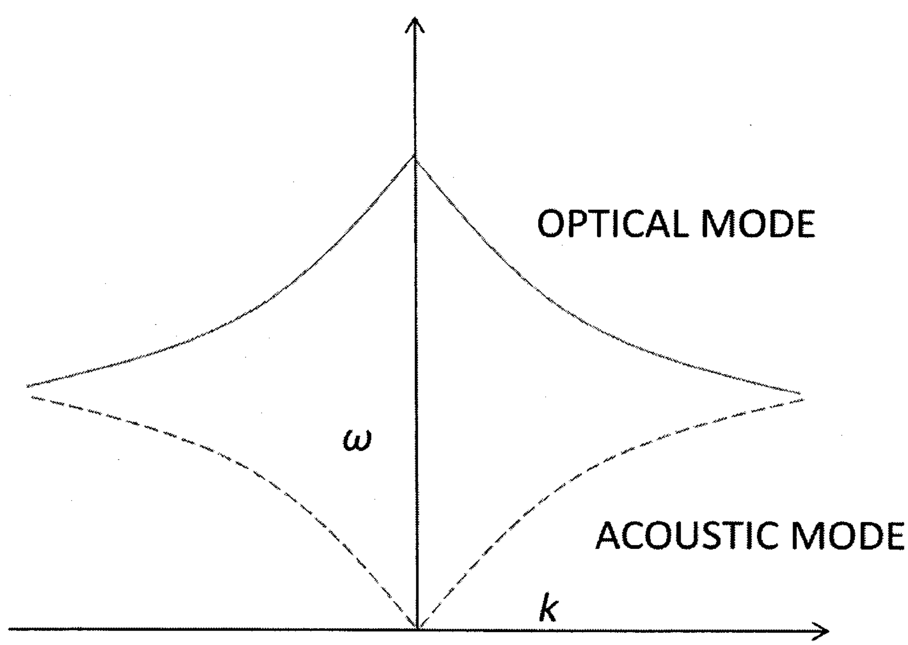
$$H_\Delta = \sum_n \frac{p_i^2}{2m}(-1)^{n+1}\Delta + \frac{1}{2}m\omega_0^2 \sum_n (x_n - x_{n+1})^2(-1)^n\Delta$$

(3) Once again, Fourier transform and use $-1 = e^{i\pi}$

$$H_\Delta = \sum_k \frac{p_k p_{\pi-k}}{2m} + \frac{1}{2}m\omega_0^2 \sum_k 2x_k x_{\pi-k}(1+\cos(k))$$

The new mode is therefore given by

$$\omega_k = \omega_0\sqrt{(1+\cos(k))} = \omega_0 \left|\cos\left(\frac{k}{2}\right)\right|$$



Expt Techniques - Pasupathy
Sec. 5 General I
3

General Experiment

In many experiments, the surface of the sample or detector being used has to be placed in a vacuum environment to avoid contamination from air molecules.

(a) Estimate the pressure in a vacuum chamber (in atmospheres) where one air molecule hits every surface atom of the walls of the chamber every second. Assume that air is composed of only nitrogen molecules (molecular weight 28) that travel at 500 m/s. Assume also that a typical atom on the wall of the chamber has a size of 1 Angstrom.

(b) Such low pressures are reached by the use of vacuum pumps. A vacuum pump operates by displacing a certain volume C per second from the chamber which is then exhausted externally (imagine a chamber where the volume of the chamber is continuously increased by C per second resulting in a continuous drop in pressure). How long will it take a vacuum pump with a displacement of 1 liter per second to reduce the pressure in a 100 liter chamber from atmosphere to the pressure required in (a)? Assume that temperature is maintained constant throughout.

Solution

(a) Momentum of nitrogen molecule = $\frac{M}{N}v$

where M= 28/1000 kg, N = 6 X 10^{23}, v = 500 m/s

If one molecule bounces off the surface atom per second, the net force per second is $\frac{2M}{N}v$ and pressure

$$P = \frac{2Mv}{NA}$$

$$P = \frac{2 * 0.028 * 500}{6 * 10^{23} * 10^{-20}} = 5 * 10^{-3} Pa = 5 * 10^{-8}\ Atmospheres$$

(b) Assume the chamber has a volume V_0. The rate of change of the number of molecules in the chamber dN is given by (assuming the pump is displacing air out of the chamber)

$$\frac{dN}{N} = -\frac{dV}{V_0} = -\frac{1}{V_0}\frac{dV}{dt}dt = -\frac{C}{V_0}dt$$

At constant temperature the number of molecules in the chamber is proportional to the pressure, so

$$\frac{dP}{P} = -\frac{C}{V_0}dt$$

Therefore

$$P(t) = P_0 e^{-\frac{C}{V_0}t}$$

Plug in the numbers

$$\frac{t}{100} = \ln(5 * 10^8) = 2000\ seconds$$

Fluids Nicolis
Sec 5
General I
4 (fluids)

Quals 2009-10

1 General: Fluids – spectrum of small fluctuations

A perfect fluid is described by the continuity and Euler equations, which govern the time-evolution of the density- and velocity-fields $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$:

$$\dot{\rho} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{1}$$

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p\,. \tag{2}$$

Here we assume that the pressure p is a given function of ρ:

$$p = p(\rho)\,. \tag{3}$$

1. Linearize the equations of motion above, for small fluctuations $\delta\rho$, $\delta\vec{v}$ about the homogeneous, static background configuration

$$\rho = \rho_0\,, \qquad \vec{v} = 0\,. \tag{4}$$

2. Consider plane wave-like configurations for $\delta\rho$ and $\delta\vec{v}$:

$$\delta\rho(\vec{x},t) = \delta\rho_*(t)\,e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}\,, \qquad \delta\vec{v}(\vec{x},t) = \delta\vec{v}_*(t)\,e^{i\vec{k}\cdot\vec{x}} + \text{c.c.} \tag{5}$$

   Solve the linear equations you derived in item 1 for $\delta\rho_*(t)$ and $\delta\vec{v}_*(t)$. Hint: decompose $\delta\vec{v}$ into a transverse part and a longitudinal one.

3. What do these solutions describe, physically?

Solution

1. At linear order in $\delta\rho$, $\delta\vec{v}$, eqs. (1,2) reduce to

$$\delta\dot{\rho} + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0 \tag{6}$$

$$\delta\dot{\vec{v}} + \frac{1}{\rho_0}c_s^2\,\vec{\nabla}\delta\rho = 0\,, \tag{7}$$

   where we used that the background has $\vec{v}=0$, that p is a function of ρ, so that

$$\vec{\nabla}p = \frac{dp}{d\rho}\vec{\nabla}\rho\,, \tag{8}$$

   and we defined c_s^2 as

$$c_s^2 = \left.\frac{dp}{d\rho}\right|_{\rho_0}\,. \tag{9}$$

2. For configurations of the form (5) the linearized equations reduce to

$$\begin{aligned} \delta\dot{\rho}_* + \rho_0\, i\vec{k}\cdot\delta\vec{v}_* &= 0 && (10)\\ \delta\dot{\vec{v}}_* + \frac{1}{\rho_0}\, c_s^2\, i\vec{k}\,\delta\rho_* &= 0\,. && (11) \end{aligned}$$

We now project $\delta\vec{v}_*$ and the second equation onto the parallel and transverse (w.r.t. $\vec{k}$) directions. We get two coupled equations for $\delta\rho$ and $\delta v_*^{\|}$

$$\begin{aligned} \delta\dot{\rho}_* + \rho_0\, ik\,\delta v_*^{\|} &= 0 && (12)\\ \delta\dot{v}_*^{\|} + \frac{1}{\rho_0}\, c_s^2\, ik\,\delta\rho_* &= 0\,, && (13) \end{aligned}$$

and a trivial equation for $\delta\vec{v}_*^{\perp}$:

$$\delta\dot{\vec{v}}_*^{\perp} = 0 \tag{14}$$

By using either of eqs. (12, 13) in the other, one gets an ordinary wave equation for $\delta\rho$ and $\delta v_*^{\|}$, with solutions

$$\delta\rho_*(t) = \delta\bar{\rho}\, e^{-i\omega t} \qquad \delta v_*^{\|} = \frac{\delta\bar{\rho}}{\rho_0} c_s\, e^{-i\omega t} \qquad \omega \equiv c_s k\,. \tag{15}$$

The relative phase and amplitude are fixed by either of eqs. (12, 13).

The solution to eq. (14) is instead

$$\delta\vec{v}_*^{\perp} = \text{const}\,. \tag{16}$$

3. The oscillatory solutions (15) obviously describe sound waves: they are longitudinal ($\delta\vec{v} \parallel \vec{k}$) compressional ($\delta\rho \neq 0$) modes. The transverse fluctuations instead describe vortices. More precisely, the linearized version thereof. Indeed in real space transversality means

$$\vec{\nabla}\cdot\delta\vec{v}^{\perp} = 0\,, \tag{17}$$

which implies that $\delta\vec{v}^{\perp}$ is a curl:

$$\delta\vec{v}^{\perp} = \vec{\nabla}\times\vec{A}\,. \tag{18}$$

The trivial dynamics (16) matches the fact that a vortex in constant rotation is a solution.

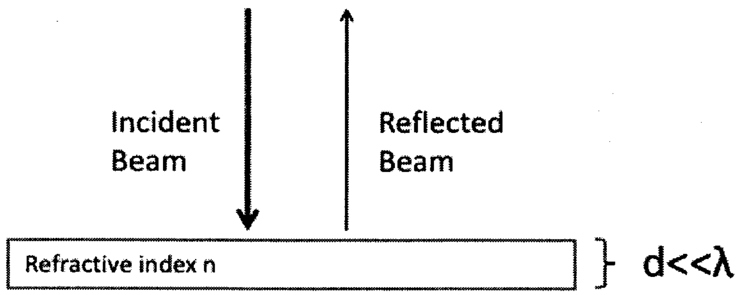
2 General: Astrophysics – the Hubble flow

Consider a self-gravitating, infinitely extended fluid. Assume that the fluid has negligible pressure. The relevant equations are the continuity and Euler ones for the fluid's dynamics, and the Poisson equation for the Newtonian potential. Call ρ the fluid's density, $\vec{v}$ its velocity field, and Φ the gravitational potential per unit mass.

Optics Heinz
Sec. 5. General I
5 (optics)

OPTICS PROBLEM

We wish to detect the presence of a thin membrane suspended in vacuum by reflection of a light beam impinging at normal incidence. Model the material as a thin slab of homogeneous, transparent material with a refractive index n and a thickness d.



(a) Find an *explicit* expression for the reflectance R of the slab in the limit of $d << \lambda$, where λ is the vacuum wavelength of light.

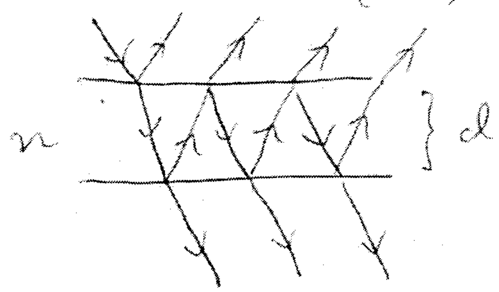
(b) Estimate the minimum effective thickness of a membrane that could in principle be detected in this fashion. Assume typical parameters for a dielectric material; that we have available a 1 μW visible laser and; that we are able to detect 10^9/s photons of reflected light. Use the relation derived above or, if unavailable, a suitable approximate expression.

<u>Solution</u>: We can analyze the problem either
(a) by a direct solution of the boundary value problem or using multiple reflections. Here we present the latter. We use reflection and transmission coefficients for the electric field for normal incidence radiation for a boundary of $n_1 \to n_2$ of

$$r_{12} = (n_1 - n_2)/(n_1 + n_2) \qquad t_{12} = 2n_1/(n_1 + n_2)$$

For our case

$$r = (1-n)/(1+n) \qquad t = 2/(1+n) \qquad \text{entering slab}$$
$$r' = (n-1)/(1+n) \qquad t' = 2n/(1+n) \qquad \text{exiting slab}$$



phase shift in propagation
$$e^{i\delta} = e^{i2\pi n d/\lambda}$$

$$\begin{aligned}
r_{tot} &= r + tt'r'e^{2i\delta}\left[1 + r'^2 e^{2i\delta} + (r'^2 e^{2i\delta})^2 + \dots\right] \\
&= r + tt'r'e^{2i\delta}/(1 - r'^2 e^{2i\delta}) = r\left[1 - \frac{nt^2 e^{2i\delta}}{1 - r^2 e^{2i\delta}}\right] \\
&= r\left[1 - e^{2i\delta}(\underbrace{r^2 + nt^2}_{1})\right]/\left[1 - r^2 e^{2i\delta}\right] \\
&= \left[r/(1-r^2)\right](2i\delta) \quad \text{to leading order in } \delta \ll 1 \\
&= \left[(1-n^2)/4n\right](i4\pi n d/\lambda) = i\frac{\pi d}{\lambda}(1-n^2) = i\frac{\pi d}{\lambda}\chi \leftarrow \text{suscept.}
\end{aligned}$$

The power reflection coefficient is

$$R_{tot} = |r_{tot}|^2 = (\pi d/\lambda)^2(1-n^2) \quad \|$$

(b) From the above, we see that for a given R_{tot} that we can measure, we have

$$d = \frac{\lambda R_{tot}^{1/2}}{\pi(n^2-1)}.$$

From the given data, we can determine R_{tot}. In the visible, a typical photon energy is 1 eV, so $1\,\mu W = 10^{-6}\,J/s \leftrightarrow 10^{13}$ photons/s. If we can detect $10^9/s$, then we can measure $R_{tot} = 10^{-4}$. For a typical $n^2 \sim 3$ and $\lambda \sim 1\,\mu m$, we obtain

$$d = 10^{-6}\,m\; 10^{-2}/2\pi \sim 1\ nm$$

Stat Mech. Heinz
Sec. 5 General I
#6

STATISTICAL MECHANICS PROBLEM

Consider the rotational degree of freedom of a dilute gas of diatomic CO molecules at temperature T. Suppose that the moment of inertia of each molecule is I.

(a) Write an explicit expression for the (quantum) partition function Z_{rot} for the rotational degree of freedom of one molecule. Although you may not be able to reduce to closed form, make sure that all quantities in Z_{rot} are defined so that it could be evaluated numerically.

(b) Write in terms of Z_{rot} a general expression for the heat capacity per molecule associated with rotational motion.

(c) Obtain an analytic expression for the asymptotic behavior of the rotational contribution to the heat capacity per molecule in the limit of low temperature.

(d) For CO molecules, approximately how low does the temperature have to be so that the relation derived in part (c) is applicable. Use suitable estimates of the relevant physical parameters.

Solution

(a) The rotational spectrum follows from $H = J^2/2I$, where I is the moment of inertia of the molecule. This yields energy levels $E_j = j(j+1)\hbar^2/2I$ with $j = 0, 1, 2, 3, \ldots$ Each level has a degeneracy of $(2j+1)$, corresponding to the allowed values of m_j

$$\therefore \quad Z_{rot} = \sum_{j=0}^{\infty} (2j+1)\, e^{-j(j+1)\hbar^2/2I\,\beta} \quad \|$$

with $\beta = (kT)^{-1}$ being the inverse temperature

(b) $\bar{E}_{rot} = -\frac{\partial}{\partial\beta} \ln Z_{rot}$ and $C_{rot} = \frac{\partial E_{rot}}{\partial T} = -k\beta^2 \frac{\partial E_{rot}}{\partial \beta}$

$$C_{rot} = k\beta^2 \frac{\partial^2}{\partial\beta^2} \ln Z_{rot} \quad \|$$

$$= k\left(T^2\frac{\partial^2}{\partial T^2} + 2T\frac{\partial}{\partial T}\right) \ln Z_{rot}$$

(c) The asymptotic behavior for low T (high β) is found by keeping the leading-order T-dep. term:

$$Z_{rot} = 1 + 3e^{-2\epsilon\beta} \quad \text{with } \epsilon = \hbar^2/2I$$

Keeping only the leading-order terms (slowest decay in T):

$$C_{rot} = 12k\left(\epsilon/kT\right)^2 e^{-2\epsilon/kT} \quad \|$$

(d) The low-temperature limit is valid for $kT \ll \epsilon = \hbar^2/2I = h^2/8\pi^2 I$

For a diatomic molecule $I = \mu R^2$ where $\mu = (m_1 m_2)/(m_1+m_2)$ is the reduced mass and R is the bond length. For CO, $R = 0.11\,nm$ and $\mu = 12 \cdot 16/(12+16)$ amu $= 7$ amu. Then

$$T \ll \frac{\epsilon}{k} = \frac{h^2}{8\pi^2 k \mu R^2} = \frac{(hc)^2}{8\pi k \mu c^2 R^2}$$

$$= \frac{(1240\ eV\text{-}nm)^2}{8\pi^2 (8.6\times10^{-5}\ eV/K)(7)(930\ MeV)(0.11\ nm)^2}$$

$$= 2.9\,K \quad ||$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Friday, January 15, 2010
3:10 PM - 5:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 6 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 or 6 problems, the highest scoring problem grade(s) will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2; Section 6 (General Physics), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. Consider a self-gravitating, infinitely extended fluid (i.e. a fluid whose individual volume elements interact gravitationally with each other). Assume that the fluid has negligible pressure. The relevant equations are the continuity and Euler ones for the fluid's dynamics, and the Poisson equation for the Newtonian potential. Call ρ the fluid's density, $\vec{v}$ its velocity field, and Φ the gravitational potential per unit mass.

   (a) Show that there is a solution to the dynamics such that ρ is homogeneous and the fluid expands radially, with $\vec{v}$ proportional to the position vector:

   $$\vec{v}(\vec{x}, t) = H(t)\vec{x} \ .$$

   (b) Determine $\rho(t)$ and $H(t)$, via a power-law ansatz $\rho \propto t^{\alpha}$, $H \propto t^{\beta}$.

   (c) Show that, despite the appearances, for this solution the origin is not a preferred point. That is, all observers comoving with the fluid see exactly the same fluid flow around them. Are there other possible solutions with the same property?

2. Rydberg atoms are highly excited atoms, usually with the principal quantum number $n \gg 1$.

   (a) Find the energy spacing between the nth and the $(n+1)$st Rydberg states of hydrogen.

   (b) Find the size of the atom in the nth energy state.

   (c) Are relativistic effects more or less important in Rydberg states than in the low-lying states? (In other words, how do the typical electron velocities compare to the speed of light in both cases?)

3. A scientist constructed a field effect transistor (FET) that employs a two-dimensional electron gas. In the FET the density of electrons is varied with an external voltage. Under the working conditions of this device, the energy states of the electrons can be represented as:

$$\mathbf{E}(j, k_x, k_y) = \left[j + \frac{1}{2}\right] \mathbf{E}_z + \frac{\hbar^2}{2m}(k_x^2 + k_y^2)$$

The two-dimensional electron gas is in the x-y plane. $\left[j + \frac{1}{2}\right] \mathbf{E}_z$ represents the energy for the motion of electrons along the direction normal to the plane. $\mathbf{E}_z = 0.001$ eV (1 eV = 1.6×10^{-19} Joules). The allowed values of the quantum number j are $j = 0, 1, 2, \ldots$. k_x and k_y are the two in-plane components of the wave vector of states for the electrons, and m is the electron rest mass ($m = 9.11 \times 10^{-28}$ gm).

When the areal electron density of the two-dimensional electron gas is controlled by an external gate voltage, the density can have two limiting values: n_{low} and n_{high}.

(a) Assume that $n_{\text{low}} = 10^{10}$ cm^{-2}. Find the difference between the energies of the lowest and highest states that are populated by the electrons when the temperature is T = 0.

(b) Repeat (a) with $n_{\text{high}} = 10^{12}$ cm^{-2}.

(c) Repeat (a) for T = 10 K.

(The Boltzmann constant is $k_B = 10^{-23}$ Joules/K.)

4. (a) How many photons per second are emitted by a typical incandescent light bulb?

   (b) How many photons per second reach your eye, if you are standing 1 km away from the light bulb?

   (c) Can you see the light bulb from 1 km away, if about 10% of the photons are in the visible portion of the spectrum, 10% of the photons reaching the eye actually hit the retina, and the minimum flux to activate the brain response is 100 photons/s?

5. A fluid in thermodynamic equilibrium at temperature T fills a rigid cubical container of volume V. For wavelenths relevant to the questions below assume the sound speed in the fluid (v_s) to be independent of wavelenght λ ($\ll$ interatomic spacing).

   (a) What is the lowest angular frequency (ω_0) for a standing sound wave in the fluid?

   (b) What is the average energy in that mode when $k_B T \gg \hbar\omega_0$?
   (k_B is the Boltzmann constant, $\hbar \equiv h/2\pi$, neglect "zero point" energy)

   (c) What is the average energy in that mode when the inequality in (b) does not hold?

   (d) What is the probability for finding no energy in this mode (neglecting "zero point" energy)?

   (e) What is the total energy in the modes whose wavelength lies between λ and $\lambda + d\lambda$ and are $\ll V^{1/3}$?

6. The relationship between the free energy F, the internal energy U, temperature T, and entropy S of a gas with a fixed number of atoms is given by:

$$F = U - TS$$

(a) Find an expression for pressure P and entropy S expressed as partial derivatives with respect to the free energy.

(b) Write an expression for $\left(\frac{\partial S}{\partial V}\right)_T$ in terms of pressure, volume, and temperature.

(c) Use the result from part (b) to show that

$$\left(\frac{\partial U}{\partial T}\right)_V = -T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S$$

(d) Show that

$$\left(\frac{\partial T}{\partial V}\right)_S = \frac{\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial P}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}$$

Astro - Nicolis

2 General: Astrophysics – the Hubble flow

Consider a self-gravitating, infinitely extended fluid. Assume that the fluid has negligible pressure. The relevant equations are the continuity and Euler ones for the fluid's dynamics, and the Poisson equation for the Newtonian potential. Call ρ the fluid's density, $\vec{v}$ its velocity field, and Φ the gravitational potential per unit mass.

Sec 6
General II
1
astrophysics

Atomic Zelevinsky
Sec. 6
general II
2 (atomic)

GENERAL PHYSICS – ATOMIC

Rydberg hydrogen atoms.

Rydberg atoms are highly excited atoms, usually with the principal quantum number $n \gg 1$.
a) Find the energy spacing between the nth and the $(n+1)$st Rydberg states of hydrogen.
b) Find the size of the atom in the nth energy state.
c) Are relativistic effects more or less important in Rydberg states than in the low-lying states? (In other words, how do the typical electron velocities compare to the speed of light in both cases?)

Finczuk
Sec. 6 General II
3 condensed matter

General-Section 6: condensed matter

A scientist constructed a field effect transistor (FET) that employs a two-dimensional electron gas. In the FET the density of electrons is varied with an external voltage. Under the working conditions of this device the energy states of the electrons can be represented as:

$$\mathbf{E}(j, k_x, k_y) = \mathbf{E_z}\,[j+ ½] + (\hbar^2/2m)\,(k_x^2 + k_y^2)$$

The two-dimensional electron gas is in the (x,y) plane.
$\mathbf{E_z}$ (j+ ½), represents the energy for motion of electrons along the direction normal to the plane. $\mathbf{E_z}$ = 0.001eV (1eV=1.6×10^{-19}Joules). The allowed values of the quantum number j are j= 0, 1, 2 .... . k_x and k_y are the two in-plane components of wave vector of the states of the electrons, and m is the electron rest mass (m=9.11×10^{-28}gm).

When the areal electron density of the two-dimensional electron gas is controlled by an external gate voltage, the density can have two limiting values: n_{low} and n_{high}.

(a) Assume that $n_{low} = 10^{10}\text{cm}^{-2}$. Find the difference between the energies of the lowest- and highest- states that are populated by the electrons when the temperature is T=0.

(b) Repeat (a) with $n_{high} = 10^{12}\text{cm}^{-2}$.

(c) Repeat (a) for T=10K (the Boltzmann constant is $k_B=10^{-23}$Joules/K).

Solution

(a) The energy difference is the Fermi energy: $E_F = \frac{\hbar^2 k_F^2}{2m}$ where $k_F^2 = (2\pi n)$

For $n_{low} = 10^{10}\ cm^{-2}$ $E_F = 3.8\times10^{-17}$ ergs $= 2.37\times10^{-2}$ meV

(b) Here there are two j-states populated. The Fermi energies of the two states have to be identical;

$$\frac{1}{2}E_2 + \frac{\hbar^2(2\pi n_0)}{2m} = \frac{3}{2}E_2 + \frac{\hbar^2}{2m}(2\pi n_1)$$

$$n_0 - n_1 = mE_2/\hbar^2\pi = 4.22\times10^{11}\,cm^{-2}$$

$$n_0 + n_1 = n_{high} = 10^{12}\ cm^{-2}$$

$$n_0 = 0.71\times10^{12}\,cm^{-2};\ n_1 = 0.29\times10^{12}\,cm^{-2}$$

$E_F = 1.68$ meV from the $j=0$ level

(c) $k_BT = 8.6\times10^{-1}$ meV $\gg E_F$

The electron gas has a classical distribution energies.

order of Magnitude Zelevinsky
Sec. 6 General II
#4
order of mag.

GENERAL PHYSICS – ORDER OF MAGNITUDE ESTIMATE

Light bulbs and photons.

a) How many photons per second are emitted by a typical incandescent light bulb?

b) How many photons per second reach your eye, if you are standing 1 km away from the light bulb?

c) Can you see the light bulb from 1 km away, if about 10% of the photons are in the visible portion of the spectrum, 10% of the photons reaching the eye actually hit the retina, and the minimum flux to activate the brain response is 100 photons/s?

\#5

A fluid in thermodynamic equilibrium at Temperature T fills a rigid cubical container of volume V. For wavelengths relevant to the questions below assume sound speed in the fluid (v_s) to be independent of wavelength λ ($\ll$ interatomic spacing).

1) What is the lowest angular frequency (ω_0) for a standing sound wave in the fluid?

2) What is the average energy in that mode when $k_B T \gg \hbar\omega_0$? (k_B is the Boltzmann constant; $\hbar \equiv h/2\pi$; neglect "zero point" energy)

3) What is the average energy in that mode when the inequality in 2) does not hold?

4) What is the probability for finding no energy in this mode (neglecting "zero point" energy)?

5) What is the total energy in the modes whose wavelengths lie between λ and $\lambda + d\lambda$ and are $\ll V^{1/3}$?

Answers To Stat. Mech. Problem

Suggested credits for correct answer out of a perfect 10

1) $\omega_0 = \frac{\pi}{V^{1/3}} \sqrt{3}\, v_s$

(2/10)

1 Dim: min $kL = \pi$

$\frac{\omega}{v_s} = k$

3 Dim $\omega = \sqrt{k_x^2 + k_y^2 + k_z^2}\, v_s$

(2/10) 2) Classical regime in which

$\overline{\varepsilon}_{\text{Harmonic oscillator}} = k_B T$

(3/10) 4) $P_n = \dfrac{e^{-n\hbar\omega_0\beta}}{\sum_{n=0}^{\infty} (e^{-\hbar\omega_0\beta})^n}$ $\left(\beta = \frac{1}{kT}\right)$

$P_{n=0} = \frac{1}{\Sigma} = 1 - e^{-\hbar\omega_0\beta}$

(3/10) 3) $\sum n\hbar\omega_0 P_n = \dfrac{\hbar\omega_0}{e^{\hbar\omega_0/kT} - 1} = \langle \epsilon \rangle$

(1/10) 5) $\dfrac{4\pi \cdot k^2 dk\, V\, \hbar\omega_0}{e^{\hbar k v_s \beta} - 1}$

Thermo - Hughes
Sec. 6 General II
#6

THERMODYNAMICS

The relationship between the free energy F and the internal energy U, temperature T and entropy S of a gas with a fixed number of atoms is given by:

$$F = U - TS$$

(a) Find an expression for pressure P and entropy S expressed as a partial derivatives with respect to the free energy.

(b) Write an expression for $\left(\frac{\partial S}{\partial V}\right)_T$ in terms of pressure, volume and temperature.

(c) Use the result from part (b) to show that $\left(\frac{\partial U}{\partial T}\right)_V = -T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S$

(d) Show that $\left(\frac{\partial T}{\partial V}\right)_S = \frac{\left(\frac{\partial V}{\partial T}\right)_P\left(\frac{\partial P}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}$

Thermo

(a) At constant N,

$$dU = dQ - pdV = TdS - pdV$$

$$dF = dU - TdS - SdT$$

$$= TdS - pdV - TdS - SdT$$

$$= -pdV - SdT$$

But $dF = \left(\frac{\partial F}{\partial V}\right)_T dV + \left(\frac{\partial F}{\partial T}\right)_V dT$

$$\boxed{p = -\left(\frac{\partial F}{\partial V}\right)_T \qquad S = -\left(\frac{\partial F}{\partial T}\right)_V}$$

b $$\frac{\partial^2 F}{\partial V \partial T} = \frac{\partial^2 F}{\partial T \partial V}$$

$$\left[\frac{\partial}{\partial V}\underbrace{\left(\frac{\partial F}{\partial T}\right)_V}_{-S}\right]_T = \left[\frac{\partial}{\partial T}\underbrace{\left(\frac{\partial F}{\partial V}\right)_T}_{-P}\right]_V$$

$$-\left(\frac{\partial S}{\partial V}\right)_T = -\left(\frac{\partial P}{\partial T}\right)_V$$

$$\boxed{\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V}$$

c $$\left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$ since $dU = TdS - pdV$

$$= -T\left(\frac{\partial S}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_S$$

Now we use: $$\left(\frac{\partial S}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_S\left(\frac{\partial V}{\partial S}\right)_T = -1$$

From part b:

$$\left(\frac{\partial U}{\partial T}\right)_V = -T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_S$$ ✓

d) $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial T}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T$

$= -\left(\frac{\partial T}{\partial S}\right)_V \left(\frac{\partial P}{\partial T}\right)_V$ part b

$= \dfrac{-\left(\frac{\partial P}{\partial T}\right)_V}{\left(\frac{\partial S}{\partial T}\right)_V}$

$= \dfrac{\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial S}{\partial T}\right)_V}$ ✓