Fall 2010

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

September 15, 2010

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Short Answer Questions

(a) A certain type of bacteria is known to double in number every 24 hours. Two cultures of these bacteria are prepared, each consisting initially of one bacterium. One culture is left on Earth and the other placed on a rocket that travels at a speed of $0.866c$ relative to Earth. At a time when the earthbound culture has grown to 256 bacteria, how many bacteria are in the culture on the rocket according to observers on Earth?

(b) Main sequence stars are stabilized by a balance between inward gravitational pressure and outward fluid pressure. By dividing the star into concentric spherical shells, this can be described by the equation of hydrostatic equilibrium

$$dP/dr = -\rho(r)g(r)$$

where $g(r)$ is the local acceleration of gravity within the star and $\rho(r)$ is the local mass density. Using this fact, estimate the order of magnitude of the central pressure of the Sun.

(c) For an experiment you are handed a custom designed electrical plate on which you find equipotentials, relative to a given spot on the plate, as follows:

r(cm)	V(mV)
1.0	0.0
2.0	-1.4
3.0	-2.2
4.0	-2.8

where r is the radial distance from the spot. Estimate the magnitude of the electric field in the plate at $r = 2.75$ cm.

(d) Consider two spheres with the same diameter and same mass, but one is solid while the other is hollow. Clearly describe quantitatively a nondestructive experiment to determine which sphere is solid and which sphere is hollow.

(e) An elevator is pulled upward by an AC motor which has a maximum rating of $I_{\text{rms}} = 20$ amps and is connected to an electric grid with a voltage of $V_{\text{rms}} = 240$ volts. Determine the maximum speed with which the elevator can ascend if its total mass is $M = 500$ kg.

Answers

(a) $N = 2^d$ where d is the number of days. On Earth $N = 256 \Rightarrow d = 8$. Relative to Earth, time on the rocket is slower by a factor of γ. For the given velocity $\gamma \approx 2$; therefore, only 4 days have passed on the rocket. So, $N_{ship} = 2^4 = 16$.

(b) Treat the whole Sun as one shell with zero pressure at the outer surface. Therefore, $dr = R_\odot$, $dP = P_c$ (the central pressure), $g = GM / R_\odot^2$, and the order of magnitude of the density is $\rho = M / R_\odot^3$. This gives

$$P_c \sim GM^2 / R_\odot^4$$

Plugging in numbers gives $P_c \sim 10^{15}\ \mathrm{N/m^2}$.

(c) The magnitude of the electric field is given by

$$E \approx \left|\frac{\Delta V}{h}\right|$$

We can estimate $E(2.75)$ by interpolation. $E(2.5)$ and $E(3.5)$ are estimated as

$$E(2.5) \approx \left|\frac{V(3.0) - V(2.0)}{3.0 - 2.0}\right| = 0.8\,\frac{\mathrm{V}}{\mathrm{cm}} \qquad E(3.5) \approx \left|\frac{V(4.0) - V(3.0)}{4.0 - 3.0}\right| = 0.6\,\frac{\mathrm{V}}{\mathrm{cm}}$$

Thus, we interpolate to get $E(2.75)$

$$\frac{E(3.5) - E(2.75)}{3.5 - 2.75} = \frac{E(3.5) - E(2.5)}{3.5 - 2.5} \quad \Rightarrow \quad E(2.75) = 0.75\,\frac{\mathrm{V}}{\mathrm{cm}}$$

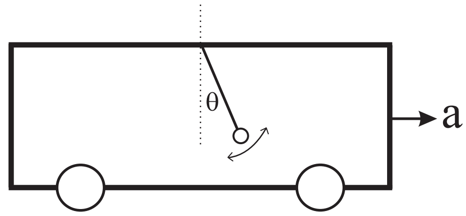
(d) Any behavior which depends on moment of inertia can distinguish the spheres. For example, if the spheres roll from rest down a ramp of height h without slipping, the change in gravitational potential energy at the top becomes kinetic energy at the bottom:

$$KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left[MR^2 + I\right]\omega^2$$

So the solid sphere, with the smaller moment of inertia, will have a larger ω and reach the bottom first.

(e) Assuming 100% efficiency, $I_{\mathrm{rms}}^{\mathrm{max}} V_{\mathrm{rms}} = Mgv_{\mathrm{max}} \quad \Rightarrow \quad v_{\mathrm{max}} = 0.98\ \mathrm{m/s}$

2. Moving Pendulum



A simple pendulum is placed in a railroad car that has a constant acceleration a in the horizontal direction. Consider the motion of the pendulum in Earth's reference frame.

(a) What is the Lagrangian describing the pendulum motion and the associated equation(s) of motion?

(b) What is the equilibrium angle for which the pendulum hangs without oscillating?

(c) Suppose that the pendulum is now perturbed slightly away from that equilibrium angle, and as a result executes small oscillations about its equilibrium angle. What is the oscillation frequency?

Moving Pendulum - Solution

(a) Choose coordinates so that the pivot of the pendulum is at $x = \frac{1}{2}at^2$, $y = 0$. Then, the position of the pendulum bob is $x = (\frac{1}{2}at^2 + \ell\sin\theta)\hat{x} + (-\ell\cos\theta)\hat{y}$. It's velocity is $\dot{\vec{x}} = (at + \ell\cos\theta\dot{\theta})\hat{x} + (\ell\sin\theta\dot{\theta})\hat{y}$. The kinetic energy is $T = \frac{1}{2}m\dot{\vec{x}}\cdot\dot{\vec{x}} = \frac{1}{2}m(a^2t^2 + 2at\ell\cos\theta\dot{\theta} + \ell^2\dot{\theta}^2)$. The potential energy is $U = mg\vec{x}\cdot\hat{y} = -mg\ell\cos\theta$. The Lagrangian, therefore, is

$$\mathcal{L} = \frac{1}{2}m(a^2t^2 + 2at\ell\cos\theta\dot{\theta} + \ell^2\dot{\theta}^2) + mg\ell\cos\theta.$$

To get the equation of motion we use the Euler-Lagrange equation, $\frac{\partial L}{\partial\theta} = \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}$

$$\begin{aligned}
\frac{1}{2}m(-2at\ell\sin\theta\dot{\theta}) - mg\ell\sin\theta &= \frac{d}{dt}\left[m(at\ell\cos\theta + \ell^2\dot{\theta})\right] - mat\ell\sin\theta\dot{\theta} - mg\ell\sin\theta \\
&= ma\ell\cos\theta - mat\ell\sin\theta\dot{\theta} + m\ell^2\ddot{\theta} \\
\ddot{\theta} &= -\frac{g}{\ell}\sin\theta - \frac{g}{\ell}\cos\theta
\end{aligned}$$

(b) At the equilibrium angle, $\theta(t) = \theta_0$. Therefore,

$$\ddot{\theta} = 0 = -\frac{g}{\ell}\sin\theta - \frac{g}{\ell}\cos\theta \implies \theta_0 = \tan^{-1}(-\frac{a}{g}).$$

(c) Write $\theta(t) = \theta_0 + \epsilon(t)$, where $\epsilon(t) \ll 1$ describes small oscillations about equilibrium. Further note that

$$\begin{aligned}
\cos\theta(t) &= \cos(\theta_0 + \epsilon) = \cos\theta_0\cos\epsilon - \sin\theta_0\sin\epsilon \\
\sin\theta(t) &= \sin(\theta_0 + \epsilon) = \sin\theta_0\cos\epsilon + \cos\theta_0\sin\epsilon
\end{aligned}$$

where

$$\cos\theta_0 = \frac{g}{\sqrt{a^2 + g^2}} \quad \text{and} \quad \sin\theta_0 = \frac{-a}{\sqrt{a^2 + g^2}}.$$

The signs must be this way because $-\frac{\pi}{2} \leq \theta_0 \leq 0$.

From the equation of motion we have

$$\frac{d^2}{dt^2}(\theta_0 + \epsilon) = -\frac{g}{\ell}\sin(\theta_0 + \epsilon) - \frac{g}{\ell}\cos(\theta_0 + \epsilon).$$

This reduces to

$$\ddot{\epsilon} = -\frac{\sqrt{a^2 + g^2}}{\ell}\epsilon + \mathcal{O}(\epsilon^3).$$

Thus, to first order, the oscillations about equilibrium are harmonic with frequency

$$\omega = \left(\frac{\sqrt{a^2 + g^2}}{\ell}\right)^{1/2}.$$

3. Band of Rubbers

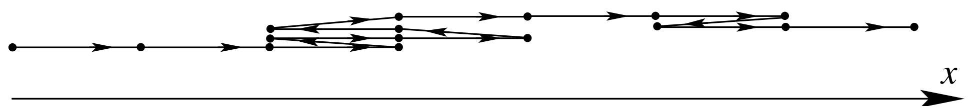


Figure 1: A simplified model of rubber. A rubber molecule is modeled by a chain consisting of rigid rods. The rigid rods are shown as arrows with circles at both ends. The rods are assumed always to be parallel to the x-axis

Rubber consists of many cross-linked polymer chains, and exhibits an unusual behavior of its elastic properties as functions of temperature T. To understand the elastic properties of rubber, consider just one polymer chain. We model the chain as being composed of N rigid rods of length l connected to each other at their ends. The angle between two adjacent rods is assumed to take only one of two values: 0 or π, independently at each junction. These two configurations are assumed to have the same energy. The model is illustrated in Fig. 1. One end of the chain is fixed at the origin $x = 0$, while the other is free to move along the axis (in discrete steps of length l).

(a) When the chain's free end is at the position $x = L$, find expressions for the number of configurations W and the entropy S of the chain in terms of N and L.

(b) Find the free energy F of the chain, and the force f necessary to hold the free end at $x = L$, as functions of N, L, and the temperature T. What is the behavior of f for $L \ll Nl$?

(c) Find the thermal expansion coefficient $\alpha = L^{-1}\partial L/\partial T|_f$. Does the rubber band expand or shrink when heated?

Hints: Use the Stirling formula $n! \sim n \log n - n$; in one dimension the distance L is analogous to the volume of a gas, and the force f is analogous to the *negative* of the pressure in a gas.

Band of Rubbers - Solution

(a) Let the number of left-oriented links be n_-, the number of right-oriented links n_+. We have

$$N = n_+ + n_- \qquad\qquad L = (n_+ - n_-)l. \tag{1}$$

Solving this for $n_\pm$, we have

$$n_+ = \frac{Nl + L}{2l}, \qquad\qquad n_- = \frac{Nl - L}{2l}. \tag{2}$$

For fixed N, the distance L between the end points of the chain is fixed by n_+. Then the total number of configurations for a given L is the number of ways of choosing n_+ out of N to be right-oriented, that is

$$W = \frac{N!}{n_+!n_-!} \tag{3}$$

Setting the Boltzmann constant to be one, we find the entropy:

$$\begin{aligned} S = \ln W &= \ln \frac{N!}{n_+!n_-!} \\ &\approx N \log N - n_- \log n_- - n_+ \log n_+ - N + n_- + n_+ \\ &= N \log N - \frac{Nl + L}{2l} \ln \frac{Nl + L}{2l} - \frac{Nl - L}{2l} \ln \frac{Nl - L}{2l}. \end{aligned} \tag{4}$$

(b) Since all the configurations have the same energy, the (non-constant part of the) free energy F is simply related to the entropy S by

$$F = -TS = T\left(\frac{Nl + L}{2l} \ln \frac{Nl + L}{2l} + \frac{Nl - L}{2l} \ln \frac{Nl - L}{2l} - N \ln N\right). \tag{5}$$

Using the analogy with pressure, the tension of the string is given by differentiating the free energy (5) with respect to the length L

$$f = \frac{\partial F}{\partial L} = \frac{T}{2l}\left[\ln \frac{Nl + L}{2l} - \ln \frac{Nl - L}{2l}\right] = \frac{T}{2l} \ln \frac{Nl + L}{Nl - L}. \tag{6}$$

When $L \ll Nl$, this expression simplifies to

$$f \approx \frac{T}{Nl^2} L, \tag{7}$$

which is the Hooke's law with the effective spring constant $k = T/Nl^2$. Notice that the polymer becomes stiffer when the temperature is raised, and this is related to the negative coefficient of thermal expansion found in the next part.

(c) Solving for L Eq. (6) we have

$$L = Nl\frac{e^{2fl/T} - 1}{e^{2fl/T} + 1} = Nl \tanh \frac{fl}{T}. \tag{8}$$

and so

$$\alpha = \frac{1}{L}\frac{\partial L}{\partial T}\bigg|_f = -\frac{fl}{T^2}\frac{1}{\cosh^2 \dfrac{fl}{T}}\frac{1}{\tanh \dfrac{fl}{T}} = -\frac{2fl}{T^2 \sinh \dfrac{2fl}{T}} \sim -\frac{1}{T}, \tag{9}$$

where the last expression is the limit of high temperature ($T \gg fl$). For any temperature the coefficient of thermal expansion is *negative*, and this means that the rubber band shrinks when heated.

4. A Rod on Two Rails

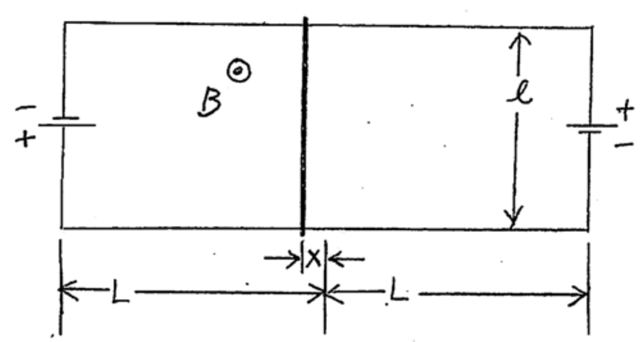
Two rails, each of length $2L$, are positioned parallel to each other and on a horizontal plane as shown in the figure. The rails have resistance of ρ per unit length, and they are located with a separation distance ℓ from each other. Their ends are connected to identical batteries with emf V. A metal rod of mass m and resistance R lies perpendicular to the rails and can slide along them. The entire system is placed in a uniform magnetic field perpendicular to the plane as shown. Neglect internal resistance of the batteries and contacts, and inductance of the circuit.

(a) What is the current that flows through the rod when it is placed in the middle of the rails?

(b) If the rod is held at a position x away from the middle of the rails as shown, what is the current that flows through the rod?

(c) The rod is released from position x of part (b); what is the equation of motion of the rod? Assume x is small compared to L.

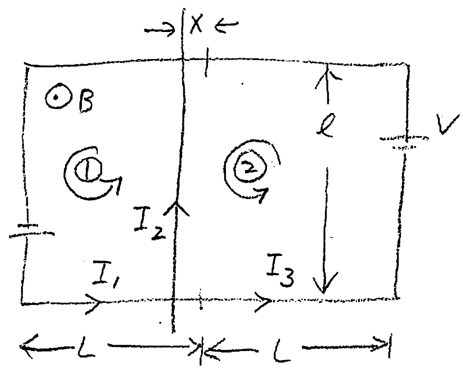
(d) Once released from position x, if the rod oscillates, find the period of oscillation; if it does not oscillate, find the characteristic time for exponential growth of its displacement.



A Rod on Two Rails - Solution

(a) The current in this situation is zero.

(b) We use Kirchhoff's laws:



$$\text{Bottom Junction: } I_1 = I_2 + I_3 \tag{10}$$

$$\text{Loop 1: } 2I_1\rho(L - x) + I_2R = V \tag{11}$$

$$\text{Loop 2: } 2I_3\rho(L + x) - I_2R = V \tag{12}$$

These equations give

$$I_2 = \frac{Vx}{\rho(L^2 - x^2) + RL} \approx \frac{Vx}{\rho L^2 + RL}.$$

(c) The force on the rod is given by

$$F = I_2\ell B = \frac{V\ell B}{L(\rho L + R)}x.$$

Therefore, the equation of motion is

$$-m\ddot{x} = \frac{V\ell B}{L(\rho L + R)}x.$$

(d) From the equation, it is clear that the motion is an oscillation with period

$$T = 2\pi\sqrt{\frac{mL(\rho L + R)}{V\ell B}}$$

If the induced emf on the rod is included, a term $B\ell\dot{x}$ needs to be added and this term will cause damping to the oscillation.

5. A Potential Step

A quantum mechanical particle of mass m and energy $E > 0$ moves in one dimension along the x axis in the potential $U(x) = 0$ for $x < 0$ (region 1), and $U(x) = -V_0$ for $x > 0$ (region 2).

The general solution of the time-independent Schrodinger equation in this case is

$$\psi(x) = \begin{cases} A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} & \text{for } x < 0 \text{ (region 1)}, \\ A_2 e^{ik_2 x} + B_2 e^{-ik_2 x} & \text{for } x > 0 \text{ (region 2)}. \end{cases} \tag{13}$$

(a) Write the time-independent Schrodinger equation and find the values of the momenta k_1 and k_2.

(b) Find the scattering matrix $\mathcal{S}$ that, by definition, relates the amplitudes of the incoming and outgoing waves:

$$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = \mathcal{S} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}. \tag{14}$$

(c) The probability flux (or current) is defined in one dimension as

$$j(x) = \frac{\hbar}{2mi}\big[\psi^*(x)\psi'(x) - \text{c.c.}\big]. \tag{15}$$

Find the currents in both regions. Check that the current conservation holds across the potential step.

(d) Find the reflection (R) and transmission (T) coefficients defined as the ratios of the reflected, transmitted, and incident currents as follows:

$$R(E) = \frac{|j_{\text{reflected}}|}{|j_{\text{incident}}|}, \qquad T(E) = \frac{|j_{\text{transmitted}}|}{|j_{\text{incident}}|}. \tag{16}$$

Do these coefficients depend on which region we choose as the region where the incident wave is coming from?

(e) Find the asymptotic behavior of the reflection coefficient $R(E)$ for large energies $E \gg V_0$, and the transmission coefficient $T(E)$ for small energies $E \ll V_0$.

A Potential step - Solution

(a) The time-independent Schrodinger equation is

$$-\frac{\hbar^2}{2m}\psi''(x) + U(x)\psi(x) = E\psi(x). \tag{17}$$

Substituting the general solution in this equation, we find

$$k_1 = \frac{1}{\hbar}\sqrt{2mE}, \qquad\qquad k_2 = \frac{1}{\hbar}\sqrt{2m(E+V_0)}. \tag{18}$$

(b) Continuity of the wave function and its derivative at $x = 0$ (the position of the potential step) gives

$$\begin{cases} A_1 + B_1 = A_2 + B_2, \\ k_1(A_1 - B_1) = k_2(A_2 - B_2). \end{cases} \tag{19}$$

Solving for B_1 and A_2, we obtain

$$B_1 = \frac{k_1 - k_2}{k_1 + k_2}A_1 + \frac{2k_2}{k_1 + k_2}B_2, \qquad A_2 = \frac{2k_1}{k_1 + k_2}A_1 - \frac{k_1 - k_2}{k_1 + k_2}B_2. \tag{20}$$

From this we read off the scattering matrix

$$\mathcal{S} = \frac{1}{k_1 + k_2}\begin{pmatrix} k_1 - k_2 & 2k_2 \\ 2k_1 & k_2 - k_1 \end{pmatrix}. \tag{21}$$

(c) For the current in region 1 we have

$$\begin{aligned} j_1 &= \frac{\hbar}{2mi}\left[ik_1\left(A_1^* e^{-ik_1x} + B_1^* e^{ik_1x}\right)\left(A_1 e^{ik_1x} - B_1 e^{-ik_1x}\right) - \text{c.c.}\right] \\ &= \frac{\hbar k_1}{2m}\left[|A_1|^2 - |B_1|^2 + A_1 B_1^* e^{2ik_1x} - A_1^* B_1 e^{-2ik_1x} + \text{c.c.}\right] \\ &= \frac{\hbar k_1}{m}\left(|A_1|^2 - |B_1|^2\right). \end{aligned} \tag{22}$$

For region 2 we have, similarly,

$$j_2 = \frac{\hbar k_2}{m}\left(|A_2|^2 - |B_2|^2\right). \tag{23}$$

To check the current conservation we have to express both currents j_1 and j_2 in terms of the same pair of amplitudes. To do this we need to use the relations (20). Thus, for region 1 we have

$$\begin{aligned} j_1 &= \frac{\hbar k_1}{m}\left(|A_1|^2 - \left|\frac{k_1 - k_2}{k_1 + k_2}A_1 + \frac{2k_2}{k_1 + k_2}B_2\right|^2\right) \\ &= \frac{\hbar k_1}{m(k_1 + k_2)^2}\Big(4k_1k_2|A_1|^2 - 4k_2^2|B_2|^2 - 2k_2(k_1 - k_2)(A_1^* B_2 + A_1 B_2^*)\Big) \\ &= \frac{2\hbar k_1 k_2}{m(k_1 + k_2)^2}\left(2k_1|A_1|^2 - 2k_2|B_2|^2 + (k_2 - k_1)(A_1^* B_2 + A_1 B_2^*)\right). \end{aligned} \tag{24}$$

Similarly, for j_2 we have

$$
\begin{aligned}
j_2 &= \frac{\hbar k_2}{m}\left(\left|\frac{2k_1}{k_1+k_2}A_1 + \frac{k_2-k_1}{k_1+k_2}B_2\right|^2 - |B_2|^2\right) \\
&= \frac{\hbar k_2}{m(k_1+k_2)^2}\Big(4k_1^2|A_1|^2 - 4k_1k_2|B_2|^2 + 2k_1(k_2-k_1)(A_1^*B_2 + A_1B_2^*)\Big) \\
&= \frac{2\hbar k_1 k_2}{m(k_1+k_2)^2}\big(2k_1|A_1|^2 - 2k_2|B_2|^2 + (k_2-k_1)(A_1^*B_2 + A_1B_2^*)\big). \qquad (25)
\end{aligned}
$$

We see that $j_1 = j_2$, that is, the current is conserved across the potential step.

(d) Let us choose region 1 as the region where the incident wave is coming from. Thus we set $A_1 = 1$ and $B_2 = 0$. For the amplitudes of the reflected and transmitted waves we have, using Eq. (20),

$$
B_1 = \frac{k_1-k_2}{k_1+k_2}, \qquad A_2 = \frac{2k_1}{k_1+k_2}. \qquad (26)
$$

The incident, reflected and transmitted currents are

$$
\begin{aligned}
j_{\text{incident}} &= \frac{\hbar k_1}{m}|A_1|^2 = \frac{\hbar k_1}{m}, \\
j_{\text{reflected}} &= -\frac{\hbar k_1}{m}|B_1|^2 = -\frac{\hbar k_1}{m}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2, \\
j_{\text{transmitted}} &= \frac{\hbar k_2}{m}|A_2|^2 = \frac{\hbar k_1}{m}\frac{4k_1k_2}{(k_1+k_2)^2}. \qquad (27)
\end{aligned}
$$

The reflection and transmission coefficients follow:

$$
R(E) = \left(\frac{k_2-k_1}{k_1+k_2}\right)^2, \qquad T(E) = \frac{4k_1k_2}{(k_1+k_2)^2} = 1 - R(E). \qquad (28)
$$

(e) Substituting expressions (18) into the reflection and transmission coefficients, we get their energy dependence:

$$
R(E) = \left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2, \qquad T(E) = \frac{4\sqrt{E(E+V_0)}}{(\sqrt{E+V_0}-\sqrt{E})^2}. \qquad (29)
$$

For large energies $E \gg V_0$ the reflection coefficient behaves as

$$
R(E) = \left(\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right)^2 \approx \left(\frac{V_0}{4E}\right)^2. \qquad (30)
$$

For small energies $E \ll V_0$ we can neglect E everywhere in the equation for T except for the first factor in the numerator:

$$
T(E) \approx 4\sqrt{\frac{E}{V_0}}. \qquad (31)
$$

6. The Friedmann Equations

The Friedmann equations for a flat Universe are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \ , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w) \tag{32}$$

where G is Newton's constant, $a(t)$ is a scale factor describing the "size" of the Universe, taken to be unity at the present time, and the parameter w describes the ratio of pressure p and energy density ρ: $p = w\rho$. For a matter-dominated Universe, $w = 0$, while for a Universe dominated by radiation, $w = 1/3$. We are taking units with $c = 1$.

(a) Suppose the energy density scales as $\rho = \rho_0/a^\alpha$, where $\alpha > 0$ and ρ_0 is the present energy density. Solve the first Friedmann equation to find the time-dependence of $a(t)$, assuming $a(0) = 0$. Does this assumption make sense if ρ is independent of $a(t)$, and hence of t? What is the solution in that case? For convenience define a constant H_0 with dimensions of inverse time such that

$$\frac{8\pi G\rho_0}{3} = H_0^2 \tag{33}$$

(b) Use consistency of the first and second Friedmann equations to relate w to the scale-dependence parameter α of the energy density. What values of α correspond to a pressure-dominated Universe ($w = 0$) and to a radiation-dominated Universe ($w = 1/3$)? What w corresponds to a Universe with energy density independent of a?

(c) Suppose you have a Universe consisting of a present fraction f of matter-dominated energy density and $1 - f$ of a-independent energy density, with total energy density at the present time equal to ρ_0. Integrate the first Friedmann equation to relate the scale factor $a(t)$ to the look-back time $\tilde{t} = t_{\text{now}} - t$.

The Friedmann Equations - Solution

(a) The first Friedmann equation may be written

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-\alpha} = H_0^2 a^{-\alpha} \ . \tag{34}$$

This may be solved to give

$$a = \left(\frac{\alpha}{2}H_0 t\right)^{2/\alpha} \ (\alpha > 0) \ ; \quad a = \exp(H_0 t) \ (\alpha = 0) \ . \tag{35}$$

(b) For $\alpha \neq 0$,

$$\frac{\ddot{a}}{a} = \frac{2}{\alpha}\left(\frac{2}{\alpha} - 1\right)\frac{1}{t^2} = -\frac{1}{2}H_0(1+3w)a^{-\alpha} \ . \tag{36}$$

Demanding consistency with $(\dot{a}/a)^2 = (2/\alpha)^2 t^{-2}$, we find

$$\frac{1+3w}{2} = \left(1 - \frac{2}{\alpha}\right)/\frac{2}{\alpha} = \frac{\alpha}{2} - 1 \ \Rightarrow \ \alpha = 3(1+w) \ . \tag{37}$$

For $\alpha = 0$ consistency between the two Friedmann equations leads directly to $w = -1$.

(c) For a matter-dominated Universe with $f = 1$, so that $\rho = \rho_0/a^3$, the first Friedmann equation may be expressed in terms of the lookback time $\tilde{t}$ with the condition $a(0) = 1$ to give $a = \left(1 - \frac{3}{2}H_0\tilde{t}\right)^{2/3}$. For a Universe where $f = 0$ so $\rho = \rho_0$, the solution is $a = \exp(-H_0\tilde{t})$. For a Universe with a present fraction f of matter and $1 - f$ of constant energy density, the first Friedmann equation becomes

$$\frac{da}{d\tilde{t}} = -H_0\left(\frac{f}{a} + (1-f)a^2\right)^{1/2} \ . \tag{38}$$

The solution of this is

$$H_0\tilde{t} = \int_a^1 \frac{\sqrt{a'}da'}{\sqrt{f + (1-f)a'^3}} \ , \tag{39}$$

which may be expressed with the substitution $\alpha \equiv a'^{3/2}$ as

$$\begin{aligned} \frac{3}{2}H_0\tilde{t} &= \int_{a^{3/2}}^1 \frac{d\alpha}{\sqrt{f + (1-f)\alpha^2}} \\ &= \frac{1}{\sqrt{1-f}}\left[\sinh^{-1}\sqrt{\frac{1-f}{f}} - \sinh^{-1}\left(\sqrt{\frac{1-f}{f}}a^{3/2}\right)\right] \ . \end{aligned} \tag{40}$$

An equivalent function for $\sinh^{-1}$ is $\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$. The limits of Eq. (40) for $f \to 0,\ 1$ are:

$$f \to 0 \ : \ \ a(t) \to \exp(-H_0\tilde{t}) \ : \ \ f \to 1 \ : \ \ a(t) \to \left(1 - \frac{2}{3}H_0\tilde{t}\right)^{2/3} \ . \qquad (41)$$

Fall 2010

DEPARTMENT OF PHYSICS
Ph.D. CANDIDACY EXAMINATION

Day 2
September 16, 2010
(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

7. Bead on a Loop

A circular horizontal loop of radius R is made of wire with coefficient of friction μ. A small bead can slide on the loop. What initial velocity should be given to the bead for it to make exactly one full turn before stopping? Hint: It is useful in this problem to look for the velocity v of the bead as a function of the angle θ around the loop.

Bead on a Loop - Solution

Let us use the cylindrical coordinates with the origin placed at the center of the loop, and the z axis vertically up. We will write components of a vector $\boldsymbol{A}$ in this coordinate system as $\boldsymbol{A} = (A_r, A_\theta, A_z)$.

The forces acting on the loop are: gravity $m\boldsymbol{g} = (0, 0, -mg)$ (vertically down), normal reaction force from the loop $\boldsymbol{N} = (N_r, 0, N_z)$ (direction is not known a priori), friction $\boldsymbol{F} = (0, -F, 0)$ (tangentially to the loop against the direction of the velocity). The acceleration $\boldsymbol{a} = (a_r, a_\theta, 0)$ of the bead has the radial component $a_r = -v^2/R$, and the tangential component $a_\theta = \dot{v}$, where the dot denotes the time derivative. Expressing the components of the acceleration in terms of the velocity $v(\theta)$ viewed as a function of the angle θ along the loop, we have

$$a_r = -\frac{v^2(\theta)}{R}, \qquad a_\theta = \dot{v}(\theta) = v'(\theta)\dot{\theta} = \frac{v(\theta)v'(\theta)}{R} = \frac{\left[v^2(\theta)\right]'}{2R}, \tag{1}$$

where prime stands for the derivative with respect to the angle θ.

The three components of the Newton's second law are

$$ma_r = N_r, \qquad ma_\theta = -F, \qquad 0 = N_z - mg. \tag{2}$$

In addition, we have the equation for the friction force $F = \mu N$ valid during the motion of the bead. These four equations give

$$\begin{aligned} &N_z = mg, \qquad N_r = -\frac{mv^2(\theta)}{R}, \\ &N = \left(N_r^2 + N_z^2\right)^{1/2} = \frac{m}{R}\left(v^4(\theta) + g^2R^2\right)^{1/2}, \\ &\frac{m}{2R}\left[v^2(\theta)\right]' = -\mu\frac{m}{R}\left(v^4(\theta) + g^2R^2\right)^{1/2}. \end{aligned} \tag{3}$$

The last equation is a first order differential equation for $x(\theta) = v^2(\theta)$:

$$x'(\theta) = -2\mu\left(x^2(\theta) + g^2R^2\right)^{1/2}. \tag{4}$$

This equation must be solved with the "initial" value $x(2\pi) = 0$ (the bead stops after exactly one turn).

Separation of variables gives

$$\frac{dx}{\sqrt{x^2 + g^2R^2}} = -2\mu d\theta. \tag{5}$$

The substitution $x = gR \sinh y$ gives $dy = -2\mu d\theta$. Thus,

$$x(\theta) = gR \sinh(C - 2\mu\theta). \tag{6}$$

The integration constant C is found from the initial condition to be $C = 4\pi\mu$. Finally, the initial velocity is found as

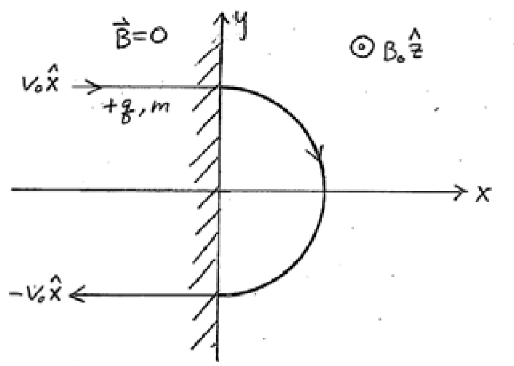
$$v_0 = \sqrt{x(0)} = \sqrt{gR \sinh(4\pi\mu)}. \tag{7}$$

8. Half Circle Radiation

A magnetic field $\vec{B}(x)$ is given by:

$$\vec{B}(x) = \begin{cases} 0 & \text{for } x < 0 \\ B_0\hat{z} & \text{for } x > 0 \end{cases} \tag{8}$$

There is no electrostatic field. A non-relativistic particle of mass m and charge $+q$, initially moves with a constant velocity $\vec{v} = v_0\hat{x}$ in the space $x < 0$. As shown in the figure, at $t = 0$ the particle enters the space $x > 0$, moves in a semi-circle for a time π/ω_0, and then exits the space $x > 0$ at velocity $\vec{v} = -v_0\hat{x}$. The quantities $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in the x, y, and z directions.



(a) What is the radius of the half circle? What is the angular frequency ω_0 of the particle in the region $x > 0$?

(b) What is the total energy emitted by the particle?

(c) An observer on the z-axis at $\vec{Z} = r\hat{z}$, $r \gg c/\omega_0$ observes the differential power $dP(t)/d\Omega$ where Ω is the solid angle. Sketch the behavior of the differential power $dP(t)/d\Omega$ as a function of time.

(d) For an observer on the x-axis at $\vec{X} = r\hat{x}$, $r \gg c/\omega_0$, what is the differential power $dP(t)/d\Omega$ observed? Again, sketch $dP(t)/d\Omega$ as a function of time.

Half-Circle Radiation - Solution

(a) We can determine the radius from

$$\frac{mv_0^2}{r} = qv_0B \quad \Longrightarrow \quad r = \frac{mv_0}{qB}.$$

The angular frequency is determined by

$$\omega_0 = \frac{v_0}{r} = \frac{qB}{m}.$$

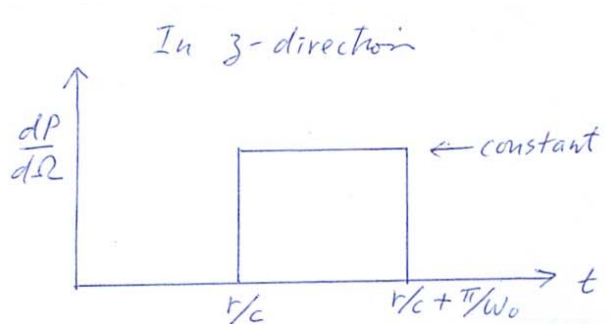
(b) We use the (non relativistic) relation for the power emitted $P = \mu_0 q^2 a^2/(6\pi c)$ where $a = r\omega_0^2$. This gives

$$P = \frac{\mu_0 q^2}{6\pi c} v_0^2 \omega_0^2.$$

Given that the particle accelerates for π/ω_0 seconds, the total energy, $E = Pt$, is

$$E = \frac{\mu_0 q^2}{6c} v_0^2 \omega_0.$$

(c) The sketch appears as follows.



(d) Here we use

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0} \frac{|\hat{\mathscr{r}} \times (\hat{u} \times \vec{a})|^2}{(\hat{\mathscr{r}} \times \vec{u})^5},$$

where $\vec{u} = c\hat{\mathscr{r}} - \vec{v} = c\hat{\mathscr{r}}$. Given that $\vec{p} = q\vec{x}$, we can write $\ddot{\vec{p}} = q\vec{a}$ and we have

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2\epsilon_0} \frac{\left|\hat{\mathscr{r}} \times (\hat{u} \times \ddot{\vec{p}})\right|^2}{(\hat{\mathscr{r}} \times \vec{u})^5}$$

In the region $x > 0$, $\vec{p} = -v_0 q/\omega_0(\sin\omega_0 t, -\cos\omega_0 t, 0)$, so that

$$\ddot{\vec{p}} = \begin{cases} qv_0\omega_0(\sin\omega_0 t, -\cos\omega_0 t, 0) & \text{for } 0 \le t \le \pi/\omega_0 \\ 0 & \text{for } t < 0,\ t > \pi/\omega_0 \end{cases}$$

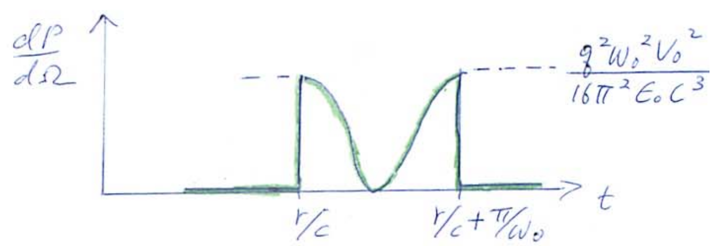
For the x direction

$$\begin{aligned} \hat{\boldsymbol{\mathscr{r}}} \times \ddot{\vec{p}} &= -\hat{z}\left[\cos\omega_0(t - r/c)\right] q\omega_0 v_0 \\ (\hat{\boldsymbol{\mathscr{r}}} \cdot \vec{u})^5 &= c^5 \\ c^2\left|\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \ddot{\vec{p}})\right|^2 &= c^2 q^2 \omega_0^2 v_0^2 \cos^2\omega_0(t - r/c) \end{aligned}$$

This gives

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2\epsilon_0 c^5} c^2 q^2 \omega_0^2 v_0^2 \cos^2\omega_0(t - r/c)$$

Hence, the sketch looks as follows



9. Statistical Mechanics of an Anharmonic Oscillator

(a) A harmonic oscillator with the Hamiltonian (energy)

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{9}$$

is in thermal equilibrium at temperature T. Using the equipartition theorem, find the RMS value $x_{\mathrm{RMS}} = \sqrt{\langle x^2 \rangle}$ of the displacement x. Here, the angular brackets denote the thermal average.

(b) The potential of the oscillator now contains anharmonic terms:

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{1}{3}\beta x^3 + \frac{1}{4}\gamma x^4. \tag{10}$$

What conditions do we have to impose on the coefficients β and γ for the anharmonic terms to be small compared to the quadratic term?

(c) Assuming that the anharmonic terms are small and treating them as perturbations, find (up to the first non-zero order of β and γ) the partition function, and the average displacement $\langle x \rangle$.

COMMENT: The integrals you need for this problem are

$$\int\limits_{-\infty}^{\infty} e^{-ax^2} x^{2k} dx = \frac{\Gamma(k+1/2)}{a^{k+1/2}} = \sqrt{\frac{\pi}{a}} \frac{(2k-1)!!}{(2a)^k}. \tag{11}$$

Statistical Mechanics of an Anharmonic Oscillator - Solution

(a) According to the equipartition theorem every quadratic term in the Hamiltonian contributes $k_BT/2$ to the average energy. Therefore, we have

$$\frac{1}{2}k_BT = \frac{1}{2}m\omega^2\langle x^2\rangle, \tag{12}$$

and

$$x_{\mathrm{RMS}} = \sqrt{\frac{k_BT}{m\omega^2}}. \tag{13}$$

(b) We should compare the values of the anharmonic terms with that of the quadratic term at x_{RMS}. The cubic term can be considered small if

$$\frac{1}{3}\beta x_{\mathrm{RMS}}^3 \ll \frac{1}{2}m\omega^2 x_{\mathrm{RMS}}^2, \qquad \Rightarrow \qquad \beta \ll \frac{m\omega^2}{x_{\mathrm{RMS}}} = \sqrt{\frac{m^3\omega^6}{k_BT}}. \tag{14}$$

Similarly, for the quartic term we have

$$\frac{1}{4}\gamma x_{\mathrm{RMS}}^4 \ll \frac{1}{2}m\omega^2 x_{\mathrm{RMS}}^2, \qquad \Rightarrow \qquad \gamma \ll \frac{m\omega^2}{x_{\mathrm{RMS}}^2} = \frac{m^2\omega^4}{k_BT}. \tag{15}$$

(c) The energy of the system is

$$H = \frac{p^2}{2m} + \frac{\alpha}{2}x^2 + \frac{\beta}{3}x^3 + \frac{\gamma}{4}x^4, \tag{16}$$

where $\alpha = m\omega^2$, and the partition function is

$$\begin{aligned} Z &= \int \frac{dpdx}{2\pi\hbar}e^{-H/k_BT} \\ &= \int_{-\infty}^{\infty} e^{-p^2/(2mk_BT)}\frac{dp}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\alpha x^2/2k_BT}e^{-\beta x^3/3k_BT}e^{-\gamma x^4/4k_BT}dx. \end{aligned} \tag{17}$$

Using Eq. (11), the momentum integral gives

$$\int_{-\infty}^{\infty} e^{-p^2/(2mk_BT)}\frac{dp}{2\pi\hbar} = \frac{(2\pi m k_BT)^{1/2}}{2\pi\hbar} = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{1/2}. \tag{18}$$

Considering β and γ to be small parameters, we can expand the exponentials in Eq. (17) as

$$e^{-\beta x^3/3k_BT} \approx 1 - \frac{\beta x^3}{3k_BT} + \frac{\beta^2 x^6}{18(k_BT)^2} \tag{19}$$

$$e^{-\gamma x^4/4k_BT} \approx 1 - \frac{\gamma x^4}{4k_BT}. \tag{20}$$

The odd correction (the x^3 term) does not contribute to the partition function. So to first (non-zero) order in each parameter, the partition function is given by

$$Z \approx \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2k_BT} \left(1 + \frac{\beta^2}{18(k_BT)^2}x^6 - \frac{\gamma}{4k_BT}x^4\right). \tag{21}$$

Integration using Eq. (11) yields the partition function

$$\begin{aligned} Z &\approx \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{1/2} \left(\frac{2\pi k_BT}{\alpha}\right)^{1/2} \left[1 + \frac{15\beta^2}{18(k_BT)^2}\left(\frac{k_BT}{\alpha}\right)^3 - \frac{3\gamma}{4k_BT}\left(\frac{k_BT}{\alpha}\right)^2\right] \\ &= \frac{k_BT}{\hbar}\sqrt{\frac{m}{\alpha}}\left[1 + \left(\frac{5\beta^2}{6\alpha} - \frac{3}{4}\gamma\right)\frac{k_BT}{\alpha^2}\right] = \frac{k_BT}{\hbar\omega}\left[1 + \left(\frac{5\beta^2}{6m\omega^2} - \frac{3}{4}\gamma\right)\frac{k_BT}{m^2\omega^4}\right]. \end{aligned} \tag{22}$$

The average position is given by

$$\langle x \rangle = \frac{1}{Z}\int \frac{dpdx}{2\pi\hbar} x e^{-H/k_BT}. \tag{23}$$

Using the same expansion as above, we only have to keep the x^3 term now, since the even terms multiplied by x in the integrand will give zero upon integration. Thus, the average position is

$$\langle x \rangle \approx -\frac{1}{Z}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{1/2} \int_{-\infty}^{\infty} x e^{-\alpha x^2/2k_BT} \frac{\beta x^3}{3k_BT}\, dx. \tag{24}$$

Since the integral here is already proportional to the small parameter β, we can replace the partition function Z by its value for the harmonic oscillator $Z_0 = k_BT/\hbar\omega$:

$$\langle x \rangle \approx -\frac{1}{Z_0}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{1/2} \frac{\beta}{3k_BT}\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2/2k_BT}\, dx = -\frac{\beta k_BT}{m^2\omega^4}. \tag{25}$$

(d) The free energy is given by

$$F = -k_BT\ln Z \approx -k_BT\ln\frac{k_BT}{\hbar\omega} + \left(\frac{3}{4}\gamma - \frac{5\beta^2}{6m\omega^2}\right)\frac{k_B^2T^2}{m^2\omega^4}. \tag{26}$$

Using the thermodynamic identities (there is no pressure or volume to worry about in this problem)

$$dE = TdS, \qquad\qquad dF = -SdT, \tag{27}$$

we have

$$E = F + TS, \qquad\qquad S = -\frac{\partial F}{\partial T}. \tag{28}$$

This gives the entropy of the oscillator:

$$S = -\frac{\partial F}{\partial T} \approx k_B \ln \frac{k_B T}{\hbar\omega} + k_B + \left(\frac{5\beta^2}{3m\omega^2} - \frac{3}{2}\gamma\right)\frac{k_B^2 T}{m^2\omega^4}, \tag{29}$$

and its average energy

$$E = F + TS \approx k_B T + \left(\frac{5\beta^2}{6m\omega^2} - \frac{3}{4}\gamma\right)\frac{k_B^2 T^2}{m^2\omega^4}. \tag{30}$$

Finally, the specific heat of the anharmonic oscillator is

$$c_v = \frac{\partial E}{\partial T} \approx k_B + \left(\frac{5\beta^2}{3m\omega^2} - \frac{3}{2}\gamma\right)\frac{k_B^2 T}{m^2\omega^4}. \tag{31}$$

Notice that γ and β contributions have different signs, so whether the specific heat is increased or decreased due to anharmonicity depends on the ratio $10\beta^2/9\gamma m\omega^2$.

10. A Bump in a Well

A quantum particle of mass m moves in a one-dimensional cavity bounded by hard walls at $x = -a$ and $x = a$.

(a) Find the properly normalized wave functions and energy levels of the particle.

(b) A weak potential of the form $V(x) = \lambda\delta(x)$ is now added. Find the shift in the energy levels to first order in λ.

(c) Find the exact solutions for the energy levels when λ is not necessarily small and show that they reduce to the answer in part (b) in the limit of small λ.

A Bump in a Well - Solution

(a) The solutions of the free-particle Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \tag{32}$$

satisfying the boundary conditions $\psi(\pm a) = 0$ are

$$\begin{aligned}
\psi^{(+)}(x) &= \frac{1}{\sqrt{a}}\cos\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{a}\right] \quad (n = 0, 1, 2, \ldots)\ , && (33)\\
\psi^{(-)}(x) &= \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi x}{a}\right) \quad (n = 1, 2, \ldots)\ . && (34)
\end{aligned}$$

The corresponding energy levels are:

$$\begin{aligned}
\psi^{(+)}\ : E_n &= \frac{\hbar^2}{2m}\left[\left(n+\frac{1}{2}\right)\frac{\pi}{a}\right]^2 \quad (n = 0, 1, 2, \ldots)\ , && (35)\\
\psi^{(-)}\ : E_n &= \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 \quad (n = 1, 2, \ldots)\ . && (36)
\end{aligned}$$

(b) If a weak perturbation $V = \lambda\delta(x)$ is added, the energy levels corresponding to $\psi^{(-)}$ are not affected, because their wave functions vanish at $x = 0$, while the energy levels corresponding to $\psi^{(+)}$ all shift by an identical amount (calculated in first-order perturbation theory)

$$\langle\psi^{(+)}|V|\psi^{(+)}\rangle = \frac{\lambda}{a}\ . \tag{37}$$

(c) The solutions for $x \neq 0$ may be written in the form

$$\psi(x) = \mathcal{N}_> \sin k(a-x)\ (x > 0)\ \ ; \quad \psi(x) = \mathcal{N}_< \sin k(a+x)\ (x < 0)\ \ . \tag{38}$$

To find the allowed values of k, we integrate the Schrödinger equation from $x = -\epsilon$ to $x = \epsilon$, where ϵ is small, and find

$$\frac{d\psi}{dx}|_{x=\epsilon} - \frac{d\psi}{dx}|_{x=-\epsilon} = \frac{2m\lambda}{\hbar^2}\psi(0)\ . \tag{39}$$

The free-particle solutions $\psi^{(-)}(x)$ continue to satisfy the Schrödinger equation with any strength of potential $\lambda\delta(x)$ because their wave functions vanish at $x = 0$. Hence their energy levels are not affected by V. The solutions $\psi^{(+)}(x)$ may be taken to have normalization factors $\mathcal{N}_> = \mathcal{N}_< = \mathcal{N}_+$ by symmetry, and one finds a transcendental equation for k,

$$-2k\cos ka = \frac{2m\lambda}{\hbar^2}\sin ka\ , \tag{40}$$

with $E = (\hbar k)^2/2m$. In the limit $\lambda \to 0$, ka approaches the roots of $\cot ka$ at $(n + \frac{1}{2})\pi$ $(n = 0, 1, 2, \ldots)$. Letting

$$k = \left(n + \frac{1}{2}\right)\frac{\pi}{a} + \eta \tag{41}$$

and expanding $k \cot ka$ to lowest order in η, one finds

$$\eta = \frac{m\lambda}{\hbar^2}\left[\left(n + \frac{1}{2}\right)\pi\right]^{-1} \quad \Rightarrow \quad E_n = \frac{(\hbar k)^2}{2m} \simeq \frac{\hbar^2}{2m}\left[\left(n + \frac{1}{2}\right)\frac{\pi}{a}\right]^2 + \frac{\lambda}{a}\ . \tag{42}$$

11. A Good Model?

A certain phenomenon, quantified by a quantity Q, is modeled to behave as

$$Q = a\xi^n \ , \tag{43}$$

where ξ is an independent quantity on which Q depends. An experimental effort makes the first measurements of this quantity with the data below:

ξ	Q
0.55	0.08 ± 0.01
1.50	1.73 ± 0.07
3.25	17.02 ± 0.15
4.75	54.01 ± 0.45

Perform an analysis to determine whether the data seem to rule out the model given. If so, give a clear explanation why. If the model remains viable, use the data to estimate values for a and n with approximate uncertainties.

A Good Model? - Solution

To test for a power-law dependence, it is best to take the logarithm, giving

$$\ln Q = \ln a + n \ln \xi \ . \tag{44}$$

The table then may be expanded to yield

ξ	$\ln \xi$	Q	$\ln Q$
0.55	-0.5978	0.08 ± 0.01	-2.526 ± 0.125
1.50	0.4055	1.73 ± 0.07	0.548 ± 0.041
3.25	1.179	17.02 ± 0.15	2.834 ± 0.0088
4.75	1.558	54.01 ± 0.45	3.989 ± 0.0083

where we have used $\ln(x \pm \delta x) = \ln x \pm (\delta x/x)$. To see whether these data satisfy the linear relation between $\ln Q$ and $\ln \xi$ (as implied by the plot on the next page), one may take differences between points:

$$\text{First two}: \quad n = 3.065 \pm 0.131 \ , \tag{45}$$

$$\text{Middle two}: \quad n = 2.955 \pm 0.054 \ , \tag{46}$$

$$\text{Last two}: \quad n = 3.047 \pm 0.032 \ . \tag{47}$$

Taking the average of these points weighted by the inverse squares of their errors, we arrive at a value $n = 3.025 \pm 0.027$. This is sufficiently close to 3 that we are encouraged to try a relation of the form $Q = a\xi^3$. If $\ln a = \ln Q - 3 \ln \xi$, we may calculate the value of $\ln a$ for each point:

$$\text{First point}: \quad \ln a = -0.733 \pm 0.125 \ , \tag{48}$$

$$\text{Second point}: \quad \ln a = -0.669 \pm 0.041 \ , \tag{49}$$

$$\text{Third point}: \quad \ln a = -0.7030 \pm 0.0088 \ , \tag{50}$$

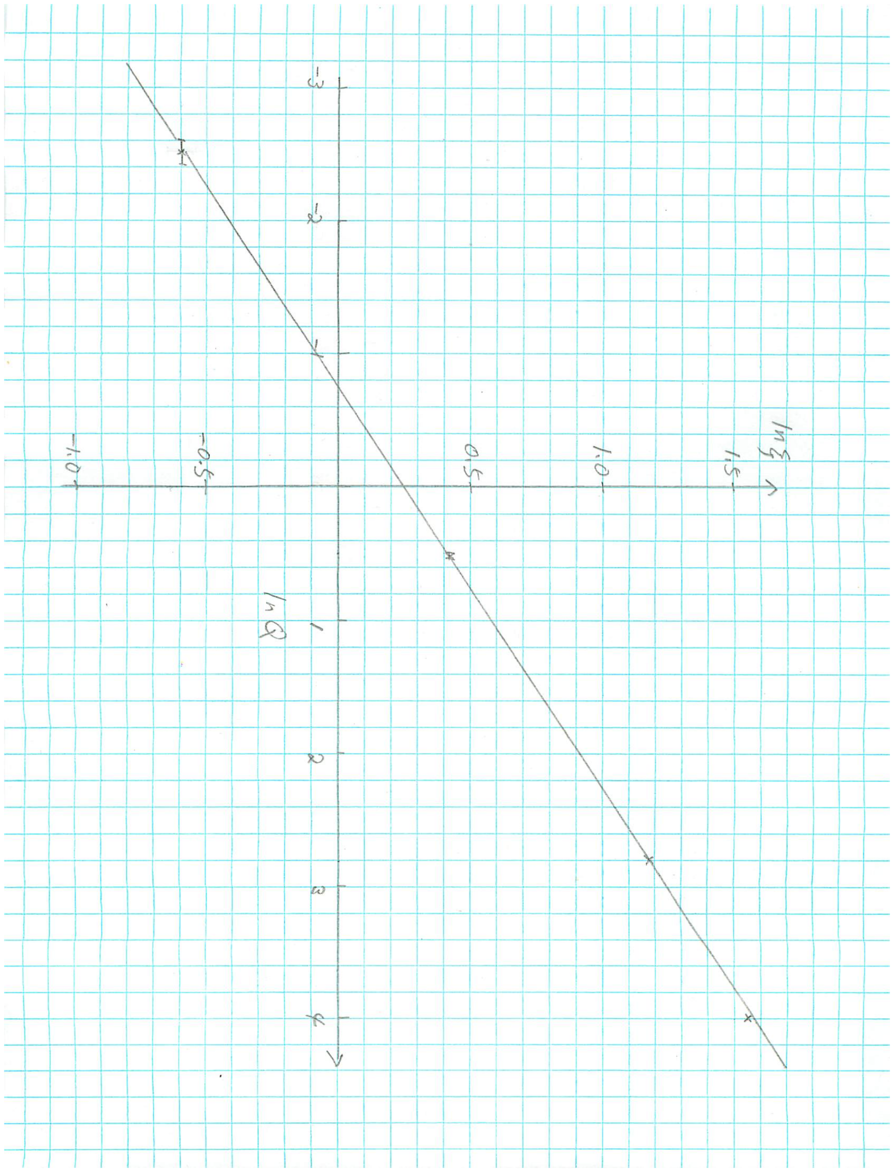
$$\text{Fourth point}: \quad \ln a = -0.6850 \pm 0.0083 \ . \tag{51}$$

Taking the average of these points weighted by the inverse squares of their errors, one finds $\ln a = 0.693 \pm 0.006$, or $a = 0.500 \pm 0.003$.

One can now calculate the quality of the model $Q = 0.5\xi^3$ using the χ^2 test:

$$\chi^2 = \sum_{i=1}^{4} \left(\frac{\text{data}_i - \text{model}_i}{\text{error}_i} \right)^2 \ , \tag{52}$$

which is calculated to be 2.28. Generally a value of χ^2 less than or equal to the number of degrees of freedom = N(data) - N(parameters), which is 2 in this case, signifies a satisfactory fit. This is close to being the case here; a more sophisticated method would have explored the values of χ^2 ranging over the different values of a and n.



12. Magnetic Dipole Interaction

Two particles with spin $s = 1/2$ and magnetic moments $\boldsymbol{\mu}_i = \mu \boldsymbol{s}_i$ are separated by a vector $\boldsymbol{r}$ and interact via the magnetic dipole-dipole interaction

$$H = \frac{1}{r^3}\left[\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 - \frac{3}{r^2}(\boldsymbol{\mu}_1 \cdot \boldsymbol{r})(\boldsymbol{\mu}_2 \cdot \boldsymbol{r})\right]. \tag{53}$$

(a) What are the possible values of the total spin S and its projection S_r on the direction of the vector $\mathbf{r}$?

(b) Consider the total spin $\boldsymbol{S} = \boldsymbol{s}_1 + \boldsymbol{s}_2$ and its projection $S_r = (\boldsymbol{S} \cdot \boldsymbol{r})/r$. Show that the Hamiltonian (53) can be written as

$$H = \frac{\mu^2}{2r^3}\left[\boldsymbol{S}^2 - 3S_r^2\right]. \tag{54}$$

Hint: express each spin-1/2 in terms of Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{55}$$

and use their properties.

(c) Find the spectrum of the Hamiltonian (53) and the spin configuration (the values of S and S_r) in the ground state. Do the particles repel or attract each other in the ground state?

Magnetic Dipole Interaction - Solution

(a) Two spins $s = 1/2$ can for either a singlet ($S = 0$) or a triplet ($S = 1$). The direction of vector $\boldsymbol{r}$ can be chosen as the quantization axis for the spin (the z axis). Then the possible values of S_r are $S_r = 0$ for the singlet, and $S_r = 0, \pm 1$ for the triplet.

(b) The square of the total spin operator

$$\boldsymbol{S}^2 = (\boldsymbol{s}_1 + \boldsymbol{s}_2)^2 = \boldsymbol{s}_1^2 + \boldsymbol{s}_2^2 + 2\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = \frac{3}{2} + 2\boldsymbol{s}_1 \cdot \boldsymbol{s}_2. \tag{56}$$

Thus,

$$\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = \frac{1}{2}\left(\boldsymbol{S}^2 - \frac{3}{2}\right). \tag{57}$$

Similarly, for the projection $S_r = (\boldsymbol{S} \cdot \boldsymbol{r})/r$ we have

$$(\boldsymbol{S} \cdot \boldsymbol{r})^2 = (\boldsymbol{s}_1 \cdot \boldsymbol{r} + \boldsymbol{s}_2 \cdot \boldsymbol{r})^2 = (\boldsymbol{s}_1 \cdot \boldsymbol{r})^2 + (\boldsymbol{s}_2 \cdot \boldsymbol{r})^2 + 2(\boldsymbol{s}_1 \cdot \boldsymbol{r})(\boldsymbol{s}_2 \cdot \boldsymbol{r}). \tag{58}$$

The first two terms here can be found as follows. Choosing the z axis along the vector $\boldsymbol{r}$, we have $(\boldsymbol{s}_1 \cdot \boldsymbol{r})^2 = (s_{1z} z)^2 = \sigma_z^2 z^2/4 = r^2/4$, where in the last equality we used the fact that with this special choice of the z axis $r = z$. The same result holds for $(\boldsymbol{s}_2 \cdot \boldsymbol{r})^2$.

An alternative derivation. Writing the components of the spin operator $\boldsymbol{s}_1$ as $\sigma_x/2, \sigma_y/2, \sigma_z/2$, and the components of $\boldsymbol{r}$ as x, y, z, we have

$$\begin{aligned}(\boldsymbol{s}_1 \cdot \boldsymbol{r})^2 = \frac{1}{4}\big(\sigma_x x + \sigma_y y + \sigma_z z\big)^2 = \frac{1}{4}\big(&\sigma_x^2 x^2 + \sigma_y^2 y^2 + \sigma_z^2 z^2 \\ &+ (\sigma_x\sigma_y + \sigma_y\sigma_x)xy + (\sigma_x\sigma_z + \sigma_z\sigma_x)xz + (\sigma_y\sigma_z + \sigma_z\sigma_y)yz\big).\end{aligned} \tag{59}$$

Using the properties of the Pauli matrices

$$\begin{aligned}\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1,\\ \sigma_x\sigma_y + \sigma_y\sigma_x = \sigma_x\sigma_z + \sigma_z\sigma_x = \sigma_y\sigma_z + \sigma_z\sigma_y = 0,\end{aligned} \tag{60}$$

we have

$$(\boldsymbol{s}_1 \cdot \boldsymbol{r})^2 = \frac{1}{4}\big(x^2 + y^2 + z^2\big) = \frac{r^2}{4}, \tag{61}$$

and the same is true for $(\boldsymbol{s}_2 \cdot \boldsymbol{r})^2$.

Using these values in Eq. (58) we get

$$(\boldsymbol{S} \cdot \boldsymbol{r})^2 = \frac{r^2}{2} + 2(\boldsymbol{s}_1 \cdot \boldsymbol{r})(\boldsymbol{s}_2 \cdot \boldsymbol{r}), \tag{62}$$

and

$$(\boldsymbol{s}_1 \cdot \boldsymbol{r})(\boldsymbol{s}_2 \cdot \boldsymbol{r}) = \frac{1}{2}\left(\boldsymbol{S} \cdot \boldsymbol{r})^2 - \frac{r^2}{2}\right) = \frac{r^2}{2}\left(S_r^2 - \frac{1}{2}\right). \tag{63}$$

Now we turn to the Hamiltonian:

$$H = \frac{1}{r^3}\Big[\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 - \frac{3}{r^2}(\boldsymbol{\mu}_1 \cdot \boldsymbol{r})(\boldsymbol{\mu}_2 \cdot \boldsymbol{r})\Big] = \frac{\mu^2}{r^3}\Big[\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 - \frac{3}{r^2}(\boldsymbol{s}_1 \cdot \boldsymbol{r})(\boldsymbol{s}_2 \cdot \boldsymbol{r})\Big]. \tag{64}$$

Using Eqs. (57, 63) we rewrite the Hamiltonian as

$$H = \frac{\mu^2}{2r^3}\Big[\boldsymbol{S}^2 - 3S_r^2\Big]. \tag{65}$$

(c) Given the preciously determined values of S and S_r we find the spectrum of the Hamiltonian. The energy levels are given in terms of S and S_r as

$$E_{S,S_r} = \frac{\mu^2}{2r^3}\Big[S(S+1) - 3S_r^2\Big]. \tag{66}$$

Thus, we get the spectrum

$$E_{0,0} = 0, \qquad E_{1,1} = E_{1,-1} = -\frac{\mu^2}{2r^3}, \qquad E_{1,0} = \frac{\mu^2}{r^3}. \tag{67}$$

We see that the ground state with the energy $-\mu^2/2r^3$ is doubly degenerate, and that the spin configuration can be either $S = 1, S_r = 1$ or $S = 1, S_r = -1$. The dependence of the ground state energy on the distance between the particles r clearly corresponds to the attraction between them.