Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part I
Saturday, March 28, 2009

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ________________
Fill in your S-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your S-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

The problems are divided into two groups. Solve

Group A: 4 problems out of 5
Group B: 3 problems out of 4

Do not turn in more than the above number $(4 + 3 = 7)$ of problems.

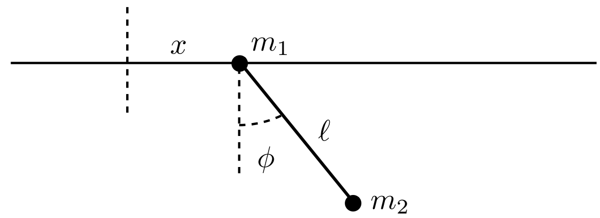
The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ A.1 ☐ A.2 ☐ A.3 ☐ A.4 ☐ A.5
☐ B.1 ☐ B.2 ☐ B.3 ☐ B.4

Group A. **Choose 4 out of 5 problems**

A.1. (Classical Mechanics)



A simple pendulum of mass m_2 and length ℓ is attached to mass m_1 which can freely slide along a horizontal line. The horizontal line lies in the plane of oscillations of the pendulum, see the figure. All motion is frictionless, and the link between the two masses is assumed straight, massless and unstretchable.

(i) Find the Lagrangian of this system.

(ii) Write down the equations of motion and identify conserved quantities.

(iii) Describe *qualitatively* the motion of the system. (You do not need to solve the equations of motion.)

A.2. (Electricity and Magnetism)

Consider a standard LRC circuit with an electromotive force $\mathcal{E}$, an inductance L, a capacitor C, a resistor R, and a switch S all connected in series.

(i) Draw this circuit.

(ii) Suppose that the switch is initially open and the charge on the capacitor is zero. At $t = 0$, the switch is closed. Calculate the current, $I(t)$, that flows in the circuit for all later times.

A.3. (Thermodynamics)

Consider a single component thermodynamic system. The specific heat at constant volume, C_V, is defined by

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N},$$

where U is the internal energy.

(i) Derive a relation between C_V and

$$\left(\frac{\partial^2 S}{\partial U^2}\right)_{V,N},$$

where $S = S(U, V, N)$ is the entropy of the system.

(ii) Using (i), prove that for an isolated system in thermodynamic equilibrium $C_V \geq 0$.

A.4. (Quantum Mechanics)

Consider the scattering problem in one dimension where the potential is

$$V(x) \;=\; \frac{\hbar^2}{2m}\lambda\,\delta(x)\,,$$

where m is the mass of a particle and λ is a real constant. There are an incoming wave and a reflected wave, $\psi_{\rm in} = Ce^{ikx}$ and $\psi_{\rm ref} = CRe^{-ikx}$, for $x < 0$, and a transmitted wave, $\psi_{\rm tr} = C\,Te^{ikx}$, for $x > 0$. The wave vector, k, is related to the particle energy, E, by $(\hbar k)^2/2m = E$. At $x = 0$ the wave function is continuous, but its derivative is discontinuous.

(i) Use the Schrödinger equation to show that the discontinuity in the derivative of the wave function is given by

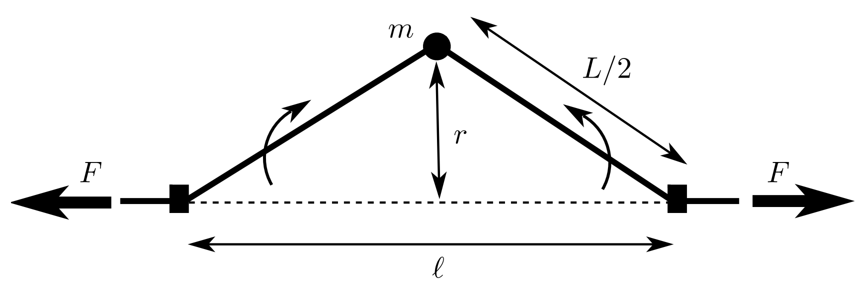
$$\lim_{\epsilon\to 0^+} \left.\frac{d\psi}{dx}\right|_{0-\epsilon}^{0+\epsilon} \;=\; \lambda\,\psi(0)\,. \tag{1}$$

(ii) Calculate the reflection and transmission coefficients R and T.

(iii) Show that $|R|^2 + |T|^2 = 1$. What is the physical meaning of this equation?

Hint: Parts (ii) and (iii) can be solved using (1) without proving it.

A.5. (Statistical Physics)



Consider a mass m fixed to the middle point of a string of length L whose extremities are a distance ℓ apart, and pulled with a tension F. The system is in thermal equilibrium, and one supposes that the only effect of thermal fluctuations is to make the system rotate about the horizontal (dashed) axis. As a result of this rotation, a tension force F arises along the string, as shown.

(i) Calculate the tension F as a function of the speed v of the mass.

(ii) Show that

$$\langle F\rangle = \frac{\ell}{L^2 - \ell^2}\,k_B T\,,$$

where k_B is the Boltzman constant and T is the temperature.

Group B. **Choose 3 out of 4 problems**

B.1. (Mathematical Methods)

Calculate the Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} dx\, e^{ikx} f(x)\,,$$

for the following functions:

(i) $$f(x) = e^{-x^2/2}\,.$$

Hint: You may use that $\tilde{f}(0) = \sqrt{2\pi}$.

(ii) $$f(x) = e^{-|x|}\,.$$

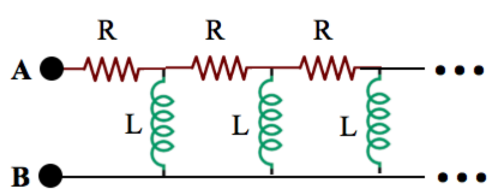
(iii) $$f(x) = \frac{1}{1+x^2}\,.$$

B.2. (Special Relativity)

Cosmic ray photons from space are bombarding your laboratory and smashing massive objects to pieces. Your detectors indicate that a stationary object of mass M has been split into two fragments, each of mass m_0, that depart a collision moving at speed $0.6\,c$ at angles of 60° relative to the photon's original direction of motion.

(i) In terms of m_0 and c, what is the energy of the cosmic ray photon?
(ii) In terms of m_0, what is the mass M of the particle being struck?

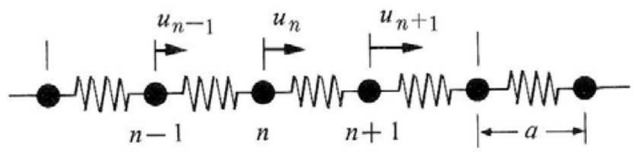
B.3. (Experimental Physics)



(i) Derive an expression for the complex impedance Z between points A and B of the infinite LR-circuit shown above.
(ii) When the frequency ω is very small, how does the impedance depend on frequency?
(iii) When the frequency ω is very large, how does the impedance depend on frequency?

Hint: Consider the impedance of the circuit when an adjacent pair of R and L is removed.

B.4. (Solid State)



Consider a longitudinal wave propagating in a monoatomic linear chain of atoms of mass M, spacing a, and nearest-neighbor elastic interaction constant C. The equation of motion for the n-th atom is

$$M\ddot{u}_n \;=\; C(u_{n+1} - u_n) + C(u_{n-1} - u_n)\,, \tag{1}$$

where u_n is the displacement of the n-th atom. The complex representation of the travelling wave solution is

$$u_n \;=\; Ae^{i(kx_n - \omega t)}\,, \tag{2}$$

where $x_n = na$ is the equilibrium position of the n-th atom, and the dispersion relation is

$$\omega^2 \;=\; \frac{2C}{M}(1 - \cos ka)\,. \tag{3}$$

(i) Show that the dispersion relation (1) becomes an acoustic wave in the long-wavelength limit. What is the speed of this wave?

(ii) Argue that the total energy of the wave is

$$E \;=\; \frac{M}{2}\sum_n \dot{u}_n^2 + \frac{C}{2}\sum_n (u_n - u_{n+1})^2\,.$$

Then use the solution (2,3) to show that the time-average total energy per atom is equal to

$$M\omega^2 A^2\,.$$

Hint: Recall that for two complex oscillating functions the time average is

$$\langle fg\rangle = \frac{1}{2}\mathrm{Re}(fg^*)\,.$$

(iii) Show how in the long-wavelength limit eq. (1) goes over into a one-dimensional wave equation. Demonstrate that the wave speed predicted by this equation agrees with the answer in (i).

Department of Physics and Astronomy
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Graduate Screening Examination
Part II
Saturday, April 4, 2009

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ________________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Partial credits for each problem are indicated.

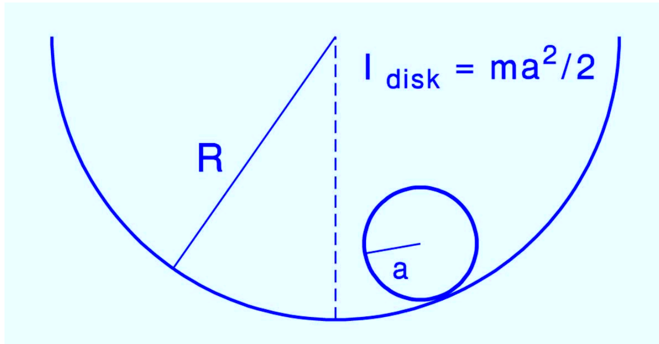
Solve 2 problems of your choice. Do not turn in more than 2 problems.

The total time allowed **2 hrs**.

Please, indicate problems you are turning in

☐ II-1 ☐ II-3 ☐ II-4

II-1. (Classical Mechanics)



A uniform thin disk of mass m and radius a is rolling without sliding inside a circular track of radius R as shown. Assume that the motion is limited to a vertical plane.

(i) (7 pts) Find the Lagrangian of the system and also the Lagrange's equation of motion.

(ii) (3 pts) Find the Hamiltonian which describes the motion of this system.

(iii) (3 pts) Write down the Hamilton-Jacobi equation.

(iv) (7 pts) If the motion of the disk is such that the center of mass moves slightly away from the vertical at the bottom of its track, find the frequency of the small oscillation with the method of *action-angle variable* by first expressing the Hamiltonian in terms of the action-angle variable J.

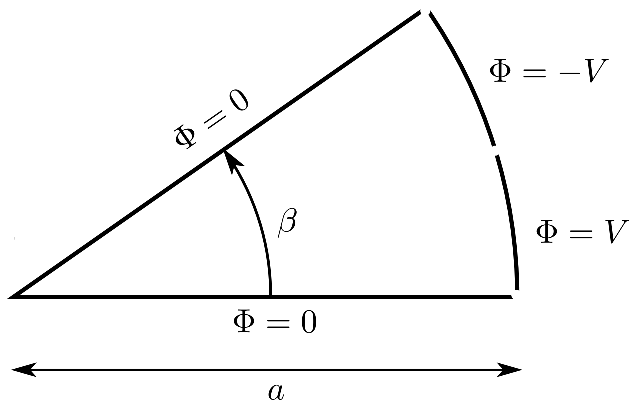
Hint: You may find useful the following integrals:

$$\int \sin^2 x\, dx = \frac{1}{2} - \frac{1}{4}\sin 2x\,, \qquad \int \cos^2 x\, dx = \frac{1}{2} + \frac{1}{4}\sin 2x\,,$$

$$\int (a^2 - x^2)^{1/2} dx = \frac{1}{2}\left[x(a^2 - x^2)^{1/2} + a^2 \arcsin\frac{x}{a}\right], \qquad a > 0\,,$$

$$\int \frac{dx}{(a^2 - x^2)^{1/2}} = \arcsin\frac{x}{a}\,, \qquad a > 0\,.$$

II-2. (Electricity and Magnetism)



Consider a two-dimensional wedge geometry. The volume enclosed is expressed in polar coordinates as $0 < \rho < a$ and $0 < \theta < \beta$. There is no charge in the volume, but there is a potential on the surface given by (see, the figure):

- $\Phi = 0$ for $\theta = 0$ and $\theta = \beta$;
- on the surface at $\rho = a$, Φ takes two different constant values,
  $\Phi = V$ for $0 < \theta < \beta/2$ and $\Phi = -V$ for $\beta/2 < \theta < \beta$.

(i) (12 pts) Find a Fourier sine-series solution in the θ variable to the Laplace equation, $\nabla^2\Phi = 0$, everywhere in the volume satisfying the above boundary conditions.

(ii) (8 pts) Then sum the series explicitly.

Hint: Potentially useful formula

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots .$$

II-3. (Quantum Mechanics)

A certain system has the Hamiltonian

$$\boldsymbol{H} \;=\; E_0 \sum_{n=1}^{N} |n\rangle\langle n| + \lambda \sum_{n=1}^{N} \left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right) \,,$$

where λ is the coupling energy and E_0 is the (degenerate) energy in the zero-coupling (i.e., $\lambda = 0$) limit. The base kets $|n\rangle$, $n = 1, 2, \ldots, N$, are complete and orthonormal, and satisfy the periodic boundary condition

$$|N+n\rangle \;=\; |n\rangle \,.$$

Define the operator

$$\boldsymbol{b} \;=\; \sum_{n=1}^{N} |n\rangle\langle n+1| \,.$$

(i) (2 pts) Express $\boldsymbol{H}$ in terms of $\boldsymbol{b}$, $\boldsymbol{b}^\dagger$, and the identity operator $\mathbf{1}$.

(ii) (2 pts) Show that $\boldsymbol{H}$, $\boldsymbol{b}$, and $\boldsymbol{b}^\dagger$ mutually commute.

(iii) (4 pts) Represent $\boldsymbol{b}$ by a $N \times N$ matrix on the basis $\{|n\rangle, n = 1, 2, \ldots, N\}$.

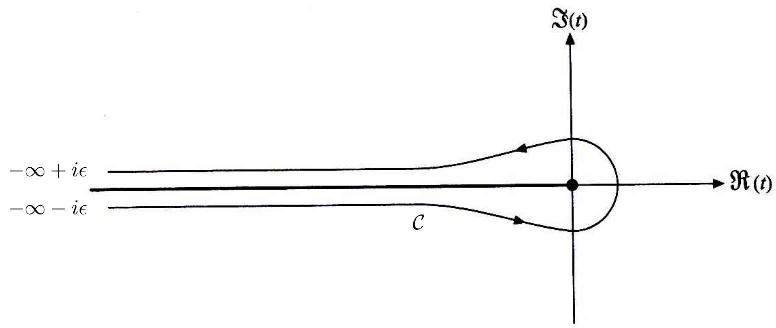
(iv) (6 pts) Determine the eigenvalues of $\boldsymbol{b}$ and $\boldsymbol{b}^\dagger$.

(v) (6 pts) Determine the eigenvalues of $\boldsymbol{H}$ and show that the eigenkets are

$$|E_m\rangle \;=\; \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i(n-1)\theta_m} |n\rangle \,,$$

where $\theta_m \;=\; 2\pi(m/N)$.

II-4. (Mathematical Methods)



The standard definition of the gamma function, $\Gamma(z)$, in terms of the Euler integral is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt\,, \qquad \mathrm{Re}\, z > 0\,. \tag{1}$$

(i) (5 pts) Show that (1) implies that for positive integers,

$$\Gamma(n+1) = n!\,, \qquad n = 0, 1, 2, \ldots.$$

Now, consider a function $G(z)$ defined by the contour integral,

$$G(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{e^t}{t^z} dt\,, \tag{2}$$

where the contour $\mathcal{C}$ is shown in the figure above. The integrand is made single valued by introducing a branch cut in the t-plane along the real axis from $-\infty$ to 0, such that $\mathrm{Arg}\, t = +\pi$ and $\mathrm{Arg}\, t = -\pi$ right above and below the cut, respectively.

(ii) (2 pts) For which complex z's does the above integral converge and hence $G(z)$ is well defined? What is the domain of analyticity of $G(z)$? (Don't worry about rigorous proofs, just give a plausible argument that supports your claims.)

(iii) (4 pts) Evaluate the complex integral (2) for $z = n$, where $n = 0, \pm 1, \pm 2, \ldots$, is a real integer.

(iv) (7 pts) Use (2) to calculate the derivative $G'(z)$ for $z = 0, -1, -2, \ldots$.

(v) (2 pts) Based on the properties of $G(z)$, suggest a relation between $G(z)$ and the gamma function $\Gamma(z)$.