QUALIFYING EXAMINATION, Part 1

1:00 PM – 4:00 PM, Thursday August 30, 2012

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Mathematical Methods

(a) (30 points) Evaluate $\nabla^2 \left(\frac{1}{r}\right)$ given

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + (\theta \text{ and } \phi \ \text{ terms}) \ ,$$

$$\nabla f = \hat{\mathbf{e}}_r \frac{\partial f}{\partial r} + (\theta \text{ and } \phi \ \text{ terms}) \ ,$$

where $\hat{\mathbf{e}}_r$ is a unit vector in the radial direction.

Hint: to obtain the limit $r \to 0$, evaluate the integral of $\nabla^2 \left(\frac{1}{r}\right)$ over the volume of a small sphere around $r = 0$.

In the following parts (b)-(e), consider the Γ function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{for} \quad x > 0 \ .$$

(b) (5 points) Evaluate $\Gamma(1)$.

(c) (20 points) Show that $\Gamma(n+1) = n!$ when n is a non-negative integer $(n \geq 0)$.

(d) (10 points) Approximate $\ln n!$ in the limit of large n $(n \gg 1)$ without using the integral representation of the Γ function by writing $\ln n!$ as a sum (using the usual definition of $n!$) and replacing the sum by an integral. Rewrite the answer as an approximation for $n!$.

(e) (35 points) Improve the approximation in (d) by evaluating the above integral for $\Gamma(n+1)$ in the saddle point approximation for $n \gg 1$ (including the gaussian correction) to show

$$n! = n^n e^{-n} \sqrt{2\pi n} \ (1 + \text{corrections} \ ) \ .$$

Hint: for $\alpha > 0$

$$\int_{-\infty}^{\infty} e^{-\alpha w^2} dw = \sqrt{\frac{\pi}{\alpha}} \ .$$

Problem 2: Classical Mechanics

Consider a point particle of mass m moving in the x-y plane.

(a) (30 points) For each Lagrangian below determine which, if any, of the following quantities are conserved: momentum in the x direction p_x, momentum in the y direction p_y, angular momentum in the z direction L_z, and total energy E. For each conserved quantity state the transformations under which the system is invariant.

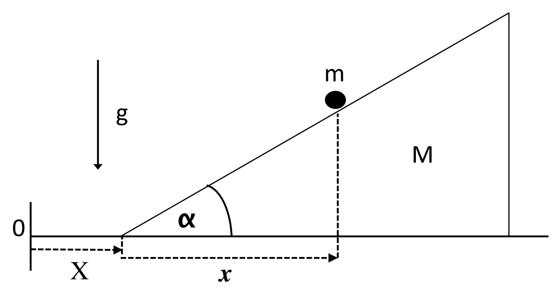
$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}ky^2 \ , \\
\mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}k(x^2+y^2) \ , \\
\mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}k(t)y^2 \ .
\end{aligned}$$

Next [in parts (b),(c)], consider the point particle to scatter in a central potential $U(r) = -k/r^2$ $(k > 0)$. The particle initially moves on the far left towards the origin with velocity $\vec{v} = v_0\hat{x}$ parallel to the x-axis but displaced by a distance b (the impact parameter) in the $+\hat{y}$ direction.

(b) (10 points) What are the constants of motion that characterize the particle subsequent motion ? Write down the explicit values of these constants.

(c) (20 points) What is the distance of closest approach of this particle to the origin (the scattering center)? Express your answer in terms of m, v_0, b, and k. Is this distance greater than or smaller than the impact parameter b? Provide a qualitative explanation of your answer to the latter question.

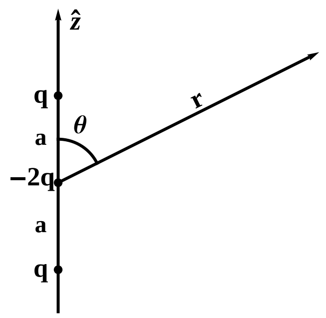
(d) (40 points) The point particle is placed on a frictionless wedge of mass M and opening angle α (see figure) and is subjected to a gravitational field g. The wedge itself slides without friction on a horizontal surface. We denote by X the horizontal coordinate of the left corner of the wedge and by x horizontal displacement of the particle from this left corner.



Write the Lagrangian of the system in terms of x and X and find the accelerations $\ddot{x}$ and $\ddot{X}$.

Problem 3: Electromagnetism I

Consider three charges q, $-2q$ and q located along the z-axis at $a\hat{z}$, 0, and $-a\hat{z}$ respectively (see figure).



(a) (15 points) Find the electrostatic potential at a general point (r, θ).

(b) (35 points) Consider the limit $a \to 0$ keeping $Q = qa^2$ constant. Show that the potential at (r, θ) is given [in SI (mks) units] by

$$V(r, \theta) = \frac{Q}{2\pi\epsilon_0 r^3} P_2(\cos\theta) ,$$

where $P_2(x) = (3x^2 - 1)/2$ is the second order Legendre polynomial.

Hint: use the law of cosines relating the lengths of the sides of a triangle a, b, and c and the the angle θ opposite to side c by $c^2 = a^2 + b^2 - 2ab\cos\theta$.

For the remaining parts of this problem (c)-(d) assume the system in (b) is surrounded with a *grounded* conducting hollow sphere of radius R with its center at the origin.

(c) (15 points) What is the electrostatic potential outside the sphere? Explain how you derive your answer.

(d) (35 points) Find the electrostatic potential inside the sphere.

Hint: the solution of Laplace's equation in spherical coordinates with azimuthal symmetry has the general form

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) ,$$

where P_l are Legendre polynomials and A_l, B_l are constants.

Use the result in (b) in the limit $r \to 0$.

Problem 4: Electromagnetism II

Consider a very long uniform cylinder of radius a made of material with permeability μ. A wire (of negligible thickness and insulated from the cylinder) is uniformly wrapped around the cylinder with n turns per unit length and carries a time-dependent current that produces a given time-dependent magnetic field of magnitude $B(t)$ inside the cylinder.

(a) (10 points) What is the voltage V per turn of the wire coil?

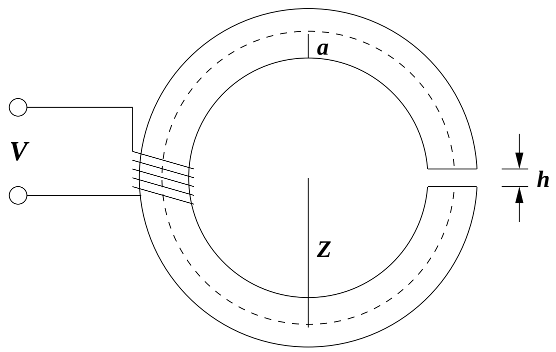
(b) (10 points) What is the magnetic field outside of the cylinder ($r > a$)? Show how you derive your answer.

(c) (15 points) Find the current in the wire as a function of time.

(d) (30 points) Find the vector potential $\vec{A}(r, t)$ outside the cylinder ($r > a$) and inside the cylinder ($r < a$).

(e) (35 points) Consider a cylinder of the same material and with radius a that is bent to form a torus of radius Z (see figure) with $a \ll Z$. The torus is not completely closed and a small gap of length h ($h \ll a$) remains between the two ends of the bent cylinder. This time, a wire with resistance R is wrapped around the torus with a total number of N turns (see figure), and a constant voltage V is applied to its ends.

Find the magnetic field B inside the gap. Assume the permeability μ of the material to be very large compared with the permeability of free space μ_0 so that the field inside the torus is uniform.



Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1\frac{\partial\psi}{\partial x_1} + \mathbf{e}_2\frac{\partial\psi}{\partial x_2} + \mathbf{e}_3\frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1\frac{\partial\psi}{\partial\rho} + \mathbf{e}_2\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3\frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1\frac{\partial\psi}{\partial r} + \mathbf{e}_2\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$

$$+\mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

QUALIFYING EXAMINATION, Part 2

1:30 PM – 4:30 PM, Friday August 31, 2012

Attempt all parts of all four problems.

Please begin your answer to each problem on a separate sheet, write your 3 digit code and the problem number on each sheet, and then number and staple together the sheets for each problem. Each problem is worth 100 points; partial credit will be given.

Calculators may NOT be used.

Problem 1: Quantum Mechanics I

Consider a one-dimensional quantum harmonic oscillator of mass m and frequency ω. Its Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \ ,$$

where x is the coordinate and p is the momentum operator. We denote by $|n\rangle$ ($n = 0, 1, 2, \ldots$) the n-th excited eigenstate of the oscillator. The annihilation operator is given by

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\omega\hbar}}p \ .$$

(a) (15 points) Show that

$$[a, a^\dagger] = 1 \ .$$

(b) (15 points) Using

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ ; \quad a|n\rangle = \sqrt{n}|n-1\rangle$$

write out the first 3×3 block of matrices representing a and $a^\dagger$ in the basis $|n\rangle$ (i.e., the upper left hand corner of each matrix).

(c) (30 points) Calculate $\langle H\rangle$, $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, and $\langle p^2\rangle$ in the eigenstate $|n\rangle$.

(d) (30 points) Consider the following normalized superposition at time $t = 0$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \ .$$

Find the wave function $|\psi(t)\rangle$ at time t (expressed in terms of $|0\rangle$ and $|1\rangle$), and $\langle x\rangle$ and $\langle x^2\rangle$ as a function of time t.

(e) (10 points) Consider the initial superposition

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |2\rangle\right) \ .$$

Find $\langle x\rangle$ as a function of time t.

Problem 2: Quantum Mechanics II

(a) (30 points) Consider two nonrelativistic, noninteracting spin-1/2 fermions each of mass m in an infinite square well potential of length L between $x = -L/2$ and $x = L/2$. Find the five lowest energy eigenstates and eigenvalues of this system.

Hint: express a two-particle eigenstate as a product of a spatial wave function and a spin wave function.

(b) (25 points) Consider the same system as in part (a), but with the two particles interacting via an exchange term

$$H_{\rm ex} = A\vec{\sigma}_1 \cdot \vec{\sigma}_2 \ ,$$

where A is a positive constant, and $\vec{\sigma}_1$, $\vec{\sigma}_2$ are, respectively, the dimensionless spin operators of particle 1 and 2. Find the *exact* (i.e., not just in perturbation theory) five lowest energy eigenstates and eigenvalues. Assume that A is much smaller than the ground-state energy you found in part (a).

(c) (15 points) Consider a nonrelativistic spinless particle of mass m in an infinite square well potential of length L between $x = -L/2$ and $x = L/2$. The following time-dependent perturbation is added to the system

$$H_1 = B \sin \omega t \ ,$$

where B and ω are positive constants and t is the time. You may assume that B is very small and that H_1 has been applied for a long time. What are the selection rules for transitions generated by this perturbation (i.e., which eigenstates of the unperturbed system will be coupled by H_1)? Consider all possible values of ω.

(d) (30 points) Consider the same situation as in part (c), but now with

$$H_1 = Bx \sin \omega t \ ,$$

where x is the position of the particle. What are the selection rules for transitions generated by this perturbation? Consider all possible values of ω.

Problem 3: Statistical Mechanics I

Consider a system of N distinguishable particles. Each particle can be in one of three states having energies ϵ_1, ϵ_2, and ϵ_3. We denote the number of particles in the state with energy ϵ_i by n_i $(i = 1, 2, 3)$.

(a) (30 points) What is the number of states of this system, Ω, given N and n_i $(i = 1, 2, 3)$?

What is the entropy S of the system? Simplify the expression for S assuming that N and n_i are all large, so that Stirling's approximation $\ln n! \approx n \log n - n$ can be applied.

(b) (10 points) State the general extremum principle that the entropy satisfies for a system in equilibrium and under what conditions.

(c) (10 points) Using the expression you derived for the entropy in part (a), show that S/N (the entropy per particle) coincides with the usual single-particle entropy if we interpret $\frac{n_i}{N}$ as the probability of the particle to occupy state i.

(d) (10 points) Write expressions for the total energy of the system, E, and the total number of particles N in terms of ϵ_i and n_i.

(e) (40 points) Assume that the system is *isolated*, i.e., its total energy and total particle number are fixed. Compute $\frac{n_i}{N}$ $(i = 1, 2, 3)$ in equilibrium using the method of Lagrange multipliers and the results from (a) and (d). Express your answer in terms of the energies ϵ_i and one of the Lagrange multipliers (the other Lagrange multiplier can be eliminated explicitly).

Write an implicit equation for the remaining Lagrange multiplier. What is the physical interpretation of this Lagrange multiplier ?

Problem 4: Statistical Mechanics II

Consider a two-dimensional ideal gas of N non-relativistic spinless fermions of mass m in a box of area A.

(a) (20 points) Find the density of single-particle states $g(\epsilon)$ at energy ϵ.

(b) (25 points) Write the equation that determines the chemical potential μ of the gas at temperature T. Show that in the limit when the fugacity $\zeta \equiv e^{\beta\mu} \ll 1$ (classical limit), the approximate solution to this equation is

$$\zeta \approx \frac{N}{N_c} \quad \text{where} \quad N_c = A\frac{mkT}{2\pi\hbar^2} \ .$$

Hint: in the limit $\zeta \ll 1$ we have $e^{\beta(\epsilon-\mu)} \gg 1$ for any $\epsilon \geq 0$.

In the following parts (c) - (d), we consider the situation where in addition to the free-particle states, the fermions can also occupy any of N bound single-particle states that have the same negative energy $-\epsilon_0$ ($\epsilon_0 > 0$). Note that the number N of single-particle bound states is equal to the number of fermions (N) so that at zero temperature the fermions fully occupy the bound states.

(c) (25 points) At temperature T, write down expressions for N_b and N_f, the average numbers of bound and free particles, respectively, in terms of the chemical potential μ of the system. Evaluate the expression for N_f in the limit $\zeta \ll 1$.

(d) (30 points) Write down the equation that determines μ. Show that in the limit $\zeta \ll 1$ this equation reduces to a quadratic equation for ζ. Solve this equation to determine ζ and N_f as a function of T and N.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$

$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. . .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1 m_2	1, +1
+1/2 +1/2	1

m_1 m_2	1, 0	0, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1, −1
−1/2 −1/2	1

1×1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

2×1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

1×1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

2×1/2

m_1 m_2	5/2, +5/2
+2 +1/2	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

3/2×1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1