Spring 2012

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

March 20, 2012

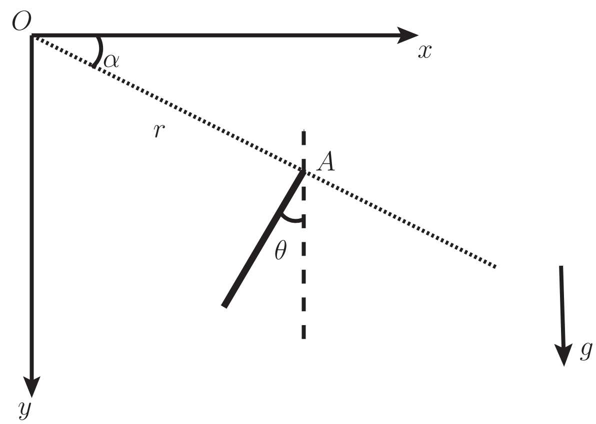
(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Stick in a Vertical Plane

A uniform stick with length ℓ and mass M is moving in a vertical plane (with gravity pointing downwards), as shown in the figure below. One end of the stick, labelled by A, is constrained to move along a straight line $y = x \tan \alpha$. The other end of the stick is free. Denote the angle between the stick and the vertical y direction by θ, and the distance between end A and the origin O by r.



(a) Write the Lagrangian and equation of motion using coordinates r and θ.

(b) Is it possible for the stick to have a translational motion without rotation? If so, what are the possible values of θ during such translational motion?

Stick in a Vertical Plane - Solution

The center of mass of the stick is

$$\begin{aligned} x_c &= r\cos\alpha - \frac{\ell}{2}\sin\theta \\ y_c &= r\sin\alpha + \frac{\ell}{2}\cos\theta \end{aligned} \tag{1}$$

Therefore,

$$\begin{aligned} \dot{x}_c &= \dot{r}\cos\alpha - \frac{\ell}{2}\cos\theta\dot{\theta} \\ \dot{y}_c &= \dot{r}\sin\alpha - \frac{\ell}{2}\sin\theta\dot{\theta} \end{aligned} \tag{2}$$

The total kinetic energy is

$$\begin{aligned} T &= \frac{M}{2}(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{24}M\ell^2\dot{\theta}^2 \\ &= \frac{M}{2}\dot{r}^2 + \frac{1}{6}M\ell^2\dot{\theta}^2 - \frac{1}{2}M\ell\cos(\alpha-\theta)\dot{r}\dot{\theta}. \end{aligned} \tag{3}$$

The potential energy is

$$V = -Mg(r\sin\alpha + \frac{\ell}{2}\cos\theta). \tag{4}$$

Now, $L = T - V$. The equation of motions are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}, \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}, \tag{5}$$

which lead to

$$\begin{aligned} &\ddot{r} - \frac{1}{2}\ell\cos(\alpha-\theta)\ddot{\theta} - \frac{1}{2}\ell\sin(\alpha-\theta)\dot{\theta}^2 - g\sin\alpha = 0 \\ &\frac{1}{3}\ell^2\ddot{\theta} - \frac{1}{2}\ell\cos(\alpha-\theta)\ddot{r} + \frac{g}{2}\ell\sin\theta = 0 \end{aligned} \tag{6}$$

For pure translational motion without rotation, we require $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. Therefore,

$$\begin{aligned} &\cos(\alpha-\theta)\ddot{r} = g\sin\theta, \\ &\ddot{r} = g\sin\alpha. \end{aligned} \tag{7}$$

which implies

$$\cos(\alpha-\theta)\sin\alpha = \sin\theta, \tag{8}$$

with solutions

$$\theta = \alpha, \qquad \theta = \alpha - \pi. \tag{9}$$

2. Electron Bubbles

When electrons are injected into liquid helium with nearly zero energy, bubbles quickly form around them. The surface tension of liquid helium at zero temperature is $\sigma \approx 0.4$ erg/cm^2. Assume that the bubble surface is impenetrable for electrons. The electron mass $m \approx 9 \cdot 10^{-28}$ g, while Planck's constant $\hbar \approx 10^{-27}$ erg·sec.

(a) Find or estimate the radius R of a bubble containing one electron at zero temperature.

(b) Find the radius $R(N)$ of a bubble containing $N \gg 1$ electrons at zero temperature. Neglect the interaction between electrons.

Electron Bubbles - Solution

(a) Let us first estimate the radius R. The electron trapped in a bubble of radius R in its ground state has an uncertainty in its momentum of order $\hbar/R$. The ground state energy can then be estimated as

$$E_0(R) = C\frac{\hbar^2}{mR^2}, \tag{10}$$

with C a constant of order unity. To create the surface of the bubble costs surface energy

$$E_{\text{surf}} = 4\pi R^2\sigma. \tag{11}$$

The total energy $E(R)$ as a function of radius,

$$E(R) = C\frac{\hbar^2}{mR^2} + 4\pi R^2\sigma, \tag{12}$$

should be minimized with respect to R. This gives $8\pi\sigma R = 2C\hbar^2/(mR^3)$, or

$$R = \left(\frac{C\hbar^2}{4\pi m\sigma}\right)^{1/4} \approx C^{1/4}\left(\frac{10^{-54}}{4\cdot 3.14\cdot 9\cdot 0.4\cdot 10^{-28}}\right)^{1/4}\text{cm} \approx C^{1/4}\left(\frac{10^{-26}}{45}\right)^{1/4}\text{cm} \approx 1.2\cdot 10^{-7}C^{1/4}\text{cm}. \tag{13}$$

The actual ground state energy E_0 of a particle of mass m trapped in an infinitely deep spherical well of radius R can be found exactly. Indeed the ground state wave function has zero angular momentum, and satisfies the radial part of the Schrodinger equation

$$\frac{1}{r}\frac{d^2}{dr^2}(r\psi(r)) + k^2\psi(r) = 0, \qquad k = \frac{1}{\hbar}\sqrt{2mE_0}. \tag{14}$$

The boundary condition at the well boundary is $\psi(R) = 0$. A solution of this equation that is finite at the origin is

$$\psi(r) = A\frac{\sin kr}{r}. \tag{15}$$

The boundary condition gives the spectrum of possible values of k:

$$k_n = \frac{\pi n}{R}, \qquad n = 1, 2, 3, \ldots \tag{16}$$

The ground state corresponds to $n = 1$, and has the energy

$$E_0(R) = \frac{\pi^2\hbar^2}{2mR^2}. \tag{17}$$

This gives the constant C introduced above as $C = \pi^2/2$. Finally, the radius of the bubble is

$$R = = \left(\frac{\pi\hbar^2}{8m\sigma}\right)^{1/4} \approx 1.8 \cdot 10^{-7}\text{cm} = 18\text{Å}. \tag{18}$$

The experimental value for the radius is about 17 Å.

(b) In this case we are dealing with a degenerate gas of non-interacting fermions. Its energy is determined by the Pauli principle which leads to filling the states up to a certain Fermi momentum k_F. The usual counting arguments go as follows. In a box of size L (volume $V = L^3$) the spacing between allowed values of any component of the particle momentum is $2\pi\hbar/L$. Then there is one allowed value of momentum per volume $(2\pi)^3/V$ in the momentum space. This gives the density of states, and the prescription for converting sums over momentum states into integrals:

$$\sum_{\boldsymbol{k}} f(\boldsymbol{k}) \to V \int \frac{d\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}). \tag{19}$$

When we sum quantities $f(k)$ that depend only on $k = |\boldsymbol{k}|$, this is further simplified to

$$V \int \frac{4\pi k^2 dk}{(2\pi)^3} f(k) = \frac{V}{2\pi^2} \int f(k) k^2 dk. \tag{20}$$

Assuming that all the states with $|\boldsymbol{k}| \leq k_F$ are filled, and accounting for their double degeneracy from electron spin, we find the equality

$$2 \cdot \frac{4}{3}\pi k_F^3 = \frac{(2\pi)^3}{V} N, \tag{21}$$

which gives the Fermi momentum in terms of the particle number N and the volume V:

$$k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}. \tag{22}$$

Now we can find the total energy of the electron gas in this volume (the factor of 2 comes from the spin):

$$E = 2\sum_{\boldsymbol{k}} \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{m}\frac{V}{2\pi^2}\int_0^{k_F} k^4 dk = \frac{\hbar^2 V k_F^5}{10\pi^2 m} = \frac{3(3\pi^2)^{2/3}}{10}\frac{\hbar^2}{m}\frac{N^{5/3}}{V^{2/3}}. \tag{23}$$

For electrons trapped in a bubble of radius R the volume is that of the bubble: $V = \frac{4}{3}\pi R^3$. Then

$$E = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2}{mR^2}N^{5/3}. \tag{24}$$

Adding to this the surface energy of the bubble and minimizing with respect to R, as in the first part, we get the radius

$$R(N) = \left(\frac{3}{40\pi}\right)^{1/4}\left(\frac{9\pi}{4}\right)^{1/6}\left(\frac{\hbar^2}{m\sigma}\right)^{1/4}N^{5/12} \approx 12.5 \cdot N^{5/12}\text{Å}. \tag{25}$$

3. Electromagnetic Transformations

An infinitely long cylindrical wire (aligned along the z-axis) of radius a is electrically neutral, but carries a nonzero current density with Cartesian (x, y, z) components given by $\vec{j} = (0, 0, j)$.

(a) How can an electrically neutral wire carry a nonzero electric current? Give an example.

(b) Determine the electric and magnetic fields outside the wire.

Let us denote the reference frame at rest with respect to the wire as S. Now view this system from an inertial frame S' moving in the $+z$ direction with speed v. Assume that the spatial axes of S and S' are parallel and that their origins coincide at $t = t' = 0$.

(c) The current density $\vec{j}$ is part of a 4-vector. Determine this 4-vector as observed in S'.

(d) Find the electric and magnetic fields outside the wire as observed in S'.

Clearly, the electric charge density in S is zero.

(e) Is the charge density in S' also zero? If not, how is this possible since the electric charge is a Lorentz invariant?

Electromagnetic Transformations - Solution

(a) Image equal numbers of electrons and protons moving in opposite directions with the same drift velocities and spatial distributions. The total electric charge is zero but each component (electrons and protons) contributes to a nonzero current in the direction of flow of the protons.

(b) The axial symmetry of the problem tells us that the fields will only depend on the distance from the wire r.

Since the total electric charge of the wire is zero, $\vec{E} = 0$.

Applying Ampere's law gives

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 j \pi a^2.$$

The magnetic field cannot have radial component (that would contradict $\nabla \cdot \vec{B} = 0$). For the tangential component B_ϕ Ampere's law gives

$$B_\phi = \mu_0 j \pi a^2 / 2\pi r = \mu_0 j a^2 / 2r.$$

(c) The time component of the current density is given by $j^0 = c\rho$, where ρ is the electric charge density. In our case, in the reference frame of the wire, $\rho = 0$. Therefore, in S the current density 4-vector is $j^\mu = (0, 0, 0, j)$. The Lorentz transformation for a boost in the z direction gives the components of the current density 4-vector in the frame S':

$$j^{\mu'} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v/c & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ j \end{pmatrix} = \begin{pmatrix} -\gamma j v/c \\ 0 \\ 0 \\ \gamma j \end{pmatrix}.$$

Here, as usual, $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(d) To solve this, one can either use $j^{\mu'}$ to solve for the fields, or having found the fields in S, one can apply the Lorentz transformations for the field components. Here, we shall do the latter. As is commonly worked out in standard texts the transformations for the fields can be conveniently written in terms of field components parallel and perpendicular to the boost.

$$\begin{aligned} \vec{B}'_\parallel &= \vec{B}_\parallel, \quad \vec{E}'_\parallel = \vec{E}_\parallel, \\ \vec{B}'_\perp &= \gamma \left(\vec{B}_\perp - \vec{v} \times \vec{E}/c^2 \right), \\ \vec{E}'_\perp &= \gamma \left(\vec{E}_\perp + \vec{v} \times \vec{B} \right). \end{aligned}$$

Using the result of part (b) the only nonzero components are B'_ϕ and E'_r obtained as

$$\begin{aligned}\vec{B}' &= \gamma\vec{B} = \frac{\gamma\mu_0 j a^2}{2r}\hat{\phi},\\ \vec{E}' &= \gamma(\vec{v}\times\vec{B}) = -\frac{\gamma\mu_0 j v a^2}{2r}\hat{r}.\end{aligned}$$

(e) The charge density is given by $\rho = j^0/c$. Therefore, $\rho' = -\gamma v j/c^2 \neq 0$ (see part (c)). To see how this can be, consider the example in the answer for part (a) of opposite charges moving in opposite directions with equal speeds in S. The speeds of the positive and negative charges will not be equal in S'. As a result, the Lorentz contraction effect for the (+) charge will be different from that for the (−) charge giving rise to separate charge densities that do not balance.

4. Higgs Boson at the LHC

Searching for the Higgs boson is the main mission of the Large Hadron Collider (LHC). The LHC is a proton-proton collider. Each proton beam runs at energy $E/2$. We can view protons has composed of constituents called partons. A given parton in a proton with four-momentum P_i carries four-momentum $p_i = x_i P_i$, where the momentum fraction $x_i < 1$. An event in which a single Higgs boson is produced can be modeled by,

$$\text{parton}_1 + \text{parton}_2 \to \text{Higgs boson},$$

where parton_1 has momentum $p_1 = x_1 P_1$, while parton_2 has momentum $p_2 = x_2 P_2$. Denote the mass of the Higgs boson by M. Choose the beam to be along the z-direction, and ignore the masses of each proton and its constituents. For simplicity, we set $c = 1$ in this problem.

(a) Consider a particular collision in which parton_1 has momentum fraction x_1. Find the minimal value of $x_2 = x_2^{\text{min}}$ needed to produce a Higgs particle of mass M.

(b) Suppose a Higgs boson is produced in an event where parton_1 has momentum fraction x_1. What is the range of x_2 so that the Higgs particle is bound to the earth? The escape velocity of the Earth is $v_e = 11.2$ km/s, or 3.73×10^{-5} in units where $c = 1$.

(c) Given any 4-momentum (E, p_x, p_y, p_z), define the rapidity y by,

$$y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right).$$

For a collision with momentum fractions x_1 and x_2, how does the rapidity transform in going between the center of mass frame and the lab frame?

(d) Each produced Higgs boson will subsequently decay into a pair of photons. In one such event, the direction of one photon is perpendicular to the direction of the beam in the lab frame. On the other hand, the same photon moves off at an angle θ_0 with respect to the direction of the beam in the rest-frame of the Higgs boson. Find the rapidity of the other photon in the lab frame and in the rest frame of the Higgs boson.

Higgs Boson at the LHC - Solution

In this solution we use the natural units where $c = 1$.

(a) The center of mass energy of the protons is E. However, the actual available energy in any collision event is the center of mass energy of the partons which participate in the collision. Using $P_1 = (E/2, 0, 0, E/2)$ and $P_2 = (E/2, 0, 0, -E/2)$, the parton center of mass energy can be obtained as

$$(E_{\rm cm}^{\rm parton})^2 = (x_1 P_1 + x_2 P_2)^2 = x_1 x_2 E^2.$$

(Remember that the square of a 4-vector is understood in the relativistic sense: $P^2 = P_0^2 - \vec{P}^2$, so that, for example, $P_1^2 = P_2^2 = 0$.) In order to produce the Higgs boson, we must require

$$E_{\rm cm}^{\rm parton} \geq M.$$

Therefore, we need

$$E_{\rm cm}^{\rm parton} = \sqrt{x_1 x_2} E \geq M, \ \rightarrow \ x_2 \geq x_2^{\rm min} = \frac{M^2}{E^2}\frac{1}{x_1}.$$

(b) The 4-momentum of the Higgs boson is

$$p_h = x_1 P_1 + x_2 P_2 = (x_1 + x_2, 0, 0, x_1 - x_2)\frac{E}{2}$$

Therefore, the velocity is

$$v_h = \frac{(p_h)_z}{E_h} = \frac{x_1 - x_2}{x_1 + x_2}.$$

To be bound to the Earth, we require

$$-v_e < v_h < v_e \quad \rightarrow \quad x_1 \frac{1 - v_e}{1 + v_e} \leq x_2 \leq x_1 \frac{1 + v_e}{1 - v_e}.$$

(c) The difference between the lab frame and the parton center of mass frame is a boost along the z-direction. We begin with (E, p_z) in the lab frame O. In the parton center of mass frame which is moving with velocity $v_{\rm cm}$ in the positive z-direction, we have

$$\begin{aligned} E' &= \gamma_c E - v_{\rm cm} \gamma_c p_z \\ p_z' &= \gamma_c p_z - v_{\rm cm} \gamma_c E \end{aligned}$$

where $\gamma_c = (1 - v_{\rm cm}^2)^{-1/2}$. The rapidity transforms as follows,

$$y' = \frac{1}{2}\ln\frac{E' + p'_z}{E' - p'_z} = y + \frac{1}{2}\ln\frac{1 + v_{\rm cm}}{1 - v_{\rm cm}}$$

(d) For a massless particle like the photon, we have $p_z = E\cos\theta$ where θ is the angle of it momentum with respect to the z-direction. Therefore, we have

$$y = \ln\left(\cot\left(\frac{\theta}{2}\right)\right).$$

Denote the rapidity of photon 1 in the lab frame as y_1. Since it is perpendicular to the z-direction, we have $y_1 = 0$. The rapidity of the same photon in the Higgs rest frame is

$$y'_1 = \ln\left(\cot\left(\frac{\theta_0}{2}\right)\right).$$

In the Higgs rest frame, the two photons must be back to back because of momentum conservation. Therefore, the rapidity of the other photon in this frame is

$$y'_2 = \ln\left(\tan\left(\frac{\theta_0}{2}\right)\right) = -\ln\left(\cot\left(\frac{\theta_0}{2}\right)\right).$$

From the transformation law obtained in part (3), we see that $y_1 - y_2 = y'_1 - y'_2$. Therefore the rapidity of the second photon in the lab frame is

$$y_2 = -2\ln\left(\cot\left(\frac{\theta_0}{2}\right)\right).$$

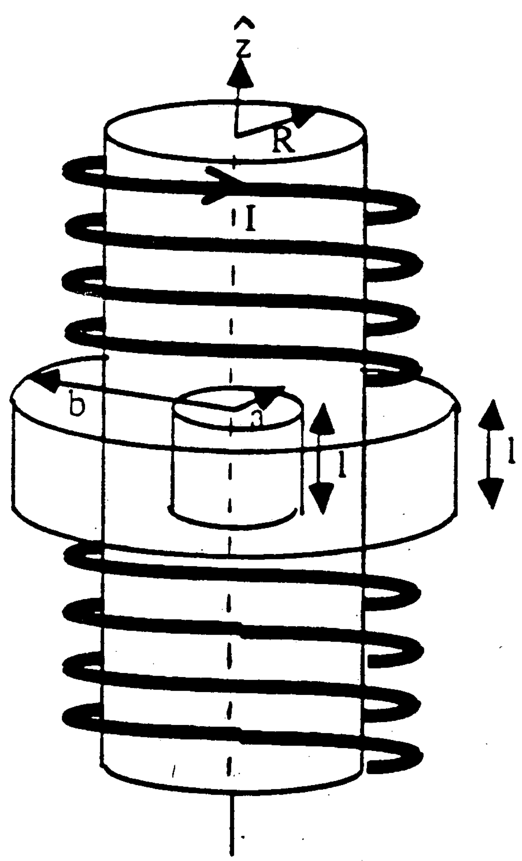
5. Solenoid and Rotating Shells

Consider a very long solenoid, of radius R and n turns per unit length, initially carrying a current I_0. Two long cylindrical shells of length l are suspended coaxially about the solenoid and are free to rotate about their common vertical axis. The inner shell, of mass M_{in} and radius a ($< R$), has a uniform surface charge $+Q$ and the outer shell, of mass M_{out} and radius b ($> R$), carries a uniform surface charge $-Q$. Assume $l \gg b$ (the figure is not to scale). As the current is reduced to zero, the cylinders begin to rotate.

(a) Find the electric fields at the surfaces of the inner and outer cylinders while the current is changing.

(b) Determine the torque on each cylinder while the current is changing.

(c) Once the current has reduced to zero, determine the final angular speed and sense of rotation of each cylinder.



Solenoid and Rotating Shells - Solution

(a) From Gauss' law we can determine that there is a radially outward electric field between the charged cylinders given by

$$E_r = \frac{Q}{2\pi\epsilon_0 l r}$$

The changing current gives rise to an electric field component around the axis. From Faraday's law we see that $E_\phi(2\pi r) = -A(\mu_0 n)dI/dt$, which gives

$$\begin{aligned} E_\phi &= -\frac{1}{2}\mu_0 n \frac{dI}{dt} r \qquad (r < R) \\ &= -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{r} \quad (r > R) \end{aligned}$$

These can be evaluated at the appropriate values of r (a and b) to get the fields at the surfaces.

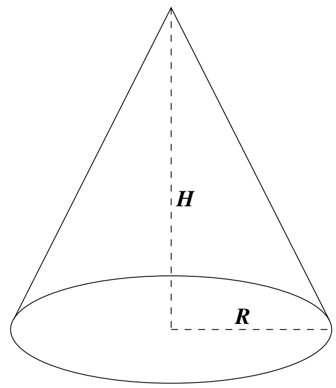
(b) The torque on each surface is given by $\vec{\tau} = \vec{r} \times q\vec{E}$ where q is the charge on the cylinder ($\pm Q$).

$$\begin{aligned} \vec{\tau} &= -\frac{1}{2}\mu_0 n Q \frac{dI}{dt} a^2 \hat{z} \qquad \text{(inner cylinder)} \\ &= +\frac{1}{2}\mu_0 n Q \frac{dI}{dt} R^2 \hat{z} \qquad \text{(outer cylinder)} \end{aligned}$$

(c) The torques give rise to angular accelerations according to $d\vec{\omega}/dt = \vec{\tau}/I_{zz}$, where I_{zz} are the moments of inertia of the cylindrical shells ($M_{in}a^2$ and $M_{out}b^2$). Integrating in time from the moment when the current is $I = I_0$ to the moment when $I = 0$ gives

$$\begin{aligned} \vec{\omega_f} &= \frac{\mu_0 n I_0 Q}{M_{in}} \hat{z} \qquad \text{(inner cylinder, counterclockwise)} \\ &= \frac{\mu_0 n I_0 Q}{M_{out}} \frac{R^2}{b^2}(-\hat{z}) \quad \text{(outer cylinder, clockwise)} \end{aligned}$$

6. Particle on a Cone



A quantum mechanical particle of mass m is confined to move on the surface of a right circular cone (without the bottom disc) of height H and the base radius R. Find the energy spectrum in terms of zeros $j_{\alpha,l}$ of the Bessel function. Here $j_{\alpha,l}$ denotes the l-th zero of the function $J_\alpha(x)$ which satisfies the Bessel differential equation

$$x^2\frac{d^2 f}{dx^2} + x\frac{df}{dx} + (x^2 - \alpha^2)f = 0.$$

Particle on a Cone - Solution

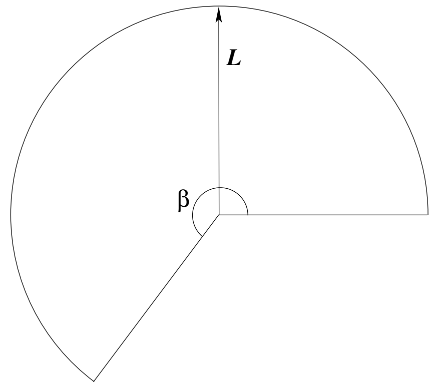


Figure 1: Unfolding of the cone.

We need to solve the the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi = E\psi \tag{26}$$

on the side surface of the cone. Using the notation

$$k = \sqrt{\frac{2mE}{\hbar^2}} \tag{27}$$

we rewrite the Schrodinger equation as

$$(\Delta + k^2)\psi = 0. \tag{28}$$

This should be solved with the boundary conditions that the wave function ψ vanishes on the circle at the base of the cone, and should be 2π periodic (invariant) upon one full rotation about the symmetry axis of the cone.

The key to a solution is to realize that there is a simple system of orthogonal coordinates on the cone. It is easiest to visualize if we cut the cone along one of its generatrices and unfold it onto a plane. This results in a wedge of radius $L = \sqrt{R^2 + H^2}$ and the opening angle β that can be found by equating the perimeter of the base circle of the cone and the length of the arc of the wedge:

$$2\pi R = \beta L \qquad \Rightarrow \qquad \beta = \frac{2\pi R}{L} = \frac{2\pi R}{\sqrt{R^2 + H^2}}. \tag{29}$$

The usual polar coordinate system (r, ϕ) on the plane is the orthogonal system on the cone. In this coordinate systems the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}, \tag{30}$$

and the Eq. (28) becomes

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0. \tag{31}$$

The boundary conditions for the wave functions are

$$\psi(R,\phi) = 0, \qquad\qquad \psi(r,\phi+\beta) = \psi(r,\phi). \tag{32}$$

In addition, the wave functions should be square integrable:

$$\int_0^\beta d\phi \int_0^R dr\, r|\psi|^2 < \infty. \tag{33}$$

We can separate the variables by looking for solutions in the form

$$\psi(r,\phi) = f(r)g(\phi). \tag{34}$$

Substituting this ansatz into (31) and dividing through by $f \cdot g$ leads to

$$r^2\frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} + \frac{g''(\phi)}{g(\phi)} + (kr)^2 = 0. \tag{35}$$

Denote the separation constant by $-\alpha^2$, then $g(\phi)$ satisfies

$$g''(\phi) = -\alpha^2 g(\phi) \tag{36}$$

with the solutions

$$g(\phi) = e^{\pm i\alpha\phi}. \tag{37}$$

The boundary condition in the angle ϕ leads to the constraint on the allowed values of α:

$$\alpha\beta = 2\pi n, \qquad \Rightarrow \qquad \alpha = \frac{2\pi n}{\beta}, \tag{38}$$

where n is an integer. Using Eq. (29), this becomes

$$\alpha = \frac{L}{R}n = \frac{\sqrt{R^2+H^2}}{R}n. \tag{39}$$

In the radial part of the Schrodinger equation we use the dimensionless radius $x = kr$:

$$x^2 f''(x) + xf'(x) + (x^2-\alpha^2)f(x) = 0. \tag{40}$$

This is the Bessel's equation whose general solution is

$$f(x) = AJ_\alpha(x) + BN_\alpha(x). \tag{41}$$

(The Bessel function of the second kind $N_\alpha(x)$ is also denoted by $Y_\alpha(x)$ in some books.)

For $n = 0$ ($\alpha = 0$) the function $N_0(x)$ has a logarithmic singularity at $x = 0$. Then the wave function $\psi(r,\phi) \propto N_0(kr)$ does not actually satisfy the Schrodinger equation, since $\Delta \log kr \propto \delta(\boldsymbol{r})$, which is easily established using the Gauss theorem (think about the potential of a point charge in two dimensions). Thus, only $J_0(x)$ should be kept.

For $n \neq 0$ the limiting behavior for small x is $J_\alpha(x) \propto x^\alpha$ and $N_\alpha(x) \propto x^{-|\alpha|}$. Thus the solution $N_\alpha(x)$ is not normalizable (the integral in Eq. (33) diverges for $\psi \propto N_\alpha(kr)$). Thus, for we only can have $J_\alpha(x)$ with $\alpha > 1$, which selects $n \geqslant 1$.

The obtained solutions

$$\psi_n(r,\phi) = e^{\pm i\alpha\phi} J_\alpha(kr), \qquad n = 0, 1, 2, \ldots \tag{42}$$

should also satisfy the boundary condition $\psi(R,\phi) = 0$, which, for each given n selects a discrete set of allowed values of the wave number k:

$$k_{n,l} = \frac{j_{\alpha,l}}{R}. \tag{43}$$

Finally, the energies of eigenstates are given by

$$E_{n,l} = \frac{\hbar^2}{2mR^2} j_{\alpha,l}^2, \qquad \alpha = \frac{\sqrt{R^2 + H^2}}{R} n. \tag{44}$$