Spring 2013

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 1

March 27, 2013

(Problems 1 - 6)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

1. Reflection

Light strikes a sheet of glass along the normal to its surface taken to lie in the $z = 0$ plane. The electric field of the incident light is given by $\vec{E}_i = E_{i0} e^{i(kz-\omega t)} \hat{x}$. Take the index of refraction of glass to be n and its permeability to be the free space value, μ_0.

(a) Find the relationships of the electric and magnetic fields of the reflected wave to E_{i0}.

(b) Determine the phase difference between the incident and reflected waves.

Reflection - Solution

(a) The boundary conditions require continuity of the component of $\vec{E}$ parallel to the boundary. Therefore, at the surface $E_{air} = E_g$ where E_{air} is the total field in air (incident + reflected) and E_g is the field in the glass (transmitted). We must also consider the boundary conditions for the magnetic fields. The magnetic field for the incident wave is given by $\vec{B} = \hat{z} \times \vec{E}/c$ so that

$$\vec{B}_i = \frac{E_{i0}}{c} e^{i(kz-\omega t)} \hat{y}.$$

The boundary conditions for $\vec{B}$ require $B_{air}/\mu_0 = B_g/\mu_g \implies B_{air} = B_g$. This is equivalent to $E_{air}/c = E_g/v_g$ where $v_g = c/n$ is the speed of light in glass.

Therefore we have

$$\begin{aligned} E_i + E_r &= E_t \\ E_i - E_r &= nE_t \end{aligned}$$

Eliminating the transmitted component gives

$$E_r = E_i \left(\frac{1-n}{1+n} \right)$$

Therefore, the reflected wave is given by

$$\vec{E}_r = \left(\frac{1-n}{1+n} \right) E_{i0} e^{i(-kz-\omega t)} \hat{x}.$$

This wave propagates in the negative-z direction so the magnetic field is given by

$$\vec{B}_r = -\hat{z} \times \vec{E}_r/c = \frac{n-1}{c(1+n)} E_{i0} e^{i(-kz-\omega t)} \hat{y}$$

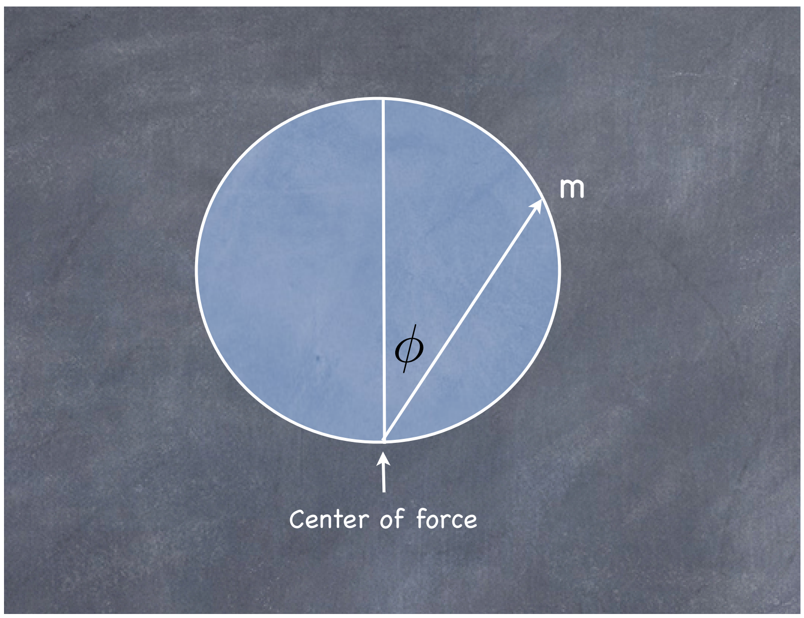
(b) Because the factor $(1-n)/(1+n)$ in the expression for $\vec{E}_r$ is negative, $\vec{E}_r$ and $\vec{E}_i$ are 180° out of phase.

2. Center of Attraction

Imagine a particle of mass m moving under the influence of a central force.

(a) Please write down a Lagrangian for this system. What symmetries does the Lagrangian possess, and what are the associated conserved quantities?

(b) If the particle moves in a circular orbit of radius R and passes through the center of the force as depicted below, what is the force law?



Center of Attraction - Solution

(a) The Lagrangian takes the form

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - V(r). \tag{1}$$

For a central force, the Lagrangian is independent of time so energy is conserved. It is also rotationally symmetric so angular momentum is conserved.

(b) Now we come to the bulk of the problem. First the angular momentum along the axis perpendicular to the plane of motion is conserved:

$$l = mr^2\dot{\phi}. \tag{2}$$

Here ϕ is the angle with respect to the center of the force.

The main point is to realize that we should consider the orbit as a function of ϕ so that $r = r(\phi)$. This is quite standard for central force problems. For a circular orbit, $r = 2R\cos(\phi)$. This makes sense at both $\phi = 0$ where $r = 2R$ and $\phi = \pi/2$ where $r = 0$. We can then differentiate

$$\dot{r} = -2R\dot{\phi}\sin(\phi) = -\frac{2lR}{mr^2}\sin(\phi). \tag{3}$$

We now observe that the energy is conserved $E = T + V$ which we can evaluate:

$$E = \frac{2l^2R^2}{mr^4}\sin^2(\phi) + \frac{1}{2}\frac{l^2}{mr^2} + V(r). \tag{4}$$

Let us write $1/r^2$ in the second term more conveniently as $r^2/r^4 = 4R^2\cos^2(\phi)/r^4$. How convenient! We note that the energy becomes

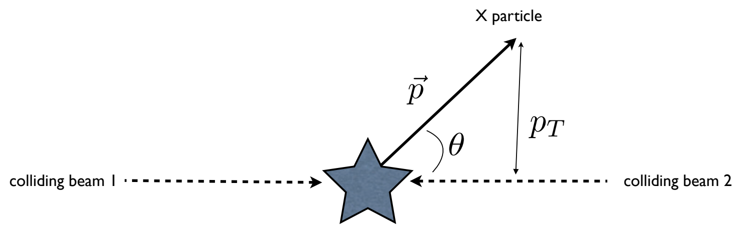
$$E = \frac{2l^2R^2}{mr^4} + V(r). \tag{5}$$

We conclude that $V(r) = -\frac{2l^2R^2}{mr^4}$ up to an irrelevant additive constant. The force is then

$$F(r) = -\frac{dV}{dr} = -\frac{8l^2R^2}{mr^5}. \tag{6}$$

3. Decay to Photons

A new particle with mass m_X has been observed at the Large Hadron Collider. Let us call it the "X particle".



(a) The X particle has been observed to decay into two photons. Recall that the photon is a massless particle with spin one. Based solely on the fact that the X particle is massive and decays into two photons, what are the possible values for the intrinsic spin of the X particle? Justify your answer.

(b) Define the helicity λ of a particle as the component of angular momentum along the particle's direction of motion. The photon helicity can take values $\lambda = +1$ or $\lambda = -1$. Suppose that X is a scalar particle, i.e., intrinsic spin zero. Consider the decay of X in its rest frame. What are the relative probabilities for the helicities of photon 1 and photon 2 to be (i) $\lambda_1 = +1$, $\lambda_2 = +1$, (ii) $\lambda_1 = +1$, $\lambda_2 = -1$, (iii) $\lambda_1 = -1$, $\lambda_2 = +1$ and (iv) $\lambda_1 = -1$, $\lambda_2 = -1$?

(c) Suppose instead that X is a particle with intrinsic spin $J = 2$, and consider a decay into two massive spin-one particles Y. Assume that $m_Y \approx m_X/2$ and neglect the velocity of the final state particles. Use angular momentum conservation to derive the relative probabilities for the final state bosons to have helicities (i) $\lambda_1 = +1$, $\lambda_2 = +1$, (ii) $\lambda_1 = +1$, $\lambda_2 = -1$, (iii) $\lambda_1 = -1$, $\lambda_2 = +1$ and (iv) $\lambda_1 = -1$, $\lambda_2 = -1$. You may use the following table of Clebsch-Gordan coefficients for adding two spin-1 objects to obtain a spin-2 object:

		J	2	2	2	2	2
		M	2	1	0	-1	-2
m	m'						
1	1		1	0	0	0	0
1	-1		0	0	$\sqrt{\frac{1}{6}}$	0	0
-1	1		0	0	$\sqrt{\frac{1}{6}}$	0	0
-1	-1		0	0	0	0	1

(d) Consider the production of X at a particle collider again decaying to photons. Suppose that X is produced with three-momentum at an angle θ with respect to the beam direction, and with boost parameter γ; the energy as seen in the laboratory frame is $\gamma m_X c^2$. What is the maximum value for the transverse momentum p_T of either photon? Here p_T, pictured in the diagram, is the magnitude of the laboratory frame three-momentum component in the plane perpendicular to the beam direction.

Decay to Photons - Solution

(a) By conservation of angular momentum, e.g. along the direction of photon momentum, the particle can only have spin $s = 0$, $s = 1$ or $s = 2$. By Bose symmetry for the final state photons, the case $s = 1$ is forbidden. Therefore X can have spin zero or two.

(b) The photon helicities must satisfy $\lambda_1 - \lambda_2 = 0$. Since there is no preferred direction, probabilities for $\lambda_1 = \lambda_2 = +1$ and $\lambda_1 = \lambda_2 = -1$ are equal:

$$P_{++} : P_{+-} : P_{-+} : P_{--} = 1 : 0 : 0 : 1.$$

(c) We are free to average over the initial state polarizations in determining the distribution of boson spin correlations. Let us take the bosons to be emitted in the $+\hat{z}$ and $-\hat{z}$ directions, and average over the spin states of the X particle quantized in the $\hat{z}$ direction. The probability for a given helicity state is determined by

$$\text{probability} \propto \frac{1}{5} \sum_{M=-2}^{2} |\langle 2, M | 1, \lambda_1; 1, -\lambda_2 \rangle|^2 ,$$

where the minus sign appears in $-\lambda_2$ since for the boson moving in the $-\hat{z}$ direction, the helicity is positive when the z component of angular momentum is negative. Using the Clebsch-Gordan coefficients for adding $j_1 = 1$ and $j_2 = 1$ to obtain $J = 2$, we find

$$P_{++} : P_{+-} : P_{-+} : P_{--} = \frac{1}{6} : 1 : 1 : \frac{1}{6}.$$

(d) p_X^μ is obtained by the Lorentz transformation, with $\gamma^2 = 1/(1 - \beta^2)$,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta\sin\theta & 0 & \gamma\beta\cos\theta \\ \gamma\beta\sin\theta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta\cos\theta & 0 & 0 & \gamma \end{pmatrix}.$$

Let the photon momentum be the boost of the rest frame momentum,

$$p^\mu_{\gamma,\text{rest}} = \frac{m_X}{2} \begin{pmatrix} 1 \\ \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}.$$

Then

$$p^\mu_\gamma = \frac{m_X}{2} \begin{pmatrix} \gamma(1 + \beta\cos(\theta - \alpha)) \\ \gamma(\beta\sin\theta + \sin\alpha) \\ 0 \\ \gamma(\beta\cos\theta + \cos\alpha) \end{pmatrix}$$

We have

$$p_T = \frac{\gamma m_X}{2}(\beta\sin\theta + \sin\alpha)$$

This is maximized for $\sin\alpha = 1$, hence

$$(p_T)_{\max} = \frac{\gamma m_X}{2}(1 + \beta\sin\theta)$$

4. Short Answer

(a) **Energy from Wind:** The MidAmerican wind turbines in Texas have a diameter of 82 meters. A typical wind speed there is 30 km/hour. The turbines are engineered so that they remove 1/2 the kinetic energy of the incident air and turn it into electrical energy. How many kilowatts does each turbine generate?

(b) **Squeezed Balloon:** A spherical balloon inflated with Argon to a volume V is squeezed in a clamp so that its volume is $V/2$. After it has equilibrated to room temperature, the clamp is quickly released with negligible escape of heat. Which of the following is closest to the final temperature? Make sure you explain your answer! i) 150K, ii) 190 K, iii) 280 K, iv) 380 K, v) 480 K.

(c) **Space Laser:** You've been commissioned to build a satellite capable of zapping a missile as it takes off. The satellite is to be placed in geosynchronous orbit (36,000 km above the surface of the earth) and must be able to target an object with a 1 meter^2 cross-section. The laser is CO_2 based and emits at approximately $\lambda = 10\,\mu m$. What is the minimum diameter of the laser?

Short Answer - Solution

(a) The power P in the wind is the energy density ϵ times speed v times area πR^2. Noting that $\epsilon = \frac{1}{2}\rho v^2$ while $\rho = 18\,\text{grams}/22.4\,\text{liters} = 18/22\,\text{grams}/10^3\text{cm}^3 \sim 1 \times 10^{-3} * 1000\ \text{SI} = 1\,\text{SI}$. The velocity $v = 30 \times 10^3/3600 = 30/36 \times 10\,\text{SI}$,

$$P = \epsilon\ v\ \ * \pi R^2 = \frac{1}{2}\rho v^3 * \pi R^2 \sim \frac{3}{2}10^3 * 40^2 \sim 2\,\text{megawatts}.$$

So the power generated is half of this—about 1 megawatt.

(b) The energy U is $\frac{3}{2}NkT$—$\frac{1}{2}kT$ per degree of freedom. When we release, work W is done, but no heat escapes. During the process $dW = -pdV$. Since $p = N\ kT/V$, $dW = -N\ T\ dV/V$. Setting $dU = dW$, we infer $\frac{3}{2}N\ k\ dT/T = -NdV/V$, or $\frac{3}{2}\ dT/T = -dV/V$. Integrating gives, $\frac{3}{2}\ \log(T_2/T_1) = -\log(V_2/V_1)$, or $T_2/T_1 = (V_2/V_1)^{-2/3}$. Since $V_2 = 2V_1$, we get $T_2 = T_1\ 2^{-2/3} = .63T_1$. The temperature $T_1 = 300\text{K}$; therefore, $T_2 \sim 188\text{K}$. The closest answer is **ii)**.

(c) Assume the object being focused on is of size a, the diameter of the satellite lens is w and the distance to the target is d. The resolving power of a microscope or imaging system is given by $a = \lambda/2NA$ where λ is the wavelength and $NA = \sin\theta \approx w/d$ is the numerical aperature. From here one can solve directly for $w = \lambda d/2a \approx 360m$.

This can also be derived based on diffraction, where a ray going through the center of the lens will have an angular uncertainty of $\Delta\theta$ given by the equation, $w \sin\Delta\theta = \lambda$. Here $\Delta\theta \approx a/d$. Solving the resulting equation gives the same result up to a factor of 2.

Finally one could use the uncertainty principle saying that $\Delta p \Delta x > \hbar/2$, where $\Delta x = w$ and $\Delta p = |p| \sin\Delta\theta$. $|p| = \hbar\omega/c = 2\pi\hbar/\lambda$. This yields the equation $2\pi w \sin\Delta\theta/\lambda = 1/2$, with $\sin\Delta\theta \approx \Delta\theta \approx a/d$. Solving for $w = \lambda d/4\pi a$ which is the same up to a factor of 2π.

5. Anomalous Magnetic Moments

(a) The quantum mechanical Hamiltonian for a non-relativistic charged spin-1/2 fermion in a magnetic field $\boldsymbol{B}$ with vector potential $\boldsymbol{A}$ is

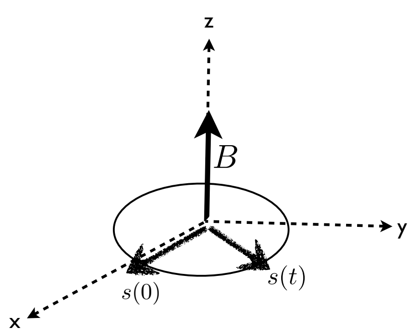
$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 - \boldsymbol{\mu}\cdot\boldsymbol{B}\,,$$

where $\boldsymbol{\mu} = \frac{g|e|}{2mc}\boldsymbol{s}$ and $\boldsymbol{s} = \frac{\hbar}{2}\boldsymbol{\sigma}$ with σ^i the 2×2 Pauli matrices. The Hamiltonian acts on two-component spinor wavefunctions,

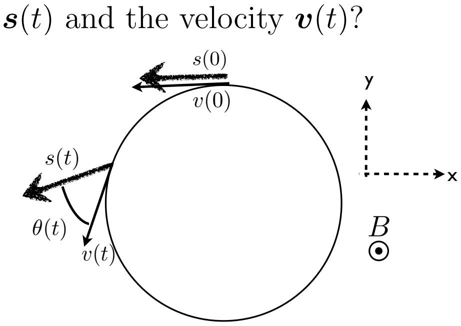
$$\psi(\boldsymbol{x}) = \begin{pmatrix} \psi_1(\boldsymbol{x}) \\ \psi_2(\boldsymbol{x}) \end{pmatrix}$$

Compute the time variation, $\frac{d}{dt}\boldsymbol{s}$, of the spin operator in the Heisenberg picture. Express your answer in terms of $\boldsymbol{s}$, $\boldsymbol{B}$, g, $|e|$ and m.

(b) Suppose that an ensemble of muons (electric charge $e = -|e|$) are prepared at rest, in a magnetic field $\boldsymbol{B} = |B|\hat{\boldsymbol{z}}$, with spin initially aligned in the $\hat{\boldsymbol{x}}$ direction. The muon decays to an electron plus neutrinos with the electron emitted preferentially in the direction of spin. As a function of the time at which the muon decays, what is the direction of the observed peak electron intensity?



(c) Suppose that a muon is maintained in a circular orbit in the x-y plane by immersing it in a magnetic field as in part (b). The spin is initially aligned along the muon velocity. At low velocity, $v \ll c$, what is the time dependence of the angle $\theta(t)$ between the directions of the spin $\boldsymbol{s}(t)$ and the velocity $\boldsymbol{v}(t)$?



Anomalous Magnetic Moments - Solution

(a) The Heisenberg picture operator is $\boldsymbol{s}(t) = e^{iHt}\boldsymbol{s}(0)e^{-iHt}$, and the time variation is

$$\frac{d}{dt}s^i = \frac{i}{\hbar}[H, s^i] = \frac{i}{\hbar}\hbar^2\left(-\frac{|e|g}{2mc}\right)B^j\left[\frac{\sigma^j}{2}, \frac{\sigma^i}{2}\right] = -\frac{|e|g}{2mc}(\boldsymbol{B}\times\boldsymbol{s})^i\,,$$

where we use the property of Pauli matrices, $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$.

(b) The electron distribution is measuring the precession of the muon spin about the magnetic field. From (a),

$$\frac{d}{dt}\boldsymbol{s}^1(t) = \omega s^2(t)\,, \quad \frac{d}{dt}\boldsymbol{s}^2(t) = -\omega s^1(t)\,, \quad \omega = \frac{g|e|B}{2mc}\,. \tag{7}$$

Suppose that $|\psi\rangle$ is the initial state representing spin in the positive x direction. At time t the component of spin in the direction of the unit vector $\boldsymbol{n} = (\cos(\alpha), \sin(\alpha), 0)$ is

$$\langle\psi(t)|\boldsymbol{n}\cdot\boldsymbol{s}|\psi(t)\rangle = \boldsymbol{n}\cdot\langle\psi|\boldsymbol{s}(t)|\psi\rangle = \boldsymbol{n}\cdot\boldsymbol{S}(t)\,, \tag{8}$$

where initial conditions give $S^1(0) = \hbar/2$, $S^2(0) = 0$, $dS^1/dt_{t=0} = 0$, $dS^2/dt|_{t=0} = -\omega\hbar/2$,

$$S^1(t) = \frac{\hbar}{2}\cos(\omega t)\,, \quad S^2(t) = -\frac{\hbar}{2}\sin(\omega t)\,, \quad \boldsymbol{n}\cdot\boldsymbol{S} = \frac{\hbar}{2}\cos(\alpha - \omega t)\,, \tag{9}$$

corresponding to spin aligned at angle $\alpha = \omega t$. The direction of maximum e^- intensity is

$$\boldsymbol{n} = \cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\,. \tag{10}$$

(c) The non-relativistic cyclotron frequency is determined by the Lorentz force. We have

$$\boldsymbol{F} = -\frac{|e|}{c}\boldsymbol{v}\times\boldsymbol{B} = -\frac{mv^2}{R}\hat{\boldsymbol{R}}\,,$$

and for a circular orbit of radius R we find the *cyclotron* frequency,

$$\omega_c = \frac{|e|B}{mc}\,.$$

If the spin did not precess, the spin would rotate clockwise as seen from above in the figure, at a frequency ω_c. However, the precession acts in the opposite direction, yielding

$$\theta(t) = (\omega_c - \omega_s)t\,, \tag{11}$$

with the total precession frequency

$$\omega_a = \omega_s - \omega_c = (g-2)\frac{|e|B}{2mc}\,.$$

In particular, for a particle with $g = 2$ the spin remains aligned with the velocity. This can be shown to hold also in the relativistic case.

6. Laser Target Practice

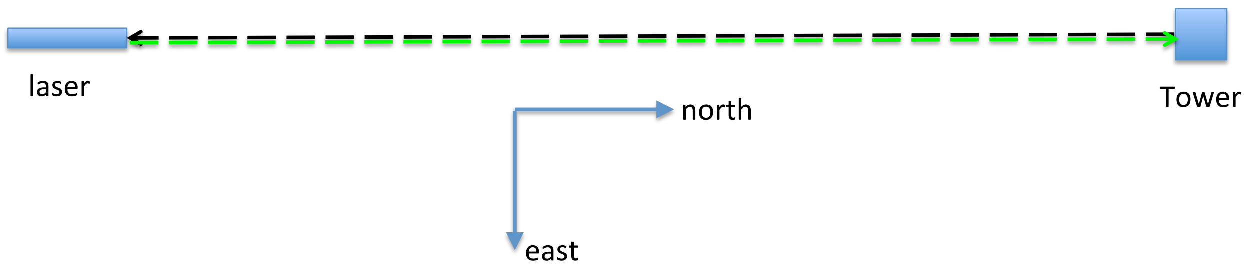


Figure 1: Light beams from the target to the laser telescope and from the laser back to the target. The deflections to be calculated are not shown.

A laser on the roof of Kersten is aimed at a distant object via a telescopic sight; for example, a mosquito on a building in the Loop. We suppose the object is directly north of the laser. After the sight is trained on the target, the laser is fired toward it. The light rays experience acceleration as viewed from the accelerating frame of the rotating earth. Because of this acceleration, the laser beam is deflected from the center of the target.

It might be useful to recall the relation between the time derivative, $\frac{d\vec{Q}}{dt}$, of a vector $\vec{Q}$ in a rotating versus a non-rotating frame. For a frame rotating at constant angular velocity $\vec{\Omega}$, $\left(\frac{d\vec{Q}}{dt}\right)_{\text{non}}$ measured in the non-rotating frame is related to $\left(\frac{d\vec{Q}}{dt}\right)_{\text{rot}}$ measured in the rotating frame by:

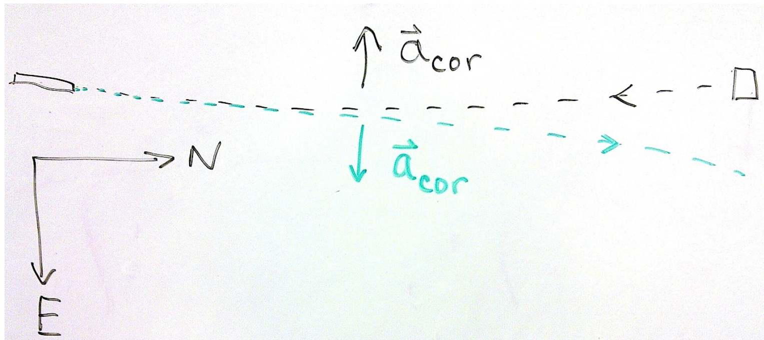
$$\left(\frac{d\vec{Q}}{dt}\right)_{\text{non}} = \left(\frac{d\vec{Q}}{dt}\right)_{\text{rot}} + \vec{\Omega} \times \vec{Q}.$$

(a) Find the east-west deflection x of the laser beam from the center of the target in terms of the distance S to the target, the latitude θ, and the angular speed Ω of the earth to lowest non-vanishing order in Ω.

(b) Estimate x to factor-of-two accuracy for $S \simeq 10$ km, $\theta \simeq 45$ degrees.

(c) If the target distance S were ten times larger, how much larger would the deflection x become?

Laser Target Practice - Solution



Applying the given transformation formula to the position vector $\vec{r}$,

$$\left(\frac{d}{dt}\right)_{\text{non}} \vec{r} \;=\; \left(\frac{d}{dt}\right)_{\text{rot}} \vec{r} \;+\; \vec{\Omega} \times \vec{r}.$$

Denoting the velocity by $\vec{v}$ the given transformation formula says

$$\vec{v}_{\text{non}} = \vec{v}_{\text{rot}} + \vec{\Omega} \times \vec{r}.$$

We differentiate both sides with respect to time to obtain the acceleration $\vec{a}_{\text{non}}$:

$$\vec{a}_{\text{non}} = \left(\frac{d}{dt}\vec{v}_{\text{non}}\right) = \left(\frac{d}{dt}\right)_{\text{non}} \left[\vec{v}_{\text{rot}} + \vec{\Omega} \times \vec{r}\right] = \left[\left(\frac{d}{dt}\right)_{\text{rot}} + \vec{\Omega} \times\right] \left[\vec{v}_{\text{rot}} + \vec{\Omega} \times \vec{r}\right].$$

The acceleration on the left is zero for the light rays. Denoting $\frac{d}{dt}_{\text{rot}}\vec{v}_{\text{rot}}$ by $\vec{a}_{\text{rot}}$, and carrying through the operations,

$$0 = \vec{a}_{\text{rot}} + 2\,\vec{\Omega} \times \vec{v}_{\text{rot}} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right).$$

To lowest order in Ω we keep only the first term, the Coriolis acceleration $\vec{a}_{\text{cor}}$:

$$\vec{a}_{\text{cor}} = -2\,\vec{\Omega} \times \vec{v}_{\text{rot}}.$$

This derivation of $\vec{a}_{\text{cor}}$ is not needed for credit.

We first consider the Coriolis acceleration of the light rays coming into the telescope from the mosquito. The direction of $\vec{v}_{\text{rot}}$ is southward. The direction of $\vec{\Omega}$ is in the north-up plane. Thus $\vec{a}_{\text{cor}}$ is perpendicular to this plane, in the east-west direction. Using the right hand rule, $\vec{a}_{\text{cor}}$ points west. (The gravitational and centrifugal accelerations are simpler to treat. These accelerations are independent of $\vec{v}_{\text{rot}}$. Thus the return ray is deflected with the same acceleration as the incoming ray. The return ray's motion is just the time-reversal of the incoming ray's motion. So these two accelerations produce no aiming error—unlike the Coriolis acceleration.)

So the south-moving ray experiences a westward Coriolis acceleration. In order to reach the telescope, directly to the south, the ray must start out directed slightly eastward. On entering the telescope, it is moving slightly westward. Now we consider the laser beam. We send it in the direction towards the light received by the telescope. Thus the beam is traveling north and slightly east. It also experiences a Coriolis acceleration $-2\,\vec{\Omega}\times\vec{v}_{\rm rot}$. Since $\vec{v}_{\rm rot}$ is now reversed, $\vec{a}_{\rm cor}$ is also reversed (green ray in figure): the acceleration is almost eastward since the outgoing ray is almost northward. The initial beam is also directed slightly eastward; the $\vec{a}_{\rm cor}$ acceleration bends it even further eastward.

One may find the magnitude of the deflection by including only the dominant east-west part of $\vec{a}_{\rm cor}$.

(a) We first calculate the deflection x_1 of the incident ray from the tower.

$$x_1 = \frac{1}{2}a_{\rm cor}t^2 \simeq \frac{1}{2}a_{\rm cor}(S/c)^2 = (\Omega\ c\sin\theta)\ (S/c)^2.$$

This deflection is to the west of the line of sight. The returning laser beam experiences a deflection x_2 equal in magnitude to x_1 in lowest order. Since the direction of the returning ray has been reversed, the acceleration is also reversed. It is directed eastward. Thus the deflections of the two beams add, and the laser spot lands east of the target by an amount:

$$x = 2\ \Omega\cos\theta\ S^2/c.$$

(b) This part tests your knowledge of the meaning of Ω in a physical context and the value of c. $|\Omega|/2\pi = 1$ revolution per day or 1 per 3600*24 sec. Thus $\Omega = 2\pi/(3600*24)$ sec^{-1}. Thus

$$x = 1.4*2\pi/(3600*24)(10^4)^2\ (1/3\times10^8) = 1.4*2*3/(3.6*2.4*3)\times10^{8-4-8}$$

$$x \simeq 1/3\times10^{-4}\ \text{meters} \simeq 30\ \text{microns}.$$

(c) Since the effect goes as S^2, a ten-fold increase in the distance gives a hundred-fold increase in the deflection: it becomes 3 millimeters, comparable to a typical laser beam width.

Spring 2013

DEPARTMENT OF PHYSICS

Ph.D. CANDIDACY EXAMINATION

Day 2

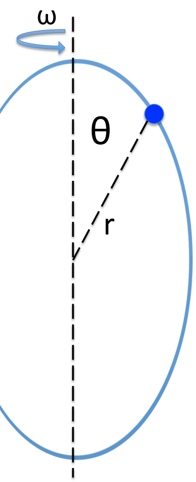
March 28, 2013

(Problems 7 - 12)

Work all six problems. Please write clearly and show all the steps of your work. Define any symbols that you introduce. Credit will be given only for significant progress toward a solution. Use clear diagrams wherever appropriate.

NO NAMES SHOULD APPEAR ON ANYTHING YOU SUBMIT; USE YOUR CODE NUMBER ONLY.

7. Mass on a Hoop



A mass m is confined to a vertical circular hoop of radius r that is rotating about a vertical axis maintained at a fixed angular speed ω. If this mass is currently at angle θ relative to the top of the hoop and if the time derivative of θ is $\dot{\theta}$, then the kinetic energy T is given by

$$T = \frac{1}{2}mr^2\,\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta$$

The gravitational potential U can be written $U = mgr\cos\theta$.

(a) Find a function of θ and $\dot{\theta}$ whose value does not change during the motion. Give the function explicitly in terms of these quantities, and not in terms of T, U or their derivatives.

(b) Suppose the mass is released from rest at $\theta = \pi/2$. For sufficiently large ω the mass reaches a maximum angle $\theta_m < \pi$, after which the angle decreases. Find θ_m and the condition on ω that is necessary for $\theta_m < \pi$.

Mass on a Hoop - Solution

(a) In this system neither T nor U depends explicitly on time. Thus the Lagrangian $\mathcal{L} \equiv T - U$ is time-independent. In such cases the Hamiltonian $\mathcal{H} \equiv \dot{\theta}\ \partial\mathcal{L}/\partial\dot{\theta} - \mathcal{L}$ is conserved.

$$\mathcal{L} = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta - mgr\ \cos\theta,$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mr^2\dot{\theta},$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\ \dot{\theta} = mr^2\dot{\theta}^2,$$

$$\mathcal{H} = \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}mr^2\omega^2\sin^2\theta + mgr\ \cos\theta.$$

This is the function that is conserved. It is not the total energy.

(b) At the beginning of the motion $\dot{\theta} = 0$ and $\sin^2\theta = 1$ and $\cos\theta = 0$. Thus $\mathcal{H} = -\frac{1}{2}mr^2\omega^2$. When θ reaches θ_m, $\dot{\theta} = 0$. Then we must have

$$-\frac{1}{2}mr^2\omega^2 = \mathcal{H} = -\frac{1}{2}m\omega^2r^2\sin^2\theta_m + mgr\cos\theta_m,$$

or setting $C \equiv \frac{\frac{1}{2}mr^2\omega^2}{mgr}$,

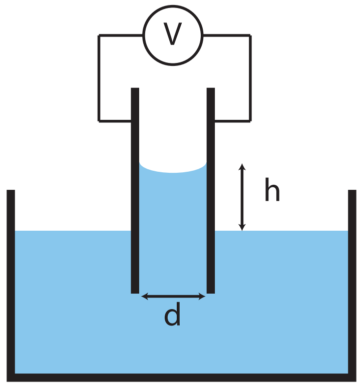
$$C(1 - \sin^2\theta_m) = -\cos\theta_m,$$

or

$$C\cos^2\theta_m = -\cos\theta_m; \qquad \cos\theta_m = -\frac{1}{C}.$$

This equation has a solution for $\theta_m < \pi$ when $C > 1$ i.e. when $\omega^2 > \frac{2g}{r}$.

8. Liquid Dielectric



A parallel-plate capacitor with spacing d is inserted vertically into a bath of dielectric insulating liquid with mass density ρ. The capillary pressure is given by $P_c = \gamma/r$ where r is the radius of curvature of the surface, and γ is the surface tension constant of the dielectric. This is shown in the figure above with d exaggerated for clarity. A voltage V can be applied to the capacitor.

(a) With no voltage applied, the liquid is observed to rise to a height $h \gg d$ in the tube. The liquid meets each plate at an angle that is independent of d or h. How does h vary with d? You can leave prefactors undetermined.

(b) Now a voltage V is applied so that h rises much further than in part (a). How does h vary with V and d? Again you may leave prefactors undetermined.

(c) If the liquid of part (b) is disturbed slightly, the level rises and falls periodically with period τ. By what factor does τ change if V is doubled?

Liquid Dielectric - Solution

(a) The liquid rises because of the negative capillary pressure p, given by γ/r where γ is the surface tension and r is the radius of curvature. The pressure just under the interface is uniform, so the curvature is uniform: the meniscus is a circular arc. Since it meets the walls at a fixed angle α, the radius must be a fixed multiple of the width d: $r = d\ f(\alpha)$. The capillary force per unit length F is then $p\ d$. This force is balanced by the weight of the column, $\rho g h d$, i.e., $\rho g h d = p\ d = \gamma/r\ d = \gamma f(\alpha)$ so that

$$h \sim 1/d.$$

(b) The electrostatic energy per unit length $E = \frac{1}{2}CV^2 = \frac{1}{2}C_0(\epsilon - 1)h/d\ V^2$. The gravitational energy $U = \frac{1}{2}\rho g h^2 d$. At equilibrium these two energies are equal up to a numerical factor: $(h/d)V^2 \sim h^2 d$, so that $V^2 \sim hd^2$ or

$$h \sim V^2/d^2.$$

(c) Near the equilibrium point the net unbalanced force per unit length $F \equiv d/dh\ (E + U)$ is $\rho g h d - \text{const.} V^2/d$. Thus $F = \rho g(h - h_0)d \equiv k(h - h_0)$. The oscillation frequency ω is $\sqrt{k/m} \sim \sqrt{d/(\rho d h_0)} \sim {h_0}^{-1/2}$. If V is doubled, h_0 quadruples and ω reduces by a factor of 2. Thus the period τ doubles.

9. Time Varying Mass

We usually treat constants of nature as truly constant in time. However, it is possible that quantities we assume are constant actually vary slowly in time. As an example, let us assume that the electron mass, m_e, changes slowly in time. We can model such a time variation as follows,

$$m_e = m\,(1 + \lambda \cos \Omega t)\,,$$

where λ is a parameter, m is a constant mass, while Ω is a frequency. Imagine we confine an electron in a two-dimensional trap. Assume the system is described by a Hamiltonian in the (x, y)-plane with,

$$H = \frac{\vec{p}^2}{2m_e} + \frac{1}{2}k\vec{x}^2,$$

with k a constant. This is an unrealistic system but, nevertheless, useful for testing your understanding of quantum mechanics. Treat the time-dependence as a small perturbation and work to leading order in the perturbation.

It may be useful to note that for a single simple harmonic oscillator, the normalized eigenstates $|n\rangle$ can be built from the ground state $|0\rangle$ using annihilation/creation operators, satisfying $[a, a^\dagger] = 1$, with $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(a) Find the lowest energy state of the unperturbed system which can decay to a lower energy unperturbed state because of the perturbation.

(b) Is any angular momentum conserved in the unperturbed system? How about the perturbed system? Are your findings consistent with your decay result from part (a)?

(c) Suppose at time $t = 0$, we start the system in the ground state of the unperturbed system. At time t, what is the probability of finding the system in an excited state of the unperturbed system?

Time Varying Mass - Solution

(a) The unperturbed system is a two-dimensional simple harmonic oscillator. The energy is determined by the occupation numbers $\vec{n} = (n_x, n_y)$ with

$$E_{\vec{n}} = \hbar\omega\left(1 + n_x + n_y\right), \qquad \omega^2 = \frac{k}{m}.$$

To leading order in λ, the time-dependent perturbation is given by

$$H' = -\frac{\vec{p}^2}{2m}\lambda\cos\Omega t.$$

Let us use H_0 to denote the unperturbed oscillator Hamiltonian. It's actually enough to recall that both $\vec{x}$ and $\vec{p}$ are linear sums of annihilation and creation operators. Precisely,

$$p_i = i\sqrt{2m\omega\hbar}\left(a_i^\dagger - a_i\right) \sim i\left(a_i^\dagger - a_i\right).$$

The normalization is unimportant. What is important is that p_i^2 acting on a state of the SHO either moves the level by ± 2, or leaves the level unchanged. That means that the ground state and single excited states are stable. However, the state which is a linear sum of occupation numbers $(2,0)$ and $(0,2)$ is unstable. This state is proportional to

$$\left(a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger\right)|(0,0)\rangle,$$

and can decay to the ground state. In polar coordinates, this state is of the form $(r^2 - c)e^{-\frac{m\omega}{2\hbar}r^2}$ for some constant c with $r^2 = x^2 + y^2$.

(b) The only possible conserved angular momentum is L_z. As an operator $L_z = xp_y - yp_x$. One can either check that $[H_0, L_z] = 0$, or just note that in polar coordinates (r,θ), H_0 does not depend explicitly on θ. For example, the potential is $\frac{1}{2}k^2r^2$. Either way confirms that the z component of angular momentum is conserved.

It is also easy to check that $[H', L_z] = 0$. The perturbation H' is proportional to the free Hamiltonian which is rotationally invariant so both the perturbed and unperturbed theories conserve angular momentum.

How is this consistent with the decay claim of part (1)? Interestingly, one combination of $(2,0)$ and $(0,2)$ is rotationally invariant. That is the combination we found in part (1), which could decay to the ground state. This is even clearer in complex coordinates with $z = x + iy$. There are 3 states with 2 oscillators excited: $(2,0)$, $(0,2)$ and $(1,1)$. Each is a degree 2 polynomial multiplied by $e^{-\frac{m\omega}{2\hbar}|z|^2}$, which is rotationally invariant. The only possible polynomials are

$$|z|^2, \quad z^2, \quad (z^*)^2.$$

The first is rotationally invariant while the remaining two carry L_z angular momentum.

(c) At leading order in the perturbation parameter λ, the ground state can only mix with the linear sum of the $(2,0)$ and $(0,2)$ excited states we found earlier. We use first order time-dependent perturbation theory. The amplitude for transition between the unperturbed ground state $|0\rangle$ and some final state $|f_0\rangle$ in time t is given by,

$$d_f(t) = -\frac{i}{\hbar}\int_0^t dt' \langle f_0|H'|0\rangle e^{i\omega(n_x^f+n_y^f)t'}.$$

It is convenient to rewrite the cosine in terms of exponentials:

$$\cos\Omega t = \frac{1}{2}\left(e^{i\Omega t} + e^{-i\Omega t}\right).$$

The time integral is now simple. Now let us write the perturbation conveniently in terms of annihilation and creation operators. Actually we only need the term in p_i^2 proportional to two creation operators,

$$H' = \frac{\lambda\cos\Omega t}{2m}(2m\omega\hbar)\left(a_x^\dagger a_x^\dagger + a_y^\dagger a_y^\dagger + \ldots\right).$$

The normalized final state is then

$$|f_0\rangle = \frac{1}{\sqrt{2}}\left(|(2,0)\rangle + |(0,2)\rangle\right).$$

The amplitude for a transition to this state is

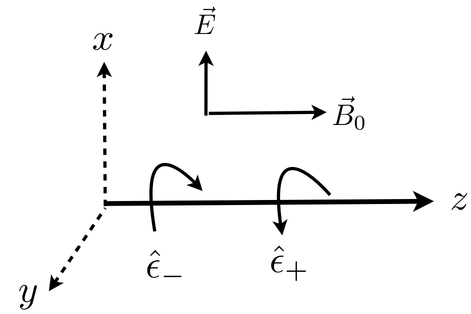
$$d_f(t) = \frac{i}{2\sqrt{2}\hbar}2\sqrt{2}\lambda\omega\hbar\int_0^t dt'\left(e^{i(2\omega+\Omega)t'} + e^{i(2\omega-\Omega)t'}\right) = \lambda\omega\left[\frac{e^{i(2\omega+\Omega)t}-1}{2\omega+\Omega} + \frac{e^{i(2\omega-\Omega)t}-1}{2\omega-\Omega}\right],$$

where one $\sqrt{2}$ comes from the relation $(a^\dagger)^2|0\rangle = \sqrt{2}|2\rangle$. We do not need to simplify this expression further. The probability for a transition to the excited state is therefore:

$$P = |d_f(t)|^2.$$

You could also have computed the probability, P_0, to stay in the ground state with a calculation along similar lines. The probability to transition to an excited state is $1 - P_0$.

10. Faraday Rotation



Consider an electromagnetic wave propagating in an electron gas. The gas consists of free electrons with charge e and number density n_e while the wave has frequency ω, and is propagating in the z-direction. Ignore self-interactions between the electrons. The incident wave is linearly polarized. There is an external magnetic field $\vec{B}_0$ pointed along the z-direction, and no external electric field.

(a) What is the net force acting on a single electron in the plasma in terms of the local electric and magnetic fields?

(b) Assume that the external magnetic field $\vec{B}_0$ is much larger than the internal magnetic field of the electromagnetic wave. Express the linearly polarized electric field $\vec{E} = E\hat{x}$ for the incident wave in terms of a basis $(\hat{\epsilon}_+, \hat{\epsilon}_-)$ of left and right circularly polarized waves, where $(\hat{\epsilon}_+, \hat{\epsilon}_-)$ are unit vectors. Make the same decomposition for the velocity of the electron. Assume that the time variation of the electron velocity is governed by the same angular frequency ω as the wave. Show that the velocity amplitude for the left and right circularly polarized components is given by:

$$v_\pm = \frac{ieE}{\sqrt{2}m\,(\omega \mp \omega_0)}, \qquad \omega_0 = \frac{eB_0}{mc}. \tag{1}$$

You might have a different definition of $v_\pm$ so please state your conventions explicitly.

(c) Given the velocity components (1) and the density of electrons n_e, calculate the induced current density. Show that the effective dielectric constant for the left and right-polarized waves is given by,

$$\epsilon_\pm^{\text{dielectric}} = 1 - \frac{\omega_p^2}{\omega\,(\omega \pm \omega_0)}, \qquad \omega_p^2 = \frac{4\pi n_e e^2}{m}. \tag{2}$$

(d) Finally, given the different dielectric constants (2) for the left and right polarized modes, calculate the amount of rotation in the polarization direction after the wave travels a distance z by computing the phase difference between the left and right circular polarizations.

Faraday Rotation - Solution

(a) Credit to Johannes Heinonen for providing this question. An electron is acted on by the Lorentz force giving a force law:

$$\vec{F} = m\frac{d\vec{v}}{dt} = e\vec{E}_{wave} + e\frac{\vec{v}}{c} \times \left(\vec{B}_{wave} + \vec{B}_0\right). \tag{3}$$

(b) Using unit vectors $\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}\left(\hat{x} \pm i\hat{y}\right)$, we express the electric field and electron velocity as follows:

$$\vec{E} = E\hat{x} = \frac{E}{\sqrt{2}}\left(\hat{\epsilon}_+ + \hat{\epsilon}_-\right), \qquad \vec{v} = v_-\hat{\epsilon}_+ + v_+\hat{\epsilon}_-, \tag{4}$$

where $v_\pm = \frac{1}{\sqrt{2}}\left(v_x \pm iv_y\right)$. The time-dependence of $\vec{v}$ implies:

$$\frac{d\vec{v}}{dt} = -i\omega\vec{v}.$$

Plugging into the force equation gives:

$$m(-i\omega)(v_-\hat{\epsilon}_+ + v_+\hat{\epsilon}_-) = e\frac{E}{\sqrt{2}}(\hat{\epsilon}_+ + \hat{\epsilon}_-) + \frac{eB_0}{c}(v_-\hat{\epsilon}_+ + v_+\hat{\epsilon}_-) \times \hat{z}. \tag{5}$$

Solving for $v_\pm$ gives the quoted expression.

(c) Ampére's law states that

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}. \tag{6}$$

The current $\vec{J} = n_e e(v_-\hat{\epsilon}_+ + v_+\hat{\epsilon}_-)$. Plugging into (6) gives

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \frac{4\pi}{c}n_e e(v_-\hat{\epsilon}_+ + v_+\hat{\epsilon}_-) + \frac{1}{c}(-i\omega)\frac{E}{\sqrt{2}}(\hat{\epsilon}_+ + \hat{\epsilon}_-), \\ &= \frac{1}{c}(-i\omega)\frac{E}{\sqrt{2}}\left[\left(1 - \frac{4\pi n_e e}{i\omega}\frac{\sqrt{2}}{E}v_-\right)\hat{\epsilon}_+ + \left(1 - \frac{4\pi n_e e}{i\omega}\frac{\sqrt{2}}{E}v_+\right)\hat{\epsilon}_-\right]. \end{aligned} \tag{7}$$

The different polarizations see different dielectric constants $\epsilon_\pm^{\text{dielectric}}$ given in terms of the plasma frequency ω_p.

(d) The phase of a plane-wave propagating through a medium with dielectric constant ϵ is proportional to $e^{i(\frac{\omega}{c}\sqrt{\epsilon}z - \omega t)}$. For our wave, we find

$$\begin{aligned}\vec{E} &= \frac{1}{\sqrt{2}}E_0\left[e^{i(\frac{\omega}{c}\sqrt{\epsilon_+}z - \omega t)}\hat{\epsilon}_+ + e^{i(\frac{\omega}{c}\sqrt{\epsilon_-}z - \omega t)}\hat{\epsilon}_-\right], \\ &= \frac{1}{\sqrt{2}}E_0 e^{i(\frac{\omega}{c}\sqrt{\epsilon_+}z - \omega t)}\left[\hat{\epsilon}_+ + e^{i(\frac{\omega}{c}(\sqrt{\epsilon_-} - \sqrt{\epsilon_+})z)}\hat{\epsilon}_-\right], \end{aligned} \tag{8}$$

which gives a phase difference

$$\Delta\phi(z) = \frac{\omega}{c}\left(\sqrt{\epsilon_-} - \sqrt{\epsilon_+}\right)z.$$

11. Magnetic Temperature Change

An insulating solid contains N_A non-magnetic atoms and N_I magnetic impurities each of which has spin 3/2. The spins are sufficiently dilute as to be approximately non-interacting. There is a very weak spin-lattice interaction which is sufficient to thermalize the spin and lattice temperatures, but which contributes negligible energy to the system.

(a) A magnetic field is applied to the system while it is held at constant temperature T. The field is strong enough to line up the spins completely. What is the magnitude and sign of the change in entropy in the system as the field is applied?

(b) Now the system is held in thermal isolation so that no heat is allowed to enter or leave. The magnetic field is slowly reduced to zero. Will the temperature of the solid increase or decrease? Justify your answer.

(c) Assume the heat capacity of the solid is given by $C = 3N_A k_{\mathrm{B}}$, where k_{B} is Boltzman's constant. What is the temperature change produced by the demagnetization procedure of part (b)? Neglect any changes in the volume of the solid.

Magnetic Temperature Change - Solution

(a) Entropy is given by $S = k_\mathrm{B}\log(\Omega)$ where Ω is the number of micro-states. The spins have 4 possible states $(+3/2, +1/2, -1/2, -3/2)$ and in the absence of the field all possible states are allowed giving $\Omega = 4^{N_I}$ possible microstates. Thus, before the external magnetic field is applied, $S = k_\mathrm{B}N_I\log(4)$. After the external magnetic field is applied all of the spins are aligned with the field in their maximum spin projection so there is only a single state $\Omega = 1$ and thus $S = 0$. Thus the the entropy change is $\Delta S = -k_\mathrm{B}N_I\log(4)$.

(b) In part (a) the external magnetic field is applied and the entropy decreases, meaning that heat ($dQ = TdS$) leaves the spin system and is deposited in the lattice. However, during this time the system is held at a constant temperature (the heat is removed). As the magnetic field is reduced entropy is transferred into the spin system. Since the system is thermally isolated this heat must be transferred from the solid lattice. Since heat is transferred out of the lattice the temperature is reduced.

(c) As described in part (b) heat is transferred from the lattice into the spin system. We start with the first law of thermodynamics:

$$dU = TdS - PdV.$$

The problem states that there is no volume change so the last term $PdV = 0$. Heat capacity (at constant volume) is defined as $C = \partial U/\partial T$. Combining these two equations gives $TdS = CdT$ or more conveniently for this problem:

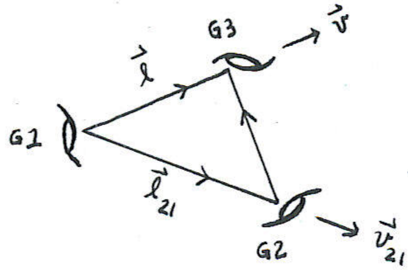
$$dS = C\,\frac{dT}{T}.$$

Integrating this equation gives $\Delta S = C(\log T_f - \log T_i) = C\log(T_f/T_i)$, where T_i and T_f are the initial and final temperatures. Solving for T_f gives

$$T_f = T_i e^{\Delta S/C} = T_i e^{-\frac{N_I}{3N_A}\log(4)}.$$

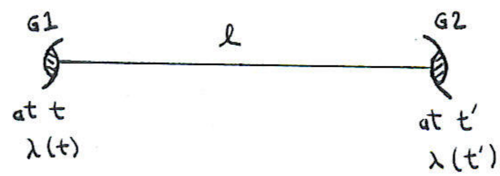
12. Expansion of the Universe

Consider two galaxies (G1 and G2) in an expanding universe; see the figure below. As a result of the expansion, G1 observes other galaxies, such as G3, receding from it with a velocity given by $\vec{v} = H_0\vec{\ell}$, where H_0 is constant and ℓ is the distance to the galaxy.



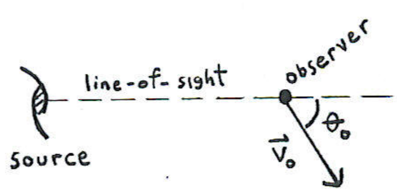
(a) Assuming flat space and velocities small compared to c, determine a relationship for the velocities of galaxies as observed from G2. Compare your result with that for G1 observers.

The expansion of the universe can be modeled by a time-dependent scale factor, $a(t)$, such that the proper distance between two points in the universe can be written as $\ell = a(t)R$, where R is a constant.



(b) Consider light emitted from G1 at time t and detected at G2 at time $t' = t + \tau$. To first order in the expansion rate, determine how the wavelength depends on the scale factor $a(t)$. [Hint: you may treat this as a Doppler effect.] The light observed from distant galaxies can be characterized by the spectral shift $z \equiv (\lambda_o - \lambda_e)/\lambda_e$, where λ_e is the emitted wavelength at t and λ_o is the observed wavelength at t'. In this context z is called the *redshift*. Please explain why this is appropriate terminology.

The observed redshift z_o depends on both the cosmological redshift z_c (due only to the expansion) and the peculiar velocities of the galaxies (due to their motions relative to the flow of the expansion).



(c) Assume the observer has a peculiar velocity $\vec{v}_0$ that makes an angle θ_0 with the line of sight to the distant emitting galaxy. Determine the relationship between the observed redshift and the cosmological redshift. Do not assume small velocities. [Hint: It might be simpler to relate the values of $1 + z$.]

Expansion of the Universe - Solution

(a) Assuming small velocities and flat space we can use nonrelativistic transformations. Let us adopt the notation x_{ab} to represent the quantity x of a relative to b. We have

$$\vec{v}_{32} = \vec{v}_{31} - \vec{v}_{21}.$$

Using the expansion law relative to G1 (Hubble's law), we obtain

$$\vec{v}_{32} = H_0\vec{\ell}_{31} - H_0\vec{\ell}_{21} = H_0\left(\vec{\ell}_{31} - \vec{\ell}_{21}\right).$$

This gives

$$\vec{v}_{32} = H_0\vec{\ell}_{32}.$$

which is equivalent to what G1 observers find. Therefore, all galaxies observe the same expansion law.

(b) The Doppler effect for light can be obtained from the Lorentz transformations. Taking $c = 1$ the transformation equation for the energy of a photon as $E_o = \gamma(E_e - vp_e)$ where v is the relative speed between observer and emitter and $\gamma = (1-v^2)^{1/2}$. For photons $E = p = hf$ so we have

$$f_o = \gamma(f_e - vf_e) = \gamma f_e(1 - v).$$

If we expand γ this becomes

$$f_o = f_e\left[\frac{1-v}{1+v}\right]^{1/2} \quad \Longrightarrow \quad \lambda_o = \lambda_e\left[\frac{1+v}{1-v}\right]^{1/2}.$$

The binomial expansion gives $\lambda_o = \lambda_e[1+v+O(v^2)]$. So, to first order in v, $\lambda_o = \lambda_e(1+v)$. The recession velocity is $v = \dot{\ell} = R\dot{a}$. Now $R = \ell/a$ and $\ell = c\tau = \tau$ because photons travel at the speed of light. Therefore, we can wright

$$\lambda(t+\tau) = \lambda(t)\left[1 + \frac{\dot{a}}{a}\tau\right].$$

A Taylor expansion gives

$$\lambda(t+\tau) = \lambda(t) + \frac{d\lambda}{dt}\tau + \ldots.$$

To first order we have

$$\frac{d\lambda}{dt} = \frac{\lambda}{a}\frac{da}{dt} \quad \Longrightarrow \quad \frac{d\lambda}{\lambda} = \frac{da}{a}.$$

From this we clearly see that

$$\lambda(t) \propto a(t).$$

The expansion causes distant galaxies to move away from each other producing spectral shifts to longer wavelengths. Since red light is at the longer wavelength end of the visible spectrum, these are called redshifts.

(c) We can relate the two redshifts via a Lorentz transformation between the frame of an observer who measures the cosmological redshift (a co-moving observer) and the frame of the peculiar velocity.

Taking the observer as the rest frame, the energy transformation is $E_c = \gamma(E_o - v_o p_o \cos\theta_0)$. Since $E = p$ this is equivalent to $p_c = \gamma p_o(1 - v_o \cos\theta_0)$. Note that

$$\frac{p_c}{p_o} = \frac{\lambda_o}{\lambda_c} = \gamma p_o(1 - v_o \cos\theta_0).$$

From the definition of z we see that $1 + z = \lambda_o/\lambda_e$ which shows that

$$\frac{\lambda_o}{\lambda_c} = \frac{1 + z_o}{1 + z_c},$$

giving

$$1 + z_o = \gamma(1 + z_c)(1 - v_o \cos\theta_0).$$