Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination
Part II
Saturday, March 28, 2015

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________________
Fill in your Lg-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your Lg-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

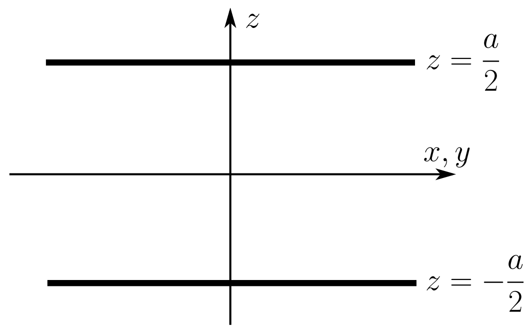
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs 30 min**.

Please, indicate problems you are turning in:

☐ **II-1** ☐ **II-2** ☐ **II-3** ☐ **II-4**

II-1. (E & M)



Consider a point charge, q, placed centrally at the origin ($\vec{r} = 0\,\hat{x} + 0\,\hat{y} + 0\,\hat{z}$) between two parallel infinitely-large grounded conducting plates as shown in the figure. The distance between the two plates is a. Using the method of images answer the following questions:

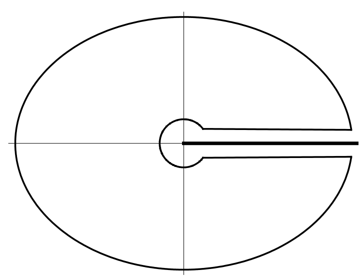
(i) If the bottom plate is removed, what is the electrostatic potential $\phi(x, y, z)$ that satisfies vanishing boundary conditions on the top plate, $\phi(x, y, \frac{a}{2}) = 0$.

(ii) What is then the electric field at the two locations, $\vec{r} = \pm a\hat{z}$?

(iii) If both plates are in place, find the location and charges of the many images and then compute the electrostatic potential $\phi\,(x, y, z)$ that satisfies vanishing boundary conditions on both plates, $\phi(x, y, \pm\frac{a}{2}) = 0$. Give your answer in the form of a sum and argue how this sum will produce the desired vanishing result at the plates.

(iv) In case (iii), what is the induced *total* charge on the top plate? How does it compare to the induced total charge on the bottom plate?

II-2. (Mathematical Methods)



The function

$$f(z) \;=\; \frac{z^{1/2}}{1+z^2}\,\log z\,,$$

is made single valued by introducing a branch cut along the positive real axis. Use the integration(s) along the key-hole contour as above to prove the following result for the real integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,\log x\,dx \;=\; \frac{\pi^2}{2\sqrt{2}}\,.$$

Hint: You may have to do the contour integration more than once.

II-3. (Classical Mechanics)

A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R. (The hoop does not spin, i.e. it stays in the vertical plane perpendicular to the axis of the cylinder.) The only external force is that of gravity. The hoop starts rolling from rest at the top of the cylinder, and while it is rolling there are two constraints, one holonomic, the other semiholonomic. (Actually, the semiholonomic constraint could be integrated to a holonomic one, but do not do this.) Use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.

II-4. (Quantum Mechanics)

Consider a two-level quantum system, e.g., a spin-$\frac{1}{2}$ particle, with the Hamiltonian and the initial state given respectively by

$$H = \frac{\hbar\omega}{2}\sigma_z, \qquad |\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

where

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \qquad \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \qquad \sigma_y = i\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right),$$

are the standard Pauli matrices.

(i) Find the expression for the time-dependent expectation-values

$$\alpha(t) := \langle\Psi|\sigma_\alpha(t)|\Psi\rangle, \qquad (\alpha = x, y, z)$$

where

$$\sigma_\alpha(t) := e^{i\frac{\omega t}{2}\sigma_z}\sigma_\alpha e^{-i\frac{\omega t}{2}\sigma_z},$$

are the Heisenberg picture evolved operators at time t.

(ii) Compute the variances of $\sigma_x(t)$ and $\sigma_z(t)$ in $|\Psi\rangle$ and discuss the associated Heisenberg Uncertainty Relations.

(iii) Compute the so-called *survival probability*

$$P_{\text{surv}}(t) := |\langle\Psi|e^{-i\frac{t}{\hbar}H}|\Psi\rangle|^2$$

What's the first time t^* such that $P_{\text{surv}}(t^*) = 0$? What is its physical meaning?

(iv) One way to model the effect of the environment on the two-level system is to assume that the initial state gets randomly *dephased* as follows

$$|\Psi\rangle \longrightarrow |\Psi_\xi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\xi}|1\rangle),$$

where ξ is a *uniformly random angle* in $[0,\, 2\pi]$. Compute the noise averages

$$\frac{1}{2\pi}\int_0^{2\pi} d\xi\, \langle\Psi_\xi|\sigma_\alpha(t)|\Psi_\xi\rangle, \qquad (\alpha = x, z)$$

and comment on the results on physical grounds.