Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 12, 2004
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

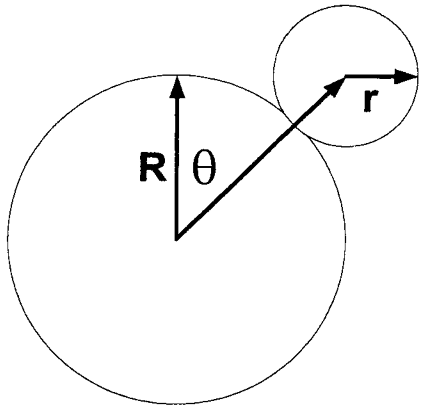
Good luck!!

Problem 1

A simple pendulum consists of a massless rod of length R with a mass m at the bottom. The rod pivots on a hinge that constrains the motion of the pendulum to the vertical plane. The hinge is attached to the lower end of a vertical shaft that is made to rotate with a constant angular frequency ω by a synchronous motor.

a) Write down the Lagrangian and the equation of motion in a stationary coordinate system.
b) What is the solution corresponding to stable small oscillations about the position of minimum energy and what is the condition for this type of motion?
c) What is the frequency of small oscillations?
d) What is the Hamiltonian, H, in the fixed coordinate system? What is the Hamiltonian in a rotating system attached to the shaft of the motor?
e) Are $T+V$ (kinetic + potential energy) and H conserved in each of the coordinate systems mentioned above?

Problem 2

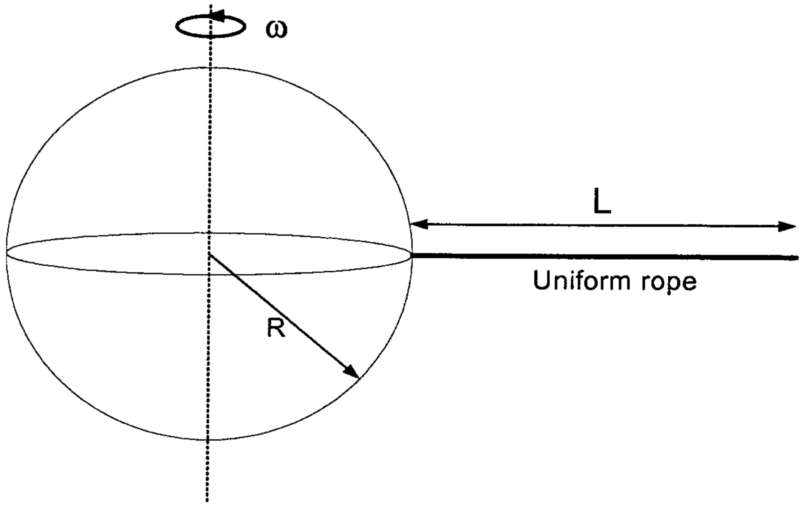


A uniform sphere of radius r and mass m rolls without sliding on the outer surface of a stationary sphere of radius R. The position of the rolling sphere is described by angle, θ, as shown in the figure. If the upper sphere starts from rest at the top of the stationary sphere $(\theta = 0)$,

a) Find the velocity of the center of mass of the moving sphere as a function of θ.
b) Determine that value of θ at which the moving sphere flies off the stationary one.
c) If the moving sphere begins at $t = 0$ with $\theta = 0$ but $\dot{\theta}(0) \neq 0$ find $\theta(t)$ in terms of $\dot{\theta}(0)$ for small t.

Problem 3

A recent Science Times article featured the concept of a "space elevator". This is a free hanging rope in stationary orbit around the earth above the equator. You could send an elevator up this rope to launch objects into space at less cost than required for shuttle flights. Imagine such a rope just reached the earth's surface. Find an expression for the tension in the rope as a function of height, y, off the earth's surface. Assume the rope has length L, and mass m, and that the earth has radius R and mass M and rotates at angular velocity ω. What length, L, allows the rope to hang freely (i.e. without being attached to the earth's surface) ?



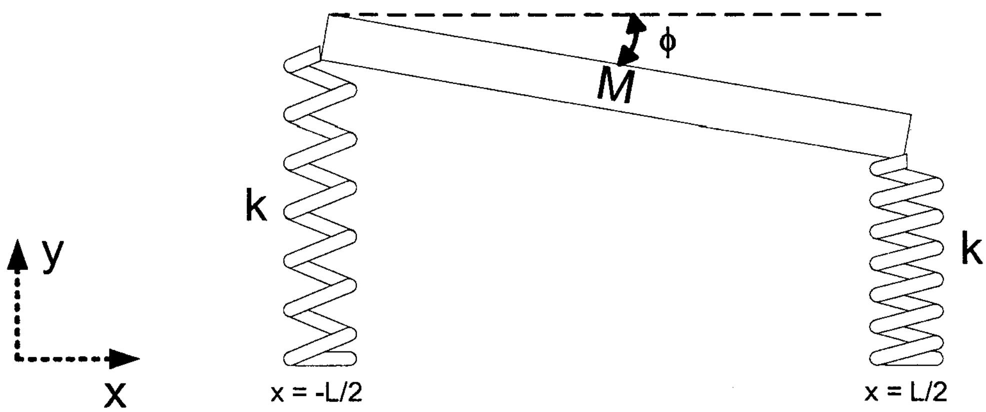
Problem 4

Consider circular orbits in the field of a central force $f(r) = -kr^n$

a) What is the relation between orbital velocity v and radius r?
b) For what values of n are these circular orbits stable to small perturbations?
c) Specifically, are the circular orbits stable or unstable for $n = -3$?

Problem 5

A thin, uniform density, bar of length L and mass M is support by two springs with identical spring constants k and unloaded lengths l. The center of mass of the bar is constrained to move vertically with displacement $y(t)$, but the bar can rotate in the x-y plane with angle $\phi(t)$.



a) Determine the Lagrangian of the system for small displacements taking gravity into account.

b) Solve the Euler-Lagrange equations of motion to determine the frequencies and eigenvectors for the normal modes of the system.

c) If at $t=0$ the end of the bar at $x=L/2$ is depressed by a *small* amount, d, while the other end is held in its equilibrium position (i.e. $y_1(t=0)=l$, $y_2(0)=l-d$) and all initial generalized velocities vanish, find expressions for the time dependence of the subsequent vertical displacement and angular rotation of the bar, $y(t)$ and $\phi(t)$.

Email

Delivered-To: lalla@phys.columbia.edu
Date: Mon, 22 Dec 2003 02:01:46 -0500
From: Elena Aprile <age@astro.columbia.edu>
Subject: Aprile_Quals2004_Mechanics Problem
To: cole@nevis1.columbia.edu
Cc: lalla@phys.columbia.edu
X-Mailer: Microsoft Outlook IMO, Build 9.0.2416 (9.0.2910.0)
Importance: Normal

Section 1-:Classical Mechanics
Question # ~~5~~ 1

Hi Brian, I attach a problem for the quals; sorry for being late. Elena

MECHANICS_PROBLEM

A simple pendulum consists of a massless rod of length R with a mass m at the bottom. The rod pivots on a hinge that constrains the motion of the pendulum to the vertical plane. The hinge is attached to the lower end of a vertical shaft that is made to rotate with a constant angular frequency w by a synchronous motor.

(1) Write down the Lagrangian and the equation of motion in a stationary coordinate system.

(2) What is the solution corresponding to stable small oscillations about the position of minimum energy and what is the condition for this type of motion? What is the frequency of small oscillations?

(3) What torque does the motor exert when the pendulum oscillates as above.

(4) What is the Hamiltonian in the fixed coordinate system? What is the Hamiltonian in a rotating system attached to the shaft of the motor?

(5) Are T+V and H conserved in each of the coordinate systems mentioned above?

NO SOLUTION

Christ Section 1, Question # 2

Suggested Solutions

1. (a) Introduce ϕ to represent rotation of the moving sphere about its center of mass. Equating the total kinetic energy with the potential energy lost gives:

$$\begin{aligned}(R+r)(1-\cos(\theta))mg &= m(r\dot{\phi})^2 + \frac{2}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mv_{\mathrm{cm}}^2\end{aligned}$$

Thus, $v_{\mathrm{cm}}(\theta) = \sqrt{\frac{5}{7}(R+r)(1-\cos(\theta))g}$.

(b) The sphere will fly off when $mv_{\mathrm{cm}}^2/(R+r) > mg\cos(\theta)$ or

$$\frac{5}{7}(1-\cos(\theta)) \quad > \quad \cos(\theta)$$

or

$$\cos(\theta) = 5/13$$

(c) Start with the equation of motion obtained by equating torque and rate of change of angular momentum around the point of contact: $\frac{7}{5}mr^2\ddot{\phi} = mgr\sin\theta$. Relate θ and ϕ by computing the velocity of the moving sphere's center of mass two ways:

$$(R+r)\dot{\theta} = r\dot{\phi}$$

Combining these equations:

$$\ddot{\theta} = \frac{5g}{7(R+r)}\theta$$

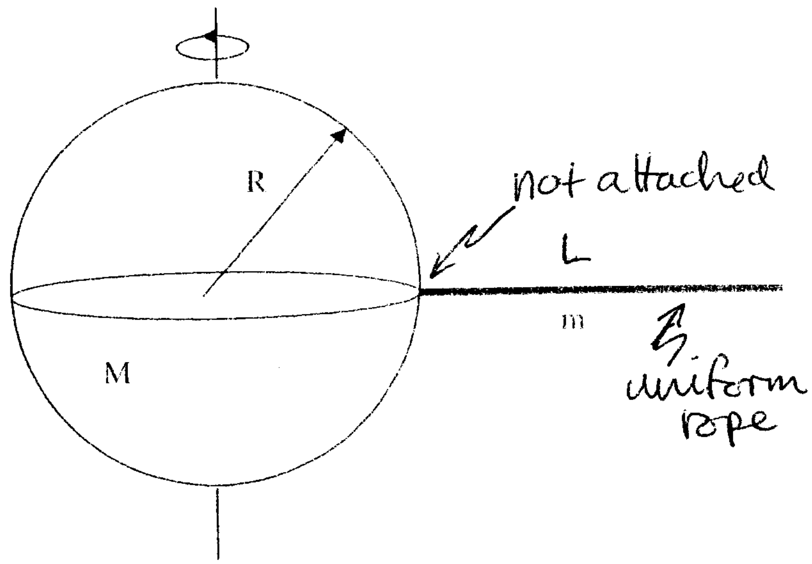
or

$$\theta(t) = \frac{\dot{\theta}(0)}{\omega}\sinh(\omega t)$$

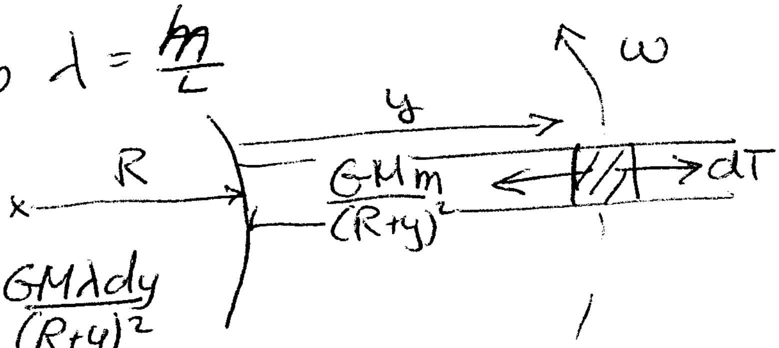
where $\omega = \sqrt{\frac{5g}{7(R+r)}}$.

Mechanics Turtz Section 1, Question #6 3

Q3. A recent Science Times article featured the concept of a space elevator. This is a "free hanging" rope in stationary orbit around the equator. You could then send an elevator up this rope to launch objects into space at less cost than the shuttle. Imagine that the rope just reached the earth's surface. What is the expression for the tension in the rope? How long does the rope have to be? Assume the rope has length L, mass m, and the earth has radius R and mass M.



A. The density is $\lambda = \frac{m}{L}$



$$\frac{GMm}{(R+y)^2} + dT = m\omega^2(R+y)$$

$$\therefore dT = \lambda dy\,\omega^2(R+y) - \frac{GM\lambda dy}{(R+y)^2}$$

$$T(y) = \frac{GM\lambda}{(R+y)} + \frac{\lambda\omega^2}{2}(R+y)^2 \Big|_y^L$$

$$\therefore T(y) = \frac{GM\lambda}{R+L} - \frac{GM\lambda}{R+y} + \frac{\lambda\omega^2(R+L)^2}{2} - \frac{\lambda\omega^2(R+y)^2}{2}$$

$$= GM\lambda\left(\frac{1}{R+L} - \frac{1}{R+y}\right) + \frac{\lambda\omega^2}{2}\left[R^2 + 2LR + L^2 - R^2 - 2Ry - y^2\right]$$

$$= GM\lambda\left(\frac{y-L}{(R+L)(R+y)}\right) + \frac{\lambda\omega^2}{2}\left(2R(L-y) + L^2 - y^2\right)$$

$$\boxed{T(y) = (L-y)\lambda\left[\frac{-GM}{(R+L)(R+y)} + \frac{\omega^2}{2}(2R+L+y)\right]}$$

So the length must be such that $T=0$ at $y=0$

$$\therefore \frac{GM}{(R+L)R} = \frac{\omega^2}{2}(2R+L)$$

$$\therefore \frac{2GM}{\omega^2 R} = (2R+L)(R+L) = 2R^2+3RL+L^2$$

$$\therefore L^2 + (3R)L + \left[2R^2 - \frac{2GM}{\omega^2 R}\right] = 0$$

$$\therefore L = \frac{-3R \pm \sqrt{9R^2 - 8R^2 - \frac{8GM}{\omega^2 R}}}{2}$$

$$\boxed{L = -\frac{3}{2}R \pm \frac{1}{2}\sqrt{R^2 - \frac{8GM}{\omega^2 R}}}$$

Section 1, Question #4

Problem $\cancel{2}$ 4 (Mechanics - Central Force)

(M.Shaevitz)

For the following questions assume that there is a central force with the form ($k > 0$)

$$f(r) = -kr^n$$

a) For circular orbits, what is the relation between velocity, v, the radius, r, and n.

b) For what values of n are these circular orbits stable to small perturbations?

c) Specifically, are the circular orbits stable or unstable for $n = -3$?

Solution:

a) From the radial equation of motion

$$m\ddot{r} = \frac{mv^2}{r} + f(r)$$

a circular orbit will have $\ddot{r} = 0$ giving (where L = angular momentum)

$$\frac{mv^2}{r} = kr^n = \frac{L^2}{mr^3}$$

$$L = mrv$$

$$v = \sqrt{\frac{k}{m} r^{n+1}}$$

b) For a circular orbit with radius equal a, let $r = a + x$, then

$$m\ddot{x} = \frac{L^2}{m(a+x)^3} + f(a+x)$$

Expanding as a power series in $\frac{x}{a}$ give

$$m\ddot{x} = \frac{L^2}{ma^3}\left(1 - 3\frac{x}{a} + \ldots\right) + (f(a) + f'(a)x + \ldots)$$

Keeping only the leading terms and using the conditions for a circular orbit then leads to

$$m\ddot{x} + \left(\frac{-3}{a}f(a) - f'(a)\right)x = 0$$

For stable orbits, coeficient of x must be greater than zero

$$-\left(f(a) + \frac{a}{3}f'(a)\right) > 0$$

Given the power law force $f(r) = -kr^n$, the stability condition then becomes

$$-ka^n - \frac{a}{3}kna^{n-1} < 0$$

or

$$n > -3$$

c) For $f(r) = -kr^{-3}$, the above equations becomes

$$f(a) + \frac{a}{3}f'(a) = 0$$

$$m\ddot{x} = 0$$

This then gives a non-stable orbit where any perturbation is not corrected by a restorative force.

Section 1, Question 1 # 5

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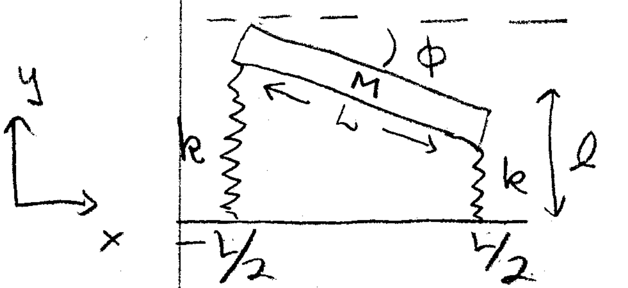


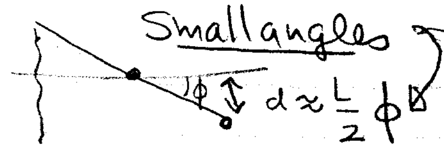
A thin uniform density bar of length L and mass M is supported by two springs with (identical) constants k and unloaded lengths ℓ

The center of mass of the rod is constrained to move vertically (with $y(t)$), but the bar can rotate with $\phi(t)$ in the xy plane

a) Determin the Lagrangian for small displacements taking gravity g into account

b) Solve the Euler Lagrange equations of motion for the normal modes and frequencies

c) If at $t=0$ the end of the bar at $x = L/2$ is depressed by a small amound d, i.e. $y = \ell - d$ while the other end at $x = -L/2$ is in its equilibrium position $y = \ell$, and if initially all generalized velocities vanish, What is the subsequent $y(t)$ and $\phi(t)$?



Solution:

1) $$T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\phi}^2$$

$$I = \int_{-L/2}^{L/2} dx\, x^2 \frac{M}{L} = 2\, \frac{1}{3}\left(\frac{L}{2}\right)^3 \frac{M}{L} = \frac{1}{12} ML^2$$

$$V = Mgy + \frac{1}{2} k \left\{ \left(y - \ell + \frac{L}{2}\phi\right)^2 + \left(y - \ell - \frac{L}{2}\phi\right)^2 \right\}$$

Equilibrium: $$\frac{\partial V}{\partial y} = 0 = \frac{\partial V}{\partial \phi}$$

a) $$Mg + k\left(\left(y_{eq} - \ell + \tfrac{L}{2}\phi\right) + \left(y_{eq} - \ell - \tfrac{L}{2}\phi\right)\right) = 0$$

$$y_{eq} = \ell - Mg/2k$$

b) $$\frac{kL}{2}\left\{\left(y - \ell + \tfrac{L}{2}\phi_{eq}\right) - \left(y - \ell - \tfrac{L}{2}\phi_{eq}\right)\right\} = 0$$

$$\phi_{eq} = 0$$

$$V(y,\phi) \cong V_{eq} + \frac{1}{2}(y - y_{eq})^2 \left(\frac{\partial^2 V}{\partial y^2}\right)_{eq} + \frac{1}{2}(\phi - \phi_{eq})^2 \left(\frac{\partial^2 V}{\partial \phi^2}\right)_{eq}$$

$$= V_{eq} + \eta^2 k + \left(\frac{kL^2}{4}\right)\phi^2$$

$\eta = y - y_{eq}$ = small vertical displacement

$$\boxed{L = \frac{1}{2} M \dot{\eta}^2 + \frac{ML^2}{24}\dot{\phi}^2 - k\eta^2 - \frac{kL^2}{4}\phi^2}$$

b) Euler Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$

1) $M\ddot{\eta} = -2k\eta \Rightarrow \ddot{\eta} + \omega_1^2\eta = 0$

2) $\left(\frac{ML^2}{12}\right)\ddot{\phi} = -\left(\frac{kL^2}{2}\right)\phi \Rightarrow \ddot{\phi} + \omega_2^2\phi = 0$

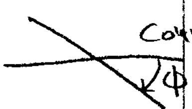
$$\omega_1^2 = \frac{2k}{M} \qquad \omega_2^2 = \frac{6k}{M}$$

$$\omega_2 = \sqrt{3}\,\omega_1$$

$$\begin{cases} \eta(t) = A\cos(\omega_1 t + \delta_1) \\ \phi(t) = B\cos(\omega_2 t + \delta_2) \end{cases}$$

c) End point height displacements (small)

Convention



$$Z_\pm(t) \equiv \eta(t) \mp L/2\,\phi(t)$$

Given $Z_+(0) = -d = \eta(0) - L/2\,\phi(0)$

$Z_-(0) = 0 = \eta(0) + L/2\,\phi(0)$

$\Rightarrow \eta(0) = -d/2\,,\quad \phi(0) = -\frac{2}{L}\eta(0) = \frac{d}{L}$

Also $\dot{Z}_+(0) = 0 = \dot{Z}_-(0)$

$\Rightarrow \dot{\eta}(0) = 0 = \dot{\phi}(0)$

Use these initial conditions to fix $A\ B\ \delta_1\ \delta_2$ from part b:

$$\left.\begin{aligned} A\cos\delta_1 &= -d/2 \\ B\cos\delta_2 &= d/L \\ A\omega_1\sin\delta_1 &= 0 \\ B\omega_2\sin\delta_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} A = -d/2 \\ B = d/L \\ \delta_1 = \delta_2 = 0 \end{cases}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 12, 2004
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

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Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1

A point-like electric dipole $\vec{p}$ is embedded at the center of a sphere of radius R. The sphere is made of linear dielectric material with dielectric constant ε_r and is surrounded by vacuum. Determine the electric potential inside and outside the sphere making sure you get the correct answer in the limit $\varepsilon_r \to 1$. *Hint:* the potential at small distances must approach the potential of a dipole in an infinitely large dielectric medium (SI units),

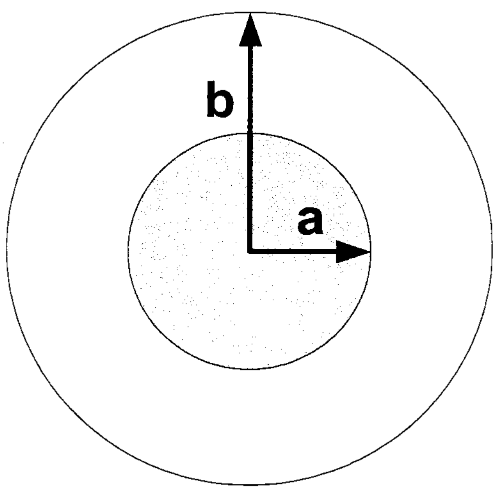
$$\Phi = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{p\cos\theta}{4\pi\varepsilon_0\varepsilon_r r^2}.$$

Problem 2

A coaxial cable consists of two cylindrical conductors. The inner conductor is a solid cylinder of radius a, and the outer conductor is a thin cylindrical shell of radius b. A current I flows in the inner conductor and current $-I$ flows in the outer conductor. Assume that the current in the inner conductor is uniformly distributed across the cross-section of the conductor.

a) Show that the inductance L per unit length l is given by $\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\frac{b}{a} + \frac{\mu_0}{4\pi}$ (you may alternatively give the result in CGS units).

b) What gives rise to the second term in the result in part a? To answer this, consider how the result changes if you assume the inner conductor is a thin cylindrical shell of radius a.



Problem 3

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a small insulating gap.

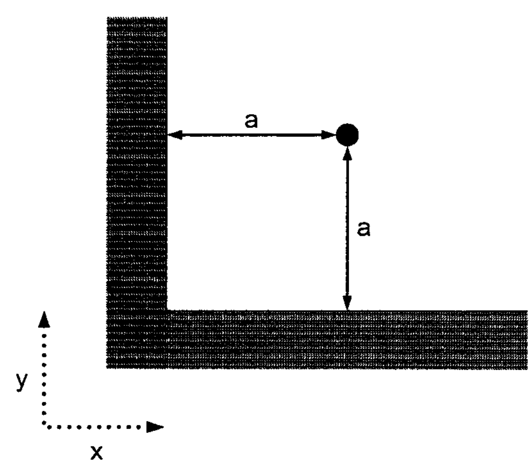
a) If the two shells are maintained at constant potentials $+V$ and $-V$, respectively find the electrostatic potential outside the hemispheres up through and including terms of order $(1/r)^2$.

b) Find the static dipole moment of the system with the conductors held at the potentials given in part a).

c) Now the potentials of the shells are made to vary slowly with time according to $\pm V \cos \omega t$. The electric dipole of the system now varies sinusoidally with time and the system gives off electric dipole radiation. Find the time-averaged power radiated by the two shells as a function of the angle θ (defined with respect to the electric dipole moment) and frequency ω.

Problem 4

A uniformly charged wire with *constant* charge per unit length, $-\lambda$ (charge density $\rho(\vec{r}) = -\lambda\delta(x-a)\delta(y-a)$), runs parallel to and is a distance, a, from the surfaces of two planar, perfect conductors oriented perpendicular to each other. The planes intersect at $x=y=0$. Find the surface charge density, $\sigma(x)$ and $\sigma(y)$ on the surfaces of the horizontal and vertical conducting planes, respectively.



Problem 5

The Proca equations describe electromagnetism with photons that have a non-zero rest mass. The electric and magnetic fields, $\vec{E}$ and $\vec{B}$, are defined in terms of a scalar potential, Φ, and a vector potential, $\vec{A}$, in the usual way, $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. The four Proca equations involve $\vec{E}$, $\vec{B}$, Φ, and $\vec{A}$ and are a generalization of the four Maxwell's equations:

(I) $\vec{\nabla} \cdot \vec{B} = 0$ (I') $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

(II) $\vec{\nabla} \cdot \vec{E} + \mu^2 \Phi = 4\pi\rho$ (II') $\vec{\nabla} \times \vec{B} + \mu^2 \vec{A} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$

The first two equations are equivalent to the definitions of $\vec{E}$ and $\vec{B}$ in terms of Φ, and $\vec{A}$. The second two equations involve a new parameter, μ, which has dimensions of reciprocal length and is related to the photon mass via, $\mu = mc/\hbar$. The sources ρ and $\vec{J}$ are the usual charge and current density that satisfy the charge/current conservation law $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$.

a) Show that the Proca equations together with charge/current conservation require that $\vec{A}$ and Φ satisfy the Lorentz condition, $\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0$.

b) Using the Proca equations together with the Lorentz condition from part a), find the generalized wave equations that Φ and the Cartesian components of $\vec{A}$ satisfy in terms of the sources ρ and $\vec{J}$.

c) Consider the plane wave solutions, $\begin{Bmatrix} \vec{A}(z,t) \\ \Phi(z,t) \end{Bmatrix} = \mathrm{Re}\left[\begin{Bmatrix} \vec{A}_0 \\ \Phi_0 \end{Bmatrix} e^{i(kz - \omega t)} \right]$

in source-free space ($\rho = 0$ and $\vec{J} = 0$). ω is a given positive constant, while $\vec{A}_0$ and Φ_0 are unspecified constants. Find the constant k in terms of ω, μ, and c. Show that there is a cut-off frequency, ω_c, such that for $\omega > \omega_c$ the wave propagates without attenuation while for $\omega < \omega_c$ the wave does not propagate but is just attenuated in the (positive or negative) z direction. Find the value for ω_c. For $\omega > \omega_c$ find the phase velocity of the given plane wave in terms of ω, μ, and c.

d) For $\omega > \omega_c$, consider a ***longitudinal*** plane wave with the vector potential, $\vec{A}(z,t)$ in the z-direction: $\vec{A}_0 = A_0\hat{z}$, where A_0 is a given positive constant. Find the plane wave's scalar potential $\Phi(z,t)$, magnetic field $\vec{B}(z,t)$, and electric field $\vec{E}(z,t)$ in terms of A_0, ω, μ, and c.

general solution of Laplace's equation in spherical coordinates

$$\phi = \sum_n \left[A_n r^n P_n(\cos\theta) + \frac{B_n}{r^{n+1}} P_n(\cos\theta)\right]$$

ϕ inside: for $r \to 0$, the potential approaches the potential of a dipole in a large dielectric,

$$\phi = \frac{1}{4\pi\varepsilon} \frac{\vec{p}\,\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon} \frac{p\cos\theta}{r^2},$$

so inside, $B_n = 0$ for $n \neq 1$, but

$$B_1 = \frac{p}{4\pi\varepsilon}$$

$$\Rightarrow \quad \phi_{in} = \frac{p}{4\pi\varepsilon} \frac{1}{r^2} P_1(\cos\theta) + \sum_n A_n r^n P_n(\cos\theta)$$

ϕ outside:

$$\phi_{out} = \sum_n \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

ϕ is continuous at $r = R$, so $\phi_{in}\big|_{r=R} = \phi_{out}\big|_{r=R}$

$$\Rightarrow \quad \frac{p}{4\pi\varepsilon} \frac{1}{R^2} P_1(\cos\theta) + \sum_n A_n R^n P_n(\cos\theta) = \sum_n \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

Coefficients of Legendre polynomials of given n must be equal for this equation to hold, so

$n = 1$
$$\frac{p}{4\pi\varepsilon} \frac{1}{R^2} + A_1 R = \frac{1}{R^2} B_1 \qquad (1)$$

$n \neq 1$
$$A_n R^n = \frac{B_n}{R^{n+1}}$$

$$\Leftrightarrow \quad A_n = \frac{B_n}{R^{2n+1}} \qquad (2)$$

Also, $$\varepsilon_r \varepsilon_0 \frac{\partial \phi_{in}}{\partial r}\Big|_{r=R} = \varepsilon_0 \frac{\partial \phi_{out}}{\partial r}\Big|_{r=R}$$

$$\Rightarrow \quad \Big[-\frac{2\rho\varepsilon_r}{4\pi\varepsilon_0\varepsilon_r}\frac{1}{R^3} + \varepsilon_r \sum_n A_n\, n\, R^{n-1} = \sum (-1)(n+1)\frac{B_n}{R^{n+2}}$$

$n=1$ $$-\frac{2\rho}{4\pi\varepsilon_0}\frac{1}{R^3} + \varepsilon_r A_1 = -\frac{2B_1}{R^3} \qquad (3)$$

$n \neq 1$ $$\varepsilon_r\, n\, A_n R^{n-1} = -(n+1)\frac{B_n}{R^{n+2}}$$

$$\Leftrightarrow \qquad A_n = -\frac{n+1}{\varepsilon_r n}\frac{B_n}{R^{2n+2}} \qquad (4)$$

(2) and (4) can not be satisfied for all n and ε_r unless

$$A_n = B_n = 0 \qquad \text{for } n \neq 1.$$

Now solve (1) and (3) for A_1 and B_1:

$$\left.\begin{aligned} \frac{\rho}{4\pi\varepsilon_0}\frac{1}{R^3} + \varepsilon_r A_1 &= \frac{\varepsilon_r B_1}{R^3} \\ -\frac{2\rho}{4\pi\varepsilon_0}\frac{1}{R^3} + \varepsilon_r A_1 &= -\frac{2B_1}{R^3} \end{aligned}\right\rangle -$$

$$\Rightarrow \quad (\varepsilon_r + 2) B_1 = \frac{3\rho}{4\pi\varepsilon_0} \qquad \Leftrightarrow \qquad B_1 = \frac{3\rho}{4\pi\varepsilon_0}\left(\frac{1}{\varepsilon_r+2}\right)$$

and $$A_1 = \frac{B_1}{R^3} - \frac{\rho}{4\pi\varepsilon_0\varepsilon_r}\frac{1}{R^3}$$

$$= \frac{\rho}{4\pi\varepsilon_0 R^3}\left(\frac{3}{\varepsilon_r+2} - \frac{1}{\varepsilon_r}\right) \quad \Leftrightarrow \quad A_1 = \frac{\rho}{4\pi\varepsilon_0 R^3}\left(\frac{2\varepsilon_r - 2}{\varepsilon_r(\varepsilon_r+2)}\right)$$

$$\Rightarrow \quad \Phi_{in} = \frac{p\cos\theta}{4\pi\varepsilon\ r^2} + \frac{p\cos\theta}{4\pi\varepsilon_0\varepsilon_r R^3}\ r\ \frac{2\varepsilon_r - 2}{\varepsilon_r + 2}$$

$$= \frac{p\cos\theta}{4\pi\varepsilon\, r^2}\left[1 + \frac{r^3}{R^3}\left(\frac{2\varepsilon_r - 2}{\varepsilon_r + 2}\right)\right]$$

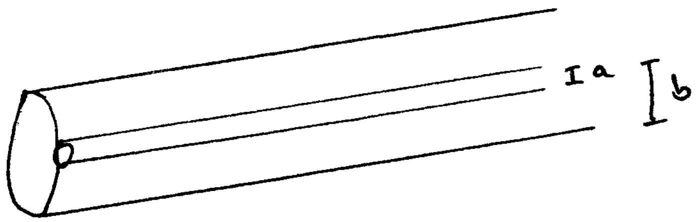
and

$$\Phi_{out} = \frac{3p\cos\theta}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\varepsilon_r + 2}\right)$$

$\varepsilon_r = 1$ (no sphere) gives $\Phi_{in} = \Phi_{out} = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ (✓)

Westerhoff
Section 2, E&M, Question #3
12/2/03

Problem 2 SOLUTION



magnetic fields using Ampère's law

$r > b \qquad \vec{B} = 0$

$a < r < b \qquad \oint \vec{B}\, d\vec{l} = \mu_0 I \Rightarrow 2\pi s B = \mu_0 I$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$r < a$

$$2\pi s B = \mu_0 I \frac{s^2}{a^2} \Rightarrow \vec{B} = \frac{\mu_0}{2\pi}\frac{I s}{a^2}\hat{\phi}$$

$W = \frac{1}{2\mu_0}\iiint B^2\, dV$, so per unit length

$$\frac{W}{l} = \frac{\mu_0}{2}\frac{I^2}{(2\pi)^2}2\pi\left\{\int_0^a \frac{s^2}{a^4} s\, ds + \int_a^b \frac{1}{s^2} s\, ds\right\}$$

$$= \frac{\mu_0}{4\pi} I^2 \left\{\left[\frac{1}{4}\frac{s^4}{a^4}\right]_0^a + \left[\ln s\right]_a^b\right\}$$

$$= \frac{\mu_0}{16\pi} I^2 + \frac{\mu_0}{4\pi}\ln\frac{b}{a} I^2$$

Now use $W = \frac{1}{2} L I^2$, so $\frac{L}{l} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln b/a$

↑

Vanishes for cylindrical shell

Delivered-To: lalla@phys.columbia.edu
X-Authentication-Warning: pegasus.phys.columbia.edu: westerhoff owned process doing -bs
Date: Tue, 13 Jan 2004 11:30:22 -0500 (EST)
From: Stefan Westerhoff <westerhoff@nevis.columbia.edu>
X-X-Sender: westerhoff@pegasus.phys.columbia.edu
To: Brian Cole <cole@nevis.columbia.edu>
Cc: "Lalla R. Grimes" <lalla@phys.columbia.edu>
Subject: qual grading

Hi Brian,

there is a tricky problem with question 2 on the
E&M part of the quals.
The self-inductance of the configuration of question 2
as given in the problem has a typo - the
contribution of the solid cylinder is mu_0/8pi,
not mu_0/4pi (the problem I submitted has the correct
inductance, but I was at a collaboration meeting in
Utah when Lalla's request for proof-reading reached me,
so I had no chance to catch this).

It would not be a big deal for most problems, but
for this one, it unfortunately is.
If you calculate the self-inductance L using the energy
of the magnetic field and then using
W = 1/2 L I^2
you get the correct answer (mu_0/8pi). However, if
you calculate the magnetic flux Phi and then use
Phi = L I
to get the self-inductance L, you get an incorrect
answer (the formula does not apply since the current
is not confined to a single path, at least not in a
trivial way -- one would have to split the finite wire
into lots of small wires etc etc ...).
This incorrect answer is unfortunately (how much more
Murphy can there be ?) the answer given in the problem...
Since lots of students used the wrong way to get to
the answer, they did not wonder why they were off
by a factor of 2 (as they should have -- that was my
reason to actually give the solution...).

Bottom line, this is somewhat tricky to grade.
I guess I cannot really subtract any points for the
wrong solution, can't I ? Maybe just a remark that
this is not the way to calculate it ?

Stefan

--

Section 2, Question # 3 A. Mueller 5

(2) E+M DEC 1 2003

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long wavelength limit find the angular distribution of radiated power and the total radiated power from the sphere.

Solution:

This is a dipole problem, because of the long wavelength limit so

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c^3}\left|\ddot{\vec{p}}(t-r/c)\times\vec{n}\right|^2$$

where $\vec{p}$ is the dipole moment and $\vec{n}=\vec{r}/r$.

Pretend for the moment that we are dealing with a static two-hemisphere problem. Then outside the sphere

$$\phi(\vec{r}) = \sum_{\ell=0}^{\infty}\phi_\ell\, P_\ell(\cos\theta)\frac{1}{r^{\ell+1}}.$$

We are interested in ϕ_1 which is determined by

$$\underbrace{\int_{-1}^{1} d\cos\theta\, P_1(\cos\theta)\, V\,\epsilon(\cos\theta)}_{V} = \frac{1}{R^2}\,\phi_1 \underbrace{\int_{-1}^{1} P_1^2(\cos\theta)\, d\cos\theta}_{2/3}$$

$$\Rightarrow \quad \phi_1 = \frac{3}{2}VR^2$$

So $\phi = \frac{3}{2}V\frac{R^2}{r^2}\cos\theta$ outside sphere

But $\phi = \frac{\vec{p}\cdot\vec{n}}{r^2} \Rightarrow \boxed{\vec{p} = \frac{3}{2}VR^2\,\vec{e}_3}$

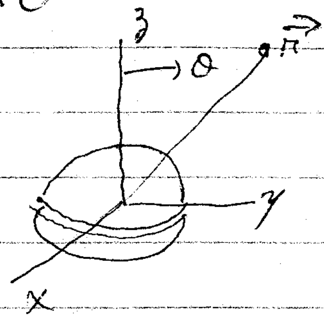
A. Mueller

Now go back to time-dependent problem

$$\vec{p} = \frac{3}{2} V R^2 \vec{e}_z \cos\omega t$$

$$\ddot{\vec{p}} = -\frac{3}{2} V R^2 \omega^2 \vec{e}_z \cos\omega t$$

$$\frac{dP(t)}{d\Omega} = \frac{9R^4V^2\omega^4}{16\pi c^3} \cos^2[\omega(t-r/c)] \sin^2\theta$$



$$P = \frac{9R^4V^2\omega^4}{16\pi c^3} \cos^2\omega(t-r/c) \underbrace{\int_{-1}^{1} \sin^2\theta \, d\cos\theta}_{4/3} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\boxed{P = \frac{3R^4V^2\omega^4}{2c^3} \cos^2\omega(t-r/c)}$$

Section 2, Question 4

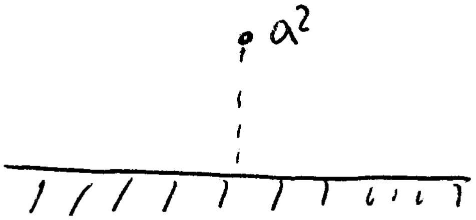
1. Solution:

Alekhel

Introduce $z = x + iy$;

By conformal transformation $u = z^2$;

the system is mapped onto



$$\varphi(u) = -2q\,\mathrm{Re}\ln\left(\frac{u - ia^2}{u + ia^2}\right);$$

The potential (charge at $u - ia^2$; image charge at $u + ia^2$)

Thus:
$$\varphi(z) = -2q\,\mathrm{Re}\ln\left(\frac{z^2 - ia^2}{z^2 + ia^2}\right):$$

Electric fields:

$$E_x = -\frac{\partial\varphi}{\partial x} = \cancel{-2q} - \mathrm{Re}\frac{\partial\varphi}{\partial z} = 4q\,\mathrm{Re}\left(\frac{z}{z^2 - ia^2} - \frac{z}{z^2 + ia^2}\right);$$

$$E_y = -\frac{\partial\varphi}{\partial y} = 4q\,\mathrm{Im}\left(\frac{z}{z^2 - ia^2} - \frac{z}{z^2 + ia^2}\right);$$

Surface charge on the horizontal segment: $y = 0$:

$$4\pi\sigma(x) = E_y = 4q\,\mathrm{Im}\left(\frac{x}{x^2 - ia^2} - \frac{x}{x^2 + ia^2}\right) = 8q\frac{xa^2}{x^4 + a^4};$$

$$\sigma(x) = \frac{2q}{\pi}\frac{xa^2}{x^4 + a^4}, \qquad x > 0;$$

Analogously (or by symmetry):

$$\sigma(y) = \frac{2q}{\pi}\frac{ya^2}{y^4 + a^4}, \qquad y > 0;$$

Dirac Electromagnetic Waves – Solution   Section 2, Question #5

(a) $\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{J} = 0$ where $\rho = \frac{1}{4\pi}\left[\operatorname{div}\vec{E} + \mu^2\Phi\right]$

& $\vec{J} = \frac{c}{4\pi}\left[\operatorname{curl}\vec{B} + \mu^2\vec{A} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right]$.

$\therefore \frac{1}{4\pi}\left[\operatorname{div}\frac{\partial \vec{E}}{\partial t} + \mu^2\frac{\partial \Phi}{\partial t}\right] + \frac{c}{4\pi}\left[\underbrace{\operatorname{div}\operatorname{curl}\vec{B}}_{0} + \mu^2\operatorname{div}\vec{A} - \frac{1}{c}\operatorname{div}\frac{\partial \vec{E}}{\partial t}\right] = 0.$

cancel

$\therefore \frac{\mu^2}{4\pi}\left\{\frac{\partial \Phi}{\partial t} + c\operatorname{div}\vec{A}\right\} = 0.$ $\therefore \boxed{\frac{1}{c}\frac{\partial \Phi}{\partial t} + \operatorname{div}\vec{A} = 0}$.

(b) $\operatorname{div}\vec{E} + \mu^2\Phi = 4\pi\rho \Rightarrow \operatorname{div}\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) + \mu^2\Phi = 4\pi\rho.$

$\therefore -\nabla^2\Phi - \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\operatorname{div}\vec{A}}_{-\frac{1}{c}\frac{\partial \Phi}{\partial t}} + \mu^2\Phi = 4\pi\rho \Rightarrow \boxed{+\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi + \mu^2\Phi = 4\pi\rho}$

$\operatorname{curl}\vec{B} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \Rightarrow \underbrace{\operatorname{curl}\operatorname{curl}\vec{A}}_{(\operatorname{grad}\operatorname{div}\vec{A} - \nabla^2\vec{A})} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$

$\operatorname{div}\vec{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$ cancel

$\therefore \boxed{+\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} + \mu^2\vec{A} = \frac{4\pi}{c}\vec{J}}.$

(c) $\rho = 0$ and $\vec{J} = 0 \Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2\right)\left[\left\{\begin{matrix}\vec{A}_0\\ \Phi_0\end{matrix}\right\}e^{i(kz-\omega t)}\right] = 0,$

$\therefore \left(\frac{-\omega^2}{c^2} + k^2 + \mu^2\right) = 0.$ $\therefore \boxed{k = \sqrt{\frac{\omega^2}{c^2} - \mu^2}}$.

$\boxed{\omega_{cutoff} = \mu c}$. $\omega > \omega_{cutoff} \Rightarrow k$ pure real (pure propagation)

$\omega < \omega_{cut\text{-}off} \Rightarrow k$ pure imaginary (pure attenuation)

$v_{phase} = \frac{\omega}{k} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \mu^2}} \Rightarrow \boxed{v_{phase} = c\frac{\omega}{\sqrt{\omega^2 - \mu^2 c^2}}}.$

(d) $\vec{A}_0 = A_0\hat{z}.$

$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \operatorname{div}\vec{A} = 0 \Rightarrow \frac{-i\omega}{c}\Phi_0 + ikA_0 = 0 \Rightarrow \boxed{\Phi_0 = \frac{kc}{\omega}A_0}$ (k given above)

$\left\{\begin{matrix}\vec{E}(z,t)\\ \vec{B}(z,t)\end{matrix}\right\} = \left\{\begin{matrix}\vec{E}_0\\ \vec{B}_0\end{matrix}\right\}e^{i(kz-\omega t)}$ — Take real part.

$\vec{B} = \operatorname{curl}\vec{A} \Rightarrow \vec{B}_0 = ik\hat{z}\times(A_0\hat{z}) = 0 \Rightarrow \boxed{\vec{B}_0 = 0}.$

$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \Rightarrow \vec{E}_0 = -ik\hat{z}\Phi_0 + \frac{i\omega}{c}A_0\hat{z}$

$\therefore \vec{E}_0 = i\hat{z}\left[\frac{\omega}{c}A_0 - k\left(\frac{kc}{\omega}A_0\right)\right] = i\hat{z}A_0\frac{\omega}{c}\left[1 - \frac{k^2c^2}{\omega^2}\right] \Rightarrow \boxed{\vec{E}_0 = i\hat{z}A_0\mu^2\frac{c}{\omega}}$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 14, 2004
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

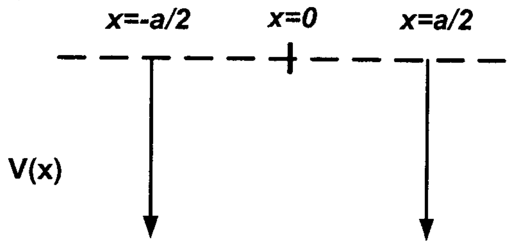
Problem 1

Consider a particle of mass m moving in one dimension over the range $-\infty < x < \infty$ and subject to the attractive two delta-function potential (see sketch),
$V(x) = -V_0\, a[\delta(x - a/2) + \delta(x + a/2)]$.

Here a is a length and V_0 is a positive constant.

a) For a sufficiently large V_0 this potential has two bound states. Sketch their wave functions.

b) For a less than some critical distance a_c, the potential has only one bound state. Find a_c in terms of m and V_0.



Problem 2

A one-dimensional harmonic oscillator with mass m and spring constant k_0 is subject to a time-independent squeezing perturbation, $V_1(x) = \frac{1}{2}k_1 x^2$, $|k_1 / k_0| << 1$.

a) Using perturbation theory, calculate the ground state energy shift to second order in k_1/k_0.

b) Compare to the exact result for the change in ground state energy in the presence of the perturbation.

c) Compute to first order the perturbed ground state wave function in terms of the unperturbed wave functions $\psi_n^{(0)}(x) = \langle x | n \rangle$.

d) What is the exact ground state wave function, $\psi_0^{exact}(x)$, in the presence of the perturbation (do not worry about the normalization).

Problem 3

The Hamiltonian for the two-dimensional harmonic oscillator is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2).$$

As for the one-dimensional harmonic oscillator, one can introduce creation and annihilation operators according to

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p_x}{\sqrt{2m\omega\hbar}}$$

$$a_y = \sqrt{\frac{m\omega}{2\hbar}}y + i\frac{p_y}{\sqrt{2m\omega\hbar}}$$

so that $H = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + 1\right)$ with $\left[a_x, a_x^\dagger\right] = \left[a_y, a_y^\dagger\right] = 1$. The angular momentum operator is $L = xp_y - yp_x$.

a) Evaluate L in terms of the creation and annihilation operators.

b) Evaluate $\left[L, a_x\right]$ and $\left[L, a_y\right]$.

c) Give a physical argument that requires $[L, H] = 0$. Use the result from part b to check that $[L, H] = 0$.

Problem 4

Consider a particle of mass *m* moving non-relativistically along the circumference of a horizontal circle of radius *R*.

a) What are the energy eigenstates and eigenvalues of this system?

b) A uniform magnetic field *B* is introduced perpendicular to the plane of the circle. If the particle has charge *q*, find the new energy eigenvalues and eigenstates.

c) Next, consider a system of two neutral particles of mass m_1 and m_2 that move on the circumference of the circle. If an infinite short-range repulsive force prevents them from passing each other, find the resulting energy eigenstates and eigenvalues.

d) Repeat your solution to part c) for the case that the repulsive force is replaced by reflectionless scattering satisfying the boundary condition, $\lim_{\theta_1 \to \theta_2^-} \psi(\theta_1, \theta_2) = e^{i\delta} \lim_{\theta_1 \to \theta_2^+} \psi(\theta_1, \theta_2)$.

Problem 5

An electron in a hydrogen atom is in a state described by the wave function

$$\Psi(\vec{r}) = A\left[\psi_{100}(\vec{r}) + 2\psi_{210}(\vec{r}) + 2\psi_{211}(\vec{r}) - \psi_{21-1}(\vec{r})\right]$$

Here $\psi_{nlm}(\vec{r})$ is a normalized wave function of hydrogen atom with the principle quantum number n, angular quantum number l and magnetic quantum number m, where the explicit functional forms are:

$$\psi_{100}(\vec{r}) = 2a_0^{-3/2}(4\pi)^{-1/2}e^{-r/a_0}$$
$$\psi_{210}(\vec{r}) = (2a_0)^{-3/2}(4\pi)^{-1/2}(r/a_0)e^{-r/2a_0}\cos\theta$$
$$\psi_{21\pm1}(\vec{r}) = \mp(2a_0)^{-3/2}(8\pi)^{-1/2}(r/a_0)e^{-r/2a_0}\sin\theta\, e^{\pm i\varphi}$$

The Bohr radius is $a_0 = \dfrac{\hbar^2}{m_e e^2}$ where m_e is the mass of electron. A is a normalization constant.

a) Neglecting the spin orbit interaction (for now), find the expectation values of the energy, L^2 and L_z, where L and L_z are the orbital angular momentum of the hydrogen atom and its z-component, respectively.
b) Now the hydrogen atom described above is placed in a weak gravitational force field, $\vec{F} = -m_e g\,\hat{z}$, where g is the gravitational acceleration constant. Compute the change in the expectation value of the energy to first order in $m_e g$. Assume that the ion core remains fixed in space.
c) We now consider a spin-orbit coupling of the electron in this problem. How many different values of energy level can be measured in the above wave function? (Note that we do not have $\psi_{200}(\vec{r})$ component in the above wave function.)

Problem 6

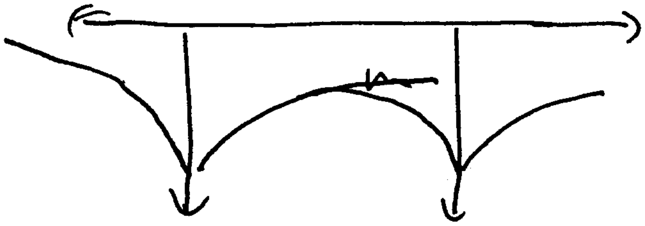
Consider a rigid rotor with moment of inertia, I, and permanent dipole moment, p, located in an external, position independent electric field E. Calculate the ground state energy and wave functions for the following cases:

a) $E=0$
b) $pE << \hbar^2/I$
c) Let the rotor be constrained to the x-y plane. Solve for the ground-state energy and wave function assuming $E = 0$.
d) Let the rotor perform small oscillations ($\phi << 1$) about the x-axis and let $\vec{E} = E_x\hat{x}$. Find the ground-state energy and wave function.

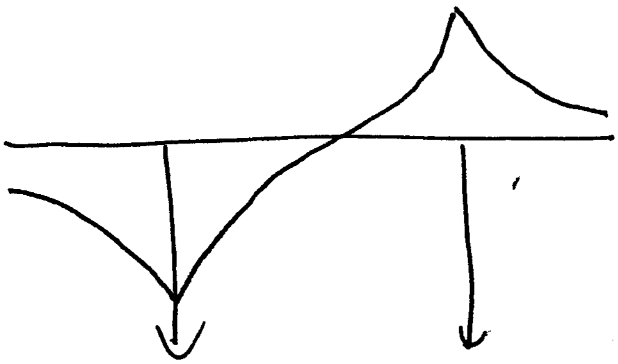
Section 3

Solution
Question #1

(i)



Lowest state (even parity)



higher lying bound state (odd parity)

(ii) We seek the odd parity solution of

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0 a\left[\delta(x-a/2) + \delta(x+a/2)\right]\psi = E\psi$$

with $E = 0$

$$E = 0 \Rightarrow \psi(x) = A + Bx$$

Soln - 2

Parity $\Rightarrow$

$x > a/2$: $\qquad \psi(x) = A$

$-a/2 < x < a/2 \qquad \psi(x) = 2A\,x/a$

$x < -a/2 \qquad \psi(x) = -A$

Boundary condition at $\delta(x - a/2)$:

$$-\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dx}\right|_{x=a/2^+} - \left.\frac{d\psi}{dx}\right|_{x=a/2^-}\right) + V_0\, a\, \psi(a/2) = 0$$

$$\Rightarrow \quad \frac{\hbar^2}{ma} + V_0\, a = 0 \quad \Rightarrow \quad a = \left(-\frac{\hbar^2}{mV_0}\right)^{1/2}$$

DEC 1 6 2003

Quantum (Quals. 04) Section 3, question # ~~3~~ 2 Gyulassy 1/5

A one dimensional harmonic oscillator with mass m and string constant k_0 is subject to a time independent squeezing perturbation $V_1(x) = \frac{1}{2} k_1 x^2$, $|k_1/k_0| \ll 1$

a) Compute the ground state energy shift through second order in k_1/k_0

b) Compare to exact result for ground state

c) Compute the first order perturbed ground state wavefunction $\psi_0^{(1)}(x)$ in terms of the unperturbed $\psi_n^{(0)}(x) = \langle x | n \rangle$.

d) What is the exact ground state $\psi_0^{exact}(x)$?
(Do not worry about normalization)

A harmonic oscillator $H = p^2/2m + \frac{k_0 x^2}{2}$ is subject to a squeezing perturbation $V_1 = \delta k \frac{x^2}{2}$ where $\delta k \equiv k_1$ (notation)

Compute the Ground state energy shift through second order in δk, Compare to exact result

---

$$H|n\rangle = E_n^0 |n\rangle \qquad E_n^0 = \hbar\omega_0 (n + 1/2)$$

$\omega_0 = \sqrt{k/m}$ is unperturbed ang. freq

$$H = \hbar\omega_0 (a^+ a + 1/2)$$

where $\hat{a} = \alpha\hat{x} + i\hat{p}\beta$

$\hat{a}^+ = \alpha\hat{x} - i\hat{p}\beta$

$$a^+ a = \alpha^2 \hat{x}^2 + \beta^2 \hat{p}^2 - i\alpha\beta \underbrace{[\hat{p}, \hat{x}]}_{-i\hbar}$$

$$\hbar\omega_0 \alpha^2 = k_0/2 \qquad \alpha\beta\hbar = 1/2$$

$$\hbar\omega_0 \beta^2 = 1/2m$$

$$\alpha^2 = \frac{k_0}{2\hbar\omega_0} = \frac{\sqrt{k_0 m}}{2\hbar} \qquad \beta^2 = \frac{1}{2\hbar\omega_0 m} = \frac{1}{2\hbar\sqrt{k_0 m}}$$

$$\alpha^2\beta^2 = \frac{k_0}{4m}\frac{1}{\hbar^2\omega_0^2} = \frac{1}{4\hbar^2} \quad ✓$$

$$\hat{x} = (a + a^+)/2\alpha \qquad \hat{p} = (\hat{a} - \hat{a}^+)/(2i\beta)$$

$$V_1 = \frac{\delta k}{2}\hat{x}^2 = \frac{1}{2}\frac{\delta k}{4\alpha^2}\left(a^2 + a^{+2} + aa^+ + a^+a\right)$$

$$\frac{\delta k}{4\alpha^2} = \frac{\delta k}{k_0}\frac{\hbar\omega_0}{2} = \left(\frac{k_1}{k_0}\right)\cdot\frac{\hbar\omega_0}{2}$$

Perturbation theory to second order

$$E_n = E_n^0 + \langle n|V_1|n\rangle + \sum_{n'\neq n}\frac{\langle n|V_1|n'\rangle\langle n'|V_1|n\rangle}{E_n^0 - E_{n'}^0}$$

Ground state to n

$$\langle n/V_1/0\rangle = \left(\frac{1}{2}\right)\frac{\delta k}{4\alpha^2}\langle n/a^2 + a^{+2} + aa^+ + a^+a/0\rangle$$

$$= \frac{\delta k}{2k_0}\left(\frac{1}{2}\hbar\omega_0\right)\langle n/a^{+2} + aa^+/0\rangle$$

where we used $\boxed{a|0\rangle = 0}$

$$\langle n|aa^+|0\rangle = \langle n|a^+a + 1|0\rangle$$

$$= \delta_{n0}\times 1$$

$$\langle n|a^{+2}|0\rangle = \langle 0|a^2|n\rangle = \sqrt{n}\,\langle 0|a|n-1\rangle$$

$$= \sqrt{n(n-1)}\,\underbrace{\langle 0|n-2\rangle}_{\delta_{n2}}$$

$$= \sqrt{2}\,\delta_{n2}$$

$$\langle n/V_1/0\rangle = \left(\frac{\hbar\omega_0}{2}\right)\left(\frac{\delta k}{2k_0}\right)\left(\delta_{n0} + \sqrt{2}\,\delta_{n2}\right)$$

$$E_0 = \frac{1}{2}\hbar\omega_0\left(1 + \frac{1}{2}\frac{\delta k}{k_0}\right) + \underbrace{\left(\frac{\hbar\omega_0}{2}\right)^2\left(\frac{\delta k}{k_0}\right)^2\frac{\sqrt{2}^2}{\frac{1}{2}\hbar\omega_0 - \frac{5}{2}\hbar\omega_0}}_{\frac{1}{2}\hbar\omega_0\left(\frac{\delta k}{2k_0}\right)^2\left(-\frac{1}{2}\right)}$$

$$E_0 = \frac{1}{2}\hbar\omega\left(1 + \frac{1}{2}\frac{\delta k}{k_0} - \frac{1}{8}\left(\frac{\delta k}{k_0}\right)^2 + \cdots\right)$$

b) Compare to exact result

$$E_0^{ex} = \frac{1}{2}\hbar\omega \qquad \text{when} \quad \omega = \sqrt{\frac{k_0 + \delta k}{m}}$$

$$\omega = \omega_0\sqrt{1 + \frac{\delta k}{k_0}} = \omega_0\left(1 + \frac{1}{2}\frac{\delta k}{k_0} - \frac{1}{8}\frac{\delta k^2}{k_0^2} + \cdots\right)$$

We recover up to second order the exact result

c.) The perturbed ground state wavefunction

$$|\Psi_0'\rangle \cong |\Psi_0\rangle + \sum_{n>0}\frac{\langle n|V_1|0\rangle}{E_0^0 - E_n^0}|\Psi_n\rangle$$

$$= |0\rangle + \frac{\sqrt{2}\left(\frac{\hbar\omega_0}{2}\right)(\delta k/2k_0)}{\frac{1}{2}\hbar\omega_0 - \frac{5}{2}\hbar\omega_0}|2\rangle$$

enough for full credit

$$\underbrace{= |0\rangle - \sqrt{2}\left(\frac{\delta k}{8k_0}\right)|2\rangle}$$

$$\psi_0^{(1)}(x) = e^{-\alpha^2x^2}\left(A_0 - \sqrt{2}\left(\frac{\delta k}{8k}\right)A_2(8\alpha^2x^2 - 2)\right)$$

$$\frac{A_1}{A_0} = \frac{1}{\sqrt{2}} \qquad \frac{A_2}{A_0} = \sqrt{\frac{1}{2^2 2!}} = \frac{1}{2^{3/2}} \qquad A_0 = \frac{\sqrt{\alpha}}{\pi^{1/4}}$$

$$= A_0 e^{-\alpha^2x^2}\left(1 + \frac{1}{8}\frac{\delta k}{k} - \frac{1}{2}\left(\frac{\delta k}{k}\right)\alpha^2x^2\right)$$

$\alpha^3 \cdots \frac{\beta^2 x^2}{2}$

Exact $\langle x|\psi_0^{ex}\rangle = \left(\frac{\alpha_{ex}^2}{\pi}\right)^{1/4} e^{-\alpha_{ex}^2 x^2}$.

$$\alpha_{ex}^2 = \frac{\sqrt{(k+\delta k)\, m}}{2\hbar} = \alpha_0^2 \underbrace{\sqrt{1 + \delta k/k}}_{1 + \frac{1}{2}\frac{\delta k}{k}}$$

$$\left(\alpha_{ex}^2\right)^{1/4} \cong \left(\alpha_0^2\right)^{1/4}\left(1 + \frac{1}{8}\frac{\delta k}{k}\right)$$

not required

d) $\psi_0^{ex}(x) = A\, e^{-\alpha^2 x^2}$

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}(k_0 + k_1)\, x^2\right)\psi_0^{ex} = \frac{1}{2}\hbar\sqrt{\frac{k_0+k_1}{m}}\,\psi_0$$

$$\partial_x \psi = -2\alpha^2 x\, \psi$$

$$\partial_x^2 \psi = (+4\alpha^4 x^2 - 2\alpha^2)\,\psi$$

(1) $-\frac{\hbar^2}{2m} 4\alpha^4 + \frac{1}{2}(k_0 + k_1) = 0$

$$\alpha^2 = \sqrt{\frac{(k_0+k_1)\, m}{4\hbar^2}} = \frac{\sqrt{k_0 m}}{2\hbar}\sqrt{1 + k_1/k_0}$$

(2) <u>check</u> $2\alpha^2 \frac{\hbar^2}{2m} = \frac{1}{2}\hbar\sqrt{\frac{k_0+k_1}{m}}$

$$\alpha^2 = \frac{1}{2\hbar}\sqrt{(k_0+k_1)m} \qquad \text{ok}$$

not needed

(Norm $\int_{-\infty}^{\infty} A^2 e^{-2\alpha^2 x^2} dx = 1$; $A^2 \frac{\sqrt{\pi}}{\sqrt{2\alpha^2}} = 1$)

section 3, Question 2 (3)

A. Mueller

DEC 1 2003

(1) Quantum (Angular Momentum)

The Hamiltonian for the two-dimensional harmonic oscillator is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2} m\omega^2 (x^2+y^2).$$

As for the one-dimensional oscillator one can introduce creation and annihilation operators according to

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{p_x}{\sqrt{2m\omega\hbar}}$$

$$a_y = \sqrt{\frac{m\omega}{2\hbar}}\, y + i \frac{p_y}{\sqrt{2m\omega\hbar}}$$

so that $H = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1)$ with $[a_x, a_x^\dagger] = [a_y, a_y^\dagger] = 1$

The angular momentum of the oscillator is

$$L = x p_y - y p_x.$$

(i) Evaluate L in terms of the a's and $a^\dagger$'s.

(ii) Evaluate $[L, a_x]$ and $[L, a_y]$.

(iii) Give a physical argument that requires $[L, H] = 0$. Use the result from (ii) & check that $[L, H] = 0$

Solution:

(i)

$$p_x = -\frac{i}{2}\sqrt{2m\omega\hbar}\,(a_x - a_x^\dagger)$$

$$x = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\,(a_x + a_x^\dagger)$$

$$L = x p_y - y p_x = \frac{-i}{4} 2\hbar \left[(a_x + a_x^\dagger)(a_y - a_y^\dagger) - (a_y + a_y^\dagger)(a_x - a_x^\dagger) \right]$$

$$= i\hbar \left[a_x a_y^\dagger - a_y a_x^\dagger \right]$$

A. Mueller

$$[L, a_x] = i\hbar[-a_y a_x^\dagger, a_x] = i\hbar a_y$$

$$[L, a_y] = i\hbar[a_x a_y^\dagger, a_y] = -i\hbar a_x$$

(iii) The Hamiltonian is invariant under rotations in the x,y plane, therefore $[L,H]=0$ since L is the generator of such rotations.

$$[L,H] = [i\hbar(a_x a_y^\dagger - a_y a_x^\dagger), \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1)]$$

$$= i\hbar^2\omega[a_x a_y^\dagger - a_y a_x^\dagger, a_x^\dagger a_x + a_y^\dagger a_y]$$

$$= i\hbar^2\omega\{\underbrace{[a_x, a_x^\dagger a_x]}_{a_x} a_y^\dagger + a_x \underbrace{[a_y^\dagger, a_y^\dagger a_y]}_{-a_y^\dagger} - a_y \underbrace{[a_x^\dagger, a_x^\dagger a_x]}_{-a_x^\dagger} - a_x^\dagger \underbrace{[a_y, a_y^\dagger a_y]}_{a_y}$$

$$= 0$$

Christ

Suggested Solutions

1. (a) Introduce ϕ to represent rotation of the moving sphere about its center of mass. Equating the total kinetic energy with the potential energy lost gives:

$$\begin{aligned}(R+r)(1-\cos(\theta))mg &= m(r\dot{\phi})^2 + \frac{2}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mr^2\dot{\phi}^2 \\ &= \frac{7}{5}mv_{\text{cm}}^2\end{aligned}$$

Thus, $v_{\text{cm}}(\theta) = \sqrt{\frac{5}{7}(R+r)(1-\cos(\theta))g}$.

(b) The sphere will fly off when $mv_{\text{cm}}^2/(R+r) > mg\cos(\theta)$ or

$$\begin{aligned}\frac{5}{7}(1-\cos(\theta)) &> \cos(\theta) \\ &\text{or} \\ \cos(\theta) &= 5/13\end{aligned}$$

(c) Start with the equation of motion obtained by equating torque and rate of change of angular momentum around the point of contact: $\frac{7}{5}mr^2\ddot{\phi} = mgr\sin\theta$. Relate θ and ϕ by computing the velocity of the moving sphere's center of mass two ways:

$$(R+r)\dot{\theta} = r\dot{\phi}$$

Combining these equations:

$$\begin{aligned}\ddot{\theta} &= \frac{5g}{7(R+r)}\theta \\ &\text{or} \\ \theta(t) &= \frac{\dot{\theta}(0)}{\omega}\sinh(\omega t)\end{aligned}$$

Section 3
Question#4

where $\omega = \sqrt{\frac{5g}{7(R+r)}}$.

2. (a) $\psi(\theta) = \frac{1}{\sqrt{2\pi}}e^{ik_n\theta}$, $E_n = \frac{(\hbar k_n)^2}{2m}$ where $k_n = n/R$ and n is an integer.

(b) The same form as in a) except $k_n = n/R + qBR/2$ where n is an integer.

(c) $\psi(\theta_1,\theta_2) = e^{i(\theta_1+\theta_2)P_n/2}\sin[(\theta_1-\theta_2)k_l]$, $E_n = \frac{(\hbar P_n)^2}{4m} + \frac{(\hbar k_l)^2}{m}$, where $P_n = n/R$, $k_l = l/(4R)$ and n, l are integers.

(d) $\psi(\theta_1,\theta_2) = e^{i(\theta_1+\theta_2)P_n/2}e^{i(\theta_1-\theta_2)k_l}$, $E_n = \frac{(\hbar P_n)^2}{4m} + \frac{(\hbar k_l)^2}{m}$, where $P_n = n/R$, $k_l = l/(2R) + \delta/(2\pi R)$ and n, l are integers.

Kim Question #5
Section 3

QM: Hydrogen (like) Atom

(a) Find the normalization constant first,

$$1 = \langle \psi | \psi \rangle = A^2 [1^2 + 2^2 + 2^2 + 1^2]$$

or $\boxed{A = \frac{1}{\sqrt{10}}.}$

Then

$$\langle E \rangle = \langle \psi | H | \psi \rangle = \frac{1}{10} [\underset{E_1}{\underset{\parallel}{E_{100}}} + 4 \underbrace{E_{210} + 4E_2 + E_2}_{\underset{E_2}{\parallel}}]$$

$$= \frac{1}{10}(E_1 + 9\underset{\frac{E_1}{2^2}}{\underset{\parallel}{E_2}}) = \underbrace{\frac{13}{40} E_1}$$

$E_1 = 13.6 eV$.

$$\langle L_z \rangle = \langle \psi | L_z | \psi \rangle = \frac{1}{10} [\hbar \cdot 0 + 4\hbar \cdot 0 + 4\hbar - \hbar]$$

$$= \underbrace{\frac{3\hbar}{10}}$$

$$\langle L^2 \rangle = \langle \psi | L^2 | \psi \rangle = \frac{1}{10} [0 + 9\hbar^2 \cdot 1 \cdot (1+1)].$$

$$= \underbrace{\frac{9\hbar^2}{5}}.$$

(b)

$$\Delta H = + m_e g z$$

Then

$$\Delta E = \langle \psi | m_e g z | \psi \rangle$$

up to the first order of $m_e g$.

Note that $[L_z, z] = 0$ & $\langle \psi_{n\ell m}|z|\psi_{n\ell m}\rangle = 0$

Therefore only non-zero contribution in $\langle\psi|z|\psi\rangle$

is the terms $\langle \psi_{210}|z|\psi_{100}\rangle$ and $\langle\psi_{100}|z|\psi_{210}\rangle$

Thus

$$\Delta E = m_e g\, A^2 \left[2\langle\psi_{210}|z|\psi_{100}\rangle + 2\langle\psi_{100}|z|\psi_{210}\rangle\right]$$

$$= \frac{m_e g}{10}\cdot 2\cdot 2\cdot \int_{-\pi}^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \int_0^{\infty} r^2 dr\, \frac{2}{2^{3/2} a_0^3}\frac{1}{4\pi}$$

$$\times \left(\frac{r}{a_0}\right) e^{-\frac{3r}{2a_0}} \cos\theta \cdot \underbrace{r\cos\theta}_{= z}$$

$$= m_e g \frac{4}{10}\frac{1}{\sqrt{2}}\frac{1}{a_0^3}\frac{1}{4\pi} a_0^4 \underbrace{\int_{-\pi}^{\pi} d\theta \sin\theta \cos^2\theta}_{2/3} \cdot 2\pi \int_0^{\infty} dt\, t^4 e^{-\frac{3}{2}t}$$

$$= m_e g \frac{a_0}{\sqrt{2}\cdot 5}\left(\frac{2}{3}\right)^5 \underbrace{\int_0^{\infty} ds\, s^4 e^{-s}}_{=4!}$$

$$= 0.447\, m_e g$$

(c)

$$H_{so} = f(r)\, \vec{L}\cdot\vec{S} \qquad f(r):\ \text{radial function only}$$

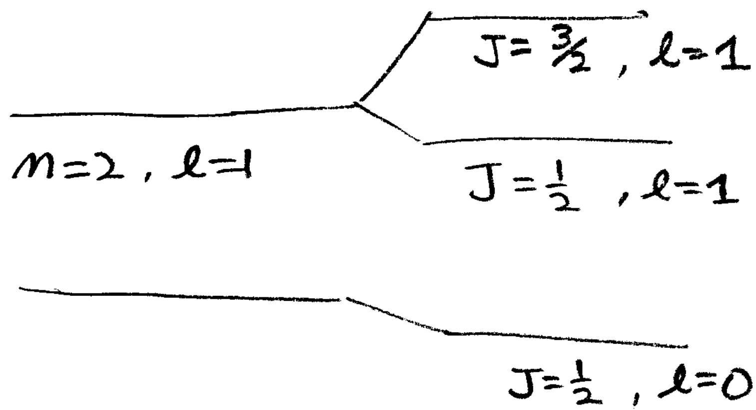
$$= f(r)\, \frac{1}{2}\left[\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right]$$

where $\vec{J} = \vec{L} + \vec{S}$

For $l=0 \Rightarrow j=\frac{1}{2}$ only

For $l=1 \Longrightarrow j=\frac{3}{2}$ & $\frac{1}{2}$

Thus



degeneracy lifted.

Note that we don't have ψ_{200} component in ψ.

Therefore $\underline{\underline{3}}$ different values of measurable energy for the wavefunction ψ.

Hailey Section 3 Question #6

Soln a.) The Hamiltonian is just

$$\frac{L_{op}^2}{2I}\psi = \varepsilon\psi \quad L_{op}^2 = L_x^2 + L_y^2 + L_z^2$$

$$\psi \sim Y_{\ell m}(\theta\phi) \quad L_{op}^2 Y_{\ell m} = \ell(\ell+1)\hbar^2$$

$$\varepsilon_{\ell m} = \frac{\ell(\ell+1)\hbar^2}{2I}, \quad \psi \sim Y_{\ell m}(\theta\phi) \quad \varepsilon_{grd} = 0 \quad \ell = 0$$

b.) perturbation theory

$$\varepsilon_{grd}^{(1)} = \langle \ell m | V | \ell m \rangle$$

$V = -pE\cos\theta = -\vec{p}\cdot\vec{E}$ for E along the z-axis

$$\varepsilon_{grd}^{(1)} = \langle \ell m | -pE\cos\theta | \ell m \rangle$$

This matrix element is 0 by parity.

$$\varepsilon_{grd}^{(2)} = \sum_{\ell m} \frac{|\langle \ell m | V | 00 \rangle|^2}{E_{00} - \varepsilon_{\ell m}}$$

$V = -pE\cos\theta = -pE\sqrt{\frac{4\pi}{3}}\,Y_{10}$ since $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$. The matrix element is

$\sim \langle \ell m | Y_{10} | 00 \rangle \sim \langle \ell m | Y_{10} \rangle$ since

$\langle \theta\phi | 00 \rangle = \frac{1}{\sqrt{4\pi}}$ so only $\ell = 1$ $m = 0$ connects ground state to higher states

$$\langle lm|V|00\rangle = -pE\sqrt{\frac{4\pi}{3}}\,\frac{\langle lm|10\rangle}{\sqrt{4\pi}} = -pE\sqrt{\frac{4\pi}{3}}$$

i.e. $\int Y_{lm}\left(-pE\sqrt{\frac{4\pi}{3}}\,Y_{10}\right)\left(\frac{1}{\sqrt{4\pi}}\right)d\Omega = -\frac{pE}{\sqrt{3}}\underbrace{\int Y_{lm}\,Y_{10}\,d\Omega}_{\delta_{l1}\delta_{m1}}$

$$= -\frac{pE}{\sqrt{3}}$$

$$E_0 - \mathcal{E}_{lm} = \mathcal{E}_0 - \mathcal{E}_{10} = 0 - \frac{l(l+1)\hbar^2}{2I} = -\frac{\hbar^2}{I}$$

$$\mathcal{E}^{(2)}_{grd} = \frac{\left(-pE/\sqrt{3}\right)^2}{-\hbar^2/I} = -\frac{p^2E^2I}{3\hbar^2}$$

well maybe they remember that

$$\psi_n^{(1)} = \psi_n^{(0)} + \sum\frac{\langle\psi_n^{(0)}|V|\psi_k^{(0)}\rangle}{E_n^0 - E_k^0}\psi_k^{(0)}$$

$$\psi_0^{(1)} = Y_{00} - \frac{pE/\sqrt{3}}{-\hbar^2/I}\,Y_{10}$$

$$\psi_0^{(1)} = Y_{00} + \frac{pEI}{\sqrt{3}\hbar^2}\,Y_{10}$$

c.) This is constrained rotator

$$-\frac{\hbar^2}{2I}\frac{\partial^2\psi}{\partial\phi^2} = \mathcal{E}\psi \qquad \psi = e^{im\phi}$$

$$\psi(\phi+2\pi) = \pm\psi(\phi)$$

$$\frac{\hbar^2m^2}{2I} = \mathcal{E}_m \qquad \Rightarrow\ m = 0, \pm1, \pm2 \ldots$$

$$e^{im\phi} \sim \psi$$

d.) $\frac{L_z^2}{2I} - pE\cos\phi = \varepsilon\psi$

$$-\frac{\hbar^2}{2I}\frac{\partial^2\psi}{\partial\phi^2} - pE(1-\phi^2/2)\psi = \varepsilon\psi$$

This is just the simple harmonic oscillator. They know the grd. state wave function is just $\psi \sim e^{-\alpha\phi^2}$

plugging this in you get

$$\left(pE/2 - \frac{2\alpha^2\hbar^2}{I}\right)\phi^2 + \alpha\frac{\hbar^2}{I} - pE = \varepsilon$$

$$\Rightarrow \quad \alpha = \frac{\sqrt{pEI}}{2\hbar} \qquad \varepsilon = \frac{\hbar}{2}\sqrt{\frac{pE}{I}} - pE$$

$$\psi \sim e^{-\sqrt{pEI}/2\hbar\,\phi^2}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 14, 2004
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2; Section 4(Relativity and Applied QM) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 4

Two particles of mass m_1 and m_2 are observed in a certain frame to have four-vectors p_1 and p_2, respectively,

a) Denote the magnitude of three-momentum each particle has in their center-of-mass frame as k. Find an expression entirely in terms of relativistic invariants for k.
b) The relative velocity between the two particles in the center-of-mass frame can also be expressed solely in terms of invariant quantities. Do so.

Hint: You may find it useful to note that the four-vector $(p_1 + p_2)/M$, where $M^2 = (p_1 + p_2)^2$ has a very simple form in the center-of-mass.

Problem 5

In neutral B-mesons, weak and mass eigenstates are not the same. These mesons therefore exhibit particle-antiparticle mixing. Because of this phenomenon, particles produced as pure B^0 or $\overline{B}^0$ weak eigenstates will evolve in time as a superposition of the two states:

$$\left|B^0(t)\right\rangle = a(t)\left|B^0\right\rangle + b(t)\left|\overline{B}^0\right\rangle$$
$$\left|\overline{B}^0(t)\right\rangle = a'(t)\left|B^0\right\rangle + b'(t)\left|\overline{B}^0\right\rangle$$

where $\left|B^0\right\rangle$ and $\left|\overline{B}^0\right\rangle$ are pure B^0 and $\overline{B}^0$ states.

The time evolution of these states is described by 2×2 hermitian mass (***M***) and decay (*Γ*) matrices.

$$i\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\Gamma\right)\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

CPT invariance requires that $M_{11} = M_{22} \equiv M$ and $\Gamma_{11} = \Gamma_{22} \equiv \Gamma$.

a) Calculate the mass and decay width differences between the two mass eigenstates of the B^0 - $\overline{B}^0$ system in terms of the elements of ***M*** and *Γ*. You may assume that CP-violation is negligible, *i.e.* that the phase difference between M_{12} and Γ_{12} is zero.
b) What are the functions *a(t), a'(t), b(t), b'(t)* in terms of the mass and decay matrix elements?

Section 4 Question #1
Hailey

Let K = rest frame

$\rho(\theta\phi)$ = # stars / solid angle

K' = moving frame $\rho(\theta'\phi')$ = # stars / solid angle

Note $\rho(\theta\phi)\,d\Omega = \rho(\theta'\phi')\,d\Omega'$

since the number of stars they see is countable and thus an invariant

$$\rho(\theta'\phi') = \rho(\theta\phi)\frac{d\Omega}{d\Omega'} = \frac{N}{4\pi}\frac{d\Omega}{d\Omega'}$$

$$\rho(\theta'\phi') = \frac{N}{4\pi}\frac{d\cos\theta}{d\cos\theta'} \quad \text{since } d\phi' = d\phi$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'}$$ ⟸ they can derive if they don't know

$$\frac{d\cos\theta}{d\cos\theta'} = \frac{1-\beta^2}{(1+\beta\cos\theta')^2}$$

$$\Rightarrow \rho(\theta'\phi') = \frac{N}{4\pi}\frac{(1-\beta^2)}{(1+\beta\cos\theta')^2}$$

This has a maximum at $\theta' = \pi$

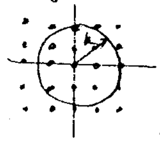
so the stars bunch up in the forward direction.

Heinz   Section 4, Question 3

Heinz - Problem 1

(a) $E(k_x, k_y, n) = \frac{\hbar^2}{2m}\left[k_x^2 + k_y^2 + \left(\frac{n\pi}{\ell}\right)^2\right]$, $n = 1, 2, 3, \dots$

(b) First find ρ_E^{2D} for free electrons by the usual state counting. Assume a lateral dimension of $L \times L$. Then, for periodic boundary conditions, values of k_x, k_y are spaced by $\Delta k = 2\pi/L$



Number of states up to energy E

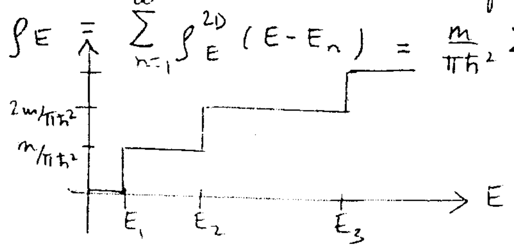
$$N = 2\frac{A}{(\Delta k)^2} = \frac{2\pi k^2}{(2\pi/L)^2} = \frac{mL^2}{\pi\hbar^2}E$$

spin deg. for $s = 1/2$

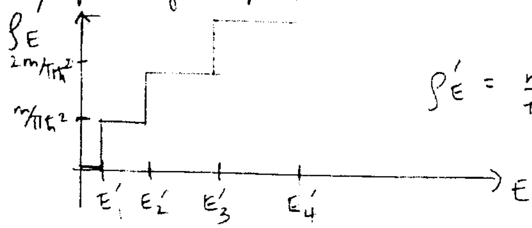
$$\Rightarrow \rho_E^{2D} = \frac{1}{L^2}\frac{dN}{dE} = \frac{m}{\pi\hbar^2}\theta(E)$$

Now include the z-axis quantization

$$\rho_E = \sum_{n=1}^{\infty} \rho_E^{2D}(E - E_n) = \frac{m}{\pi\hbar^2}\sum_n \theta(E - E_n), \quad E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$



(c) The only effect of a finite well is to lower the values of E_n'



$$\rho_E' = \frac{m}{\pi\hbar^2}\sum_n \theta(E - E_n')$$

(d) $n_s = \int_0^\infty \rho_E(E) f(E)\, dE = \int_0^\infty \rho_E(E)\left[e^{(E-\mu)/kT} + 1\right]^{-1} dE$ (*)

F-D distribution.

This eqn implicitly defines the chemical potential μ.

Then $E_{1/3}$ is given by $f(E_{1/3}) = \left[e^{(E_{1/3}-\mu)/kT} + 1\right]^{-1} = 1/3$

or $E_{1/3} = kT \ln 2 + \mu$ with μ defined by (*).

Quals 2004
Section 4 – Modern Physics: Relativity (Four Vector Rel Vel)
Question #~~5~~ 4
Bill Zajc
12/29/03

Two particles of mass m_1 and m_2 are observed in a certain frame to have four-vectors p_1 and p_2, respectively,

a) Denote the magnitude of three-momentum each particle has in their center-of-mass frame as k. Find an expression entirely in terms of relativistic invariants for k.
b) The relative velocity between the two particles in the center-of-mass frame can also be expressed solely in terms of invariant quantities. Do so.

Hint: You may find it useful to note that the four-vector $(p_1 + p_2)/M$, where $M^2 = (p_1 + p_2)^2$, has a very simple form in the center-of-mass.

NO SOLUTION PROVIDED

Section 4, Question 6
Evans

Qual's – 2004: Quantum Solutions (B-Mixing)

(H. Evans)

Part a)

First find the eigenvalues ($\lambda_\pm$) and eigenvectors ($\mathbf{x}_\pm$) of the mass/decay matrix:

$$\mathbf{D} \equiv \left(\mathbf{M} - \frac{i}{2}\Gamma\right)$$

Eigenvalues:

$$\mathbf{Dx} = \lambda\mathbf{x} \Rightarrow \begin{vmatrix} d-\lambda & d_{12} \\ d_{21} & d-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2d\lambda + (d^2 - d_{12}d_{21}) = 0$$

$$\lambda_\pm = d \pm \sqrt{d_{12}d_{21}} = \left(M - \frac{i}{2}\Gamma\right) \pm \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(M_{21} - \frac{i}{2}\Gamma_{21}\right)}$$

$$= \left(M - \frac{i}{2}\Gamma\right) \pm \sqrt{\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)}$$

since M_{12} and Γ_{12} have the same phase, ϕ:

$$\lambda_\pm = \left(M - \frac{i}{2}\Gamma\right) \pm \left(|M_{12}| - \frac{i}{2}|\Gamma_{12}|\right)$$

The time evolution equation for the mass eigenstates is then:

$$i\frac{\partial}{\partial t}\mathbf{x}_\pm(t) = \lambda_\pm \mathbf{x}_\pm(t) \Rightarrow \mathbf{x}_\pm(t) \propto exp(-i\lambda_\pm t) = exp\left(-iM_\pm t - \frac{\Gamma_\pm}{2}t\right)$$

where $M_\pm = Re(\lambda_\pm)$ and $\Gamma_\pm = Im(\lambda_\pm)$.
The mass and width differences are then:

$$\Delta m \equiv M_+ - M_- = 2|M_{12}|$$

$$\Delta\Gamma \equiv \Gamma_+ - \Gamma_- = 2|\Gamma_{12}|$$

Part b)

Eigenvectors:

$$\mathbf{Dx}_\pm = \lambda_\pm \mathbf{x}_\pm$$

$$\Rightarrow dx_{\pm 1} + d_{12}x_{\pm 2} = \lambda_\pm x_{\pm 1} \text{ and } d_{21}x_{\pm 1} + dx_{\pm 2} = \lambda_\pm x_{\pm 2}$$

$$\Rightarrow x_{\pm 2} = \pm x_{\pm 1}\sqrt{\frac{d_{21}}{d_{12}}}$$

If we defined the weak basis as:

$$\left|B^0\right\rangle \propto \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \left|\overline{B}^0\right\rangle \propto \begin{pmatrix}0\\1\end{pmatrix}$$

Then the mass eigenstates are:

$$\left|B_\pm\right\rangle = p\left|B^0\right\rangle \pm q\left|\overline{B}^0\right\rangle$$

with

$$\frac{q}{p} = \sqrt{\frac{d_{21}}{d_{12}}} = \sqrt{\frac{M_{12}^* - i\Gamma_{12}^*/2}{M_{12} - i\Gamma_{12}/2}}$$

These mass eigenstates evolve separately as:

$$\left|B_\pm(t)\right\rangle = exp(-i\lambda_\pm t)\left|B_\pm\right\rangle$$

Writing the weak eigenstates in terms of the mass eigenstates:

$$\left|B^0\right\rangle = \frac{1}{2p}\left(\left|B_+\right\rangle + \left|B_-\right\rangle\right) \qquad \text{and} \qquad \left|\overline{B}^0\right\rangle = \frac{1}{2q}\left(\left|B_+\right\rangle - \left|B_-\right\rangle\right)$$

The the mass eigenstates time evolution become:

$$p\left|B^0(t)\right\rangle + q\left|\overline{B}^0(t)\right\rangle = exp(-i\lambda_+ t)\left[p\left|B^0\right\rangle + q\left|\overline{B}^0\right\rangle\right]$$

$$p\left|B^0(t)\right\rangle - q\left|\overline{B}^0(t)\right\rangle = exp(-i\lambda_- t)\left[p\left|B^0\right\rangle - q\left|\overline{B}^0\right\rangle\right]$$

Solving these simultaneous equations gives:

$$\left|B^0(t)\right\rangle = g_+(t)\left|B^0\right\rangle + \frac{q}{p}g_-(t)\left|\overline{B}^0\right\rangle$$

$$\left|\overline{B}^0(t)\right\rangle = g_+(t)\left|\overline{B}^0\right\rangle + \frac{p}{q}g_-(t)\left|B^0\right\rangle$$

where the time evolution coefficients are:

$$g_\pm(t) = \frac{1}{2}\left[exp(-i\lambda_+ t) \pm exp(i\lambda_- t)\right]$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 16, 2004
9:00 AM – 11:00 AM

General Physics (Part I)
Section 5. Thermodynamics, Statistical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5(General Physics) Question 7, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1

An ideal gas of N non-interacting fermions in a volume V is in thermal equilibrium at temperature T.

Denote the mean occupation number of a state with energy E as $\langle n(E)\rangle$, and the corresponding number fluctuations as

$$\langle \Delta n(E)^2\rangle = \langle n^2(E)\rangle - \langle n(E)\rangle^2 .$$

Find the value of E that maximizes this quantity, compute the maximal value, and comment on the physical interpretation.

Problem 2

A spherical black body of radius R and temperature T is surrounded by a larger co-centered reflecting spherical shell of radius R_R.

a) What is the thermal equilibrium energy density per unit angular frequency interval, $u(\omega, T)$, of the electromagnetic radiation in the space, $R < r < R_R$. Hint: you might want to use $\dfrac{\sum_{n=0}^{\infty} ne^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}} = \dfrac{1}{e^x - 1}$ to get $u(\omega, T)$.

b) Suppose the $R < r < R_R$ region is filled with a non-absorbing medium with index of refraction n. What is the new electromagnetic energy density $u(\omega, T)$?

c) A thin layer reflecting a fraction $\hat{f}$ of any electromagnetic radiation incident upon it is put onto the surface of the blackbody sphere in a). How is $u(\omega, T)$ changed?

d) Suppose the region $R < r < R_R$ is filled with a liquid whose sound speed is C_s. What is the equilibrium energy density per unit frequency, $u_s(\omega, T)$, of the sound waves there (assume $\dfrac{\hbar c_s}{k_B T}$ >> inter-atomic spacing)?

e) A very tiny hole of area A is opened in the larger reflecting spherical shell and is covered by the same reflecting material used in part c) to cover the central blackbody. What is the power in escaping radiation?

Problem 3

Consider an ideal gas of N particles of mass m confined to a cubic box of volume V ($V=L^3$). The box is in a uniform gravitational field $-g\hat{y}$ and is at equilibrium at temperature T. Assume that the potential energy of a particle is $U(y)=mgy$ where y is the vertical coordinate inside the box ($-L/2 < y < L/2$).

a) Write down the partition function of the system using momentum and coordinates.

b) Find the energy of the system.

c) Find the heat capacity of the system.

d) Find the density and pressure distribution as a function of the height y.

Problem 4

Consider a system of non-interacting particles of spin ½. Each particle has mass m, and is constrained to move with non-relativistic momentum on the very flat surface of superfluid liquid He kept at very low temperature $T\to 0$. The components of the particles' momenta are restricted to the (x,y) plane that defines the surface of the liquid. The interactions between particles and the liquid He substrate are negligible. The area of the surface is $A=L^2$, where L is the length of the surface along the x and y-directions. The density of particles per unit area is n.

a) Assume cyclic boundary conditions for the wavefunctions of momentum and energy to obtain the allowed energy states. [In cyclic boundary conditions the wavefunctions at position (x,y) are identical to those at position (x+L, y+L)].

b) Use the results in a) to obtain an expression for $g(E)$ such that $g(E)dE$ gives the number of energy states of the particles in the range $[E,E+dE]$.

c) Consider the limit T=0. Obtain the energy difference between the lowest and highest energy states occupied by the particles.

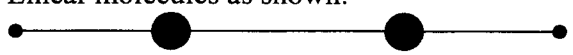
d) What is the average energy per particle at T=0?

e) The temperature is raised slightly so that ΔT is much smaller than the average energy per particle. Describe in words the changes that occur in the system. What is the expected temperature dependence of the change in the total energy of the system?

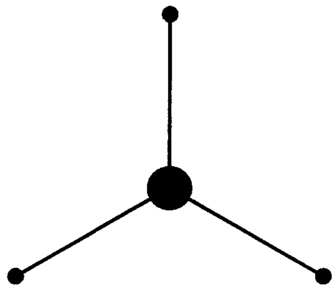
Problem 5

Calculate the classical specific heat, C_V, of the ideal gas consisting of N

a) Linear molecules as shown:



b) Planar molecules as shown:



Problem 6

A long (length d). solid, and well-insulated rod has one end attached to a cold temperature reservoir so that the entire rod is initially at the reservoir temperature. The rod is characterized by a known specific heat C, density ρ, and thermal conductivity κ. If the other end of the rod is suddenly connected to a high temperature reservoir a temperature ΔT higher, then

a) At $t=0$, find the heat intensity (power per unit area) that flows into the low temperature reservoir.
b) At thermal equilibrium, find the heat intensity that flows into the low temperature reservoir.
c) **Estimate** the time it takes for this equilibrium to be established.

Ruderman Section 5 Question 2

Ruderman solution

Answers

a)

$$u(\omega)\,d\omega = \frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\;\frac{2 \times 4\pi k^2\,dk}{(2\pi)^3}$$

with $k = \frac{\omega}{c}$

b) as above with $k = \frac{n\omega}{c}$

c) T is unchanged

d) as in a) with $c \to c_s$

and $(2) \to (1)$ [one mode]

e)

$$u(\omega)\,d\omega = \frac{\hbar\omega^3\,d\omega}{e^{\hbar\omega/k_BT}-1}\;\frac{4\pi}{(2\pi)^3}\left\{\frac{1}{c_1^3}+\frac{1}{c_2^3}\right\}$$

Note that in a 2-D layer there is only a single kind of shear wave (+ the usual compressional wave)

f) $A(1-\hat{f})\,\frac{c}{4}\,u(\omega,T)$

Kim section 5, question 3

Thermo / Stat Canonical Ensemble

(a) $H = \frac{p^2}{2m} + mgz$, $\beta = \frac{1}{k_B T}$

$$Z_1 = \left[\frac{1}{h^3}\int d\vec{p}^3 \int d\vec{r}^3\, e^{-\beta H}\right] = \left[\frac{1}{h^3}\left[\int_{-\infty}^{\infty} dp\, e^{-\frac{\beta p^2}{2m}}\right]^3 L^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-\beta m g z}\, dz\right]$$

$$= \left[\frac{2\pi m}{\beta h^2}\right] L^2 \frac{2}{\beta m g} \sinh\left[\frac{\beta m g L}{2}\right].$$

$$Z_{tot} = Z_1^N . \qquad \beta = \frac{1}{k_B T}$$

(b)
$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z_{tot}$$

$$= \frac{3}{2}\frac{N}{\beta} - N\frac{\partial}{\partial\beta}\left[-\ln\beta + \log\left\{\sinh\left(\frac{\beta m g L}{2}\right)\right\}\right]$$

$$= \frac{5}{2} N k_B T - \frac{N m g L}{2} \coth\left[\frac{m g L}{2 k_B T}\right]$$

(c)
$$C = \frac{1}{N}\frac{\partial}{\partial T}\langle E \rangle = \frac{5}{2} k_B - \frac{m g L}{2}\left(\frac{g m L}{2 k_B}\frac{-1}{T^2}\right)\underbrace{\coth'\frac{m g L}{2 k_B T}}_{= -\sinh^{-2}}$$

$$= k_B\left[\frac{5}{2} - \left(\frac{m g L}{2 k_B T}\right)^2 \sinh^{-2}\left(\frac{m g L}{2 k_B T}\right)\right]$$

(d)

2

$$n(z) = \frac{N e^{-mgz\cdot\beta}}{L^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} dz\, e^{-mgz\beta}} = N mg\beta \frac{e^{-mgz\beta}}{2\sinh\left(\frac{mg\beta L}{2}\right)}$$

$$= \frac{N}{L^3}\left(\frac{mgL}{2k_BT}\right) \frac{e^{-mgz/k_BT}}{\sinh\left[\frac{mgL}{2k_BT}\right]} \quad //.$$

$$P(z) = k_BT \cdot n(z)$$

$$= \frac{N}{L^3}\left(\frac{mgL}{2}\right) \frac{e^{-mgz/k_BT}}{\sinh\left[\frac{mgL}{2k_BT}\right]} . \quad //.$$

Pinczuk Section 5, Question # 4

General: particle statistics / thermal

JAN 13 2004

(a) momentum : $\vec{p} = \hbar \vec{k}$

The wave functions are

$$\psi = A\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}$$

where $E = \hbar\omega$

In cyclic boundary conditions

$$e^{ik_x L} = e^{ik_y L} = 1$$

$$k_x = n_x \frac{2\pi}{L} \; ; \quad k_y = n_y \frac{2\pi}{L}$$

where n_x, n_y are integers $|n_x| \le \infty$

$|n_y| \le \infty$

The energies are

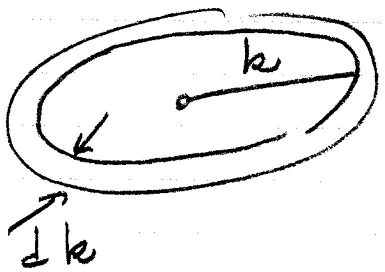
$$E = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} \equiv$$

JAN 13 20

$$E(n_x, n_y) = \frac{2\pi^2\hbar^2}{mL^2}\left(n_x^2 + n_y^2\right)$$

(b)

In wavevector space the area is



$$\pi k^2 = A_k \qquad \text{where } k^2 = k_x^2 + k_y^2$$

$$dA_k = 2\pi k\, dk$$

The area assigned to each point is

$$\left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2}{L^2}$$

The number of states in dA_k is

$$dN = \frac{L^2}{4\pi^2}\,\overset{\text{spin degeneracy}}{2}\;2\pi\, k\, dk$$

and per unit area we have

$$dn = \frac{1}{\pi}(k\, dk) = g(E)\, dE$$

$$(k\, dk) = \frac{m}{\hbar^2}\, dE$$

$$dn = \frac{m}{\pi\hbar^2}\, dE$$

JAN 13

$$g(E) = \frac{m}{\pi \hbar^2} \qquad E \geq 0$$

(c) The particles are fermions so that

$$n = \int_0^{E_{max}} g(E)\, dE$$

$$n = \frac{m}{\pi \hbar^2} E_{max}$$

$$E_{max} = \frac{\pi \hbar^2 n}{m}$$

(d)

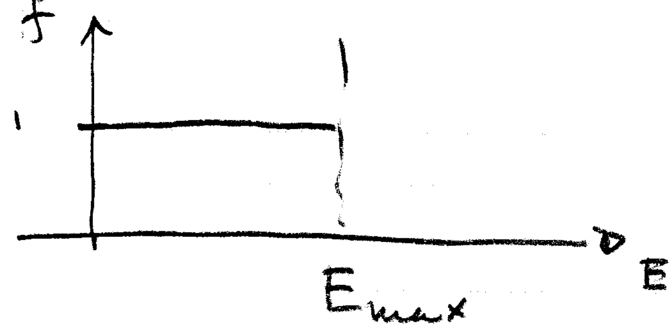
$$\langle E \rangle = \int_0^{E_{max}} E\, g(E)\, dE = \frac{m}{\pi \hbar^2} \frac{E_{max}^2}{2}$$

$$= \frac{1}{2} \frac{m}{\pi \hbar^2} E_{max} E_{max} = \frac{1}{2} \frac{m}{\pi \hbar^2} \frac{\pi \hbar^2}{m} n E_{max}$$

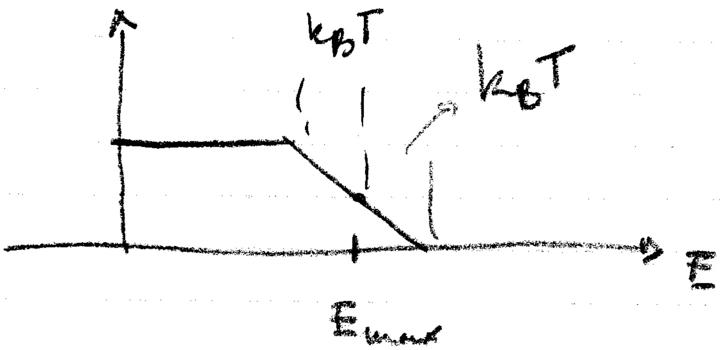
$$= \frac{1}{2} n E_{max}$$

JAN 13 2004

(e) At $T = 0$ the prob. to find a particle at energy E is



at ΔT

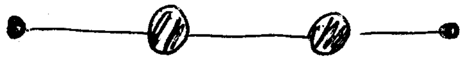


The number of particles $\Delta n \sim (k_BT)$ are excited above E_{max}

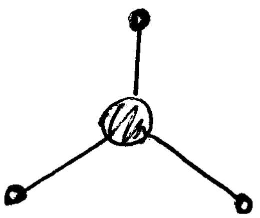
$$\Delta E \sim k_BT \, \Delta n \sim (k_BT)^2$$

#2. Calculate classical specific heat (C_V) of the ideal gas consisting of N

a) Linear molecules as shown



b) Planar molecules as shown:



Solution:

$$C_V = \frac{N}{2}\left(n_{cm} + n_r + 2n_v\right)$$

where n_{cm} is the number degrees of freedom associated with the center of mass, n_r is the number of rotational degrees of freedom and n_v is the number of vibrational degrees of freedom

a) $n_{cm} = 3$; $n_r = 2$; $n_v = 7$; $C_V = \frac{19N}{2}$;

b) $n_{cm} = 3$; $n_r = 3$; $n_v = 6$; $C_V = 9N$

Sciulli Section 5
Question # 6

Solution – heat transport problem:

F. Sciulli
Qual 2004

Using freshman physics

definition k:

$$\frac{Q}{t} = kA\frac{\Delta T}{d} \overset{1D}{\Rightarrow} I \equiv \frac{1}{A}\frac{dQ}{dt} = k\frac{dT}{dx}$$

definition C

$$Q = mC\Delta T \overset{1D}{\Rightarrow} \frac{dI}{dx} = \rho C\frac{dT}{dt}$$

$$\frac{d^2T}{dx^2} - \frac{\rho C}{k}\frac{dT}{dt} = 0$$

(diffusion equation)

1. At $t=0$, no heat has yet reached low temp reservoir

$$I_{low}\big|_{t=0} = 0$$

2. At $t=\infty$, equilibrium established

$$I_{low}\big|_{t=\infty} = k\frac{\Delta T}{d}$$

3. An <u>estimate</u> of the average $\left\langle \frac{d^2T}{dx^2}\right\rangle_{t=0} = \frac{1}{d}\left[\left\langle\frac{dT}{dx}\right\rangle_{hi} - \left\langle\frac{dT}{dx}\right\rangle_{lo}\right]$

$$= \frac{1}{d}\left[\frac{\Delta T}{d} - 0\right]$$

$$= \frac{1}{d^2}\Delta T$$

Δt = time to reach equilibrium

$$\frac{\Delta T}{\Delta t} = \frac{k}{\rho C}\left\langle\frac{d^2T}{dx^2}\right\rangle = \frac{k}{\rho C}\frac{\Delta T}{d^2}$$

$$\Delta t \approx \frac{\rho C}{k} d^2$$

(Also can get from dimensional analysis!)

Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 16, 2004
11:10 AM – 1:10 PM

General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6(General Physics) Question 6, etc.)

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Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted. Questions should be directed to the proctor.

Good luck!!

Problem 1

A panel of US physicists and astronomers has identified a list of eleven fundamental questions about the nature of the universe that will require the combined skills of particle physicists and astrophysicists to answer. The questions are in "From quarks to the cosmos", the first report from the committee on the physics of the universe set up by the National Academy of Sciences. Several of the questions were discussed by speakers in the Fall, 2003 Colloquium series. The eleven questions are:

- What is dark matter?
- What are the masses of the neutrinos, and how have they shaped the evolution of the universe?
- Are there additional spacetime dimensions?
- What is the nature of the dark energy?
- Are protons unstable?
- How did the Universe begin?
- Did Einstein have the last word on gravity?
- How do cosmic accelerators work and what are they accelerating?
- Are there new states of matter at exceedingly high density and temperature?
- Is a new theory of matter and light needed at the highest energies?
- How were the elements from iron to uranium made?

For full credit: Choose one of the questions above which was also discussed by a colloquium speaker. Briefly describe the topic of the talk. Explain the larger mystery behind the question and what types of future experiments will address the issue.

For 80% credit: You need not refer to a specific colloquium. Choose one of the questions above. Explain the larger mystery behind the question and what types of future experiments will address the issue.

NOTE: You will only be graded on content. You will not be graded on grammar, spelling or composition.

Problem 2

In an Ar-ion laser, green light (515 nm) is emitted by a transition of Ar^+ ions in a discharge. The radiative lifetime of the relevant transition is $\tau_{sp} = 10$ ns, and its measured line-width is $\Delta\nu = 3$ GHz. The discharge has an effective temperature of T=1000 K and is at a pressure of 0.1 atmospheres. The laser cavity consists of parallel mirrors separated by 1 m.

a) Estimate the contribution to the line-width from the radiative decay (the natural line-width).

b) Estimate the contribution to the line-width from Doppler broadening.

c) Estimate the contribution to the line-width from pressure broadening (collisions).

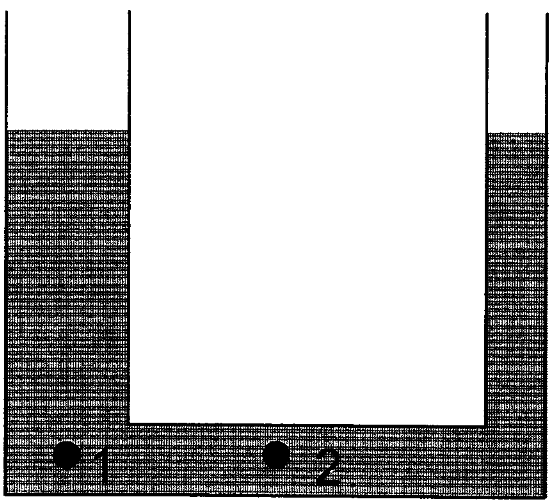
d) Estimate in how many different longitudinal modes can the laser oscillate.

e) If the laser is operated in pulsed mode, estimate the duration of the shortest pulse it can support.

f) If the average power if the laser is 10 W, what would be the peak power achievable for the laser operating in pulsed mode in which a single pulse travels back and forth in the cavity? What would be the peak intensity (irradiance) of such a pulse focused to a diffraction-limited spot?

Problem 3

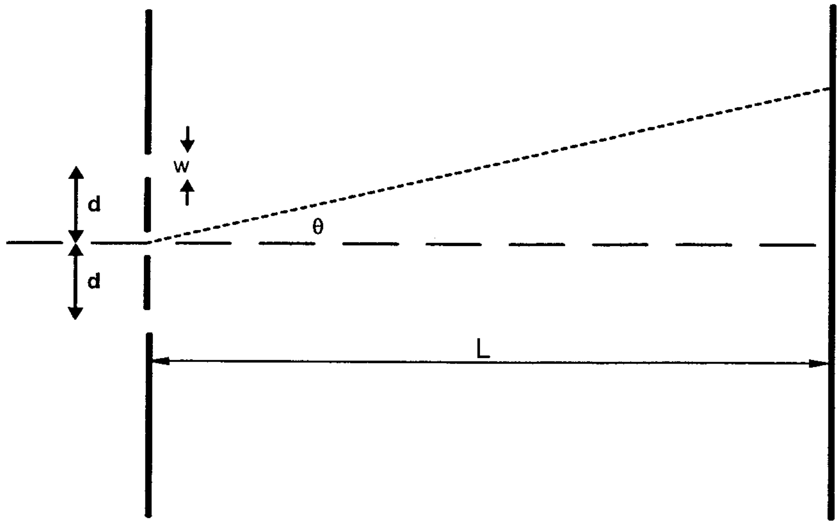
A freshman comes to you and is puzzled by the following argument regarding the pressure in a U-tube containing water. Her physics professor has told her that the pressure at points 1 and 2 are the same <u>and</u> that pressure at a point in a fluid is due to the weight of the fluid above it. The student argues that the pressure at point 1 and point 2 must be different because they have different amounts of fluid above them. Provide a simple argument that explains why the student's reasoning is incorrect.



Problem 4

A beam of laser light of intensity I_0 and wavelength λ shines on a perfect absorber with three long slits of width w spaced by a distance d. A flat screen is located a distance L from the absorber.

a) Under what conditions (express as an inequality involving λ) does the finite size of the slits have negligible influence on the diffraction pattern observed on the screen?
b) Find an approximate expression for the angular dependence of the intensity of light observed on the screen for $L/d >> 1$ under the conditions in part a).
c) Describe qualitatively how the shape of the central (θ=0) peak in the diffraction pattern changes when the finite size of the slits is considered.
d) Find an approximate expression for the angular dependence of the intensity of light observed on the screen for $L/d >> 1$ and $L/w >> 1$ taking into account the finite width of the slits.



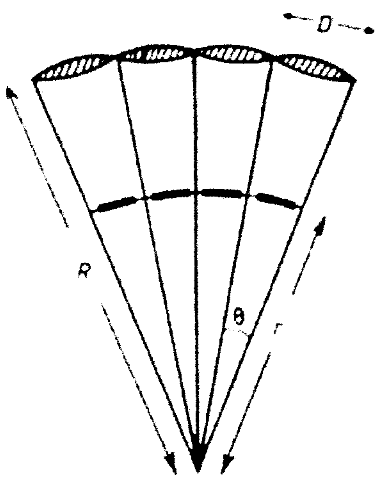
(not to scale)

Problem 5

Consider a steady rain of particles falling freely from infinity onto a gravitating sphere of mass M and radius R. The rain is spherically symmetric (radial) and its velocity at infinity is zero. The rate of the rain dM/dt [g/s] is given. The falling particles collide inelastically with the sphere and heat it. M grows very slowly as the rain material is accumulated on the sphere, so that the process is quasi-steady.

a) Find the emitted power L [erg/s] and temperature T of the sphere, assuming that it emits as a blackbody.
b) Suppose the sphere rotates with angular velocity omega. What is the spin-down rate $d\Omega/dt$? In the calculation assume that the sphere with the accumulated rain material is a solid body of constant density ρ.
c) What is the luminosity L of the rotating sphere?

Problem 6



A simple model of an insect eye consists of a spherical array of closely packed conical receptors, each optically isolated from its neighbors. The radius of the spherical array is R. Each cone has a small lens, diameter D, which focuses parallel light onto a photosensitive cell within the cone at a distance r from the center.

a) The light and the angular resolution for each element is limited by the diffraction associated with aperture D. Show that if the insect's eye is simultaneously to achieve maximum sensitivity to light and maximum angular resolving power, the photosensitive cell should lie at r = 2R/3 from the center.
b) Estimate the diameter of a single lens in the optimized eye and the minimum resolvable angle between two point sources of green light ($\lambda = 500$ nm) if R = 1 mm.

Hi Lalla.

JAN 19 2004

Conrad
Section 6
Question #1

FYI: Grading:

Max = 15 if a colloq is listed
Max = 12 if no colloq listed

top score has:

$\frac{1}{3}$ 1) Explanation of issue
$\frac{1}{3}$ 2) Reference to theories that might explain
$\frac{1}{3}$ 3) Experiments listed which have already or might address question

Section 6, Question # 2

Heinz - Problem 2

(a) $\Delta\nu_{sp} = \frac{1}{2\pi\tau_{sp}} = 16\,MHz$

(b) $\Delta\nu_D \approx 2\frac{\bar{v}_z}{c}\nu = \frac{2\bar{v}_z}{\lambda} \approx 2\frac{\sqrt{k_BT/M}}{\lambda} = \frac{2\times 680\,m/s}{511\,nm} = 2.7\,GHz$

(c) $\Delta\nu_{coll} = \frac{f_{coll}}{\pi}$

$f_{coll} \approx \bar{v}\pi d^2 n = \sqrt{3}\,\bar{v}_z \pi d^2 n$, with $d \approx 1\text{Å}$ atomic diameter

$\approx 110\,MHz$

$n = 6\times10^{23}/210\,\ell$

$= 3\times10^{24}\,m^{-3}$

$\Delta\nu_{coll} \approx 35\,MHz$

(d) $\Delta\nu_{mode} = \frac{c}{2L} = 150\,MHz$

Number of accessible modes ~ $\frac{\Delta\nu}{\Delta\nu_{mode}} = 20$ (2x if you include polarization)

(e) By the time-frequency uncertainty relation

$\tau \geq (4\pi\Delta\nu)^{-1} = 26\,ps$

(f) The longest separation possible between the pulses is one cavity roundtrip of $2m/3\times10^8 m/s = 6.7\,ns$. Then

$\bar{P} = P_{peak}\frac{25\,ps}{6.7\,ns} \Rightarrow P_{peak} = \bar{P}\frac{6.7\,ns}{25\,ps} = 2.7\,kW$

$I_{peak} = \frac{P_{peak}}{\lambda^2} = 1\times10^{16}\,W/m^2$

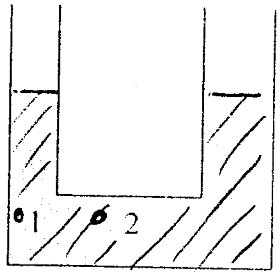
(Approximate answers are fine for all questions. Relations above are approximately correct for $\Delta\nu$ as FWHM)

Quals Questions
M. Tuts
Dec 16, 2003

Section 6, Question# 3

General?

Q1. A freshman comes to you and is puzzled by the following argument regarding the pressure in a U-tube containing water. His physics professor has told him that the pressure at points 1 and 2 are the same, and that the pressure at a point in a fluid is due to the weight of the fluid above it. The student argues that the pressure at point 1 and point 2 must be different because they have different amounts of fluid above them. Provide a simple argument that explains why the student's argument is wrong.



A. One simple way to explain this apparent discrepancy is to imagine that picture actually represents a vat of liquid with a cup pushed down into it (displacing the liquid). If we ask what is the force of the water on the cup, it is equal to the weight of the water displaced. So by the third law, the force of the cup on the water (i.e. the pressure) is equal to the weight of the fluid displaced. That means that the pressure on the water at point 2 is the equivalent of the pressure at that point if the cup were replaced by water, so indeed it is the equivalent of the weight of the column of water above it!

Shaevitz Section 6, Question 6

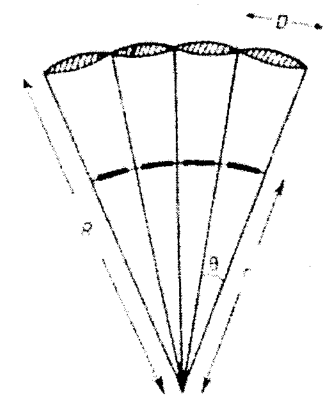
Problem 1 (Optics):

(M. Shaevitz)

A simple model of an insect eye consists of a spherical array of closely packed conical receptors, each optically isolated from its neighbours. The radius of the spherical array is R. Each cone has a small lens, diameter D, which focuses parallel light onto a photosensitive cell within the cone at a distance r from the center.

a) The light and the angular resolution for each element is limited by the diffraction associated with aperture D. Show that if the insect's eye is simultaneously to achieve maximum sensitivity to light and maximum angular resolving power, the photosensitive cell should lie at $r \bullet\bullet 2R/3$ from the center.

b) Estimate the diameter of a single lens in the optimised eye and the minimum resolvable angle between two point sources of green light (• •• •500 nm) if R • •1 mm.



Solution:

a) The focal length of the lens is $R \bullet\bullet r$. For maximum sensitivity to light, one wants the diffraction disc to equal the size of the photosensitive cell whose diameter is $d_{disc} \bullet\bullet rD/R$. The angle of the diffraction disc is $\bullet_{\min} \bullet\bullet 1.22\bullet /D$ giving a diameter at the photosensitive cell of

$$d_{disc} \bullet\bullet 2.44\bullet\bullet R \bullet\bullet r\bullet /D \bullet\bullet rD/R$$

For good angular resolution, the fields of view of the cones should be equal to the diffraction limit

$$\bullet\bullet\bullet\bullet D/R \bullet\bullet\bullet_{\min} \bullet\bullet 1.22\bullet /D$$

Solving the two equations gives $r \bullet\bullet 2R/3$.

b) For • •• •500 nm and $R \bullet\bullet 1$ mm, $D \bullet\bullet \sqrt{1.22\bullet R} \bullet\bullet 2.5 \bullet\bullet 10^{\bullet 5} m \bullet\bullet 25 \bullet m$ and $\bullet\bullet\bullet\bullet \sqrt{1.22\bullet /R} \bullet\bullet 0.025 \bullet\bullet 1.4^{\bullet}$.