Department of Physics and Astronomy
University of Southern California

Graduate Screening Examination

Part I

Saturday, April 13, 2013

Do not separate this page from the problem pages.
Fill out and turn in at the end of the exam.

Student ____________
Fill in your S-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your S-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

Solve **eight** problems of your choice. Do not turn in more than this number (8) of problems!

The total time allowed **3 hrs**.

Please, indicate problems you are turning in:

☐ 1	☐ 2	☐ 3	☐ 4	☐ 5
☐ 6	☐ 7	☐ 8	☐ 9	☐ 10

Problems that are not checked above, will not be graded. If you check more than 8 problems, only the lowest 8 scores will count towards your total score.

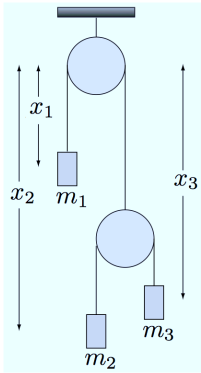
I-1. (Classical Mechanics)

Assume that the strings and pulleys in the Atwood machine are massless.

(i) Write down the constraints.

(ii) Find the equations of motion of the system using any method you like.

(iii) Determine the tension in the string on which the mass m_1 is hanging.



I-2. (Electricity and Magnetism)

A coaxial cable consists of two long coaxial conducting cylindrical tubes oriented along the z-direction. The radius of the inner tube is a, and the radius of the outer tube is c. Between the tubes there are two different linearly magnetizable insulating materials. The relative permeability is

$$\frac{\mu}{\mu_0} = 1 + \chi_m = \begin{cases} 3 & \text{for } a < s < b \\ 2 & \text{for } b < s < c \end{cases}$$

where s is the distance from the axis of the cylinder.

A free current I_{free} flows up the inner conductor and returns down the outer one. The current is uniformly distributed over the surface of each tube.

(i) Calculate the $\vec{H}$ field in the region between the two tubes using

$$\oint \vec{H} \cdot d\vec{\ell} = (I_{\text{free}})_{\text{enclosed}} \,.$$

Then evaluate the magnetic field, $\vec{B}$, and the magnetization, $\vec{M}$, in each region using

$$\vec{B} = \mu \vec{H} = \mu_0(\vec{H} + \vec{M}) \,.$$

(ii) Evaluate the magnetization current, $\vec{I}_{\text{m}}$, in the volume using $I_{\text{m}} = \vec{\nabla} \times \vec{M}$, and the surface magnetization current at the three material surfaces, $s = a, b$, and c, using $\vec{K}_m = \vec{M} \times \widehat{n}$, where $\widehat{n}$ is the normal unit vector pointing out of the material volume. Show that $\vec{B}$ calculated above satsifies

$$\oint \vec{B} \cdot d\vec{\ell} = (I_{\text{total}})_{\text{enclosed}} \,,$$

in each region, $a < s < b$ and $b < s < c$.

Hint: In cylindrical coordinates the curl of a vector field $\vec{A}$ is given by

$$\vec{\nabla} \times \vec{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\widehat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\widehat{\varphi} + \frac{1}{s}\left[\frac{\partial (sA_\varphi)}{\partial s} - \frac{\partial A_s}{\partial \varphi}\right]\widehat{z} \,.$$

I-3. (Quantum Mechanics)

A stream of particles moving in three-dimensional space is scattered by a spherically symmetric potential

$$V(r) \;=\; \begin{cases} V_0 & \text{if } r \le a \\ 0 & \text{if } r > a \end{cases}$$

where V_0 is a positive constant and a is a positive radius. Consider the Schrödinger equation for angular momentum $\ell = 0$ only! For $r > a$, the wave function is given by

$$\psi(r) \;=\; N\,\frac{\sin(kr + \delta_0)}{r}\,,$$

where δ_0 is the phase shift, k is related to the energy of the particles, $E = (\hbar k)^2/2m$, and N is a constant related to the intensity of the particle stream. The value of E is adjusted so that $E = V_0$.

(i) Solve the Schrödinger equation for $r < a$ under the condition that $E = V_0$ and ψ does not diverge anywhere. There should be one additional constant in your solution, call it C.

(ii) At $r = a$, the wave function ψ and its radial derivative are continuous. Use those conditions to determine the two unknown constants, δ_0 and C.

Hint: The Laplacian is spherical coordinates is

$$\Delta \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

I-4. (Thermodynamics)

The entropy of a certain gas is given by

$$S \;=\; \frac{4}{3}a^{1/4}E^{3/4}V^{1/4}\,,$$

where a is a constant. Note that the gas is quite special in that S does not depend explicitly on the number of particles, N.

(i) Show that the energy density, E/V, of this gas depends only on its temperature.

(ii) Show that the energy density of the gas is proportional to its pressure. Find the proportionality constant.

(iii) Show that during a quasistatic adiabatic compression of the gas,

$$PV^{\gamma} \;=\; \text{const}\,.$$

What is the value of the adiabatic constant γ?

(iv) Show that the Helmholtz free energy, F, is proportional to E. Find the proportionality constant.

(v) Calculate the temperature of the gas when

$$\frac{E}{V} \;=\; 2.5\times 10^5\,\frac{\text{eV}}{\text{m}^3}\,, \qquad a = 4.7\times 10^3\,\text{eV/K}^4\cdot\text{m}^3\,.$$

(vi) Any suggestions as to what this gas might be?

I-5. (Statistical Mechanics)

A cylindrical centrifuge of radius R and height L contains a *classical* ideal gas. The centrifuge rotates at angular frequency Ω. If the z-axis is along the axis of the cylinder, the dynamics of one particle can be described by the Hamiltonian

$$H_1 = \frac{\vec{p}^{\,2}}{2m} - \Omega(xp_y - yp_x),$$

where m is the mass of the particles.

(i) The partition function for one particle is given by

$$Z_1 = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2\pi L}{m\beta\Omega^2}\left(e^{m\beta\Omega^2 R^2/2} - 1\right).$$

Set up a calculation of this partition function, but do not evaluate intermediate steps.

(ii) Using the partition function in (i), calculate the partition function if there are N identical, non interacting particles.

(iii) Calculate the free energy, and obtain the pressure on the wall of the centrifuge. What happens if $\Omega \to 0$?

I-6. (Solid State)

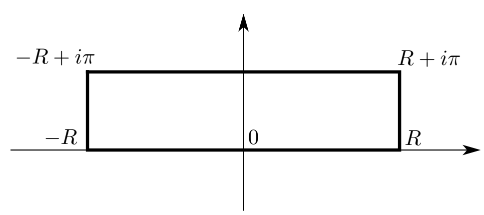
The specific heat of europium oxide (EuO) at low temperatures is proportional to $T^{3/2}$. To receive credit, you must justify your answers to the following questions in clear detail.

(i) Is EuO a metal?

(ii) What type of simple dispersion curve $E(k)$ gives rise to this specific heat?

Comment: "Simple" means that $E(k)$ has a simple functional form, and also that it depends only on the magnitude of the crystal momentum and not its direction.

I-7. (Math Methods)



Evaluate

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}\, dx\,.$$

Hint: Change variables $x = e^t$ and use the contour in the figure. Identities of the type

$$te^t \;=\; \frac{d}{d\alpha}e^{\alpha t}\Big|_{\alpha=1}\,,$$

might also be useful.

I-8. (Particle/Special Relativity)

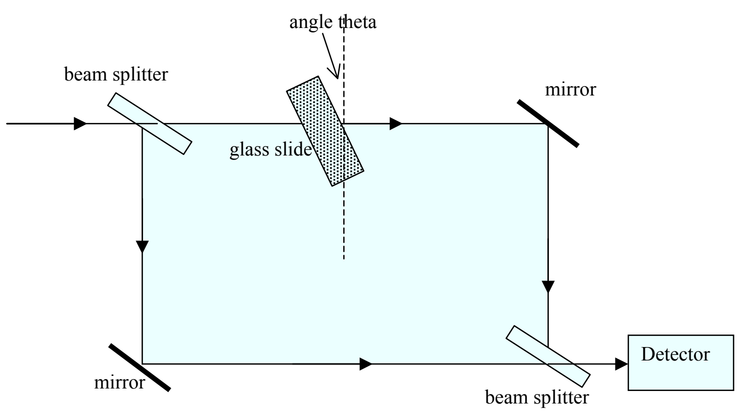
A rocket propels itself by emitting radiation in the direction opposite to its motion. If V is the final velocity of the rocket relative to its initial rest frame, prove that the ratio of the initial to the final rest mass is given by:

$$\frac{M_{\mathrm{i}}}{M_{\mathrm{f}}} = \left(\frac{c+V}{c-V}\right)^{1/2} \quad .$$

I-9. (Optics)

A soap bubble is illuminated by white light. The index of refraction of the soap film is 1.36, and the soap film is 250 nm thick. For light rays reflected at normal incidence, which colors (visible wavelengths) are not seen and which colors appear strong in the reflected light? What color approximately does the soap bubble appear?

I-10. (Experimental)



Two light beams of wavelength λ in a Mach-Zehnder interferometer interfere to create an output beam. A glass slide of uniform thickness t and refractive index n is inserted in one of the beams. As the slide is rotated through an angle theta as shown, the intensity of the output beam is seen to vary periodically. Assume that the refractive index of air is unity, that the two beams entering the detector have the same intensity, and that when the angle theta equals zero the total intensity into the detector is at its maximum value. Derive an expression for the smallest non-zero value of the angle theta that gives an intensity maximum.

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Graduate Screening Examination
Part II
Saturday, April 13, 2013

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Student ______________
Fill in your L-#

The exam is **closed book**. Use only the paper provided and *make sure that each page is signed with your L-number*. Do not write answers to different problems on the same page. Mark each page with the problem number. Staple *separately* your answers to *each* problem.

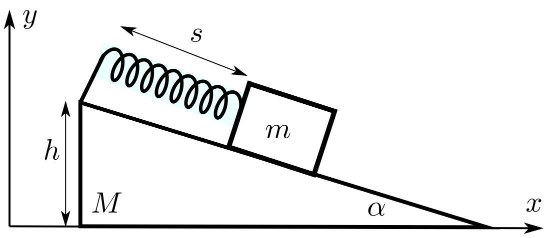
Solve 3 problems of your choice. Do not turn in more than 3 problems.

The total time allowed **2 hrs.**

Please, indicate problems you are turning in:

☐ **II-1** ☐ **II-2** ☐ **II-3** ☐ **II-4**

II-1. (Classical Mechanics)



A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface, as shown above.

(i) Given the relaxed length of the spring alone is d, find the value s_0 when both the block and the wedge are at rest.

(ii) Find the Lagrangian of the system as a function of the x coordinate of the wedge and the length of the spring s. Write the equations of motion.

(iii) What is the natural frequency of small oscillations?

II-2. (E & M)

A hollow sphere of inner radius R is initially uncharged, but a pure point dipole $\vec{p}$ sits at its center. Subsequently, the sphere is grounded.

(i) Calculate the final potential $\Phi(r, \theta)$ for $r < R$.

(ii) Calculate the final surface charge density on the sphere.

(iii) How much charge flows from ground to the sphere?

(iv) What would be the surface charge density if the sphere were not grounded?

II-3. (Quantum Mechanics)

Find the energy levels and the wave function of two harmonic oscillators of masses m_1 and m_2, having identical frequencies ω, and coupled by the interaction $\frac{1}{2}k(x_1 - x_2)^2$. Your answer should include:

(i) the full Hamiltonian,

(ii) a desription in words of the path you will follow to solve the problem, the final goal, and the steps to answer parts (iii) and (iv),

(iii) the computation of the spectrum,

(iv) the full set of eigenstates.

II-4. (Mathematical Methods)

The gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt\,, \qquad \mathrm{Re}\, z > 0\,.$$

(i) Prove that for positive integer n,

$$\Gamma(n+1) = n!\,.$$

(ii) Show that

$$\Gamma(z+1) = z^{z+1} \int_0^\infty e^{z(\ln s - s)} ds\,.$$

(iii) For large z, the dominant contribution to the integration in (ii) comes from the maximum of function $g(s) = \ln s - s$. Use this observation to prove the Stirling's formula

$$n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}\,, \qquad n \gg 1\,.$$

Hint: Perform the Taylor series expansion of $g(s)$ around its maximum and use the leading terms to evaluate the integration.