Exercise 6.2.7

Suppose that Charlie and Alice possess a bipartite state $|\psi\rangle_{CA}$, then Alice teleports her share of $|\psi\rangle_{CA}$ to Bob. In this case, before the teleportation, the joint state of C, A and the shared ebit of Alice system A' and Bob's system B becomes

$$|\psi\rangle_{CA}|\Phi^+\rangle_{AB} \tag{1}$$

If we write $|\psi\rangle_{CA}$ as an arbitrary state, then

$$\begin{aligned}|\psi\rangle_{CA}|\Phi^+\rangle_{A'B} =&(a|00\rangle_{CA}+b|01\rangle_{CA}+c|10\rangle_{CA}+d|11\rangle_{CA})\left(\frac{|00\rangle_{A'B}+|11\rangle_{A'B}}{\sqrt{2}}\right)\\ =&\frac{1}{\sqrt{2}}(a|0000\rangle_{CAA'B}+b|0100\rangle_{CAA'B}+c|1000\rangle_{CAA'B}+d|1100\rangle_{CAA'B}\\ &+a|0011\rangle_{CAA'B}+b|0111\rangle_{CAA'B}+c|1011\rangle_{CAA'B}+d|1111\rangle_{CAA'B})\end{aligned} \tag{2}$$

Note that we can express two qubits state with the Bell basis,

$$\begin{aligned}|00\rangle_{AA'} &= \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{AA'}+|\Phi^-\rangle_{AA'})\\ |01\rangle_{AA'} &= \frac{1}{\sqrt{2}}(|\Psi^+\rangle_{AA'}+|\Psi^-\rangle_{AA'})\\ |10\rangle_{AA'} &= \frac{1}{\sqrt{2}}(|\Psi^+\rangle_{AA'}-|\Psi^-\rangle_{AA'})\\ |11\rangle_{AA'} &= \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{AA'}-|\Phi^-\rangle_{AA'})\end{aligned} \tag{3}$$

Substitute eq. (3) into eq. (2), we have

$$\begin{aligned}|\psi\rangle_{CA}|\Phi^+\rangle_{AB} =&(a|00\rangle_{CA}+b|01\rangle_{CA}+c|10\rangle_{CA}+d|11\rangle_{CA})\left(\frac{|00\rangle_{A'B}+|11\rangle_{A'B}}{\sqrt{2}}\right)\\ =&\big[a|0\rangle_C(|\Phi^+\rangle_{AA'}+|\Phi^-\rangle_{AA'})|0\rangle_B+b|0\rangle_C(|\Psi^+\rangle_{AA'}-|\Psi^-\rangle_{AA'})|0\rangle_B\\ &+c|1\rangle_C(|\Phi^+\rangle_{AA'}+|\Phi^-\rangle_{AA'})|0\rangle_B+d|1\rangle_C(|\Psi^+\rangle_{AA'}-|\Psi^-\rangle_{AA'})|0\rangle_B\\ &+a|0\rangle_C(|\Psi^+\rangle_{AA'}+|\Psi^-\rangle_{AA'})|1\rangle_B+b|0\rangle_C(|\Phi^+\rangle_{AA'}-|\Phi^-\rangle_{AA'})|1\rangle_B\\ &+c|1\rangle_C(|\Psi^+\rangle_{AA'}+|\Psi^-\rangle_{AA'})|1\rangle_B+d|1\rangle_C(|\Phi^+\rangle_{AA'}-|\Phi^-\rangle_{AA'})|1\rangle_B]\end{aligned} \tag{4}$$

We can re-write eq. (4) as

$$\begin{aligned}|\psi\rangle_{CA}|\Phi^+\rangle_{AB} =&\big[a|0\rangle_C(|\Phi^+\rangle_{AA'}+|\Phi^-\rangle_{AA'})|0\rangle_B+b|0\rangle_C(|\Psi^+\rangle_{AA'}-|\Psi^-\rangle_{AA'})|0\rangle_B\\ &+c|1\rangle_C(|\Phi^+\rangle_{AA'}+|\Phi^-\rangle_{AA'})|0\rangle_B+d|1\rangle_C(|\Psi^+\rangle_{AA'}-|\Psi^-\rangle_{AA'})|0\rangle_B\\ &+a|0\rangle_C(|\Psi^+\rangle_{AA'}+|\Psi^-\rangle_{AA'})|1\rangle_B+b|0\rangle_C(|\Phi^+\rangle_{AA'}-|\Phi^-\rangle_{AA'})|1\rangle_B\\ &+c|1\rangle_C(|\Psi^+\rangle_{AA'}+|\Psi^-\rangle_{AA'})|1\rangle_B+d|1\rangle_C(|\Phi^+\rangle_{AA'}-|\Phi^-\rangle_{AA'})|1\rangle_B]\\ =&a|0\Phi^+0\rangle_{CAA'B}+b|0\Phi^+1\rangle_{CAA'B}+c|1\Phi^+0\rangle_{CAA'B}+d|1\Phi^+1\rangle_{CAA'B}\\ &+a|0\Phi^-0\rangle_{CAA'B}-b|0\Phi^-1\rangle_{CAA'B}+c|1\Phi^-0\rangle_{CAA'B}-d|1\Phi^-1\rangle_{CAA'B}\\ &+a|0\Psi^+1\rangle_{CAA'B}+b|0\Psi^+0\rangle_{CAA'B}+c|1\Psi^+1\rangle_{CAA'B}+d|1\Psi^+0\rangle_{CAA'B}\\ &+a|0\Psi^-1\rangle_{CAA'B}-b|0\Psi^-0\rangle_{CAA'B}+c|1\Psi^-1\rangle_{CAA'B}-d|1\Psi^-0\rangle_{CAA'B}\\ =&a|0\Phi^+0\rangle_{CAA'B}+b|0\Phi^+1\rangle_{CAA'B}+c|1\Phi^+0\rangle_{CAA'B}+d|1\Phi^+1\rangle_{CAA'B}\\ &+(I_{CAA'}\otimes Z_B)(a|0\Phi^-0\rangle_{CAA'B}+b|0\Phi^-1\rangle_{CAA'B}+c|1\Phi^-0\rangle_{CAA'B}+d|1\Phi^-1\rangle_{CAA'B})\\ &+(I_{CAA'}\otimes X_B)(a|0\Psi^+0\rangle_{CAA'B}+b|0\Psi^+1\rangle_{CAA'B}+c|1\Psi^+0\rangle_{CAA'B}+d|1\Psi^+1\rangle_{CAA'B})\\ &+(I_{CAA'}\otimes X_BZ_B)(a|0\Psi^-0\rangle_{CAA'B}+b|0\Psi^-1\rangle_{CAA'B}+c|1\Psi^-0\rangle_{CAA'B}+d|1\Psi^-1\rangle_{CAA'B})\end{aligned} \tag{5}$$

From eq. (5) we conclude that, if Charlie and Alice possess a bipartite state $|\psi\rangle_{CA}$ and Alice teleports her share of state $|\psi\rangle_{CA}$ to Bob, then Bob will receive his part of eq. (5), and he can use Alice's measurement of Bell states to recover his part and finally get the following joint state,

$$|\psi\rangle_{CAA'B} = a|0\Phi^i 0\rangle_{CAA'B} + b|0\Phi^i 1\rangle_{CAA'B} + c|1\Phi^i 0\rangle_{CAA'B} + d|1\Phi^i 1\rangle_{CAA'B} \tag{6}$$

where Φ^i denotes one of four Bell basis that Alice decides to measure, and obviously, now Charlie and Bob share the state

$$|\psi\rangle_{CB} = a|00\rangle_{CB} + b|01\rangle_{CB} + c|10\rangle_{CB} + d|11\rangle_{CB} \tag{7}$$

A special case of eq. (7) is ebit when $b = c = 0$ and $a = d = 1/\sqrt{2}$.