Exercise 3.5.7

Suppose we have two quantum state $|\psi\rangle$ and $|\psi^\perp\rangle$ with $\langle\psi|\psi^\perp\rangle = 0$, we want to find a unitary U that

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle \tag{1a}$$
$$U(|\psi^\perp\rangle \otimes |0\rangle) = |\psi^\perp\rangle \otimes |\psi^\perp\rangle \tag{1b}$$

A unitary matrix U can be written in the form $U = \sum_i \lambda_i |i\rangle\langle i|$, which inspire us to find a linear combination of something in the form $|i\rangle\langle j|$.

From eq. (1a), we notice that

$$(|\psi\rangle \otimes |\psi\rangle)(\langle\psi| \otimes \langle 0|)(|\psi\rangle \otimes |0\rangle) = |\psi\rangle\langle\psi|\psi\rangle \otimes |\psi\rangle\langle 0|0\rangle = |\psi\rangle \otimes |\psi\rangle \tag{2}$$

From eq. (1b), we notice that

$$(|\psi^\perp\rangle \otimes |\psi^\perp\rangle)(\langle\psi^\perp| \otimes \langle 0|)(|\psi^\perp\rangle \otimes |0\rangle) = |\psi^\perp\rangle\langle\psi^\perp|\psi^\perp\rangle \otimes |\psi^\perp\rangle\langle 0|0\rangle = |\psi^\perp\rangle \otimes |\psi^\perp\rangle \tag{3}$$

We also notice that

$$(|\psi^\perp\rangle \otimes |\psi\rangle)(\langle\psi| \otimes \langle 1|)(|\psi\rangle \otimes |0\rangle) = |\psi^\perp\rangle\langle\psi|\psi\rangle \otimes |\psi\rangle\langle 1|0\rangle = 0 \tag{4a}$$
$$(|\psi\rangle \otimes |\psi^\perp\rangle)(\langle\psi^\perp| \otimes \langle 1|)(|\psi^\perp\rangle \otimes |0\rangle) = |\psi\rangle\langle\psi^\perp|\psi^\perp\rangle \otimes |\psi^\perp\rangle\langle 1|0\rangle = 0 \tag{4b}$$

From eq. (2), eq. (3) and eq. (4a) $-$ (4b), we can construct a linear combination of them as U to make eq. (1a) $-$ (1b) valid,

$$\begin{aligned} U =& (|\psi\rangle \otimes |\psi\rangle)(\langle\psi| \otimes \langle 0|) + (|\psi^\perp\rangle \otimes |\psi\rangle)(\langle\psi| \otimes \langle 1|) \\ &+ (|\psi^\perp\rangle \otimes |\psi^\perp\rangle)(\langle\psi^\perp| \otimes \langle 0|) + (|\psi\rangle \otimes |\psi^\perp\rangle)(\langle\psi^\perp| \otimes \langle 1|) \\ =& |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle 0| + |\psi^\perp\rangle\langle\psi| \otimes |\psi\rangle\langle 1| + |\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 0| + |\psi\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 1| \end{aligned} \tag{5}$$

The reason why we need to add eq. (4a) $-$ (4b) is to make sure U is a unitary matrix (see below). To verify eq. (5), we need to check whether it is unitary and it satisfies eq. (1a) $-$ (1b).

- To verify whether eq. (5) is unitary, we need to adopt the property

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger, (A + B)^\dagger = A^\dagger + B^\dagger \tag{6}$$

  From eq. (5), we know that

$$\begin{aligned} U^\dagger =& (|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle 0|)^\dagger + (|\psi^\perp\rangle\langle\psi| \otimes |\psi\rangle\langle 1|)^\dagger \\ &+ (|\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 0|)^\dagger + (|\psi\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 1|)^\dagger \\ =& (|\psi\rangle\langle\psi|)^\dagger \otimes (|\psi\rangle\langle 0|)^\dagger + (|\psi^\perp\rangle\langle\psi|)^\dagger \otimes (|\psi\rangle\langle 1|)^\dagger \\ &+ (|\psi^\perp\rangle\langle\psi^\perp|)^\dagger \otimes (|\psi^\perp\rangle\langle 0|)^\dagger + (|\psi\rangle\langle\psi^\perp| \otimes (|\psi^\perp\rangle\langle 1|)^\dagger \\ =& |\psi\rangle\langle\psi| \otimes |0\rangle\langle\psi| + |\psi\rangle\langle\psi^\perp| \otimes |1\rangle\langle\psi| + |\psi^\perp\rangle\langle\psi^\perp| \otimes |0\rangle\langle\psi^\perp| + |\psi^\perp\rangle\langle\psi| \otimes |1\rangle\langle\psi^\perp| \end{aligned} \tag{7}$$

  Then we could have

$$
\begin{aligned}
U^\dagger U =&(|\psi\rangle\langle\psi| \otimes |0\rangle\langle\psi| + |\psi\rangle\langle\psi^\perp| \otimes |1\rangle\langle\psi| + |\psi^\perp\rangle\langle\psi^\perp| \otimes |0\rangle\langle\psi^\perp| + |\psi^\perp\rangle\langle\psi| \otimes |1\rangle\langle\psi^\perp|)\\
&\cdot (|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle 0| + |\psi^\perp\rangle\langle\psi| \otimes |\psi\rangle\langle 1| + |\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 0| + |\psi\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 1|)\\
\\
=&|\psi\rangle\langle\psi|\psi\rangle\langle\psi| \otimes |0\rangle\langle\psi|\psi\rangle\langle 0| + |\psi\rangle\langle\psi|\psi^\perp\rangle\langle\psi| \otimes |0\rangle\langle\psi|\psi\rangle\langle 1|\\
&+ |\psi\rangle\langle\psi|\psi^\perp\rangle\langle\psi^\perp| \otimes |0\rangle\langle\psi|\psi^\perp\rangle\langle 0| + |\psi\rangle\langle\psi|\psi\rangle\langle\psi^\perp| \otimes |0\rangle\langle\psi|\psi^\perp\rangle\langle 1|\\
\\
&+ |\psi\rangle\langle\psi^\perp|\psi\rangle\langle\psi| \otimes |1\rangle\langle\psi|\psi\rangle\langle 0| + |\psi\rangle\langle\psi^\perp|\psi^\perp\rangle\langle\psi| \otimes |1\rangle\langle\psi|\psi\rangle\langle 1|\\
&+ |\psi\rangle\langle\psi^\perp|\psi^\perp\rangle\langle\psi^\perp| \otimes |1\rangle\langle\psi|\psi^\perp\rangle\langle 0| + |\psi\rangle\langle\psi^\perp|\psi\rangle\langle\psi^\perp| \otimes |1\rangle\langle\psi|\psi^\perp\rangle\langle 1|\\
\\
&+ |\psi^\perp\rangle\langle\psi^\perp|\psi\rangle\langle\psi| \otimes |0\rangle\langle\psi^\perp|\psi\rangle\langle 0| + |\psi^\perp\rangle\langle\psi^\perp|\psi^\perp\rangle\langle\psi| \otimes |0\rangle\langle\psi^\perp|\psi\rangle\langle 1|\\
&+ |\psi^\perp\rangle\langle\psi^\perp|\psi^\perp\rangle\langle\psi^\perp| \otimes |0\rangle\langle\psi^\perp|\psi^\perp\rangle\langle 0| + |\psi^\perp\rangle\langle\psi^\perp|\psi\rangle\langle\psi^\perp| \otimes |0\rangle\langle\psi^\perp|\psi^\perp\rangle\langle 1|\\
\\
&+ |\psi^\perp\rangle\langle\psi|\psi\rangle\langle\psi| \otimes |1\rangle\langle\psi^\perp|\psi\rangle\langle 0| + |\psi^\perp\rangle\langle\psi|\psi^\perp\rangle\langle\psi| \otimes |1\rangle\langle\psi^\perp|\psi\rangle\langle 1|\\
&+ |\psi^\perp\rangle\langle\psi|\psi^\perp\rangle\langle\psi^\perp| \otimes |1\rangle\langle\psi^\perp|\psi^\perp\rangle\langle 0| + |\psi^\perp\rangle\langle\psi|\psi\rangle\langle\psi^\perp| \otimes |1\rangle\langle\psi^\perp|\psi^\perp\rangle\langle 1|\\
\\
=&|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0| + |\psi\rangle\langle\psi| \otimes |1\rangle\langle 1| + |\psi^\perp\rangle\langle\psi^\perp| \otimes |0\rangle\langle 0| + |\psi^\perp\rangle\langle\psi^\perp| \otimes |1\rangle\langle 1|
\end{aligned} \tag{8}
$$

Since $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ are all orthonormal basis, we should have

$$
|0\rangle\langle 0| + |1\rangle\langle 1| = I,\ \ |\psi\rangle\langle\psi| + |\psi^\perp\rangle\langle\psi^\perp| = I \tag{9}
$$

Therefore, from eq. (8) we know that $U^\dagger U = I$.

- To verify whether eq. (5) satisfies eq. (1a), we have

$$
\begin{aligned}
U(|\psi\rangle \otimes |0\rangle) =&|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle 0|(|\psi\rangle \otimes |0\rangle) + |\psi^\perp\rangle\langle\psi| \otimes |\psi\rangle\langle 1|(|\psi\rangle \otimes |0\rangle)\\
&+ |\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 0|(|\psi\rangle \otimes |0\rangle) + |\psi\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 1|(|\psi\rangle \otimes |0\rangle)\\
=&|\psi\rangle\langle\psi|\psi\rangle \otimes |\psi\rangle\langle 0|0\rangle + |\psi^\perp\rangle\langle\psi|\psi\rangle \otimes |\psi\rangle\langle 1|0\rangle\\
&+ |\psi^\perp\rangle\langle\psi^\perp|\psi\rangle \otimes |\psi^\perp\rangle\langle 0|0\rangle + |\psi\rangle\langle\psi^\perp|\psi\rangle \otimes |\psi^\perp\rangle\langle 1|0\rangle\\
=&|\psi\rangle \otimes |\psi\rangle
\end{aligned} \tag{10}
$$

- To verify whether eq. (5) satisfies eq. (1b), we have

$$
\begin{aligned}
U(|\psi\rangle \otimes |0\rangle) =&|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle 0|(|\psi^\perp\rangle \otimes |0\rangle) + |\psi^\perp\rangle\langle\psi| \otimes |\psi\rangle\langle 1|(|\psi^\perp\rangle \otimes |0\rangle)\\
&+ |\psi^\perp\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 0|(|\psi^\perp\rangle \otimes |0\rangle) + |\psi\rangle\langle\psi^\perp| \otimes |\psi^\perp\rangle\langle 1|(|\psi^\perp\rangle \otimes |0\rangle)\\
=&|\psi\rangle\langle\psi|\psi^\perp\rangle \otimes |\psi\rangle\langle 0|0\rangle + |\psi^\perp\rangle\langle\psi|\psi^\perp\rangle \otimes |\psi\rangle\langle 1|0\rangle\\
&+ |\psi^\perp\rangle\langle\psi^\perp|\psi^\perp\rangle \otimes |\psi^\perp\rangle\langle 0|0\rangle + |\psi\rangle\langle\psi^\perp|\psi^\perp\rangle \otimes |\psi^\perp\rangle\langle 1|0\rangle\\
=&|\psi^\perp\rangle \otimes |\psi^\perp\rangle
\end{aligned} \tag{11}
$$