Exercise 9.1.2

Suppose that we have two single qubit state ρ and σ, where

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}(I + r_x X + r_y Y + r_z Z) \tag{1a}$$

$$\sigma = \frac{1}{2}(I + \vec{s}\cdot\vec{\sigma}) = \frac{1}{2}(I + s_x X + s_y Y + s_z Z) \tag{1b}$$

We can calculate the trace distance of ρ and σ as

$$\begin{aligned}
\|\rho - \sigma\|_1 =& \left\| \frac{1}{2}[(r_x - s_x)X + (r_y - s_y)Y + (r_z - s_z)Z] \right\|_1 \\
=& \frac{1}{2}\mathrm{Tr}\left\{ \sqrt{[(r_x - s_x)X + (r_y - s_y)Y + (r_z - s_z)Z][(r_x - s_x)X + (r_y - s_y)Y + (r_z - s_z)Z]^\dagger} \right\} \\
=& \frac{1}{2}\mathrm{Tr}\left\{ \left[(r_x - s_x)^2X^2 + (r_x - s_x)(r_y - s_y)XY + (r_x - s_x)(r_z - s_z)XZ \right.\right. \\
& + (r_y - s_y)(r_x - s_x)YX + (r_y - s_y)^2Y^2 + (r_y - s_y)(r_z - s_z)YZ \\
& \left.\left. + (r_z - s_z)(r_x - s_x)ZX + (r_z - s_z)(r_y - s_y)ZY + (r_z - s_z)^2Z^2\right]^{1/2} \right\} \\
=& \frac{1}{2}\mathrm{Tr}\left\{ \left[\left[(r_x - s_x)^2 + (r_y - s_y)^2 + (r_z - s_z)^2\right] I\right]^{1/2} \right\} \\
=& \frac{1}{2}\mathrm{Tr}\left\{ \left[\left[(r_x - s_x)^2 + (r_y - s_y)^2 + (r_z - s_z)^2\right]^{1/2} I\right] \right\} \\
=& \frac{1}{2}\cdot 2\left[(r_x - s_x)^2 + (r_y - s_y)^2 + (r_z - s_z)^2\right]^{1/2} = \|\vec{r} - \vec{s}\|_2
\end{aligned} \tag{2}$$

where we use the information in the appendix to simplify the fourth identity.

Appendix

Here is some relations that are useful to simplify $\sqrt{MM^\dagger}$ in the fourth identity of eq. (2).

- For XY and YX, we have

$$\begin{aligned}
XY &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ \\
YX &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -iZ
\end{aligned} \tag{A1}$$

- For XZ and ZX, we have

$$\begin{aligned}
XZ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -iY \\
ZX &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = iY
\end{aligned} \tag{A2}$$

- For YZ and ZY, we have

$$\begin{aligned}
YZ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = iX \\
ZY &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -iX
\end{aligned} \tag{A3}$$

Substitute eq. $(\mathrm{A1}) - (\mathrm{A3})$ to eq. (2), we find that only X^2, Y^2 and Z^2 remain in the third lines of eq. (2)