Exercise 5.2.2

Suppose that we have isometry

$$
\begin{aligned}
V_{AE}|\psi\rangle_A|0\rangle_E &= \sqrt{1-p}|\psi_A\rangle|0\rangle_E + \sqrt{p}(X|\psi_A\rangle)|1\rangle_E && \text{(1a)} \\
V_{AE}|\psi\rangle_A|1\rangle_E &= \sqrt{p}|\psi_A\rangle|0\rangle_E - \sqrt{1-p}(X|\psi_A\rangle)|1\rangle_E && \text{(1b)}
\end{aligned}
$$

From eq. $(1a) - (1b)$, we could get the following equations,

$$
\begin{aligned}
V_{AE}|00\rangle_{AE} &= \sqrt{1-p}|00\rangle_{AE} + \sqrt{p}|11\rangle_{AE} \\
V_{AE}|01\rangle_{AE} &= \sqrt{p}|00\rangle_{AE} - \sqrt{1-p}|11\rangle_{AE} \\
V_{AE}|10\rangle_{AE} &= \sqrt{1-p}|10\rangle_{AE} + \sqrt{p}|01\rangle_{AE} \\
V_{AE}|11\rangle_{AE} &= \sqrt{p}|10\rangle_{AE} - \sqrt{1-p}|01\rangle_{AE}
\end{aligned} \tag{2}
$$

We can simply check that

$$
\begin{aligned}
\langle 00|_{AE}V_{AE}^{\dagger}V_{AE}|00\rangle_{AE} &= (\sqrt{1-p}\langle 00|_{AE} + \sqrt{p}\langle 11|_{AE})(\sqrt{1-p}|00\rangle_{AE} + \sqrt{p}|11\rangle_{AE}) \\
&= (1-p)\langle 00|00\rangle + p\langle 11|11\rangle = 1
\end{aligned} \tag{3}
$$

Note that $\langle 00|00\rangle = 1 = \langle 00|_{AE}V_{AE}^{\dagger}V_{AE}|00\rangle_{AE}$, so we should have $V_{AE}^{\dagger}V_{AE} = I$ and V_{AE} is unitary. To check whether four output in eq. (2) are othonormal basis, we could check

$$
\begin{aligned}
\langle 00|_{AE}V_{AE}^{\dagger}V_{AE}|00\rangle_{AE} &= (\sqrt{1-p}\langle 00|_{AE} + \sqrt{p}\langle 11|_{AE})(\sqrt{1-p}|00\rangle_{AE} + \sqrt{p}|11\rangle_{AE}) \\
&= (1-p)\langle 00|00\rangle + p\langle 11|11\rangle = 1 \\
\langle 01|_{AE}V_{AE}^{\dagger}V_{AE}|01\rangle_{AE} &= (\sqrt{p}\langle 00|_{AE} - \sqrt{1-p}\langle 11|_{AE})(\sqrt{p}|00\rangle_{AE} - \sqrt{1-p}|11\rangle_{AE}) \\
&= p\langle 00|00\rangle + (1-p)\langle 11|11\rangle = 1 \\
\langle 10|_{AE}V_{AE}^{\dagger}V_{AE}|10\rangle_{AE} &= (\sqrt{1-p}\langle 10|_{AE} + \sqrt{p}\langle 01|_{AE})(\sqrt{1-p}|10\rangle_{AE} + \sqrt{p}|01\rangle_{AE}) \\
&= (1-p)\langle 10|10\rangle + p\langle 01|01\rangle = 1 \\
\langle 11|_{AE}V_{AE}^{\dagger}V_{AE}|11\rangle_{AE} &= (\sqrt{p}\langle 10|_{AE} - \sqrt{1-p}\langle 01|_{AE})(\sqrt{p}|10\rangle_{AE} - \sqrt{1-p}|01\rangle_{AE}) \\
&= p\langle 10|10\rangle + (1-p)\langle 01|01\rangle = 1
\end{aligned} \tag{4}
$$

and also,

$$
\begin{aligned}
\langle 01|_{AE}V_{AE}^{\dagger}V_{AE}|00\rangle_{AE} &= (\sqrt{p}\langle 00|_{AE} - \sqrt{1-p}\langle 11|_{AE})(\sqrt{1-p}|00\rangle_{AE} + \sqrt{p}|11\rangle_{AE}) \\
&= \sqrt{p(1-p)}\langle 00|00\rangle - \sqrt{p(1-p)}\langle 11|11\rangle = 0 \\
\langle 10|_{AE}V_{AE}^{\dagger}V_{AE}|00\rangle_{AE} &= (\sqrt{1-p}\langle 10|_{AE} + \sqrt{p}\langle 01|_{AE})(\sqrt{1-p}|00\rangle_{AE} + \sqrt{p}|11\rangle_{AE}) = 0 \\
\langle 11|_{AE}V_{AE}^{\dagger}V_{AE}|00\rangle_{AE} &= (\sqrt{p}\langle 10|_{AE} - \sqrt{1-p}\langle 01|_{AE})(\sqrt{1-p}|00\rangle_{AE} + \sqrt{p}|11\rangle_{AE}) = 0 \\
\langle 10|_{AE}V_{AE}^{\dagger}V_{AE}|01\rangle_{AE} &= (\sqrt{1-p}\langle 10|_{AE} + \sqrt{p}\langle 01|_{AE})(\sqrt{p}|00\rangle_{AE} - \sqrt{1-p}|11\rangle_{AE}) = 0 \\
\langle 11|_{AE}V_{AE}^{\dagger}V_{AE}|01\rangle_{AE} &= (\sqrt{p}\langle 10|_{AE} - \sqrt{1-p}\langle 01|_{AE})(\sqrt{p}|00\rangle_{AE} - \sqrt{1-p}|11\rangle_{AE}) = 0 \\
\langle 11|_{AE}V_{AE}^{\dagger}V_{AE}|10\rangle_{AE} &= (\sqrt{p}\langle 10|_{AE} - \sqrt{1-p}\langle 01|_{AE})(\sqrt{1-p}|10\rangle_{AE} + \sqrt{p}|01\rangle_{AE}) \\
&= \sqrt{p(1-p)}\langle 10|10\rangle - \sqrt{p(1-p)}\langle 01|01\rangle = 0
\end{aligned} \tag{5}
$$

From eq. $(4) - (5)$, we conclude that four outputs in eq. (2) form an othonormal basis.

Appendix

There is another way to prove V_{AE} is unitary. Suppose that the qubit-environment system starts initially with $|\psi\rangle_A|0\rangle_E$, and the interaction between qubit and environment can be described by an isometry V_{AE} such that

$$V_{AE}|\psi\rangle_A|0\rangle_E = \sqrt{1-p}|\psi_A\rangle|0\rangle_E + \sqrt{p}(X|\psi_A\rangle)|1\rangle_E \tag{A1a}$$

$$V_{AE}|\psi\rangle_A|1\rangle_E = \sqrt{p}|\psi_A\rangle|0\rangle_E - \sqrt{1-p}(X|\psi_A\rangle)|1\rangle_E \tag{A1b}$$

In this case, we need to check whether V_{AE} is unitary. We can do the following

$$\begin{aligned}\langle\psi|_A\langle 0|_E V_{AE}^\dagger V_{AE}|\psi\rangle_A|0\rangle_E =&\left[\sqrt{1-p}\langle\psi_A|\langle 0|_E + \sqrt{p}(\langle\psi_A|X^\dagger)\langle 1|_E\right]\left[\sqrt{1-p}|\psi_A\rangle|0\rangle_E + \sqrt{p}(X|\psi_A\rangle)|1\rangle_E\right]\\ =&(1-p)\langle\psi_A|\psi_A\rangle\langle 0|0\rangle_E + \sqrt{p(1-p)}\langle\psi_A|X|\psi_A\rangle\langle 0|1\rangle_E\\ &+\sqrt{p(1-p)}\langle\psi_A|X^\dagger|\psi_A\rangle\langle 1|0\rangle_E + p\langle\psi_A|X^\dagger X|\psi_A\rangle\langle 1|1\rangle_E\\ =&(1-p)\langle\psi_A|\psi_A\rangle + p\langle\psi_A|X^\dagger X|\psi_A\rangle = 1\end{aligned} \tag{A2}$$

Note also that

$$\langle\psi_A|\langle 0|_E|\psi\rangle_A|0\rangle_E = \langle\psi_A|\psi_A\rangle\langle 0|0\rangle_E = 1 \tag{A3}$$

So $||V_{AE}(|\psi_A\rangle|0\rangle|)| = ||(|\psi_A\rangle|0\rangle)||$ is norm preserving and it should be unitary (from exer 3.3.1).