Exercise 2.2.1

Suppose that for message $m \in [M]$ and a certain code $\mathcal{C}$, we have following relation

$$\bar{p}_e = \frac{1}{M} \sum_m p_e(m, \mathcal{C}) \leq \epsilon \tag{1}$$

Let random variable $X = \{p_e(m, \mathcal{C})\}$ with size M, the expectation value of random variable X is given by

$$E(X) = \frac{1}{M} \sum_m p_e(m, \mathcal{C}) = \bar{p}_e \leq \epsilon \tag{2}$$

In this problem, we need to prove that whether there is at least half of message m with $p_e \leq 2\epsilon$, that is, we need to prove that the probability of getting $X \leq 2\epsilon$ is at least $1/2$. According to the Markov's inequality, we have

$$P(X \geq 2\epsilon) \leq \frac{E(X)}{2\epsilon} = \frac{1}{2} \tag{3}$$

Equivalently, we have

$$P(X \leq 2\epsilon) = 1 - P(X \geq 2\epsilon) \geq \frac{1}{2} \tag{4}$$

Namely, we conclude from eq. (4) that if we have eq. (1), we should have at least half of the message m satisfies the condition $p_e(m, \mathcal{C}) \leq 2\epsilon$.