Exercise 3.3.5

The Pauli matrices are given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

For commutator part, we have

- The Pauli matrices commute with itself, that is

$$[X, X] = XX - XX = 0 \tag{2a}$$
$$[Y, Y] = YY - YY = 0 \tag{2b}$$
$$[Z, Z] = ZZ - ZZ = 0 \tag{2c}$$

- The Pauli matrices does not commute with other Pauli matrices, that is,
  - For $[X, Y]$, we have

$$\begin{aligned} [X, Y] = XY - YX &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2iZ \end{aligned} \tag{3}$$

  - For $[Y, Z]$, we have

$$\begin{aligned} [Y, Z] = YZ - ZY &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\ &= 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2iX \end{aligned} \tag{4}$$

  - For $[Z, X]$, we have

$$\begin{aligned} [Z, X] = ZX - XZ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2iY \end{aligned} \tag{5}$$

For anticommute part, we have

- The Pauli matrices does not anticommute with itself, that is

$$\{X, X\} = XX + XX = 2I \tag{6a}$$
$$\{Y, Y\} = YY + YY = 2I \tag{6b}$$
$$\{Z, Z\} = ZZ + ZZ = 2I \tag{6c}$$

since X, Y, Z square to the identity (exercise 3.3.3).

- The Pauli matrices anticommute with other Pauli matrices, that is,
  - For $\{X, Y\}$, we have

$$\begin{aligned} \{X, Y\} = \{Y, X\} = XY + YX \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0 \end{aligned} \tag{7}$$

  - For $\{Y, Z\}$, we have

$$\begin{aligned} \{Y, Z\} = \{Z, Y\} = YZ + ZY \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0 \end{aligned} \tag{8}$$

  - For $\{Z, X\}$, we have

$$\begin{aligned} \{Z, X\} = \{X, Z\} = ZX + XZ \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 \end{aligned} \tag{9}$$

Therefore, from eq. $(2a) - (2c)$ and eq. $(7) - (9)$, we conclude that the Pauli matrices either commute or anticommute, that is,

$$[X, X] = [Y, Y] = [Z, Z] = 0 \tag{10a}$$
$$\{X, Y\} = \{Y, X\} = \{Y, Z\} = \{Z, Y\} = \{Z, X\} = \{X, Z\} = 0 \tag{10b}$$