Exercise 4.1.3

The unnormalized maximally entagled vector is defined by

$$|\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} |i\rangle_R \otimes |i\rangle_S \tag{1}$$

where d is the dimension of vector space $R \otimes S$. $|i\rangle_R$ and $|i\rangle_S$ are orthonormal basis in vector space R and S, respectively. We need to use the properties of tensor product to prove the statement. The definition of operator $A \otimes B$ on vector space $R \otimes S$ is given by

$$(A \otimes B) \sum_i |r_i\rangle \otimes |s_i\rangle = \sum_i A|r_i\rangle \otimes B|s_i\rangle \tag{2}$$

where $|r_i\rangle$and $|s_i\rangle$ are arbitrary vectors in vector space R and S, respectively. Therefore,

$$\begin{aligned}
\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} &= \langle\Gamma|_{RS} I_R \otimes A_S \left(\sum_{i=0}^{d-1} |i\rangle_R \otimes |i\rangle_S\right) \\
&= \langle\Gamma|_{RS} \left(\sum_{i=0}^{d-1} I_R|i\rangle_R \otimes A_S|i\rangle_S\right) \\
&= \left(\sum_{j=0}^{d-1} \langle j|_R \otimes \langle j|_S\right) \left(\sum_{i=0}^{d-1} |i\rangle_R \otimes A_S|i\rangle_S\right)
\end{aligned} \tag{3}$$

where we use i and j to clarify contribution from $|\Gamma\rangle_{RS}$ and $\langle\Gamma|_{RS}$. Note also that for the inner product of $R \otimes S$, we have

$$\left(\sum_i a_i|r_i\rangle \otimes |s_i\rangle, \sum_j b_j|r'_j\rangle \otimes |s'_j\rangle\right) = \left(\sum_i a_i^* \langle r_i| \otimes \langle s_i|\right) \left(\sum_j b_j|r'_j\rangle \otimes |s'_j\rangle\right) = \sum_{i,j} a_i^* b_j \langle r_i|r'_j\rangle \otimes \langle s_i|s'_j\rangle \tag{4}$$

From eq. (4), we can simplify eq. (3) and get

$$\begin{aligned}
\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} &= \sum_{i,j} \langle j|i\rangle_R \langle j|A_S|i\rangle_S \\
&= \sum_i \langle i|i\rangle_R \langle i|A_S|i\rangle_S \\
&= \sum_i \langle i|A_S|i\rangle_S \\
&= \mathrm{Tr}\{A_S\}
\end{aligned} \tag{5}$$

where the second and third identity comes from the orthonormality of $|i\rangle_R$.