Exercise 6.2.6

In the noisy teleportation case, if Alice prepares $|\Phi^+\rangle$ and sends one share to Bob via dephasing channel $\mathcal{N}(\rho)$, where

$$\mathcal{N}(\rho) = (1-p)I\rho I + pZ\rho Z \tag{1}$$

then when Alice and Bob performs usual teleportation protocol, the joint state of system A', A and B becomes

$$|\psi\rangle_{A'}\langle\psi|_{A'} \otimes (I_A \otimes \mathcal{N}_B)(|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}) \tag{2}$$

From eq. (1) we notice that $\mathcal{N}$ can be written in Kraus representation form, according to appendix, we can write the map $I_A \otimes \mathcal{N}_B$ as

$$\begin{aligned}(I_A \otimes \mathcal{N}_B)(\rho) = &\sum_\alpha (I_A \otimes K_{B,\alpha})\rho(I_A \otimes K_{B,\alpha})^\dagger \\ &= (1-p)(I_A \otimes I_B)\rho(I_A \otimes I_B) + p(I_A \otimes Z_B)\rho(I_A \otimes Z_B) \\ &= (1-p)\rho + p(I_A \otimes Z_B)\rho(I_A \otimes Z_B)\end{aligned} \tag{3}$$

We can use eq. (3) to re-write eq. (2) as

$$\begin{aligned}|\psi\rangle_{A'}\langle\psi|_{A'} \otimes (I_A \otimes \mathcal{N}_B)(|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}) =&(1-p)|\psi\rangle_{A'}\langle\psi|_{A'} \otimes |\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} \\ &+ p|\psi\rangle_{A'}\langle\psi|_{A'} \otimes (I_A \otimes Z_B)|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}(I_A \otimes Z_B) \\ =&(1-p)|\psi\rangle_{A'}\langle\psi|_{A'} \otimes |\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} \\ &+ p|\psi\rangle_{A'}\langle\psi|_{A'} \otimes |\Phi^-\rangle_{AB}\langle\Phi^-|_{AB}\end{aligned} \tag{4}$$

The first term of eq. (4) is the noiseless teleportation, and we can write it in following way,

$$|\psi\rangle_{A'}|\Phi^+\rangle_{AB} = |\Phi^+\rangle_{A'A}|\psi\rangle_B + |\Phi^-\rangle_{A'A}Z|\psi\rangle_B + |\Psi^+\rangle_{A'A}X|\psi\rangle_B + |\Psi^-\rangle_{A'A}XZ|\psi\rangle_B \tag{5}$$

For the second term of eq. (4), we could follow the analysis of noiseless teleportation, and re-write the joint state in the following way,

$$\begin{aligned}|\psi\rangle_{A'}|\Phi^-\rangle_{AB} =&(\alpha|0\rangle_{A'} + \beta|1\rangle_{A'})\left(\frac{|00\rangle_{AB} - |11\rangle_{AB}}{\sqrt{2}}\right) \\ =&\frac{1}{\sqrt{2}}(\alpha|000\rangle_{A'AB} - \alpha|011\rangle_{A'AB} + \beta|100\rangle_{A'AB} - \beta|111\rangle_{A'AB})\end{aligned} \tag{6}$$

Note that we can express two qubits state with Bell basis,

$$\begin{aligned}|00\rangle_{A'A} &= \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{A'A} + |\Phi^-\rangle_{A'A}) \\ |01\rangle_{A'A} &= \frac{1}{\sqrt{2}}(|\Psi^+\rangle_{A'A} + |\Psi^-\rangle_{A'A}) \\ |10\rangle_{A'A} &= \frac{1}{\sqrt{2}}(|\Psi^+\rangle_{A'A} - |\Psi^-\rangle_{A'A}) \\ |11\rangle_{A'A} &= \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{A'A} - |\Phi^-\rangle_{A'A})\end{aligned} \tag{7}$$

Substitute eq. (7) into eq. (6), we have

$$
\begin{aligned}
|\psi\rangle_{A'}|\Phi^-\rangle_{AB} =& \frac{1}{\sqrt{2}}(\alpha|000\rangle_{A'AB} - \alpha|011\rangle_{A'AB} + \beta|100\rangle_{A'AB} - \beta|111\rangle_{A'AB}) \\
=& \big[\alpha(|\Phi^+\rangle_{A'A} + |\Phi^-\rangle_{A'A})|0\rangle_B - \alpha(|\Psi^+\rangle_{A'A} + |\Psi^-\rangle_{A'A})|1\rangle_B \\
& + \beta(|\Psi^+\rangle_{A'A} - |\Psi^-\rangle_{A'A})|0\rangle_B - \beta(|\Phi^+\rangle_{A'A} - |\Phi^-\rangle_{A'A})|1\rangle_B\big] \\
=& |\Phi^+\rangle_{A'A}(\alpha|0\rangle_B - \beta|1\rangle_B) + |\Phi^-\rangle_{A'A}(\alpha|0\rangle_B + \beta|1\rangle_B) \\
& - |\Psi^+\rangle_{A'A}(\alpha|1\rangle_B - \beta|0\rangle_B) - |\Psi^-\rangle_{A'A}(\alpha|1\rangle_B + \beta|0\rangle_B) \\
=& |\Phi^+\rangle_{A'A}Z|\psi\rangle_B + |\Phi^-\rangle_{A'A}ZZ|\psi\rangle_B - |\Psi^+\rangle_{A'A}XZ|\psi\rangle_B - |\Psi^-\rangle_{A'A}XZZ|\psi\rangle_B
\end{aligned} \tag{8}
$$

From eq. (4), (5) and eq. (8), we know that after Alice measures A' and A and transmit two classcial bits to Bob for recovery, Bob will finally get a mixed state,

$$
\rho_B = (1-p)|\psi\rangle_B\langle\psi|_B + pZ|\psi\rangle_B\langle\psi|_B Z \tag{9}
$$

which is equivalent as sending Bob $|\psi\rangle$ directly from the dephasing channel.

Appendix

Given a CPTP map $\mathcal{N}(\rho)$, we can write it in the form of Kraus operator sum representation,

$$
\mathcal{N}(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger \tag{A1}
$$

Then for the map $(I_A \otimes \mathcal{N}_B)\rho_{AB}$, if $\rho_{AB} = \rho_A \otimes \rho_B$, we have

$$
\begin{aligned}
(I_A \otimes \mathcal{N}_B)\rho_{AB} &= (I_A \otimes \mathcal{N}_B)(\rho_A \otimes \rho_B) \\
&= I_A\rho_A \otimes \mathcal{N}_B(\rho_B) \\
&= I_A\rho_A I_A \otimes \sum_\alpha K_\alpha \rho_B K_\alpha^\dagger \\
&= \sum_\alpha (I_A \otimes K_\alpha)\rho_{AB}(I_A \otimes K_\alpha)^\dagger
\end{aligned} \tag{A2}
$$