Exercise 9.2.2

Similar with the hypothesis-tesing scenario in the textbook for trace norm, we can do the following to get the probability of passing the test:

- Suppose that Bob has a pure state $|\varphi\rangle$ and Alice has a binary POVM with elements $\Lambda = \{\Lambda_0 = |\varphi\rangle\langle\varphi|, \Lambda_1 = I - |\varphi\rangle\langle\varphi|\}$ with measurement outcome as φ and $I - \varphi$, respectively. If Alice gets φ from her measurement, then we say it passes the test; if Alice gets $I - \varphi$ from her measurement, then we say the test failed.

Following the above rules, the probability of Alice getting φ is

$$p(\varphi) = \mathrm{Tr}\{\Lambda_0 \sigma\} = \mathrm{Tr}\{|\varphi\rangle\langle\varphi|\sigma\} \tag{1}$$

If we write $|\varphi\rangle = \sum_j \varphi_j |j\rangle$ and σ into the form of spectral decomposition $\sigma = \sum_x p_X(x)|\phi_x\rangle\langle\phi_x|$, eq. (1) becomes

$$\begin{aligned}
p(\varphi) = \mathrm{Tr}\{|\varphi\rangle\langle\varphi|\sigma\} &= \sum_i \langle i| \left(|\varphi\rangle\langle\varphi|\sigma\right)|i\rangle \\
&= \sum_i \langle i| \left[\left(\sum_j \varphi_j |j\rangle\right)\left(\sum_k \varphi_k^* \langle k|\right)\left(\sum_x p_X(x)|\phi_x\rangle\langle\phi_x|\right)\right]|i\rangle \\
&= \sum_{i,j,k,x} p_X(x)\varphi_j\varphi_k^* \langle i|j\rangle\langle k|\phi_x\rangle\langle\phi_x|i\rangle \\
&= \sum_{j,k,x} p_X(x)\varphi_j\varphi_k^* \langle j|j\rangle\langle k|\phi_x\rangle\langle\phi_x|j\rangle \\
&= \left(\sum_k \varphi_k^* \langle k|\right)\left(\sum_x p_X(x)|\phi_x\rangle\langle\phi_x|\right)\left(\sum_j \varphi_j |j\rangle\right) \\
&= \langle\varphi|\sigma|\varphi\rangle
\end{aligned} \tag{2}$$

where $\{|i\rangle\}$ is some orthonormal basis for $\mathcal{H}$. From eq. (2) we conclude that the probability of passing the test is equal to the fidelity $F(\varphi, \sigma)$.