Exercise 4.1.11

For a single qubit state

$$|\psi\rangle \equiv \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \tag{1}$$

according to the textbook, the density matrix of $|\psi\rangle$ is given by

$$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z) \tag{2}$$

where $r_x = \sin\theta\cos\varphi$, $r_y = \sin\theta\sin\varphi$ and $r_z = \cos\theta$. If we want to compute r_x, r_y and r_z via $\mathrm{Tr}\{X\rho\}, \mathrm{Tr}\{Y\rho\}, \mathrm{Tr}\{Z\rho\}$, we need to use the result from Exercise 3.3.6 that $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$ where $\sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z$. From the statement above and eq. (2), we have

$$\begin{aligned}
\mathrm{Tr}\{X\rho\} &= \mathrm{Tr}\left\{X\left[\frac{1}{2}(I + r_x X + r_y Y + r_z Z)\right]\right\} \\
&= \frac{1}{2}[\mathrm{Tr}\{XI\} + r_x\mathrm{Tr}\{XX\} + r_y\mathrm{Tr}\{XY\} + r_z\mathrm{Tr}\{XZ\}] \\
&= \frac{1}{2}[2r_x] = r_x \\
\mathrm{Tr}\{Y\rho\} &= \mathrm{Tr}\left\{Y\left[\frac{1}{2}(I + r_x X + r_y Y + r_z Z)\right]\right\} \\
&= \frac{1}{2}[\mathrm{Tr}\{YI\} + r_x\mathrm{Tr}\{YX\} + r_y\mathrm{Tr}\{YY\} + r_z\mathrm{Tr}\{YZ\}] \\
&= \frac{1}{2}[2r_y] = r_y \\
\mathrm{Tr}\{Z\rho\} &= \mathrm{Tr}\left\{Z\left[\frac{1}{2}(I + r_x X + r_y Y + r_z Z)\right]\right\} \\
&= \frac{1}{2}[\mathrm{Tr}\{ZI\} + r_x\mathrm{Tr}\{ZX\} + r_y\mathrm{Tr}\{ZY\} + r_z\mathrm{Tr}\{ZZ\}] \\
&= \frac{1}{2}[2r_z] = r_z
\end{aligned} \tag{3}$$