Exercise 3.3.6

If we label Pauli matrices as

$$\sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \tag{1}$$

we need to verify $\mathrm{Tr}\{\sigma_i\sigma_j\} = 2\delta_{ij}$.

- For $i = j$, we should verify $\mathrm{Tr}\{\sigma_i\sigma_j\} = 2$. Since $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are Hermitian and unitary operator, we do have

  $$\sigma_0\sigma_0 = \sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = I \tag{2}$$

  and thus,

  $$\mathrm{Tr}(\sigma_0\sigma_0) = \mathrm{Tr}(\sigma_1\sigma_1) = \mathrm{Tr}(\sigma_2\sigma_2) = \mathrm{Tr}(\sigma_3\sigma_3) = \mathrm{Tr} I = 2 \tag{3}$$

- For $i \neq j$, we should verify $\mathrm{Tr}\{\sigma_i\sigma_j\} = 0$.
  - For $\mathrm{Tr}\{\sigma_0\sigma_1\}$ and $\mathrm{Tr}\{\sigma_1\sigma_0\}$,

    $$\mathrm{Tr}\{\sigma_0\sigma_1\} = \mathrm{Tr}\{\sigma_1\sigma_0\} = \mathrm{Tr}(\sigma_1) = \mathrm{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 \tag{4}$$

  - For $\mathrm{Tr}\{\sigma_0\sigma_2\}$ and $\mathrm{Tr}\{\sigma_2\sigma_0\}$,

    $$\mathrm{Tr}\{\sigma_0\sigma_2\} = \mathrm{Tr}\{\sigma_2\sigma_0\} = \mathrm{Tr}(\sigma_2) = \mathrm{Tr}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0 \tag{5}$$

  - For $\mathrm{Tr}\{\sigma_0\sigma_3\}$ and $\mathrm{Tr}\{\sigma_3\sigma_0\}$,

    $$\mathrm{Tr}\{\sigma_0\sigma_3\} = \mathrm{Tr}\{\sigma_3\sigma_0\} = \mathrm{Tr}(\sigma_3) = \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 \tag{6}$$

  - For $\mathrm{Tr}\{\sigma_1\sigma_2\}$,

    $$\mathrm{Tr}\{\sigma_1\sigma_2\} = \mathrm{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 0 \tag{7}$$

  - For $\mathrm{Tr}\{\sigma_1\sigma_3\}$,

    $$\mathrm{Tr}\{\sigma_1\sigma_3\} = \mathrm{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 \tag{8}$$

  - For $\mathrm{Tr}\{\sigma_2\sigma_1\}$,

    $$\mathrm{Tr}\{\sigma_2\sigma_1\} = \mathrm{Tr}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0 \tag{9}$$

  - For $\mathrm{Tr}\{\sigma_2\sigma_3\}$,

$$\mathrm{Tr}\{\sigma_2\sigma_3\} = \mathrm{Tr}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = 0 \tag{10}$$

- For $\mathrm{Tr}\{\sigma_3\sigma_1\}$,

$$\mathrm{Tr}\{\sigma_3\sigma_1\} = \mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = 0 \tag{11}$$

- For $\mathrm{Tr}\{\sigma_3\sigma_2\}$,

$$\mathrm{Tr}\{\sigma_3\sigma_2\} = \mathrm{Tr}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = 0 \tag{12}$$