Exercise 3.5.8

Following the proof of no-cloning theorem, we can also prove the no-deletion theorem. Suppose for a contradiction that there exists an operator U such that

$$\begin{aligned} U|\psi\rangle|\psi\rangle|A\rangle &= |\psi\rangle|0\rangle|A'\rangle \\ &= (\alpha|0\rangle + \beta|1\rangle)|0\rangle|A'\rangle \end{aligned} \tag{1}$$

Since the deletion is universal, it also deletes the state $|0\rangle$ and $|1\rangle$,

$$\begin{aligned} U|0\rangle|0\rangle|A\rangle &= |0\rangle|0\rangle|A_0\rangle \\ U|1\rangle|1\rangle|A\rangle &= |1\rangle|0\rangle|A_1\rangle \end{aligned} \tag{2}$$

Due to the linearity of quantum theory, from eq. (2) we then have for a superposition state $\alpha|0\rangle + \beta|1\rangle$ as

$$\begin{aligned} U|\psi\rangle|\psi\rangle|A\rangle &= U(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)|A\rangle \\ &= U(\alpha^2|0\rangle|0\rangle + \alpha\beta|0\rangle|1\rangle + \alpha\beta|1\rangle|0\rangle + \beta^2|1\rangle|1\rangle)|A\rangle \\ &= \alpha^2 U|0\rangle|0\rangle|A\rangle + \alpha\beta U|0\rangle|1\rangle|A\rangle + \alpha\beta U|1\rangle|0\rangle|A\rangle + \beta^2 U|1\rangle|1\rangle|A\rangle \\ &= \alpha^2|0\rangle|0\rangle|A_0\rangle + \alpha\beta U|0\rangle|1\rangle|A\rangle + \alpha\beta U|1\rangle|0\rangle|A\rangle + \beta^2|1\rangle|0\rangle|A_1\rangle \end{aligned} \tag{3}$$

Compare eq. (1) and eq. (3), we find that these two equations will not equal for all cases, since we have in general case,

$$\alpha^2|0\rangle|0\rangle|A_0\rangle + \alpha\beta U|0\rangle|1\rangle|A\rangle + \alpha\beta U|1\rangle|0\rangle|A\rangle + \beta^2|1\rangle|0\rangle|A_1\rangle \neq \alpha|0\rangle|0\rangle|A'\rangle + \beta|1\rangle|0\rangle|A'\rangle \tag{4}$$

Thus, our result contradicts the existence of a universal quantum deletion.