Exercise 9.1.7

If we modify the rule of hypothesis-tesing scenario in the textbook and it becomes the following steps:

- Bob has probability $p_X(0) = p_0$ to prepare ρ_0 and probability $p_X(1) = p_1$ to prepare ρ_1. He sends his state to Alice and Alice will guess which state Bob has prepared
- Alice has a binary POVM with elements $\Lambda = \{\Lambda_0, \Lambda_1\}$ with measurement outcome as 0 and 1, respectively. If Alice gets 0 from her measurement, she will guess the state that Bob prepared is ρ_0; if Alice gets 1 from her measurement, she will guess the state that Bob prepared is ρ_1.

Following the above rules, the probability of Alice getting outcome 0 and 1 are shown below,

$$\begin{aligned} p_Y(0) &= p_{Y|X}(0|0)p_X(0) + p_{Y|X}(0|1)p_X(1) \\ p_Y(1) &= p_{Y|X}(1|0)p_X(0) + p_{Y|X}(1|1)p_X(1) \end{aligned} \tag{1}$$

Note that here $p_{Y|X}(i|j)p_X(j)$ means the probability of getting i in Alice measurement if Bob prepares ρ_j. From eq. (1) we can see the probability of making correct guess is

$$\begin{aligned} p_{\text{succ}}(\Lambda) &= p_{Y|X}(0|0)p_X(0) + p_{Y|X}(1|1)p_X(1) \\ &= p_0 \text{Tr}\{\Lambda_0\rho_0\} + p_1 \text{Tr}\{\Lambda_1\rho_1\} \end{aligned} \tag{2}$$

Note that for POVM operators we have $\Lambda_0 + \Lambda_1 = I$, so eq. (2) becomes

$$\begin{aligned} p_{\text{succ}}(\Lambda_0) = &\; p_0 \text{Tr}\{\Lambda_0\rho_0\} + p_1 \text{Tr}\{\Lambda_1\rho_1\} \\ &= p_0 \text{Tr}\{\Lambda_0\rho_0\} + p_1 \text{Tr}\{(I - \Lambda_0)\rho_1\} \\ &= p_0 \text{Tr}\{\Lambda_0\rho_0\} + p_1 \text{Tr}\{\rho_1\} - p_1 \text{Tr}\{\Lambda_0\rho_1\} \\ &= p_1 + \text{Tr}\{\Lambda_0(p_0\rho_0 - p_1\rho_1)\} \end{aligned} \tag{3}$$

Now Alice has freedom to choose the POVM such that $p_{\text{succ}}(\Lambda_0)$ can be maximized. If we define the success probability with respect to all measurement as follows:

$$p_{\text{succ}} \equiv \max_{0 \leq \Lambda_0 \leq I} (p_1 + \text{Tr}\{\Lambda_0(p_0\rho_0 - p_1\rho_1)\}) = p_1 + \max_{0 \leq \Lambda_0 \leq I} \text{Tr}\{\Lambda_0(p_0\rho_0 - p_1\rho_1)\} \tag{4}$$

In order to find the maximum of the second term in eq. (4), we need to follow the proof of Lemma 9.1.1 in the textbook. Let us consider the difference operator $p_0\rho_0 - p_1\rho_1$ directly. Since ρ_0 and ρ_1 are density matrix, we have $\rho_0^\dagger = \rho_0$ and $\rho_1^\dagger = \rho_1$, and

$$\begin{aligned} (p_0\rho_0 - p_1\rho_1)^\dagger(p_0\rho_0 - p_1\rho_1) &= (p_0\rho_0^\dagger - p_1\rho_1^\dagger)(p_0\rho_0 - p_1\rho_1) \\ &= (p_0\rho_0 - p_1\rho_1)(p_0\rho_0^\dagger - p_1\rho_1^\dagger) \\ &= (p_0\rho_0 - p_1\rho_1)(p_0\rho_0 - p_1\rho_1)^\dagger \end{aligned} \tag{5}$$

Then we find that $p_0\rho_0 - p_1\rho_1$ is a normal operator and we could decompose $p_0\rho_0 - p_1\rho_1$ into diagonalized form,

$$p_0\rho_0 - p_1\rho_1 = \sum_i (p_0\mu_i - p_1\nu_i)|i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i| \tag{6}$$

where $\{|i\rangle\}$ is an orthonormal basis. Note that $\{|i\rangle\}$ is a set of eigenvector of $p_0\rho_0 - p_1\rho_1$ but do not have to be the eigenvector of ρ_0 and ρ_1. Let us define two operator as

$$P=\sum_{i:\lambda_i\geq 0}\lambda_i|i\rangle\langle i|, Q=\sum_{i:\lambda<0}\lambda_i|i\rangle\langle i| \tag{7}$$

From the definition of P and Q, and the diagonalized form of $p_0\rho_0-p_1\rho_1$, we could find $p_0\rho_0-p_1\rho_1=P-Q$ and,

$$\begin{aligned}|p_0\rho_0-p_1\rho_1| &= \sqrt{(p_0\rho_0-p_1\rho_1)^\dagger(p_0\rho_0-p_1\rho_1)} \\ &= \sqrt{\sum_i|\lambda_i|^2|i\rangle\langle i|}=\sum_i|\lambda_i||i\rangle\langle i|=P+Q\end{aligned} \tag{8}$$

Therefore, we have the trace norm of $p_0\rho_0-p_1\rho_1$ as

$$\|p_0\rho_0-p_1\rho_1\|_1=\mathrm{Tr}\{|p_0\rho_0-p_1\rho_1|\}=\mathrm{Tr}\{P\}+\mathrm{Tr}\{Q\} \tag{9}$$

We can also find the relation of P and Q and $p_0\rho_0-p_1\rho_1$ as below,

$$\mathrm{Tr}(P-Q)=\mathrm{Tr}\{p_0\rho_0-p_1\rho_1\}=p_0-p_1 \tag{10}$$

Note that eq. (10) is the main difference from Lemma 9.1.1. From eq. (9) and (10), we have

$$\|p_0\rho_0-p_1\rho_1\|_1=2\mathrm{Tr}\{P\}+2p_1-1 \tag{11}$$

Here we consider a projection operator $\Pi_P=\sum_{i:\lambda_i\geq 0}|i\rangle\langle i|\leq I$ and check Π_P is the operator that maxmizes $\Lambda_0(p_0\rho_0-p_1\rho_1)$. From the definition, we find that

$$\mathrm{Tr}\{\Pi_P(p_0\rho_0-p_1\rho_1)\}=\mathrm{Tr}\{\Pi_P(P-Q)\}=\mathrm{Tr}\{P\}=\frac{1}{2}\|p_0\rho_0-p_1\rho_1\|_1-p_1+\frac{1}{2} \tag{12}$$

Then for $0\leq\Lambda_0\leq I$, we have

$$\mathrm{Tr}\{\Lambda_0(p_0\rho_0-p_1\rho_1)\}=\mathrm{Tr}\{\Lambda_0(P-Q)\}\leq\mathrm{Tr}\{\Lambda_0P\}\leq\mathrm{Tr}\{P\}=\frac{1}{2}\|p_0\rho_0-p_1\rho_1\|_1-p_1+\frac{1}{2} \tag{13}$$

Combine eq. (4) and eq. (13), we finally conclude our proof and obtain

$$p_{\mathrm{succ}}=p_1+\max_{0\leq\Lambda_0\leq I}\mathrm{Tr}\{\Lambda_0(p_0\rho_0-p_1\rho_1)\}=\frac{1}{2}(1+\|p_0\rho_0-p_1\rho_1\|_1) \tag{14}$$