Exercise 3.5.1

Here we would show the matrix representation of $I_1 X_2$ and $X_1 X_2$ are the same as tensor product of $I \otimes X$ and $X \otimes X$.

- For $I_1 X_2$ case, the matrix representation can be shown as

$$\begin{aligned}&\begin{pmatrix}\langle 00|I_1X_2|00\rangle & \langle 00|I_1X_2|01\rangle & \langle 00|I_1X_2|10\rangle & \langle 00|I_1X_2|11\rangle\\ \langle 01|I_1X_2|00\rangle & \langle 01|I_1X_2|01\rangle & \langle 01|I_1X_2|10\rangle & \langle 01|I_1X_2|11\rangle\\ \langle 10|I_1X_2|00\rangle & \langle 10|I_1X_2|01\rangle & \langle 10|I_1X_2|10\rangle & \langle 10|I_1X_2|11\rangle\\ \langle 11|I_1X_2|00\rangle & \langle 11|I_1X_2|01\rangle & \langle 11|I_1X_2|10\rangle & \langle 11|I_1X_2|11\rangle\end{pmatrix}\\ =&\begin{pmatrix}\langle 00|01\rangle & \langle 00|00\rangle & \langle 00|11\rangle & \langle 00|10\rangle\\ \langle 01|01\rangle & \langle 01|00\rangle & \langle 01|11\rangle & \langle 01|10\rangle\\ \langle 10|01\rangle & \langle 10|00\rangle & \langle 10|11\rangle & \langle 10|10\rangle\\ \langle 11|01\rangle & \langle 11|00\rangle & \langle 11|11\rangle & \langle 11|10\rangle\end{pmatrix} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\end{aligned} \tag{1}$$

  The tensor product of $I \otimes X$ is given by

$$\begin{aligned}I \otimes X &= \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes X = \begin{pmatrix}X&0\\0&X\end{pmatrix}\\ &= \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\end{aligned} \tag{2}$$

  Compare eq. (1) and eq. (2), we conclude that the matrix representation of $I_1 X_2$ is the same as tensor product $I \otimes X$.

- For $I_1 X_2$ case, the matrix representation can be shown as

$$\begin{aligned}&\begin{pmatrix}\langle 00|X_1X_2|00\rangle & \langle 00|X_1X_2|01\rangle & \langle 00|X_1X_2|10\rangle & \langle 00|X_1X_2|11\rangle\\ \langle 01|X_1X_2|00\rangle & \langle 01|X_1X_2|01\rangle & \langle 01|X_1X_2|10\rangle & \langle 01|X_1X_2|11\rangle\\ \langle 10|X_1X_2|00\rangle & \langle 10|X_1X_2|01\rangle & \langle 10|X_1X_2|10\rangle & \langle 10|X_1X_2|11\rangle\\ \langle 11|X_1X_2|00\rangle & \langle 11|X_1X_2|01\rangle & \langle 11|X_1X_2|10\rangle & \langle 11|X_1X_2|11\rangle\end{pmatrix}\\ =&\begin{pmatrix}\langle 00|11\rangle & \langle 00|10\rangle & \langle 00|01\rangle & \langle 00|00\rangle\\ \langle 01|11\rangle & \langle 01|10\rangle & \langle 01|01\rangle & \langle 01|00\rangle\\ \langle 10|11\rangle & \langle 10|10\rangle & \langle 10|01\rangle & \langle 10|00\rangle\\ \langle 11|11\rangle & \langle 11|10\rangle & \langle 11|01\rangle & \langle 11|00\rangle\end{pmatrix} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\end{aligned} \tag{3}$$

  The tensor product of $I \otimes X$ is given by

$$
\begin{aligned}
I \otimes X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes X = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \\
= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{4}
$$

Compare eq. (3) and eq. (4), we conclude that the matrix representation of $X_1 X_2$ is the same as tensor product $X \otimes X$.