Exercise 3.3.9

The Hadmard gate is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{1}$$

We can check that Hadmard gate given in eq. (1) is its own inverse,

$$HH = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I \tag{2}$$

We can also check that,

$$\begin{aligned} |0\rangle\langle+| + |1\rangle\langle-| &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H \end{aligned} \tag{3}$$

That is, Hadmard gate can be re-written as $H = |0\rangle\langle+| + |1\rangle\langle-|$. Also, we can check

$$\begin{aligned} |+\rangle\langle0| + |-\rangle\langle1| &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H \end{aligned} \tag{4}$$

From eq. (3) and eq. (4), we have

$$H = |0\rangle\langle+| + |1\rangle\langle-| = |+\rangle\langle0| + |-\rangle\langle1| \tag{5}$$

We can also prove the identity without the matrix form of Hadmard gate. The Hadmard gate can take $|+\rangle$ and $|-\rangle$ into $|0\rangle$ and $|1\rangle$, respectively. The definition of Hadmard gate

$$H = |0\rangle\langle+| + |1\rangle\langle-| \tag{6}$$

satisfies the definition,

$$\begin{aligned} H|+\rangle &= |0\rangle\langle+|+\rangle + |1\rangle\langle-|+\rangle = |0\rangle\\ H|-\rangle &= |0\rangle\langle+|-\rangle + |1\rangle\langle-|-\rangle = |1\rangle \end{aligned} \tag{7}$$

For $|+\rangle\langle0| + |-\rangle\langle1|$, note that $|0\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ and $|1\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$, we have

$$
\begin{aligned}
(|+\rangle\langle 0| + |-\rangle\langle 1|)|+\rangle &= |+\rangle\langle 0|+\rangle + |-\rangle\langle 1|+\rangle \\
&= \frac{1}{\sqrt{2}}|+\rangle(\langle +| + \langle -|)|+\rangle + \frac{1}{\sqrt{2}}|-\rangle(\langle +| - \langle -|)|+\rangle \\
&= \frac{1}{\sqrt{2}}(|+\rangle\langle +|+\rangle + |+\rangle\langle -|+\rangle + |-\rangle\langle +|+\rangle - |-\rangle\langle -|+\rangle) \\
&= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle
\end{aligned} \tag{8}
$$

Also,

$$
\begin{aligned}
(|+\rangle\langle 0| + |-\rangle\langle 1|)|-\rangle &= |+\rangle\langle 0|-\rangle + |-\rangle\langle 1|-\rangle \\
&= \frac{1}{\sqrt{2}}|+\rangle(\langle +| + \langle -|)|-\rangle + \frac{1}{\sqrt{2}}|-\rangle(\langle +| - \langle -|)|-\rangle \\
&= \frac{1}{\sqrt{2}}(|+\rangle\langle +|-\rangle + |+\rangle\langle -|-\rangle + |-\rangle\langle +|-\rangle - |-\rangle\langle -|-\rangle) \\
&= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = |1\rangle
\end{aligned} \tag{9}
$$

From eq. $(7) - (9)$, we conclude that

$$
H = |0\rangle\langle +| + |1\rangle\langle -| = |+\rangle\langle 0| + |-\rangle\langle 1| \tag{10}
$$