Exercise 6.2.3

Suppose Alice hopes to prepare state

$$|\psi\rangle = \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \tag{1}$$

in Bob's system. In this case we should have $|\psi^*\rangle$ and $|\psi^{*\perp}\rangle$ as

$$\begin{aligned} |\psi^*\rangle = \frac{|0\rangle + e^{-i\phi}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\phi} \end{pmatrix}, |\psi^{*\perp}\rangle = \frac{|0\rangle - e^{-i\phi}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{-i\phi} \end{pmatrix} \\ \iff |0\rangle = \frac{1}{\sqrt{2}}(|\psi^*\rangle + |\psi^{*\perp}\rangle), |1\rangle = \frac{e^{i\phi}}{\sqrt{2}}(|\psi^*\rangle - |\psi^{*\perp}\rangle) \end{aligned} \tag{2}$$

Consider the composite quantum state $|\Phi^+\rangle_{AB}$, we have

$$\begin{aligned} |\Phi^+\rangle_{AB} &= \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \\ &= \frac{1}{2}\left[(|\psi^*\rangle_A + |\psi^{*\perp}\rangle_A)|0\rangle_B + e^{i\phi}(|\psi^*\rangle_A - |\psi^{*\perp}\rangle_A)|1\rangle_B\right] \\ &= \frac{1}{\sqrt{2}}\left[|\psi^*\rangle_A \frac{|0\rangle_B + e^{i\phi}|1\rangle_B}{\sqrt{2}} + |\psi^{*\perp}\rangle_A \frac{|0\rangle_B - e^{i\phi}|1\rangle_B}{\sqrt{2}}\right] \end{aligned} \tag{3}$$

From above information, we can design a protocol as follow to make sure Bob will finally get the state $|\psi\rangle$:

- Alice performs a measurement of her system A under basis $\{|\psi^*\rangle, |\psi^{*\perp}\rangle\}$, and suppose that if she gets 1 when measuring $|\psi^*\rangle$ and gets -1 when measuring $|\psi^{*\perp}\rangle$.
- Alice sends Bob a classical bit 1 when measuring $|\psi^*\rangle$ and sends 0 when measuring $|\psi^{*\perp}\rangle$.
- If Bob receives 1 from Alice, it means that the state on his side is $|\psi\rangle$ and he doesn't need to do anything; if he receives 0 from Alice, he need to perform a Z gate on his side and then will get $|\psi\rangle$.