

## Exercise 2.2.1

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Suppose that for message  $m \in [M]$  and a certain code  $\mathcal{C}$ , we have following relation

$$\bar{p}_e = \frac{1}{M} \sum_m p_e(m, \mathcal{C}) \leq \epsilon \quad (1)$$

Let random variable  $X = \{p_e(m, \mathcal{C})\}$  with size  $M$ , the expectation value of random variable  $X$  is given by

$$E(X) = \frac{1}{M} \sum_m p_e(m, \mathcal{C}) = \bar{p}_e \leq \epsilon \quad (2)$$

In this problem, we need to prove that whether there is at least half of message  $m$  with  $p_e \leq 2\epsilon$ , that is, we need to prove that the probability of getting  $X \leq 2\epsilon$  is at least  $1/2$ . According to the Markov's inequality, we have

$$P(X \geq 2\epsilon) \leq \frac{E(X)}{2\epsilon} = \frac{1}{2} \quad (3)$$

Equivalently, we have

$$P(X \leq 2\epsilon) = 1 - P(X \geq 2\epsilon) \geq \frac{1}{2} \quad (4)$$

Namely, we conclude from eq. (4) that if we have eq. (1), we should have at least half of the message  $m$  satisfies the condition  $p_e(m, \mathcal{C}) \leq 2\epsilon$ .