Exercise 3.3.11

Suppose that A is Hermitian and it can be decomposed into the form

$$A = \sum_i a_1 |i\rangle\langle i| \tag{1}$$

where $|i\rangle$ is a set of orthonormal basis, the function of an operator A is defined by

$$f(A) = \sum_i f(a_i)|i\rangle\langle i| \tag{2}$$

Also, for sine and cosine function, we have

$$\cos\frac{\phi}{2} = \frac{e^{i\phi/2} + e^{-i\phi/2}}{2}, \sin\frac{\phi}{2} = \frac{e^{i\phi/2} - e^{-i\phi/2}}{2i} \tag{3}$$

Here I will use eq. $(1) - (3)$ to prove the rotation operator expression.

- For $R_X(\phi) = \exp\{iX\phi/2\}$, since $|+\rangle$ and $|-\rangle$ are eigenvector of X With eigenvalue $+1$ and -1, respectively, we can write X into the form as eq. (1),

  $$\begin{aligned} X &= |+\rangle\langle+| - |-\rangle\langle-| \\ &= \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} \\ &= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \end{aligned} \tag{4}$$

  According to eq. (2), the rotation operator is then given by

  $$\exp\{iX\phi/2\} = e^{i\phi/2}|+\rangle\langle+| + e^{-i\phi/2}|-\rangle\langle-| \tag{5}$$

  Note that

  $$|+\rangle\langle+| = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = \frac{1}{2}(I + X) \tag{6a}$$

  $$|-\rangle\langle-| = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} = \frac{1}{2}(I - X) \tag{6b}$$

  Use eq. $(6a) - (6b)$, we can re-write eq. (5) as

  $$\begin{aligned} \exp\{iX\phi/2\} &= e^{i\phi/2}|+\rangle\langle+| + e^{-i\phi/2}|-\rangle\langle-| \\ &= \frac{1}{2}e^{i\phi/2}(I + X) + \frac{1}{2}e^{-i\phi/2}(I - X) \end{aligned} \tag{7}$$

  Use eq. (3), we can re-write eq. (7) as

$$
\begin{aligned}
\exp\{iX\phi/2\} &= \frac{1}{2}e^{i\phi/2}(I+X) + \frac{1}{2}e^{-i\phi/2}(I-X) \\
&= \frac{1}{2}(e^{i\phi/2}+e^{-i\phi/2})I + \frac{1}{2i}(e^{i\phi/2}-e^{-i\phi/2})iX \\
&= \cos\frac{\phi}{2}I + i\sin\frac{\phi}{2}X
\end{aligned} \tag{8}
$$

- For $R_Y(\phi) = \exp\{iY\phi/2\}$, since

$$
|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, |\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} \tag{9}
$$

are eigenvector of Y with eigenvalue $+1$ and -1, respectively, we can write Y into the form as eq. (1),

$$
\begin{aligned}
Y &= |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \\
&= \frac{1}{2}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\ -i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} \\
&= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}
\end{aligned} \tag{10}
$$

According to eq. (2), the rotation operator is then given by

$$
\exp\{iY\phi/2\} = e^{i\phi/2}|\psi\rangle\langle\psi| + e^{-i\phi/2}|\phi\rangle\langle\phi| \tag{11}
$$

Note that

$$
|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} = \frac{1}{2}(I+Y) \tag{12a}
$$

$$
|\phi\rangle\langle\phi| = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} = \frac{1}{2}(I-Y) \tag{12b}
$$

Use eq. $(12a) - (12b)$, we can re-write eq. (11) as

$$
\begin{aligned}
\exp\{iY\phi/2\} &= e^{i\phi/2}|\psi\rangle\langle\psi| + e^{-i\phi/2}|\phi\rangle\langle\phi| \\
&= \frac{1}{2}e^{i\phi/2}(I+Y) + \frac{1}{2}e^{-i\phi/2}(I-Y)
\end{aligned} \tag{13}
$$

Use eq. (3), we can re-write eq. (13) as

$$
\begin{aligned}
\exp\{iY\phi/2\} &= \frac{1}{2}e^{i\phi/2}(I+Y) + \frac{1}{2}e^{-i\phi/2}(I-Y) \\
&= \frac{1}{2}(e^{i\phi/2}+e^{-i\phi/2})I + \frac{1}{2i}(e^{i\phi/2}-e^{-i\phi/2})iY \\
&= \cos\frac{\phi}{2}I + i\sin\frac{\phi}{2}Y
\end{aligned} \tag{14}
$$

- For $R_Z(\phi) = \exp\{iZ\phi/2\}$, since $|0\rangle$ and $|1\rangle$ are eigenvector of Z With eigenvalue $+1$ and -1, respectively, we can write Z into the form as eq. (1),

$$
\begin{aligned}
Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{15}
$$

According to eq. (2), the rotation operator is then given by

$$
\exp\{iZ\phi/2\} = e^{i\phi/2}|0\rangle\langle 0| + e^{-i\phi/2}|1\rangle\langle 1| \tag{16}
$$

Note that

$$
|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(I + Z) \tag{17a}
$$

$$
|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(I - X) \tag{17b}
$$

Use eq. (17a) $-$ (17b), we can re-write eq. (16) as

$$
\begin{aligned}
\exp\{iZ\phi/2\} &= e^{i\phi/2}|+\rangle\langle +| + e^{-i\phi/2}|-\rangle\langle -| \\
&= \frac{1}{2}e^{i\phi/2}(I + Z) + \frac{1}{2}e^{-i\phi/2}(I - Z)
\end{aligned} \tag{18}
$$

Use eq. (3), we can re-write eq. (18) as

$$
\begin{aligned}
\exp\{iZ\phi/2\} &= \frac{1}{2}e^{i\phi/2}(I + Z) + \frac{1}{2}e^{-i\phi/2}(I - Z) \\
&= \frac{1}{2}(e^{i\phi/2} + e^{-i\phi/2})I + \frac{1}{2i}(e^{i\phi/2} - e^{-i\phi/2})iZ \\
&= \cos\frac{\phi}{2}I + i\sin\frac{\phi}{2}Z
\end{aligned} \tag{19}
$$

From eq. (8), eq. (14) and eq. (19), we can conclude that

$$
\exp\{iX\phi/2\} = \cos\frac{\phi}{2}I + i\sin\frac{\phi}{2}X \tag{20a}
$$

$$
\exp\{iY\phi/2\} = \cos\frac{\phi}{2}I + i\sin\frac{\phi}{2}Y \tag{20b}
$$

$$
\exp\{iZ\phi/2\} = \cos\frac{\phi}{2}I + i\sin\frac{\phi}{2}Z \tag{20c}
$$