Exercise 3.3.1

In this problem, we need to prove two statements,

- If an operator T is unitary, then T is norm preserving.
- If T is norm preserving, then T is unitary.

From the definition of norm preserving operator, if $||T|\psi\rangle||_2 = |||\psi\rangle||_2$ for any quantum state $|\psi\rangle$, where $|||\psi\rangle||_2 = \sqrt{\langle\psi|\psi\rangle}$, then we say T is norm preserving. Thus, for unitary operator T, for any quantum state we should have

$$||T|\psi\rangle||_2 = \sqrt{\langle\psi|T^\dagger T|\psi\rangle} = \sqrt{\langle\psi|I|\psi\rangle} = \sqrt{\langle\psi|\psi\rangle} = |||\psi\rangle||_2 \tag{1}$$

Thus, the first statement is proved.

For the second statement, if we assume that for any unit vector $|\psi\rangle$, we have $||T|\psi\rangle||_2 = |||\psi\rangle||_2$. According to the definition of the norm, we should have

$$||T|\psi\rangle||_2 = |||\psi\rangle||_2 \iff \sqrt{\langle\psi|T^\dagger T|\psi\rangle} = \sqrt{\langle\psi|\psi\rangle} = 1 \iff \langle\psi|T^\dagger T|\psi\rangle = 1 \tag{2}$$

Namely, we can write

$$\langle\psi|T^\dagger T|\psi\rangle - \langle\psi|\psi\rangle = \langle\psi|(T^\dagger T - I)|\psi\rangle = 0 \tag{3}$$

Notice that

$$(T^\dagger T - I)^\dagger = (T^\dagger T)^\dagger - I = T^\dagger T - I \tag{4}$$

so $T^\dagger T - I$ is Hermitian operator and thus normal operator. We can re-write operator $T^\dagger T - I$ using spectral decomposition,

$$T^\dagger T - I = \sum_i \lambda_i |i\rangle\langle i| \tag{5}$$

where $\{|i\rangle\}$ is a set of orthonormal basis. Note that eq. (3) is valid for any state, so for eigenstate $|i\rangle$ we have from eq. (3),

$$\langle i|(T^\dagger T - I)|i\rangle = \langle i|\lambda_i|i\rangle = 0 \tag{6}$$

Thus, for any eigenvector of $T^\dagger T - I$, the corresponding eigenvalue is 0, and from eq. (5) we know that

$$T^\dagger T - I = 0 \iff T^\dagger T = I \tag{7}$$

In conclusion, if T is norm preserving, then T is unitary operator.