Exercise 4.1.13

Consider an ensemble $\{p(j), |\psi_j\rangle\}$ where $|\psi_j\rangle$ is single-qubit pure state, its corresponding density matrix is given by

$$\rho = \sum_j p(j)|\psi_j\rangle\langle\psi_j| \tag{1}$$

Note that for each single-qubit pure state $|\psi_j\rangle$, its Bloch vector $\mathbf{r}_j = (r_{j,x}, r_{j,y}, r_{j,z})$, and its density matrix is given by

$$\rho_j = \frac{1}{2}\begin{pmatrix} 1 + r_{j,z} & r_{j,x} - ir_{j,y} \\ r_{j,x} + ir_{j,y} & 1 - r_{j,z} \end{pmatrix} \tag{2}$$

Combine eq. (1) and eq. (2), we should have the density matrix for mixed state as

$$\begin{aligned}
\rho &= \frac{1}{2}\sum_j p(j)\begin{pmatrix} 1 + r_{j,z} & r_{j,x} - ir_{j,y} \\ r_{j,x} + ir_{j,y} & 1 - r_{j,z} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \sum_j p(j)(1 + r_{j,z}) & \sum_j p(j)(r_{j,x} - ir_{j,y}) \\ \sum_j p(j)(r_{j,x} + ir_{j,y}) & \sum_j p(j)(1 - r_{j,z}) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \sum_j p(j) + \sum_j p(j)r_{j,z} & \sum_j p(j)r_{j,x} - i\sum_j p(j)r_{j,y} \\ \sum_j p(j)r_{j,x} + i\sum_j p(j)r_{j,y}) & \sum_j p(j) - \sum_j p(j)r_{j,z}) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 + \sum_j p(j)r_{j,z} & \sum_j p(j)r_{j,x} - i\sum_j p(j)r_{j,y} \\ \sum_j p(j)r_{j,x} + i\sum_j p(j)r_{j,y}) & 1 - \sum_j p(j)r_{j,z}) \end{pmatrix}
\end{aligned} \tag{3}$$

Following eq. (2), if we define the density matrix of mixed state as

$$\rho_j = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} \tag{4}$$

where $\mathbf{r} = (r_x, r_y, r_z)$ is considered as the Bloch vector of mixed state. From eq. (3) we should have for $\mathbf{r}$,

$$\begin{aligned}
\mathbf{r} = (r_x, r_y, r_z) &= \left[\sum_j p(j)r_{j,x}, \sum_j p(j)r_{j,y}, \sum_j p(j)r_{j,z}\right] \\
&= \sum_j p(j)(r_{j,x}, r_{j,y}, r_{j,z}) = \sum_j p(j)\mathbf{r}_j
\end{aligned} \tag{5}$$