Exercise 3.5.4

Consider a two-qubit operation as $H_1H_2(\mathrm{CNOT})H_1H_2$, if we have for CNOT gate,

$$\mathrm{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X \tag{1}$$

and for the two-qubit Hadmard gate, $H_1H_2 = H \otimes H$, the given operation can be re-written as

$$\begin{aligned} H_1H_2(\mathrm{CNOT})H_1H_2 &= (H \otimes H)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(H \otimes H) \\ &= (H \otimes H)(|0\rangle\langle 0|H \otimes IH + |1\rangle\langle 1|H \otimes XH) \\ &= H|0\rangle\langle 0|H \otimes HIH + H|1\rangle\langle 1|H \otimes HXH \end{aligned} \tag{2}$$

For the Hadmard gate,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3}$$

we could see that H is a Hermitian operator, $H^\dagger = H$. Also, we could have $HXH = Z$ and

$$HIH = HH = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I \tag{4}$$

When Hadmard gate peforms on computational basis, we have

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle \\ H|1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle \end{aligned} \tag{5}$$

Using eq. $(4) - (5)$, and the given properties of Hadmard gate above, we could simplify eq. (2) as

$$\begin{aligned} H_1H_2(\mathrm{CNOT})H_1H_2 &= H|0\rangle\langle 0|H \otimes HIH + H|1\rangle\langle 1|H \otimes HXH \\ &= (H|0\rangle)(\langle 0|H^\dagger) \otimes I + (H|1\rangle)(\langle 1|H^\dagger) \otimes Z \\ &= |+\rangle\langle +| \otimes I + |-\rangle\langle -| \otimes Z \end{aligned} \tag{6}$$