Exercise 4.1.5

For ensemble $\{\{1/2, |0\rangle\}, \{1/2, |1\rangle\}\}$, the corresponding density matrix is given by

$$\begin{aligned}\rho &= \sum_x p_X(x)|\psi_x\rangle\langle\psi_x| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\ &= \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\end{aligned} \tag{1}$$

For ensemble $\{\{1/2, |+\rangle\}, \{1/2, |-\rangle\}\}$, the corresponding density matrix is given by

$$\begin{aligned}\rho &= \sum_x p_X(x)|\psi_x\rangle\langle\psi_x| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \\ &= \frac{1}{4}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\end{aligned} \tag{2}$$

From eq. (1) and eq. (2), we see that two ensembles have same density operator.