Exercise 3.4.2

In quantum mechanics, the measurement result is the eigenvalues of the observable. Suppose we have an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and

$$\alpha = a + ib,\ \beta = c + id \quad \alpha, \beta \in \mathbb{C} \tag{1}$$

we can check the expectation value of measuring X and Y.

- For observable X, the eigenvalues are $+1$ and -1, and the corresponding eigenvectors are $|+\rangle$ and $|-\rangle$. Note that

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \tag{2a}$$

$$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \tag{2b}$$

We can re-write the quantum state as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\frac{|+\rangle + |-\rangle}{\sqrt{2}} + \beta\frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{(\alpha+\beta)|+\rangle + (\alpha-\beta)|-\rangle}{\sqrt{2}} \tag{3}$$

Thus, the probability of getting $|+\rangle$ is $|\alpha+\beta|^2/2$, the probability of getting $|-\rangle$ is $|\alpha-\beta|^2/2$, and the expectation value of measuring X is given by

$$\mathbb{E}(X) = \frac{|\alpha+\beta|^2}{2} \times 1 + \frac{|\alpha-\beta|^2}{2} \times (-1) = \frac{|\alpha+\beta|^2 - |\alpha-\beta|^2}{2} \tag{4}$$

According to eq. (1), we can re-write eq. (4) as

$$\begin{aligned}
\mathbb{E}(X) &= \frac{|\alpha+\beta|^2 - |\alpha-\beta|^2}{2} \\
&= \frac{|a+ib+c+id|^2 - |a+ib-c-id|^2}{2} \\
&= \frac{|(a+c)+i(b+d)|^2 - |(a-c)+i(b-d)|^2}{2} \\
&= \frac{(a+c)^2 + (b+d)^2 - (a-c)^2 - (b-d)^2}{2} \\
&= 2(ac+bd)
\end{aligned} \tag{5}$$

Meanwhile, the $\langle\psi|X|\psi\rangle$ is given by

$$\langle\psi|X|\psi\rangle = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}\beta^* & \alpha^*\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\beta^* + \alpha^*\beta \tag{6}$$

According to eq. (1), we can re-write eq. (6) as

$$\langle\psi|X|\psi\rangle = \alpha\beta^* + \alpha^*\beta = (a+ib)(c-id) + (a-ib)(c+id) = 2(ac+bd) \tag{7}$$

From eq. (5) and eq. (7), we have $\mathbb{E}(X) = \langle\psi|X|\psi\rangle$

- For observable Y, the eigenvalues are $+1$ and -1, and the corresponding eigenvectors are

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} \tag{8}$$

Note that

$$|0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right] = \frac{|v_1\rangle + |v_2\rangle}{\sqrt{2}} \tag{9a}$$

$$|1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\right] = \frac{-i(|v_1\rangle - |v_2\rangle)}{\sqrt{2}} \tag{9b}$$

We can re-write the quantum state as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\frac{|v_1\rangle + |v_2\rangle}{\sqrt{2}} - i\beta\frac{|v_1\rangle - |v_2\rangle}{\sqrt{2}} = \frac{(\alpha - i\beta)|v_1\rangle + (\alpha + i\beta)|v_2\rangle}{\sqrt{2}} \tag{10}$$

Thus, the probability of getting $|v_1\rangle$ is $|\alpha - i\beta|^2/2$, the probability of getting $|v_2\rangle$ is $|\alpha + i\beta|^2/2$, and the expectation value of measuring Y is given by

$$\mathbb{E}(Y) = \frac{|\alpha - i\beta|^2}{2} \times 1 + \frac{|\alpha + i\beta|^2}{2} \times (-1) = \frac{|\alpha - i\beta|^2 - |\alpha + i\beta|^2}{2} \tag{11}$$

According to eq. (1), we can re-write eq. (11) as

$$\begin{aligned}\mathbb{E}(Y) &= \frac{|\alpha - i\beta|^2 - |\alpha + i\beta|^2}{2}\\ &= \frac{|a + ib - ic + d|^2 - |a + ib + ic - d|^2}{2}\\ &= \frac{|(a+d) + i(b-c)|^2 - |(a-d) + i(b+c)|^2}{2}\\ &= \frac{(a+d)^2 + (b-c)^2 - (a-d)^2 - (b+c)^2}{2}\\ &= 2(ad - bc)\end{aligned} \tag{12}$$

Meanwhile, the $\langle\psi|Y|\psi\rangle$ is given by

$$\langle\psi|Y|\psi\rangle = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}i\beta^* & -i\alpha^*\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = i\alpha\beta^* - i\alpha^*\beta \tag{13}$$

According to eq. (1), we can re-write eq. (13) as

$$\langle\psi|Y|\psi\rangle = i\alpha\beta^* - i\alpha^*\beta = i(a+ib)(c-id) - i(a-ib)(c+id) = 2(ad-bc) \tag{14}$$

From eq. (12) and eq. (14), we have $\mathbb{E}(Y) = \langle\psi|Y|\psi\rangle$