Exercise 5.2.4

The bit flip channel is defined as

$$\mathcal{E}(\rho) = (1-p)I\rho I + pX\rho X \tag{1}$$

and its purification is given by

$$|\psi\rangle_A \to \sqrt{1-p}|\psi\rangle_A|0\rangle_E + \sqrt{p}X|\psi\rangle_A|1\rangle_E \tag{2}$$

We will compare the output state received by a receiver and environment. If the input of the channel is $|\psi\rangle_A$, the receiver would obtain a state ρ_A as

$$\begin{aligned}
\mathrm{Tr}_E\rho_{AE} =& \mathrm{Tr}_E\left\{\left(\sqrt{1-p}|\psi\rangle_A|0\rangle_E + \sqrt{p}X|\psi\rangle_A|1\rangle_E\right)\left(\sqrt{1-p}\langle\psi|_A\langle 0|_E + \sqrt{p}(\langle\psi|_A X^\dagger)\langle 1|_E\right)\right\} \\
=& \mathrm{Tr}_E\left\{(1-p)|\psi\rangle\langle\psi|_A \otimes |0\rangle\langle 0|_E\right\} + \mathrm{Tr}_E\left\{\sqrt{p(1-p)}X|\psi\rangle\langle\psi|_A \otimes |1\rangle\langle 0|_E\right\} \\
&+ \mathrm{Tr}_E\left\{\sqrt{p(1-p)}|\psi\rangle\langle\psi|_A X^\dagger \otimes |0\rangle\langle 1|_E\right\} + \mathrm{Tr}_E\left\{pX|\psi\rangle\langle\psi|_A X^\dagger \otimes |1\rangle\langle 1|_E\right\} \\
=& \sum_{l=0}^{1}(I_A\otimes\langle l|_E)\left\{(1-p)|\psi\rangle\langle\psi|_A \otimes |0\rangle\langle 0|_E\right\}(I_A\otimes|l\rangle_E) \\
&+ \sum_{l=0}^{1}(I_A\otimes\langle l|_E)\left\{\sqrt{p(1-p)}X|\psi\rangle\langle\psi|_A \otimes |1\rangle\langle 0|_E\right\}(I_A\otimes|l\rangle_E) \\
&+ \sum_{l=0}^{1}(I_A\otimes\langle l|_E)\left\{\sqrt{p(1-p)}|\psi\rangle\langle\psi|_A X^\dagger \otimes |0\rangle\langle 1|_E\right\}(I_A\otimes|l\rangle_E) \\
&+ \sum_{l=0}^{1}(I_A\otimes\langle l|_E)\left\{pX|\psi\rangle\langle\psi|_A X^\dagger \otimes |1\rangle\langle 1|_E\right\}(I_A\otimes|l\rangle_E) \\
=& (1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X^\dagger
\end{aligned} \tag{3}$$

If we write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we have for eq. (3),

$$\begin{aligned}
\mathrm{Tr}_E\rho_{AE} &= (1-p)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} + pX\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}X^\dagger \\
&= (1-p)\begin{pmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{pmatrix} + p\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\
&= (1-p)\begin{pmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{pmatrix} + p\begin{pmatrix}|\beta|^2 & \alpha^*\beta\\ \alpha\beta^* & |\alpha|^2\end{pmatrix}
\end{aligned} \tag{4}$$

Meanwhile, the state received by the environment ρ_E is given by

$$
\begin{aligned}
\mathrm{Tr}_A\rho_{AE} =& \mathrm{Tr}_A\left\{\left(\sqrt{1-p}|\psi\rangle_A|0\rangle_E+\sqrt{p}X|\psi\rangle_A|1\rangle_E\right)\left(\sqrt{1-p}\langle\psi|_A\langle 0|_E+\sqrt{p}(\langle\psi|_A X^\dagger)\langle 1|_E\right)\right\} \\
=& \mathrm{Tr}_A\left\{(1-p)|\psi\rangle\langle\psi|_A\otimes|0\rangle\langle 0|_E\right\}+\mathrm{Tr}_A\left\{\sqrt{p(1-p)}X|\psi\rangle\langle\psi|_A\otimes|1\rangle\langle 0|_E\right\} \\
&+\mathrm{Tr}_A\left\{\sqrt{p(1-p)}|\psi\rangle\langle\psi|_A X^\dagger\otimes|0\rangle\langle 1|_E\right\}+\mathrm{Tr}_A\left\{pX|\psi\rangle\langle\psi|_A X^\dagger\otimes|1\rangle\langle 1|_E\right\} \\
=& \sum_{l=0}^{1}(\langle l|_A\otimes I_E)\left\{(1-p)|\psi\rangle\langle\psi|_A\otimes|0\rangle\langle 0|_E\right\}(|l\rangle_A\otimes I_E) \\
&+\sum_{l=0}^{1}(\langle l|_A\otimes I_E)\left\{\sqrt{p(1-p)}X|\psi\rangle\langle\psi|_A\otimes|1\rangle\langle 0|_E\right\}(|l\rangle_A\otimes I_E) \\
&+\sum_{l=0}^{1}(\langle l|_A\otimes I_E)\left\{\sqrt{p(1-p)}|\psi\rangle\langle\psi|_A X^\dagger\otimes|0\rangle\langle 1|_E\right\}(|l\rangle_A\otimes I_E) \\
&+\sum_{l=0}^{1}(\langle l|_A\otimes I_E)\left\{pX|\psi\rangle\langle\psi|_A X^\dagger\otimes|1\rangle\langle 1|_E\right\}(|l\rangle_A\otimes I_E)
\end{aligned} \tag{5}
$$

If we write $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$, so $X|\psi\rangle=\alpha|1\rangle+\beta|0\rangle$ and we have

$$
\begin{aligned}
\langle 0|\psi\rangle=\alpha,\ \langle\psi|0\rangle=\alpha^*,\ \langle 1|\psi\rangle=\beta,\ \langle\psi|1\rangle=\beta^* \\
\langle 0|X|\psi\rangle=\beta,\ \langle\psi|X|0\rangle=\beta^*,\ \langle 1|X|\psi\rangle=\alpha,\ \langle\psi|X|1\rangle=\alpha^*
\end{aligned} \tag{6}
$$

and thus eq. (5) becomes

$$
\begin{aligned}
\mathrm{Tr}_A\rho_{AE} =& \left[(|\alpha|^2+|\beta|^2)(1-p)|0\rangle\langle 0|_E\right]+\left[\sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*)|1\rangle\langle 0|_E\right] \\
&+\left[\sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*)|0\rangle\langle 1|_E\right]+\left[p(|\alpha|^2+|\beta|^2)|1\rangle\langle 1|_E\right] \\
=& (1-p)|0\rangle\langle 0|_E+\sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*)(|1\rangle\langle 0|_E+|0\rangle\langle 1|_E)+p|1\rangle\langle 1|_E
\end{aligned} \tag{7}
$$

If we write eq. (7) into matrix form, we then have

$$
\begin{aligned}
\mathrm{Tr}_A\rho_{AE} &= (1-p)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+p\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \\
&= \begin{pmatrix}1-p & \sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*)\\ \sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*) & p\end{pmatrix}
\end{aligned} \tag{8}
$$

From eq. (4), when $p=1/2$, we should have

$$
\begin{aligned}
\mathrm{Tr}_E\rho_{AE} &= \frac{1}{2}\begin{pmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{pmatrix}+\frac{1}{2}\begin{pmatrix}|\beta|^2 & \alpha^*\beta\\ \alpha\beta^* & |\alpha|^2\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}|\alpha|^2+|\beta|^2 & \alpha\beta^*+\alpha^*\beta\\ \alpha^*\beta+\alpha\beta^* & |\alpha|^2+|\beta|^2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & \alpha\beta^*+\alpha^*\beta\\ \alpha^*\beta+\alpha\beta^* & 1\end{pmatrix}
\end{aligned} \tag{9}
$$

From eq. (8), when $p=1/2$, we should have $\sqrt{p(1-p)}=1/2$ and

$$
\mathrm{Tr}_A\rho_{AE}=\begin{pmatrix}1-p & \sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*)\\ \sqrt{p(1-p)}(\alpha^*\beta+\alpha\beta^*) & p\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & \alpha^*\beta+\alpha\beta^*\\ \alpha^*\beta+\alpha\beta^* & 1\end{pmatrix} \tag{10}
$$

From eq. $(9) - (10)$, we conclude that the receiver output density operator for a bit-flip channel with $p = 1/2$ is the same as what the environment obtains.