Exercise 4.7.2

The Pauli channel is defined by

$$\mathcal{E}(\rho) = p_I\rho + p_X X\rho X + p_Y Y\rho Y + p_Z Z\rho Z \tag{1}$$

For an arbitrary single qubit state ρ, we can write the density matrix in Bloch sphere coordinate (r_x, r_y, r_z),

$$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z) \tag{2}$$

Substitute eq. (2) to eq. (1), we have

$$\begin{aligned}
\mathcal{E}(\rho) =& \frac{1}{2}p_I(I + r_x X + r_y Y + r_z Z) + \frac{1}{2}p_X X(I + r_x X + r_y Y + r_z Z)X \\
&+ \frac{1}{2}p_Y Y(I + r_x X + r_y Y + r_z Z)Y + \frac{1}{2}p_Z Z(I + r_x X + r_y Y + r_z Z)Z \\
=& \frac{1}{2}p_I I + \frac{1}{2}p_I r_x X + \frac{1}{2}p_I r_y Y + \frac{1}{2}p_I r_z Z \\
&+ \frac{1}{2}p_X XIX + \frac{1}{2}p_X r_x XXX + \frac{1}{2}p_X r_y XYX + \frac{1}{2}p_X r_z XZX \\
&+ \frac{1}{2}p_Y YIY + \frac{1}{2}r_x p_Y YXY + \frac{1}{2}r_y p_Y YYY + \frac{1}{2}p_Y r_z YZY \\
&+ \frac{1}{2}p_Z ZIZ + \frac{1}{2}r_x p_Z ZXZ + \frac{1}{2}r_y p_Z ZYZ + \frac{1}{2}r_z p_Z ZZZ
\end{aligned} \tag{3}$$

Substitute everything in the Appendix into eq. (3) and re-organize it, we have

$$\begin{aligned}
\mathcal{E}(\rho) =& \frac{1}{2}p_I I + \frac{1}{2}p_I r_x X + \frac{1}{2}p_I r_y Y + \frac{1}{2}p_I r_z Z \\
&+ \frac{1}{2}p_X I + \frac{1}{2}r_x p_X X - \frac{1}{2}r_y p_X Y - \frac{1}{2}r_z p_X Z \\
&+ \frac{1}{2}p_Y I - \frac{1}{2}r_x p_Y X + \frac{1}{2}r_y p_Y Y - \frac{1}{2}r_z p_Y Z \\
&+ \frac{1}{2}p_Z I - \frac{1}{2}r_x p_Z X - \frac{1}{2}r_y p_Z Y + \frac{1}{2}r_z p_Z Z \\
=& \frac{1}{2}[(p_I + p_X + p_Y + p_Z)I + (p_I + p_X - p_Y - p_Z)r_x X \\
&+ (p_I - p_X + p_Y - p_Z)r_y Y + (p_I - p_X - p_Y + p_Z)r_z Z]
\end{aligned} \tag{4}$$

Compare eq. (2) and eq. (4), we could have

$$(r_x, r_y, r_z) = ((p_I + p_X - p_Y - p_Z)r_x, (p_I - p_X + p_Y - p_Z)r_y, (p_I - p_X - p_Y + p_Z)r_z) \tag{5}$$

Appendix

The Pauli matrices are given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A1}$$

Here is detailed calculation of $\sigma_i \sigma_j \sigma_i$.

- For $\sigma_i = X$, we have

$$\begin{aligned} XXX &= XI = X \\ XYX &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -Y \\ XZX &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -Z \end{aligned} \tag{A2}$$

- For $\sigma_i = Y$, we have

$$\begin{aligned} YXY &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -X \\ YYY &= YI = Y \\ YZY &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -Z \end{aligned} \tag{A3}$$

- For $\sigma_i = Z$, we have

$$\begin{aligned} ZXZ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -X \\ ZYZ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -Y \\ ZZZ &= ZI = Z \end{aligned} \tag{A4}$$