Exercise 3.6.4

Suppose that Alice and Bob share the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \tag{1}$$

The rule of playing CHSH game is, the referee send Alice and Bob a bit x and y, respectively, then Alice and Bob send back a bit a and b to the referee, Alice and Bob win the game if $x \wedge y = a \oplus b$, where $\oplus$ means exclusive OR. The strategy is that:

- If Alice get 0 and Bob get 0, Alice measure $|\Phi^+\rangle$ in Z basis and send $a = 0$ if measuring $+1$, and $a = 1$ when measure -1. For Bob, he needs to measure $|\Phi^+\rangle$ in the $(X+Z)/\sqrt{2}$ basis and send $b = 0$ if measuring $+1$, and $b = 1$ when measure -1.
- If Alice get 0 and Bob get 1, Alice measure $|\Phi^+\rangle$ in Z basis and send $a = 0$ if measuring $+1$, and $a = 1$ when measure -1. For Bob, he needs to measure $|\Phi^+\rangle$ in the $(Z-X)/\sqrt{2}$ basis and send $b = 0$ if measuring $+1$, and $b = 1$ when measure -1.
- If Alice get 1 and Bob get 0, Alice measure $|\Phi^+\rangle$ in X basis and send $a = 0$ if measuring $+1$, and $a = 1$ when measure -1. For Bob, he needs to measure $|\Phi^+\rangle$ in the $(X+Z)/\sqrt{2}$ basis and send $b = 0$ if measuring $+1$, and $b = 1$ when measure -1.
- If Alice get 1 and Bob get 1, Alice measure $|\Phi^+\rangle$ in X basis and send $a = 0$ if measuring $+1$, and $a = 1$ when measure -1. For Bob, he needs to measure $|\Phi^+\rangle$ in the $(Z-X)/\sqrt{2}$ basis and send $b = 0$ if measuring $+1$, and $b = 1$ when measure -1.

Here we will estimate the probability of winning the game for each case. Firstly, let's get the basis of $(X+Z)/\sqrt{2}$ and $(Z-X)/\sqrt{2}$. The eigenvalue of $(X+Z)/\sqrt{2}$ can be obtained by

$$\left|\frac{X+Z}{\sqrt{2}} - \lambda I\right| = \begin{vmatrix} 1/\sqrt{2} - \lambda & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} - \lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{2}$$

The corresponding eigenvectors are given by

$$|v_1\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle \text{ for } \lambda = 1 \tag{3a}$$

$$|v_2\rangle = \sin\frac{\pi}{8}|0\rangle - \cos\frac{\pi}{8}|1\rangle \text{ for } \lambda = -1 \tag{3b}$$

The eigenvalue of $(Z-X)/\sqrt{2}$ can be obtained by

$$\left|\frac{Z-X}{\sqrt{2}} - \lambda I\right| = \begin{vmatrix} 1/\sqrt{2} - \lambda & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} - \lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{4}$$

The corresponding eigenvectors are given by

$$|v_1\rangle = -\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle \text{ for } \lambda = 1 \tag{5a}$$

$$|v_2\rangle = \sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle \text{ for } \lambda = -1 \tag{5b}$$

In the appendix I will verify eq. (3a) $-$ (3b) and eq. (5a) $-$ (5b) are the eigenvectors for $(X+Z)/\sqrt{2}$ and $(Z-X)/\sqrt{2}$, respectively.

Then we need to calculate the probability of winning the game for each case.

- If $x=0$ and $y=0$, $x \wedge y = 0$
    - If Alice get $|0\rangle$ and $a=0$, then Bob should measure $|0\rangle$ in $(X+Z)\sqrt{2}$ basis. From eq. (3a) $-$ (3b), we could solve the equation and get

      $$|0\rangle = \cos\frac{\pi}{8}|v_1\rangle + \sin\frac{\pi}{8}|v_2\rangle \tag{6}$$

      To make sure $a \oplus b = 0$, we need to measure $|v_1\rangle$ with $\lambda = 1$ then $b=0$ to win the game; so the probability of getting $|v_1\rangle$ and winning the game is $\cos^2(\pi/8)$.
    - If Alice get $|1\rangle$ and $a=1$, then Bob should measure $|1\rangle$ in $(X+Z)\sqrt{2}$ basis. From eq. (3a) $-$ (3b), we could solve the equation and get

      $$|1\rangle = \sin\frac{\pi}{8}|v_1\rangle - \cos\frac{\pi}{8}|v_2\rangle \tag{7}$$

      To make sure $a \oplus b = 0$, we need to measure $|v_2\rangle$ with $\lambda = -1$ then $b=1$ to win the game; so the probability of getting $|v_2\rangle$ and winning the game is $\cos^2(\pi/8)$.
- If $x=0$ and $y=1$, $x \wedge y = 0$
    - If Alice get $|0\rangle$ and $a=0$, then Bob should measure $|0\rangle$ in $(Z-X)\sqrt{2}$ basis. From eq. (5a) $-$ (5b), we could solve the equation and get

      $$|0\rangle = -\cos\frac{\pi}{8}|v_1\rangle + \sin\frac{\pi}{8}|v_2\rangle \tag{8}$$

      To make sure $a \oplus b = 0$, we need to measure $|v_1\rangle$ with $\lambda = 1$ then $b=0$ to win the game; so the probability of getting $|v_1\rangle$ and winning the game is $\cos^2(\pi/8)$.
    - If Alice get $|1\rangle$ and $a=1$, then Bob should measure $|1\rangle$ in $(Z-X)\sqrt{2}$ basis. From eq. (5a) $-$ (5b), we could solve the equation and get

      $$|1\rangle = \sin\frac{\pi}{8}|v_1\rangle + \cos\frac{\pi}{8}|v_2\rangle \tag{9}$$

      To make sure $a \oplus b = 0$, we need to measure $|v_2\rangle$ with $\lambda = -1$ then $b=1$ to win the game; so the probability of getting $|v_2\rangle$ and winning the game is $\cos^2(\pi/8)$.
- If $x=1$ and $y=0$, $x \wedge y = 0$
    - If Alice get $|+\rangle$ and $a=0$, then Bob should measure $|+\rangle$ in $(Z-X)\sqrt{2}$ basis. From eq. (3a) $-$ (3b), eq. (6) $-$ (7) and eq. (A2), we could get

      $$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right]|v_1\rangle + \frac{1}{\sqrt{2}}\left[\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\right]|v_2\rangle \\ &= \cos\frac{\pi}{8}|v_1\rangle - \sin\frac{\pi}{8}|v_2\rangle \end{aligned} \tag{10}$$

To make sure $a \oplus b = 0$, we need to measure $|v_1\rangle$ with $\lambda = 1$ then $b = 0$ to win the game; so the probability of getting $|v_1\rangle$ and winning the game is $\cos^2(\pi/8)$.

- If Alice get $|-\rangle$ and $a = 1$, then Bob should measure $|-\rangle$ in $(Z - X)\sqrt{2}$ basis. From eq. $(3a) - (3b)$, eq. $(6) - (7)$ and eq. $(A2)$, we could get

$$
\begin{aligned}
|-\rangle &= \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\right]|v_1\rangle + \frac{1}{\sqrt{2}}\left[\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right]|v_2\rangle \\
&= \sin\frac{\pi}{8}|v_1\rangle + \cos\frac{\pi}{8}|v_2\rangle
\end{aligned} \tag{11}
$$

To make sure $a \oplus b = 0$, we need to measure $|v_2\rangle$ with $\lambda = -1$ then $b = 1$ to win the game; so the probability of getting $|v_2\rangle$ and winning the game is $\cos^2(\pi/8)$.

- If $x = 1$ and $y = 1$, $x \wedge y = 1$
  - If Alice get $|+\rangle$ and $a = 0$, then Bob should measure $|+\rangle$ in $(Z - X)\sqrt{2}$ basis. From eq. $(5a) - (5b)$, eq. $(8) - (9)$ and eq. $(A2)$, we could get

$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}\left[-\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right]|v_1\rangle + \frac{1}{\sqrt{2}}\left[\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right]|v_2\rangle \\
&= -\sin\frac{\pi}{8}|v_1\rangle + \cos\frac{\pi}{8}|v_2\rangle
\end{aligned} \tag{12}
$$

To make sure $a \oplus b = 1$, we need to measure $|v_2\rangle$ with $\lambda = -1$ then $b = 1$ to win the game; so the probability of getting $|v_2\rangle$ and winning the game is $\cos^2(\pi/8)$.

  - If Alice get $|-\rangle$ and $a = 1$, then Bob should measure $|-\rangle$ in $(Z - X)\sqrt{2}$ basis. From eq. $(5a) - (5b)$, eq. $(8) - (9)$ and eq. $(A2)$, we could get

$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}\left[-\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\right]|v_1\rangle + \frac{1}{\sqrt{2}}\left[\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\right]|v_2\rangle \\
&= -\cos\frac{\pi}{8}|v_1\rangle - \sin\frac{\pi}{8}|v_2\rangle
\end{aligned} \tag{13}
$$

To make sure $a \oplus b = 1$, we need to measure $|v_1\rangle$ with $\lambda = 1$ then $b = 0$ to win the game; so the probability of getting $|v_1\rangle$ and winning the game is $\cos^2(\pi/8)$.

Therefore, the probability of winning the game is $\cos^2(\pi/8)$.

Appendix

To verify eq. $(3a)$, we have

$$
\frac{X + Z}{\sqrt{2}}|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}\cos\pi/8 \\ \sin\pi/8\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\pi/8 + \sin\pi/8 \\ \cos\pi/8 - \sin\pi/8\end{pmatrix} \tag{A1}
$$

Note that

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}\right)=\left(\cos\frac{\pi}{4}\cos\frac{\pi}{8}+\sin\frac{\pi}{4}\sin\frac{\pi}{8}\right)=\cos\left(\frac{\pi}{4}-\frac{\pi}{8}\right)=\cos\left(\frac{\pi}{8}\right)\\ \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}-\sin\frac{\pi}{8}\right)=\left(\cos\frac{\pi}{4}\cos\frac{\pi}{8}-\sin\frac{\pi}{4}\sin\frac{\pi}{8}\right)=\cos\left(\frac{\pi}{4}+\frac{\pi}{8}\right)=\sin\left(\frac{\pi}{8}\right)\end{aligned} \tag{A2}$$

So eq. (A1) becomes

$$\frac{X+Z}{\sqrt{2}}|v_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\cos\pi/8\\ \sin\pi/8\end{pmatrix}=\begin{pmatrix}\cos\pi/8\\ \sin\pi/8\end{pmatrix} \tag{A3}$$

and thus we verify eq. (3a).

To verify eq. (3b), we have

$$\frac{X+Z}{\sqrt{2}}|v_2\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\sin\pi/8\\ -\cos\pi/8\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\sin\pi/8-\cos\pi/8\\ \cos\pi/8+\sin\pi/8\end{pmatrix} \tag{A4}$$

From eq. (A2), eq. (A4) becomes

$$\frac{X+Z}{\sqrt{2}}|v_2\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\sin\pi/8\\ -\cos\pi/8\end{pmatrix}=\begin{pmatrix}-\sin\pi/8\\ \cos\pi/8\end{pmatrix} \tag{A5}$$

and thus we verify eq. (3b).

To verify eq. (5a), we have

$$\frac{Z-X}{\sqrt{2}}|v_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}-\cos\pi/8\\ \sin\pi/8\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-\sin\pi/8-\cos\pi/8\\ \cos\pi/8-\sin\pi/8\end{pmatrix} \tag{A6}$$

From eq. (A2), eq. (A6) becomes

$$\frac{Z-X}{\sqrt{2}}|v_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}-\cos\pi/8\\ \sin\pi/8\end{pmatrix}=\begin{pmatrix}-\cos\pi/8\\ \sin\pi/8\end{pmatrix} \tag{A7}$$

and thus we verify eq. (5a).

To verify eq. (5b), we have

$$\frac{Z-X}{\sqrt{2}}|v_2\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}\sin\pi/8\\ \cos\pi/8\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\sin\pi/8-\cos\pi/8\\ -\cos\pi/8-\sin\pi/8\end{pmatrix} \tag{A8}$$

From eq. (A2), eq. (A8) becomes

$$\frac{Z-X}{\sqrt{2}}|v_2\rangle\quad=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}\sin\pi/8\\ \cos\pi/8\end{pmatrix}=\begin{pmatrix}-\sin\pi/8\\ -\cos\pi/8\end{pmatrix} \tag{A9}$$

and thus we verify eq. (5b).