Exercise 3.3.3

The Pauli matrices are given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

In this problem, we need to verify X, Y, Z are all Hermitian, unitary, square to identity and their eigenvalue are ± 1.

- For X, according to eq. (1),

$$X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \tag{2}$$

which shows that X is Hermitian. Also,

$$X^\dagger X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{3}$$

which shows that X is unitary. Meanwhile, since $X^\dagger = X$ we have $X^\dagger X = X^2 = I$, which shows that X square to identity. Finally, the eigenvalue λ of X is the solution of the following equation,

$$|X - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{4}$$

which shows eigenvalues of X are ± 1.

- For Y, according to eq. (1),

$$Y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y \tag{5}$$

which shows that Y is Hermitian. Also,

$$Y^\dagger Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{6}$$

which shows that Y is unitary. Meanwhile, since $Y^\dagger = Y$ we have $Y^\dagger Y = Y^2 = I$, which shows that Y square to identity. Finally, the eigenvalue λ of Y is the solution of the following equation,

$$|Y - \lambda I| = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{7}$$

which shows eigenvalues of Y are ± 1.

- For Z, according to eq. (1),

$$Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z \tag{8}$$

which shows that Z is Hermitian. Also,

$$Z^\dagger Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{9}$$

which shows that Z is unitary. Meanwhile, since $Z^\dagger = Z$ we have $Z^\dagger Z = Z^2 = I$, which shows that Z square to identity. Finally, the eigenvalue λ of Z is the solution of the following equation,

$$|Z - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{vmatrix} = \lambda^2 - 1 = 0 \iff \lambda = \pm 1 \tag{10}$$

which shows eigenvalues of Z are ± 1.