Exercise 9.2.1

Suppose we have pure state $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ which are defined as

$$|\psi\rangle = \sum_x \sqrt{p(x)}|x\rangle \tag{1a}$$

$$|\phi\rangle = \sum_x \sqrt{q(x)}|x\rangle \tag{1b}$$

where $\{|x\rangle\}$ is some orthonormal basis for $\mathcal{H}$. The fidelity $|\psi\rangle$ and $F(|\psi\rangle, |\phi\rangle)$ is obtained by

$$\begin{aligned}
F(|\psi\rangle, |\phi\rangle) &= |\langle\psi|\phi\rangle|^2 \\
&= \left|\left[\sum_y \left(\sqrt{p(y)}\right)^* \langle y|\right]\left(\sum_x \sqrt{q(x)}|x\rangle\right)\right|^2 \\
&= \left|\sum_{x,y} \left(\sqrt{p(y)}\right)^* \sqrt{q(x)}\langle y|x\rangle\right|^2 \\
&= \left|\sum_x \left(\sqrt{p(x)}\right)^* \sqrt{q(x)}\right|^2
\end{aligned} \tag{2}$$

Note that $\sqrt{p(x)}$ and $\sqrt{q(x)}$ are non-negative real number since $p(x)$ and $q(x)$ are two probability distribution, then eq. (2) becomes

$$F(|\psi\rangle, |\phi\rangle) = \left[\sum_x \sqrt{p(x)q(x)}\right]^2 \tag{3}$$