Exercise 5.1.2

The canonical purification is defined by

$$(I_R \otimes \sqrt{\rho_A})|\Gamma\rangle_{RA} \tag{1}$$

where the unnormalized maximally entangled vector $|\Gamma\rangle_{RA}$ for is given by

$$|\Gamma\rangle_{RA} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_A \tag{2}$$

Note that $\{|i_R\rangle\}$ and $\{|i_A\rangle\}$ are orthonormal basis. If we need to check whether $|\psi\rangle_{RA} = (I_R \otimes \sqrt{\rho_A})|\Gamma\rangle_{RA}$ is a purification of ρ_A, we need to check whether $\mathrm{Tr}_R\{|\psi\rangle_{RA}\langle\psi|_{RA}\} = \rho_A$. Note that density matrix ρ_A should be Hermitian, we have

$$\rho_A^\dagger = (\sqrt{\rho_A}\sqrt{\rho_A})^\dagger = (\sqrt{\rho_A})^\dagger(\sqrt{\rho_A})^\dagger = \rho_A = \sqrt{\rho_A}\sqrt{\rho_A} \tag{3}$$

Note also that $\sqrt{\rho_A}$ is unique positive semi-define square root, so we should have $(\sqrt{\rho_A})^\dagger = \sqrt{\rho_A}$. With help of this, we can calculate the partial trace of canonical purification,

$$\begin{aligned}
\mathrm{Tr}_R\{|\psi\rangle_{RA}\langle\psi|_{RA}\} &= \mathrm{Tr}_R\left\{(I_R \otimes \sqrt{\rho_A})\left(\sum_{i=0}^{d-1}|i\rangle_R \otimes |i\rangle_A\right)\left(\sum_{j=0}^{d-1}\langle j|_R \otimes \langle j|_A\right)(I_R \otimes \sqrt{\rho_A})^\dagger\right\} \\
&= \mathrm{Tr}_R\left\{\sum_{i,j} I_R|i\rangle_R\langle j|_R I_R \otimes \sqrt{\rho_A}|i\rangle_A\langle j|_A\sqrt{\rho_A}\right\}
\end{aligned} \tag{4}$$

If $\{|l\rangle\}$ is a set of orthonormal basis of $\mathcal{H}_R$, then we should have

$$\begin{aligned}
\mathrm{Tr}_R\{|\psi\rangle_{RA}\langle\psi|_{RA}\} &= \sum_{l=0}^{d-1}(\langle l|_R \otimes I_A)\left(\sum_{i,j}|i\rangle_R\langle j|_R \otimes \sqrt{\rho_A}|i\rangle_A\langle j|_A\sqrt{\rho_A}\right)(|l\rangle_R \otimes I_A) \\
&= \sum_{i,j,l}\langle l|i\rangle_R\langle j|l\rangle_R \otimes \sqrt{\rho_A}|i\rangle_A\langle j|_A\sqrt{\rho_A} \\
&= \sum_{i,j}\langle j|i\rangle_R \otimes \sqrt{\rho_A}|i\rangle_A\langle j|_A\sqrt{\rho_A} \\
&= \sum_{i}\sqrt{\rho_A}|i\rangle_A\langle i|_A\sqrt{\rho_A}
\end{aligned} \tag{5}$$

If we write ρ_A into spectral decomposition form such that $\rho_A = \sum_a \lambda_a|a\rangle\langle a|$ where $\{|a\rangle\} = \{|i\rangle_A\}$ is orthonormal basis of $\mathcal{H}_A$, then we have for eq. (5),

$$
\begin{aligned}
\mathrm{Tr}_R\{|\psi\rangle\langle\psi|_{RA}\} &= \sum_i \sqrt{\rho_A}|i\rangle_A\langle i|_A\sqrt{\rho_A} \\
&= \sum_{i,a,b} \left(\sqrt{\lambda_a}|a\rangle\langle a|\right)|i\rangle_A\langle i| \left(\sqrt{\lambda_b}|b\rangle\langle b|\right) \\
&= \sum_{i,a,b} \sqrt{\lambda_a\lambda_b}|a\rangle\langle a|i\rangle\langle i|b\rangle\langle b| \\
&= \sum_a \lambda_a|a\rangle\langle a| = \rho_A
\end{aligned} \tag{6}
$$