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MASTER'S THESIS

# Complexity of threshold functions

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**Abstract**

We consider the complexity of computing Boolean functions by threshold functions. We are interested in lower bounds of maximum coefficients (its absolute values) of this functions. In this work we study two functions: odd-max-bit and decision list (more precisely 1-decision-list − 1-DL for short). We obtain precise lower bounds for coefficients of this functions and compare them.

## 1 Introduction

Threshold functions have been studied in many areas of computer science since 60-s [11, 10].

A threshold function is a function that determines whether a value equality of its inputs exceeded a certain threshold. More precise, a threshold function is specified by coefficients $a_1, a_2, ..., a_n$ and $t$. And it determines whether $a_1x_1 + a_2x_2 + ... + a_nx_n$ bigger than threshold $t$.

There's great interest in exact threshold function to implement different Boolean function [6, 9, 3, 1]. In most works, people are interested in the coefficients of threshold function ($a_i$). The main subject of study is growth rate of absolute value of coefficients depending on the size of the input. In the beginning the main question was: is it polynomial or exponential growth rate? [4]. Later in Johan Hastad's work [8] we can find that there are threshold functions with coefficients at least $n^{\Omega(n)}$.

There're different types of Boolean function representation. But in this work we study decision lists.

Decision list − representation for Boolean functions. We may think of a decision list as an extended "if - then - elseif - ... else -" rule. Much attention is paid to decision lists, as representation for Boolean functions [12, 7]. In this work we will see that any decision list can be represented by threshold function. And for them the question of growth rate of coefficients is quite natural. As a consequence, the question arises about the function that maximizes the coefficients. And this is what we study in this work.

We prove the largest growth of coefficients is growth at the same rate as for Fibonacci numbers, and it is achieved on Odd-max-bit function. Odd-max-bit is a Boolean function whose value is 1 if and only if in the input vector the last 1 is in odd position. This function was considered in different works [5, 2], in this papers the main interest is coefficients of threshold function which represents Odd-max-bit.

In the beginning we consider Odd-max-bit: give definition, lower bounds for coefficients and obtain recursive formula for coefficients of threshold function, which represents odd-max-bit, and based on that we can get precise bound for them.

In second part we consider decision list(1-DL - as its subset). We try to find function which can be represented by 1-DL and which requires maximum coefficients for threshold function's representation $-$ it turns out that this function is Odd-max-bit.

# 2 ODD-MAX-BIT function.

## 2.1 Definition

Let ODD-MAX-BIT (OMB for short) be a function: $\{0,1\}^n \to \{0,1\}$. Let $x \in \{0,1\}^n$ and

$$OMB(x) = 1 \Leftrightarrow max\{i|x_i = 1\} \text{ } is \text{ } odd$$

## 2.2 Threshold function

Let

$$f(x) = \sum_{i=1}^{n} a_i x_i + t, \text{ } where \text{ } a_i \in \mathbb{Z} \text{ } and \text{ } t \in \mathbb{Z}$$

be our threshold function. And

$$\forall x \text{ } f(x) > 0 \Leftrightarrow OMB(x) = 1$$

So we have to answer:

- What is the tight upper bound for a function of the form:

$$g(x) = \sum_{i=1}^{n} a_i + |t|,$$

where $a_i$ - coefficient of our OMB function

## 2.3 Additional notations

For further reasoning, we will use next notations:

- $e_i$ - unit vectors, through which we would represent our input vector $x$
- $e_0$ - zero vector

| $x_1$ | $x_2$ | $x_3$ | $OMB(X)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

Table 1: OMB-function truth table.

### 2.4 Example of threshold function

Consider case for $n = 3$ and $t = -1$. According to Table 1, we can find the corresponding function:

$$f(x) = 2x_1 - x_2 + 3x_3 - 1$$

### 2.5 Bound for coefficients

At the beginning we will define the signs of the coefficients.

**Lemma 1 (Sign of coefficients):** Let

$$a_1,\ a_2,\ \dots\ a_{n-1},\ a_n,\ t$$

$-$ coefficients of threshold function which implements OMB, then following statement is true:

$$a_i = \begin{cases} > 0 & \text{if } i \text{ is odd} \\ < 0 & \text{if } i \text{ is even} \end{cases} \quad \textit{and } t \leq 0$$

**Proof:** Let's show this by considering some special cases:

- $f(e_0)$; since $OMB(e_0) = 0 \Rightarrow f(e_0) \leq 0 \Rightarrow t \leq 0$; for convenience, we will write our function in the following way

$$f(x) = \sum_{i=1}^{n} a_i x_i - t$$

- $f(e_i)$ where $i$ *is odd*, since our function $f(x)$ for this case should be greater than zero, our coefficients with odd index have to be greater than $t(a_i - t > 0)$;

- $f(e_{2k-1} + e_{2k})$ to coincide with $OMB(x)$ our function should be smaller than zero, based on that we get $a_{2k-1} + a_{2k} - t \leq 0$; $a_{2k-1} > t \Rightarrow a_{2k} < 0$ ■

Now, when we determine sign of coefficients, we can obtain the "strongest" inequality for our coefficients.

**Lemma 2 (Lower bound of coefficients):** If

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

− coefficients of threshold function for OMB, the following inequalities are true:

$$a_{2k+1} > \sum_{i-even}^{2k} |a_i| + t \quad (1.a)$$

$$|a_{2k}| \geq \sum_{i-odd}^{2k-1} a_i - t \quad (1.b)$$

**Proof:** From Lemma 1 we know the signs of coefficients, by this we can consider inputs, where coefficient reaches the biggest value:

- For odd current coefficient ($a_{2k+1} > 0$):

$$f(e_2 + e_4 + e_6 + ... + e_{2k} + e_{2k+1}) \Rightarrow a_{2k+1} > \sum_{i-even}^{2k} |a_i| + t$$

- For even current coefficient ($a_{2k} < 0$):

$$f(e_1 + e_3 + e_5 + ... + e_{2k-1} + e_{2k}) \Rightarrow |a_{2k}| \geq \sum_{i-odd}^{2k-1} a_i - t$$

■

Let's consider how we can choose $OMB$-coefficients based on $(1.a)$ and $(1.b)$. It turns out that if

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

− coefficients of threshold function, and satisfy:

$$a_{2k+1} = \sum_{i-even}^{2k} |a_i| + t + 1 \quad (2.a)$$

$$|a_{2k}| = \sum_{i-odd}^{2k-1} a_i - t \quad (2.b)$$

then using them we can implement $OMB$ and minimise the coefficients of threshold function.

More details in the following Lemmas 3 and 4:

**Lemma 3:** If we choose coefficients for threshold function based on $(2.a)$ and $(2.b)$, this function implements $OMB$.

**Proof:** Assume we choose coefficients $a_1,\ a_2,\ ...\ a_{n-1},\ a_n$ based on $(2.a)$ and $(2.b)$, and let's consider some input vector $x$.

Let's assume that on $2k$-position we have latest 1:

$$x_{2k} = 1,\ i > 2k : x_i = 0$$

$$S = \{i \mid i < 2k,\ x_i = 1\}$$

Choice of this coefficient coincides with $(1.b)$. If in previous positions of input vector we don't have 1 in each odd position and do have 1 in some even position, our statement is all the more true:

$$|a_{2k}| = \sum_{\substack{i<2k \\ i-odd}} |a_i| - t = |\sum_{\substack{i<2k \\ i-odd}} a_i| - t \geq |\sum_{i \in S} a_i| - t$$

If latest 1 in $2k+1$-position:

$$x_{2k+1} = 1,\ i > 2k+1 : x_i = 0$$

$$S = \{i \mid i < 2k+1,\ x_i = 1\}$$

For this case coefficient coincides with $(1.a)$. If in previous positions of input vector we don't have 1 in each even position and do have 1 in some odd position, our statement is all the more true:

$$a_{2k+1} = \sum_{i-even}^{2k} |a_i| + t + 1 = |\sum_{i-even}^{2k} a_i| + t + 1 \geq |\sum_{i \in S} a_i| + t + 1$$

■

**Lemma 4:** If we choose $a_i$ based on $(2.a)$, $(2.b)$ and $a'_i$ implements $OMB$ then:

$$\forall i\ :\ |a'_i| \geq |a_i| \quad (3)$$

, and if $|a'_i| > |a_i|$ then:

$$\forall j \geq i\ :\ |a'_j| > |a_j| \quad (4)$$

.

**Proof:** Notation: $(a_1, a_2, .., a_n)$ be coefficients satisfying $(2.a)$, $(2.b)$ and $(a'_1, a'_2, .., a'_n)$ - implement $OMB$.

Let's prove (3)-inequality by induction:

Base: for $n = 1$ by $(1.a)$ and $(2.a)$:

$$a_1 = t + 1$$

$$a'_1 > t$$

Inductive step: Let's assume that for $k$-position (3) is true. Let's assume for simplicity $k$ is even(for odd $k$ - the same situation). For $k+1$-position we have:

$$a_{k+1} = \sum_{i-even}^{2k} |a_i| + t + 1$$

$$a'_{k+1} > \sum_{i-even}^{2k} |a'_i| + t$$

By assumption each coefficient under sums satisfies (3). As a result we can conclude:

$$a'_{k+1} \geq a_{k+1}$$

As for (4)-inequality we'll prove it by induction on $j \geq i$ :

Base case: $j = i$, if $a'_j > a_j$, (4) is obviously true.

Inductive step: Approach is the same as for (3)-inequality, but let's assume that in $k$-position we have $|a'_k| > |a_k|$. Here we have the same bounds for $a_{k+1}$:

$$a_{k+1} = \sum_{i-even}^{2k} |a_i| + t + 1$$

$$a'_{k+1} > \sum_{i-even}^{2k} |a'_i| + t$$

By assumption each coefficient under sums satisfies (3), and one in position $k$ satisfy (4). As a result we can conclude, that for $(k+1)$-position the following inequality holds: $|a'_i| > |a_i|$ ■

Based on Lemma 3 and Lemma 4, which we've proved, further we can consider only equations $(2.a)$ and $(2.b)$ to choose coefficients for OMB.

**Lemma 5:** To fulfill (2a) and (2b), the coefficients must satisfy following equation:

$$\forall n \geq 3 \ : \ \ |a_n| = |a_{n-1}| + |a_{n-2}| \ \ (5)$$

**Proof:**

Base case: n=3 (because we need two previous coefficients for the current):

$$f(x) = (1+t)x_1 - (1)x_2 + (2+t)x_3 - t$$

For $a_1$ there's no previous coefficients so by $(2.a)$ we have only $a_1 = 1 + t$. For $a_2$, now we have previous coefficient $a_1$ and by $(2.b)$:

$$|a_2| = a_1 - t = 1 + t - t = 1$$

So by (7) we have:

$$a_3 = (2+t) = 1 + t + |-1|$$

and by $(2.a)$ we have the same value:

$$a_3 = |a_2| + t + 1 = 2 + t$$

For this our statement is true.

Inductive step:

1. Current coefficient is even. For $2k-1$ it's true. Let's show for $2k$ We need to show $|a_{2k}| = \sum_{i-odd}^{2k-1} a_i - t$ taking into account that $|a_{2k}| = |a_{2k-2}| + a_{2k-1}$. Assume the induction hypothesis $|a_{2k-2}| = \sum_{i-odd}^{2k-3} a_i - t$:

$$|a_{2k}| = |a_{2k-2}| + a_{2k-1} = a_{2k-1} + \sum_{i-odd}^{2k-3} |a_i| - t = |a_{2k}| = \sum_{i-odd}^{2k-1} |a_i| - t$$

2. Current coefficient is odd. For $2k$ it's true. Let's show for $2k+1$. We need to show $a_{2k+1} = \sum_{i-even}^{2k} |a_i| + t + 1$ taking into account that $a_{2k+1} = |a_{2k}| + a_{2k-1}$

   Assume the induction hypothesis $a_{2k-1} = \sum_{i-even}^{2k-2} |a_i| + t + 1$:

$$a_{2k+1} = |a_{2k}| + a_{2k-1} = |a_{2k}| + \sum_{i-even}^{2k-2} |a_i| + t + 1 = a_{2k+1} = \sum_{i-even}^{2k} |a_i| + t + 1 \blacksquare$$

There's another question: **How do we choose t to minimise coefficients?**

As we can see in previous proof (base case) first coefficient contains t:

$$a_1 = (1 + t),\ a_2 = -1$$

and it's initial values for our sequence of coefficients. And taking into account the dependence current coefficient on the previous(Lemma 5) we can conclude that any value of $t$ other than zero would lead us to faster growth of $g(x)$.

Let's formulate theorem, as a result of lemmas proved above:

**Theorem 1:** Let

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

$-$ optimal coefficients of threshold function which implements OMB (satisfy $(2.a)$ and $(2.b)$), then

$$|a_1|,\ |a_2|,\ ...\ |a_{n-1}|,\ |a_n|$$

represent the Fibonacci's sequence.

**Proof:** As we said for our coefficients we have to use $t = 0$ and based on that our $(2.a)$ and $(2.b)$ become:

$$a_{2k+1} = \sum_{i-even}^{2k} |a_i| + 1 \quad (7.a)$$

$$|a_{2k}| = \sum_{i-odd}^{2k-1} a_i \quad (7.b)$$

So we have:

$$a_1 = 1, \ a_2 = -1$$

After we can use result of Lemma 5 - (5), and we get Fibonacci's sequence.

■

# 3 Decision list

In this part we consider decision lists, represent OMB by them. Also we study bounds for threshold function's coefficients to implement decision list, and try to determine Boolean function with which we maximise coefficients.

## 3.1 Definition

A decision list of length r is a list $L$ of pairs:

$$L = (f_1, v_1), ..., (f_r, v_r) \qquad (8)$$

where $f_i$ is the i-th formula and $v_i$ is the i-th boolean for $i \in \{1...r\}$, and the last function $f_i$ is the constant function true. A decision list $L$ defines a Boolean function as follows: for any assignment $x \in \{0,1\}^n$ $L(x)$ is defined to be equal to $v_i$ where $i$ is the least index such that $f_i(x) = 1$. (Such an item always exists, since the last function is always true.)

We may think of a decision list as an extended "if - then - elseif - ... else -" rule.

## 3.2 1-DL

A k-DL is a decision list where all of formulas have at most k terms. Sometimes "decision list" is used to refer to a 1-DL, where all of the formulas are either a variable or its negation. In this article we consider 1-DL. Let's assume that $v_0 \neq v_1$, otherwise for any value of $x_1$ we have the same value for 1-DL-function.

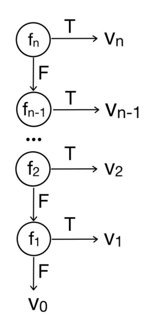


Figure 1: 1-DL

## 3.3 OMB

We can represent $OMB$ function, which was described in previous part, by 1-DL.

As a vector $v$, for odd n, for example, we have:

$$v_{OMB} = (0, 1, 0, 1, ..., 0, 1, 0, 1)$$

and as $f_i - x_i$(Figure 2).

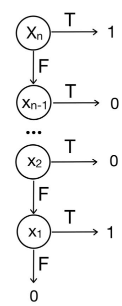


Figure 2: OMB as 1-DL

We've considered threshold functions and how we can describe $OMB$-function by them. Also we've given tight bound for $OMB$ threshold function's coefficients.

## 3.4 Coefficients of 1-DL function

**Lemma 6 (Sign of coefficients for 1-DL):** Let's

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

− coefficients of threshold function which implements 1-DL, then following statement is true:

$$a_i = \begin{cases} > 0 & \text{if } v_i \text{ is } 1 \\ < 0 & \text{if } v_i \text{ is } 0 \end{cases} \quad for\ \forall t$$

**Proof:** Here we consider two cases: $v_0 = 0$ and $v_1 = 1$.

For $v_0 = 0$:

- $f(e_0)$; since 1-DL($e_0$) $= 0 \Rightarrow f(e_0) \leq 0 \Rightarrow t \leq 0$;
- $f(e_i)$ where $v_i = 1$, since our function $f(x)$ for this case should be greater than zero, our coefficients have to be greater than $t(a_i - t > 0)$;
- $f(e_i + e_j)$ where $v_i = 1$ and $v_j = 0$ $i < j$ to coincide with 1-DL($x$) our function should be smaller than zero, based on that we get $a_i + a_j - t \leq 0$; $a_i > t \Rightarrow a_j < 0$

For $v_0 = 1$:

- $f(e_0)$; since 1-DL$(e_0) = 1 \Rightarrow f(e_0) > 0 \Rightarrow t > 0$;
- $f(e_1)$; by our assumption $(v_0 \neq v_1) \Rightarrow v_1 = 0 \Rightarrow a_1 \leq -t$;
- $f(e_1 + e_i)$ where $v_i = 1$, since our function $f(x)$ for this case should be greater than zero, our coefficients have to be greater than $-t$ $(a_1 + a_i + t > 0)$ and $a_1 \leq -t \Rightarrow a_i > 0$;
- $f(e_i)$ where $v_i = 0$, since our function $f(x)$ for this case should be less or equal to zero, our coefficients have to be smaller or equal to $-t$ $(a_i + t \leq 0)$ $\Rightarrow a_i < 0$;

■

**Lemma 7 (Lower bound of coefficients for 1-DL):** If

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

– coefficients of threshold function for 1-DL, the following inequalities are true:
For $t \leq 0$:

- For $v_n = 1$:

$$a_n > \sum_{\substack{i<n \\ v_i=0}} |a_i| + |t| \quad (9.a)$$

- For $v_n = 0$:

$$|a_n| \geq \sum_{\substack{i<n \\ v_i=1}} a_i - |t| \quad (9.b)$$

For $t > 0$:

- For $v_n = 1$:

$$a_n > \sum_{\substack{i<n \\ v_i=0}} |a_i| - t \quad (10.a)$$

- For $v_n = 0$:

$$|a_n| \geq \sum_{\substack{i<n \\ v_i=1}} a_i + t \quad (10.b)$$

**Proof:** Based on Lemma 6 and special inputs when $a_n$ reaches biggest value.(The same approach as we did for OMB - Lemma 2):

$$x_n = 1,\ v_n = 1,\ \forall i\ v_i = 0 \Rightarrow x_i = 1$$

$$x_n = 1,\ v_n = 0,\ \forall i\ v_i = 1 \Rightarrow x_i = 1$$

As a result we have to consider all coefficients with opposite sign. ■

Let's consider how we can choose 1-DL-coefficients based on (9) – (10). If

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

- coefficients of threshold function for 1-DL, and for $v_n = 1$ and $v_n = 0$ satisfy:

for $t \leq 0$:

$$a_n = \sum_{\substack{i<n \\ v_i=0}} |a_i| + |t| + 1 \quad (11.a)$$

$$|a_n| = \sum_{\substack{i<n \\ v_i=1}} a_i - |t| \quad (11.b)$$

for $t > 0$:

$$a_n = \sum_{\substack{i<n \\ v_i=0}} |a_i| - t + 1 \quad (12.a)$$

$$|a_n| = \sum_{\substack{i<n \\ v_i=1}} a_i + t \quad (12.b)$$

then using them we can implement 1-DL and minimise coefficients.

More detail in the following Lemmas 8 and 9:

**Remark:**In Lemmas 8-11 we'll proof only for (11) equation, as for (12) proof is absolutely the same.

**Lemma 8:** If we choose coefficients for threshold function based on (11) − (12), they implements 1-DL.

**Proof:** Assume we choose coefficients $a_1,\ a_2,\ ...\ a_{n-1},\ a_n$ based on (11), and let's consider some input vector $x$.

Let's assume that on $k$-position we have latest 1 and $v_k = 0$:

$$x_k = 1,\ i > k : x_i = 0$$

$$S = \{i \mid i < k,\ x_i = 1\}$$

Choice of this coefficients coincides with $(9.b)$. If in previous positions of input vector we don't have 1 in each position where $v_i = 1$ and do have 1 where $v_i = 0$, our statement is all the more true:

$$|a_k| = \sum_{\substack{i<k \\ v_i=1}} |a_i| - |t| = |\sum_{\substack{i<k \\ v_i=1}} a_i| - |t| \geq |\sum_{i \in S} a_i| - |t|, \quad t \leq 0$$

If latest 1 in $k$-position where $v_k = 1$:

$$x_k = 1,\ i > k : x_i = 0$$

$$S = \{i \mid i < k,\ x_i = 1\}$$

For this case coefficients coincides with $(9.a)$. If in previous positions of input vector we don't have 1 in each position where $v_i = 0$ and do have 1 where $v_i = 1$, our statement is all the more true:

$$a_k = \sum_{\substack{i<k \\ v_i=0}} |a_i| + |t| + 1 = |\sum_{\substack{i<k \\ v_i=0}} a_i| + |t| + 1 \geq |\sum_{i \in S} a_i| + |t| + 1, \quad t \leq 0$$

$\blacksquare$

**Lemma 9:** If we choose $a_i$ based on $(11)-(12)$ and $a'_i$ implements 1-DL then:

$$\forall i \ : \ |a'_i| \geq |a_i| \quad (13)$$

, and if $|a'_i| > |a_i|$ then:

$$\forall j > i \ : \ |a'_j| > |a_j| \quad (14)$$

.

**Proof:** Notation: $(a_1, a_2, .., a_n)$ be coefficients satisfying $(11.a)$, $(11.b)$ and $(a'_1, a'_2, .., a'_n)$ - implement $OMB$.

Let's prove (13)-inequality by induction:

Base: for $k = 1,\ v_1 = 1$ by $(9.a)$ and $(11.a)$:

$$a_1 = |t| + 1$$

$$a'_1 > t$$

Inductive step: Let's assume that for $k$-position (13) is true. For $k+1$-position we have:

$$a_{k+1} = \sum_{\substack{i<k+1 \\ v_i=0}} |a_i| + |t| + 1$$

$$a'_{k+1} > \sum_{\substack{i<k+1 \\ v_i=0}} |a'_i| + |t|$$

By assumption each coefficient under sums satisfies (13). As a result we can conclude:

$$a'_{k+1} \geq a_{k+1}$$

As for (14)-inequality we'll prove it by induction on $j \geq i$ :

Base case: $j = i$, if $a'_j > a_j$, (14) is obviously true.

Inductive step: Approach is the same as for (13)-inequality, but let's assume that in $k$-position we have $|a'_k| > |a_k|$. Here we have the same bounds for $a_{k+1}$:

$$a_{k+1} = \sum_{\substack{i<k+1 \\ v_i=0}} |a_i| + |t| + 1$$

$$a'_{k+1} > \sum_{\substack{i<k+1 \\ v_i=0}} |a'_i| + t$$

By assumption each coefficient under sums satisfies (13), and one in position $k$ satisfy (14). As a result we can conclude, that for $(k+1)$-position the following inequality holds: $|a'_i| > |a_i|$ $\blacksquare$

Based on Lemma 8 and Lemma 9, which we've proved, further we can consider only equations $(11)-(12)$ to choose coefficients for 1-DL.

Now we can determine the dependence of the current coefficient on the previous ones:

**Lemma 10:** If

$$a_1,\ a_2,\ \dots\ a_{n-1},\ a_n,\ t$$

− optimal coefficients of threshold function for 1-DL (satisfy (11) and (12)), and if for some position $k$ and $k+1$ we have:

$$v_k = v_{k+1}$$

, then:

$$a_k = a_{k+1}$$

**Proof:** We have to consider two cases − $v_k = v_{k+1} = 1$ and $v_k = v_{k+1} = 0$. For the first we use (11.$a$):

$$a_k = \sum_{\substack{i<k \\ v_i=0}} |a_i| + |t| + 1$$

$$a_{k+1} = \sum_{\substack{i<k+1 \\ v_i=0}} |a_i| + |t| + 1$$

As $v_k = v_{k+1} = 1$ numbers of $a_i$ where $v_i = 0$ for $a_k$ and $a_{k+1}$ are equal, and under the sum we have the same value.

For the second ($v_k = v_{k+1} = 0$) we use (11.$b$) and we can see the same situation. ■

**Lemma 11:** If

$$a_1,\ a_2,\ \dots\ a_{n-1},\ a_n,\ t$$

− optimal coefficients of threshold function for 1-DL (satisfy (11) and (12)), and if:

$$v_{k-1} \neq v_k,$$

$$i \in [k; k+\alpha),\ v_i = z$$

$$v_{k-1} = v_{k+\alpha}$$

, then to satisfy (11.$a$) and (11.$b$) for $v_{k+\alpha} = 1$:

$$a_{k+\alpha} = a_{k-1} + \alpha|a_k| \quad (15)$$

, and for $v_{k+\alpha} = 0$:

$$|a_{k+\alpha}| = |a_{k-1}| + \alpha a_k \quad (16)$$

**Proof**:

Base case: To check our statement we need two changes of values in vector $v$. By our assumption for 1-DL($v_1 \neq v_0$), first change we have in first position, next change in $(1+\alpha)$-position ($\alpha \geq 1$). It our base case. There're two cases:

- $v_1 = 1$ Choose $a_1$ and $a_2$ based on (11):

$$a_1 = (t+1) \quad a_2 = (-1)$$

By Lemma 10 till the next change of values ($(1+\alpha)$-position) we have:

$$i \in (1, 1+\alpha] : \quad a_i = a_2$$

For $(1+\alpha)$-position by (15):

$$a_{1+\alpha+1} = a_1 + \alpha(|a_{2|}) = t + 1 + \alpha(|-1|))$$

And it coincides with (11.$a$), as we take into account all coefficients where $v_i = 0$ ($(1, 1+\alpha]$) and $t+1$;

- $v_1 = 0$ Choose $a_1$ and $a_2$ based on (12):

$$a_1 = (-t) \quad a_2 = (1)$$

By Lemma 10 till the next change of values ($(1+\alpha)$-position) we have:

$$i \in (1, 1+\alpha] : \quad a_i = a_2$$

For $(1+\alpha)$-position by (16):

$$|a_{1+\alpha+1}| = |a_1| + \alpha(a_2) = t + \alpha(1)$$

And it coincides with (12.$b$), as we take into account all coefficients where $v_i = 1$ ($(1, 1+\alpha]$) and $t$.

Induction step:
For $v_{k+\alpha} = 1$:
For $a_{k-1}$ assume (11.$a$) holds, so rewrite (15):

$$a_{k+\alpha} = \sum_{\substack{i<k-1 \\ v_i=0}} |a_i| + |t| + 1 + \alpha|a_k|$$

By Lemma 10 $a_i$, where $i \in [k; k+\alpha)$, are equal, let's denote it as $a_k$. As in this position $v_i = 0$ if we insert this coefficients under the sum we will take into account all coefficients where $v_i = 0$, and this is (11.$a$)
For $v_{k+\alpha} = 0$:
The same way but we use (11.$b$) ■

Now we can try to find Boolean function, which can be represented by 1-DL, and requires to maximise coefficients of threshold function. For the start introduce the following type of change, denote it as **extra-change**:

1. Take any sequence 0/1 with length 2 or longer;

2. Invert last 0/1 in this sequence, and invert all 0/1 after this position.

**Lemma 12:** After applying the extra-change to any vector $v$ (if such change can be used) the coefficients of the corresponding threshold function satisfy the following inequality:

$$|a_i'| \geq |a_i| \quad (17)$$

**Notation:**
$v$- original vector;
$v'$ - after the change;
$a_i$ and $a_i'$ - corresponding coefficients;
$y = \sum_{\substack{i \leq k-1 \\ v_i=0}} |a_i| + 1 = \sum_{\substack{i \leq k-1 \\ v_i=0}} |a_i'| + 1$;
$x = \sum_{\substack{i \leq k-1 \\ v_i=1}} |a_i| = \sum_{\substack{i \leq k-1 \\ v_i=1}} |a_i'|$.

**Proof:** Make it by induction. Prove it for $v_0 = 0$,as for the opposite we would have identical proof. We take any sequence of 0 (or 1), with length equals to 2 or longer, let's length would be 2, and let's this sequence starts in $k$-position. Our change begins from $(k+1)$-position. After this position 0 and 1 replace each other in the same position(Table 1 and 2), but in opposite direction. Until $k$-position we have the same $v_i$, and as a consequence the coefficients are also equal. Based on Lemma 10 since $(k+1)$-position we need to analyse difference of our coefficients only in position of changing value. It would be our step. And we need to show that our coefficients satisfy inequality (17) in "change-value"-position in vector $v$

1. Base case:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a_i$ | $x$ | $x$ | $2x+y$ | $2x+y$ | $2x+y$ | $3(2x+y)+x$ | $3(2x+y)+x$ |
| $v_i$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| $a_i$ | $x$ | $x+y$ | $2x+y$ | $2x+y$ | $2x+y$ | $3(2x+y)+x+y$ | $3(2x+y)+x+y$ |
| $v_i'$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| pos | $k$ | $k+1$ | $k+2$ | $k+3$ | $k+4$ | $k+5$ | $k+6$ |

Table 2: Changes in $v$ for sequence of 0.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a_i$ | $y$ | $y$ | $2y+x$ | $2y+x$ | $2y+x$ | $3(2y+x)+y$ | $3(2y+x)+y$ |
| $v_i$ | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $a_i$ | $x$ | $y+x$ | $2y+x$ | $2y+x$ | $2y+x$ | $3(2y+x)+y+x$ | $(2y+x)+y+x$ |
| $v_i'$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| pos | $k$ | $k+1$ | $k+2$ | $k+3$ | $k+4$ | $k+5$ | $k+6$ |

Table 3: Changes in $v$ for sequence of 1.

Let's consider sequence of 0, described above. Based on **Lemma 10** there's no growth for $v$ and increasing coefficient for $v'$ as a result ($a_{k+1} \leq a'_{k+1}$) in $(k+1)$-position (Table 1). Let's analyze the subsequent behavior of our coefficients in first common position of changing value in vector $v$ and $v'$ - $(k+2)$-position. As we can see from Table 1 we satisfy (15)-inequality.

2. Inductive step: Until $n-1$ our statement is true, and based on **Lemma 11** for n we have:

$$a_n = |a_{n-1}|\alpha + a_{n-1-\alpha}$$
$$a'_n = |a'_{n-1}|\alpha + a'_{n-1-\alpha}$$

, where $\alpha$ - length of previous sequence of same values. As by induction $a_{n-1} \leq a'_{n-1}$ and $a_{n-1-\alpha} \leq a'_{n-1-\alpha}$ we have $a_n \leq a'_n$. We can see in Table 2 - in case of sequence of 1 - we have the same situation. ■

Applying extra-changes a certain number of times, we will come to the moment when our vector $v$ become a sequence of alternation of zeros and ones. And there're two cases:

- $i - odd : v_i = 1, \quad i - even : v_i = 0$ – It's OMB, and for this we already know precise lower bound;
- $i - odd : v_i = 0, \quad i - even : v_i = 1$ – It's inverse of OMB, Lemma 13 will be about this function.

**Lemma 13:** If

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

– optimal coefficients of threshold function for 1-DL (satisfy (12)), and if:

$$v_i = \begin{cases} = 1 & \text{if } i \text{ is even} \\ = 0 & \text{if } i \text{ is odd} \end{cases} \quad and\ t > 0$$

then the coefficients must satisfy following equation:

$$\forall n \geq 3\ :\ \ |a_n| = |a_{n-1}| + |a_{n-2}| \quad (18)$$

**Proof:**
Base case: n=3 (because we need two previous coefficients for the current):

$$f(x) = (-t)x_1 + (1)x_2 - (t+1)x_3 + t$$

For $a_1$ there's no previous coefficients so by (12.$b$) we have only $a_1 = -t$. For $a_2$, now we have previous coefficient $a_1$ and by (12.$a$):

$$a_2 = |a_1| - t + 1 = t - t + 1 = 1$$

So by (18) we have:

$$a_3 = -(t+1) = -(|-t| + |1|)$$

and by (12.$b$) we have the same value:

$$|a_3| = |a_2| + t = 1 + t$$

For this our statement is true.
Inductive step:

1. $v_k = 0$. For $k-1$ it's true. Let's show for $k$ We need to show $|a_k| = \sum_{\substack{i<k \\ v_i=1}} a_i + t$ taking into account that $|a_k| = |a_{k-2}| + a_{k-1}$. Assume the induction hypothesis $|a_{k-2}| = \sum_{\substack{i<k-2 \\ v_i=1}} a_i + t$:

$$|a_k| = |a_{k-2}| + |a_{k-1}| = |a_{k-1}| + \sum_{\substack{i<k-2 \\ v_i=1}} a_i + t = \sum_{\substack{i<k \\ v_i=1}} a_i + t$$

2. $v_k = 1$. For $k-1$ it's true. Let's show for $k$ We need to show $a_k = \sum_{\substack{i<k \\ v_i=0}} a_i - t + 1$ taking into account that $|a_k| = |a_{k-2}| + a_{k-1}$. Assume the induction hypothesis $|a_{k-2}| = \sum_{\substack{i<k-2 \\ v_i=0}} a_i - t + 1$:

$$|a_k| = |a_{k-2}| + |a_{k-1}| = |a_{k-1}| + \sum_{\substack{i<k-2 \\ v_i=0}} a_i - t + 1 = \sum_{\substack{i<k \\ v_i=0}} a_i - t + 1$$

$\blacksquare$

Let's formulate theorem, as a result of Lemma 13:

**Theorem 2:** Let

$$a_1,\ a_2,\ ...\ a_{n-1},\ a_n,\ t$$

− optimal coefficients of threshold function which implements inverse of OMB (satisfy (12)), 1-DL with:

$$v_i = \begin{cases} = 1 & \text{if } i \text{ is even} \\ = 0 & \text{if } i \text{ is odd} \end{cases} \quad and\ t > 0$$

then

$$|a_1|,\ |a_2|,\ ...\ |a_{n-1}|,\ |a_n|$$

represent the Fibonacci's sequence with shift equals $t$.

**Proof:** As we said for our coefficients we have to use $t > 0$ and based on that our $(12.a)$ and $(12.b)$ become:

$$a_1 = -t,\ a_2 = 1$$

After we can use result of Lemma 12 - (18), and we get Fibonacci's sequence with shift $t$. $\blacksquare$

In the end let's formulate theorem about maximum coefficients of function represented by 1-DL:

**Theorem 3:** There's no function which can be represented by 1-DL and its threshold function's coefficients would be bigger than for inverse of $OMB$-function.

**Proof:** Let's assume that there's 1-DL function(represented by $v_{max}$) and its threshold function's coefficients are maximum possible values. As for OMB and its inverse we have $v_{omb}$ as alternation of zeros and ones, our $v_{max}$ must contain a sequence of consecutive 0 or 1. But we can apply the extra-change, described above, to our $v_{max}$, and based on Lemma 12 its corresponding threshold function coefficients can't decrease, and thus we can apply extra-change to our original vector without decreasing of coefficients. But in some moment $v_{max}$ turns into $v_{omb}$ or the inverse of $v_{omb}$, alternation of zeros and ones. As was said in Theorem 2 for inverse of $v_{omb}$ coefficients are Fibonacci sequence with shift $t$.

How do we choose $t$ to minimise our coefficients? For this we have to consider two cases:

- $v_0 = 1$: For this we use (12):

$$a_1 = -t, \ a_2 = 1$$

- $v_0 = 0$: For this we use (11):

$$a_1 = 1 + t, \ a_2 = -1$$

and it's initial values for our sequences of coefficients. And taking into account the dependence current coefficient on the previous(Lemma 11) we can conclude that to minimise coefficients we have to minimise $t$. For $v_0 = 0$, as for $OMB$, we can choose $t = 0$, but for $v_0 = 1$ as $t > 0$, the best we can make is to set $t = 1$. In both case it's Fibonacci sequence. And this is maximum for coefficients of 1-DL functions.

■

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