Lecture 3

In the previous lecture we derived two crucial properties of the $T\bar{T}$ operator, namely the finiteness and factorization of its expectation value. From this analysis we learned that any Poincaré invariant QFT defined on the plane or the cylinder has a composite operator with scaling dimension $\Delta = 4$. We then showed that in theories with higher spin conserved currents it's possible to find an infinite number of irrelevant operators analogous to $T\bar{T}$ with integer scaling dimension $\Delta \geq 3$. In this lecture we will show that the $T\bar{T}$ deformation of an integrable QFT preserves its integrability. Then we will exploit the factorizability of the $T\bar{T}$ operator to derive a universal differential equation for the spectrum of $T\bar{T}$-deformed QFTs.

3.3 Integrability

An integrable QFT (IQFT) is characterized by *an infinite number of commuting conserved charges* which we denote by P_s and $\bar{P}_s$ such that

$$[P_s, P_{s'}] = [P_s, \bar{P}_{s'}] = [\bar{P}_s, \bar{P}_{s'}] = 0. \tag{3.1}$$

A generic deformation of an IQFT breaks this property, i.e. it breaks integrability. Interestingly, deforming an IQFT by any of the X_s operators preserves integrability, at least infinitesimally in the deformation. We will now show how this comes about.

Let us consider a deformation of an IQFT such that its action I_{IQFT} is deformed by

$$I_{IQFT} \to I_{IQFT} + \delta I, \qquad \delta I = \mu_s \int d^2 z X_s. \tag{3.2}$$

Since the X_s operator has scaling dimension $\Delta = 2(s+1)$ the deformation is irrelevant and $[\mu_s] = L^{2s}$. The conserved charges P_σ, $\bar{P}_\sigma$ where defined in the previous lecture to be given by

$$P_\sigma = \int_c \star j_{\sigma+1} = \int_c \big(T_{\sigma+1} dz + \theta_{\sigma-1} d\bar{z}\big), \qquad \bar{P}_\sigma = \int_c \star \bar{j}_{\sigma+1} = \int_c \big(\bar{\theta}_{\sigma-1} dz + \bar{T}_{\sigma+1} d\bar{z}\big). \tag{3.3}$$

The fact that all of the P_σ currents commute implies that their commutator with the higher spin currents must be given by derivatives of local operators such that

$$\begin{aligned} [P_\sigma, T_{s+1}(z)] &= \partial_z A_{\sigma+s}(z), \qquad & [P_\sigma, \theta_{s-1}(z)] &= \partial_{\bar{z}} A_{\sigma+s}(z), \\ [P_\sigma, \bar{T}_{s+1}(z)] &= \partial_{\bar{z}} B_{\sigma,s}(z), \qquad & [P_\sigma, \theta_{s-1}(z)] &= \partial_z B_{\sigma,s}(z). \end{aligned} \tag{3.4}$$

where $A_{\sigma+s}(z) \equiv A_{\sigma+s,0}(z)$. Similar expressions exist for $\bar{P}_\sigma$.

The commutators (3.4) imply that the commutator between the charges P_σ, $\bar{P}_\sigma$ and the irrelevant operator X_s is also a total derivative. In order to see this we first note that

$$[P_\sigma, X_s(z')] = \lim_{z\to z'} \underbrace{[P_\sigma, T_{s+1}]\bar{T}_{s+1} - [P_\sigma, \theta_{s-1}]\bar{\theta}_{s-1}}_{I_1(z,z')} + \underbrace{T_{s+1}[P_\sigma, \bar{T}_{s+1}] - \theta_{s-1}[P_\sigma, \bar{\theta}_{s-1}]}_{I_2(z,z')}, \tag{3.5}$$

where we dropped the coordinate dependence of the currents for convenience. Using the commutators (3.4) we can write $I_1(z,z')$ as follows

$$I_1(z,z') = \partial_z A_{\sigma+s}(z)\bar{T}_{s+1}(z') - \partial_{\bar{z}} A_{\sigma+s}(z)\bar{\theta}_{s-1}(z') + A_{\sigma+s}(z)\big(\underbrace{\partial_{z'}\bar{T}_{s+1}(z') - \partial_{\bar{z}'}\bar{\theta}_{s-1}(z')}_{0}\big) \tag{3.6}$$

$$= (\partial_z + \partial_{z'})\big(A_{\sigma+s}(z)\bar{T}_{s+1}(z')\big) - (\partial_{\bar{z}} + \partial_{\bar{z}'})\big(A_{\sigma+s}(z)\bar{\theta}_{s-1}(z')\big), \tag{3.7}$$

where we used the conservation law of the currents $\partial_z \bar{T}_{s+1}(z) = \partial_{\bar{z}}\theta_{s-1}(z)$ to add the third term in the first line. Similarly, we find that $I_2(z,z')$ is a total derivative

$$I_2(z,z') = (\partial_{\bar{z}} + \partial_{\bar{z}'})\big(B_{\sigma,s}(z)T_{s+1}(z')\big) - (\partial_z + \partial_{z'})\big(B_{\sigma,s}(z)\theta_{s-1}(z'). \tag{3.8}$$

As a result, we find that in the limit $z \to z'$, the commutator $[P_\sigma, X_s(z)]$ is a combination of total derivative terms, meaning that it can be written as

$$[P_\sigma, X_s(z)] = \partial_{\bar{z}}\hat{T}_{\sigma+s+1,s}(z) - \partial_z\hat{\theta}_{\sigma+s,s+1}(z), \tag{3.9}$$

for some $\hat{T}_{\sigma+s+1,s}(z)$ and $\hat{\theta}_{\sigma+s,s+1}(z)$.

Exercise 3.1: the operators $\hat{T}_{\sigma+s+1,s}$ and $\theta_{\sigma+s,s+1}$ can be determined explicitly from $I_1(z,z')$ and $I_2(z,z')$. Verify that the spin and scaling dimension of these operators are consistent with those of the commutator appearing on the left-hand side of (3.9).

In order to see that the X_s deformations preserve integrability, we must show that the conservation law of the higher spin currents $j_{\sigma+1}$ and $\bar{j}_{\sigma+1}$ continues to hold after the deformation. This is equivalent to showing that the contour integrals $\oint \star j_{\sigma+1}$ and $\oint \star \bar{j}_{\sigma+1}$ vanish within correlation functions. Focusing on $j_{\sigma+1}$ we will show that

$$\big\langle \Pi_i \mathcal{O}_i(z_i)\, d\star j_{\sigma+1}(z)\big\rangle = 0 \qquad \Longrightarrow \qquad \mathcal{J}_{\sigma+1} \equiv \big\langle \Pi_i \mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle = 0, \tag{3.10}$$

where $\partial\mathcal{D}$ denotes a closed contour that encloses a region $\mathcal{D}$ that is free of operator insertions. In other other words, the contour $\partial\mathcal{D}$ is chosen so that the $\mathcal{O}_i(z_i)$ operators are inserted in the complement $\bar{\mathcal{D}}$ of $\mathcal{D}$, as shown in figure 1.

Before the deformation, $\mathcal{J}_{\sigma+1} = 0$ since the current $j_{\sigma+1}$ is conserved. After the deformation (3.2), the correlation function becomes

$$\begin{aligned} \delta\mathcal{J}_{\sigma+1} = {}& \big\langle \Pi_i \delta_\mu \mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle + \big\langle \Pi_i \mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star \delta_\mu j_{\sigma+1}(z)\big\rangle \\ & - \mu \int d^2w \big\langle X_s(w) \Pi_i \mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle + \mathcal{O}(\mu^2), \end{aligned} \tag{3.11}$$

where $\delta_\mu \mathcal{O}_i$ and $\delta_\mu j_{\sigma+1}$ denote the change of the operators $\mathcal{O}_i$ and the currents $j_{\sigma+1}$ under the deformation. Since $\oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z) = 0$ before the deformation, i.e. to zeroth order in μ, the first term vanishes at $\mathcal{O}(\mu)$. We will see that we can always choose the flow of the current $\delta_\mu j_{\sigma+1}$

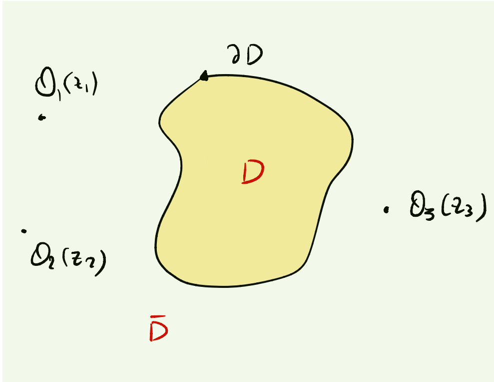

Figure 1: The contour $\partial\mathcal{D}$.

such that the RHS of (3.11) vanishes. In order to see this note that the integral in the last term of (3.11) can be spit into

$$\int d^2w \big\langle X_s(w)\Pi_i\mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle = \int_{\bar{\mathcal{D}}} d^2w\langle\ldots\rangle + \int_{\mathcal{D}} d^2w\langle\ldots\rangle. \tag{3.12}$$

In the first term, the irrelevant operator $X_s(w)$ is inside $\bar{\mathcal{D}}$, i.e. outside the region bounded by $\partial\mathcal{D}$. Hence $\oint \star j_{\sigma+1}(z) = 0$ implies that the first term vanishes

$$\int_{\bar{\mathcal{D}}} d^2w \big\langle X_s(w)\Pi_i\mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle = 0. \tag{3.13}$$

In the second term, the $X_s(w)$ operator is inside $\mathcal{D}$ so we can write

$$\int_{\mathcal{D}} d^2w \big\langle X_s(w)\Pi_i\mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle = \big\langle\Pi_i\mathcal{O}_i(z_i) \int_{\mathcal{D}} d^2w \underbrace{\oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)X_s(w)}\big\rangle \tag{3.14}$$

$$= \big\langle\Pi_i\mathcal{O}_i(z_i) \int_{\mathcal{D}} d^2w \qquad [P_\sigma, X_s(w)]\big\rangle. \tag{3.15}$$

Exercise 3.2: Verify that the commutator $[P_\sigma, X_s(w)]$ is indeed given by the contour integral $\oint_c (T_{\sigma+1}(z)X_s(w)dz + \theta_{\sigma-1}(z)X_s(w)d\bar{z})$.

We have previously shown that the commutator $[P_\sigma, X_s(w)]$ is a total derivative (3.9). Introducing the following notation[2]

$$\hat{j}_{\sigma+2s+1} \equiv \hat{T}_{\sigma+s+1,s}dz - \hat{\theta}_{\sigma+s,s+1}d\bar{z}, \tag{3.16}$$

this commutator can be written as $[P_\sigma, X_s(w)] = \partial_\mu \hat{j}^\mu_{\sigma+s}$. We thus have,

$$\int_{\mathcal{D}} d^2w \big\langle X_s(w)\Pi_i\mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle = \big\langle\Pi_i\mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star\hat{j}_{\sigma+2s+1}\big\rangle. \tag{3.17}$$

[2]Note that this is not necessarily a conserved current.

Altogether, the deformed correlation function (3.11) is given to linear order in μ_s by

$$\delta J_{\sigma+1} = \delta\big\langle \Pi_i \mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star j_{\sigma+1}(z)\big\rangle = \big\langle \Pi_i \mathcal{O}_i(z_i) \oint_{\partial\mathcal{D}} \star(\delta j_\sigma - \mu_s \hat{j}_{\sigma+2s+1})\big\rangle. \tag{3.18}$$

Thus, we can guarantee that each of the conserved currents j_σ of the original IQFT is conserved after the deformation of any of the irrelevant $X_s(z)$ operators provided that we choose the flow of the current to be given by

$$\delta j_{\sigma+1} = \delta T_{s+1} dz - \delta\theta_{\sigma-1} d\bar{z} = \mu_s \hat{T}_{\sigma+s+1,s} dz - \mu_s \theta_{\sigma+s,s+1} d\bar{z} = \mu_s \hat{j}_{\sigma+2s+1}. \tag{3.19}$$

We have not shown that the IQFT remains integrable after the deformation, i.e. that $[P_\sigma, P_s]$. This is still expected to be the case, however, since $Q_{\sigma,s} = [P_\sigma, P_s]$ must either be a new conserved charge of spin $\sigma + s$ or an old charge such that $Q_{\sigma,s} = P_{\sigma+s}$. New conserved currents or a new algebraic structure are not expected to appear for infinitesimal μ such that $[P_\sigma, P_s] = 0$.

3.4 The spectrum

One of the crucial properties of the $T\bar{T}$ operator is the factorizability of its expectation value into the product of one-point functions of the stress tensor. These one-point functions vanish when the undeformed QFT is defined on the plane. On the other hand, when the theory is put on the cylinder, one-point functions of the stress tensor are related to the energy and angular momentum of the state. The $T\bar{T}$ flow equation then implies a universal differential equation for the deformed spectrum. Moreover, when the undeformed theory is scale invariant, the differential equation for the spectrum can be solved explicitly and we are able to obtain universal expressions for the deformed spectrum of $T\bar{T}$-deformed CFTs.

Let us consider an eigenstate $|n\rangle$ of the energy and angular momentum in a $T\bar{T}$-deformed QFT on a cylinder of size R such that

$$z \sim z + R, \tag{3.20}$$

or equivalently $x \sim x + R$. The state $|n\rangle$ satisfies

$$H|n\rangle = E_n|n\rangle, \qquad P|n\rangle = P_n|n\rangle, \tag{3.21}$$

where H is the deformed Hamiltonian. The definition of the $T\bar{T}$ deformation implies that that the energy eigenvalues flow according to

$$\partial_\mu E_n = \partial_\mu \langle n|H|n\rangle = -4 \int dx \langle n|T\bar{T}|n\rangle = -4R\langle n|T\bar{T}|n\rangle, \tag{3.22}$$

where the minus sign discrepancy with (**??**) originates from the fact that we are working in Euclidean signature. Note that in the last term of (3.22) we used the fact that the expectation value $\langle n|T\bar{T}|n\rangle$ is a constant. Consequently, we learn that the flow of the energy eigenvalues is determined by the expectation value of the $T\bar{T}$ operator.

We would like to use (3.22) together with the factorizability of $T\bar{T}$ to obtain a differential equation for the spectrum. In order to do this we note that the energy and angular momentum of a $T\bar{T}$-deformed QFT are related to the T_{tt} and T_{tx} components of the stress tensor via

$$\langle n|T_{tt}|n\rangle = -\frac{E_n}{R}, \qquad \langle n|T_{tx}|n\rangle = \frac{iP}{R}, \qquad \langle n|T_{xx}|n\rangle = -\frac{\partial E_n}{\partial R}. \tag{3.23}$$

Exercise 3.3: The first two terms in the above equation can be obtained from the usual expression for the conserved charge Q_ξ associated with translations $x^\mu \to x^\mu + \xi^\mu$ via $Q_\xi = i\int dx \langle n|T_{t\mu}|n\rangle \xi^\mu$. Derive the third term.

The factorizability of the stress tensor then implies

$$\langle n|T\bar{T}|n\rangle = \langle n|T_{zz}|n\rangle\langle n|T_{\bar{z}\bar{z}}|n\rangle\langle n|T_{z\bar{z}}|n\rangle^2 \tag{3.24}$$

$$= -\frac{1}{4}\left(E_n\frac{\partial E_n}{\partial R} + \frac{P_n^2}{R^2}\right), \tag{3.25}$$

where we used

$$T_{zz} = \frac{1}{4}\big(T_{xx} - T_{tt} - 2iT_{xt}\big), \qquad T_{\bar{z}\bar{z}} = T^*_{zz}, \qquad T_{z\bar{z}} = \frac{1}{4}\big(T_{xx} + T_{tt}\big). \tag{3.26}$$

Altogether, we find that the deformed energy and angular momentum satisfy

$$\partial_\mu E_n = E_n\frac{\partial E_n}{\partial R} + \frac{P_n^2}{R}, \qquad \partial_\mu P_n = 0. \tag{3.27}$$

The partial differential equation is known as the inviscid Burger's equation with an additional driving force corresponding to the P_n^2/R term. Note that the angular momentum is not changed by the deformation due to the fact that it must be quantized in units of R^{-1}, that is

$$P_n = \frac{m}{R}, \qquad m \in \mathbb{Z}. \tag{3.28}$$

When the undeformed QFT is scale invariant, it's possible to solve (3.27) and obtain a universal expression for the deformed spectrum. In this case, the differential equation (3.27) turns out to be equivalent to an algebraic (quadratic) equation that can be easily solved. This follows from the fact that in a CFT the only dimensionful scale is the size of the cylinder so that RE_n is dimensionless. Consequently, RE_n can only depend on dimensionless variables such that

$$RE_n(\mu) = e(\hat{\mu}), \qquad RP_n(\mu) = p(\hat{\mu}), \qquad \hat{\mu} = \frac{\mu}{R^2}, \tag{3.29}$$

where e, p, and $\hat{\mu}$ are the dimensionless energy, angular momentum, and deformation parameter. We then have

$$\partial_\mu E_n = \frac{1}{R}\partial_\mu(RE_n) = \frac{1}{R^3}\partial_{\hat{\mu}} e, \tag{3.30}$$

$$\partial_R E_n = \partial(R^{-1}e) = -\frac{1}{R^2}(e + 2\hat{\mu}\partial_{\hat{\mu}}e). \tag{3.31}$$

Consequently, the inviscid Burguer's equation (3.27) can be recast into

$$\partial_{\hat{\mu}}(e + \hat{\mu}e^2 - \hat{\mu}p^2) = 0, \tag{3.32}$$

which yields a quadratic equation for the deformed dimensionless energy

$$e(\hat{\mu}) + \hat{\mu}e(\hat{\mu})^2 - \hat{\mu}p^2 = e(0). \tag{3.33}$$

The spectrum of $T\bar{T}$-deformed CFTs is therefore given by

$$E_n(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu E_n(0)}{R} + \frac{4\mu^2 P_n(0)^2}{R^2}}\right), \qquad P(\mu) = P(0). \tag{3.34}$$

The spectrum is plotted in figure 2.

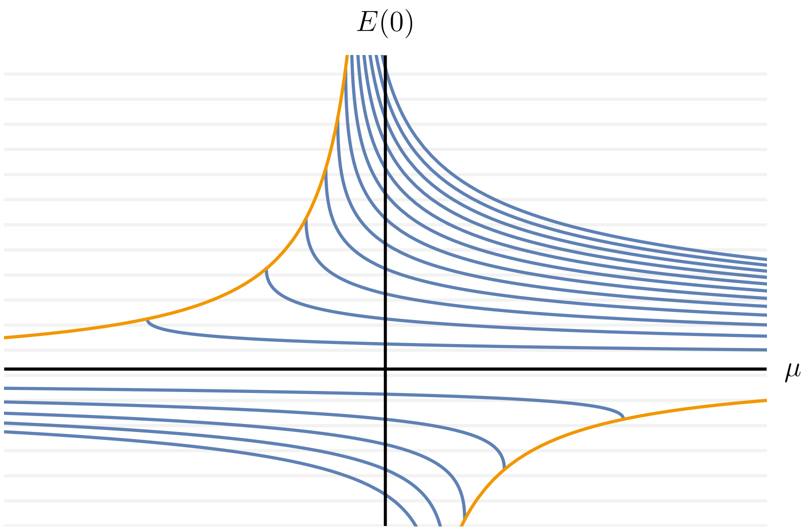

Figure 2: The deformed energies of $T\bar{T}$-deformed CFTs as a function of the deformation parameter μ. The orange lines denote the maximum/minimum value of the deformed energies beyond which the spectrum becomes complex.

The deformed spectrum (3.34) is universal, meaning that it's valid for any $T\bar{T}$-deformed CFT on the cylinder. This is one way in which the $T\bar{T}$ deformation is solvable. In particular, for an undeformed CFT with central charge c, the undeformed energy and angular momentum are given by

$$E_n(0) = \frac{1}{R}\left(h_n + \bar{h}_n - \frac{c}{12}\right), \qquad P_n(0) = \frac{1}{R}(h_n - \bar{h}_n), \tag{3.35}$$

where $(h_n, \bar{h}_n)$ are the conformal weights of the state $|n\rangle$. The conformal weights are the eigenvalues of the $(L_0, \bar{L}_0)$ modes of the Virasoro algebra we encountered earlier

$$L_0|n\rangle = \left(h_n - \frac{c}{12}\right)|n\rangle, \qquad \bar{L}_0|n\rangle = \left(\bar{h}_n - \frac{c}{12}\right)|n\rangle. \tag{3.36}$$

In terms of these variables the $T\bar{T}$ spectrum becomes

$$E_n(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu}{R^2}\left(h_n + \bar{h}_n - \frac{c}{12}\right) + \frac{4\mu^2}{R^4}(h_n - \bar{h}_n)^2}\right) \tag{3.37}$$

There are several interesting features about this spectrum. First, let us consider a string propagating in a D-dimensional flat spacetime where one of the coordinates is compact with size R. The mass of this string can be obtained by solving the Virasoro constraints and is given in terms of the excitation levels $(N, \bar{N})$ and winding w of the string by (see eq. (8.3.2a) of Polchinski's String Theory Volume I),

$$m = \sqrt{\frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \bar{N} - \frac{D-2}{12}\right) + \frac{(N - \bar{N})^2}{w^2 R^2}}. \tag{3.38}$$

Here $1/\alpha'$ is proportional to the string tension and has dimensions $[\alpha'] = L^2$. The first term is the potential energy of a winding string, the second term is the contribution to the excitations of the string, and the third corresponds to the contribution of the angular momentum. This formula is remarkably similar to (3.37). In fact, up to an overall constant shift, the $T\bar{T}$ energy (3.37) and the mass of the string (3.38) agree upon the identification

$$(h_n, \bar{h}_n) = (N, \bar{N}), \qquad w = 1, \qquad c = D - 2, \qquad \mu = \frac{\alpha'}{2}. \tag{3.39}$$

Exercise 3.4: How would you modify the dictionary to match the mass of a string with a different amount of winding? There are actually two ways, a simple way and a slightly more difficult one. Give a physical interpretation to the simple one. The more difficult one will reappear in the context of holography later in the class.

This is the first hint that the $T\bar{T}$-deformation of a CFT, and more generally of a QFT, is related to quantum gravity. In fact, we will see that the $T\bar{T}$ deformation of a free scalar field yields the classical Nambu-Goto action for a $D = 3$ dimensional string in the static gauge. This model has several gravitational features and has been dubbed "the simplest theory of quantum gravity".

Another interesting feature of (3.37) is that it depends critically on the sign of the deformation parameter. When $\mu > 0$, the spectrum is well behaved at all energies provided that μ satisfies the bound

$$\mu\left(h_0 + \bar{h}_0 - \frac{c}{12}\right) \geq -R^2 \tag{3.40}$$

where $(h_0, \bar{h}_0)$ denote the minimum values of the conformal weights. For a unitary CFT with a normalizable vacuum we have $h_0 = \bar{h}_0 = 0$ and there's a maximum value of the deformation parameter that is given by

$$\mu \leq \frac{3R^2}{c}. \tag{3.41}$$

We will see in the next lecture that this bound implies the existence of a maximum (Hagedorn) temperature. On the other hand, when $\mu < 0$, the spectrum becomes complex at high energies where

$$E_n(0) > \frac{R}{4|\mu|} + \frac{|\mu|}{R} P_n(0)^2. \tag{3.42}$$

This feature of $T\bar{T}$-deformed CFTs has a geometrical interpretation in holography where it corresponds either to the existence of a cutoff or the emergence of closed timelike curves (CTCs).

Finally, let us return to a statement made in the first lecture about the nature of the $T\bar{T}$ deformation. There, we mentioned that the $T\bar{T}$ deformation is not an RG flow and it does not add any new degrees of freedom. In particular, the theory does not flow to a UV fixed point and is in fact not a local QFT. This features of $T\bar{T}$-deformed QFTs is already visible in the spectrum (3.37). In order to see it we note that in a CFT, the ground state energy is given by

$$RE_0 = h_0 + \bar{h}_0 - \frac{c}{12}. \tag{3.43}$$

In a unitary CFT with a normalizable vacuum, the ground state energy is proportional to the central charge $RE_0 = -c/12$. It is not difficult to verify that in the IR limit $R \to \infty$, the deformed energy reduces to

$$RE_0(\mu)\big|_{R\to\infty} \to -\frac{c}{12}. \tag{3.44}$$

One way to see this is that the $R \to \infty$ limit is equivalent to the $\mu \to 0$, i.e. to the limit where we recover the undeformed CFT. In contrast, in the UV limit $R \to 0$ is not sensible, as the condition (3.41) is violated. In this limit the ground state energy becomes complex and we don't recover a CFT

$$RE_0(\mu)\big|_{R\to 0} \to \text{complex}. \tag{3.45}$$