1 Canonical quantization of scalar fields

1.1 Klein-Gordon field

In QM, the evolution of the state is governed by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\phi, t\rangle = H|\phi, t\rangle \tag{1}$$

where $c = \hbar = 1$For a non-relativistic free particle, the Hamiltonian $H = \frac{1}{2m}\mathbf{P}^2 = -\frac{\hbar^2}{2m}\nabla^2$ To extend it to the relativistic case, we need to use a relativistic Hamiltonian, which is $H = \sqrt{\mathbf{P}^2 + m^2}$, and the Schrödinger equation now is

$$i\frac{\partial}{\partial t}|\phi, t\rangle = \sqrt{-\nabla^2 + m^2}|\phi, t\rangle. \tag{2}$$

However, we see that the space and time are treated on a different footing as the space derivative appears in the square root. This problem can be alleviated by squaring both sides, which results in

$$-\frac{\partial^2}{\partial t^2}\phi(x, t) = \left(-\nabla^2 + m^2\right)\phi(x, t). \tag{3}$$

Rewrite it using the four-vector and rearrange it, we get

$$\left(-\gamma^\mu \gamma_\mu + m^2\right)\phi(x) = 0 \tag{4}$$

which is the famous Klein-Gordon equation. In QM, the position is an operator while the time is a parameter, therefore when we write $\phi(x, t)$, the space and time are actually treated in a different way. In the relativistic case, we require Lorentz symmetry, which will mix up the space and time. Therefore we get a new problem. There are two ways to solve this problem, one is to demote the position to a parameter, and the other is to promote time to an operator. The second approach turn out to be too complicated, therefore here we treat the position as parameters.

As we are generalizing the Schrodinger equation to the relativistic case, we still want to explain the $|\phi(x, t)|^2$ as a probability density. Then we will find that the total probability $\int d^3x|\phi|^2$ is not conserved, the conserved quantity is $\int d^3x \left(\phi^\dagger \partial_t \phi - \phi \partial_t \phi^\dagger\right)$. However, we then find that the conserved quantity is not positive definite which means there is a negative probability density in our theory. Now we know that $\phi(x, t)$ actually represents a field which contains an infinite number of degrees of freedom. To find the general form of $\phi(x, t)$, insert the Fourier transformation

$$\phi(x)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}e^{-ip\cdot x}\phi(p) \tag{5}$$

into the KG equation, we get

$$\frac{1}{(2\pi)^4}\left(-\partial_\mu\partial^\mu+m^2\right)\int d^4pe^{-ipx}\phi(p)=\frac{1}{(2\pi)^4}\int d^4p\left(-p^2+m^2\right)e^{-ipx}\phi(p)=0. \tag{6}$$

Then we can see unless $-p^\mu p_\nu=(p^0)^2-\mathbf{p}^2=m^2$, we have $\phi(p)=0$, therefore we can write $\phi(p)=\delta(p^2-m^2)C(p)$. Insert it into our Fourier expansion, and we get

$$\begin{aligned}\phi(x)&=\int\frac{d^4p}{(2\pi)^4}\delta\left(\left(p^0\right)^2-E(p)^2\right)A(p)e^{-ipx}\\&=\int\frac{d^4p}{(2\pi)^4}\left(A(p)e^{-ipx}\frac{1}{2E_p}\left[\delta\left(p^0-E_p\right)+\delta\left(p^0+E_p\right)\right]\right)\\&=\int\frac{d^3p}{(2\pi)^42E_p}\left(A(p)e^{-iE_pt+\mathbf{px}}+A(p)e^{iE_pt+\mathbf{px}}\right)\\&=\int\frac{d^3p}{(2\pi)^32E_p}\left(A(p)e^{-iE_pt+\mathbf{px}}+A(-p)e^{iE_pt-\mathbf{px}}\right),\end{aligned} \tag{7}$$

where in the last step we make a change of variable $-\mathbf{p}\to\mathbf{p}$ in the second term and absorb a 2π into $A(\pm p)$ to match the usual convention of commutation relations[1] As $\phi(p)$ here is a real scalar field, we also have $\phi^\dagger(p)=\phi(p)$, this indicate $A^\dagger(p)=A(-P)$. The expression is Lorentz-invariant 3-momentum integral, as we derive it from the first line of (7) which is manifestly Loerntz-invariant.

To obtain the coefficient, we need to define the Klein-Gordon inner product first. In flat spacetime, the Klein-Gordon inner product is defined on the space of solutions of the Klein-Gordon equation by

$$\langle\phi_1,\phi_2\rangle_{\mathrm{KG}}=i\int_{\mathbb{R}^3}d^3x\left[\phi_1^\dagger\dot{\phi}_2-\dot{\phi}_1^\dagger\phi_2\right] \tag{8}$$

[1] In some textbooks, this mode expansion is given by

$$\phi(x)=\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(A(p)e^{-iE_pt+\mathbf{px}}+A(-p)e^{iE_pt-\mathbf{px}}\right)$$

or

$$\phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\left(A(p)e^{-iE_pt+\mathbf{px}}+A(-p)e^{iE_pt-\mathbf{px}}\right).$$

. These different conventions will lead to the same commutation relation up to a normalization constant. For the former one, it's given by $\left[a(\mathbf{k}),a^\dagger\left(\mathbf{k}'\right)\right]=(2\pi)^32\delta^3\left(\mathbf{k}-\mathbf{k}'\right)$ and for the latter it's given by $\left[a(\mathbf{k}),a^\dagger\left(\mathbf{k}'\right)\right]=\delta^3\left(\mathbf{k}-\mathbf{k}'\right)$.

where the dot represents the time derivative, and ϕ_1, ϕ_2 are both Klein-Gordon field. The coefficient is simply

$$A(\mathbf{p}) = \left\langle \phi, e^{-ikx} \right\rangle_{\mathrm{KG}} = i \int d^3x e^{-ikx} \overleftrightarrow{\partial_0} \phi(x) \tag{9}$$

where we define $f \overleftrightarrow{\partial_0} g := f\partial_0 g - g\partial_0 f$.

1.2 Canonical quantization of free scalar field

To canonical quantize a real scalar field, we need to introduce the equal time commutation relations:

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = 0, \quad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^3(\mathbf{x} - \mathbf{x}'), \quad [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0 \tag{10}$$

where $\pi = -\partial^0\phi = \partial_0\phi$ is the generalized momentum correspond to ϕ. Moreover, we require the quantum field to still satisfy the KG equation which means the general form is represented by (**??**). The quantity in the exponent is all scalar and therefore could not lead to the equal time commutation relations, therefore we need to promote the expansion coefficient to an operator. We also have $[\phi(x, t), \partial_0\phi(x, t)] = i\delta^3(x - x')$ from the equal time commutation relations, and now we can calculate the commutation relation of the coefficient operator use (9):

$$\begin{aligned}
\left[a(\mathbf{k}), a^\dagger(\mathbf{k})\right] &= -\int d^3x \int d^3x' e^{-ikx+ik'x'} \left[\partial_0\phi(\mathbf{x}, t) - iE_{\mathbf{p}}\phi(\mathbf{x}, t), \partial_0\phi(\mathbf{x}', t) + iE_{\mathbf{p}}\phi(\mathbf{x}', t)\right] \\
&= \int d^3x \int d^3x' e^{-ikx+ik'x'} \left(2E_{\mathbf{p}}\delta^3(\mathbf{x} - \mathbf{x}')\right) \\
&= (2\pi)^3 2E_{\mathbf{p}}\delta^3(\mathbf{k} - \mathbf{k}').
\end{aligned} \tag{11}$$

Other commutators can be obtained in a similar way, finally, we have

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \quad \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0, \quad \left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = (2\pi)^3 2E_{\mathbf{k}}\delta^3(\mathbf{k} - \mathbf{k}') \tag{12}$$

The operator we obtain is actually time-independent which means if we plug in a field defined at t or t' into (9), we will get the same operator. Consider a field operator defined at $t = 0$, we can expand it as

$$\phi(\mathbf{x}, 0) = \int \widetilde{dk} \left(a(\mathbf{k})e^{ik\cdot x} + a^\dagger(\mathbf{k})e^{-ikx}\right). \tag{13}$$

A field operator defined at $t \neq 0$ can be obtained by

$$\phi(\mathbf{x}, t) = e^{iHt}\phi(\mathbf{x}, 0)e^{-iHt}, \tag{14}$$

its expansion is given by

$$\phi(\mathbf{x},t) = \int \widetilde{dk} \left(a(\mathbf{k}) e^{ik\cdot x} e^{-iE_{\mathbf{k}}t} + a^\dagger(\mathbf{k}) e^{-ikx} e^{iE_{\mathbf{k}}t} \right). \tag{15}$$

We can see this is the mode expansion of $\phi(\mathbf{x},t)$. Therefore we see operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are invariant under time evolution.

Then, we are going to show these operators are the creation and annihilation operators. The Lagrangian density of a free real scalar field is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{16}$$

The corresponding Hamiltonian is

$$H = \int d^3x \left(\frac{1}{2}\partial^0\phi\partial_0\phi + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \tag{17}$$

Plug the mode expansion into this expression, we get

$$H = \frac{1}{2}\int \widetilde{dk} E_{\mathbf{k}} \left(a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) \right) = \int \widetilde{dk} E_{\mathbf{k}} N(\mathbf{k}) + \frac{1}{2}\int d^3k E_{\mathbf{k}} \delta^3(0) \tag{18}$$

where $N(k) = a^\dagger(\mathbf{k})a(\mathbf{k})$ is the number operator and we donate the Lorentz invariant measure by

$$\widetilde{dk} := \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}. \tag{19}$$

The Hamiltonian contain a divergent term, as the divergent term is a constant and physics is independent of the choice of the zero point of energy, we are free to discard the divergent term. The momentum at the classic level is given by the conserved charge of the Poincare group. We can also plug the mode expansion into its expression which results in

$$\mathbf{P} = -\int d^3x \partial_0\phi\nabla\phi = \int \widetilde{dk}\mathbf{k}N(\mathbf{k}). \tag{20}$$

Now we just obtained the expression of Hamiltonian and momentum operators at the quantum level, but we haven't shown that these operators still represent energy and momentum. In QM, the energy and momentum are the generators of spacetime translation. Here we are going to show that the Hamiltonian and momentum operators we just obtained is also the generator of spacetime translation. The commutator of the field operator and momentum operator is

$$[\phi(x), P^\mu] = \left[\phi(x), \int \widetilde{dk} k^\mu a^\dagger(\mathbf{k})a(\mathbf{k})\right] = \int \widetilde{dk} k^\mu \left[a(\mathbf{k})e^{ikx} - a^\dagger(\mathbf{k})e^{-ikx}\right] = -i\partial^\mu\phi(x). \tag{21}$$

Then using the Baker-Hausdorf formula, we get[2]

$$e^{iPa}\phi(x)e^{-iPa} = \phi(x-a), \tag{22}$$

therefore we see (20) generate space translation for a quantum state. Actually, one can show that this hold for all generator of the Poincare group. After quantization, the conserved charge of the Poincare group becomes the generator of a representation of the Poincare group in Hilbert space. The energy, momentum and angular momentum at the classic level are those at the quantum level. The commutator of $a^\dagger(k)$ and Hamiltonian is given by

$$\left[H, a^\dagger(k)\right] = E_{\mathbf{k}} a^\dagger(k), \quad \left[\mathbf{P}, a^\dagger(k)\right] = k a^\dagger(k). \tag{23}$$

We see that acting the $a^\dagger(k)$ on a state will increase the eigenvalue of energy and momentum by $E_{\mathbf{k}}$ and k respectively. Therefore we regard the operator $a^\dagger(k)$ as the creation operator of a particle with energy $E_{\mathbf{k}}$ and momentum k, $a(k)$ is the annihilation operator of such a particle.

The ground state $|0\rangle$ is defined as the state that is annihilated by any annihilation operator. Act a creation operator on the ground state will give a one-particle state $|k\rangle = a^\dagger(k)|0\rangle$. A multi-particle state is given by

$$|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = \hat{a}^\dagger(\mathbf{k}_1) \ldots \hat{a}^\dagger(\mathbf{k}_n)|0\rangle \tag{24}$$

which is also an eigenstate of Hamiltonian, The eigenvalue of this state is given by $E(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \sum_i E_{\mathbf{k}_i}$. These states are complete in fock space[3]

$$1 = |0\rangle\langle 0| + \int \tilde{dk} |\mathbf{k}\rangle\langle\mathbf{k}| + \int \tilde{dk} \sum_\alpha |\mathbf{k}, \alpha\rangle\langle\mathbf{k}, \alpha| \tag{25}$$

where $|\mathbf{k}\rangle$ is the one-particle state, $|\mathbf{k}, \alpha\rangle$ is the multi-particle state and α represents all other labels of the multi-particle state except momentum, like relative velocity.

In summary, we work in the Heisenberg picture, starting with two assumptions: the equal-time commutation relation and the quantum field still satisfy the classic equation of motion (it indicates the field could be expanded as (**??**)); using the Klein-Gordon inner product, we can calculate the commutator of the operator $a(p)$ and $a^\dagger(p)$; using the commutation of $a(p)$ and $a^\dagger(p)$, we then found that the conserved charge of Poincare group at the classic level

[2]We are working in the Heisenberg picture.

[3]The Fock space is the direct sum of tensor products of copies of a single-particle Hilbert.

becomes the generator of Poincare group at the quantum level; then we saw the coefficient in the mode expansion carries exactly energy and momentum of one particle, using these operators, we can construct a single particle state and also multi-particle state, these states are complete and therefore span the fock space, we then completed the canonical quantization.

Finally, therefore are two things worth mentioning. First, these $a^{\dagger}(\mathbf{k})$ and $a(\mathbf{k})$ we obtained are actually not operators because acting these on a state will not give a non-normalizable result (For $|\mathbf{K}\rangle = a^{\dagger}(\mathbf{k})|\mathbf{k}\rangle$, it's norm $\langle \mathbf{k}|\mathbf{k}\rangle = \delta^{(3)}(\mathbf{0}) = \infty$). In the language of mathematicians, they are referred to as "operator-valued distributions". In the subsequent subsection, I will simply refer to them as operators. Second, one might get confused that we derive the KG equation by combining the Schrödinger equation with special relativity, why we call it a classic equation of motion. This is because when demote the position to a parameter, we lose the canonical commutation relation, and therefore the equation we obtained is no longer "quantum". We introduce a field when we demote the position to a parameter, therefore the KG equation is a classic equation of motion for a field.

1.3 Two-point correlation function

The correlation function is an important quantity in QFT. On the one hand, they determine the scattering amplitudes through the LSZ reduction formula. On the other hand, from an axiomatic point of view, the Wightman reconstruction theorem tells us that obtaining a complete set of n-point functions is equivalent to giving the actual quantum fields.

For a real free scalar field, the free propagator, which is the 2-pt correlation function, is given by

$$\Delta_F(x-y) := i\langle 0|\ \mathrm{T}\{\phi(x)\phi(y)\}|0\rangle \tag{26}$$

where T represent the time ordered. It could be given more explicitly by

$$\mathrm{T}\{\phi(x)\phi(y)\} = \theta\left(x^0 - y^0\right)\phi(x)\phi(y) + \theta\left(y^0 - x^0\right)\phi(y)\phi(x). \tag{27}$$

Plug the mode expansion into this expression, we get

$$\begin{aligned}
\langle 0|\ \mathrm{T}\{\phi(x)\phi(y)\}|0\rangle &= \theta\left(x^0-y^0\right)\langle 0|\phi(x)\phi(y)|0\rangle + \theta\left(y^0-x^0\right)\langle 0|\phi(y)\phi(x)|0\rangle \\
&= \theta\left(x^0-y^0\right)\langle 0|\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left[a(\mathbf{k})e^{ik\cdot x}\right]\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}'}}\left[a^\dagger\left(\mathbf{k}'\right)e^{-ik'\cdot y}\right]|0\rangle \\
&+ \theta\left(y^0-x^0\right)\langle 0|\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left[a(\mathbf{k})e^{ik\cdot y}\right]\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}'}}\left[a^\dagger\left(k'\right)e^{-ik'\cdot x}\right]|0\rangle \\
&= \theta\left(x^0-y^0\right)\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}e^{ik(x-y)} + \theta\left(y^0-x^0\right)\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}e^{ik(y-x)}.
\end{aligned} \tag{28}$$

The Heaviside step function has an integral representation:

$$\theta(x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dt e^{-ixt}}{t+i\varepsilon}. \tag{29}$$

Insert it into (28), we get

$$\begin{aligned}
&\langle 0|\{\phi(x)\phi(y)\}|0\rangle \\
&= -\frac{1}{2\pi i}\left[\int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\frac{1}{t+i\varepsilon}e^{ik(x-y)-it\left(x^0-y^0\right)} + \int\frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\frac{1}{t+i\varepsilon}e^{-ik(x-y)+it\left(x^0-y^0\right)}\right] \\
&= -\int\frac{d^4q}{2(2\pi)^4\sqrt{\mathbf{q}^2+m^2}}e^{iq(x-y)}\left(\frac{1}{q^0-\sqrt{\mathbf{q}^2+m^2}+i\varepsilon} + \frac{1}{-q^0-\sqrt{\mathbf{q}^2+m^2}+i\varepsilon}\right) \\
&= -\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2+m^2-i\varepsilon}e^{iq(x-y)}
\end{aligned} \tag{30}$$

where we define a new variable $q = (k^0+t, \mathbf{k})$ at the second step. We can see the 4-momentum q in the last line is no longer on-shell, and an off-shell 4-momentum is usually regarded as corresponding to a virtual particle. Off-shell particles come to our expression after we introduce the time ordered. The 2-pt correlation, which is also the free propagator, in momentum space is represented by

$$\tilde{\Delta}_F(k) = \frac{1}{k^2+m^2-i\varepsilon}. \tag{31}$$

We can force a Klein-Gorden operator on the free propagator:

$$\left(-\partial^2+m^2\right)\Delta_F(x-y) = \int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)} = \delta^4(x-y). \tag{32}$$

Therefore we see that the free propagator is also the Green function of the Klein-Gorden equation.

2 Poincare group and its representation

2.1 Revisit the Lagrangian

In the last subsection, we derived the KG equation from the perspective of historical development. The Lagrangian correspond to the KG equation is given by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{33}$$

Let's investigate the features of this Lagrangian[4]. First, this Lagrangian preserves the locality. Locality is a requirement of relativity which forbids interaction between any spacelike point, therefore a Lagrangian defined on a spacelike hypersurface can only be a function of x. Second, this Lagrangian is translation invariant. Actually, any local Lagrangian are translation invariant due to the property of the integral measure

$$\int d^4x\mathcal{L}(x+\delta x) = \int d^4(x+\delta x)\mathcal{L}(x+\delta x) = \int d^4x\mathcal{L}(x). \tag{34}$$

This shows that if we ensure the locality of Lagrangian, then there is no need to worry about the space translation symmetry. The Lagrangian (33) does not depend on time explicitly, therefore we also have time translation symmetry.

Another feature of the Lagrangian of the massive free scalar field is it's a Lorentz scalar. A Lorentz scalar is invariant under Lorentz transformation, Lagrangian is Lorentz scalar is equivalent to saying that Lagrangian is invariant under Lorentz transformation[5]. It is also important to ensure the Lagrangian only contains a first-order derivative as higher-order derivatives will lead to a Hamiltonian unbounded from below, this is called the Ostrogradsky instability. Let's summarize our finding:

1. The Lagrangian must be in the form of $\mathcal{L}(x)$ to preserve the locality.
2. The Lagrangian must be a Lorentz scalar to preserve the Lorentz invariant.
3. The Lagrangian cannot contain a high derivative term according to Ostrogradsky instability.

[4]This is actually the Lagrangian density, we will always call it a Lagrangian in this note.

[5]Usually, we want our action to be Lorentz transformation invariant, the integral measure is obviously invariant under Lorentz transformation as Lorentz group is $SO(3,1)$, therefore we also require the Lagrangian density to be Lorentz transformation invariant.

4. Usually, we want energy conservation in our system, therefore the Lagrangian cannot depend on time explicitly.

Using this condition as our constraint, we want to find if is there any other possible Lagrangian to satisfy these conditions. Constraints $(1),(3)$ and (4) are easy to impose, so the problem of constructing a Lagrangian now becomes using different fields to construct a Lorentz scalar.

2.2 Poincare group and its representation

The symmetry of (33) forms the Poincare group. Before constructing more Lagrangian, let's first investigate the properties of this group and its representation.

2.2.1 Poincare group

Poincare group is defined to be the isometry group of the Minkowski spacetime, which is the transformation that leaves the metric invariant:

$$\eta_{\alpha\beta}\frac{dx'^{\alpha}}{dx^{\mu}}\frac{dx'^{\beta}}{dx^{\nu}} = \eta_{\mu\nu}. \tag{35}$$

The coordinate transformation satisfies the above condition is given by

$$x'^{\alpha} = \Lambda^{\mu}_{\alpha}x^{\nu} + a^{\mu} \tag{36}$$

with $\eta_{\alpha\beta}\Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu} = \eta_{\mu\nu}$. Let's show how to drive it, the process is similar to the procedure for deriving the conformal symmetry. First, consider an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x). \tag{37}$$

Plug it into (35), we get

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 0. \tag{38}$$

Constructing the following linear combination

$$\partial_{\omega}\left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right) + \partial_{\mu}\left(\partial_{\nu}\epsilon_{\omega} + \partial_{\omega}\epsilon_{\nu}\right) - \partial_{\nu}\left(\partial_{\omega}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\omega}\right) = 2\partial_{\omega}\partial_{\mu}\epsilon_{\nu} = 0, \tag{39}$$

we deduce that ϵ could only be linear in x:

$$\epsilon_{\mu}(x) = a_{\mu} + \omega_{\mu\nu}x^{\nu}. \tag{40}$$

Plug this back to (38) gives

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0, \tag{41}$$

therefore $\omega_{\mu\nu}$ is an antisymmetrical tensor. The constant a^μ corresponds to translation while Λ^μ_ν represent the Lorentz transformation. The Poincare group is given by the semi-direct product of translation and Lorentz transformation

$$\begin{aligned} x'^\alpha &= \Lambda^\alpha{}_\mu x^\mu + a^\alpha \\ &= \lambda^\alpha{}_\mu \left(\Lambda^\mu{}_\nu x^\nu + a^\mu\right) + a^\alpha \\ &= \Lambda^\alpha_\mu \Lambda^\mu{}_\nu x^\nu + \Lambda^\alpha_\mu a^\mu + a^\alpha, \end{aligned}$$

which means if we donate an element of the Poincare group by $T(\Lambda, a)$, then we have

$$T\left(\Lambda', a'\right) T(\Lambda, a) = T\left(\Lambda'\Lambda, \Lambda' a + a'\right). \tag{42}$$

Sometimes it's written as

$$ISO(3,1) = T(4) \rtimes O(3,1), \tag{43}$$

where the Poincare group is represented by $ISO(3,1)$[6], translations group is represented by $T(4)$ and the Lorentz group is represented by $O(3,1)$[7]. Our Minkowski signature is $(-,+,+,+)$.

2.2.2 Generator of Poincare group

In group theory, we have a Lie group–Lie algebra correspondence, which means we can use the exponential map to build the Lie group from the elements of the Lie algebra, which is the tangent space at the identity. Physically, the Lie algebra generates infinitesimal symmetry transformations. Therefore it's worth investigating the Lie algebra structure of the Poincare group.

For an infinitesimal Lorentz transformation, we can expand it as

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu + O\left(\omega^2\right). \tag{44}$$

where we define $\omega_{\sigma p} \equiv \eta_{\mu\sigma}\omega^\mu{}_\rho$ and $\omega^\mu{}_p \equiv \eta^{\mu\sigma}\omega_{\sigma\rho}$. As ω^μ_ν is a real antisymmetrical tensor, it has 6 degrees of freedom in four dimensions. Translation has 4 degrees of freedom. Therefore

[6] The Poincare group was called the inhomogeneous Lorentz group in the time of Wigner, this is where the notion $ISO(3,1)$ comes from.

[7] Here, we refer to $O(3,1)$ as Lorentz group, $SO(3,1)$ as proper Lorentz Group and $SO^+(3,1)$ as Restricted Lorentz Group. You should be careful that some authors refer to $SO(3,1)$ as the Lorentz group.

the Poincare group has 10 generators. If we donate the generator of Lorentz group and translation by $J^{\mu\nu}$ and P^μ, then an infinitesimal Poincare group is given by

$$T(\Lambda, a) = \exp\left(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - ia_\mu P^\mu\right) \approx 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - ia_\mu P^\mu. \tag{45}$$

After a tedious calculation, we get the Lie algebra of the Poincare group:

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= i\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\sigma}J^{\mu\rho}\right) \\ [P^\mu, J^{\rho\sigma}] &= i\left(\eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}P^\sigma\right) \\ [P^\mu, P^\nu] &= 0. \end{aligned} \tag{46}$$

To make the Lorentz subalgebra $\mathfrak{so}(3,1)$ manifest, we can define

$$J_i := \frac{1}{2}\varepsilon_{ijk}J^{jk}, \quad K_i := J^{i0}, \quad P_i := P^i, \quad H := P^0, \tag{47}$$

where J_i is the generator of $SO(3)$ rotation and K_i is the generator of boosts. The Lie algebra then takes the form

$$\begin{aligned} &[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, P_j] = i\varepsilon_{ijk}P_k, \quad [P_i, P_j] = 0, \\ &[J_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, P_j] = i\delta_{ij}H, \quad [J_i, H] = 0, \\ &[K_i, K_j] = -i\varepsilon_{ijk}J_k, \quad [K_i, H] = iP_i, \quad [P_i, H] = 0. \end{aligned} \tag{48}$$

We can see H commutes with rotations and spatial translations but not with boosts, therefore in quantum theory, we do not label state with the eigenvalue of boost. Moreover, for the Lorentz algebra spanned by $\{J^i, K^i\}$, we can define a set of new basis by

$$\begin{aligned} M_i &= (J_i + iK_i)/2 \\ N_i &= (J_i - iK_i)/2. \end{aligned} \tag{49}$$

The Lorentz algebra reduces to two $\mathfrak{su}(2)$ subalgebras on this basis:

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k, \\ [N_i, N_j] &= i\epsilon_{ijk}N_k, \\ [M_i, N_j] &= 0. \end{aligned} \tag{50}$$

This relation is sometimes written as

$$\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)^\star \tag{51}$$

where the complex conjugate arises from

$$(M_i)^\star = -N_i. \tag{52}$$

We will see this complex conjugate guarantee that we must have both right-handed spinor and left-handed spinor at the same time.

2.2.3 Unitary irreducible representation of Poincare groups

Einstein's special relativity tells us that different inertial frames should be equivalent, and our physics should be the same No matter which inertial frame you're in. In terms of relativistic quantum theory, we still need the Poincare symmetry and it plays an important role when we construct the Hilbert space because the one-particle state transforms under irreducible unitary representations of the Poincare group. In this subsection, we will explain what's the meaning of it.

In Hilbert space, a symmetry transformation is represented by operators, if there is a homomorphism between those operators and the abstractly defined Poincare group, then we say this forms a representation of the Poincare group in Hilbert space. The first question is why we need a unitary representation. Consider a static observer measures a system, it's possible for the system to collapse to a series of states[8] represented by rays $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_n$. Now, let the observer accelerate to a certain speed, and measure the system again, he will get a set of states represented by $\mathscr{B}'_1, \mathscr{B}'_2, \ldots, \mathscr{B}'_n$. However, for example, if he is measuring the energy. the value of the corresponding state would be different. You might wonder why there are different energies, but it's not a characteristic of quantum systems, in classic systems, the kinetic energy of an object varies in different reference frames. Now, the equivalence of physics in different reference frames requires

$$P\left(\mathscr{B} \to \mathscr{B}_n\right) = P\left(\mathscr{B}' \to \mathscr{B}'_n\right). \tag{53}$$

The fundamental theorem proved by Eugene Wigner tells us that a symmetry that satisfies the above constraints must be either a unitary and linear operator or an antiunitary and antilinear operator. The detailed proof can be found in Appendix A of Chapter Two in Weinberg's QFT. The representation of a group in a Hilbert space is actually a projective

[8] The state means physical state rather than certain quantum eigenstate, the physical state here is represented by an eigenstate up to a phase factor $e^{i\alpha}$, which form a ray, see my CFT note for more detail.

representation due to the fact that physical condition can only determine a state up to a phase factor. What is lucky is that the antiunitary, antilinear branch and unitary, linear branches are disconnected for the Projective Hilbert space, and the identity transformation is only contained in the unitary, linear branch, therefore in most cases, there is no need to worry about the antiunitary.

The next question is why we need an irreducible representation for a particle. The irreducible representation is the smallest subspace that is invariant under the Poincare transformation. For a particle, the invariant subspace is spanned by all possible states you can obtain by rotating, translating or boosting a particle. Moreover, the invariant subspace span by, for example, electron and photon should be different as an electron is massive and therefore cannot reach the speed of light. Therefore we see different kinds of particles are distinguished by their intrinsic properties, which are some Poincare transformation invariant quantum numbers, including mass, and spin plus some additional internal quantum numbers like electric charge. From a group-theoretical perspective, irreducible representations are uniquely characterised by Casimir elements. For the Poincare group, the Casimir elements are

$$P^\mu P_\mu, \quad W^\mu W_\mu, \text{ with } W_\mu := \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}M^{\nu\rho}P^\lambda, \tag{54}$$

where P^μ is the momentum, M is the Lorentz generator and W is the so-called Pauli-Lubanski pseudovector. In a rest frame, we know $P^2 = m^2$, therefore the first Casimir operator gives the mass. The Pauli-Lubanski pseudovector is a bit complicated and one can show that it depends on the spin. Therefore the mass and spin are actually given by the quadratic Casimir elements of the Poincare group. Other internal quantum numbers will depend on what internal symmetry you have. The actual symmetry group is given by $ISO(3,1) \bigotimes I$ where I represents an internal symmetry group like $U(1)$ and $SU(2)$. As the spacetime symmetry doesn't mix with the internal symmetry, their generators commute and we can also regard the generator of internal symmetry as Casimir elements of the Poincare group.

This definition of a particle can be easily generalized. In the non-relativistic case, you just need to replace the Poincare group with the Galilean group. For the anti-de Sitter or de Sitter spacetime, the spacetime isometry groups are $SO(3,2)$ and $SO(4,1)$ respectively. All these are special cases of what is known as kinematical groups. The kinematical group is a group of automorphisms of spacetime which satisfied

1. Space is isotropic and spacetime is homogeneous,
2. Parity and time-reversal are automorphisms of the kinematical group,
3. The one-dimensional subgroups generated by the boosts are non-compact.

There are in total 11 possible kinematical groups as shown in Figure 1. The particle in corresponding spacetime can be defined in the same way.

Symbol	Name
dS_1	de Sitter group $SO(4,1)$
dS_2	de Sitter group $SO(3,2)$
P	Poincaré group
P'_1	Euclidean group $SO(4)$
P'_2	Para-Poincaré group
C	Carroll group
N_+	Expanding Newtonian Universe group
N_-	Oscillating Newtonian Universe group
G	Galilei group
G'	Para-Galilei group
St	Static Universe group

Figure 1: The 11 possible kinematical groups, Figure taken from [1].

2.2.4 Casimir operator of Poincare algebra

In this subsection, we are going to systematically introduce the Casimir operator, especially the quadratic Casimir operator from a group-theoretical perspective.

The Casimir operator is not an element of our Lie algebra $\mathfrak{g}$; it lives in the so-called Universal enveloping algebra. The reason is that there is no multiplication defined on a Lie algebra, and we see the Casimir operator (54) is built on the product of a Lie algebra generator. Therefore, we need a larger space with multiplication defined. The Universal enveloping algebra, which is an associative algebra containing the identity, is the space we are looking for. The formal definition of it is shown in Appendix A.4.

The Casimir operators are elements of the centre of the universal enveloping algebra of a Lie algebra. In physics, the Casimir invariant we always use is the quadratic Casimir element like (54)[9]. The quadratic Casimir element is the element of the universal enveloping algebra given by

$$\Omega=\sum_{i=1}^{n} X_i X^i \tag{55}$$

where X_i and X^i are set of basis and dual basis on $\mathfrak{g}$. The basis and dual basis are linked by the metric-like thing on a Lie algebra. For a semisimple Lie algebra, the typical choice of the metric-like thing is the Killing form. Here we give an example using the quadratic Casimir element of $\mathfrak{su}_2(\mathbb{C})$ and $\mathfrak{so}(3,1)$.

su(2)

The commutation relations of $\mathfrak{su}_2(\mathbb{C})$ are given by

$$[X_1,X_2]=iX_3,\quad [X_2,X_3]=iX_1,\quad [X_3,X_1]=iX_2. \tag{56}$$

Then we need to calculate the Killing form, the definition of it is attached to the appendix B. Based on (56), we get

$$\begin{aligned}
&\operatorname{ad}(x_1)(x_1)=0, && \operatorname{ad}(x_2)(x_1)=-ix_3, && \operatorname{ad}(x_3)(x_1)=ix_2,\\
&\operatorname{ad}(x_1)(x_2)=ix_3, && \operatorname{ad}(x_2)(x_2)=0, && \operatorname{ad}(x_3)(x_2)=-ix_1,\\
&\operatorname{ad}(x_1)(x_3)=-ix_2, && \operatorname{ad}(x_2)(x_3)=ix_1, && \operatorname{ad}(x_3)(x_3)=0.
\end{aligned} \tag{57}$$

Therefore $ad(X_a)$ in the basis $\{X_1,X_2,X_3\}$ are represented by following matrices

$$\operatorname{ad}(X_1)=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \operatorname{ad}(X_2)=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},\quad \operatorname{ad}(X_3)=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}. \tag{58}$$

The Killing form of $\mathfrak{su}_2(\mathbb{C})$ therefore is

$$\mathcal{K}(x_a,x_b)=\operatorname{tr}(\operatorname{ad}(x_a)\operatorname{ad}(x_b))=2\delta_{ab} \tag{59}$$

where δ_{ab} is Kronecker delta. The Casimir operator is given by

$$C_2=\frac{1}{2}\delta^{ab}X_aX_b=\frac{1}{2}X^2 \tag{60}$$

where $\frac{1}{2}\delta^{ab}$ is the inverse matrix of the Killing form. This is exactly the S^2 operator in the spin system up to a normalization factor which gives the spin s by $C_2=\frac{1}{2}S^2=\frac{1}{2}s(s+1)$[10].

[9] The Pauli-Lubanski pseudovector is a quartic Casimir operator

[10] We set $\hbar=1$

For a spin 1/2 system, the generators of $\mathfrak{su}_2(\mathbb{C})$ are represented $X_a = \frac{1}{2}\sigma_a$ (a=1,2,3), where σ_a are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{61}$$

The Casimir operator gives $C_2 = \frac{1}{2}\sigma^2 = \frac{3}{8}$ for spin 1/2 system.

so(3,1)

The commutation relation of the Lorentz algebra is

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \tag{62}$$

Its Killing form can be computed in the same way shown in the last subsection. However, the Killing form of $\mathfrak{so}(q,p)$ is already derived by mathematician, which is given by

$$\mathcal{K}(X,Y) = (q+p-2)\,\mathrm{tr}(XY). \tag{63}$$

Moreover, the generator of the Lorentz group can be represented by the following matrices

$$\begin{gathered} J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ K_1 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}. \end{gathered} \tag{64}$$

Therefore the Killing form of Lorentz algebra is given by

$$\mathcal{K}(J_i, J_j) = 4\delta_{ij}, \quad \mathcal{K}(K_i, K_j) = -4\delta_{ij}. \tag{65}$$

The quadratic Casimir operator is given by

$$C_2 = \frac{1}{4}(J^2 - K^2) = \frac{1}{2}M^{\mu\nu}M_{\mu\nu}. \tag{66}$$

One can in principle also construct higher-order Casimir invariants like the Pauli-Lubanski pseudovector. For a complex semisimple lie algebra, the number of independent Casimir invariants is equal to the rank of the lie algebra according to Racah's theorem. The Casimir invariant is important because according to Schur's lemma, any Casimir element is proportional to the identity in any irreducible representation of the Lie algebra. Therefore the Casimir operator acts like a scalar on an irreducible representation and can be used to classify the irreducible representation. Another property of the Casimir invariant is that it commutes with all other elements in the algebra, therefore we can always check whether we get the right result by showing it commutes with the generator of the group. The formal aspect of the Casimir invariant is attached to the appendix B.

2.2.5 Induced representation

Now, let's try to construct an irreducible representation of the Poincare group explicitly. Recall the irreducible representation of the Poincare group is the collection of all states that you can obtain by performing arbitrary translation and Lorentz transformation. It's natural to start with an eigenstate of the momentum operator because it makes the translation part easier:

$$P^{\mu}\Psi_{p,\sigma} = p^{\mu}\Psi_{p,\sigma} \tag{67}$$

where all other degrees of freedom are donated by σ[11]. We know a finite translation is given by

$$U(1,a) = \exp\left(-iP^{\mu}a_{\mu}\right). \tag{68}$$

Therefore we have

$$U(1,a)\Psi_{p,\sigma} = e^{-ipa}\Psi_{p,\sigma}. \tag{69}$$

As translation is generated by the momentum operator, performing an arbitrary translation transformation on the eigenstates of momentum just results in a constant change. Therefore we only need to consider the Lorentz transformation, the effect of a Lorentz transformation operator on the momentum eigenstate is given by

$$\begin{aligned}
P^{\mu}U(\Lambda)\Psi_{p,\sigma} &= U(\Lambda)\left[U^{-1}(\Lambda)P^{\mu}U(\Lambda)\right]\Psi_{p,\sigma} \\
&= U(\Lambda)\left((\Lambda^{-1})_{\rho}{}^{\mu}P^{\rho}\right)\Psi_{p,\sigma} \\
&= \Lambda^{\mu}{}_{\rho}p^{\rho}U(\Lambda)\Psi_{p,\sigma}
\end{aligned} \tag{70}$$

[11]We could assume σ to be purely discrete for one particle state.

where in the second step we use $U(\Lambda, a)P^\rho U^{-1}(\Lambda, a) = \Lambda_\mu{}^\rho P^\mu$[12] and in the third step we use $(\Lambda^{-1})_\rho{}^\mu = \Lambda^\mu{}_\rho$. This implies that the state $U(\Lambda)\Psi_{p,\sigma}$ is a linear combination of momentum eigenstate with eigenvalue ΛP:

$$U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\Psi_{\Lambda p,\sigma'}. \tag{72}$$

If we can explicitly find the coefficient $C_{\sigma'\sigma}$ for a given state, then we construct the irreducible representation. The way to find it is called the induced representation method.

Notice the only function of p^μ that is invariant under proper orthochronous Lorentz transformation is $p^2 = \eta_{\mu\nu}p^\mu p^\nu$. Moreover, if $p^2 < 0$, the sign of p^0 is also invariant[13]. These two invariants classify momentum into several classes. Let's take a "standard" four-momentum k^μ in one of the classed and express any momentum p^μ of this class by

$$p^\mu = L^\mu{}_\nu(p)k^v \tag{73}$$

where $L^\mu{}_\nu(p)$ is some Lorentz transformation depend on p^μ. Then a state $\Psi_{p,\sigma}$ can be defined by

$$\Psi_{p,\sigma} \equiv N(p)U(L(p))\Psi_{k,\sigma} \tag{74}$$

where $N(p)$ is a normalization factor. Apply a Lorentz transformation on this state, and we find

$$\begin{aligned} U(\Lambda)\Psi_{p,\sigma} &= N(p)U(\Lambda L(p))\Psi_{k,\sigma} \\ &= N(p)U(L(\Lambda p))U\left(L^{-1}(\Lambda p)\Lambda L(p)\right)\Psi_{k,\sigma}. \end{aligned} \tag{75}$$

Notice $(\Lambda p)^\mu = L^\mu{}_\nu(\Lambda p)k^v$ and $(\Lambda p)^\mu = \Lambda^\mu{}_\kappa L^\kappa{}_\nu(p)k^v$, we find the transformation $L^{-1}(\Lambda p)\Lambda L(p)$ leave k^μ invariant, therefore form a subgroup of homogeneous Lorentz group. This subgroup is called the *little group* which consist of transformation $W^\mu{}_\nu$ that leave k^μ invariant:

$$W^\mu{}_\nu k^\nu = k^\mu. \tag{76}$$

Operating a element of the little group on a state $\Psi_{k,\sigma}$ gives

$$U(W)\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma'\sigma}(W)\Psi_{k,\sigma'}. \tag{77}$$

[12] This can be proved by expanding and matching both sides of

$$U(\Lambda, a)U(1+\omega, \epsilon)U^{-1}(\Lambda, a) = U\left(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\right) \tag{71}$$

to first order in ω and ϵ.

[13] Recall for $(p^0)^2 = E^2 = (p)^2 + m^2$ split into two disconnected branches for $m^2 = -p^2 > 0$. The $p^0 > 0$ branches will not turn to $p^0 < 0$ branch under a proper orthochronous Lorentz transformation.

where

$$D_{\sigma'\sigma}(\bar{W}W) = \sum_{\sigma''} D_{\sigma'\sigma''}(\bar{W}) D_{\sigma''\sigma}(W) \tag{78}$$

because of

$$\begin{aligned} \sum_{\sigma'} D_{\sigma'\sigma}(\bar{W}W)\Psi_{k,\sigma'} &= U(\bar{W}W)\Psi_{k,\sigma} = U(\bar{W})U(W)\Psi_{k,\sigma} \\ &= U(\bar{W}) \sum_{\sigma''} D_{\sigma''\sigma}(W)\Psi_{k,\sigma''} = \sum_{\sigma'\sigma''} D_{\sigma''\sigma}(W) D_{\sigma'\sigma''}(\bar{W})\Psi_{k,\sigma'}. \end{aligned} \tag{79}$$

The (78) tells us that the coefficient forms a representation of the little group. Rewrite (75) using W and (74), we finally get

$$U(\Lambda)\Psi_{p,\sigma} = \left(\frac{N(p)}{N(\Lambda p)}\right) \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))\Psi_{\Lambda p,\sigma'}, \tag{80}$$

and we can determine the coefficient in (72) by finding the representation of the little group. For example, if we take the standard momentum k to be the momentum of some static massive particle, then the little group is the three-dimensional rotation group $SO(3)$, and we can use the representation of $SO(3)$ to calculate the coefficient in (72), and construct the irreducible representation of Poincare group. The little groups of different types of particles are different, a massless particle always travels at the speed of light and can't be static, its corresponding little group is $E(2) = ISO(2) \cong T(2) \rtimes O(2)$. One way to understand it physically is assuming there is a massless particle whose momentum is represented by a lightlike vector toward the z direction, and then you will have rotation and reflection symmetry about the momentum vector in the $x-y$ plane. Moreover, you also have translation symmetry in the $x-y$ plane. The reason you don't have space translation symmetry in the z direction is the particle travels at the speed of light in the z direction, and the space in this direction is suppressed, therefore it doesn't make sense to talk about translation in this direction. You may doubt that the Lorentz group shouldn't contain translation, however by defining

$$\begin{aligned} A &= -J^{13} + J^{10} = J_2 + K_1, \\ B &= -J^{23} + J^{20} = -J_1 + K_2 \end{aligned} \tag{81}$$

we find their commutation relation is given by

$$\begin{aligned} [J_3, A] &= +iB, \\ [J_3, B] &= -iA, \\ [A, B] &= 0. \end{aligned} \tag{82}$$

Therefore, we see that the Lorentz group does contain a $E(2)$ subgroup.

3 Schwinger-Dyson equation and Ward-Takahashi identity

3.1 Schwinger–Dyson equation in canonical formalism

In canonical formalism, the Schwinger-Dyson equation originates from that the time derivative doesn't commute with time ordering. Recall for the two-point correlation function, we have

$$T\{\phi(x)\phi(x')\} = \theta\left(x-x'\right)\phi(x)\phi\left(x'\right)+\theta\left(x'-x\right)\phi\left(x'\right)\phi(x). \tag{83}$$

Let's take a time derivative of this time-ordered operator product

$$\begin{aligned}
&\partial_t T\left\{\phi(x)\phi\left(x'\right)\right\} \\
=&\partial_t\left(\theta\left(x-x'\right)\phi(x)\phi\left(x'\right)+\theta\left(x'-x\right)\phi\left(x'\right)\phi(x)\right) \\
=&\delta\left(t-t'\right)\phi(x)\phi\left(x'\right)+\theta\left(x-x'\right)\left(\partial_t\phi(x)\right)\phi\left(x'\right) \\
&-\delta\left(t'-t\right)\phi\left(x'\right)\phi(x)+\theta\left(x'-x\right)\phi\left(x'\right)\partial_t\phi(x) \\
=&T\left\{\partial_t\phi\left(x\right),\phi\left(x'\right)\right\}+\delta\left(t-t'\right)\left[\phi(x),\phi\left(x'\right)\right].
\end{aligned} \tag{84}$$

Using equal time commutation relation

$$\left[\phi(x,t),\phi\left(x',t\right)\right]=0, \quad \left[\phi(x,t),\partial_t\phi\left(x',t\right)\right]=i\delta^{(3)}\left(x-x'\right), \tag{85}$$

we obtain

$$\partial_t T\left\{\phi(x)\phi\left(x'\right)\right\} = T\left\{\partial_t\phi(x),\phi\left(x'\right)\right\} \tag{86}$$

which shows that the first-order time derivative commutes with time ordering. Now let's take the second-order derivative on a time-ordered operator product, after a similar calculation, we get

$$\begin{aligned}
\partial_t T\left\{\partial_t\phi(x),\phi\left(x'\right)\right\} &= T\left\{\partial_t^2\phi\left(x,\phi\left(x'\right)\right\}+\delta(t-t')\left[\partial_t\phi(x),\phi\left(x'\right)\right]\right. \\
&= T\left\{\partial_t^2\phi(x)\phi\left(x'\right)\right\}-i\delta^{(4)}\left(x-x'\right)
\end{aligned} \tag{87}$$

where in the second step we again use the equal time commutation relation. As the space derivative commutes with time ordering, we finally obtain

$$\left(\partial_x^2+m^2\right)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \left\langle\Omega\right|\left(\partial_x^2+m^2\right)\phi(x)\phi(y)|\Omega\rangle - i\delta^{(4)}(x-y) \tag{88}$$

where the vacuum $|\Omega\rangle$ could be either the vacuum of an interacting theory or a free theory. For a free theory, we have $(\partial^2+m^2)\,\phi(x)=0$, therefore the above formula reduced to

$$\left(\partial_x^2+m^2\right)\langle 0|T\{\phi(x)\phi(y)\}|0\rangle=-i\delta^{(4)}(x-y), \tag{89}$$

which looks similar to the KG equation, but now has an extra contact term on the RHS. The Schwinger–Dyson equation is regarded as the equation of motion of correlation function, different from the classic equation of motion, we get extra contact terms which means the Schwinger–Dyson equation encodes the difference between the classical and quantum theories. As the only thing we use to derive the Schwinger–Dyson equation is the equal time commutation relation, this result is non-perturbative.

3.2 Schwinger–Dyson equation in path integral formalism

Let's use path integral formalism to derive the general form of the Schwinger-Dyson equation. Recall that in path integral formalism, we have

$$\langle\Omega\,|T\left\{\phi\left(x_1\right)\cdots\phi\left(x_n\right)\right\}|\,\Omega\rangle=\frac{\int\mathcal{D}\phi\phi\left(x_1\right)\cdots\phi\left(x_n\right)e^{iS[\phi]}}{\int\mathcal{D}\phi e^{iS[\phi]}} \tag{90}$$

where on the LHS we have a correlation function in canonical formalism and it's given by path integral on the RHS. Notice the fields on the LHS are non-commutating operators, different from the fields on the RHS which are c-number functions. (90) is always rewritten using the generating functional, which gives

$$\langle\Omega\,|\mathrm{T}\left\{\phi\left(x_1\right)\ldots\phi\left(x_n\right)\right\}|\,\Omega\rangle=\frac{1}{Z[0]}\left[\prod_{i=1}^{n}\left(\frac{1}{i}\frac{\delta}{\delta J\left(x_i\right)}\right)Z[J]\right]_{J=0} \tag{91}$$

where the generating function $Z[J]=\int\mathcal{D}\phi\exp\left(i\int d^4x(\mathcal{L}(x)+J(x)\phi(x))\right)$. To derive the Schwinger-Dyson equation, let's first perform a transformation which leaves the measure of path integral invariant $\phi_a\to\phi_a'=\phi_a+\delta\phi_a$, the generating functional becomes

$$\begin{aligned}Z'[J]&=\int\mathcal{D}\phi'\exp\left(iS\left[\phi'\right]+i\int d^4xJ(x)\phi'\right)\\&=\int\mathcal{D}\phi\exp\left(iS\left[\phi_a+\delta\phi_a\right]+i\int d^4xJ\left(x\right)\left(\phi_a+\delta\phi_a\right)\right).\end{aligned} \tag{92}$$

Expand it to first order of $\delta\phi_a$, we get

$$Z'[J]=Z[J]+\delta Z[J] \tag{93}$$

where $Z'[J] = Z[J]$ after we rename the field variable and

$$\delta Z[J] = i \int \mathcal{D}\phi e^{i(S+\int d^4 y J_a \phi_a)} \int dx^4 \left(\frac{\delta S}{\delta \phi_a} + J_a(x) \right) \delta \phi_a(x) = 0. \tag{94}$$

Taking a functional derivative of (93), we then find [14]

$$\frac{\delta \delta Z}{\delta J_{a'}(x')} = i \int \mathcal{D}\phi e^{i(S+\int d^4 x J_a \phi_a)} \left(i\phi_{a'} \int dx^4 \left(\frac{\delta S}{\delta \phi_a} + J_a(x) \right) + \int d^4 x \delta_{a'a} \delta^{(4)'}(x-x') \right) = 0 \tag{95}$$

where we drop $\delta\phi_a$ as it's arbitrary. Taking n functional derivatives of (93) with respect to $J_{a_j}(x_j)$ will give

$$\begin{aligned} \frac{(-i)^{-n}}{Z_0} \prod_{j=1}^{n} \frac{\delta Z}{\delta J_{a_j}} \bigg|_{J=0} &= \frac{1}{Z_0} \int \mathcal{D}\phi e^{iS} \int d^4 x \left[i \left(\frac{\delta S}{\delta \phi_a} \phi_1 \cdots \phi_n \right) \right. \\ &\left. + \sum_{j=1}^{n} \phi_1 \cdots \delta_{aa_j} \delta(x - x_j) \ldots \phi_n \right]. \end{aligned} \tag{96}$$

Write it in terms of the correlation function, we finally obtain the general form of the Schwinger-Dyson equation for the scalar field, which is

$$\langle \Omega | T\{ \frac{\delta S}{\delta \phi_a(x)} \phi_1 \ldots \phi_n \} | \Omega \rangle = i \sum_{j=1}^{n} \left\langle \Omega \left| \phi_1 \ldots \delta_{aa_j} \delta^{(4)}(x - x_j) \cdots \phi_n \right| \Omega \right\rangle. \tag{97}$$

One thing that needs to be careful is that if $\frac{\delta S}{\delta \phi_a(x)}$ contains a derivative, then it should be outside of the correlation function. To explain this, let's first define a correlation function in path integral by

$$\langle \phi_1 \cdots \phi_n \rangle = \frac{1}{Z[0]} \int \mathcal{D}\phi \phi_1 \cdots \phi_n e^{iS}. \tag{98}$$

You may be confused that haven't the correlation function in the path integral already given in (90)? The answer is that when there are no derivatives involved, we have

$$\langle \phi_1 \cdots \phi_n \rangle = \langle \Omega | \mathrm{T} \{ \phi(x_1) \ldots \phi(x_n) \} | \Omega \rangle \tag{99}$$

[14]Functional derivative of $Z'[J]$ cancel with the functional derivative of $Z[J]$, therefore we only left with the functional derivative of $\delta Z[J]$.

which leads to (90). The case is different when there is a derivative operator, in this case, we have

$$\begin{aligned}
\langle\partial_\mu\phi(x)\phi_1\ldots\phi_n\rangle &= \frac{1}{Z[0]}\int\mathcal{D}\phi\partial_\mu\phi\phi_1\ldots\phi_n e^{iS} \\
&= \frac{1}{Z[0]}\int\mathcal{D}\phi\bigg|_{\varepsilon\to 0}\frac{\phi\left(x^\mu+\varepsilon\right)-\phi\left(x^\mu\right)}{\varepsilon}\phi_1\ldots\phi_n e^{iS} \\
&= \lim_{\varepsilon\to 0}\frac{\langle\Omega|T\left\{\phi\left(x^\mu+\varepsilon\right)\phi_1\ldots\phi_n|\Omega\rangle-\langle\Omega|T\left\{\phi\left(x^\mu\right)\phi_1\ldots|\Omega\rangle\right.\right.}{\varepsilon} \\
&= \partial_\mu\left\langle\Omega\left|T\left\{\phi(x)\phi_1,\ldots\phi_n\right\}\right|\Omega\right\rangle .
\end{aligned}\tag{100}$$

This shows that in path integral formalism, we can only obtain the derivative of the correlation function rather than the correlation function of the derivative operator. In (97), if there is a derivative in the LHS, then it should be outside of the correlation function according to our analysis above. For a free scalar field theory, take two field operators correlation function, and we arrive at the same formula as (89). For an interacting theory $\mathcal{L}=-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2+\mathcal{L}_{\text{int}}(\phi)$, we have

$$\frac{\delta S}{\delta\phi}=\left(\partial^2-m^2\right)\phi+\mathcal{L}_{\text{int}}^{\prime} .\tag{101}$$

The equation of motion gives $\left(\partial^2-m^2\right)\phi+\mathcal{L}_{\text{int}}^{\prime}=0$, therefore we get $\mathcal{L}_{\text{int}}^{\prime}=-\left(\partial^2-m^2\right)\phi$, plug these into Schwinger-Dyson equation, we finally obtain

$$\begin{aligned}
&\left(\partial^2-m^2\right)\left\langle\Omega\left|\mathrm{T}\left\{\phi(x)\phi\left(x_1\right)\ldots\phi\left(x_n\right)\right\}\right|\Omega\right\rangle \\
&\quad=\left\langle\Omega\left|\mathrm{T}\left\{\left(\partial^2-m^2\right)\phi(x)\phi\left(x_1\right)\ldots\phi\left(x_n\right)\right\}\right|\Omega\right\rangle \\
&\quad+i\sum_{j=1}^{n}\left\langle\Omega\left|\mathrm{T}\left\{\phi\left(x_1\right)\ldots\delta^4\left(x-x_j\right)\ldots\phi\left(x_n\right)\right\}\right|\Omega\right\rangle ,
\end{aligned}\tag{102}$$

which tells us that you will get extra contact terms when we take the derivative operator out of the correlation function.

3.3 Ward-Takahashi identity

In this subsection, we will derive the Ward-Takahashi identity which is a reflection of the current conservation in a quantum theory. In a classic theory, if perform an infinitesimal symmetry transformation $\phi_\alpha\to\phi_\alpha+\delta\phi_\alpha$, then the Lagrangian will change as

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a+\frac{\delta L}{\partial\left(\partial_\mu\phi_a\right)}\partial\mu\delta\phi_a \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_\alpha\right)}\delta\phi_\alpha\right)+\left(\frac{\partial\mathcal{L}}{\partial\phi_\alpha}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_\alpha\right)}\right)\right)\delta\phi_\alpha
\end{aligned}\tag{103}$$

If the field is physical, then the second term vanishes, and we derive $\partial_\mu j^\mu = 0$[15] where $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)}\delta\phi_\alpha$. However, in the path integral fields don't need to be on-shell, therefore the current conservation becomes

$$\partial_\mu J^\mu(x) = -\frac{\delta S}{\delta \phi_\alpha(x)}\delta\phi_\alpha(x), \tag{104}$$

plug this relation into (96) and translate it into the correlation function, we get

$$\partial_\mu \left\langle \Omega \left| T\left\{ j^\mu(x)\phi_1(x_1)\cdots\phi_n(x_n)\right\}\right| \Omega \right\rangle = -i\sum_{j=1}^{n}\langle\Omega|\phi_1\cdots\delta_{aa_j}\delta^{(4)}\left(x-x_j\right)\delta\phi_a\cdots\phi_n|\Omega\rangle \tag{105}$$

which is the general form of Ward-Takahashi identity in QFT. In quantum theory, as there are not only on-shell configurations but off-shell configurations, the current conservation doesn't strictly hold, and there are extra contact terms just as in the Schwinger-Dyson equation.

3.4 Ward-Takahashi identity in spinor QED

The Lagrangian of spinor QED is[16]

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\partial - m)\psi + q\bar{\psi}A_\mu\gamma^\mu\psi. \tag{106}$$

It has global $U(1)$ symmetry, which means the Lagrangian is invariant under transformation

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}. \tag{107}$$

The corresponding conserved current of $U(1)$ global symmetry is

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{108}$$

where we ignore the constant factor. Now we want to plug it into the general Ward-Takahashi identity. (105) works for scalar fields, for fermionic fields, instead, we have

$$\begin{aligned}\partial_\mu\langle 0|T\{\bar{\psi}(x)\gamma^\mu\psi\left(x\right),\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\}|0\rangle = &-\delta\left(x-x_1\right)\langle 0|T\{\psi\left(x_1\right)\bar{\psi}\left(x_1\right)\mid 0)\\ &+\delta\left(x-x_2\right)\langle 0|T\left\{\psi(x_1)\overline{\psi}\left(x_1\right)|0\right\rangle\end{aligned} \tag{109}$$

[15] We perform a symmetry transformation which keeps Lagrangian invariant, therefore we also have $\delta\mathcal{L} = 0$.

[16] This, of course, is not renormalized Lagrangian. Here we simply ignore the renormalization as it will not affect our result.

Defining the following momentum space correlation

$$
\begin{aligned}
M^{\mu}(k,p,q) &= \int dx^4 \int dx_1^4 \int dx_2^4 e^{ikx} e^{ipx_1} e^{-iqx_2} \left\langle j^{\mu}(x)\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right\rangle , \\
M_0(p,q) &= \int dx_1^4 dx_2^4 e^{ipx_1} e^{-iqx_2} \left\langle \psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right\rangle , \\
M_0(p+k,q) &= \int dx^4 \int dx_1^4 \int dx_2^4 e^{ikx} e^{ipx_1} e^{-iqx_2(4)} \delta^{(4)}(x-x_1) \left\langle \psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right\rangle ,
\end{aligned}
\tag{110}
$$

then (109) can be written as

$$
ik_{\mu} M^{\mu}(k,p,q) = -M_0(k+p,q) + M_0\left(p,q-k\right). \tag{111}
$$

According to the LSZ reduction formula, if we want the RHS to contribute to the S-matrix, then both q and p should be on-shell. However, if q and p are on-shell momentum, then for nonzero k, $k+p$ and $q-k$ are both off-shell, which means the RHS doesn't contribute to the S-matrix. Donate the scattering amplitude correspond to the LHS to be $\mathcal{M}^{\mu}$, we finally get

$$
k_{\mu}\mathcal{M}^{\mu} = 0. \tag{112}
$$

Now, there is only one more question left. The Ward identity states that if the amplitude can be written as $\epsilon_{\mu}\mathcal{M}^{\mu}$ where ϵ_{μ} is the polarization vector of a photon, then we have $k_{\mu}\mathcal{M}^{\mu} = 0$. So the question is how do we know the k_{μ} we find in (112) is the momentum vector of a photon? To see this, recall the LSZ reduction formula of the photon shows that the S-matrix involves objects such as

$$
i\varepsilon^{\mu}(k) \int d^4x e^{-ikx} \left(-\partial^2\right) \langle \Omega | T\{A_{\mu}(x)\cdots\} | \Omega \rangle. \tag{113}
$$

Using the Schwinger-Dyson equation of a photon field, we get

$$
\epsilon_{\mu}\partial^2 \langle \Omega | T\{A_{\mu}\cdots\} | \Omega \rangle = \epsilon_{\mu} \langle \Omega | T\{J_{\mu}\cdots\} | \Omega \rangle + \text{Contact term}. \tag{114}
$$

As those contact terms don't have the required pole to contribute to S-matrix, we can simply ignore these terms. Now, replace the polarization vector ϵ_{μ} with the momentum of the photon k_{μ}, we can see it's exactly the LHS of (111). We therefore derived the Ward identity in spinor QED from the Ward-Takahashi identity of the $U(1)$ global symmetry.

A Universal enveloping algebra

A.1 Associative algebra

An algebra $\mathcal{A}$ over a field $\mathbb{K}$[17][18] is a vector space over $\mathbb{K}$ equipped with a bilinear product $*$: $\mathcal{A} * \mathcal{A} \to \mathcal{A}$ We define an algebra to be *associative* if

$$x * (y * z) = (x * y) * z, \quad \forall x, y, z \in \mathcal{A}. \tag{115}$$

Moreover, if it contains the unit element a, which satisfy

$$x * a = x = a * x, \tag{116}$$

we say it's an *unital* algebra. An associative algebra is a vector space with multiplication defined, or, in general, an algebraic structure with an addition, a multiplication, and a scalar multiplication defined.

Any associative algebra can also be viewed as a Lie algebra because we can naturally define a Lie bracket by

$$[a, b] := a * b - b * a. \tag{117}$$

This is why the associative algebra $M_n(\mathbb{K})$ of $n \times n$-matrices can be viewed as a Lie algebra. Moreover, let $\mathcal{A}$ be a unital associative algebra of finite dimension over the field $\mathbb{K}$. Then $\mathcal{A}$ is isomorphic to a subalgebra of the algebra $M(n; \mathbb{K})$ of $n \times n$ matrices for some non-negative integer $n \in \mathbb{N}$. Therefore, we can always represent a unital associative algebra by matrix.

A.2 Tensor algebra

For a vector space $V(\mathbb{K})$, its tensor algebra $T(V)$ is a unital associative algebra defined on

$$\bigoplus_{k=0}^{\infty} T^k V = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots \tag{118}$$

where $\oplus$ is direct sum and $\otimes$ is direct product. The direct product gives the product map required by the associative algebra.

[17] We denote by $\mathbb{K}$ the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

[18] An algebra over $\mathbb{K}$ is also called a K-algebra

A free (unital, associative) algebra over $\mathbb{K}$ on a set $\{X_1, X_2, \ldots, X_n\}$, which we donated as $\mathbb{K}\langle X_i \rangle$, is defined to be the algebra spanned by

$$\sum_{i_1,i_2,\cdots,i_k \in \{1,2,\cdots,n\}} a_{i_1,i_2,\cdots,i_k} X_{i_1} X_{i_2} \cdots X_{i_k} \tag{119}$$

where $a_{i_1,i_2,\cdots,i_k} \in \mathbb{K}$ and $k \in (0, 1, \ldots, n)$. Let $\{e_i\}$ to be a basis of a vector space over $\mathbb{K}$, then we have the following isomorphism of algebras

$$T(V) \cong \mathbb{K} < e_i > . \tag{120}$$

This means any elements of tensor algebra can be written as a linear combination of the polynomial product of the basis vector of the vector space.

A.3 Quotient algebra

A left (or right) ideal is a subalgebra $\mathcal{I} \subseteq \mathcal{A}$ satisfy $\mathcal{A} * \mathcal{I} \subseteq \mathcal{I}$ (or $\mathcal{I} * \mathcal{A} \subseteq \mathcal{I}$). A subalgebra $\mathcal{I}$ is called a two-side ideal if it's both a right and left ideal. A quotient algebra is defined to be $\mathcal{A}/\mathcal{I}$, which is an algebra of the equivalent class defined by a two-side ideal $\mathcal{I}$. An equivalent class of an element a in $\mathcal{A}$ is given by

$$[a] = a + \mathcal{I} := \{a + r : r \in \mathcal{I}\}. \tag{121}$$

We need the $\mathcal{A}$ to be a two-sided ideal because we need

$$[a][b] = [ab], \tag{122}$$

which requires

$$(a + \mathcal{I})(b + \mathcal{I}) \tag{123}$$

$$= ab + a\mathcal{I} + \mathcal{I}b + \mathcal{I}^2 \tag{124}$$

$$= ab + \mathcal{I}. \tag{125}$$

We can see this requires $\mathcal{A} * \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} * \mathcal{A} \subseteq \mathcal{I}$ which means $\mathcal{I}$ is needed to be a two-sided ideal.

A.4 Universal enveloping algebra

The tensor algebra $T(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is an unital associative algebra, this means we naturally have a Lie bracket $[\cdot,\cdot]$ defined on the tensor algebra, which is given by

$$a \otimes b - b \otimes a \tag{126}$$

where $\otimes$ is the tensor product. Therefore the tensor Lie algebra is also a Lie algebra. However, the map $\mathfrak{g} \to T(\mathfrak{g})$ is not a Lie algebra homomorphism. A map between two Lie algebras $\mathfrak{g}$ and η is a *Lie algebra homomorphism* if

$$\varphi([a, b]) = [\varphi(a), \varphi(b)], \quad \forall a, b \in \mathfrak{g}. \tag{127}$$

We can see the map between $\mathfrak{g}$ and $T(\mathfrak{g})$ is not a Lie algebra homomorphism because $\mathbf{[}a, b\mathbf{]} = a \otimes b - b \otimes a \neq [a, b]$ based on the current setup. To impose this constrain, we need a map $\sigma : T(\mathfrak{g}) \to T'(\mathfrak{g})$ which is a Lie algebra homomorphism satisfy

$$\sigma(a)\sigma(b) - \sigma(b)\sigma(a) = \sigma([a, b]). \tag{128}$$

We did nothing but just identify $\mathbf{[}a, b\mathbf{]}$ with $[a, b]$. Then, according to the fundamental theorem of homomorphism, we get

$$\text{im } \sigma = T(\mathfrak{g})/\text{ker } \sigma. \tag{129}$$

For a Lie algebra $\mathfrak{g}$, the above formula gives its universal enveloping algebra $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I} \tag{130}$$

where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and $\mathcal{I}$ is the two-sided ideal generated by elements of the form

$$a \otimes b - b \otimes a - [a, b] \tag{131}$$

for all $a, b \in \mathfrak{g}$. Given a Lie algebra $\mathfrak{g}$, its universal enveloping algebra is unique: suppose we have a map $\varphi : \mathfrak{g} \to A$ which satisfy

$$\varphi([X, Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) \tag{132}$$

for all $X, Y \in \mathfrak{g}$. The algebra A is a unital associative algebra with the Lie bracket given by the commutator. Then there exists a unique unital algebra homomorphism

$$\widehat{\varphi} : U(\mathfrak{g}) \to A \tag{133}$$

such that

$$\varphi = \widehat{\varphi} \circ \tau. \tag{134}$$

where $\tau : \mathfrak{g} \to U(\mathfrak{g})$ is the embedding map. This is called the universal property of universal enveloping algebra. It shows that all universal enveloping algebras of $\mathfrak{g}$ are isomorphic, therefore guaranteeing the uniqueness of universal enveloping algebra.

A.5 Poincaré–Birkhoff–Witt theorem

The map $\tau : \mathfrak{g} \to U(\mathfrak{g})$ is obtained by embedding $\mathfrak{g}$ into $T(\mathfrak{g})$[19] then composing with the quotient map (129). We state the following theorem without proof.

Theorem 1 (PBW theorem)**.** *If $\mathfrak{g}$ is a finite-dimensional Lie algebra with basis $X_1, \ldots, X_k$, then its universal enveloping algebra $U(\mathfrak{g})$ is spanned by linearly independent element of the form*

$$\tau\left(X_1\right)^{n_1} \tau\left(X_2\right)^{n_2} \cdots \tau\left(X_k\right)^{n_k} \tag{135}$$

where each $nnew_k$ is a non-negative integer.

The linear independence in PBW theorem implies that the map τ is injective.

B Casimir invariant

B.1 Killing form

Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, every element $a \in \mathfrak{g}$ define an adjoint endomorphism[20] $ad(a)$ with the help of the Lie bracket, as

$$ad(a)(b) = [a, b]. \tag{136}$$

The Killing form is the bilinear map$\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by

$$\mathcal{K}(a, b) = \operatorname{tr}(\operatorname{ad}(a) \operatorname{ad}(b)). \tag{137}$$

A key property of K is "invariance". That is,

$$\mathcal{K}([g, a], b) + \mathcal{K}(a, [g, b]) = 0, \quad \forall a, b, g \in \mathfrak{g}. \tag{138}$$

B.2 Casimir operator

For a finite-dimensional semisimple Lie algebra, its Casimir operators are a distinguished basis of the centre $\mathcal{Z}(U(\mathfrak{g}))$ of the universal enveloping algebra made of homogeneous polynomials

$$\mathcal{C}_k = d^{i_1 \ldots i_k} T_{i_1} \ldots T_{i_k} \tag{139}$$

[19] The embedding is achieved by the map $h : \mathfrak{g} \to T^1(\mathfrak{g})$.

[20] The endomorphism is, in short, a homomorphism from an object to itself.

with $d^{i_1 \dots i_k}$ suitable symmetric tensors which are invariant under adjoint action[21]. For quadratic Casimir element, the typical choice of the symmetric tensors is the Killing form.

As elements in the centre of $U(\mathfrak{g})$ commute with all elements of $U(\mathfrak{g})$, this holds for any subset of $U(\mathfrak{g})$. Therefore Casimir operator commute with all generator of (embedded) Lie algebra $\mathfrak{g}$.

B.3 Schur's lemma

If V and W are vector space over $\mathbb{K}$, let ϕ and ψ be a representation of group G over V and W respectively, then the intertwining (or equivalent) map is a linear map $\alpha : V \to W$ which satisfy

$$\alpha(g \cdot v) = g \cdot \alpha(v) \tag{140}$$

for all $g \in G$ and $v \in V$. An intertwining definition is, for $phi : G \to GL(V)$ and $\psi : G \to GL(W)$, an equivalent map must satisfy

$$\alpha \circ \varphi(g) = \psi(g) \circ \alpha \tag{141}$$

for all $g \in G$, which ensures the following diagram commutes[22]:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ {\scriptstyle \phi(g)}\big\downarrow & & \big\downarrow{\scriptstyle \psi(g)} \\ V & \xrightarrow{\alpha} & W \end{array}$$

If $V = W$ then an intertwiner $\phi : V \to V$ is called a self-intertwiner.

Theorem 2 (Schur's Lemma)**.** 1. *Let V and W be irreducible real or complex representations of a group or Lie algebra and let $\phi : V \to W$ be an intertwining map. Then either $\phi = 0$ or ϕ is an isomorphism.*

2. *Let V be an irreducible complex representation of a group or Lie algebra and let $\phi : V \to V$ be a self-intertwiner map of V. Then $\phi = \lambda I$, for some $\lambda \in \mathbb{C}$.*

3. *Let V and W be irreducible complex representations of a group or Lie algebra and let $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then $\phi_1 = \lambda \phi_2$, for some $\lambda \in \mathbb{C}$.*

[21] I haven't fully understand this point, I might add an explanation in the future.

[22] You can simply understand it as all directed paths in the diagram with the same start and endpoints lead to the same result.

Let R be a finite-dimensional irreducible representation of a complex algebra, according to Schur's Lemma, if z is in the centre of the complex algebra, then $R(z) = \lambda I$, with $\lambda \in \mathbb{C}$ because $R(z)$ is a self-intertwiner. The Casimir invariants belong to the centre of the universal enveloping algebra, it's also the centralizer of the Lie algebra $\mathfrak{g}$, this is why the Casimir operator must be in the form of λI for an irreducible representation.

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