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1 Introduction

The first exact solution to Einstein's field equations was obtained by Karl Schwarzschild in 1916 [1], now known as the Schwarzschild solution. This metric describes the external spacetime of a static, spherically symmetric gravitating body in the absence of a cosmological constant. However, since most celestial bodies in the universe possess angular momentum, a more general description of rotating spacetimes was required. Such solutions were not found until 47 years later, primarily because the nonlinearity of Einstein's equations makes it exceedingly difficult to obtain exact axisymmetric solutions.

In 1963, Roy Kerr succeeded in deriving the exact metric for a rotating, uncharged mass through an ingenious approach formulated within the Newman–Penrose formalism. Ordinarily [2], one would expect that the exterior metric of a rotating and charged body could be derived by applying the same analytical procedure to solve the Einstein equations coupled to Maxwell theory. Nevertheless, only two years after the discovery of the Kerr metric, in 1965, Newman and his collaborators, by employing a complex coordinate transformation, almost miraculously obtained a solution from the Schwarzschild metric that proved to be equivalent to the Kerr solution [3]. In the same year, by applying a similar transformation to the Reissner–Nordström metric, they derived what is now known as the Kerr–Newman solution, which represents the spacetime geometry exterior to a general rotating and charged gravitating body [4].

The complex coordinate transformation method, now known as the Newman-Janis (NJ) algorithm, is a type of off-shell[1] solution-generating technique for the Einstein equation. Although a naive application of this algorithm often fails to yield a reasonable solution [5, 6], it has nevertheless produced valid results in a variety of theoretical models [7–13]. The underlying mechanism responsible for its effectiveness, as well as the precise conditions required for it to work, remains poorly understood. Even though in practice this algorithm rarely provides genuinely new solutions, it offers valuable insights into the structure of the phase space of gravitational theories and may ultimately contribute to a deeper understanding of quantum gravity.

In this note, we review various explanations and extensions of the NJ algorithm. We first show the basic setup and its application to flat spacetime and KN spacetime in Section 2 and Section 3 respectivily. Then we reviewed an equivalent but computationally more efficient method, introduced by Giampieri in Section 4. The NJ algorithm for the gauge field is reviewed in Section 5, using both tetrad formalism and Giampieri's approach. Several explanations about the effectiveness of this algorithm are reviewed in Section 6. In section 7, we show the recent development of the generalization of the NJ algorithm proposed by Harold Erbin [13–16]. A concise introduction to several concepts that appear in this note and may be unfamiliar to the reader is presented in the Appendix.

[1] Unlike most on-shell solution-generating techniques, which rely on the hidden symmetry of the Einstein equation and therefore preserve the Einstein equation, off-shell techniques do not necessarily preserve the Einstein equation.

2 The setup

The original Newman-Janis algorithm can be summarized as the following:

1. Start from seed metric

$$ds^2 = -f(r)dt^2 - f(r)dr^2 + f_\Omega(r)d\Omega^2, \tag{2.1}$$

Change to the retarded Eddington-Finkelstein [2] (EF) coordinate $\{u, r, \theta, \phi\}$ in which $g_{rr} = 0$,

$$ds^2 = -f(r)du^2 - 2dr^2 + f_\Omega(r)d\Omega^2, \tag{2.2}$$

one could also use advanced Eddington-Finkelstein coordinates, but later the sign of complex coordinate transformation would be different.

2. Decompose the metric (in retarded Eddington-Finkelstein coordinate) into four null tetrads $\{n^a, l^a, m^a, \bar{m}^a\}$, where $n^a l_a = -1, m^a \bar{m}_a = 1$ and m^a is the complex conjugate of m^a:

$$\ell^a = \partial_r^a, \quad n^a = \partial_u^a - \frac{f}{2}\partial_r^a, \quad m^a = \frac{1}{\sqrt{2f_\Omega}}\left(\partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \tag{2.3}$$

where $\partial_\mu^a = (\frac{\partial}{\partial x^\mu})^a$ is the basis vector.

3. Take u and r to be complex and apply the following transformation rule to the tetrads:

$$\begin{aligned} &r \longrightarrow \tfrac{1}{2}(r + \bar{r}) = \mathrm{Re}(r), \\ &\frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) = \frac{\mathrm{Re}(r)}{|r|^2}, \\ &\frac{1}{r^2} \longrightarrow \frac{1}{|r|^2}, \\ &r^2 \longrightarrow |r|^2. \end{aligned} \tag{2.4}$$

A more complicated formula should be factorized to multiplication of quadratic forms in order to apply the rule.

4. Perform a complex coordinate transformation

$$u \to u' = u - ia\cos\theta, \quad r \to r' = r + ia\cos\theta, \quad \theta \to \theta', \quad \phi \to \phi' \tag{2.5}$$

to n^a, l^a, m^a. Taking $\bar{m}^a$ to be the complex conjugate of m^a, restricting $\{u', r', \theta', \phi'\}$ to be real, and removing the prime, the following tetrad is obtained,

$$\begin{aligned} &\ell'^a = \partial_r^a, \quad n'^a = \partial_u^a - \frac{\tilde{f}}{2}\partial_r^a, \\ &m'^a = \frac{1}{\sqrt{2\tilde{f}_\Omega}}\left(ia\sin\theta\,(\partial_u^a - \partial_r^a) + \partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \end{aligned} \tag{2.6}$$

where $\tilde{f}$ and $\tilde{f}_\Omega$ is the function in seed metric after applying the transformation rule (2.4) and complex transformation (2.5).

[2] In the following we always use retarded EF coordinate and directly refer to it as the EF coordinate.

5. Calculate the inverse metric using (A.15) and then convert it to metric, applying a coordinate transformation

$$du' = du - g(r)dr, \quad d\phi = d\varphi - h(r)dr, \tag{2.7}$$

g where

$$g(r) = \frac{\tilde{f}_\Omega + a\sin^2\theta}{\Delta}, \quad h(r) = \frac{a}{\Delta} \tag{2.8}$$

and

$$\Delta = \tilde{f}_\Omega \tilde{f} + a^2 \sin^2\theta. \tag{2.9}$$

This transformation leads to a metric having only one non-zero off-diagonal term $g_{t\phi}$ which is called *Boyer-Lindquist (BL) coordinates.*

There are several things worth mentioning. First, The rule in step 3 is an "empirical" rule, one shouldn't mix the rule for r^{-1} and r^{-2} because it will lead to a different expression:

$$\frac{1}{r^2} \longrightarrow \frac{1}{|r|^2}, \quad \frac{1}{r}\frac{1}{r} \longrightarrow \frac{(\mathrm{Re}\, r)^2}{|r|^4}. \tag{2.10}$$

Second, the operation of keeping $\bar{m}^a$ conjugate to m^a in step four is nontrivial as $\bar{m}^a$ is no longer conjugate to m^a after the complex coordinate transformation. Third, the transformation (2.7) does not always exist because f and g could be a function of both r and θ, which makes it not a valid coordinate transformation[3].

3 Example

In this section, we are going to illustrate how the NJ algorithm is applied to metrics using the Minkowski metric and Kerr-Newman metric. By setting the charge Q to zero, the RN metric to KN metric process will reduce to the Schwarzschild to Kerr process.

3.1 Flat spacetime

The Minkowski metric in advanced Eddington-Finkelstein coordinates is given by

$$ds^2 = -du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.1}$$

We can decompose it into a null tetrad:

$$\begin{aligned} l^a &= \partial_r^a, \\ n^a &= \partial_u^a - \frac{1}{2}\partial_r^a, \\ m^a &= \frac{1}{\sqrt{2}r}\left(\partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \\ \bar{m}^a &= \frac{1}{\sqrt{2}r}\left(\partial_\theta^a - \frac{i}{\sin\theta}\partial_\phi^a\right). \end{aligned} \tag{3.2}$$

[3] Some works [17, 18] don't check the validity of BL transformation; therefore, they obtain an (at least partly) unreliable result.

Next, take u and r to be complex-valued, and then take $\bar{m}^a$ to be complex conjugate to m, we get

$$
\begin{aligned}
l^a &= \partial_r^a \\
n^a &= \partial_u^a - \frac{1}{2}\partial_r^a, \\
m^a &= \frac{1}{\sqrt{2}\bar{r}}\left(\partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \\
\bar{m}^a &= \frac{1}{\sqrt{2}r}\left(\partial_\theta^a - \frac{i}{\sin\theta}\partial_\phi^a\right).
\end{aligned}
\tag{3.3}
$$

After performing the complex transformation (2.5) and then take $\bar{m}^a$ conjugate to m^a, a new null tetrad is obtained[4],

$$
\begin{aligned}
l^a &= \partial_r^a, \\
n^a &= \partial_u^a - \frac{1}{2}\partial_r^a, \\
m^a &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(\partial_\theta^a + ia\sin\theta\left(\partial_u^a - \partial_r^a\right) + \frac{i}{\sin\theta}\partial_\phi^a\right), \\
\bar{m}^a &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(\partial_\theta^a - ia\sin\theta\left(\partial_u^a - \partial_r^a\right) - \frac{i}{\sin\theta}\partial_\phi^a\right).
\end{aligned}
\tag{3.4}
$$

Calculating the inverse metric use the new null tetrad and convert it to metric, we end up with

$$
\begin{aligned}
ds^2 = &- du^2 - 2dudr + 2a\sin^2\theta drd\phi\ \left(r^2 + a^2\cos^2\theta\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
&+ a^2\sin^4\theta^d\phi^2.
\end{aligned}
\tag{3.5}
$$

After performing coordinate transformation (2.7), we get the Minkowski metric in the oblate spheroidal coordinates

$$
ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + \left(r^2 + a^2\cos^2\theta\right)d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta d\varphi^2, \tag{3.6}
$$

which is connected with the original Minkowski metric by the transformation

$$
\begin{aligned}
x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi, \\
y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi, \\
z &= r\cos\theta.
\end{aligned}
\tag{3.7}
$$

Therefore, the NJ algorithm is nothing more than a diffeomorphism for flat spacetime.

3.2 Kerr-Newman spacetime

The Kerr-Newman metric can be derived by coupling Maxwell's theory to Einstein's theory and solving the Einstein equation, which is quite lengthy. The NJ algorithm allows you to derive it very quickly from the Reissner–Nordström (RN) metric. One starts with RN metric in spherical coordinates,

$$
ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.8}
$$

[4]The prime is omitted

The advanced Eddington-Finkelstein form of the Reissner–Nordström metric is

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.9}$$

The corresponding null tetrad is

$$\begin{aligned}
l^a &= \partial_r^a, \\
n^a &= \partial_u^a - \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)\partial_r^a, \\
m^a &= \frac{1}{\sqrt{2}r}\left(\partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \\
\bar{m}^a &= \frac{1}{\sqrt{2}r}\left(\partial_\theta^a - \frac{i}{\sin\theta}\partial_\phi^a\right).
\end{aligned} \tag{3.10}$$

After applying the rule (2.4), the tetrad becomes

$$\begin{aligned}
l^a &= \partial_r^a, \\
n^a &= \partial_u^a - \frac{1}{2}\left(1 - \frac{2m \operatorname{Re} r}{|r|^2} + \frac{Q^2}{|r|^2}\right)\partial_r^a, \\
m^a &= \frac{1}{\sqrt{2}\bar{r}}\left(\partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \\
\bar{m}^a &= \frac{1}{\sqrt{2}r}\left(\partial_\theta^a - \frac{i}{\sin\theta}\partial_\phi^a\right).
\end{aligned} \tag{3.11}$$

Next, perform the complex coordinate transformation (2.5), a new tetrad is obtained, which is

$$\begin{aligned}
l'^a &= \partial_r^a, \\
n'^a &= \partial_u^a - \frac{1}{2}\left(1 - \frac{2mr - Q^2}{R^2}\right)\partial_r^a, \\
m'^a &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta\partial_u^a - ia\sin\theta\partial_r^a + \partial_\theta^a + \frac{i}{\sin\theta}\partial_\phi^a\right), \\
\bar{m}'^a &= \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(-ia\sin\theta\partial_u^a + ia\sin\theta\partial_r^a + \partial_\theta^a - \frac{i}{\sin\theta}\partial_\phi^a\right),
\end{aligned} \tag{3.12}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$. The inverse and thus the metric can be calculated using this tetrad, finally, we obtain the Kerr-Newman metric in the Eddington-Finkelstein coordinate:

$$\begin{aligned}
ds^2 = &-\left(1 - \frac{2mr - Q^2}{\rho^2}\right) du^2 - 2dudr - \frac{2a\sin^2\theta}{\rho^2}\left(2mr - Q^2\right) dud\phi \\
&+ 2a\sin^2\theta drd\phi + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(\Delta a^2\sin^2\theta - \left(a^2 + r^2\right)^2\right) d\phi^2,
\end{aligned} \tag{3.13}$$

where $\Delta = r^2 + a^2 - 2mr + Q^2$. In the zero Q or zero a limit, it reduces to Kerr or Reissner–Nordström metric respectively. This coordinate system makes calculations cumbersome because there are too many off-diagonal terms. To reduce the number of off-diagonal terms, one performs the transformation

$$du = dt - \frac{r^2 + a^2}{\Delta} dr, \quad d\varphi = d\phi - \frac{a}{\Delta} dr, \tag{3.14}$$

which results in the Boyer-Lindquist form of the Kerr-Newman metric,

$$ds^2 = -\frac{\Delta}{\rho^2} \left(a \sin^2\theta d\varphi - dt\right)^2 + \frac{\sin^2\theta}{\rho^2} \left(\left(r^2 + a^2\right) d\varphi - a dt\right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \tag{3.15}$$

4 An alternative approach: Giampieri's method

The original NJ algorithm relies on the null tetrad, making the calculation tedious. An alternative approach, equivalent to the NJ algorithm but computationally more efficient, was proposed by Giacomo Giampieri in 1990[19]. Here we use the Kerr-Newman metric as an example to illustrate this approach.

The new approach also starts with static metric (here we use RN metric as example) in advanced Eddington-Finkelstein coordinates (3.9) Instead of constructing null tetrad, we directly take u and r to be complex value and apply the transformation rule (2.4) for (3.9), which result in

$$ds^2 = -\left(1 - \frac{2m \operatorname{Re} r}{|r|^2} + \frac{Q^2}{|r|^2}\right) du^2 - 2du dr + r\bar{r} \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.1}$$

Next, we perform a complex coordinate transformation similar to (2.5), but replace θ by $\theta*$ which is an extra coordinate

$$u \to u' = u - ia\cos\theta^*, \quad r \to r' = r + ia\cos\theta^*. \tag{4.2}$$

This gives a five-dimensional metric

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta^*} + \frac{Q^2}{r^2 + a^2\cos^2\theta^*}\right) du^2 - 2du dr + 2ia\sin\theta^* d\theta^* dr \\ & - \left(\frac{4mr}{r^2 + a^2\cos^2\theta^*} - \frac{2Q^2}{r^2 + a^2\cos^2\theta^*}\right) ia\sin\theta^* du d\theta^* \\ & - \left(1 + \frac{2mr}{r^2 + a^2\cos^2\theta^*} - \frac{Q^2}{r^2 + a^2\cos^2\theta^*}\right) a^2\sin^2\theta^* d\theta^{*2} \\ & + \left(r^2 + a^2\cos^2\theta^*\right) \left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{4.3}$$

In order to obtain the Kerr-Newman metric, one needs to perform the following identification

$$i\frac{d\theta^*}{\sin\theta^*} = d\phi \tag{4.4}$$

together with the substitution $\theta^* = \theta$. This would lead to

$$\begin{aligned}
ds^2 =& -\left(1-\frac{2mr}{r^2+a^2\cos^2\theta}+\frac{Q^2}{r^2+a^2\cos^2\theta}\right)du^2 - 2dudr + 2a\sin^2\theta d\phi dr \\
&-\left(\frac{4mr}{r^2+a^2\cos^2\theta}-\frac{2Q^2}{r^2+a^2\cos^2\theta}\right)a\sin^2\theta dud\phi \\
&+\left(r^2+a^2\cos^2\theta\right)d\theta^2 + \left(\left(r^2+a^2\cos^2\theta\right)a^2\sin^2\theta + 2mra^2\sin^2\theta\right. \\
&\left.-Q^2a^2\sin^2\theta + \left(r^2+a^2\cos^2\theta\right)^2\right)\frac{\sin^2\theta}{(r^2+a^2\cos^2\theta)}d\phi^2,
\end{aligned} \tag{4.5}$$

which is the Kerr-Newman metric in the advanced Eddington-Finkelstein coordinate. One can check this is exactly (3.13). One thing worth noticing is the mysterious identification (4.4) corresponding to the operation that takes $\bar{m}^a$ conjugate to m^a after the complex coordinate transformation in the original NJ algorithm.

5 Apply to gauge field

An interesting discovery is that the NJ algorithm works not only for the metric but for the gauge field [15]. It means in the case of Einstein's equation coupled with a gauge field [5], the NJ algorithm will transform both the static metric and static gauge field into their rotating counterpart. In this section, we use electromagnetic fields as an example. We will show both Giampieri's approach and the tetrad approach.

5.1 Giampieri's approach

For Giampieri's approach, we start with 4-potential corresponding to RN black hole

$$A_a = \frac{Q}{r}(dt)_a. \tag{5.1}$$

In the $\{u, r\}$ coordinate, it becomes

$$A_a = \frac{Q}{r}((du)_a + f^{-1}(dr)_a). \tag{5.2}$$

As of now, $A_r = A_r(r)$, the second term doesn't contribute to the field strength; we are free to perform a gauge transformation to remove this term. Next, perform the transformation (4.2) after complexification (2.4), followed by substitution (4.4) and $\theta^* = \theta$, we obtained

$$A'_a = \frac{Qr}{\rho^2}\left((du)_a - a\sin^2\theta(d\phi)_a\right), \tag{5.3}$$

where the prime is omitted. Change to the Boyer-Lindquist coordinate using transformation

$$du' = du - \frac{r^2+a^2}{r^2+a^2+Q^2-2mr}dr, \quad d\phi = d\varphi - \frac{a}{r^2+a^2+Q^2-2mr}dr, \tag{5.4}$$

[5] Both abelian and non-abelian gauge field

one obtains a gauge potential in the following form

$$A'_a = \frac{qr}{\rho^2}\left((du)_a - \frac{\rho^2}{\Delta}(dr)_a - a\sin^2\theta(d\phi)_a\right). \tag{5.5}$$

As A'_r now is just a function of r, we are again free to perform a gauge transformation to remove this term. After the transformation (3.14), we end up with

$$A'_a = \frac{qr}{\rho^2}\left((dt)_a - a\sin^2\theta(d\varphi)_a\right), \tag{5.6}$$

which is the electromagnetic gauge field corresponding to the Kerr-Newman black hole.

5.2 Tetrad approach

For the tetrad approach, we start with

$$A_a = \frac{Q}{r}(du)_a, \tag{5.7}$$

which is (5.2) after gauge transformation removing the second term [6]. One can raise the index using the inverse metric with respect to (3.9), which results in

$$A^a = -\frac{q}{r}\partial_r^a = -\frac{q}{r}\ell^a, \tag{5.8}$$

where ℓ^a is the same tetrad component defined in (3.10). Next, perform steps 3 and steps 4 shown in section 2; the above vector becomes

$$A'^a = -\frac{qr}{\rho^2}\ell'^a, \tag{5.9}$$

where ℓ'^a is defined in (5.2) and $\ell'^a = \partial_r^a = \ell^a$. Lowering the index use (3.13) result in

$$A'_a = \frac{qr}{\rho^2}\left(du - a\sin^2\theta d\phi\right) \tag{5.10}$$

which equivalent to (5.5) up to a gauge transformation.

6 Various explanations

Although a geometric or symmetry-related interpretation is lacking, numerous efforts have been made to explain why it works since its discovery. Talbot [7]shows this algorithm work for a subclass of Kerr-Schild type metric [21] (also see [22]). A partial explanation is provided by Newman based on Newman-Penrose formalism [23]. Schiffer et al. show that the Schwarzschild metric and Kerr metric can be obtained using two generating functions respectively, and their generating functions are related by a complex coordinate transformation [24] (for RN metric to KN metric, see [25]). Flaherty found the Schwarzschild and

[6] NJ algorithm would not lead to the correct answer without performing a gauge transformation to remove the radial component. Without the gauge transformation, a null rotation after step 4 could also lead to the rotating gauge field [20].

[7] It was also shown in Newman's original paper that in private communication Roy Kerr showed this argument.

Kerr metrics are different real slices of a four-complex-dimensional Hermitian manifold [26]. Drake and Szekeres prove the uniqueness of obtaining the Kerr metric for the NJ algorithm under some conditions [27]. Recently, some attempts have been made to understand this algorithm from the scattering amplitude perspective [28–30]. This section will mainly introduce the work of Talbot, Newman, Drake and Szekeres. The interpretations in this section are all for Schwarzschild to Kerr process

6.1 Kerr-Talbot explanation

In this section, we are going to show Talbot's discovery that the NJ algorithm is *on-shell* for a subclass of Kerr-Schild metric[8].

For the Minkowski metric in Cartesian coordinates $\{t, x, y, z\}$, one can perform the transformation

$$\begin{aligned}\zeta &= \frac{1}{\sqrt{2}}(x+iy),\ \bar{\zeta} = \frac{1}{\sqrt{2}}(x-iy),\\ u &= \frac{1}{\sqrt{2}}(z+t),\quad v = \frac{1}{\sqrt{2}}(z-t),\end{aligned} \tag{6.1}$$

which turn the Minkowski metric into

$$ds_0^2 = 2dudv + 2d\zeta d\bar{\zeta}. \tag{6.2}$$

One property of the Kerr-Schild metric is that they admit at least one Killing vector [31]. If that Killing vector is timelike, the Kerr-Schild metric can be expressed as

$$ds^2 = 2dudv + 2d\zeta d\bar{\zeta} - 4\sqrt{2}m\,\mathrm{Re}\left(\frac{1}{F_{,Y}}\right)\left[\frac{du + Yd\bar{\zeta} + \bar{Y}d\zeta - Y\bar{Y}dv}{1+Y\bar{Y}}\right]^2, \tag{6.3}$$

where m is a real constant, Y is a complex variable, F is a function defined as

$$F(Y,\zeta,\bar{\zeta},u+v) = \phi(Y) + \left[Y^2\bar{\zeta} - \zeta + (u+v)Y\right], \tag{6.4}$$

and $\phi(Y)$ is holomorphic function. The Y is given implicitly by $F=0$ which is

$$\phi(Y) = -Y^2\bar{\zeta} + \zeta - (u+v)Y. \tag{6.5}$$

Both Schwarzschild and Kerr belong to the subclass (6.3), the Schwarzschild correspond to $\phi(Y)=0$ and Kerr correspond to $\phi(Y) = -i\sqrt{2}aY$. To transform the Schwarzschild metric to the Kerr metric, one performs the transformation

$$\hat{\phi}(Y) = \phi(Y) - \sqrt{2}iaY \tag{6.6}$$

where $\phi(Y)$ is the function of Schwarzschild and $\phi(\hat{Y})$ is function of Kerr. Plug this expression into the LHS of (6.5) and rearrange it, one find

$$\phi(Y) = -Y^2\bar{\zeta} + \zeta - [(u+\sqrt{2}ia)+v]Y. \tag{6.7}$$

[8]Talbot's original explanation is lengthy as they explain from the NP formalism perspective why the complex transformation leads to a solution still satisfies the Einstein's equation. Instead, we will show Flaherty's approach, which is much more brief [26].

From this, we see the Kerr metric could be obtained from Schwarzschild by a transformation

$$u \to u + \sqrt{2} ia. \tag{6.8}$$

This approach shows two Kerr-Schild metrics are linked by a complex transformation. There is one problem with this explanation. If one expresses the Schwarzschild metric in the $(u, v, \zeta, \bar{\zeta})$ coordinate and performs the transformation (6.8), the resulting metric is not Kerr metric. To obtain the Kerr metric, $\bar{Y}(u)$ needs to be transformed into $\bar{Y}(\bar{u})$, which is a non-holomorphic transformation.

6.2 Newman's explanation

Newman's explanation [23] (also see [32]) starts with the Kerr-Schild form of the inverse metric of Schwarzschild (C.11); it has the form of

$$g^{ab} = \eta^{ab} - Hk^a k^b, \tag{6.9}$$

where $k^a = \partial_r^a$ and $H = \frac{2m}{r}$. We can see the NJ algorithm can be separated into two parts, the first part is to apply the NJ algorithm to the Minkowski metric, which is already shown in section 3.1. The second part is to apply our empirical rule to the function H:

$$\frac{2m}{r} \to m\left(\frac{1}{r} + \frac{1}{\bar{r}}\right), \tag{6.10}$$

follow by applying the complex coordinate transformation,

$$m\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) \to \frac{2mr'}{r'^2 + a^2 \cos^2\theta'}. \tag{6.11}$$

After removing the prime, we end up with the Kerr metric [9]. What motivates Newman is the similarity between the Weyl scalar of the Schwarzschild and Kerr metrics. The five Weyl scalars of RN metric are

$$\Psi_0 = 0, \quad \Psi_1 = 0, \quad \Psi_2 = \frac{m}{r^3}, \quad \Psi_3 = 0, \quad \Psi_4 = 0, \tag{6.12}$$

while for the Kerr metric, the five Weyl scalars are

$$\Psi_0 = 0, \quad \Psi_1 = 0, \quad \Psi_2 = \frac{m}{(r - ia\cos\theta)^3}, \quad \Psi_3 = 0, \quad \Psi_4 = 0. \tag{6.13}$$

This suggests the replacement $r = r' - ia\cos\theta$.

From the Wyel scalars above, it is obvious both Schwarzschild and Kerr spacetime are type D. Actually, RN and KN spacetimes are also type D as the only non-vanishing Wyel scalar is $\Psi_2 = \frac{m}{r^3} - \frac{Q^2}{r^4}$ and $\Psi_2 = \frac{m}{(r-ia\cos\theta)^3} - \frac{Q^2}{(r+ia\cos\theta)(r-ia\cos\theta)^3}$ respectively.

[9] Analysis of NJ algorithm within the framework of Cartan calculus would lead to a similar explanation [33]

6.3 Drake-Szekeres' explanation

The empirical rule is one of the most confusing steps in the NJ algorithm. A natural question is can we still get the Kerr metric from Schwarzschild with a different rule? A series of important results related to this is proved by S.P. Drake and P. Szekeres [27]. They start with a static spherically symmetric seed metric in EF coordinate,

$$ds^2 = e^{2\Phi(r)}du^2 + e^{\Phi(r)+\lambda(r)}dudr - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6.14}$$

Then use the same complex coordinate transformation,

$$\begin{aligned} u' &= u - ia\cos\theta, \\ r' &= r + ia\cos\theta, \\ \theta' &= \theta, \\ \phi' &= \phi, \end{aligned} \tag{6.15}$$

but leave the transformation rule arbitrary, in which the null tetrad after transformation is

$$\begin{aligned} \tilde{l}^\mu &= \partial^\mu_{r'} \\ \tilde{n}^\mu &= e^{-\lambda(r',\bar{r'})-\Phi(r',\bar{r'})}\partial^\mu_{u'} - \frac{1}{2}e^{-2\lambda(r',\bar{r'})}\partial^\mu_{r'} \\ \tilde{m}^\mu &= \frac{1}{\sqrt{2}}\bar{r'}\left(\partial^\mu_{\theta'} + \frac{i}{\sin\theta^{prime}}\partial^\mu_{\phi'}\right). \end{aligned} \tag{6.16}$$

Based on the above setting, they prove the following theorems.

Theorem 6.1. *The only perfect fluid generated by the Newman-Janis trick is the vacuum. [27].*

Theorem 6.2. *The only algebraically special spacetimes generated by the Newman-Janis algorithm is Petrov type D [27].*

Theorem 6.3. *The only Petrov type D spacetime generated by the Newman-Janis algorithm with a vanishing Ricci scalar is the Kerr-Newman spacetime [27].*

The Kerr spacetime is also included for the last theorem, as it is a special type of KN spacetime with zero charges.

7 Extension of NJ algorithm

Two pieces of evidence suggest that the NJ algorithm reveals an underlying structure. First, NJ has been successfully applied in many other scenarios besides the KN metric. Recently, it was found that it could generate rotating interior solutions [7], Sen's axion–dilaton rotating black hole [8], rotating BTZ black hole [9, 10] and Myers-Perry solution with one angular momentum parameter [11, 12]. It was also found by Newman that complex Poincaré transformation can be used to generate solutions for the Maxwell equation [34]. Moreover, Newman studied the relationship between complex Minkowski spacetime and the NJ algorithm, which led to a so-called "Heavenly" spacetime theory [35]. One might want to extend the NJ algorithm to various modified gravity theories. However, it was found that this algorithm is invalid for Brans-Dicke theory [5] and leads to pathologies in spacetime in quadratic gravity theory [6], which suggest one should carefully check the validity when applying NJ algorithm outside of GR. Moreover, Erbin proposed the five-dimensional NJ algorithm and derived the full Myers-Perry solution [13]. Second, the NJ algorithm has been successfully extended to include additional parameters beyond angular momentum. In this section, we are going to show how other parameters use complex coordinate transformation. We adapt Giampieri's approach in this section for simplification[10].

7.1 Extension to add NUT charge

We are going to show how a modified NJ algorithm can add NUT charge to the Schwarzschild metric proposed by Demiański [11][36]. We start with Schwarzschild metric in advanced EF coordinate

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{7.1}$$

To add the NUT charge, one needs to perform a complex shift to the parameters. To achieve it, the complexify rule for r^{-1} term is modified first,

$$\frac{2m\,\mathrm{Re}(r)}{|r|^2} \to \frac{2\,\mathrm{Re}(m\bar{r})}{|r|^2}. \tag{7.2}$$

Then perform the following complex coordinate transformation

$$u = u' - 2in\ln\sin\theta^*, \quad r = r' + in, \quad m = m' + in, \tag{7.3}$$

which gives

$$\begin{aligned} ds^2 = &\left(\frac{r'^2 - 2mr - n^2}{r'^2 + n^2}\right)\left(-du'^2 + 4in\cot\theta^* du' d\theta^* + 4n^2\cot^2\theta^* d\theta^{*^2}\right) \\ &- 2\left(du'dr' - 2in\cot\theta^* d\theta^* dr'\right) + \left(r'^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{7.4}$$

The fifth dimension θ^* needs to be remove by identification (4.4) followed by substituting $\theta^* = \theta$, after the coordinate transformation

$$du = dt' - \frac{r'^2 + n^2}{r'^2 - n^2 - 2mr} dr' \quad d\varphi = d\phi \tag{7.5}$$

and remove the prime, a black hole solution which has three parameters m, n, Q is obtained:

$$\begin{aligned} ds^2 = &-\left(\frac{r^2 - 2mr - n^2}{r^2 + n^2}\right) dt^2 + \left(\frac{r^2 + n^2}{r^2 - 2mr - n^2}\right) dr^2 + \left(r^2 + n^2\right) d\theta^2 \\ &+ \frac{1}{r^2 + n^2}\left(\left(r^2 + n^2\right)\sin^2\theta - 4n^2(\cos\theta)^2\left(r^2 - 2mr - n^2\right)\right) d\varphi^2 \\ &- \frac{1}{r^2 + n^2}\left(2n\cos\theta\left(r^2 - 2mr - n^2\right)\right) dt d\varphi \end{aligned} \tag{7.6}$$

[10] All examples given here can also be equivalently derived using tetrad formalism.

[11] The NJ algorithm involves a complex shift to the parameter is also referred to as Demiański-Newman-Janis (DNJ) algorithm

This is actually the Taub-NUT solution in Boyer-Lindquist coordinates [37, 38]. The Schwarzschild mass m here represents an electric mass while the NUT charge n represents a magnetic mass [12]. It is also possible to add angular momentum for the metric (7.6) [13], one just needs to change (7.3) into

$$u = u' + i(a\cos\theta^* - 2n\ln\sin\theta^*), \quad r = r' + i(n - a\cos\theta^*), \quad m = m' + in, \tag{7.7}$$

and keep other procedures unchanged, the resulting metric is the KN-Taub-NUT metric [40]. Recently, it was found that the complex shift for the parameter can be interpreted as a duality rotation while the complex transformation of u and r are complex BMS supertranslations [29].

7.2 Extension to non-zero cosmological constant

Usually, the original NJ algorithm is usually considered not to work for a spacetime with a non-vanishing cosmological constant. However, Erbin shows that the DNJ algorithm can be extended to non-zero cosmological constant Λ cases to add the NUT charge [16]. The seed metric they use is

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Omega^2 \tag{7.8}$$

where

$$f(r) = \kappa - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \tag{7.9}$$

and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The κ here is the sign of the surface curvature of $d\Omega^2$. For sphere S^2 we have $\kappa = 1$ while for hyperboloid H^2, we have $\kappa = -1$. The line element of H^2 is $d\Omega^2 = d\theta^2 + \sinh^2\theta d\phi^2$. The S^2 corresponds to the Schwarzschild solution and H^2, corresponds to a topological black hole. To apply the NJ algorithm, instead of taking $\kappa = 1$ in the beginning, we take it as an undetermined parameter throughout the algorithm. We start with the metric in EF form (2.2). Then the complexification (2.4) gives

$$\tilde{f} = \kappa - \frac{2\,\mathrm{Re}(m\bar{r})}{|r|^2} - \frac{\Lambda}{3}|r|^2. \tag{7.10}$$

To add the NUT charge, first, perform the same complex coordinate transformation for u and r

$$u = u' - 2in\ln\sin\theta^*, \quad r = r' + in. \tag{7.11}$$

Then one needs to shift the parameter as

$$m = m' + i\kappa n, \quad \kappa = \kappa' - \frac{4\Lambda}{3}n^2, \tag{7.12}$$

and therefore the mass is transformed as

$$m = m' + in\left(\kappa' - \frac{4\Lambda}{3}n^2\right). \tag{7.13}$$

[12] The concept of electric mass and magnetic mass can be understood by analogy with electric charge and magnetic charge [39]; therefore the Taub-NUT solution is known as a gravitational analogue of an electromagnetic dyon.

It is worth to notice even though the parameter is shifted, the κ and κ' should take the same value. Plug these transformations into f, take $\kappa' = 1$ and remove the prime on the coordinate, we obtain

$$\tilde{f} = 1 - \frac{2mr + 2n^2}{r^2 + n^2} - \frac{\Lambda}{3}\left(r^2 + 5n^2\right) + \frac{8\Lambda}{3}\frac{n^4}{r^2 + n^2}. \tag{7.14}$$

Then the resulting metric in EF form is obtained after the identification (4.4) and $\theta^* = \theta$. The corresponding Boyer-Lindquist coordinates transformation is

$$du = dt - \frac{r^2 + n^2}{\Delta}dr, \quad d\varphi = d\phi, \tag{7.15}$$

$$\Delta = r^2 - 2mr + \Lambda n^4 - \frac{\Lambda}{3}r^4 - n^2\left(1 + 2\Lambda r^2\right). \tag{7.16}$$

Finally, the resulting metric is

$$\begin{aligned} ds^2 &= -\tilde{f}dt^2 + \left(\frac{r^2 + n^2}{\Delta}\right)dr^2 + 4n\tilde{f}\cos\theta dt d\varphi \\ &+ \left(r^2 + n^2\right)d\theta^2 + \left(\left(r^2 + n^2\right)\sin^2\theta - 4n^2\tilde{f}\cos^2\theta\right)d\varphi^2, \end{aligned} \tag{7.17}$$

which is Taub-NUT-AdS in BL coordinates [41]. In the zero n limit, it reduces to the Schwarzschild-AdS metric. Moreover, Erbin found that angular momentum[13] can't be added when processing a non-zero cosmological constant [16].

7.3 More extensions

The complex coordinate transformation (7.3) is actually found by Demiański by solving the Einstein equation [36]. Demiański's idea is reformulated and generalized by Erbin [16]. In this section, we are going to show how to derive the general complex transformation from the equation of motion.

In the Maxwell-Einstein theory with a non-zero cosmological constant Λ, the Einstein equation becomes

$$G_{ab} + \Lambda g_{ab} = 2T_{ab}, \tag{7.19}$$

in which we have set the Einstein coupling constant to be one following Erbin's convention. The energy-momentum tensor is given by

$$T_{ab} = F_{ac}F^c_b - \frac{1}{4}g_{ab}F^{de}F_{de}, \tag{7.20}$$

where the field strength $F_{ab} = \nabla_{[a}A_{b]} = \partial_{[a}A_{b]}$ and A_b is the 4-potential of electromagnetic field. The field strength of an electromagnetic field also needs to satisfy the vacuum Maxwell equation

$$\nabla_a F^{ab} = 0. \tag{7.21}$$

[13] It was claimed that Kerr-AdS is obtained with a generalized complexify rule [42],

$$r^p \longrightarrow \frac{(\mathrm{Re}(r))^{p+2}}{|r|^2}, \tag{7.18}$$

however, the solution they obtained is not the well-known metric obtained by Cater [43].

For a general static seed configuration

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{7.22}$$

$$A = f_A dt, \tag{7.23}$$

and

$$d\Omega^2 = d\theta^2 + H(\theta)^2 d\phi^2, \quad H(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{7.24}$$

One can determine the function f and f_A by plugging (7.23) and (7.23) into (7.19) and (7.22) respectively, and then solving the equation. The solution of the equation of motion is

$$f(r) = \kappa - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3}r^2, \tag{7.25}$$

$$f_A(r) = \alpha + \frac{q}{r}. \tag{7.26}$$

The seed configuration in null coordinate reads

$$ds^2 = -f(r)du^2 - 2dudr + r^2 d\Omega^2, \tag{7.27}$$

$$A = f_A du. \tag{7.28}$$

The complex transformation is given by

$$r = r' + iF(\theta), \quad u = u' + iG(\theta). \tag{7.29}$$

Giampieri's approach converts (7.27) and (7.28) into (the prime on the coordinate is omitted)

$$ds^2 = -\tilde{f}(du + \tilde{f}^{-1}dr + \omega H d\phi)^2 + 2\tilde{f}^{-1}F'Hdrd\phi + \rho^2\left(d\theta^2 + \sigma^2 H^2 d\phi^2\right), \tag{7.30}$$

$$A = \tilde{f}_A\left(du + G'Hd\phi\right), \tag{7.31}$$

where $\tilde{f}$ is (7.25) after complexification and complex transformation, $\rho^2 = r^2 + F^2$, $\omega = G' + \tilde{f}^{-1}F'$, $\sigma^2 = 1 + \frac{F'^2}{\tilde{f}\rho^2}$, $\sigma^2 = 1 + \frac{F'^2}{\tilde{f}\rho^2}$, and G', F' are derivative of G, F with respect to θ. One can again put (7.30) and (7.31) into our equation of motion, solving the equation of motion gives

$$G'' + \frac{H'}{H}G' = 2F \tag{7.32}$$

$$\Lambda F' = 0 \tag{7.33}$$

$$\tilde{f} = \kappa - \frac{2mr - q^2 + 2F(\kappa F + K)}{r^2 + F^2} - \frac{\Lambda}{3}\left(r^2 + F^2\right) - \frac{4\Lambda}{3}F^2 + \frac{8\Lambda}{3}\frac{F^4}{r^2 + F^2}, \tag{7.34}$$

$$\tilde{f}_A = \frac{qr}{r^2 + F^2} + \alpha\frac{r^2 - F^2}{r^2 + F^2}, \tag{7.35}$$

where $\alpha = 0$ if $F' \neq 0$ and

$$2K = F'' + \frac{H'}{H}F'. \tag{7.36}$$

We only consider the $\alpha = 0$ case in the rest of this section. The original f is only a function of r; therefore all θ dependent terms should come from F, and F and r should be of the same order [14]. Therefore we have one extra constrain from the numerator of the second term of $\tilde{f}$, which is

$$\kappa F' + K' = 0 \quad \Longrightarrow \quad \kappa F + K = \kappa n. \tag{7.37}$$

Then we discuss the $\Lambda = 1$ case, solving the (7.37) gives

$$F(\theta) = n - aH'(\theta) + c\left(1 + H'(\theta)\ln\frac{H(\theta/2)}{H'(\theta/2)}\right) \tag{7.38}$$

where a and c are integration constants. With the help of this formula, the (7.32) can also be solved, which gives

$$\begin{aligned} G(\theta) = c_1 + \kappa a H'(\theta) - \kappa c H'(\theta)\ln\frac{H(\theta/2)}{H'(\theta/2)} - 2\kappa n \ln H(\theta) \\ + (a + c_2)\ln\frac{H(\theta/2)}{H'(\theta/2)} \end{aligned} \tag{7.39}$$

where c_1 and c_2 are also integration constants. Here we set $c_1 = 0$ and remove the last term by transformation $du = du' - (c_2 + a)\, d\phi$. Take $\kappa = 1, H(\theta) = \sin\theta$, we get

$$F(\theta) = n - a\cos\theta + c\left(1 + \cos\theta \ln\tan\frac{\theta}{2}\right), \tag{7.40}$$

$$G(\theta) = a\sin\theta - c\sin\theta\ln\tan\frac{\theta}{2} - 2n\ln\sin\theta. \tag{7.41}$$

We could see the term contains n and a coincident with the NUT charge term and angular term in our complex coordinate transformation (7.7) [15]. The parameter c is called the Demianski parameter, which appears in Demianski's original paper and doesn't receive any physical interpretation. To go to the BL coordinate, this parameter needs to be discarded as the BL coordinate is not well-defined for $c \neq 0$.

In summary, one can leave the complex transformation arbitrary and then solve the equation for seed and resulting metric to determine its specific form. Following the same idea, the general form of complex coordinate transformation can be found for non-zero cosmological constant cases. Moreover, it was found by Erbin that the magnetic charge can be added by a complex shift of electric charge [14]. Therefore, the parameter that could be added using the DNJ algorithm included the angular momentum, NUT charge and magnetic charge. By complex shifting the parameter, the Yang–Mills Kerr–Newman black hole together with its gauge counterpart can also be derived [13]. Furthermore, the DNJ algorithm is generalized by Erbin to the complex scalar field and supergravity theory [14]. Therefore, it is possible to extend the DNJ algorithm to make it work for all bosonic fields (spin ≤ 2).

---

[14] One might doubt that the numerator shouldn't contain θ according to (2.4), however, the complexity rule used here is (7.2).

[15] The sign before them is different as we adapt different metric signatures in this section

7.4 Adding angular momentum to de Sitter

It was mentioned in [[13]] that it's impossible to add angular momentum to the nonzero cosmological constant case, where they take this as a no-go theorem. However, it's worth noticing that when applying the algorithm to the Schwarzschild black hole, the only r^2 term only appears in the angular coordinate part. In terms of the null tetrad, the complexification of this term arises from r and $\bar{r}$ in the denominator of m^a and $\bar{m}^a$ in (3.10), which is different from the complexification of r^{-1} and r^{-2}. When considering the Kerr-Schild form of the Schwarzschild solution,

$$g^{ab} = \eta^{ab} - Hk^a k^b, \tag{7.42}$$

The r^2 term comes from the flat spacetime part. The NJ algorithm for the flat spacetime is just equivalent to a coordinate transformation. This indicates that when considering the nonzero cosmological constant, we should treat the de Sitter background and deviation separately.

A Null tetrad

A.1 Tetrad

A tetrad [16] is a set of four linearly independent vector fields that are defined locally. We donated it by $\{(e_\mu)^a\}$ where $\mu = 0$, 1, 2 or 3 is the component index which represents which component of tetrad it is. We can write the tetrad and its dual tetrad in terms of coordinate basis vector:

$$e^a_\mu = \alpha^i_\mu (\frac{\partial}{\partial x^i})^a, \quad e^\mu_a = \beta^\mu_i (dx^i)_a \tag{A.1}$$

The tetrad and its dual tetrad satisfy

$$(e_\mu)^a (e^\nu)_a = \delta^\nu_\mu, \quad (e_\mu)^a (e^\mu)_b = \delta^a_b \tag{A.2}$$

For a given metric g^{ab}, we could rewrite it using tetrad

$$g^{ab} = e^a_\mu e^b_\nu g^{\mu\nu} \tag{A.3}$$

or equivalently

$$g^{\mu\nu} = e^\mu_a e^\nu_b g^{ab}, \tag{A.4}$$

where $g^{\mu\nu}$ is the frame metric and e^a_μ is the tetrad. For the tetrad and its dual tetrad, we have $(e_\mu)^a (e^\nu)_a = \delta^\nu_\mu$. If the tetrad corresponds to an orthonormal frame, the frame metric is the Minkowski metric, which is a reflection of the equivalence principle. Moreover, define

$$(a) \quad \left(e_\mu\right)_a \equiv g_{ab} \left(e_\mu\right)^b, \quad (b) \quad \left(e^\mu\right)^a \equiv g^{ab} \left(e^\mu\right)_b, \tag{A.5}$$

then we have the following relation

$$(a) \quad \left(e^\mu\right)_a = g^{\mu\nu} \left(e_v\right)_a, \quad (b) \quad \left(e_\mu\right)^a = g_{\mu\nu} \left(e^v\right)^a. \tag{A.6}$$

[16] If not specified, the tetrad considered in this thesis are all rigid tetrad. The rigid tetrad is the tetrad in which the component of frame metric $g_{\mu\nu}$ is constant ($\nabla_a g_{\mu\nu} = 0$).

The proof of these two relations is easy, one only needs to act $(e_\sigma)^a$ on both sides of (a) and act $(e^\sigma)_a$ on both sides of (b), the two sides of the (a) and (b) are equal. We could also act g^{ab} and g_{ab} on (a) and (b) of (A.6) respectively to raise and lower the abstract index

$$(a) \quad (e^\mu)^a = g^{\mu v} (e_v)^a \,, \quad (b) \quad (e_\mu)_a = g_{\mu v} (e^v)_a \,. \tag{A.7}$$

This means the component index μ and ν could be raised and lowered using $g^{\mu\nu}$ and $g_{\mu\nu}$.

A.2 Null tetrad

The null tetrad $\{n^a, l^a, m^a, \bar{m}^a\}$ used for NP formalism consists of a pair of real null component $\{n^a, l^a\}$ and a pair of complex null component $\{m^a, \bar{m}^a\}$. These components need to satisfy the following relation

$$l^a n_a = -, \quad n^a l_a = -1, \quad m^a \bar{m}_a = 1, \quad \bar{m}^a m_a = 1 \tag{A.8}$$

$$l^a l_a = 0, \quad n^a n_a = 0, \quad m^a m_a = 0, \quad \bar{m}^a \bar{m}_a = 0. \tag{A.9}$$

In terms of an orthonormal basis $\{e_0^a, e_1^a, e_2^a, e_3^a\}$ at point p of a four-dimensional spacetime (M, g_{ab}), we could construct the following null tetrad at the same point:

$$l^a = \frac{1}{\sqrt{2}} (e_0^a + e_1^a) \tag{A.10}$$

$$n^a = \frac{1}{\sqrt{2}} (e_0^a - e_1^a) \tag{A.11}$$

$$m^a = \frac{1}{\sqrt{2}} (e_2^a + i e_3^a) \tag{A.12}$$

$$\bar{m}^a = \frac{1}{\sqrt{2}} (e_2^a - i e_3^a) \,. \tag{A.13}$$

The frame metric $g^{\mu\nu}$ with respect to this null tetrad now is

$$g^{ij} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \tag{A.14}$$

The metric g^{ab} can be expressed as

$$g^{ab} = -\ell^a n^b - n^a \ell^b + m^a \bar{m}^b + \bar{m}^a m^b. \tag{A.15}$$

B Petrov classification

In general relativity, the Einstein field equation just determines how matter distribution determines the Ricci tensor. The Ricci tensor does not totally fix the Riemann tensor; for example, Schwarzschild spacetime has vanished Ricci tensor but is not a flat spacetime. The curvature of spacetime that is not caused by the matter field in that region is characterised by the Weyl tensor — the traceless part of the Riemann tensor.

The Weyl tensor has 10 independent components, one can express them into five complex numbers, which are called *Weyl scalars*

$$
\begin{aligned}
\Psi_0 &= C_{1313} = C_{abcd} l^a m^b l^c m^d, \\
\Psi_1 &= C_{1213} = C_{abcd} l^a n^b l^c m^d, \\
\Psi_2 &= C_{1342} = C_{abcd} l^a m^b \bar{m}^c n^d, \\
\Psi_3 &= C_{1242} = C_{abcd} l^a n^b \bar{m}^c n^d, \\
\Psi_4 &= C_{2424} = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d.
\end{aligned}
\tag{B.1}
$$

There are physical interpretations for five Weyl scalars. Ψ_1 and Ψ_3 characterise the effect of ingoing and outgoing longitudinal waves[44]. Ψ_0 and Ψ_4 characterise the effect of ingoing and outgoing transverse waves [44]. Ψ characterizes the influence of the mass source [44]. For a more formal aspect of the Weyl scalar, see [45]. Based on five Weyl scalars, a criterion to classify spacetime is proposed [46, 47] which is now called the Petrov classification. Based on which of the Weyl scalars vanish, this classification divided spacetime into six types:

$$
\begin{aligned}
&\text{Type I} \to \Psi_0 = 0, \\
&\text{Type II} \to \Psi_0 = \Psi_1 = 0, \\
&\text{Type D} \to \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \\
&\text{Type III} \to \Psi_0 = \Psi_1 = \Psi_2 = 0, \\
&\text{Type N} \to \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \\
&\text{Type O} \to \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0.
\end{aligned}
\tag{B.2}
$$

As we have physical interpretations of five Weyl scalars, the physical meaning of different types of spacetime can be directly obtained. Type D represent the exterior field of an isolated mass source. Type O represent the conformally flat region. Type I represent near-field radiation zone[17]. Type N represents the pure radiation zone. Type II and Type III represent the transition zone between Type I and Type N [48]. Different regions in spacetime can be different types. The Schwarzschild and Kerr spacetime are everywhere in type D. FLRW metric is conformally flat and therefore is type O. The type I spacetime is called algebraically general while other types are called algebraically special.

C Kerr-Schild metric

In 1965, Kerr and Schild discovered a class of algebraically special solutions, which is now called the Kerr-Schild metric. The metric is given by

$$g_{ab} = \eta_{ab} + S k_a k_b, \tag{C.1}$$

where η_{ab} is the Minkowski metric, S is a scalar function and k_a is a null dual vector satisfying

$$g_{ab} k^a k^b = \eta_{ab} k^a k^b = 0. \tag{C.2}$$

[17]The gravitational radiation is nonzero and the influence of mass source can not be neglected

The inverse metric is in the form of

$$g^{ab} = \eta^{ab} - S k^a k^b, \tag{C.3}$$

we could easily prove it using $g^{ab}g_{ab} = 4$ and (C.2) . One property of the Kerr-Schild metric is its determinant $g = -1$ Moreover, if the energy-momentum tensor satisfy $T^{ab}k_a k_b = 0$, then we have

$$k^a \nabla_a k_b = k^a \partial_a k_b = 0, \tag{C.4}$$

and corresponding Kerr-Schild metric could only be type II or type D[18].

The Kerr-Schild can also be extended to non-flat backgrounds, with the flat metric replaced by a non-flat metric $\bar{g}_{ab}$ [49, 50] :

$$g_{ab} = \bar{g}_{ab} + S k_a k_b, \tag{C.5}$$

and k_a is still a null vector with respect to both metrics. Many well-known metrics can be written in the Kerr-Schild and generalized Kerr-Schild form; here, we use the Schwarzschild and Taub-NUT solution as examples. The proof of some of the properties is shown in appendix **??**. For a detailed review, see [45].

C.1 Example I: Schwarzschild metric

The Schwarzschild black hole solution in Schwarzschild coordinate $\{t, r, \theta, \phi\}$ is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{C.6}$$

where M is the geometric mass. This is an asymptotically flat, static and spherically symmetric spacetime derived from the vacuum Einstein's equation. Next, perform the coordinate transformation

$$u = t - r - 2m \ln\left(\frac{r}{2M} - 1\right), \tag{C.7}$$

we obtain the Schwarzschild solution in retarded Eddington-Finkelstein coordinates:

$$ds^2 = -du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{2M}{r}du^2. \tag{C.8}$$

The first four-term is actually the Minkowski metric,

$$\begin{gathered} ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \\ \downarrow u = t - r \\ ds^2 = -du^2 - 2dudr + r^2 d\Omega^2. \end{gathered} \tag{C.9}$$

Therefore (C.8) is already in the Kerr-Schild form, where $k_a = (1, 0, 0, 0)$ [19]. We can check the null property of k_a, the inverse metric of g_{ab} is

$$\partial_s^2 = -\partial_u\partial_r + \partial_r^2 + \frac{1}{r^2}(\partial_\theta^2 + \frac{1}{\sin\theta}\partial_\phi^2) - \frac{2M}{r}\partial_r^2. \tag{C.11}$$

[18] We only consider Kerr-schild metric that satisfies this condition in the subsequent chapter.

[19] In some paper [51, 52], a additional transformation $u = \bar{t} - r$ is used, which turns the metric (C.8) into

$$ds^2 = -d\bar{t}^2 - dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{2M}{r}(d\bar{t}^2 - 2d\bar{t}dr + dr^2), \tag{C.10}$$

The null condition $g^{ab}k_a k_b = \eta^{ab}k_a k_b = 0$ is satisfied. We also have $k^a\nabla_a k_b = k^a\partial_a k_b = 0$, because u is an affine parameter of the radial null geodesic. Therefore, the Schwarzschild metric is type II or type D; in the later section, we will see Schwarzschild is type D spacetime.

C.2 Example II: Taub-NUT metric

The Taub-NUT solution is a stationary axially symmetric solution of the Einstein equation. It carries two parameters: the mass M and the NUT charge N. The line element of this metric in polar coordinates (t, r, θ, ϕ) is given by

$$ds^2 = -f(r)(dt - 2N\cos\theta d\varphi)^2 + f^{-1}(r)dr^2 + \left(r^2 + N^2\right)d\Omega_2^2 \tag{C.12}$$

where $f(r) = \frac{(r-r_+)(r-r_-)}{r^2+N^2}$, $d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2$, $r_\pm = M \pm r_0$ and $r_0^2 = M^2 + N^2$. This metric reduces to the Schwarzschild metric with a vanished NUT charge. Different from other well-known solutions, the Taub-NUT solution is not asymptotically flat as $g_{t\varphi} \neq 0$ at infinity. It has wire singularities at $\theta = 0$ and $\theta = \pi$. For other properties of this metric, see [53]. This metric processes a so-called double Kerr-Schild structure in (2,2) signature [54], which means it could be written as

$$g_{ab} = \bar{g}_{ab} + \kappa\left(\phi k_a k_b + \psi l_a l_b\right) \tag{C.13}$$

where k_a and l_a satisfy

$$g^{ab}k_a k_b = 0, \quad \bar{g}_{ab}l_a l_b = 0, \quad k^a\nabla_a k_b = l^a\nabla_a l_b = 0 \tag{C.14}$$

and $\bar{g}_{ab}$ is the background metric. To show it explicitly, start from the Taub-NUT-de Sitter solutions in Plebanski coordinate [55]:

$$ds^2 = \frac{p^2+q^2}{X}dp^2 + \frac{p^2+q^2}{Y}dq^2 + \frac{X}{p^2+q^2}\left(d\tau + q^2 d\sigma\right)^2 - \frac{Y}{p^2+q^2}\left(d\tau - p^2 d\sigma\right)^2 \tag{C.15}$$

where

$$X = \gamma - \epsilon p^2 - \lambda p^4 + 2Np, \quad Y = \gamma + \epsilon q^2 - \lambda q^4 - 2Mq \tag{C.16}$$

The physical parameter included M and N, which is mass and NUT charge, respectively. The cosmological constant is represented by λ. The nonphysical parameter ϵ can take -1, 0 or 1. To write it in Kerr-Schild form, one needs to do the following analytic continuation to change the signature,

$$\begin{aligned} &p \longrightarrow ip, \quad N \longrightarrow i^{-1}N, \quad \gamma \longrightarrow -\gamma, \\ &X \longrightarrow -\Delta_p, \quad Y \longrightarrow \Delta_q, \end{aligned} \tag{C.17}$$

where

$$\Delta_p = \gamma - \epsilon p^2 + \lambda p^4 - 2Np, \quad \Delta_q = -\gamma + \epsilon q^2 - \lambda q^4 - 2Mq \tag{C.18}$$

The resulting metric is

$$\begin{aligned} ds^2 =& \frac{q^2-p^2}{\Delta_n}dp^2 + \frac{q^2-p^2}{\Delta_\sigma}dq^2 - \frac{\Delta_p}{q^2-p^2}\left(d\tau + q^2 d\sigma\right)^2 \\ &- \frac{\Delta_q}{q^2-p^2}\left(d\tau + p^2 d\sigma\right)^2 + \frac{p^2 q^2}{\gamma}d\Omega_k^2. \end{aligned} \tag{C.19}$$

Then, defining two new coordinate $\tilde{\tau}$ and $\tilde{\sigma}$ by

$$d\tilde{\tau} = d\tau + \frac{p^2 dp}{\Delta_p} - \frac{q^2 dq}{\Delta_q}, \quad d\tilde{\sigma} = d\sigma - \frac{dp}{\Delta_p} + \frac{dq}{\Delta_q}. \tag{C.20}$$

then the resulting metric admits a double Kerr-Schild form,

$$ds^2 = d\bar{s}^2 - \frac{2Mq}{q^2 - p^2}\left[d\tilde{\tau} + p^2 d\tilde{\sigma}\right]^2 - \frac{2Np}{q^2 - p^2}\left[d\tilde{\tau} + q^2 d\tilde{\sigma}\right]^2, \tag{C.21}$$

where

$$\begin{aligned} d\bar{s}^2 =& \frac{\bar{\Delta}_p}{q^2 - p^2}\left[d\tilde{\tau} + q^2 d\tilde{\sigma}\right]^2 + \frac{\bar{\Delta}_q}{q^2 - p^2}\left[d\tilde{\tau} + p^2 d\tilde{\sigma}\right]^2 \\ &+ 2\left[d\tilde{\tau} + q^2 d\tilde{\sigma}\right] dp + 2\left[d\tilde{\tau} + p^2 d\tilde{\sigma}\right] dq \end{aligned} \tag{C.22}$$

where $\bar{\Delta}_p = \gamma - \epsilon p^2 + \lambda p^4$ and $\bar{\Delta}_q = -\gamma + \epsilon q^2 - \lambda q^4$. The $d\bar{s}^2$ is actually the 4-dimensional de Sitter metric [20].Setting the cosmological constant λ to zero (C.21) reduces to a double Kerr-Schild metric with a Minkowski metric background.

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