Lecture 4

In this lecture we will consider the torus partition function of two-dimensional CFTs. We will show that modular invariance implies a universal asymptotic density of states known as Cardy's formula. We will also consider CFTs with large central charges and show that, provided the spectrum of light states is sparse in a way we shall make precise, the partition function is universal. We will use the universality of partition function to extend the regime of validity of Cardy's formula.

4 Partition function and modular invariance

Let us begin by considering a CFT on a torus parametrized by complex coordinates $(z, \bar{z})$ satisfying the identification

$$z \to z + R, \qquad z \to z + \tau, \qquad \tau \equiv \frac{\theta + i\beta}{2\pi}. \tag{4.1}$$

We now set $R = 1$ for convenience. Using $z = x + it$, we see that the first identification is the *spatial circle* of the cylinder (3.20). The second identification corresponds to $(t, x) \sim (t + \beta/2\pi, \theta/2\pi)$ and is referred to as the *thermal circle*. The latter instructs us to glue the cylinder up to a rotation of the disk at $t = \beta/2\pi$ by a shift in x by $\theta/2\pi$, as illustrated in figure 3. The variables (β, θ) are associated with translations along the t and x coordinates. They are

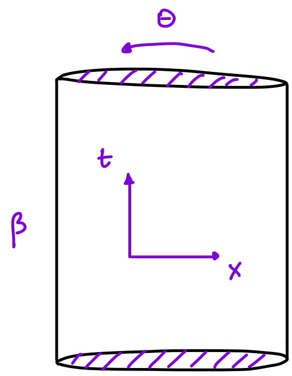

Figure 3: The torus satisfying $z \sim z + R$ and $z \sim z + \tau$ is obtained by gluing the disk at $t = 0$ with the disk at $t = i\beta/2\pi$. Note that the $t = i\beta/2\pi$ disk is rotated by $\theta/2\pi$ before the gluing.

naturally identified with the potentials conjugate to the generators of these symmetries, namely, with the temperature and angular potential. Consequently, the partition function takes the form

$$Z(\theta, \beta) = \mathrm{Tr}\big(e^{-\beta H + i\theta P}\big) = \sum_{E_n, P_n} c(E_n, P_n)\big(e^{-\beta E_n + i\theta P_n}\big), \tag{4.2}$$

where the trace is taken over all the states in the Hilbert space and $c(E_n, P_n)$ is the density of states with energy E_n and angular momentum P_n. It is convenient to write the partition function directly in terms of τ and the conformal weights introduced in (3.35) such that

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\left(q^{h_n - c/24}\,\bar{q}^{\bar{h}_n - c/24}\right), \qquad q = e^{2\pi i \tau}. \tag{4.3}$$

Conformal symmetry implies that the partition function of any 2d CFT must be invariant under large diffeomorphisms of the torus known as modular transformations, namely

$$Z(\gamma\tau,\gamma\bar{\tau}) = Z(\tau,\bar{\tau}), \tag{4.4}$$

where $\gamma\tau$ is given by

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}). \tag{4.5}$$

Any modular transformation can be obtained from combinations of the following two elements of PSL(2,$\mathbb{Z}$),

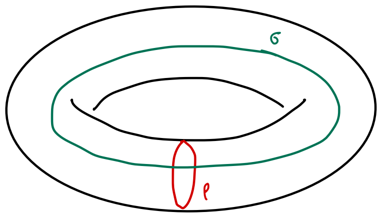

Figure 4: The ρ and σ cycles of the torus.

(*i*) $\mathcal{T}: \tau \to \tau + 1$. This transformation corresponds to a so-called Dehn twist of the torus along the spatial coordinate x. Let ρ and σ denote the two cycles of the torus shown in figure 4. A Dehn twist consist of cutting open the torus along ρ, rotating one end of the cylinder by 2π, and gluing the cylinder back together. Modular invariance under $\mathcal{T}$ transformations implies that the angular momentum is quantized, namely

$$P_n = h_n - \bar{h}_n \in \mathbb{Z}. \tag{4.6}$$

(*ii*) $\mathcal{S}: \tau \to -1/\tau$. This transformation correspond to flipping the two cycles of the torus such that $\rho \leftrightarrow \sigma$. In other words, this exchanges the spatial and thermal circles of the torus. Modular invariance under $\mathcal{S}$ transformations implies that the first excited state in any 2d CFT is bounded from above by

$$h_1 + \bar{h}_1 \lesssim \frac{c}{9.08}, \qquad c \gg 1. \tag{4.7}$$

Modular invariance under $\mathcal{S}$ transformations also implies a universal density of high energy states as we now describe.

Exercise 4.1: Verify that any combination of $\mathcal{T}$ and $\mathcal{S}$ transformations is an element of $PSL(2,\mathbb{Z})$. The proof of the converse can be obtained by following the steps in exercise 10.2 of Di Francesco et. al. Conformal Field Theory book.

4.1 Universality of the partition function at large c

The invariance of the partition function under modular $\mathcal{S}$ transformations implies that the high energy spectrum of any CFT is related to the low energy spectrum in a universal way. In order to see this, we use the invariance of the partition function under $\tau \to -1/\tau$ to write

$$Z(\tau,\bar{\tau}) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right). \tag{4.8}$$

At high temperatures $|\tau| \to 0$, the RHS of (4.8) is dominated by the lowest energy state of the theory, which in a unitary CFT is the vacuum. As a result we find that

$$Z(\tau,\bar{\tau}) \approx e^{-\frac{i\pi c}{12}\frac{\tau-\bar{\tau}}{\tau\bar{\tau}}} + \mathcal{O}\big(e^{\frac{2\pi i(\tau-\bar{\tau})}{\tau\bar{\tau}}}\big). \tag{4.9}$$

This is a remarkable consequence of modular invariance. It's telling us that the high energy spectrum of the theory is related to the low energy spectrum, and more precisely, to the contribution of the vacuum!

Using the definition of the partition function (4.2), we can extract the asymptotic density of states $c(E_n, P_n)$ in the following way. First, let us assume that E_n is a continuous variable. In addition, we perform an analytic continuation of the modular parameter such that

$$\tau = i\gamma, \qquad \bar{\tau} = i\bar{\gamma}, \tag{4.10}$$

where γ and $\bar{\gamma}$ are two *real and independent* variables. The partition function can then be written as

$$Z(\gamma,\bar{\gamma}) = \int dh_n \int d\bar{h}_n\, \rho(h_n,\bar{h}_n) e^{-2\pi\gamma(h_n-c/24)-2\pi\bar{\gamma}(\bar{h}_n-c/24)+\frac{\pi c}{12}\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}}}. \tag{4.11}$$

Here $\rho(h_n,\bar{h}_n)$ is a sum of delta functions such that, for $\epsilon \sim 0$,

$$\int_{h_n-\epsilon}^{h_n+\epsilon} dh \int_{\bar{h}_n-\epsilon}^{\bar{h}_n+\epsilon} d\bar{h}\, \rho(h,\bar{h}) = c(h_n,\bar{h}_n), \tag{4.12}$$

for every state n in the Hilbert space.

The value of $\rho(h,\bar{h})$ can be obtained by an inverse Laplace transformation of (4.8), namely

$$\rho(h,\bar{h}) = \int d\gamma \int d\bar{\gamma}\, e^{2\pi\gamma(h-c/24)+2\pi\bar{\gamma}(\bar{h}-c/24)+\frac{\pi c}{12}\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}}}. \tag{4.13}$$

We can evaluate this integral using the saddle-point approximation. The advantage of working with the variables introduced above is that the saddle $(\gamma_*, \bar{\gamma}_*)$ is simply located at

$$\gamma_* = \sqrt{\frac{c}{24(h - \frac{c}{24})}}, \qquad \bar{\gamma}_* = \sqrt{\frac{c}{24(\bar{h} - \frac{c}{24})}}. \tag{4.14}$$

Plugging these equations into (4.13) we reproduce the celebrated Cardy formula for the asymptotic density of states in two-dimensional CFTs

$$S \equiv \log \rho(h, \bar{h}) = 2\pi \sqrt{\frac{c}{6}\left(h - \frac{c}{24}\right)} + 2\pi \sqrt{\frac{c}{6}\left(\bar{h} - \frac{c}{24}\right)}, \tag{4.15}$$

where we have ignored logarithmic corrections that arise in the evaluation of the integral (4.13). It is important to note the regime of validity of Cardy's formula (4.15). The approximation (4.9) is valid at high temperatures. In terms of the $(\gamma, \bar{\gamma})$ variables, this corresponds to $(\gamma, \bar{\gamma}) \to 0$. Thus, we see that Cardy's formula (4.15) is valid provided that

$$h \gg c, \qquad \bar{h} \gg c. \tag{4.16}$$

Exercise 4.2: Verify that the saddle-point approximation of (4.13) yields Cardy's formula (4.15) and compute the log corrections to this formula.

As we will review later on, the energy of black holes in three-dimensional theories of gravity with a negative cosmological constant scales linearly with the central charge of the dual CFT. Up to a constant shift, the holographic dictionary identifies the energy of the black hole with the sum of the conformal weights in the dual theory. More precisely, we have

$$E_{BH} = h + \bar{h} - \frac{c-1}{12} > 0. \tag{4.17}$$

Consequently, the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence predicts that, for very massive black holes satisfying $E_{BH} \gg c$, the log of the number of black hole microstates, i.e. the black hole entropy, is given by the Cardy formula (4.15). We will show later on that this is indeed the case! In order to reproduce the entropy of lighter black holes we need to consider a different regime from that of Cardy (4.16). The desired regime is one where the central charge of the dual CFT is semiclassical, meaning that

$$c \gg 1. \tag{4.18}$$

This follows from the holographic dictionary which identifies $c \sim 1/G_N$ where G_N is Newton's constant. Hence, a semiclassical description of three-dimensional gravity requires a CFT that, among other important conditions — one of which will be derived below — satisfies (4.18).

Let us show how Cardy formula's (4.15) can be extended to the semiclassical regime (4.18) with $h, \bar{h}$ greater than c but not much greater. For convenience we will show how the argument works in the case $\theta = 0$ as this illustrates the essential steps in the derivation. Our goal will be to show that the partition function of any large-c CFT is universally given by the vacuum at low temperatures and by its $\mathcal{S}$-modular image at high temperatures. The universality of the partition function allows us to extend the validity of Cardy's formula to the holographic regime.

We begin by assuming that $\beta > 2\pi$ and distinguish between light (L) and heavy (H) states

$$\text{Light states: } E \leq \epsilon, \tag{4.19}$$

$$\text{Heavy states: } E > \epsilon, \tag{4.20}$$

where epsilon is a small positive number. The distinction between light and heavy states is related to the fact that black holes have energies that are greater than zero.[3] Consequently, black hole microstates are heavy from the point of view of the dual CFT. In addition, it is useful to define light and heavy partition functions which receive contributions only from the light and heavy states, as well as the $\mathcal{S}$-modular images of these light and heavy partition functions

$$\begin{aligned} Z[L] &\equiv \mathrm{Tr}_L(e^{-\beta E}), \qquad & Z'[L] &\equiv \mathrm{Tr}_L(e^{-\beta' E}), \\ Z[H] &\equiv \mathrm{Tr}_H(e^{-\beta E}), \qquad & Z'[H] &\equiv \mathrm{Tr}_H(e^{-\beta' E}), \end{aligned} \tag{4.21}$$

where $\beta' \equiv 4\pi^2/\beta$ (this follows from $\tau' = -1/\tau$). In terms of the light and heavy contributions (4.21), the full partition function the CFT satisfies

$$Z(\beta) = Z[L] + Z[H] = Z'[L] + Z'[H] = Z(\beta'). \tag{4.22}$$

The assumption that $\beta > 2\pi$ implies that

$$\beta' E - \beta E < 0. \tag{4.23}$$

As a result, the heavy part of the partition function is bounded by its modular image

$$Z[H] = \mathrm{Tr}_H\left(e^{\beta' E - \beta E - \beta' E}\right) < \alpha Z'[H], \tag{4.24}$$

where α is some number of $\mathcal{O}(c^0)$ satisfying $0 < \alpha < 1$. We can now use the modular invariance of the partition function, as written in (4.22), to show that

$$\frac{1}{\alpha} Z[H] < Z'[H] < Z[H] + Z[L]. \tag{4.25}$$

This means that the contribution of the heavy states to the partition function is bounded from above by the contribution of the light states, that is

$$Z[H] < \frac{\alpha}{1-\alpha} Z[L]. \tag{4.26}$$

This is again a remarkable consequence of modular invariance, which relates the high spectrum of the theory to the low energy one. As a result, we find that the partition function is bounded

[3] In contrast, empty AdS_3 has energy $E_{\mathrm{vac}} = -c/12$.

from above and below by the contribution of the light states, that is

$$\log Z[L] < \log Z(\beta) < \log Z[L] - \log(1-\alpha). \tag{4.27}$$

In the large-c limit, the contribution of the light states on the RHS of this equation is exponentially larger than the $\log(1-\alpha)$ term which is of $\mathcal{O}(1)$. Consequently, the partition function of any large-c 2d CFT is dominated by the light states at low temperatures

$$\log Z(\beta) \approx \log \mathrm{Tr}_L\big(e^{-\beta E}\big). \tag{4.28}$$

The light state dominance (4.28) of the partition function is not yet sufficient to derive a universal expression for the partition function. This follows from the fact that different CFTs have different distributions of light states. Nevertheless, it is not difficult to find a condition that guarantees that the dominant contribution to the partition function is that of the vacuum. Expanding the RHS of (4.28) we find

$$Z(\beta) \approx e^{-\beta E_{vac}} + \sum_{E_{\mathrm{vac}}<E\leq\epsilon} c(E) e^{-\beta E}, \tag{4.29}$$

where we have split the contribution of the vacuum from that of the other light states and $c(E)$ is the degeneracy of states at energy E.[4] Since $E_{\mathrm{vac}} = -c/12$ for a unitary CFT with a normalizable vacuum, and we furthermore assumed $\beta > 2\pi$, we see that the vacuum dominates over all other states provided that

$$c(E) \leq e^{\beta(E-E_{\mathrm{vac}})} < e^{2\pi(E-E_{\mathrm{vac}})}, \qquad E \leq \epsilon. \tag{4.30}$$

This condition on the density of light states is known as the *sparseness condition*.

The sparseness condition (4.30) guarantees that the partition function is dominated by the vacuum when c is large and $\beta > 2\pi$,

$$Z(\beta) \approx e^{-\beta E_{\mathrm{vac}}}, \qquad \beta > 2\pi. \tag{4.31}$$

The critical temperature $\beta = 2\pi$ is a fixed point of the $\mathcal{S}$ modular transformation $\beta \to 4\pi^2/\beta$. Consequently, the value of the partition function at high temperatures $\beta < 2\pi$ is simply obtained by performing an $\mathcal{S}$-modular transformation on (4.31) such that

$$Z(\beta) \approx e^{-\frac{4\pi^2}{\beta}E_{\mathrm{vac}}}, \qquad \beta < 2\pi. \tag{4.32}$$

Generalizing these results to include the angular potential θ, one finds the following sparseness condition

$$c(E_L, E_R) \leq e^{4\pi\sqrt{(E_L-\frac{1}{2}E_{\mathrm{vac}})(E_R-\frac{1}{2}E_{\mathrm{vac}})}}, \tag{4.33}$$

[4]We assume that the vacuum is unique so that $c(E_{\mathrm{vac}}) = 1$.

where the left and right-moving energies E_L and E_R are defined by

$$E_L \equiv E + P, \qquad E_R \equiv E - P. \tag{4.34}$$

The generalized sparseness condition (4.33) implies that the partition function of any modular invariant and unitary 2d CFT is universal in the large-c limit: it is dominated by the vacuum at low temperatures and by its $\mathcal{S}$-modular image at high temperatures such that

$$Z(\tau,\bar{\tau}) \approx \begin{cases} e^{\pi i(\tau-\bar{\tau})E_{vac}}, & |\tau| > 1, \\ e^{-\pi i\left(\frac{1}{\tau}-\frac{1}{\bar{\tau}}\right)E_{\text{vac}}}, & |\tau| < 1. \end{cases} \tag{4.35}$$

We will see later in the class that the partition function (4.35) matches the contributions of empty AdS_3 and BTZ black holes to the gravitational path integral. We learn that a holographic CFT, i.e. one capable of describing semiclassical Einstein gravity in three dimensions, requires a large central charge c and a spectrum of light states that is sparse (4.33), among other conditions. Using standard thermodynamics relations, it is not difficult to show that the log of the density of states at high temperatures $|\tau| < 1$ is given by

$$S(\tau,\bar{\tau}) = (1 - \tau\partial_\tau - \bar{\tau}\partial_{\bar{\tau}}) \log Z(\tau,\bar{\tau}), \tag{4.36}$$

$$= \frac{i\pi c}{6}\Big(\frac{1}{\tau} - \frac{1}{\bar{\tau}}\Big). \tag{4.37}$$

This formula depends on the temperature and chemical potential, i.e. it's valid in the canonical ensemble. Using once again standard thermodynamic relations, it is not difficult to show that this expression reproduces Cardy's formula (4.15),

$$S \equiv \log \rho(h,\bar{h}) = 2\pi\sqrt{\frac{c}{6}\Big(h - \frac{c}{24}\Big)} + 2\pi\sqrt{\frac{c}{6}\Big(\bar{h} - \frac{c}{24}\Big)}. \tag{4.38}$$

This version of Cardy's formula is valid in the semiclassical limit for states whose energies satisfy

$$E_L E_R > \Big(\frac{c}{24}\Big)^2, \qquad c \gg 1. \tag{4.39}$$

As a result, the universal partition function (4.35) allows us to extend the derivation of Cardy's formula, and consequently the holographic derivation of black hole entropy, to a larger family of black holes with energies $E_{BH} > \frac{c}{12}$.

Exercise 4.3: Compute the thermal expectation values of the energies $E_L \equiv \langle E_L\rangle_{\tau,\bar{\tau}} = (2\pi i)^{-1}\partial_\tau \log Z(\tau,\bar{\tau})$, $E_R \equiv \langle E_R\rangle_{\tau,\bar{\tau}} = -(2\pi i)^{-1}\partial_{\bar{\tau}} \log Z(\tau,\bar{\tau})$ and verify that you obtain (4.38). How do the expressions for E_L and E_R compare to (4.14)? Finally, use the thermal expectation values of the energies to obtain the regime of validity (4.39).