SHIV NADAR INSTITUTION OF EMINENCE

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Classification of Essentially Normal Operators

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Abstract

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It was discovered through the collaborative efforts of Weyl, von Neumann, Berg, and Sikonia over several years that any two normal operators are essentianlly equivalent if and only if they possess the same essential spectrum. However, when it comes to the class of essentially normal operators, this theorem did not generalize.

Nonetheless, Brown, Douglas, and Fillmore discovered that the essential spectrum, along with certain index data, could form a set of invariants that provide us with an analogous theorem.

This report provides brief insight into their theory and discovered theorem. We will incorporate new ideas later discovered by Arveson, Davie, and O'Donovan into this report. Therefore, we will deviate from the original exposition presented in the 1970s.

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Contents

Preliminaries

Before we begin exploring the subject matter of this report, it is important to establish some ground rules. This includes listing the necessary knowledge required to understand the report and standardizing the notation that we will use.

Prerequisites

Given that this is a report, it cannot possibly cover everything that has been studied in the past year. Therefore, we must assume that the reader has familiarity with the concepts listed below. The following exposition is heavily reliant on Spectral theory and C*-algebras, so we assume that the reader is familiar with these. Additionally, a basic understanding of Functional Analysis and Topology is also required, as they serve as the foundation for the preceding topics.

Furthermore, we also assume that the reader has a working knowledge of Operator Theory and basic Fredholm theory, such as Hilbert-Schmidt operators and Fredholm index. Although a reader with knowledge of algebraic topology and K-homology will be able to better appreciate the intuition behind these ideas, such knowledge is not presumed or required.

As a general guide, readers may refer to [Bha09] and Chapters 1-2 of [Sun97] for functional analysis. For topology, [Mun18] may be referred to, and Chapters 3-4 of [Sun97] shall suffice for C*-algebra and introductory spectral theory. The appendices in [CM21] contain all this information and some further generalized spectral theory in a compact form. The basics of operator theory can be gained from Chapter 1 [CM21]. For Fredholm theory, Chapters 1-2 of [Bre16] also serve as a very good introduction and contain some novel ideas for proof.

During my study of these topics, I have used the resources mentioned here. I do not presume to have any knowledge outside of what is covered in these resources. In case of any results that are not mentioned in these resources, I will state them explicitly either in the following sections or in the appendix. Additionally, I will cite the appropriate origin, where the proof is also available, of any results in brackets.

Notation

In the discussion that follows, we will consider $\mathcal{H}$ as a separable complex Hilbert space and X as a compact metric space. We will also denote the space of bounded linear operators, normal operators, essentially normal operators, Fredholm operators, and compact operators on $\mathcal{H}$ by $\mathcal{L}$, $\mathcal{N}$, $\mathcal{N}_{\mathcal{E}}$, $\mathcal{F}$, and $\mathcal{K}$, respectively.

Furthermore, we will use $\mathcal{Q}$ to refer to the Calkin algebra and π to refer to the Calkin map. If T is an element of $\mathcal{L}(\mathcal{H})$, we will use $\sigma(T)$ and $\sigma_{ess}(T)$ to denote the spectrum and essential spectrum of T, respectively. It is worth noting that $\sigma(T) = \sigma_{ess}(\pi(T))$. Lastly, if $F \in \mathcal{F}$, we will use $index(F)$ to represent the Fredholm index.

Γ shall be used to represent the Gelfand map. If $\rho : X \to Y$, then ρ^* shall denote $\rho^* : \mathcal{C}(Y) \to \mathcal{C}(X)$, where $\mathcal{C}(A)$ shall refer to the set of all continuous functions on a given set A. ∂A shall be used to denote topological boundary of a set A. We shall denote by $C^*(A)$ the C*-algebra given by set A. We shall also frequently just write τ which will mean $[\tau]$.

In case there are multiple elements of the same kind present in the text, we will provide appropriate subscripts. However, we will adhere to the above conventions.

For the time being, we will use the symbol $\sim$ to indicate essential equivalence between two operators. However, in the subsequent section, we will use this symbol in other contexts as well and will define it accordingly.

Further, in the spirit, of being concise and revising we shall define some terms and what they mean to us.

1. T shall be called an essentially normal operator if $T^*T - TT^*$ is compact.
2. $T_1 \sim T_2$ if there exists a unitary U such that $U^*T_1U - T_2$ is compact.
3. ($\mathcal{S}$,ψ) is called an extension of $\mathcal{A}$ by $\mathcal{M}$ if

$$0 \longrightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathcal{M} \longrightarrow 0$$

   is a short exact sequence, where $\mathcal{S}, \mathcal{A}, \mathcal{M}$ are C*-algebras.
4. By a *-monomorphism ψ, we mean that it is a homomorphism and injective. Further, $\psi(x^*) = \psi(x)^*$.

Chapter 1

Introduction

The fact that normal operators on finite-dimensional inner product spaces can be diagonalized is well-known. Hermann Weyl, while, researching stability for a differential operator under changes in boundary conditions was motivated to find the set of invariants for self-adjoint operators in a more general setting. In 1909, he proved that if two self-adjoint operators are essentially equivalent, they have the same essential spectrum. Roughly 20 years later, von Neumann proved the converse.

In 1970, Halmos [Hal70] asked if the preceding result could be extended to normal operators; Berg and Sikonia independently confirmed this, yielding the following theorem.

Theorem 1.1 (Weyl-von Neumann-Berg-Sikonia) *For $N_1, N_2 \in \mathcal{N}$, $N_1 \sim N_2$ if and only if $\sigma_{ess}(N_1) = \sigma_{ess}(N_2)$.*

A proof of the above using a lemma by Halmos [Hal72] is provided in Chapter 2 of [CM21]. Further, if $T \in \mathcal{N} + \mathcal{K}$, then we notice the following,

$$\sigma_{ess}(T) = \sigma_{ess}(N + K) = \sigma(\pi(N + K)) = \sigma(\pi(N)) = \sigma_{ess}(N)$$

Using this realization we see that the theorem can be extended to $\mathcal{N} + \mathcal{K}$.

In Operator Theory, essentially normal operators are quite commonly encountered, and as such, it was deemed “essential" to have a theorem that applies to this class. However, a problem arises when we attempt to do so. It should be noted that $\mathcal{N} + \mathcal{K} \neq \mathcal{N}_{\mathcal{E}}$. If this were the case, we wouldn't face any issues. To justify this we provide the following example.

Example 1.1 ([BDF73, Example 1.1]) *Let U be the unilateral shift operator. It is known that $index(U) = -1$. Suppose U can be expressed as $U = N + K$, where $N \in \mathcal{N}$ and $K \in \mathcal{K}$. Then N must be a Fredholm operator, and hence its index is zero, but the decomposition of U gives us $index(N) = -1$, creating a contradiction.*

Next, we provide an example that proves that two essentially normal operators with the same essential spectrum may not be essentially equivalent.

Example 1.2 ([BDF73, Example 1.2]) *Let U and Y be unilateral and bilateral shift operators, respectively. It can be observed that $\sigma_{ess}(U) = \Delta = \sigma_{ess}(Y)$, where Δ denotes the unit disc. Moreover, $index(Y) = 0$ and $index(U) = -1$.*
Suppose $U \sim Y$, i.e., there exists a unitary O such that $YO = OU + K$. It follows that $\pi(O)$ is invertible, and Atkinson's theorem implies that O is Fredholm, and hence its index is 0.

By the properties of the index, we must have index$(Y) =$ index(U). However, this contradicts the index information stated earlier.

It is evident from the above two examples that Theorem 1.1 does not extend directly to the class of essentially normal operators.

Remark 1.1 *It must be noted, however, that for any two operators T_1, T_2, $T_1 \sim T_2$ implies that they have the same index data and essential spectrum, thus converse if what poses the problem in generalization as seen in Example 1.2 .*

Let us consider the set of all essentially normal operators that have the same essential spectrum X, which is denoted by $\mathcal{N}_{\mathcal{E}}^{X}$. Now, if quotient this by the equivalence relation $\sim$, we get the set of equivalence classes of essentially normal operators with essential spectrum X, which we shall call Ext(X). It is worth noting that the classification of elements of Ext(X) will help us determine whether two essentially normal operators say T_1 and T_2, are equivalent or not. This can be done by checking if $[T_1] = [T_2]$ in Ext(X).

Brown, Douglas, and Filmore developed a remarkable theory combining the above observation that focuses on finding an analogous theorem for essentially normal operators. Their work connects the classification of essentially normal operators to the classifications of certain types of extensions and *-monomorphisms by viewing Ext(X) as a set of equivalences classes formed by the mathematical objects just mentioned.

They demonstrated that Ext(X) is a group, which implies that by using essential spectrum information together with certain index data, we can determine when $[T_1] = [T_2]$, thereby giving us $T_1 \sim T_2$. We aim to illustrate the same by following the structure closely of the original paper [BDF73], while substituting later developments and notes adapted from [CM21], [BDF77] and other papers which shall be mentioned in due course. To ensure that we maintain our focus, we will place results whose proof is deemed unnecessary in the appendix and refer to them as needed.

Chapter 2

Ext(X) is a group

Earlier, it was stated that the proof for the BDF theorem relies on proving that Ext(X) is a group. To do this, we need to demonstrate a well-defined binary operation on it. Additionally, we must prove the existence and uniqueness of a zero element, as well as the further existence of inverses.

To achieve this, we will characterize Ext(X) using various mathematical objects as mentioned in the Introduction. This will be the primary goal of this chapter - to present alternate characterizations of Ext and show that it is a group.

Before we begin, however, we will show a useful result that will help us now and in the future - the Absorption lemma. To prove this, we shall present two lemmas which are interesting in their own right and prove to be useful later on.

2.1 Absorption Lemma

Lemma 2.1.1 ([BDF77, Lemma 1.3]) *Let $\mathfrak{F}$ be a countable family of operators such that $T_i^*T_j - T_iT_j^*$ is compact for $i, j \in \mathbb{N}$. Let $\{\lambda_k\}$ be in joint spectrum of $\{\pi(T_k)\}$, where $T_k \in \mathfrak{F}$, for all $k \in \mathbb{N}$. Then there exists an orthonormal sequence $\{\psi_n\}$ in $\mathcal{H}$, such that*

$$\|T_k\psi_n - \lambda_k\psi_n\| \to 0 \textit{ for all } k \in \mathbb{N}$$

Proof: By Theorem A.1 $\{\pi(T_k)\}$ is a family of mutually commuting normal elements. Let

$$S = \sum_{k\in\mathbb{N}} \frac{(T_k - \lambda_k)^*(T_k - \lambda_k)}{2^k\|T_k - \lambda_k\|^2}$$

discarding the terms where $T_k = \lambda_k I$. Let Ψ be a complex *-homomorphism such that $\Psi(\pi(T_k)) = \lambda_k$, then $\Psi(\pi(S)) = 0$ and $0 \in \sigma(\pi(S))$. We can conclude that $\pi(S)$ is not invertible based on the given information. Atkinson's theorem then implies that S is not Fredholm, indicating that either its range is not closed or it has an infinite dimensional kernel. By referring to Theorem A.2 , we can find an orthonormal sequence $\{\psi_n\}$ such that $\|S\psi_n\| \to 0$. From here, our desired result follows. $\square$

An alternative proof for the above exists in [CM21, Lemma 2.4.1]. The following lemma builds on this result and will be used to prove the Absorption Lemma.

Lemma 2.1.2 ([BDF73, Lemma 2.2]) *Let $\mathfrak{F}$ be a countable family of operators such that $T_i^*T_j - T_iT_j^*$ is compact for $i, j \in \mathbb{N}$. Let $\lambda^r = \{\lambda_k^r\}$ be in joint spectrum of $\{\pi(T_k)\}$, where $T_k \in \mathfrak{F}$, for all $k, r \in \mathbb{N}$. Then there exists an orthonormal sequence $\{\xi_r\}$ in $\mathcal{H}$, such that*

$$T_k = (D_k \oplus R_k) + L_k \textit{ for all } k \in \mathbb{N}$$

with regards to $\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_\in$, where $\mathcal{H}_\infty = span\{\xi_r\}, \mathcal{H}_\in = \mathcal{H}_\infty{}^\perp, L_k$ is compact, D_k is diagonal with eigenvectors $\{\xi_r\}$ and corresponding eigenvalues $\{\lambda_k^r\}$, and $\sigma(\pi(R_k)) = \sigma(\pi(T_k))$.

Proof: Suppose first that T_k's as given are self-adjoint. We shall inductively construct an orthonormal sequence $\{\xi_r\}$ such that

$$\|T_k\xi_r - \lambda_k^r\xi_r\| < \frac{1}{2^r}, \text{ for } k \le r$$

Assume we have $\xi_1 \ldots \xi_{s-1}$ satisfying the above inequality for $k \le r \le s$, then by Lemma 2.1.1 there exists an orthonormal sequence $\{\psi_n\}$, such that

$$\lim_{n\to\infty} \|T_k\psi_n - \lambda_k^s\psi_n\| = 0 \text{ for all } k \in \mathbb{N}$$

let $0 < \delta < 1/2$ and choose n_0 such that

$$|\langle\psi_{n_0}, \xi_r\rangle| < \frac{\delta}{\sqrt{s-1}}, \text{ for } r < s$$

$$\|T_k\psi_{n_0} - \lambda_k^s\psi_{n_0}\| < \delta, \text{ for } k \le s$$

then set

$$\xi_s' = \psi_{n_0} - \sum_{r=1}^{s-1} \langle\psi_{n_0}, \xi_r\rangle\xi_r$$

and the resulting normalized vector $\xi_s = \xi_s'/\|\xi_s'\|$ is a unit vector orthogonal to $\xi_1 \ldots \xi_{s-1}$ satisfying

$$\|T_k\xi_s - \lambda_k^s\xi_s\| < G\lambda, \text{ for } k \le s$$

where G depends only on $\|T_k\|$, for $k \le s$. Hence take δ sufficiently small for $r = s$ and we are done.
Now as stated in the statement of the Lemma, let $\mathcal{H}_\infty = \text{span}\{\xi_r\}, \mathcal{H}_\in = \mathcal{H}_\infty{}^\perp$, and let

$$T_k = \begin{bmatrix} X_k & Y_k^* \\ Y_k & R_k \end{bmatrix}$$

be the decomposition of T_k with regards to $\mathcal{H}_\infty \oplus \mathcal{H}_\in$ and so,

$$T_k - D_k \oplus R_k = \begin{bmatrix} X_k - D_k & Y_k^* \\ Y_k & 0 \end{bmatrix}$$

where $D_k\xi_r = \lambda_k^r\xi_r$ for $r \in \mathbb{N}$. Now we see that

$$\|(X_k - D_k)\xi_r\|^2 + \|Y_k\xi_r\|^2 = \|(T_k - D_k)\xi_r\|^2 < \frac{1}{4^r}$$

So $X_k - D_k$ and Y_m are Hilbert-Schmidt Operators so $T_k - D_k \oplus R_k$ is compact. If in the above construction we use each λ^r twice in succession, R_k as constructed shall be a compact pertubation of $D_k \oplus S_k$ and hence $\sigma(\pi(R_k)) = \sigma(\pi(T_k))$. $\square$

Remark 2.1.1 *By using λ^r twice in succesion we mean applying Lemma 2.1.1 to $\{\lambda^r | r \geq 1\} \cup \{\lambda^r | r \geq 1\}$.*

Our next result the Absorption Lemma, although ironically labelled a lemma and derived as a corollary in the original paper [BDF73] turns out to be pretty useful hence here we label it as a theorem.

Theorem 2.1.1 (Absorption Lemma [BDF73, Corollary 2.3],[CM21, Lemma 2.4.2]) *If $T \in \mathcal{N}_{\mathcal{E}}$ and $N \in \mathcal{N}$, such that $\sigma_{ess}(N) \subseteq \sigma_{ess}(T)$, then $T \oplus N \sim T$ and $N \oplus T \sim T$*

Proof: Assuming the hypothesis, consider a sequence $\{\lambda^r\}$ dense in $\sigma_{ess}(T)$, such that isolated points are counted infinitely often. By Lemma 2.1.2 , we have

$$T = (D \oplus R) + L$$

where D is diagonal with entries $\{\lambda^r\}$ and L is compact. Then due to our construction $\sigma_{ess}(T) = \sigma_{ess}(D) = \sigma_{ess}(D \oplus N)$. By Weyl-von Neumann-Berg-Sikonia $D \oplus N \sim D$, hence $T \oplus N \sim T$, similarly for $N \oplus T \sim T$. $\square$

2.2 Characterisations of Ext(X)

As we promised earlier, we will now discuss the relationship between the equivalence class formed by essentially normal operators and the equivalence class formed by certain types of extensions and *-monomorphisms. By understanding these characterisations, we can determine the identity and inverse for the elements in Ext(X), which in turn makes Ext(X) a group. The following was understood by reading [CM21, Chapter 2].

2.2.1 Characterization via Extensions

If we are given an essentially normal operator T, we can create an extension $(\mathcal{S}_T, \psi_T)$. Here, ψ_T is the composition of the Gelfand map, Γ, and the projection map, π, and $\mathcal{S}_T$ is the C*-algebra generated by $T, \mathcal{K}(\mathcal{H})$ and $I_{\mathcal{H}}$. This pair, $(\mathcal{S}_T, \psi_T)$, will act as an extension of $\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(X)$.

Conversely, if we are given such an extension, $(\mathcal{S}, \psi)$, we can find an operator $T_0 \in \mathcal{S}$ such that $\psi(T_0) = id|_X$. In this case, we will have $\mathcal{S}_{T_0} = \mathcal{S}$. The diagram below shows the relationship between the operator T and the extension that arises from it.

$$\begin{tikzcd}
\mathcal{S}_T \arrow[r] \arrow[d, "\pi"] & \mathcal{C}(X) \\
\mathcal{S}_T/\mathcal{Q}(\mathcal{H}) \arrow[r, hook] & \mathcal{Q}(\mathcal{H}) \arrow[u, "\Gamma_{\pi(\mathcal{S})}"']
\end{tikzcd}$$

We will now only consider extensions of $\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(X)$. We will omit repeating this multiple times in the following sections, and it will be understood what is meant by this.

Remark 2.2.1 *Due to the explanation above, it can be deduced that the C*-algebra $\mathcal{S}$ considered in a extension pair $(\mathcal{S}, \psi)$ is then always seperable.*

The concept of equivalence between extensions is based on the idea of equivalence between the corresponding essentially normal operators for the given extensions. Let $(\mathcal{S}_1, \psi_1)$ and $(\mathcal{S}_2, \psi_2)$ be two extensions with corresponding operators T_1 and T_2, such that $T_1 \sim T_2$. There exists a unitary operator U and a compact operator K such that $U^*T_2U = T_1 + K$. Due to the continuity of the map $T \mapsto U^*TU$ and the fact that S_1 and S_2 are generated by T_1, T_2, and the compacts respectively, we have $U^*\mathcal{S}_2U = \mathcal{S}_1$ and $\psi_2(T) = \psi_1(U^*TU)$. Hence, we give the following definition.

Definition 2.2.1 (Equivalence b/w Extensions) *We say $(\mathcal{S}_1, \psi_1) \sim (\mathcal{S}_2, \psi_2)$, if there exists a unitary U, such that $U^*\mathcal{S}_2U = \mathcal{S}_1$ and $\psi_1(U^*TU) = \psi_2(T)$, for all $T \in \mathcal{S}_2$.*

The set of all such equivalence classes of extensions is referred to as Ext(X).

Remark 2.2.2 *Note that we use the symbol $\sim$ to represent equivalence between extensions, essential equivalence between operators, and later on between *-monomorphisms. As we will use a capital letter for operators, a pair for extension, and an indexed τ for *-monomorphisms, it should be clear which equivalence $\sim$ corresponds to in the given context without any misunderstandings. From now on, $\sim$ will denote all such equivalences as described.*

Next, we present a “trivial" example (pun intended) to demonstrate how in certain cases, the unavailability of index data does not restrict us from determining the equivalence classes.

Lemma 2.2.1 (A “trivial" example [CM21, Example 2.3.4],[BDF77, Lemma 1.14]) *Ext(Ω) = 0 for Ω a compact subset of $\mathbb{R}$.*
Let $T_1, T_2 \in \mathcal{N}_{\mathcal{E}}$ such that $\sigma_{ess}(T_1) = \Omega = \sigma_{ess}(T_2)$, then by spectral theorem, both $\pi(T_1), \pi(T_2)$ are self-adjoint. Hence

$$S_j = \mathfrak{R}(S_j) + K$$

where K is compact. As a consequence $\sigma_{ess}(\mathfrak{R}(T_1)) = \sigma_{ess}(T_1) = \sigma_{ess}(T_2) = \sigma_{ess}(\mathfrak{R}(T_2))$, so $\mathfrak{R}(T_1) \sim \mathfrak{R}(T_2)$, where $\mathfrak{R}$ denotes the real part, and further $T_1 \sim T_2$.

2.2.2 Characterization via *-monomorphisms

We will now show that an extension $(\mathcal{S}, \psi)$ leads to a *-monomorphism, providing an additional characterization of Ext.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r, hook] \arrow[d, equal] & \mathcal{S} \arrow[r, "\psi"] \arrow[d, hook] & \mathcal{C}(X) \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \mathcal{K}(\mathcal{H}) \arrow[r, hook] & \mathcal{L}(\mathcal{H}) \arrow[r, "\pi"] & \mathcal{Q}(\mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

Observe the commutative diagram above and notice that $\tau : \mathcal{C}(X) \to \mathcal{Q}(\mathcal{H})$ where, $\tau \circ \psi(A) = \pi(A), A \in \mathcal{S}$, τ is a *-monomorphism. Further,$\pi(A) = 0 \iff A$ is compact $\iff \psi(A) = 0$. Thus as promised we have ourselves a *-monomorphism

On the other hand, given a unital *-monomorphism $\tau : \mathcal{C}(X) \to \mathcal{Q}(H)$, one may set $\mathcal{S} = \pi^{-1}(Im\tau), \psi = \tau^{-1} \circ \pi$. It turns out the pair $(\mathcal{S}, \psi)$ obtained in this manner is an extension of $\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(X)$.

τ can also be related directly to the underlying essentially normal operator T, and we deduce that the map τ induces generalized functional calculus of T such that $\tau(f) = f(\pi(T))$, where T is the underlying operator. The details are present in [CM21][2.5].

Given $T_1, T_2 \in \mathcal{N}_{\mathcal{E}}, T_1 \sim T_2$ let $\tau_1 : \mathcal{C}(X) \to \mathcal{Q}(\mathcal{H}_1), \tau_2 : \mathcal{C}(X) \to \mathcal{Q}(\mathcal{H}_2)$ be the associated *-monomorphism we notice the following, for any $f \in \mathcal{C}(X)$,

$$\tau_1(f) = f(\pi(T_1)) = f(\pi(U^*T_2U + K)) = f(\pi(U^*T_2U))$$

now setting $\alpha_U(\pi(A)) = \pi(U^*AU)$, for $A \in \mathcal{L}((H)_2)$, we get that $\tau_1(f) = (\alpha_U\tau_2)(f)$. So we attain the following definition of equivalence.

Definition 2.2.2 (Equivalence between *-monomorphisms) *We say $\tau_1 \sim \tau_2$, if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\tau_1 = \alpha_U \circ \tau_2$, where $\alpha_U : \mathcal{Q}(\mathcal{H}_2) \to \mathcal{Q}(\mathcal{H}_1)$ is a unital *-monomorphism induced by U such that $\alpha_U(\pi(A)) = \pi(U^*AU)$, for $A \in \mathcal{L}(\mathcal{H}_2)$.*

We shall avoid here mentioning the concept of weak equivalence as in otherwise pathological cases where C*-algebras are considered non-abelian, the concept of weak equivalence implies our concept of equivalence. An interested reader may look at [CM21, Proposition 2.5.1], [BDF73, Theorem 4.3] for details.

We now have three different but equivalent ways to describe Ext(X). It can be seen as:

1. A set of equivalence classes of essentially normal operators with essential spectrum X.
2. A set of equivalence classes of extensions of $\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(X)$.
3. A set of equivalence classes of *-monomorphisms from $\mathcal{C}(X)$ to $\mathcal{Q}(\mathcal{H})$.

Each of these characterizations provides a unique perspective on Ext(X) and can be used interchangeably.

2.3 Existence of Identity

Before we move forward, we need to establish a binary operation on Ext(X). A suitable option is to define the operation such that for any two elements $[T_1]$ and $[T_2]$,

$$[T_1] + [T_2] = [T_1 \oplus T_2]$$

We use this as always to motivate our addition operation when Ext is viewed in the above-mentioned characterisations and obtain the following definitions.

Definition 2.3.1 *We define the sum $(\tau_1 + \tau_2)(f) := \rho(\tau_1(f) \oplus \tau_2(f))$ for $f \in \mathcal{C}(X)$, where $\rho : \mathcal{Q}(\mathcal{H}_1) \oplus \mathcal{Q}(\mathcal{H}_2) \to \mathcal{Q}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a injective *-homomorphism between the spaces as written.*

Remark 2.3.1 *ρ just takes $\pi_1(T_1) + \pi_2(T_2)$ to $\pi_3(T_1 \oplus T_2)$,where T_1, T_2 are in $\mathcal{H}_1, \mathcal{H}_2$ respectively. Further π_i for $i = 1, 2$ also corresponds to Calkin map on the respective Hilbert spaces and for $i = 3$*

on $\mathcal{H}_1 \oplus \mathcal{H}_2$.
This allows us to view $\mathcal{Q}(\mathcal{H}_1) \oplus \mathcal{Q}(\mathcal{H}_2)$ as subalgebra of $\mathcal{Q}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

In a similar manner as above we can define addition for extension, it is presented below.

Definition 2.3.2 *We define the sum $(S_1, \psi_1) + (S_2, \psi_2) := (S, \psi)$, where S is the C*-subalgebra of $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ generated by all $(T_1 + T_2) + K$, such that $T_i \in S_i$ and $\psi_1(T_1) = \psi_2(T_2)$ and K is compact and further defining ψ as*

$$\psi((T_1 \oplus T_2) + K) = \psi_1(T_1) = \psi_2(T_2)$$

It is easy to verify that these operations are well-defined. One can examine a short argument presented in [CM21, Lemma 2.6.1] to convince themselves.

We will now start searching for the zero element. Thanks to the Absorption Lemma (Theorem 2.1.1), we can see that the class $\mathcal{N} + \mathcal{K}$ acts as an identity for the class of essentially normal operators with the same spectrum. In other words, for any $N \in \mathcal{N} + \mathcal{K}$ and $T \in \mathcal{N}_{\mathcal{E}}$, we have

$$[T] = [T \oplus N] = [T] + [N].$$

Therefore, the class $\mathcal{N} + \mathcal{K}$ is considered *trivial*.

This property of being "trivial" can be expressed in terms of *-monomorphisms as stated below.

Definition 2.3.3 (Trivial *-monomorphism) *We shall call a unital *-monomorphism $N_X : \mathcal{C}(X) \to \mathcal{Q}(H)$ trivial if there esists a unital *-monomorphism $N_0 : \mathcal{C}(X) \to \mathcal{L}(\mathcal{H})$, such that $N_X = \pi \circ N_0$.*
In this case we say N_0 trivializes N_X and N_X lifts to $\mathcal{L}(\mathcal{H})$.

It turns out that this trivial *-monomorphism acts as the zero element but before that we will now show that for any X, there exists a trivial *-monomorphism $\tau_0(X)$. We prove that any 2 trivial *-monomorphisms lie in the same equivalence class giving us uniqueness of identity.

Theorem 2.3.1 (Existence and Uniqueness [BDF77, Thm 1.13], [CM21, Thm 2.6.5]) *Given X a compact metric space, there exists a trivial *-monomorphism τ_0 and any two trivial *-monomorphisms are equivalent.*

Proof: Consider $\varrho : \mathcal{C}(X) \to \mathcal{Q}(l^2)$, such that $\varrho(f) = \{f(x_n)\}_n$, where $\{x_n\}$ is a dense subset of X; isolated pointed being counted infinitely often. Set $\tau_0 = \pi \circ \varrho$. Then,

$$\tau_0(f) = \pi(\operatorname{diag}(f(x_n)))$$

where $\operatorname{diag}(f(x_n))$ denotes the diagonal operator with entries $\{f(x_n)\}$ on l^2. It can be seen that τ_0 is a *-monomorphism by observing that if $\tau_0(f) = 0$, then $f(x_n) \to 0$, and due to density of $\{x_n\}$, $f \equiv 0$.

Now suppose τ_0^2 is another *-monomorphism arising in the same way as above with a different dense sequence $\{y_n\}$. We invoke Theorem A.3 to say there exists a permutation α, such that $d(x_n, y_{\alpha(n)}) \to 0$. Define U unitary such that $Ux_n = e_{\alpha(n)}$, where $\{e_n\}$ denotes the standard basis of l^2. It can be seen that

$$U\mathrm{diag}(f(x_n)) - \mathrm{diag}(f(y_n))U = \mathrm{diag}(f(x_n) - f(y_{\alpha(n)})) = K$$

for some compact operator $K \in \mathcal{L}(l^2)$. So $\tau_0^2 \sim \tau_0$

However, we must also consider cases of trivial extension that do not necessarily arise through the diagonal operator. So let τ_0' be a trivial *-monomorphism, then there exists a trivalizing map ϖ such that $\tau_0' = \pi \circ \varpi$. Let $\mathbb{E}$ be the spectral measure on Borel subsets of X such that

$$\varpi(f) = \int_X f \mathrm{d}\mathbb{E}$$

for all $f \in \mathcal{C}(X)$. Let $\{U_n\}$ be a basis of open sets for X and let $\mathfrak{C}$ be the C*-algebra generated by projections $P_n = \mathbb{E}(U_n)$. Then $\varpi(\mathcal{C}(X)) \subseteq \mathfrak{C}$. Further, by Theorem A.4 $\mathfrak{C}$ is generated by H where H is self-adjoint.
Suppose $\sigma(H) = \Omega$, and $\varkappa : \mathcal{C}(\Omega) \to \mathfrak{C}$ is a *-isomorphism such that $\varkappa(f) = f(H)$. Since $\varpi(\mathcal{C}(X)) \subseteq \mathfrak{C}$, we get a surjection $\eta : \Omega \to X$ such that $\varpi = \eta^* \circ \varkappa$, where η^* is a mapping from $\mathcal{C}(X)$ to $\mathcal{C}(\Omega)$.

$$\begin{tikzcd} \mathcal{C}(X) \arrow[r, "\eta^*"] \arrow[dr, "\varpi"] & \mathcal{C}(\Omega) \arrow[d, "\varkappa"] \\ & \mathfrak{C} \end{tikzcd}$$

We note that

$$\varpi = \eta^* \circ \varkappa \implies \varpi(f) = \eta^* \circ \varkappa(f) = \eta^* f(H) = (f \circ \eta)(H)$$

By Weyl-von Neumann, $H = D + K$, where D has diagonal enteries λ_m and K is compact and further $\sigma(H) = \sigma(D)$. Thus,

$$\varpi(f) = (f \circ \eta)(D) + K_f, K_f \text{ compact}$$

for all $f \in \mathcal{C}(X)$ as $\pi(f \circ \eta)(D) = (f \circ \eta)(\pi(D))$, same for H. Note $\{\lambda_m\}$ is dense on Ω so $\{\eta(\lambda_m)\}$ is dense in X.
Since $(f \circ \eta)(D) = \mathrm{diag}\{f(x_n)\}$,

$$\pi(f \circ \eta)(D) = \pi(\mathrm{diag}\{f(x_n)\})$$

So,

$$\tau_0'(f) = \pi \circ \varpi(f) = \pi(\circ f \circ \eta(D)) = \pi(\mathrm{diag}\{f(x_n)\})$$

□

Theorem 2.3.2 (Trivial Class as the zero element [BDF77, Thm 1.17], [CM21, Thm 2.7.1])
The trivial element acts as the zero element in Ext(X).

Proof: Let $\tau : \mathcal{C} \to \mathcal{Q}(\mathcal{H})$ be a *-monomorphism. Let $\{f_k\}$ be dense in $\mathcal{C}(X)$ and $\{x_r\}$ be dense in X; as always isolated points being counted infinitely often. Define $\lambda_k^r = f_k(x_r)$ and choose T_k such that $\pi(T_k) = \tau(f_k)$. Then this family of T_k satisfies the hypothesis of Lemma 2.1.2 with $\lambda^r = \{\lambda_k^r | k \in \mathbb{N}\}$ being in the joint spectrum. So, there exists an orthonormal sequence $\{\psi_r\}$. Let $\mathcal{H}_\infty, \mathcal{H}_\in$ be as defined in the lemma, let P be the projection on $\mathcal{H}_\infty$ and $\phi = \pi(P)$. It can be seen that due to our construction ϕ and $\mathrm{im}\tau$ commute and that

$$\tau_1 : f \mapsto \tau(f)\phi, \tau_2 : f \mapsto \tau(f)(1-\phi)$$

are both *-monomorphisms.
Indeed, if the $T_k = D_k \oplus R_k + L_k$ is the decomposition achieved from Lemma 2.1.2 , $\tau_1(f_k), \tau_2(f_k)$ are just $\pi(D_k), \pi(R_k)$ respectively. Further if $f_n \to f$, then $\tau_1(f_n) \to \tau_1(f) = \pi(D_f)$, similarly for τ_2. Since π is multiplicative, maps are well defined and homomorphisms. It can further be seen by applying Remark 2.1.1 and continuity of π that

$$\|\pi(D_f)\| = \|f\|_\infty = \|\pi(R_f)\|$$

and we are through, full details in [CM21]. □

2.4 Existence of Inverse

In this section, we depart from the structure outlined in [BDF73]. The original proof for demonstrating the existence of an inverse was a complex and laborious process, which involved splitting and Mayer-Vietoris Sequence from Algebraic Topology.

However, in 1974, Arveson [Arv74] provided a much simpler proof that relied on a lifting theorem by Anderson and Naimark's Dilation Theorem. Later, it was noticed that Anderson's lifting theorem could be substituted for Davie's Lifting Theorem without affecting the main proof. This substitution was adapted and printed long with a full proof in [BDF77].

In this section, we will rely on these two theorems to prove our result.

Theorem 2.4.1 (Davie's Lifting Theorem [BDF77, Theorem 1.19]) *Let (X, d) be compact,let $\mathfrak{C}$ be an unital C^*-algebra, $\mathfrak{J}$ be a closed 2-sided *-ideal of $\mathfrak{C}$ and $\pi_\mathfrak{J}$ be the corresponding quotient map, and let $\tau_\mathfrak{J} : \mathcal{C}(X) \to \mathfrak{C}/\mathfrak{J}$ be a unital positive linear map. Then there exists a unital positive linear map $\kappa : \mathcal{C}(X) \to \mathfrak{C}$ such that $\pi_\mathfrak{J} \circ \kappa = \tau_\mathfrak{J}$.*

Theorem 2.4.2 (Naimark's Dilation Theorem) *Given $\kappa : \mathcal{C}(X) \to \mathcal{L}(\mathcal{H})$, a unital positive linear map, there exists a Hilbert space $\mathcal{H}'$ and a unital *-homomorphism $\eta : \mathcal{C}(X) \to \mathcal{L}(\mathcal{H}')$, such that $\mathcal{H} \subseteq \mathcal{H}'$ and $\kappa(f) = \mathcal{P}_\mathcal{H} \circ \eta(f)$ for $f \in \mathcal{C}(X)$, where $\mathcal{P}_\mathcal{H}$ is the projection of $\mathcal{H}'$ onto $\mathcal{H}$.*

Theorem 2.4.3 ([BDF77, Theorem 1.23], [Arv74]) *Ext(X) is a group.*

Proof: Let $\tau : \mathcal{C}(X) \to \mathcal{Q}(\mathcal{H})$ be a *-monomorphism. By Theorem 2.4.1 and subsequently Theorem 2.4.2 , we get a unital positive linear map $\kappa : \mathcal{C}(X) \to \mathfrak{C}$ such that $\pi \circ \kappa = \tau$. Further, there exists $\mathcal{H}'$ such that $\mathcal{H} \subseteq \mathcal{H}'$ and $\kappa(f) = \mathcal{P}_\mathcal{H} \circ \eta(f)$ for $f \in \mathcal{C}(X)$, where $\mathcal{P}_\mathcal{H}$ is the projection of $\mathcal{H}'$ onto $\mathcal{H}$.
Now let,

$$\eta_f = \begin{bmatrix} \kappa(f) & K_f \\ L_f & M_f \end{bmatrix}$$

be decomposition relative to $\mathcal{H} \oplus \mathcal{H}^{\perp}$. Then it can be checked that $\mathcal{P}_{\mathcal{H}}T - T\mathcal{P}_{\mathcal{H}}$ is compact for every $T \in \kappa(\mathcal{C}(X))$ and off-diagonal terms are compact [Arv74].
Now let $\tau'(f) = \pi(M_f)$, due to multiplicative nature of π this is a homomorphism. Note,

$$(\tau + \tau')(f) = (\pi \circ \kappa)(f) + \pi(M_f) = \pi(\kappa(f) \oplus M_f) = (\pi \circ \eta)(f)$$

Let τ_0 be a trivial extension and define $\tau_1(f) = \tau'(f) \oplus \tau_0(f)$, then,

$$\tau + \tau_1 = \tau + \tau' + \tau_0 = \tau + \tau' = \pi \circ \eta$$

□

Before we end this chapter we shall state here few lemmas which can help us determine Ext(X) depending on just topological properties of X.

Lemma 2.4.1 ([BDF77, Theorem 1.15]) *Ext(X) = 0, if X is totally disconnected.*

Lemma 2.4.2 ([BDF77, Proposition 2.3.1]) *Let X, Y be two metric spaces. If X is homeomorphic to Y, then there is a bijection from Ext(X) to Ext(Y)*

Chapter 3

BDF Theorem

We have already developed that Ext is a group, but we need one more ingredient to proceed to the final proof. We need to show

$$\mathrm{Ext}(A) \xrightarrow{i_*} \mathrm{Ext}(X) \xrightarrow{q_*} \mathrm{Ext}(X/A)$$

is exact, hence we must show that $\mathrm{im} i_* = \ker q_*$, where i_*, q_* are maps induced by $i : A \to X, q : X \to X/A$, respectively.

Remark 3.0.0.1 *It should be noticed that $q \circ i$ is a constant map and so, $(q_* \circ i_*)([\tau_A]) = (q \circ i)_*([\tau_A])$ is always trivial for any $[\tau_A] \in Ext(A)$. Thus im $i_* \subseteq$ ker q_*, the following section will be devoted to proving the other inclusion.*

3.1 Ext(A) → Ext(X) → Ext(X/A) is exact

In [O'D80], a direct proof of this is given using operator-theoretic terms. However, this proof seems less straightforward to understand than the one presented by Davie. Davie's proof also avoids tools from algebraic topology, but relies on two lemmas. Both lemmas and Davie's proof were later adapted and proved in [BDF77]. This concept is also presented in [CM21, Proposition 3.5.1], with any gaps filled and explored in further detail. In this text, we will provide an outline of the same concept.

First, we shall state the three results we require.

Lemma 3.1.0.1 ([BDF77, Corollary 1.16]) *If $\tau : \mathcal{C}(X) \to \mathcal{Q}(\mathcal{H})$ is an extension such that im τ is contained in a commutative C*-algebra of $\mathcal{E}$ of $\mathcal{Q}(\mathcal{H})$ generated by projections, then τ is trivial.*

Lemma 3.1.0.2 ([BDF77, Lemma 2.2]) *If X/A is totally disconnected then i_* is an isomorphism.*

Lemma 3.1.0.3 ([BDF77, Lemma 2.3]) *Let $q : X \to Y$be surjectice, let $\tau : \mathcal{C}(X) \to \mathcal{Q}(\mathcal{H})$ be a *-monomorphism such that $q_*(\tau)$ is trivial. Let $\phi : \mathcal{C}(Y) \to \mathcal{L}(\mathcal{H})$ be such that $\tau \circ q^* = \pi \circ \phi$ and let $\mathbb{E}$ be the projection-valued spectral measure on Y associated with ϕ. If C is a closed subset of Y and ∂C contains no point with multiple preimage in X, then $\pi(\mathbb{E}(C))$ commutes with imτ.*

Now we shall sketch the proof of our main theorem.

Theorem 3.1.0.1 ([BDF77, Theorem 2.4],[CM21, Proposition 3.5.1]) *Let $q : X \to Y$ be surjective and let C be a closed subset of Y such that it contains all points with multiple preimages in X. Let $A = q^{-1}(C)$, let $q' : A \to C$ be the restriction of q, and let $i : A \to X$ be inclusion. Then $ker q_* \subset i_*(ker\, q'_*)$.*

Proof: Let $\tau : \mathcal{C}(X) \to \mathcal{Q}$ be such that $[\tau]$ is in $\ker q_*$, and let ϕ and $\mathbb{E}$ be as in Lemma 3.1.0.3 . Choose a basis $\{U_n\}$ for the topology of X/A such $D_n = \overline{U_n}$ is disjoint from A. Let $\mathfrak{C}$ be the C*-algebra generated by im τ and the projections $\pi(\mathbb{E}(q(C_n)))$, then this is commutative by Lemma 3.1.0.3 . Further we have a $\tilde{X} = s^{-1}(X)$- a compact metric space, and a surjection s from $\tilde{X}$ to X. This in turn gives rises to a *-monomorphism $\tilde{\tau} : \mathcal{C}(\tilde{X}) \to \mathfrak{C}$ such that $\tau = s_*(\tilde{\tau})$. This is summarised in the following diagram, where Γ is the Gelfand Map.

$$\begin{tikzcd}
\mathcal{C}(Y) \arrow[r, "q^*"] & \mathcal{C}(X) \arrow[r, "s^*"] \arrow[d] & \mathcal{C}(\tilde{X}) \\
& \mathrm{Im}\tau \arrow[r, hook] & \mathfrak{C} \arrow[u, "\Gamma_{\mathfrak{C}}"]
\end{tikzcd}$$

It can be seen that s is a homeomorphism on $\tilde{A} = s^{-1}(A)$ and $\tilde{X}\backslash A, \tilde{X}/A$ is totally disconnected details in [CM21, Proposition 3.5.1].

Invoking Lemma 3.1.0.2 , we get an extension τ' for $\mathcal{C}(A)$, such that $k_*(\tau') = \tilde{\tau}$, where $k : A \to \tilde{X}$ is inclusion. Now we must show that $q'_*(\tau')$ is trivial.

For this consider the subalgebra $\mathfrak{D}$ generated now by im $q_*(\tau)$ and the projections $\pi(\mathbb{E}(q(C_n)))$, then in the same way as above we get a space $\tilde{Y}$ containing C such that $\tilde{Y}\backslash C$ is totally disconnected. Now consider the maps $\tilde{q} : \tilde{X} \to \tilde{Y}$ and the inclusion map $l : C \to \tilde{Y}$, then we have $\tilde{q} \circ k = l \circ q'$. We depict all this in the following diagram.

$$\begin{tikzcd}
& \tilde{X} \arrow[r, "\tilde{q}"] \arrow[d, "s"] & \tilde{Y} \arrow[d] & \\
& X \arrow[r, "q"] & Y & \\
A \arrow[uur, "k"] \arrow[ur, "i"] \arrow[rrr, "q'"] & & & C \arrow[ul, "l"'] \arrow[uul]
\end{tikzcd}$$

By Lemma 3.1.0.1 $\tilde{q}_* k_*(\tau')$ is trivial so $l_* q'_*(\tau')$ is trivial, and now by Lemma 3.1.0.2 $q'_*(\tau')$ is trivial. ☐

Theorem 3.1.0.2 *The following is exact for A a closed subset of compact metric space X.*

$$Ext(A) \xrightarrow{i_*} Ext(X) \xrightarrow{q_*} Ext(X/A)$$

Proof: To prove the exactness of the sequence as required simply take $Y = X/A$ and $C = q(A)$ in the above theorem along with Remark 3.0.0.1 . ☐

The following is a particular case of the Mayer-Vietoris sequence which was independently derived as a corollary of Theorem 3.1.0.2 in [O'D80].

Corollary 3.1.0.1 ([O'D80]) *Let X, Y be compact metric spaces such that Ext($X \cap Y$) =0. Let $(\mathcal{S}_i, \psi_i)$ be an extension by $\mathcal{C}(X_i), i = 1, 2$. Let $(\mathcal{S}, \psi)$ denote the sum of the two extensions. Then if $\mathcal{S}$ is of the form $\mathcal{N} + \mathcal{K}$, then so are both $\mathcal{S}_1, \mathcal{S}_2$.*

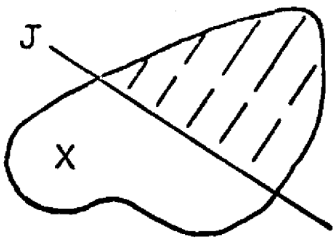
3.2 Proof of BDF Theorem

As mentioned earlier the original theory developed was heavily reliant on tools from algebraic topology and not very easy to develop. O'Donovan sought a remedy to this and presented a theorem whose proofs were intricate but set mostly in operator theoretic terms in [O'D80]. His theorem was equivalent to the original theorem formulated by Brown, Douglas and Fillmore. We shall follow this route by first showing the O'Donovan theorem and then proving its equivalence to the BDF Theorem.

3.2.1 O'Donovan's Theorem

Theorem 3.2.1.1 ([O'D80]) *Let $T \in \mathcal{N}_{\mathcal{E}}$ such that index($T - \lambda$) = 0 for $\lambda \notin X = \sigma_{ess}(T)$, then $T = N + K, N \in \mathcal{N}, K \in \mathcal{K}$.*

Proof: Assume the hypothesis and let J be a finite line passing through X. The following depiction has been taken from [O'D80].



In the above diagram, we shall denote by H_1, H_2, the left side and right side of $X \cup J$ respectively. Now let $(\mathfrak{B}, \psi)$ be an extension generated by T, refer to 2.2.1.

Let $T_1 \in C^*(\{T\})$ be such that $\psi(T_1)$ is a function in $\mathcal{C}(X)$ which is $f(z) = z$ for $z \in H_1$ and constant on lines prependicular to J in H_2. Further let N be such that $\sigma_{ess}(N) = X \cup J$. Since Ext($X \cup J$) is a group there exists $R, \sigma_{ess}(R) = X \cup J$, such that,

$$T_1 \oplus N \oplus R \sim T \oplus N \tag{3.1}$$

However, we now specificially demand N_1, R_1 such that $\psi(N_1) = \psi(R_1) = \psi(T_1)$, giving us

$$T_1 \oplus N_1 \oplus R_1 \sim T_1 \oplus N_1 \tag{3.2}$$

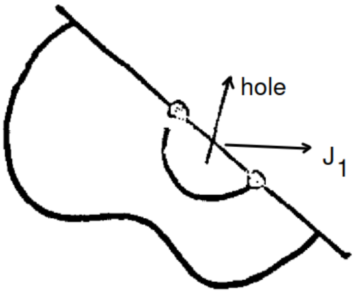
It should be noticed that by the explicit definition of mapping of ψ for T_1, N_1, R_1 both $T_1 \oplus N_1, R_1$ give extensions of $\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(H_1)$ if C*-algebras are generated through them respectively, and mapping taken is a restriction of ψ. Also $C^*(R_1)$ is of the form $N + K$ deducing from 3.2.

Consider the subalgebra of $C^*(R)$ which is constant on H_2, it can be seen that this is a subset of $\mathfrak{C}^*(R_1)$. Now apply Theorem 3.1.0.2 to $H_2 \to X \to X/H_2$ to find that $R \sim N_2 \oplus T_2$,such that

$$T \oplus N \sim T_1 \oplus N \oplus N_2 \oplus T_2 \sim T_1 \oplus T_2$$

where $\sigma_{ess}(T_1) = H_1, \sigma_{ess}(T_2) = H_2$

If λ is in a hole in in H_1 it is not in one in H_2 and vice versa, so index$(T_1 - \lambda) = 0, \lambda \notin H_1$ and index$(T_2 - \lambda) = 0, \lambda \notin H_2$. By a hole we mean as depicted in the image below, also taken from [O'D80].



Now we shall begin our job of removing the line J, firstly decompose J as

$$J = \bigcup_{i \in \mathbb{N}} J_i$$

where each J_i is an interval lying in one of the components of X^c.

Consider $\mathfrak{D}$, the subalgebra of $C^*(T_1)$ which is constant on $H_1 \backslash J_1$. It shall turn out this has essential spectrum a circle or a line. In both cases, it may be seen that $\mathfrak{D}$ can be written as Abelian + compacts, full details in [O'D80].

Now we apply Theorem 3.1.0.2 to pair $(H_1, H_1 \backslash J_1)$ to write $T_1 = T_{1,1} \oplus D_{1,1} + K_{1,1}$,such that $\sigma_{ess}(T_{1,1}) = H_1 \backslash J_1, \sigma_{ess}(D_{1,1}) = J_1, \|K_{1,1}\| < 1/2$. We similarly procced by induction with other J_i such that $\|K_{1,i}\| < 1/2^i$.

We shall repeat the process for T_2 with indexation pair $(2, i)$. Let,

$$D_J^1 = \bigoplus_{i \in \mathbb{N}} D_{1,i}, D_J^2 = \bigoplus_{i \in \mathbb{N}} D_{2,i}$$

Then we have,

$$T \oplus N \sim T_1 \oplus T_2 \oplus D_J^1 \oplus D_J^2$$

where $\sigma_{ess}(D_J^1) = \sigma_{ess}(D_J^2)$, so they are essentialy equivalent by Theorem 1.1 .

Decompose N as $N' \oplus D_J + K$ where $\sigma_{ess}(N') = X, D_J = D_J^1$. Since $X \cap J$ is homeomorphic to a compact subset of $\mathbb{R}$ so by Lemma 2.4.2 and Lemma 2.2.1 , Ext$(X \cap J) = 0$. Now by Corollary 3.1.0.1 , $T \sim T \oplus N^1 \sim T_1 \oplus T_2$. So,

$$C^*(T) + \mathcal{K}(\mathcal{H}) \subset \mathfrak{A}_1 = C^*(T_1) \oplus C^*(T_2) + \mathcal{K}(\mathcal{H})$$

Now do this whole splitting again inductively for T_1, T_2 and subsequent operators such that the diameter of corresponding spaces tends to zero. So, we obtain a chain of essentially normal C* algebras,

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \cdots \subset \mathfrak{A}_n \cdots \subset \mathfrak{K} = \cup \bar{\mathfrak{A}}_n$$

where $\sigma_{ess}(\mathfrak{K})$ is disconnected and by Lemma 2.4.1 Ext($\sigma_{ess}(\mathfrak{K})$) = 0. So $\mathfrak{K}$ is of the form $\mathcal{N} + \mathcal{K}$, it follows that $C^*(\{T\})$ must have the same form. □

3.2.2 Equivalence to BDF Theorem

We state below the original statement of the BDF theorem.

Theorem 3.2.2.1 *Let $T_1, T_2 \in \mathcal{N}_\mathcal{E}$ then $T_1 \sim T_2$ if and only if $\sigma_{ess}(T_1) = X = \sigma_{ess}(T_2)$ and $index(T_1 - \lambda) = index(T_2 - \lambda)$ for all $\lambda \in X^c$.*

A direct proof of this is given in [O'D80] using the following result stated in [BDF77].

Lemma 3.2.2.1 (Consequence of Berger-Shaw Theorem [BDF77]) *Given an operator T, we can obtain an operator R, with $\sigma_{ess}(T) = \sigma_{ess}(R)$ with arbitrary index in bounded component of $\sigma_{ess}(T)^c$.*

Theorem 3.2.2.2 *We now show that Theorem 3.2.2.1 if and only if Theorem 3.2.1.1 .*

Proof: Let $T \in \mathcal{N}_\mathcal{E}, \sigma_{ess}(T) = X$ and D be the diagonal operator with $\sigma_{ess}(D) = X$. By Theorem 3.2.2.1 , $T \sim D$ if and only if its index function is identically zero.

Conversely, by Remark 1.1 one side is clear. For the other assume $T_1, T_2 \in \mathcal{N}_\mathcal{E}$ such that

$$\sigma_{ess}(T_1) = X = \sigma_{ess}(T_2)$$

$$index(T_1 - \lambda) = index(T_2 - \lambda) = k_\lambda \text{ for all } \lambda \in X^c.$$

Then, by Lemma 3.2.2.1 we can find an operator $R, \sigma_{ess}(R) = X$, such that $T_i \oplus R$ has index 0 in each hole of the essential spectrum. So by Theorem 3.2.1.1 , both $T_1 \oplus R, T_2 \oplus R$ is of the form $\mathcal{N} + \mathcal{K}$, hence,

$$T_1 \oplus R \sim T_2 \oplus R$$

Now since Ext(X) is a group let R^- be an inverse for R, then,

$$T_1 \sim T_1 \oplus R \oplus R^- \sim T_2 \oplus R \oplus R^- \sim T_2$$

□

Epilogue

The reader may have noticed that we have tried to use operator-theoretic proofs as much as possible. However, the original paper [BDF73] used tools from algebraic topology. Even though simpler proofs exist, the theory itself is remarkable and deserves to be read for further abstract generalizations. The primary book used by me to understand this was [CM21], which acts as an excellent supplement to the papers. Chapter 2 of this book covers what we have presented in 2.1-2.3, while 2.4 is adapted from [BDF77]. Similarly, 3.1 has been borrowed from [BDF77] and section 3.5 in [CM21]. For 3.2 [O'D80] was used.

It should be noted that the theorem has had an impact on various areas of operator theory. BDF makes several results in Operator Theory tractable, such as ones relating to Bergman Operators, Hyponormal Operators, and m-Isometries [CM21]. Furthermore, the BDF theory gives us a concrete realization of Ext(X) for X planar. However, in many cases, X is not planar and it may not always be a group [And78]. Further there remain unanswered questions regarding Essentially Normal Tuples, Essentially Homogenous Tuples, and the Arveson-Douglas Conjecture [CM21], [BDF73], [BDF77].

Appendix

Theorem A.1 (Fuglede-Putnam-Rosenblum [Ros58], [Noa]) *Let $\mathcal{A}$ be a unital C*-algebra. Let two normal elements $a, b \in A$ be given and $c \in \mathcal{A}$ with $ac = cb$. Then it follows that $a^*c = cb^*$.*

Theorem A.2 ([FSW72, Section 1]) *Let T be a closed linear transformation with a dense domain in a complex Hilbert space H. Then range is non-closed or kernel is inifinite dimensional if and only if there exists an orthonormal sequence $\{\xi_n\}$ in domain of T such that $T\xi_n \to 0$ strongly.*

Theorem A.3 ([CM21, Lemma 2.1.1]) *Let (X, d) be a compact metric space. Suppose $\{x_n\}$ and $\{y_n\}$ are sequence contained in L such that there limit points are also in L. Then for $\epsilon > 0$, theres exists a permutation $\eta : \mathbb{N} \to \mathbb{N}$, such that*

$$\sum_{n=0}^{\infty} d(x_n, y_{\eta(n)}) < \epsilon$$

Theorem A.4 (Arveson Lemma [CM21, Lemma 2.6.2]) *If $\mathfrak{C}$ is an abelian C*-algebra generated by a countable family of orthogonal projections, then the maximal ideal space $\mathfrak{M}$ of $\mathfrak{L}$ is totally disconnected. Moreover, $\mathfrak{L}$ has a single self-adjoint generator.*

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