Combinatorics

Pelatnas 1 TOKI 2016

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IOI Syllabus

- Sum & Product Rule
- Permutations & Combinations
- Binomial Coefficients
- Inclusion-exclusion Principle
- Pigeonhole Principle
- Pascal's Identity, Binomial Theorem

- http://people.ksp.sk/~misof/ioi-syllabus/ioi-syllabus.pdf

Sum & Product Rule

- Sum Rule: If A and B are disjoint, then
$$|A \cup B| = |A| + |B|$$

- Product Rule:
$$|A \times B| = |A| \times |B|$$

If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

Sum & Product Rule

- OSK Problem:
  - Find the number of even three-digits integer such that all of the digits are different, and it is greater than 400

Permutations & Combinations

> In mathematics, the notion of **permutation** relates to the act of rearranging, or permuting, all the members of a set into some sequence or order (unlike combinations, which are selections of some members of the set where order is disregarded).
>
> Permutation - Wikipedia, the free encyclopedia
> en.wikipedia.org/wiki/**Permutation** Wikipedia

- $n!$ ways to permute n objects

Permutations & Combinations

- Combinations:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 1}$$

- Combination with repetitions:

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

  – The number of nonnegative integer solution to

$$x_1 + x_2 + \cdots + x_n = k$$

Pascal’s Identity

- In selecting k objects, there are two ways:
  - Select the first object
  - Don’t select the first object

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- Tadaaa!

Pascal's Identity

- Version 1

```
for (int i = 0; i <= MAXN; i++) {
    for (int j = 0; j <= i; j++) {
        c[i][j] = (j && j < i) ? c[i-1][j-1] + c[i-1][j] : 1;
    }
}
```

- Version 2

```
for (int i = 0; i <= MAXN; i++) {
    for (int j = 0; j <= MAXN; j++) {
        c[i][j] = (i && j) ? c[i-1][j] + c[i][j-1] : 1;
    }
}
```

Binomial Coefficients

- (tulis di papan aja deh)

Multinomial Coefficients

- The number of ways to partition n objects into k classes:

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \dots n_k!}$$

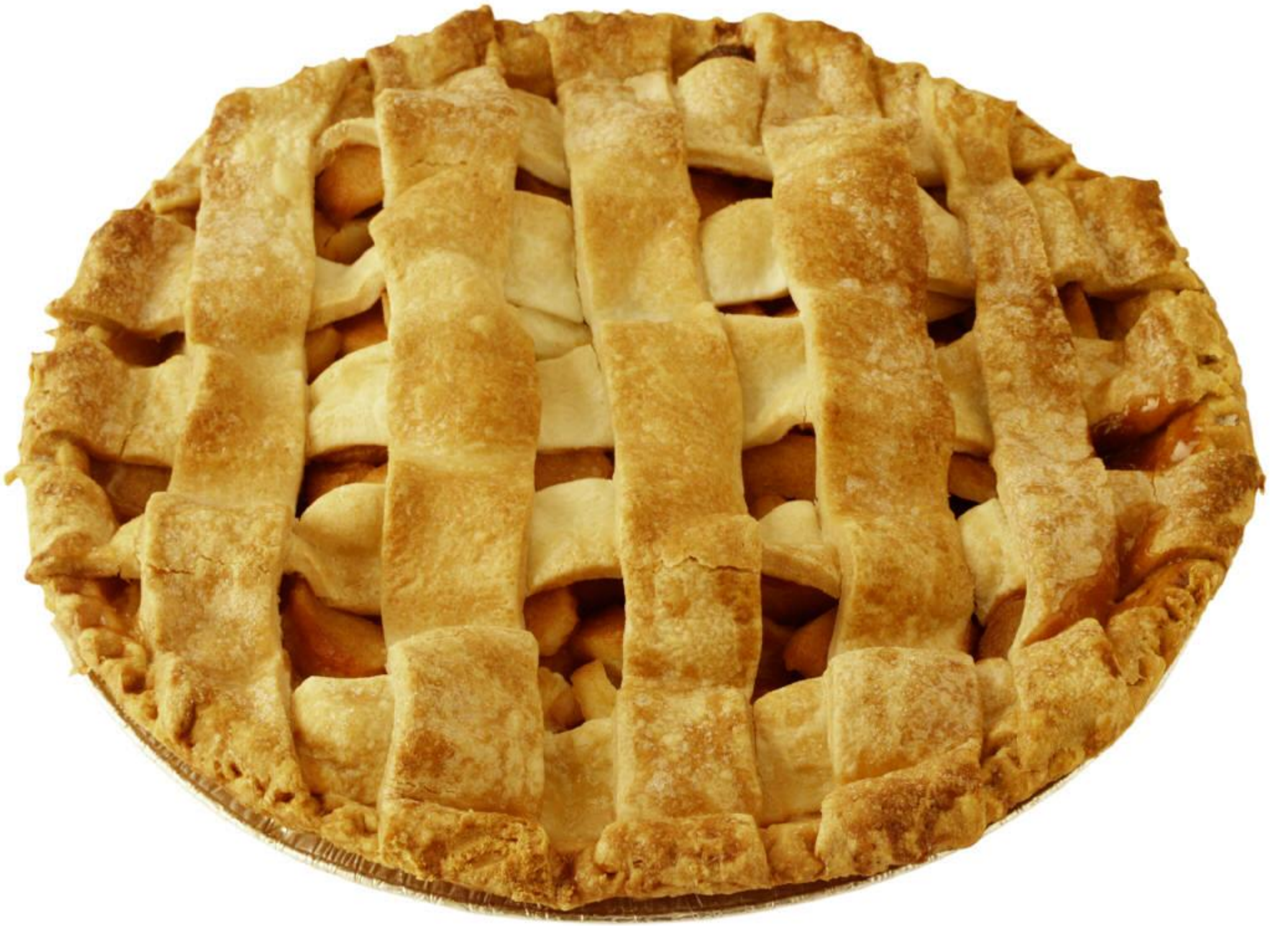
- Another formula:

$$\binom{n}{n_1, n_2, \dots, n_k} = \binom{n}{n_k} \times \binom{n - n_k}{n_1, \dots, n_{k-1}}$$

Many Ways to Calculate $\binom{n}{k}$

- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \rightarrow O(nk)$
- $\binom{n}{k} = \frac{n}{k} \times \binom{n-1}{k-1} \rightarrow O(k)$
- $\frac{n!}{k!(n-k)!} \rightarrow O(1)$ atau $O(\log P)$

Inclusion-exclusion Principle



Inclusion-exclusion Principle

- $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-exclusion Principle

- General form:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} (-1)^{|J|-1} \left|\bigcap_{j \in J} A_j\right|$$

Pigeonhole Principle

Problem: Card Derangement

- How many ways of shuffling n cards such that no card is in the correct position?

Problem: Kesetiawakanan Lebah (P1 2015)

- Find a subset of $\{a_1, a_2, \ldots, a_n\}$ for which their sum is divisible b $(n \geq b)$
- What if $n < b$?

Problem: Play with GCDs (IEEExtreme 8.0)

- Find the number of subset of A such that the greatest common divisor of its elements is equal to X
- $|A|, A_i, X \leq 10000$

Problem: Playing with Boxes (INC 2010)

- (tulis di papan aja deh)

Appendix: Generating Function

- $\frac{1}{1-x} = 1 + x + x^2 + \cdots$
- $\frac{1}{1-x^n} = 1 + x^n + x^{2n} + \cdots$
- $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$

Appendix: Generating Function

- How many nonnegative integer solutions to the equation $x_1 + \cdots + x_n = k$?
  - Coefficient of x^k of $\frac{1}{(1-x)^n} = \binom{n+k-1}{k}$
- There are n boxes, each of them contains A_i balls, and balls from one box have the same color. How many ways to take k balls?
  - Similar to former problem