Himpunan, Fungsi dan Logika

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Sets

- Group of objects represented as a unit
- May contain any type of object: numbers, symbols, other sets,…
  - Set membership: $\in$
  - Non-membership: $\notin$
  - Subset: $\subseteq$
    - Proper subset
  - Empty set: $\emptyset$
  - Infinite set contains infinitely many elements
    - E.g., set of integers {…-2, -1, 0, 1, 2,…}

Sets

- Common sets:
  - **N** the set of natural numbers = {1,2,3,...}
  - **Z** the set of integers = {...,-3,-2,-1,0,1,2,3,...}
  - **Q** the set of rational numbers = {0.25, 1/7, 0.3333..., etc.}
  - **R** the set of real numbers = {-2.4, sqrt(5), Pi, etc.}
- To denote the set of all objects of discourse (universal set) we use **U.**

Set Operations

Given two sets A and B, we can also define new sets by applying ***set operations***:

- The ***intersection*** of sets A and B is the set

  $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

- The ***union*** of sets A and B is the set

  $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

- The ***difference*** between two sets A and B is the set

  $A - B = \{x : x \in A \text{ and } x \notin B\}$.

Set Operations

- The complement of a set is defined as

$$A^C = U - A = \{x \mid x \notin A\}$$

- E.g., if **U = N**, and A = the set of all even natural numbers, then A^C = the set of all odd natural numbers.

Set Operations

- Set Operations obey the following laws

Empty set laws:

$A \cap \emptyset = \emptyset$,

$A \cup \emptyset = A$.

Idempotency laws:

$A \cap A = A$,

$A \cup A = A$.

Commutative laws:

$A \cap B = B \cap A$,

$A \cup B = B \cup A$.

Set Operations

Associative laws:

$A \cap (B \cap C) = (A \cap B) \cap C,$

$A \cup (B \cup C) = (A \cup B) \cup C.$

Distributive laws:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C). \end{aligned}$$

Absorption laws:

$A \cap (A \cup B) = A,$

$A \cup (A \cap B) = A.$

DeMorgan's laws:

$$\begin{aligned} A - (B \cap C) &= (A - B) \cup (A - C), \\ A - (B \cup C) &= (A - B) \cap (A - C). \end{aligned}$$

Set Operations

- Laws for complement

$$(A^C)^C = A$$

$$A^C \cap A = \varnothing$$

$$A^C \cup A = U$$

- Write De Morgan's Law for complement!

Pairwise Disjoint & Partition

- Two sets A and B are called pairwise disjoint if they share no common element, i.e.,

$$A \cap B = \emptyset$$

- A collection of sets, S_i $1 \le i \le n$ is called **a partition** of a set **S**, iff:
  - Each pair of sets in the collection are pairwise disjoint, i.e., $S_i \cap S_j = \emptyset,\ \text{if } i \neq j$
  - The Union of all the sets equals S

Cardinality

- Cardinality of a set is the number of (distinct) elements that the set has
  - Cardinality of A is denoted by |A| or card(A)

$$|\emptyset| = 0$$

$$|A| \geq 0$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B| \leq |A| + |B|$$

with equality if A and B are disjoint

Subsets, Proper Subsets

- The subset B of a set A is all sets such that
  - if $x \in B$ then $x \in A$
  - Denoted by $B \subseteq A$
- A proper subset B of A is a subset B of A such that B≠A, denoted by B Ø A

$\varnothing \subseteq A$ for any A

$|B| \leq |A|$ if $B \subseteq A$

Power Sets

- The Power set of a set A is the set of all subsets of A.
  - Denoted by 2^X
- E.g., A = {a, b, c}, then

  $2^X = \{\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$
- What is the value of $|2^X|$?

Principle of Inclusion-Exclusion

- Simple form

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- General form

$$|A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| - |A_1 \cap A_2| - \ldots + |A_1 \cap A_2 \cap A_3| + \ldots + (-1)^{n+1} |A_1 \cap \ldots A_n|$$

Problems

1. Prove that : R ∪ (S ∩ T) = (R ∪ S) ∩ (R ∪ T)
2. Determine the number of natural numbers 1 to 1000, which are not divisible by 2, 3 and 5
3. Let S = {1,2,...,100}. Determine the cardinality of the the largest subset T of S such that $x \in T \Rightarrow 2x \notin T$

Generalize and formalize into an algorithm!

Cartesian Product and Relations

Cartesian product builds a set consisting of ordered pairs of elements from two or more existing sets.
$X \times Y = \{[x, y] \mid x \in X \text{ and } y \in Y\}$

A **binary relation** on sets X and Y is a subset of $X \times Y$.

An **n-ary relation** on sets $X_1, X_2, \ldots, X_n$ is a subset of $X_1 \times X_2 \times \ldots \times X_n$

What is the cardinality of $X \times Y$?

Relations

- If $(a,b) \in R$, we write aRb
- A relation is **reflexive** iff for any *a*, aRa
- A relation is **symmetric** iff $aRb \Rightarrow bRa$
- A relation is **transitive** iff aRb and $bRc \Rightarrow aRc$
- A relation which is reflexive, symmetric and transitive is called the **equivalence** relation

Problem

1. Which ones are reflexive, symmetric or transitive?
   - $<$, $>$, $\leq$, $\neq$, $=$, $=$ (mod), $\subseteq$, divisor of, has common divisor
   - Child of, brother of, knows, friend of, ancestor of, likes, married to
2. Which ones are equivalence relation?
3. Is it true: symmetric + transitive ➔ reflexive?

Functions

A **function** from a set X to a set Y is a mapping of elements of X to elements of Y such that each element of X maps to exactly one element of Y.

$f : X \to Y$

X is the **domain** of *f*.

The **range** of $f : X \to Y$ is the set $\{y \in Y \mid y = f(x)$ for some $x \in X\}$.

A **total function** *f* from X to Y is a binary relation on

$X \times Y$ such that

i. For each $x \in X$ there is a $y \in Y$ such that $[x, y] \in f$.
ii. If $[x, y] \in f$ and $[x, z] \in f$, then $y = z$.

Functions

- A function is **surjective** iff its range is its codomain
- A function is **injective** iff $a \neq b \Rightarrow f(a) \neq f(b)$
- A function that is both surjective and injective is called **bijective**
- If a function **f** is bijective, it has an inverse, denoted by $\mathbf{f^{-1}}$**,** such that whenever f(a) = b, $f^{-1}(b) = a$

Problem

1. Create a bijective function from **N** (the set of natural numbers) to the set of all odd natural numbers!
2. Create a bijective function from **N**x**N** to **N** and determine its inverse!

Logika

- Logika proposional: komponen dasar = proposisi
- Bernilai benar (1) atau salah (0)
- Pernyataan kompleks disusun dari beberapa proposisi dan operator
- Makna (semantik) sebuah pernyataan kompleks adalah sebuah fungsi dari semantik masing-masing proposisi komponen penyusunnya dan operator-operator yg digunakan

Operator dasar

Nama Operator	Pola	Notasi	Istilah Inggris
Negasi	"Tidak benar adanya bahwa p."	$\neg p$	*not*
Konjungsi	"p dan q."	$p \wedge q$	*and*
Disjungsi inklusif	"p atau q."	$p \vee q$	*or*
Disjungsi eksklusif	"p atau q (tapi tidak keduanya)."	$p \oplus q$	*xor*
Implikasi	"Jika p, maka q."	$p \rightarrow q$	*implies*

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$
B	B	S	S	B	B	S	B
B	S	S	B	S	B	B	S
S	B	B	S	S	B	B	B
S	S	B	B	S	S	S	B

Problem

- Buatlah Tabel kebenaran untuk setiap pernyataan berikut

$$p \wedge \neg q$$
$$\neg(p \vee \neg q)$$
$$p \rightarrow (q \vee \neg p)$$
$$(p \wedge \neg q) \rightarrow \neg r$$

Validity, Contradiction, Satisfiability

- A proposition is **valid** iff in every model, it is true
- A proposition is a **contradiction** iff in every model, it is false
- A proposition is said to be **satisfiable** iff in **some** model, it is true

Equivalensi

- A proposition P are equivalent to Q iff for every model in the truth table, the truth value of P is equal to the truth value of Q
- E.g.,

  $\neg p \vee q \equiv p \rightarrow q$

  $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $P \equiv Q$ iff P $\leftrightarrow$ Q is valid

Equivalences

$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$ commutativity of $\wedge$
$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$ commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma))$ associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma))$ associativity of $\vee$
$\neg(\neg\alpha) \equiv \alpha$ double-negation elimination
$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha)$ contraposition
$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \vee \beta)$ implication elimination
$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$ biconditional elimination
$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$ De Morgan
$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$ De Morgan
$(\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$ distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$ distributivity of $\vee$ over $\wedge$

Problem

1. Misalkan diketahui proposisi-proposisi berikut:
   – p = Andi memacu motornya lebih dari 100 km/jam
   – q = Andi kena tilang

Nyatakan pernyataan-pernyataan berikut sebagai proposisi dalam p dan q

- Andi memacu motornya maksimal 100 km/jam
- Andi memacu motornya lebih dari 100 km per jam, tapi dia tidak ditilang.

Problem

- Andi akan ditilang jika dia memacu motornya lebih dari 100 km per jam.
- Jika Andi tidak memacu motornya lebih dari 100 km per jam, maka dia tidak ditilang.
- Memacu motor lebih dari 100 km per jam adalah syarat cukup untuk ditilang bagi Andi.
- Andi ditilang, tapi dia tidak memacu motornya lebih dari 100 km per jam.
- Kapanpun Andi ditilang, dia memacu motornya lebih dari 100 km per jam.

Problem

- Empat buah kartu dapat ditulis sebuah huruf atau sebuah angka pada masing-masing sisinya
- Di atas meja terdapat empat kartu, masing-masing bertuliskan “1”, “A”, “2” dan “Z”.
- Saya berkata: “Jika sisi sebuah kartu bertuliskan angka genap, maka di sisi lainnya pasti bertuliskan huruf vokal”
- Kartu yang mana di antara keempat kartu di meja yang perlu Anda balik untuk memeriksa kebenaran pernyataan saya?

Inference

- Standard Inference method: Ponens, Tollens and Syllogism
- Ponens:

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Tollens

- Contrapositive inference

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Syllogism

- Transitive inference

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

General Entailment

- A set of propositions K entails a proposition P, denoted $K \models P$ iff
  - For every model for which K is true, P is also true

E.g.,

$$K = \{p \vee q, q \rightarrow r\}$$
$$P = \{\neg p \rightarrow r\}$$

Prove it!

Predicate logic (FOL)

- Components of FOL

Constants	$KingJohn,\ 2,\ UCB, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Rightarrow\ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Universal Quantification

$\forall \langle variables \rangle \; \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \;\; At(x, Berkeley) \;\Rightarrow\; Smart(x)$

$\forall x \;\; P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$\begin{array}{l} \quad (At(KingJohn, Berkeley) \;\Rightarrow\; Smart(KingJohn)) \\ \wedge \; (At(Richard, Berkeley) \;\Rightarrow\; Smart(Richard)) \\ \wedge \; (At(Berkeley, Berkeley) \;\Rightarrow\; Smart(Berkeley)) \\ \wedge \; \ldots \end{array}$$

Universal Quantification

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \;\; At(x, Berkeley) \wedge Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

Existential Quantification

$\exists \langle variables \rangle \ \langle sentence \rangle$

Someone at Stanford is smart:
$\exists x \ \ At(x, Stanford) \wedge Smart(x)$

$\exists x \ \ P$ is true in a model m iff P is true with x being **some** possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\
\vee \ & (At(Richard, Stanford) \wedge Smart(Richard)) \\
\vee \ & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\
\vee \ & \ldots
\end{aligned}
$$

Quantification contd.

Pernyataan	Kapan benar?	Kapan salah?
$\forall x.P(x)$	$P(x)$ benar untuk setiap x	Ada x sehingga $P(x)$ salah
$\exists x.P(x)$	Ada x sehingga $P(x)$ benar	Untuk semua x $P(x)$ salah
$\forall x.\forall y.P(x,y)$ $\forall y.\forall x.P(x,y)$	Untuk setiap pasang x dan y, $P(x,y)$ benar	Ada setidaknya sepasang x dan y sehingga $P(x,y)$ salah
$\forall x.\exists y.P(x,y)$	Untuk setiap x, ada setidaknya satu y sehingga $P(x,y)$ benar	Ada setidaknya satu x, sehingga $P(x,y)$ salah, apapun y
$\exists x.\forall y.P(x,y)$	Ada setidaknya satu x, sehingga $P(x,y)$ selalu benar, apapun y	Untuk semua x, selalu ada y sehingga $P(x,y)$ salah
$\exists x.\exists y.P(x,y)$ $\exists y.\exists x.P(x,y)$	Ada sepasang x dan y sehingga $P(x,y)$ benar	Untuk setiap pasang x dan y, $P(x,y)$ salah

Examples

- Example
  - Constants = {Andi, Budi, Cici, Erna}
  - Predicates/Relations = {like/2, knows/2, male/1, female/2}
  - Variabels = {x, y, ...}
- Sentences

$like(Andi, Cici) \quad \forall x, knows(Andi, x) \quad \forall x, like(x, Andi)$

$\exists x, knows(Andi, x) \quad \forall y \exists x, like(y, x) \quad \neg female(Andi)$

Quantifiers contd.

$\forall x\ \forall y$ is the same as $\forall y\ \forall x$ (why??)

$\exists x\ \exists y$ is the same as $\exists y\ \exists x$ (why??)

$\exists x\ \forall y$ is **not** the same as $\forall y\ \exists x$

$\exists x\ \forall y\ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y\ \exists x\ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x\ Likes(x, IceCream) \qquad \neg\exists x\ \neg Likes(x, IceCream)$

$\exists x\ Likes(x, Broccoli) \qquad \neg\forall x\ \neg Likes(x, Broccoli)$

Problems

1. Tentukan negasi dari masing-masing pernyataan ini
   - Semua orang yang sukses pasti rajin atau beruntung
   - Ada orang yang jika tidak tidur maka pada siang harinya tidak merasa mengantuk
   - $\forall x(p(x) \vee q(x)) \rightarrow \neg r(x)$
   - $\forall x \exists y, [p(x, y) \rightarrow (q(x, x) \vee r(y, x))]$