CS4231
Parallel and Distributed Algorithms

Solution for Homework 3

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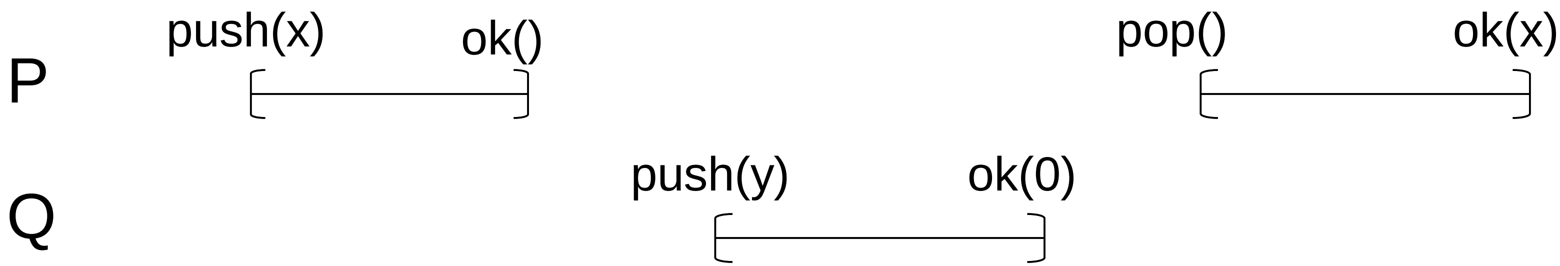
Homework Assignment

- Page 62:
  - Problem 4.1
  - If you believe a history is linearizable/serializable, give the equivalent sequential history
  - If you believe a history is not linearizable/serializable, prove that no equivalent sequential history exists

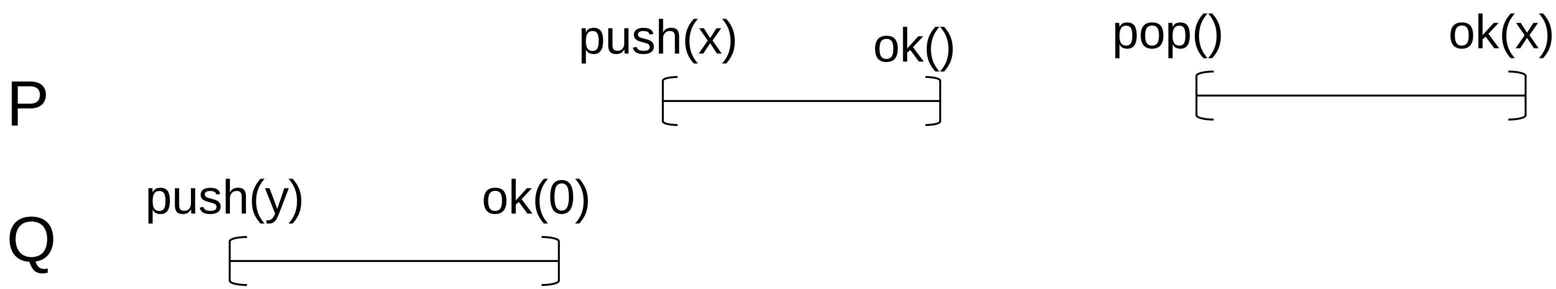
- Slide 26: Prove that linearizability is a local property using the definition on slide 26 (formalize the definition first)

- Think about Slide 21: Prove that the two definitions of linearizability are equivalent

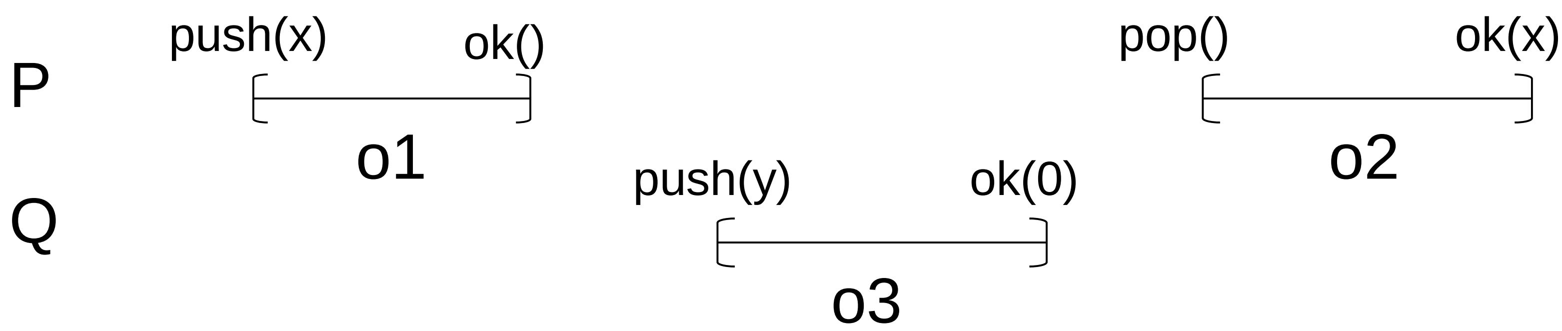
Problem 4.1 (a)



sequentially consistent because equivalent to:



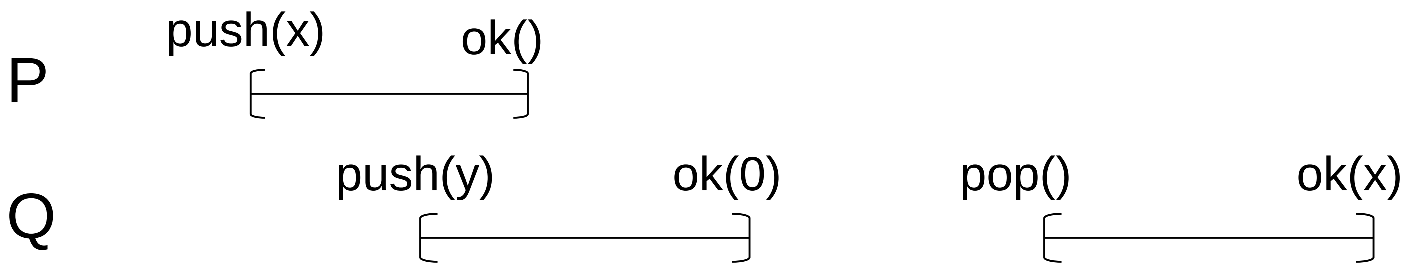
Problem 4.1 (a)



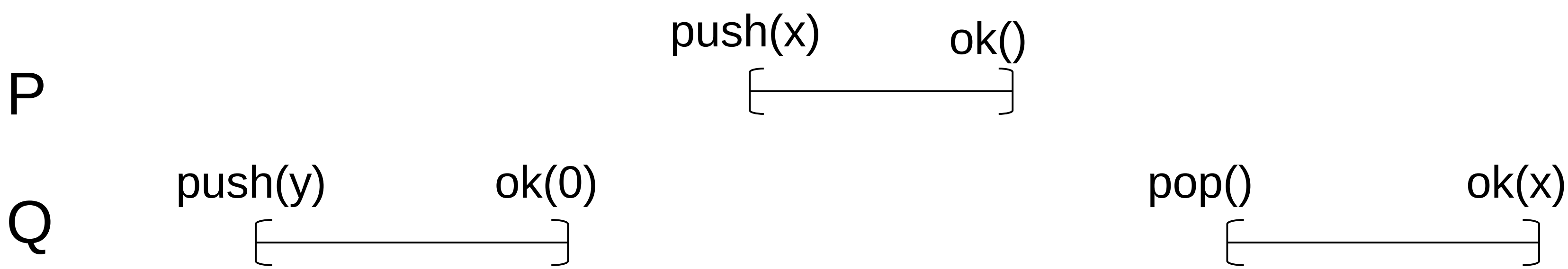
Not linearizable

- Method 1: Enumerate all 3! possible sequential histories
- Method 2: o1 < o2, o1< o3, o3 < o2, so the only sequential history that preserves the partial order induced by H is o1 o3 o2 – But o1 o3 o2 not equivalent to H

Problem 4.1 (b)



sequentially consistent and linearizable because equivalent to:



Proving Linearizability is a Local Property

- Definition #1: The execution is equivalent to some execution such that each operation happens instantaneously at some point of time (called linearization point) between the invocation and response

- Prove that H is linearizable if and only if H | x is linearizable for all object x
  - "only if" part is trivial
  - "if" part: Since H | x is linearizable, we will be able to assign linearization points to the operations in H | x and obtain an equivalent execution. Now we want to show that H is linearizable, by assigning linearization point to each operation in H. For each operation, if the operation is on object x, we simply use the linearization point for that operation when we linearize H | x.
  - One can easily show that the assigned linearization points satisfy the properties needed by the definition…

Prove two definitions for linearizability are the same

- Definition #1: The execution is equivalent to some execution such that each operation happens instantaneously at some point between the invocation and response
- Definition #2: History H is linearizable if
  - H is equivalent to some legal sequential history S, and
  - S preserves the external order in H
- If H is linearizable by Definition #1, then H is linearizable by Definition #2. Proof:
  - By ordering all operations by their linearization points, we have a sequential history S. Furthermore, S is legal.
  - Need to show the external order induced by H is preserved in S. If O1 is before O2 in external order, then O1's response is before O2's invocation. Since a linearization point is always between invocation and response, O1's linearization point must be before O2's. Hence O1 is before O2 in S.

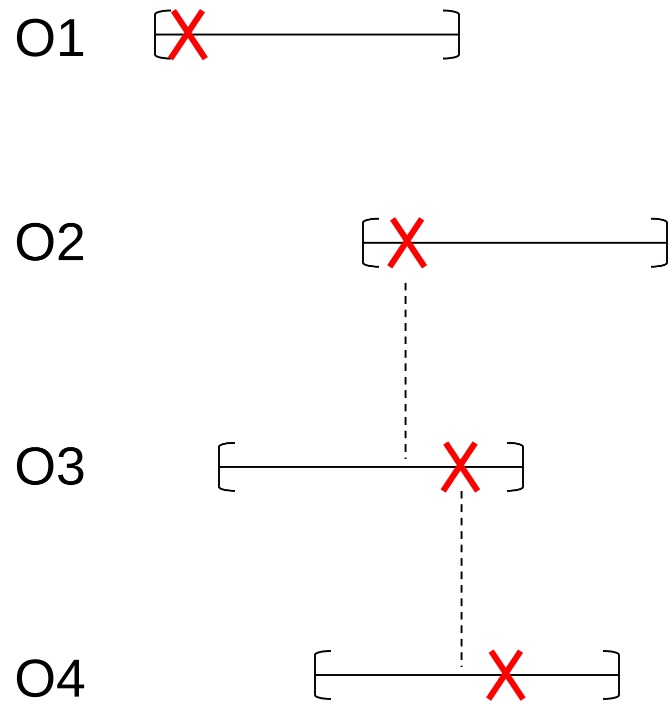
Prove two definitions for linearizability are the same

- Definition #1: The execution is equivalent to some execution such that each operation happens instantaneously at some point between the invocation and response
- Definition #2: History H is linearizable if
  - H is equivalent to some legal sequential history S, and
  - S preserves the external order in H

- If H is linearizable by Definition #2, then H is linearizable by Definition #1. Proof:
  - We already have an equivalent legal sequential history S
  - We need to assign linearization points to operations

Prove two definitions for linearizability are same

- Let the legal sequential history S be O1 O2 O3 …, with n operations total
- Order all invocation and response by their time, and let $T > 0$ be the smallest gap between any two consecutive events
- We assign the linearization point of the operations one by one:
  - O1's linearization point lp(O1) is inv(O1) + T/n
  - lp(O2) is max{ inv(O2), lp(O1) } + T/n
  - lp(O3) is max{ inv(O3), lp(O2) } + T/n
  - lp(O4) is max{ inv(O4), lp(O3) } + T/n
  - ….
- Blue parts only serve to ensure that lp(O1) ≠ lp(O2) ≠ lp(O3) …

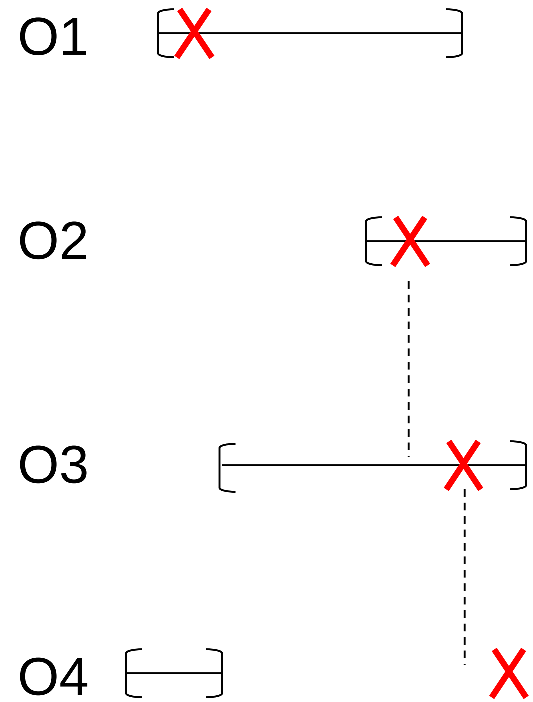


Prove two definitions for linearizability are same

- We have lp(O1) < lp(O2) < lp(O3) < lp(O4) ….
- We must have
  - Inv(O1) < lp(O1)
  - Inv(O2) < lp(O2)
  - Inv(O3) < lp(O3)
  - …
- We must have lp(O1) < resp(O1), by the definition of T.
- Next, need to show
  - lp(O2) < resp(O2)
  - lp(O3) < resp(O3)
  - …

Prove two definitions for linearizability are same

- Prove by contradiction, suppose the above does not hold.
- Let $i \geq 2$ be such that lp(O_i) > resp(O_i)
- Let $j < i$ be the largest j such that lp(O_j) = inv(O_j) + T/n. Such j must exist (it is important to prove this).
- lp(O_i) = lp(O_j) + $(i - j)$T/n ≤ lp(O_j) + (n–1)T/n
- Hence lp(O_i) > inv(O_j) = lp(O_j) – T/n ≥ lp(O_i) – T (we will use this inequality later)



- inv(O_j) and resp(O_i) are at least T apart:
- If inv(O_j) ≥ resp(O_i) + T, then O_i is before O_j in external order. But in S, O_j is before O_i . This means that S does not preserve external order. Contradiction.
- If inv(O_j) ≤ resp(O_i) – T, then we have resp(O_i) – T ≥ inv(O_j) ≥ lp(O_i) – T and hence resp(O_i) ≥ lp(O_i). Contradiction.

i = 4 and j = 2 in this example