1.

$$
\begin{aligned}
g_1(n) &= \Theta(n!) \\
g_2(n) &= \Theta(n^3 \lg n) \\
g_3(n) &= \Theta(n^{\lg n}) \\
g_4(n) &= \Theta(\sqrt{n}) \\
g_5(n) &= \Theta(2^{\lg n}) \\
g_6(n) &= \Theta(\lg^3 n) \\
g_7(n) &= \Theta(2^n) \\
g_8(n) &= \Theta(n^3 \lg \lg n)
\end{aligned}
$$

$$g_6 << g_4 << g_5 < < g_8 << g_2 << g_3 << g_7 << g_1$$

2. (a)

$$
\begin{aligned}
T(n) &= 4 \cdot T(n/2) + \Theta(n^2 \lg^3 n) \\
a &= 4 \\
b &= 2 \\
f(n) &= \Theta(n^2 \lg^3 n) \\
n^{\log_b a} &= n^{\log_2 4} \\
&= n^2 \\
Case2 &: \\
f(n) &= \Theta(n^{log_2 4} \lg^1 n) \\
T(n) &= \Theta(n^2 \lg^2 n)
\end{aligned}
$$

(b)

$$T(n) = 3 \cdot T(n/2) + 5n$$

$$
\begin{array}{ccccccccc}
 & & & & T(n) + 5n & & & & \\
 & T(n/2) + \frac{5n}{2} & & & T(n/2) + \frac{5n}{2} & & & T(n/2) + \frac{5n}{2} & \\
... & ... & ... & ... & ... & ... & ... & ... & ... \\
| & | & | & | & | & | & | & | & | \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

$$
\begin{aligned}
T(n) &= 1 \cdot 3^{\lg n} + 5n + \frac{3}{2}^1 5n + \frac{3}{2}^2 5n + \dots \\
T(n) &= 3^{\lg n} + 5n \cdot \sum_{i=0}^{\lg n - 1} \frac{3}{2}^i \\
T(n) &= ((2^{\lg 3})^{\lg n}) + 5n \cdot \frac{\frac{3}{2}^{\lg n} - 1}{\frac{3}{2} - 1} \\
T(n) &= 2^{\lg n \cdot \lg 3} + 10n \cdot (\frac{3^{\lg n}}{2^{\lg n}} - 1) \\
T(n) &= n^{\lg 3} + 10n \cdot (\frac{n^{\lg 3}}{n} - 1) \\
T(n) &= 11n^{\lg 3} - 10n \\
T(n) &= O(n^{\lg 3})
\end{aligned}
$$

(c)

$$
\begin{aligned}
U(n) &= \frac{2}{n-1}\sum_{k=1}^{n-1} U(k) + 5n \\
&= \frac{2}{n-1}U(n-1) + 10 + \frac{2}{n-1}\sum_{k=1}^{n-2} U(k) + 5(n-2) \\
&= \frac{2}{n-1}U(n-1) + \frac{n-2}{n-1}U(n-1) + 10 \\
&= (\frac{2}{n-1} + \frac{n-2}{n-1})U(n-1) + 10 \\
U(n+1) &= (\frac{2}{n} + \frac{n-1}{n})U(n) + 10 \\
&= (\frac{n+1}{n})U(n) + 10 \\
&= (\frac{n+1}{n})((\frac{n}{n-1})U(n-1) + 10) + 10 \\
&= n + 1 + 10(1 + \frac{n+1}{n} + \frac{n+1}{n-1} + \cdots + \frac{n+1}{1}) \\
&= n + 1 + 10(1 + (n+1) * \sum_{x=1}^{n} \frac{1}{x}) \\
&= n + 1 + 10(1 + (n+1) * \ln n) \\
&= n + 1 + 10 + 10n\ln n + 10\ln n \\
&= O(n\log n)
\end{aligned}
$$

3. (a)

$$f(n) \leq O(g(n)) < o(g(n))$$
$$f(n) \leq kn^{c-\epsilon} < k'n^c$$
$$\text{Proving } kn^{c-\epsilon} < k'n^c :$$
$$\lim_{n\to\infty} \frac{n^c}{n^{c-\epsilon}} = \lim_{n\to\infty} n^\epsilon \to \infty$$
$$\infty > \frac{k}{k'} \text{ because k \& k' is finite}$$
$$\text{Proven}$$

(b)

$$\text{let } f(n) = \frac{n^c}{\lg n}$$
$$f(n) \in o(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{n^c}{\lg n}}{n^c}$$
$$= \lim_{n\to\infty} \frac{1}{\lg n} \to 0$$

$$f(n) \in O(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{n^c}{\lg n}}{n^{c-\epsilon}}$$
$$= \lim_{n\to\infty} \frac{\frac{1}{\lg n}}{n^{-\epsilon}}$$
$$= \lim_{n\to\infty} \frac{n^\epsilon}{\lg n}$$
$$= \lim_{n\to\infty} \frac{\epsilon n^{\epsilon-1}}{\frac{1}{n}} \text{ (L'Hopital)}$$
$$= \lim_{n\to\infty} \epsilon n^\epsilon \to \infty$$
$$f(n) \notin O(g(n))$$
$$\text{Disproven}$$