Solution of Fuzzy Differential Equations using Fuzzy Sumudu Transforms

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***Abstract*—This paper highlights the concepts related to the fuzzy Sumudu transform (FST). Few important theorems are illustrated for uncovering the properties of FST. By utilizing the generalized FST, the fuzzy differential equations (FDEs) are resolved. The suggested technique is validated by laying down two real examples.**

***Keywords*—fuzzy Sumudu transform; fuzzy differential equation; fuzzy solution**

I. Introduction

Several physical and dynamical process can be modeled as a deterministic initial and boundary value problems. In practical the boundary values may not be crisp and they have some unknown parameters [30]. The qualitative behavior of solutions of the equations are associated with the errors. If the errors are random, then we have stochastic differential equation with the random boundary value. Also, if the errors are not probabilistic, the fuzzy numbers are replaced by random variables [20], [30]. The fuzzy derivative and FDE have been studied in [9], [12]. The Peano-like theorems for FDEs and system of FDE on R (Real line) is demonstrated in [21]. The fuzzy first-order initial value problem and fuzzy partial differential equation have been discussed in [21]. The simulation of fuzzy system is illustrated in [16], [17], [18], [19]. The application of numerical method in order to solve FDEs has been mentioned in [15]. The properties associated to FDE is analyzed by using H-differentiability. The condition of Lipschitz along with the existence and uniqueness theorem related to FDEs have been mentioned in [2], [13], [28]. The fractional fuzzy Laplace transformation has been discussed in [2].

An innovative methodology for solving FDEs is laid down on the basis of Sumudu transform. Sumudu transform with wide spread applications has been applied in the field of mathematics and control engineering. In recent advancement, Sumudu transform is popularized to resolve fractional local differential equations [3], [8], [11], [14], [26]. In [1], Sumudu transform is proposed for solving fuzzy partial differential equations. Some fundamental theorems as well as properties for Sumudu transform are generalized in [7]. In [22] the variational iteration method is suggested by means of the Sumudu transform in order to resolve ordinary equations.

In this paper FST with fundamental property concepts are imparted. Using FST it is possible to reduce FDE into an algebraic problem which is termed as operation calculus. Also, by inverse process it is easy to find the solutions of the FDEs. By utilizing the FST, the problems which are linked to the no resorting of the frequency domain can be solved. Applying the mentioned methodology, fuzzy boundary value problem can be solved directly without determining a general solution. Furthermore, in non-homogeneous problems it can solve the requirements without determining the corresponding homogeneous equation.

The main contents of the paper are highlighted as follows: some preliminary definitions which are useful throughout this paper are demonstrated in section 2. Section 3 portrays the properties related to FST. Section 4 depicts the process require for solving FDEs by using the methodology of FST. Two real examples are solved and discussed in section 5. The conclusion is given in section 6.

II. Preliminaries

Prior to the introduction of the FST, the initial phase is associated with the definitions related to the fuzzy calculations [6], [25].

Definition 1: A fuzzy number z is considered to be a function of $z \in R_f : R \to [0,1]$, in such a manner, 1) z is normal, (there exists $\zeta_0 \in R$ in such a manner $z(\zeta_0) = 1$); 2) z is convex, $z(\lambda\zeta + (1-\lambda)\xi) \geq \min\{z(\zeta), z(\xi)\}$, $\forall \zeta, \xi \in R, \forall \lambda \in [0,1]$; 3) z is upper semi-continuous on R, i.e., $z(\zeta) \leq z(\zeta_0) + \varepsilon$, $\forall \zeta \in N(\zeta_0)$, $\forall \zeta_0 \in R$, $\forall \varepsilon > 0$, $N(\zeta_0)$ is a neighborhood; 4) The set $z^+ = \{\zeta \in R, z(\zeta) > 0\}$ is compact.

Definition 2: The fuzzy number z in association to the r-level is illustrated as

$$[z]^r = \{a \in R : z(a) \geq r\} \tag{1}$$

where $0 < r \leq 1$, $z \in R_f$.

Definition 3: Let $w, z \in R_f$ and $\gamma \in R$, we lay down addition, subtraction, multiplication and scalar multiplication as follows

$$[w \oplus z]^r = [w]^r + [z]^r = [\underline{w}^r + \underline{z}^r, \overline{w}^r + \overline{z}^r]$$

$$[w \ominus z]^r = [w]^r - [z]^r = [\underline{w}^r - \underline{z}^r, \overline{w}^r - \overline{z}^r]$$

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$$[w \odot z]^r = \left(\begin{array}{c} \min\{\underline{w}^r \underline{z}^r, \underline{w}^r \overline{z}^r, \overline{w}^r \underline{z}^r, \overline{w}^r \overline{z}^r\} \\ \max\{\underline{w}^r \underline{z}^r, \underline{w}^r \overline{z}^r, \overline{w}^r \underline{z}^r, \overline{w}^r \overline{z}^r\} \end{array} \right)$$

$$[\gamma w]^r = \gamma [w]^r = \begin{cases} (\gamma \underline{w}^r, \gamma \overline{w}^r), & \gamma \geq 0 \\ (\gamma \overline{w}^r, \gamma \underline{w}^r), & \gamma \leq 0 \end{cases}$$

Definition 4: The Hausdroff distance between two fuzzy numbers w and z is illustrated as [24], [29]

$$D(w, z) = \sup_{0 \leq r \leq 1} \{\max(|\underline{w}^r - \underline{z}^r|, |\overline{w}^r - \overline{z}^r|)\}$$

$D(w, z)$ is incorporated with the following possessions

(i) $D(w \oplus m, z \oplus m) = D(w, z), \qquad \forall w, z, m \in R_f$

(ii) $D(\gamma w, \gamma z) =| \gamma | D(w, z), \qquad \forall \gamma \in R, w, z, m \in R_f$

(iii) $D(w \oplus z, m \oplus n) \leq D(w, m) + D(z, n),$

$\forall w, z, m, n \in R_f$

(iv) (D, R_f) is stated as complete metric space.

Definition 5: A function $\varphi : [a, \varrho] \longrightarrow R_f$ is termed as integrable on $[a, \varrho]$, if it satisfies in the following relation

$$\int_a^\infty \varphi(x) dx = (\int_a^\infty \underline{\varphi}(x, r) dx, \int_a^\infty \overline{\varphi}(x, r) dx)$$

If $\varphi(x)$ be a fuzzy value function and $k(x)$ be a fuzzy Riemann integrable on $[a, \infty]$ then $\varphi(x) \oplus k(x)$ can be a fuzzy Riemann integrable on $[a, \infty]$. Hence it follows

$$\int_a^\infty (\varphi(x) \oplus k(x)) dx = \int_a^\infty \varphi(x) dx \oplus \int_a^\infty k(x) dx$$

In reference to fuzzy or the case concerned to interval arithmetic, equation $w = z \oplus u$ is not equivalent with the phase $u = w \ominus z = w \oplus (-1)z$ or to $z = w \ominus u = w \oplus (-1)u$ and this is the major factor for introducing the following Hukuhara difference (H-difference).

Definition 6: The definition of H-difference [4], [5], is suggested by $w \ominus_H z = u \Longleftrightarrow w = z \oplus u$. Suppose $w \ominus_H z$ prevails, its r-level is $[w \ominus_H z]^r = [\underline{w}^r - \underline{z}^r, \overline{w}^r - \overline{z}^r]$. Precisely, $w \ominus_H w = 0$ but $w \ominus w \neq 0$.

Definition 7: Let $\varphi : [a, \varrho] \longrightarrow R_f$ as well as $x_0 = [a, \varrho]$. φ is strongly generalized differentiable at x_0, if for all $l > 0$ adequately minute, $\varphi'(x_0) \in R_f$ prevails in such a manner that

(i) $\exists\ \varphi(x_0 + l) \ominus_H \varphi(x_0)$, $\varphi(x_0) \ominus_H \varphi(x_0 - l)$ and

$\lim_{l \to 0^+} \frac{\varphi(x_0+l) \ominus_H \varphi(x_0)}{l} = \lim_{l \to 0^+} \frac{\varphi(x_0) \ominus_H \varphi(x_0-l)}{l}$
$= \varphi'(x_0)$

or (ii) $\exists\ \varphi(x_0) \ominus_H \varphi(x_0 + l)$, $\varphi(x_0 - l) \ominus_H \varphi(x_0)$ and

$\lim_{l \to 0^+} \frac{\varphi(x_0) \ominus_H \varphi(x_0+l)}{(-l)} = \lim_{l \to 0^+} \frac{\varphi(x_0-l) \ominus_H \varphi(x_0)}{(-l)}$
$= \varphi'(x_0),$

or (iii) $\exists\ \varphi(x_0 + l) \ominus_H \varphi(x_0)$, $\varphi(x_0 - l) \ominus_H \varphi(x_0)$ and

$\lim_{l \to 0^+} \frac{\varphi(x_0+l) \ominus_H \varphi(x_0)}{l} = \lim_{l \to 0^+} \frac{\varphi(x_0-l) \ominus_H \varphi(x_0)}{(-l)}$
$= \varphi'(x_0)$

or (iv) $\exists\ \varphi(x_0) \ominus_H \varphi(x_0 + l)$, $\varphi(x_0) \ominus_H \varphi(x_0 - l)$ and

$\lim_{l \to 0^+} \frac{\varphi(x_0) \ominus_H \varphi(x_0+l)}{(-l)} = \lim_{l \to 0^+} \frac{\varphi(x_0) \ominus_H \varphi(x_0-l)}{l}$
$= \varphi'(x_0)$

Remark 1:It is obvious that case (i) is H-derivative. It is justified that a function is (i)-differentiable only when it is H-derivative.
Remark 2: According to [4], the definition of differentiability is non contradictory [10].

Let us consider $\varphi : R \to R_f$ where $\varphi(t)$ has a parametric form as $[\varphi(t, r)] = [\underline{\varphi}(t, r), \overline{\varphi}(t, r)]$, for all $0 \leq r \leq 1$, then [10]
(i) Suppose φ is considered to be (i)-differentiable, hence $\underline{\varphi}(t, r)$ as well as $\overline{\varphi}(t, r)$ are differentiable functions, also $\varphi'(t) = (\underline{\varphi}'(t, r), \overline{\varphi}'(t, r))$.
(ii) Suppose φ is considered to be (ii)-differentiable, hence $\underline{\varphi}(t, r)$ as well as $\overline{\varphi}(t, r)$ are differentiable functions, also $\varphi'(t) = (\overline{\varphi}'(t, r), \underline{\varphi}'(t, r))$.

Let $k : (a, \varrho) \to R$ is considered to be differentiable on (a, ϱ), also φ' holds finite root in (a, ϱ) as well as $c \in R_f$, hence $\varphi(x) = ck(x)$ is strongly generalized differentiable on (a, ϱ) along with $\varphi'(x) = ck'(x)$, $\forall x \in (a, \varrho)$.

III. Fuzzy Sumudu transform

Fuzzy initial and boundary value constraints can be solved by using fuzzy Laplace transform [2]. Here, we explain the FST method and also the attributes of this method is presented. The FST reduces the FDE to an algebraic equation. A very important property of the FST is that it can solve the equation without resorting to a new frequency domain. The procedure of switching FDEs to an algebraic equation is cited in [2] and is stated as an operational calculus.

Definition 8: Assume $\varphi(t)$ is considered to be a continuous fuzzy value function. Let us assume that $\varphi(wt) \odot e^{-t}$ is taken to be an improper fuzzy Riemann integrable on $[0, \infty)$. So $\int_0^\infty \varphi(wt) \odot e^{-t} dt$ is termed as FST, also it is illustrated by $Q(w) = \mathbf{S}[\varphi(t)] = \int_0^\infty \varphi(wt) \odot e^{-t} dt$, where $0 \leq w < M$, $M \geq 0$ and e^{-t} is real valued function. From Theorem 3 we have

$$\int_0^\infty \varphi(wt) \odot e^{-t} dt = \Big(\int_0^\infty \underline{\varphi}(wt, r) e^{-t} dt, \int_0^\infty \overline{\varphi}(wt, r) e^{-t} dt \Big) \tag{2}$$

Let

$$\mathbf{S}[\underline{\varphi}(t, r)] = \int_0^\infty \underline{\varphi}(wt, r) e^{-t} dt \tag{3}$$

and

$$\mathbf{S}[\overline{\varphi}(t, r)] = \int_0^\infty \overline{\varphi}(wt, r) e^{-t} dt \tag{4}$$

So we get

$$\mathbf{S}[\varphi(t)] = \big(\mathbf{S}[\underline{\varphi}(t, r), \mathbf{S}\overline{\varphi}(t, r)]\big) \tag{5}$$

Theorem 2: Consider $\varphi'(t)$ as a fuzzy value integrable function, also $\varphi(t)$ is considered to be the primitive of $\varphi'(t)$ on $[0, \infty)$. Hence

$$\mathbf{S}[\varphi'(t)] = \frac{1}{w} \odot \mathbf{S}[\varphi(t)] \ominus (\frac{1}{w} \odot [\varphi(0)])$$

where φ is (i)-differentiable, or

$$\mathbf{S}[\varphi'(t)] = \frac{-1}{w} \odot [\varphi(0)] \ominus (\frac{-1}{w} \odot \mathbf{S}[\varphi(t)])$$

where φ is (ii)-differentiable.

Proof. For arbitrary fixed $r \in [0,1]$ we obtain

$$\begin{aligned} &\tfrac{1}{w} \odot \mathbf{S}[\varphi(t)] \ominus (\tfrac{1}{w} \odot \varphi(0)) \\ &= (\tfrac{1}{w}\mathbf{S}[\underline{\varphi}(t,r)] - \tfrac{1}{w}\mathbf{S}[\underline{\varphi}(0,r)], \tfrac{1}{w}\mathbf{S}[\overline{\varphi}(t,r)] - \tfrac{1}{w}\mathbf{S}[\overline{\varphi}(0,r)]) \end{aligned}$$

We have

$$\mathbf{S}[\overline{\varphi}'(t,r)] = \frac{1}{w}\mathbf{S}[\overline{\varphi}(t,r)] - \frac{1}{w}[\overline{\varphi}(0,r)] \tag{6}$$

$$\mathbf{S}[\underline{\varphi}'(t,r)] = \frac{1}{w}\mathbf{S}[\underline{\varphi}(t,r)] - \frac{1}{w}[\underline{\varphi}(0,r)] \tag{7}$$

As a result, we conclude

$$\frac{1}{w} \odot \mathbf{S}[\varphi(t)] \ominus (\frac{1}{w} \odot \varphi(0)) = (\mathbf{S}[\underline{\varphi}'(t,r)], \mathbf{S}[\overline{\varphi}'(t,r)]) \tag{8}$$

If φ is (i)-differentiable then

$$\frac{1}{w} \odot \mathbf{S}[\varphi(t)] \ominus (\frac{1}{w} \odot \varphi(0)) = \mathbf{S}[\varphi'(t)] \tag{9}$$

We suppose that φ is (ii)-differentiable. For arbitrary fixed $\alpha \in [0,1]$ we have

$$\begin{aligned} &\tfrac{-1}{w} \odot [\varphi(0)] \ominus (\tfrac{-1}{w} \odot \mathbf{S}[\varphi(t)]) \\ &= (\tfrac{-1}{w}\overline{\varphi}(0,r) + \tfrac{1}{w}\mathbf{S}[\overline{\varphi}(t,r)], \tfrac{-1}{w}\underline{\varphi}(0,r) + \tfrac{1}{w}\mathbf{S}[\underline{\varphi}(t,r)]) \end{aligned}$$

The above equation is equivalent to the following relation

$$\begin{aligned} &\tfrac{-1}{w} \odot [\varphi(0)] \ominus (\tfrac{-1}{w} \odot \mathbf{S}[\varphi(t)]) \\ &= (\tfrac{1}{w}\mathbf{S}[\overline{\varphi}(t,r)] - \tfrac{1}{w}\overline{\varphi}(0,r), \tfrac{1}{w}\mathbf{S}[\underline{\varphi}(t,r)] - \tfrac{1}{w}\underline{\varphi}(0,r)) \end{aligned}$$

We have

$$\mathbf{S}[\overline{\varphi}'(t,r)] = \frac{1}{w}\mathbf{S}[\overline{\varphi}(t,r)] - \frac{1}{w}\overline{\varphi}(0,r) \tag{10}$$

and

$$\mathbf{S}[\underline{\varphi}'(t,r)] = \frac{1}{w}\mathbf{S}[\underline{\varphi}(t,r)] - \frac{1}{w}\underline{\varphi}(0,r) \tag{11}$$

Therefore we obtain

$$(\frac{-1}{w}\varphi(0)) \ominus (\frac{-1}{w} \odot \mathbf{S}[\varphi(t)]) = (\mathbf{S}[\overline{\varphi}'(t,r)], \mathbf{S}[\underline{\varphi}'(t,r)]) \tag{12}$$

So

$$(\frac{-1}{w}\varphi(0)) \ominus (\frac{-1}{w} \odot \mathbf{S}[\varphi(t)]) = \mathbf{S}([\overline{\varphi}'(t,r)], [\underline{\varphi}'(t,r)]) \tag{13}$$

Taking into consideration that φ is (ii)-differentiable, hence

$$(\frac{-1}{w}\varphi(0)) \ominus (\frac{-1}{w} \odot \mathbf{S}[\varphi(t)]) = \mathbf{S}[\varphi'(t)] \tag{14}$$

Theorem 3: Since Sumudu transform is a linear transformation, then if we suppose that both $\varphi(t)$ as well as $k(t)$ be continuous fuzzy valued functions, also c_1 and c_2 be constant, it can be obtained

$$\mathbf{S}[(c_1 \odot \varphi(t)) \oplus (c_2 \odot k(t))] = (c_1 \odot \mathbf{S}[\varphi(t)]) \oplus (c_2 \odot \mathbf{S}[k(t)]) \tag{15}$$

Proof. We have

$$\begin{aligned} &\mathbf{S}[(c_1 \odot \varphi(t)) \oplus (c_2 \odot k(t))] \\ &= \textstyle\int_0^\infty (c_1 \odot \varphi(wt) \oplus c_2 \odot k(wt)) \odot e^{-t}dt \\ &= \textstyle\int_0^\infty c_1 \odot \varphi(wt) \odot e^{-t}dt \oplus \int_0^\infty c_2 \odot k(wt) \odot e^{-t}dt \\ &= c_1 \odot (\textstyle\int_0^\infty \varphi(wt) \odot e^{-t}dt) \oplus c_2 \odot (\int_0^\infty k(wt) \odot e^{-t}dt) \\ &= c_1 \odot \mathbf{S}[\varphi(t)] \oplus c_2 \odot \mathbf{S}[k(t)] \end{aligned}$$

Hence, we obtain

$$\mathbf{S}[(c_1 \odot \varphi(t)) \oplus (c_2 \odot k(t))] = (c_1 \odot \mathbf{S}[\varphi(t)]) \oplus (c_2 \odot \mathbf{S}[k(t)]) \tag{16}$$

Lemma 1: Suppose $\varphi(t)$ is taken to be a continuous fuzzy value function on $[0,\infty)$ as well as $\lambda \geq 0$, hence

$$\mathbf{S}[\lambda \odot \varphi(t)] = \lambda \odot \mathbf{S}[\varphi(t)] \tag{17}$$

Proof. Fuzzy Sumudu transform $\lambda \odot \varphi(t)$ is denoted as

$$\mathbf{S}[\lambda \odot \varphi(t)] = \int_0^\infty \lambda \odot \varphi(wt) \odot e^{-t}dt \tag{18}$$

also we have

$$\int_0^\infty \lambda \odot \varphi(wt) \odot e^{-t}dt = \lambda \odot \int_0^\infty \varphi(wt) \odot e^{-t}dt \tag{19}$$

then

$$\mathbf{S}[\lambda \odot \varphi(t)] = \lambda \odot \mathbf{S}[\varphi(t)] \tag{20}$$

Lemma 2: Consider $\varphi(t)$ to be a continuous fuzzy value function, also $k(t) \geq 0$. In addition if we assume that $(\varphi(t) \odot k(t)) \odot e^{-t}$ is improper fuzzy Reiman integrable on $[0,\infty)$, so

$$\begin{aligned} &\textstyle\int_0^\infty (\varphi(wt) \odot k(wt)) \odot e^{-t}dt \\ &= (\textstyle\int_0^\infty k(wt)\underline{\varphi}(wt,r)e^{-t}dt, \int_0^\infty k(wt)\overline{\varphi}(wt,r)e^{-t}dt) \end{aligned}$$

IV. Solution Procedure of the Fuzzy Initial Value Problem

We incorporate the mentioned fuzzy initial value constraint

$$\begin{cases} \psi'(t) = \varphi(t, \psi(t)), \\ \psi(0) = (\underline{\psi}(0,r), \overline{\psi}(0,r)), \quad 0 < r \leq 1 \end{cases} \tag{21}$$

where $\varphi(t, \psi(t))$ is taken to be a fuzzy function. It is quite obvious that the fuzzy function $\varphi(t, \psi(t))$ is the mapping of $\varphi : R \times R_f \to R_f$. By applying FST technique, we extract

$$\mathbf{S}[\psi'(t)] = \mathbf{S}[\varphi(t, \psi(t))] \tag{22}$$

The solving of Eq. (22) is on the basis of the following cases.

Case 1: Let $\psi'(t)$ be (i)-differentiable. According to Theorem 3 we have

$$\psi'(t) = (\underline{\psi}'(t,r), \overline{\psi}'(t,r)) \tag{23}$$

$$\mathbf{S}[\psi'(t)] = (\frac{1}{w} \odot \mathbf{S}[\psi(t)]) \ominus \frac{1}{w}\psi(0) \tag{24}$$

Eq. (24) can be portrayed as follows

$$\begin{cases} \mathbf{S}[\underline{\varphi}(t, \psi(t), r)] = \frac{1}{w}\mathbf{S}[\underline{\psi}(t,r)] - \frac{1}{w}\underline{\psi}(0,\alpha) \\ \mathbf{S}[\overline{\varphi}(t, \psi(t), r)] = \frac{1}{w}\mathbf{S}[\overline{\psi}(t,r)] - \frac{1}{w}\overline{\psi}(0,r) \end{cases} \tag{25}$$

where

$$\begin{cases} \underline{\varphi}(t, \psi(t), r) = min\{\varphi(t,w) | w \in (\underline{\psi}(t,r), \overline{\psi}(t,r))\} \\ \overline{\varphi}(t, \psi(t), r) = max\{\varphi(t,w) | w \in (\underline{\psi}(t,r), \overline{\psi}(t,r))\} \end{cases} \tag{26}$$

Now Eq. (26) can be solved based on the following assumptions

$$\mathbf{S}[\underline{\psi}(t,r)] = H_1(w,r) \tag{27}$$

$$\mathbf{S}[\overline{\psi}(t,r)] = H_2(w,r) \tag{28}$$

where $H_1(w,r)$ and $H_2(w,r)$ are considered to be the solutions of the Eq. (26). By incorporating inverse Sumudu transform, $\underline{\psi}(t,r)$ as well as $\overline{\psi}(t,r)$ are calculated as mentioned below

$$\underline{\psi}(t,r) = \mathbf{S}^{-1}[H_1(w,r)] \tag{29}$$

$$\overline{\psi}(t,r) = \mathbf{S}^{-1}[H_2(w,r)] \tag{30}$$

Case 2: Let $\psi'(t)$ be (ii)-differentiable. According to Theorem 3 we have

$$\psi'(t) = (\overline{\psi}'(t,r), \underline{\psi}'(t,r)) \tag{31}$$

$$\mathbf{S}[\psi'(t)] = (\frac{-1}{w} \odot \psi(0)) \ominus (\frac{-1}{w} \odot \mathbf{S}[\psi(t)]) \tag{32}$$

Eq. (32) can be portrayed as follows

$$\begin{cases} \mathbf{S}[\underline{\varphi}(t,\psi(t),r)] = \frac{1}{w}\mathbf{S}[\underline{\psi}(t,r)] - \frac{1}{w}\underline{\psi}(0,r) \\ \mathbf{S}[\overline{\varphi}(t,\psi(t),r)] = \frac{1}{w}\mathbf{S}[\overline{\psi}(t,r)] - \frac{1}{w}\overline{\psi}(0,r) \end{cases} \tag{33}$$

where

$$\begin{cases} \underline{\varphi}(t,\psi(t),r) = min\{\varphi(t,w) | w \in (\underline{\psi}(t,r), \overline{\psi}(t,r))\} \\ \overline{\varphi}(t,\psi(t),r) = max\{\varphi(t,w) | w \in (\underline{\psi}(t,r), \overline{\psi}(t,r))\} \end{cases} \tag{34}$$

Now Eq. (34) can be solved based on the following assumptions

$$\mathbf{S}(\underline{\psi}(t,r) = K_1(w,r) \tag{35}$$

$$\mathbf{S}(\overline{\psi}(t,r) = K_2(w,r) \tag{36}$$

where $K_1(w,r)$ as well as $K_2(w,r)$ are considered to be the extracted solutions associated to the Eq. (34). By utilizing inverse Sumudu transform, $\underline{\psi}(t,r)$ as well as $\overline{\psi}(t,r)$ are calculated as follows

$$\underline{\psi}(t,r) = \mathbf{S}^{-1}[K_1(w,r)] \tag{37}$$

$$\overline{\psi}(t,r) = \mathbf{S}^{-1}[K_2(w,r)] \tag{38}$$

V. Examples

The following examples have been used to narrate the methodology proposed in this paper.

Example 1 A tank with a heating system is shown in Figure 1, where $R = 0.5$ and the thermal capacitance is considered to be $C = 2$. The temperature is ψ. The model is [2], [23],

$$\begin{cases} \psi'(t) = -\frac{1}{RC}\psi(t), \quad 0 \le t \le T \\ \psi(0) = (\underline{\psi}(0,r), \overline{\psi}(0,r) \end{cases} \tag{39}$$

By incorporating the FST technique, we get

$$\mathbf{S}[\psi'(t)] = \mathbf{S}[-\psi(t)] \tag{40}$$

$$\mathbf{S}[\psi'(t)] = \int_0^\infty \psi'(wt) \odot e^{-t} dt \tag{41}$$

where $0 \le w < M$. If $\psi(t)$ is (i)-differentiable and case 1 holds, we extract

$$\mathbf{S}[\psi'(t)] = \frac{1}{w} \odot (\mathbf{S}[\psi(t)] \ominus \psi(0)) = \frac{1}{w}\mathbf{S}[\psi(t)] \ominus \frac{1}{w}\psi(0) \tag{42}$$

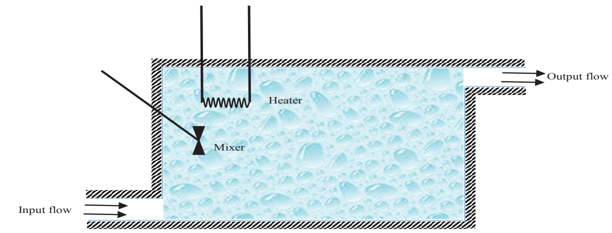

Fig. 1. Thermal system

Henceforth

$$-\mathbf{S}[\psi(t)] = \frac{1}{w}\mathbf{S}[\psi(t)] \ominus \frac{1}{w}\psi(0) \tag{43}$$

According to Eq. (25), we will have the relation mentioned below

$$\begin{cases} -\mathbf{S}[\overline{\psi}(t,r)] = \frac{1}{w}\mathbf{S}[\underline{\psi}(t,r)] - \frac{1}{w}\underline{\psi}(0,r) \\ -\mathbf{S}[\underline{\psi}(t,r)] = \frac{1}{w}\mathbf{S}[\overline{\psi}(t,r)] - \frac{1}{w}\overline{\psi}(0,r) \end{cases} \tag{44}$$

Hence, the solution of Eq. (44) is as follows

$$\begin{cases} \mathbf{S}[\overline{\psi}(t,r)] = (\frac{-1}{w^2-1})\overline{\psi}(0,r) + (\frac{w}{w^2-1})\underline{\psi}(0,r) \\ \mathbf{S}[\underline{\psi}(t,r)] = (\frac{-1}{w^2-1})\underline{\psi}(t,r) + (\frac{w}{w^2-1})\overline{\psi}(0,r) \end{cases} \tag{45}$$

Thus, by utilizing the inverse Sumudu transform we extract

$$\begin{cases} \mathbf{S}[\overline{\psi}(t,r)] = \overline{\psi}(0,r)\mathbf{S}^{-1}(\frac{-1}{w^2-1}) + \underline{\psi}(0,r)\mathbf{S}^{-1}(\frac{w}{w^2-1}) \\ \mathbf{S}[\underline{\psi}(t,r)] = \underline{\psi}(0,r)\mathbf{S}^{-1}(\frac{-1}{w^2-1}) + \overline{\psi}(0,r)\mathbf{S}^{-1}(\frac{u}{w^2-1}) \end{cases} \tag{46}$$

where

$$\begin{cases} \overline{\psi}(t,r) = e^t(\frac{\overline{\psi}(0,r)-\underline{\psi}(0,r)}{2}) + e^{-t}(\frac{\overline{\psi}(0,r)+\underline{\psi}(0,r)}{2}) \\ \underline{\psi}(t,r) = e^t(\frac{\underline{\psi}(0,r)-\overline{\psi}(0,r)}{2}) + e^{-t}(\frac{\underline{\psi}(0,r)+\overline{\psi}(0,r)}{2}) \end{cases} \tag{47}$$

Now if $\psi(t)$ be (ii)-differentiable and case 2 holds, it follows

$$\mathbf{S}[\psi'(t)] = (\frac{-1}{w}\mathbf{S}[\psi(t)]) \ominus (\frac{-1}{w}\psi(0)) \tag{48}$$

Therefore

$$-\mathbf{S}[\psi(t)] = (\frac{-1}{w}\mathbf{S}[\psi(t)]) \ominus (\frac{-1}{w}\psi(0)) \tag{49}$$

By the above relations, Eq. (39) can be written as follows

$$\begin{cases} -\mathbf{S}[\underline{\psi}(t,r)] = \frac{1}{w}\mathbf{S}[\underline{\psi}(t,r)] - \frac{1}{w}\underline{\psi}(0,r) \\ -\mathbf{S}[\overline{\psi}(t,r)] = \frac{1}{w}\mathbf{S}[\overline{\psi}(t,r)] - \frac{1}{w}\overline{\psi}(0,r) \end{cases} \tag{50}$$

Hence, solution of Eq. (50) can be portrayed as

$$\begin{cases} \mathbf{S}[\underline{\psi}(t,r)] = \underline{\psi}(0,r)(\frac{1}{w+1}) \\ \mathbf{S}[\overline{\psi}(t,r)] = \overline{\psi}(t,r)(\frac{1}{w+1}) \end{cases} \tag{51}$$

If we use inverse Sumudu transform, then

$$\begin{cases} \underline{\psi}(t,r) = \underline{\psi}(0,r)\mathbf{S}^{-1}(\frac{1}{w+1}) \\ \overline{\psi}(t,r) = \overline{\psi}(0,r)\mathbf{S}^{-1}(\frac{1}{w+1}) \end{cases} \tag{52}$$

where

$$\begin{cases} \underline{\psi}(t,r) = e^{-t}\underline{\psi}(0,r) \\ \overline{\psi}(t,r) = e^{-t}\overline{\psi}(0,r) \end{cases} \tag{53}$$

If the initial condition be a symmetric triangular fuzzy number as $\psi(0) = (-a_1(1-r), a_1(1-r))$, then the following cases will be hold

Case 1 :

$$\begin{cases} \underline{\psi}(t,r) = e^t(-a_1(1-r)) \\ \overline{\psi}(t,r) = e^t(a_1(1-r)) \end{cases} \tag{54}$$

Case 2:

$$\begin{cases} \underline{\psi}(t,r) = e^{-t}(-a_1(1-r)) \\ \overline{\psi}(t,r) = e^{-t}(a_1(1-r)) \end{cases} \tag{55}$$

Corresponding solution plots are displayed in Figure 2 and Figure 3.

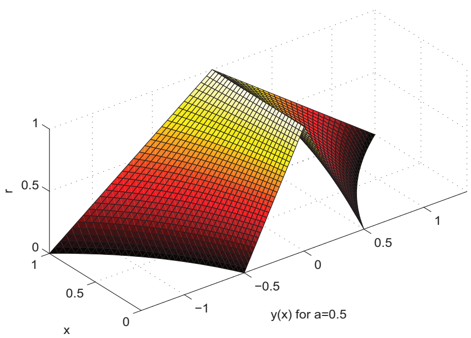

Fig. 2. The solution related to the fuzzy initial value constraint under case 1 consideration

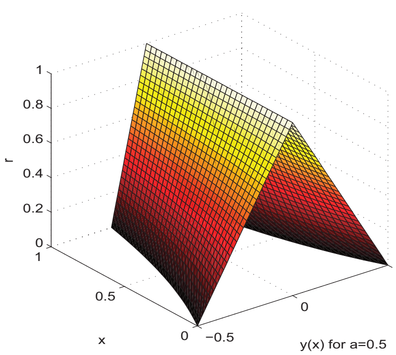

Fig. 3. The solution related to the fuzzy initial value constraint under case 2 consideration

Example 2 A tank system is shown in Figure 4. Assume $I = t+1$ to be inflow disturbances of the tank, which generates vibration in liquid level ψ, where $R = 1$ is the flow obstruction that can be curbed using the valve. $A = 1$ is the cross section of the tank. The liquid level can be described as [27],

$$\begin{cases} \psi'(t) = -\frac{1}{AR}\psi(t) + \frac{I}{A}, \quad 0 \le t \le T \\ \psi(0) = (\underline{\psi}(0,r), \overline{\psi}(0,r) \end{cases} \tag{56}$$

By the application of FST technique we get

$$-\mathbf{S}[\psi(t)] = (\frac{1}{w} \odot \mathbf{S}[\psi(t)]) \ominus (\frac{1}{w}\mathbf{S}[\psi(0)]) \tag{57}$$

$$\mathbf{S}[\psi'(t)](\int_a^\infty \psi'(wt)e^{-t}dt \tag{58}$$

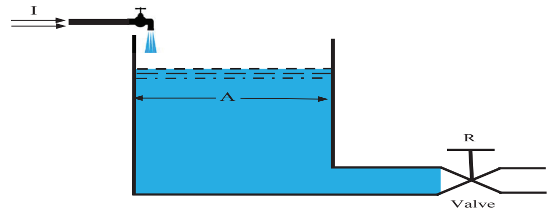

Fig. 4. Liquid tank system

The following is extracted taking into consideration case 2

$$\mathbf{S}[\psi'(t)] = (\frac{-1}{w} \odot \mathbf{S}[\psi(t)]) \ominus (\frac{-1}{w}\mathbf{S}[\psi(0)]) \tag{59}$$

Therefore

$$-\mathbf{S}[\psi(t)] + \mathbf{S}[t] + \mathbf{S}[1] = (\frac{-1}{w} \odot \mathbf{S}[\psi(t)]) \ominus (\frac{-1}{w}\mathbf{S}[\psi(0)]) \tag{60}$$

According to Eq. (25), we will have the relation mentioned below

$$\begin{cases} -\mathbf{S}[\underline{\psi}(t,r)] + \mathbf{S}[t] + \mathbf{S}[1] = \frac{1}{w}\mathbf{S}[\underline{\psi}(t,r)] - \frac{1}{w}\underline{\psi}(0,r) \\ -\mathbf{S}[\overline{\psi}(t,r)] + \mathbf{S}[t] + \mathbf{S}[1] = \frac{1}{w}\mathbf{S}[\overline{\psi}(t,r)] - \frac{1}{w}\overline{\psi}(0,r) \end{cases} \tag{61}$$

Hence, the solution of Eq. (61) is as follows

$$\begin{cases} \mathbf{S}[\underline{\psi}(t,r)] = \mathbf{S}[t] + \mathbf{S}[1] + \frac{-1}{w}\mathbf{S}[\underline{\psi}(t,r)] - \frac{1}{w}\underline{\psi}(0,r) \\ \mathbf{S}[\overline{\psi}(t,r)] = \mathbf{S}[t] + \mathbf{S}[1] + \frac{1}{w}\mathbf{S}[\overline{\psi}(t,r)] - \frac{1}{w}\overline{\psi}(0,r) \end{cases} \tag{62}$$

so

$$\begin{cases} \mathbf{S}[\underline{\psi}(t,r)] = (\frac{1}{w+1})\underline{\psi}(0,r) + w \\ \mathbf{S}[\overline{\psi}(t,r)] = (\frac{1}{w+1})\overline{\psi}(t,r) + w \end{cases} \tag{63}$$

If we use the inverse Sumudu transform, we have

$$\begin{cases} \underline{\psi}(t,r) = \underline{\psi}(0,r)\mathbf{S}^{-1}(\frac{1}{w+1}) + \mathbf{S}^{-1}(w) \\ \overline{\psi}(t,r) = \overline{\psi}(0,r)\mathbf{S}^{-1}(\frac{1}{w+1}) + \mathbf{S}^{-1}(w) \end{cases} \tag{64}$$

where

$$\begin{cases} \underline{\psi}(t,r) = e^{-t}\underline{\psi}(0,r) + t \\ \overline{\psi}(t,r) = e^{-t}\overline{\psi}(0,r) + t \end{cases} \tag{65}$$

If the initial condition be a symmetric triangular fuzzy number as $\psi(0) = (-a_1(1-r), a_1(1-r))$, then

$$\begin{cases} \underline{\psi}(t,r) = e^{-t}(-a_1(1-r)) + t \\ \overline{\psi}(t,r) = e^{-t}(a_1(1-r)) + t \end{cases} \tag{66}$$

Corresponding solution plot is displayed in Figure 5.

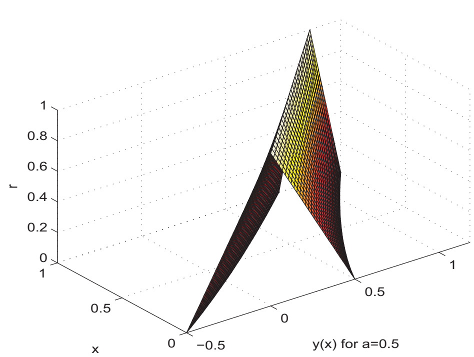

Fig. 5. The solution related to the fuzzy initial value constraint under case 2 consideration

VI. CONCLUSION

In this paper the utilization of FST is resulted in the solution of the first order FDEs in such a manner that, it is clarified by using the notion of strongly generalized differentiability. By implementing the methodology of FST, the FDE reduces to an algebraic problem. Some theorems are given to illustrate the properties of the FST. The novel method is validated by two real examples. This work has a significant contribution in initializing a superior starting point for such extensions. Future work is to study the application of this method in solving FDEs where the uncertainties are in the sense of Z-numbers.

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