INTRODUCTION

Mathematical modelling

Is a process of using (applying) mathematics to represent, analyze and understand real world phenomena.

Involves creating a mathematical representation of a system, process or problem that ~~are~~ we are interested in studying in order to better understand its behaviour and make predictions about its future behaviour.

- Mathematical modelling can be applied in a wide range of fields including physics, engineering, economics, biology and ecology etc.

The modelling process typically involves identifying the key variables and parameters that affect the system or process, developing equations or algorithms that capture the relationships between these variables and then using computational methods to simulate the behaviour of the system over time.

The main aim/goal of mathematical modelling is to gain insight into complex phenomena that may be difficult or impossible to study directly, and to use this understanding to make predictions and inform decision making in a wide range of applications.

SPACE OF VECTORS

A space of vectors also known as vector space is a collection of vectors that satisfy certain properties. which include closure under addition, scalar multiplication as well as the existence of a zero vector and additive inverse.

A vector space V over a field F is a set of vectors equipped with two operations;

1. Vector addition (+)

Any two vectors u and v in V which produces another vector $u+v$ in V

2. Scalar multiplication (·)

This takes any scalar λ in F and any vector u in V and produces λu in V.

and must also satisfy the following properties;

(exercise).

MESH FUNCTIONS

Are functions defined on a mesh or grid that is used to discretize the domain of the Partial differential equation. The mesh is a collection of points or cells that cover the domain and mesh function assign value to these points or cells.

Mesh functions are used in numerical methods as the finite element method or finite difference method which involves apploximating the solution of a PDE by discretizing the domain and solving the resulting system of equations numerically.

Norms of vectors

Norm is a mathematical function that assigns a non-negative scalar value to vectors in a vector space. A norm measures the size or magnitude of a vector.

The mostly common used norm of a vector is the Euclidean norm, also known as the L2 norm denoted by $\| \, \|_2$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

There are also other norms such as the L1 norm and L∞ norm.

$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$ is defined as the sum of the absolute values of its components

$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots |x_n|\}$ is defined as the maximum absolute value of its components.

Examples

Find the Euclidean norm of the vector
$X = [3, 4]$

Solution

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

Let $x_1 = 3$, $x_2 = 4$

$$\|x\|_2 = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5 \text{ units}$$

so the Euclidean norm of the vector x is 5.

Example

Find the L_1 norm of the vector $X = [-2, 3, -4]$

solution

$$\|x\|_1 = |x_1| + |x_2| + |x_3|$$

Let $x_1 = -2$, $x_2 = 3$, $x_3 = -4$

$$\|x\|_1 = |-2| + |3| + |-4|$$
$$= 9.$$

so the L_1 norm of the vector x is 9.

Example

Find the $L\infty$ (infinity) norm of the vector
$X = [1, -5, 2, 7]$

Solution

Using the formula for the L-infinity norm, we have

$$\|x\|_\infty = \max\{|x_1|, |x_2|, |x_3|, |x_4|\} = \max\{1, 5, 2, 7\} = 7$$

So the L-infinity norm of the vector x is 7.

EQUIVALENT NORMS (exercise).

INTERPOLATION

Definition

Consider a set D_n of $n+1$ data points/nodal points in the (x,y) plane.

$D_n = \{(x_i, y_i), i = 0, 1, \dots n; n \in \mathbb{N}$ with $x_i \neq x_j$ for $i \neq j\}$.

We assume D_n represents the values of a function $y = f(x)$ i.e $f(x_i) = y_i$ $i = 0, 1, \dots n$.

In this topic we will construct a continuous function $P(x)$ that represents $f(x)$. Thus $P(x)$ would represent $f(x)$ ~~foa~~ for all values of x. Such a function $P(x)$ is said to interpolate the set of data D_n if it satisfies the interpolation conditions.

LAGRANGES INTERPOLATION FORMULA

Let $f(x_0), f(x_1), \dots f(x_n)$ be $(n+1)$ entries of a function $y = f(x)$ where $f(x)$ is assumed to be a polynomial corresponding to the arguments $x_0, x_1, x_2, \dots x_n$

The polynomial $f(x)$ may be written as

$$f(x) = A_0(x-x_1)(x-x_2)\dots(x-x_n) + A_1(x-x_0)(x-x_2)\dots(x-x_n) + A_2(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n) + \dots + A_n(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1}) \quad \text{--- (1)}$$

Where $A_0, A_1, A_2, \dots A_n$ are constants to be determined.

Substitution $x = x_0, x_1, x_2, \dots x_n$ we get.

$x = x_0$

$$f(x_0) = A_0 (x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)$$

$$A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

$x = x_1$

$$f(x_1) = A_1 (x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)$$

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

$x = x_2$

$$f(x_2) = A_2 (x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \cdots (x_2 - x_n)$$

$\vdots$

$x = x_n$

$$f(x_n) = A_n (x_n - x_0)(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})$$

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})}$$

Substitute $A_0, A_1, A_2, \ldots A_n$ into 1 we get

$$f(x) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} \times (x - x_1)(x - x_2) \cdots (x - x_n) + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} \times (x - x_0)(x - x_2)$$

$$+ \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1) \cdots x_2 - x_n} \times (x - x_0)(x - x_1) \cdots (x - x_n) + \cdots$$

Hence

$$f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots x_1 - x_n} f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}f(x_2)+\cdots,$$

$$+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$$

This is called Lagrange's interpolation polynomial.

<u>Examples</u>

Find the polynomial of degree two given

x	-1	0	2
$y=f(x)$	-8	3	1

and hence find $f(1)$.

<u>Solution</u>

$x_0=-1$, $x_1=0$, $x_2=2$

$f(x_0)=-8$, $f(x_1)=3$, $f(x_2)=1$

By using Lagrange's polynomial

$$P(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$=\frac{(x-0)(x-2)}{(-1-0)(-1-2)}(-8)+\frac{x-(-1)(x-2)}{(0-(-1))(0-2)}(3)+\frac{(x-(-1))(x-0)}{(2-(-1))(2-0)}(1)$$

$$=\frac{x(x-2)}{}\left(\frac{-8}{3}\right)+\frac{(x+1)(x-2}{}\left(\frac{3}{-2}\right)+\frac{(x+1)(x)}{}\left(\frac{1}{6}\right)$$

$$=-\tfrac{8}{3}(x^2-2x)-\tfrac{3}{2}[(x^2-x+(-2)]+\tfrac{1}{6}(x^2+x)$$

$$=\left(-\tfrac{8}{3}x^2-\tfrac{3}{2}x^2+\tfrac{1}{6}x^2\right)+\tfrac{16}{3}x+\tfrac{3}{2}x+\tfrac{1}{6}x+3=\underline{-4x^2+7x+3}$$

APPROXIMATION OF FUNCTIONS.

Can be done piecewisely by using
1. PIECEWISE INTERPOLATION
2. SPLINE INTERPOLATION

PIECEWISE INTERPOLATION.

Let the interval $[a,b]$ be divided into a number of subintervals of the form $[x_i, x_{i+1}]$ for $i=0,1,2,\ldots$

In each subinterval we can fit either a linear, quadratic, cubic, or polynomial of degree four etc. depending on the number of nodal points.

- The fitted polynomial/function can take any form of the interpolating polynomials such as Lagrange's form, Newton's divided form etc.

PIECEWISE LINEAR INTERPOLATION.

Given $n+1$ nodal points. Let the nodal points be grouped into a number of subintervals, each contains two nodal points i.e $[x_0,x_1]$ $[x_1,x_2]$ $[x_2,x_3]$ ... $[x_{n-1},x_n]$.

We use Langrange's interpolating form to ~~fixe~~ fix the linear function in each sub-interval we have;

$[x_0, x_1]: P_{1,1}(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$

$[x_1, x_2]: P_{2,1}(x) = \frac{x-x_2}{x_1-x_2} f(x_1) + \frac{x-x_1}{x_2-x_1} f(x_2)$

$[x_2, x_3]: P_{3,1}(x) = \frac{x-x_3}{x_2-x_3} f(x_2) + \frac{x-x_2}{x_3-x_2} f(x_3)$.

$\vdots$

$[x_{n-1}, x_n]: P_{n,1}(x) = \frac{x-x_n}{x_{n-1}-x_n} f(x_{n-1}) + \frac{x-x_{n-1}}{x_n-x_{n-1}} f(x_n)$

Hence we obtain

$$P(x) = \begin{cases} P_{1,1}(x); & x_0 \le x \le x_1 \\ P_{2,1}(x); & x_1 \le x \le x_2 \\ \vdots & \\ P_{n,1}(x); & x_{n-1} \le x \le x_n \end{cases}$$

Example

Obtain piecewise linear interpolating polynomial for

x	0	1	3
$f(x)$	0	1	0

solution

Let $x_0 = 0$, $x_1 = 1$, $x_2 = 3$

$f(x_0) = 0$, $f(x_1) = 1$, $f(x_2) = 0$

By Lagrange's interpolating form

$x_0 = 0,\ x_1 = 1$ $\quad P_{1,1}(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$

$$= \frac{x-1}{0-1}(0) + \frac{x-0}{1-0}(1)$$

$$= 0 + x$$

$$= x$$

$x_1 = 1,\ x_2 = 3$ $\quad P_{2,1}(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$

$$= \frac{x-3}{1-3}(1) + \frac{x-1}{3-1}(0)$$

$$= \frac{x-3}{-2}$$

Hence we obtain

$$P(x) = \begin{cases} x, & 0 \le x \le 1 \\ \frac{x-3}{-2}, & 1 \le x \le 3 \end{cases}$$

Example

Obtain piecewise linear interpolating polynomial for

x	0.5	1.5	2.3
$f(x)$	0.125	3.375	15.625