# Technical note: orbit fitting using linear maps

The problem is to fit an orbit at a specific location, $s_0$, to a set of $N$ BPM readings. The simplest case is when the other $N-1$ BPMs are downstream of $s_0$

- $R^{(a:b)}$ is the linear map from position $s_a$ to $s_b$[1]
- $\vec{b}$ is an $N \times 1$ column vector of our BPM readings
- $\vec{x_0}$ is a $3 \times 1$ column vector of the orbit parameters at $s_0$ we are trying to calculate

$$\vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \qquad \vec{x_0} = \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix} \tag{1}$$

Assuming there is no coupling, we can relate downstream BPM readings to the initial conditions with the following set of equations:

$$\{x_k = R_{11}^{(0:k)} x_0 + R_{12}^{(0:k)} x_0' + R_{16}^{(0:k)} \delta_0 \mid k \in [0, N]\} \tag{2}$$

or, more compactly

$$\vec{b} = \underline{\mathbf{M}} \vec{x_0} \tag{3}$$

where $\underline{\mathbf{M}}$ is a $N \times 3$ matrix, which encodes the set of N equations[2]:

Written all out we get:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ R_{11}^{(0:1)} & R_{12}^{(0:1)} & R_{16}^{(0:1)} \\ R_{11}^{(0:2)} & R_{12}^{(0:2)} & R_{16}^{(0:2)} \\ \vdots & \vdots & \vdots \\ R_{11}^{(0:N)} & R_{12}^{(0:N)} & R_{16}^{(0:N)} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ \delta_0 \end{pmatrix} \tag{4}$$

$\underline{\mathbf{M}}$ is not guaranteed to be square or generally invertible/nondegenerate, however we can pretty reliably calculate a pseudo-inverse. From there the target orbit parameters are just an LLS fit away.

$$\vec{b} = \underline{\mathbf{M}} \vec{x_0} \quad \rightarrow \quad \underline{\mathbf{M}}^T \vec{b} = (\underline{\mathbf{M}}^T \underline{\mathbf{M}}) \vec{x_0} \quad \rightarrow \quad \vec{x_0} = (\underline{\mathbf{M}}^T \underline{\mathbf{M}})^{-1} \underline{\mathbf{M}}^T \vec{b} \tag{5}$$

[1] maps must be fetched from the live accelerator model for best results

[2] for fitting orbits to the vertical plane, just swap row 1 for row 3 when taking elements from the linear maps