Derivations

First stage gain derivation,Setup & Notes

$$G_m = \frac{i_{\text{out}}}{v_{\text{in}}}, \qquad \text{take } v_{\text{in}} = 1 \text{ (AC)}.$$

- R_S: source impedance from the $50.0\,\Omega$ signal source.
- Remove L_D, C_D, L_P: replace the load with an **AC short to ground** to characterize only the transistor stack.
- Cut out $L_{11}, L_{12}, C_{11}, C_{12}$: those belong to the transformer tank (Eq. 4), not Eq. (3).

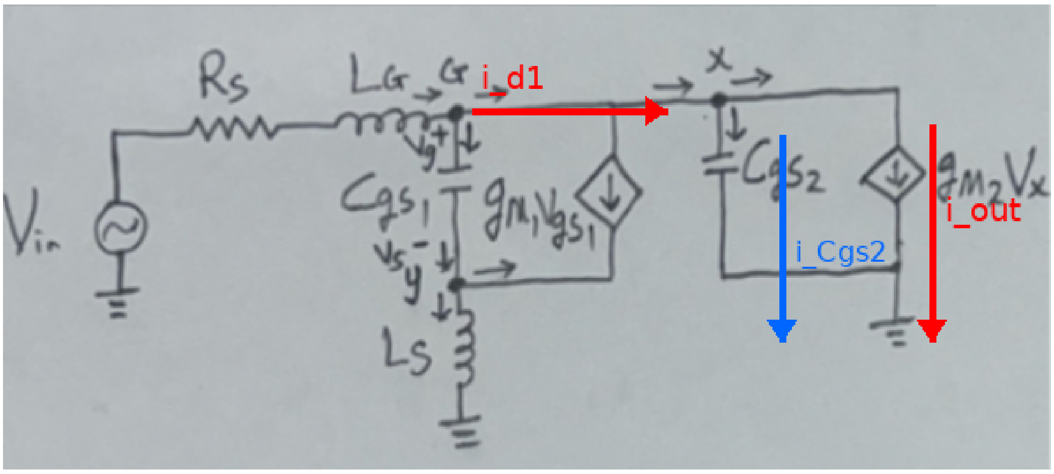

Figure 1: small signal equivalent circuit for effective transconductance of the CS stage

Variable definitions

Work in the phasor domain with $j\omega$. Voltages:

$$v_g,\ v_s,\ v_x, \qquad v_{gs1} = v_g - v_s.$$

KCL at the Gate Node

Only two currents touch the gate: through the series feed $R_S + j\omega L_G$ and into C_{gs1}.

$$\begin{aligned} &\frac{v_g - v_{\text{in}}}{R_S + j\omega L_G} + j\omega C_{gs1}(v_g - v_s) = 0 \\ &\qquad v_{\text{in}} = v_g + (R_S + j\omega L_G)\, j\omega C_{gs1}\,(v_g - v_s) \\ &\qquad\quad = v_g + (R_S + j\omega L_G)\, j\omega C_{gs1}\, v_{gs1} \end{aligned} \tag{1}$$

KCL at the Source Node

Currents leaving node s: through L_S and C_{gs1}; current entering from M1 is $g_{m1}v_{gs1}$.

$$\frac{v_s}{j\omega L_S} + j\omega C_{gs1}(v_s - v_g) - g_{m1}v_{gs1} = 0$$

$$\begin{aligned} v_s &= j\omega L_S\big(j\omega C_{gs1} + g_{m1}\big)\, v_{gs1} \\ v_g &= v_s + v_{gs1} = \Big[1 + j\omega L_S\big(j\omega C_{gs1} + g_{m1}\big)\Big]\, v_{gs1}. \end{aligned} \tag{2}$$

KCL at the Cascode Node x

Current pushed in by M1 equals currents to ground via C_{gs2} and M2:

$$\begin{aligned} &g_{m1}v_{gs1} = (j\omega C_{gs2} + g_{m2})\, v_x \\ &\qquad v_x = \frac{g_{m1}}{j\omega C_{gs2} + g_{m2}}\, v_{gs1}, \qquad i_{\text{out}} = g_{m2}v_x = \frac{g_{m1}g_{m2}}{j\omega C_{gs2} + g_{m2}}\, v_{gs1}. \end{aligned} \tag{3}$$

Relate v_{gs1} to v_{in}

Substitute the expression for v_g equation 2 into the gate equation 1, to find $v$in:

$$\begin{aligned} v_{\text{in}} &= \Big[1 + j\omega L_S(j\omega C_{gs1} + g_{m1})\Big] v_{gs1} + (R_S + j\omega L_G)\, j\omega C_{gs1}\, v_{gs1} \\ &= \Big[1 + j\omega\big(g_{m1}L_S + C_{gs1}R_S\big) \ - \ \omega^2 C_{gs1}(L_S + L_G)\Big]\, v_{gs1}. \end{aligned} \tag{4}$$

Hence

$$\frac{v_{gs1}}{v_{\text{in}}} = \frac{1}{1 + j\omega\big(g_{m1}L_S + C_{gs1}R_S\big) - \omega^2 C_{gs1}(L_S + L_G)} \tag{5}$$

$$D(\omega) = 1 + j\omega\big(g_{m1}L_S + C_{gs1}R_S\big) - \omega^2 C_{gs1}(L_S + L_G) \tag{6}$$

$$= 1 + j\omega\big(g_{m1}L_S + C_{gs1}R_S\big) + (j\omega)^2 C_{gs1}(L_S + L_G) \tag{7}$$

$$= 1 + j\omega[g_{m1}L_S + C_{gs1}R_S + j\omega C_{gs1}(L_S + L_G)] \tag{8}$$

$$= 1 + j\omega\big[C_{gs1}\big(R_S + j\omega(L_S + L_G)\big) + g_{m1}L_S\big] \tag{9}$$

$$D(\omega) \approx j\omega\big[C_{gs1}\big(R_S + j\omega(L_S + L_G)\big) + g_{m1}L_S\big], \qquad |j\omega(\cdot)| \gg 1 \tag{10}$$

$$\frac{v_{gs1}}{v_{in}} \approx \frac{1}{j\omega C_{gs1}\big(R_S + j\omega(L_S + L_G)\big) + j\omega g_{m1}L_S} \tag{11}$$

$$\frac{i_{d1}}{v_{in}} = g_{m1}\frac{v_{gs1}}{v_{in}} \approx \frac{g_{m1}}{j\omega C_{gs1}\big(R_S + j\omega(L_S + L_G)\big) + j\omega g_{m1}L_S} \tag{12}$$

$$\frac{i_{out}}{i_{d1}} = \frac{g_{m2}}{g_{m2} + j\omega C_{gs2}} \tag{13}$$

$$G_m(j\omega) = \frac{i_{out}}{v_{in}} = \frac{g_{m2}}{j\omega C_{gs2} + g_{m2}} \cdot \frac{g_{m1}}{j\omega C_{gs1}\big(R_S + j\omega(L_S + L_G)\big) + j\omega g_{m1}L_S} \tag{14}$$

Final Transconductance $G_m(j\omega)$ (paper form)

The complete expression becomes this:

$$G_m(j\omega) = \frac{g_{m2}}{j\omega C_{gs2} + g_{m2}} \cdot \frac{g_{m1}}{j\omega C_{gs1}\big(R_S + j\omega(L_S + L_G)\big) + j\omega g_{m1} L_S}. \tag{15}$$

Poles and zeros of $G_m(j\omega)$

Starting from equation 15, the overall denominator of transconductance can be written as

$$D(j\omega) = (j\omega C_{gs2} + g_{m2})\left[1 + j\omega B - \omega^2 A\right], \tag{16}$$

with

$$A = C_{gs1}(L_S + L_G), \qquad B = g_{m1} L_S + C_{gs1} R_S. \tag{17}$$

The poles are obtained from $D(j\omega) = 0$.

M1/input poles. The second factor gives

$$1 + j\omega B - \omega^2 A = 0. \tag{18}$$

Let $s = j\omega$. Then (18) becomes

$$As^2 + Bs + 1 = 0, \tag{19}$$

so

$$s_{p1,p2} = \frac{-B \pm \sqrt{B^2 - 4A}}{2A}. \tag{20}$$

Since $s = j\omega$, the corresponding positive frequencies are

$$\omega_{p1,p2} = \frac{B \mp \sqrt{B^2 - 4A}}{2A}, \qquad f_{p1,p2} = \frac{\omega_{p1,p2}}{2\pi} = \frac{B \mp \sqrt{B^2 - 4A}}{4\pi A}, \tag{21}$$

(upper sign for p_1, lower sign for p_2).

Cascode pole. The first factor in (16) gives

$$j\omega C_{gs2} + g_{m2} = 0 \;\Rightarrow\; \omega_{p,\mathrm{CG}} = \frac{g_{m2}}{C_{gs2}}, \qquad f_{p,\mathrm{CG}} = \frac{\omega_{p,\mathrm{CG}}}{2\pi} = \frac{g_{m2}}{2\pi C_{gs2}}. \tag{22}$$

Zeros. The numerator of $G_m(j\omega)$ is the constant $g_{m1} g_{m2}$, so there are no finite zeros .

Numerical pole locations

Given

$$g_{m1} = 0.220 \text{ S}, \qquad g_{m2} = 0.100 \text{ S}, \qquad C_{gs1} = 0.240 \text{ pF}, \qquad C_{gs2} = 0.240 \text{ pF},$$
$$L_S = 0.250 \text{ nH}, \qquad L_G = 0.850 \text{ nH}, \qquad R_S = 50\ \Omega,$$

the coefficients in (17) are

$$A = C_{gs1}(L_S + L_G) \approx 2.64 \times 10^{-22},$$
$$B = g_{m1} L_S + C_{gs1} R_S \approx 6.70 \times 10^{-11}.$$

Using (21) and (22),

$$
\begin{aligned}
f_{p1} &\approx 2.53 \text{ GHz},\\
f_{p2} &\approx 37.9 \text{ GHz},\\
f_{p,\mathrm{CG}} &\approx 66.3 \text{ GHz}.
\end{aligned}
$$

Singularity	Expression	Value [GHz]
M_1 pole p_1	$f_{p1} = \dfrac{B - \sqrt{B^2 - 4A}}{4\pi A}$	≈ 2.53
M_1 pole p_2	$f_{p2} = \dfrac{B + \sqrt{B^2 - 4A}}{4\pi A}$	≈ 37.9
Cascode pole p_{CG}	$f_{p,\mathrm{CG}} = \dfrac{g_{m2}}{2\pi C_{gs2}}$	≈ 66.3
Zeros	none (finite)	–

Coupled-Tank Resonant Frequencies derivation

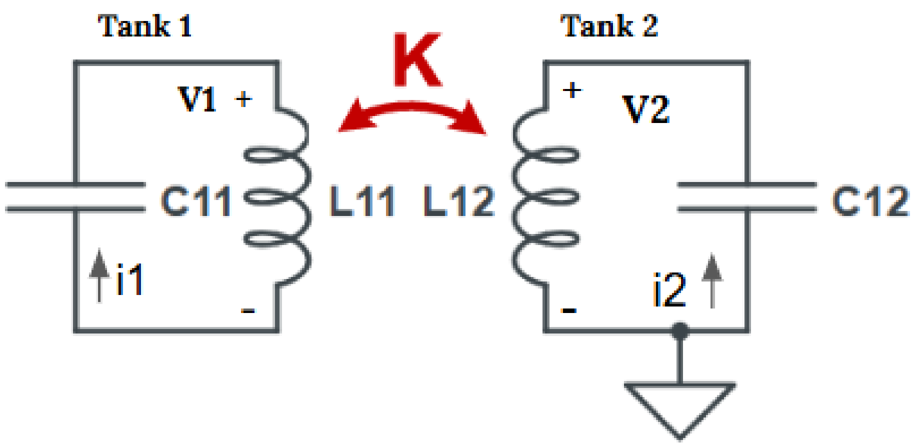

Figure 2: Resonant frequency for magnetically coupled tank

For sinusoidal steady state we use the Laplace variable $s = j\omega$. From Fig. 2 the inductor voltages are

$$
\begin{aligned}
v_1 &= sL_{11}\, i_1 + sM\, i_2,\\
v_2 &= sM\, i_1 + sL_{12}\, i_2,
\end{aligned}
\tag{23}
$$

and because the capacitors are in parallel with the inductors,

$$
i_1 + sC_{11}v_1 = 0, \qquad i_2 + sC_{12}v_2 = 0. \tag{24}
$$

Eliminating i_1 and i_2 between (23) and (24) gives

$$
\begin{bmatrix} 1 - s^2L_{11}C_{11} & -s^2MC_{11} \\ -s^2MC_{12} & 1 - s^2L_{12}C_{12} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}. \tag{25}
$$

Nontrivial solutions require

$$
(1 - s^2L_{11}C_{11})(1 - s^2L_{12}C_{12}) - s^4M^2C_{11}C_{12} = 0. \tag{26}
$$

Let

$$
x = s^2 = \omega^2, \tag{27}
$$

so that (26) becomes

$$(1 - xL_{11}C_{11})(1 - xL_{12}C_{12}) - x^2M^2C_{11}C_{12} = 0. \tag{28}$$

Expanding and collecting powers of x,

$$x^2C_{11}C_{12}(L_{11}L_{12} - M^2) - x(L_{11}C_{11} + L_{12}C_{12}) + 1 = 0. \tag{29}$$

Define

$$M = k\sqrt{L_{11}L_{12}}, \qquad \eta = \frac{L_{12}C_{12}}{L_{11}C_{11}}, \tag{30}$$

so that

$$L_{11}L_{12} - M^2 = L_{11}L_{12}(1 - k^2), \qquad \frac{1}{L_{11}C_{11}} = \frac{\eta}{L_{12}C_{12}}, \qquad \frac{1}{L_{11}L_{12}C_{11}C_{12}} = \frac{\eta}{(L_{12}C_{12})^2}. \tag{31}$$

Dividing (29) by $L_{11}L_{12}C_{11}C_{12}$ and using (30) yields a quadratic in x:

$$(1 - k^2)x^2 - \frac{1+\eta}{L_{12}C_{12}}\,x + \frac{\eta}{(L_{12}C_{12})^2} = 0. \tag{32}$$

With $A = (1 - k^2)$, $B = -\frac{1+\eta}{L_{12}C_{12}}$, $C = \frac{\eta}{(L_{12}C_{12})^2}$, the discriminant is

$$\begin{aligned} B^2 - 4AC &= \frac{(1+\eta)^2}{(L_{12}C_{12})^2} - 4(1-k^2)\frac{\eta}{(L_{12}C_{12})^2} \\ &= \frac{(1+\eta)^2 - 4\eta(1-k^2)}{(L_{12}C_{12})^2} = \frac{(1-\eta)^2 + 4\eta k^2}{(L_{12}C_{12})^2}. \end{aligned} \tag{33}$$

Hence the two roots $x = \omega^2$ are

$$\omega_{1,2}^2 = \frac{(1+\eta) \mp \sqrt{(1-\eta)^2 + 4\eta k^2}}{2L_{12}C_{12}(1-k^2)}. \tag{34}$$

Finally, using $f = \omega/(2\pi)$ we obtain the squared mode frequencies:

$$f_{1,2}^2 = \frac{1 + \eta \mp \sqrt{(1-\eta)^2 + 4\eta k^2}}{8\pi^2 L_{12}C_{12}(1-k^2)} \tag{35}$$

where f_1 corresponds to the in-phase (lower) mode and f_2 to the out-of-phase (upper) mode of the coupled tanks. derived directly from the transformer-coupled LC-tank analysis presented in:

> M. Babaie and R. B. Staszewski, *"A Class-F CMOS Oscillator," IEEE Journal of Solid-State Circuits*, vol. 48, no. 12, pp. 3120–3133, Dec. 2013 — see Eq. (6) on p. 3123 and the discussion below Fig. 3.

Poles and zeros of the coupled–tank network

Starting from the characteristic equation obtained for the coupled LC tanks,

$$(1 - s^2L_{11}C_{11})(1 - s^2L_{12}C_{12}) - s^4M^2C_{11}C_{12} = 0, \tag{36}$$

THis can be viewed as the denominator of a generic transfer function

$$H(s) = \frac{N(s)}{D(s)}, \tag{37}$$

with

$$D(s) = (1 - s^2L_{11}C_{11})(1 - s^2L_{12}C_{12}) - s^4M^2C_{11}C_{12}. \tag{38}$$

Expanding (38) gives

$$D(s) = 1 - s^2\big(L_{11}C_{11} + L_{12}C_{12}\big) + s^4(L_{11}L_{12} - M^2)C_{11}C_{12}. \tag{39}$$

Even polynomial and quadratic in s^2. The denominator is an even polynomial in s. Let

$$y = s^2. \tag{40}$$

Then (39) becomes

$$(L_{11}L_{12} - M^2)C_{11}C_{12}\, y^2 - \big(L_{11}C_{11} + L_{12}C_{12}\big)\, y + 1 = 0. \tag{41}$$

Define the coefficients

$$A_y = (L_{11}L_{12} - M^2)C_{11}C_{12}, \quad B_y = -\big(L_{11}C_{11} + L_{12}C_{12}\big), \quad C_y = 1. \tag{42}$$

The two roots in y are

$$y_{1,2} = \frac{-B_y \pm \sqrt{B_y^2 - 4A_yC_y}}{2A_y}. \tag{43}$$

For a lossless tank, $s = j\omega$, so $y = s^2 = -\omega^2$ and the resonant (natural) frequencies satisfy

$$\omega_{1,2}^2 = y_{1,2}, \tag{44}$$

with $y_{1,2}$ given by (43).

Poles. The denominator (39) therefore factors as

$$D(s) \propto (s^2 + \omega_1^2)(s^2 + \omega_2^2), \tag{45}$$

so the four poles of the network are

$$s = \pm j\omega_1, \quad s = \pm j\omega_2, \tag{46}$$

where ω_1 and ω_2 are the two natural frequencies of the coupled tanks.

Zeros. For the natural-mode problem, transfer function of the form

$$H(s) = \frac{K}{D(s)}, \qquad K \neq 0, \tag{47}$$

so the numerator is a nonzero constant:

$$N(s) = K \quad \Rightarrow \quad N(s) \neq 0 \text{ for all finite } s. \tag{48}$$

Hence there are

$$\text{no finite zeros; only the four poles } \pm j\omega_1,\ \pm j\omega_2. \tag{49}$$

1 Derivation of the Two–Tank Load $Z_L(j\omega)$

1.1 Circuit and element definitions

Two shunt LC tanks coupled magnetically, as in Fig. 3. Node 1 is the input node; node 2 is floating (no external drive).

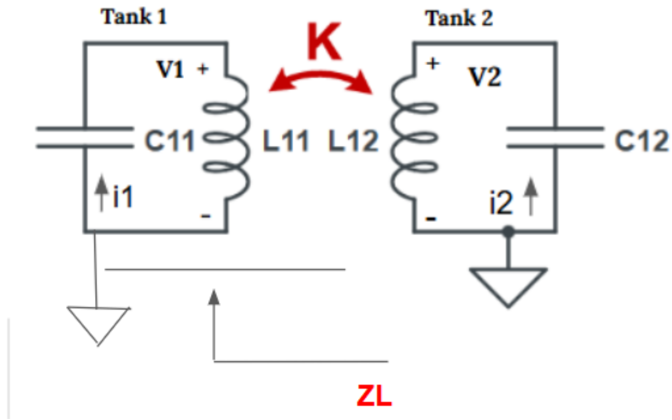

Figure 3: Two tank load

- Tank 1 (node 1): L_{11} in parallel with C_{11} and R_{p1}.
- Tank 2 (node 2): L_{12} in parallel with C_{12} and R_{p2}.
- Inductors are coupled with mutual inductance

$$M = k\sqrt{L_{11}L_{12}}, \qquad 0 \le k < 1. \tag{50}$$

The inductance matrix and its determinant are

$$\mathbf{L} = \begin{bmatrix} L_{11} & M \\ M & L_{12} \end{bmatrix}, \qquad \Delta_L = \det \mathbf{L} = L_{11}L_{12} - M^2 = L_{11}L_{12}(1-k^2). \tag{51}$$

Losses are represented by parallel conductances

$$G_1 = \frac{1}{R_{p1}}, \qquad G_2 = \frac{1}{R_{p2}}. \tag{52}$$

the frequency response is obtained with $s = j\omega$.

1.2 Nodal admittance matrix of the two tanks

The admittance matrix of the coupled inductors is

$$\mathbf{Y}_L(s) = s\,\mathbf{L}^{-1}, \qquad \mathbf{L}^{-1} = \frac{1}{\Delta_L}\begin{bmatrix} L_{12} & -M \\ -M & L_{11} \end{bmatrix}, \tag{53}$$

so that

$$\mathbf{Y}_L(s) = \frac{s}{\Delta_L}\begin{bmatrix} L_{12} & -M \\ -M & L_{11} \end{bmatrix}. \tag{54}$$

Local shunt elements at each node give

$$\mathbf{Y}_{\text{sh}}(s) = \begin{bmatrix} G_1 + sC_{11} & 0 \\ 0 & G_2 + sC_{12} \end{bmatrix}. \tag{55}$$

The total 2×2 nodal admittance matrix is

$$\mathbf{Y}(s) = \mathbf{Y}_{\text{sh}}(s) + \mathbf{Y}_L(s) = \begin{bmatrix} Y_{11}(s) & Y_{12}(s) \\ Y_{21}(s) & Y_{22}(s) \end{bmatrix}, \tag{56}$$

with entries

$$Y_{11}(s) = G_1 + sC_{11} + \frac{sL_{12}}{\Delta_L}, \tag{57}$$

$$Y_{22}(s) = G_2 + sC_{12} + \frac{sL_{11}}{\Delta_L}, \tag{58}$$

$$Y_{12}(s) = Y_{21}(s) = -\frac{sM}{\Delta_L}. \tag{59}$$

It is often convenient to separate real and imaginary parts on the $j\omega$ axis. For $s = j\omega$ we can define

$$X_1(\omega) = \omega C_{11} - \frac{L_{12}}{\omega \Delta_L}, \qquad X_2(\omega) = \omega C_{12} - \frac{L_{11}}{\omega \Delta_L}, \tag{60}$$

so that

$$Y_{11}(j\omega) = G_1 + jX_1(\omega), \tag{61}$$

$$Y_{22}(j\omega) = G_2 + jX_2(\omega), \tag{62}$$

$$Y_{12}(j\omega) = Y_{21}(j\omega) = j\,\frac{M}{\omega \Delta_L}. \tag{63}$$

1.3 Input admittance at node 1 (Schur complement)

Let V_1, V_2 and I_1, I_2 be the node voltages and injected currents. The nodal equation is

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{64}$$

Node 2 is floating (no external source), so $I_2 = 0$ and

$$Y_{21}V_1 + Y_{22}V_2 = 0 \quad \Rightarrow \quad V_2 = -\frac{Y_{21}}{Y_{22}}\,V_1. \tag{65}$$

Substituting into the equation for I_1 gives

$$I_1 = \big(Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22}}\big)V_1 \equiv Y_{\text{in}}V_1. \tag{66}$$

Thus the input admittance at node 1 is the Schur complement of Y_{22}:

$$Y_{\text{in}}(s) = Y_{11}(s) - \frac{Y_{12}(s)Y_{21}(s)}{Y_{22}(s)} = \frac{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)}{Y_{22}(s)}. \tag{67}$$

The load seen by the rest of the circuit is

$$Z_L(s) = \frac{1}{Y_{\text{in}}(s)} = \frac{Y_{22}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)}. \tag{68}$$

1.4 Explicit form of $Z_L(j\omega)$

On the $j\omega$ axis we have

$$Y_{12}^2(j\omega) = \left(j\,\frac{M}{\omega\Delta_L}\right)^2 = -\frac{M^2}{\omega^2\Delta_L^2}. \tag{69}$$

Furthermore,

$$\begin{aligned} Y_{11}Y_{22} &= (G_1 + jX_1)(G_2 + jX_2) \\ &= \big(G_1G_2 - X_1X_2\big) + j\big(G_1X_2 + G_2X_1\big), \end{aligned} \tag{70}$$

so that

$$Y_{11}Y_{22} - Y_{12}^2 = \Big(G_1G_2 - X_1X_2 + \frac{M^2}{\omega^2\Delta_L^2}\Big) + j\big(G_1X_2 + G_2X_1\big). \tag{71}$$

Since $Y_{22} = G_2 + jX_2$, the final compact expression is

$$Z_L(j\omega) = \frac{G_2 + jX_2(\omega)}{\Big(G_1G_2 - X_1(\omega)X_2(\omega) + \frac{M^2}{\omega^2\Delta_L^2}\Big) + j\big(G_1X_2(\omega) + G_2X_1(\omega)\big)}. \tag{72}$$

Equivalently, with the definitions substituted explicitly written as ;

$$Z_L(j\omega) = \frac{G_2 + j\left(\omega C_{12} - \frac{L_{11}}{\omega\Delta_L}\right)}{\left(G_1G_2 - \left(\omega C_{11} - \frac{L_{12}}{\omega\Delta_L}\right)\left(\omega C_{12} - \frac{L_{11}}{\omega\Delta_L}\right) + \frac{M^2}{\omega^2\Delta_L^2}\right) + j\left(G_1\left(\omega C_{12} - \frac{L_{11}}{\omega\Delta_L}\right) + G_2\left(\omega C_{11} - \frac{L_{12}}{\omega\Delta_L}\right)\right)}. \tag{73}$$

1.5 Finite-Q inductors

For high–Q inductors it is convenient to express loss via quality factors Q_1, Q_2 and approximate each inductor by a series R converted to an equivalent parallel resistance:

$$R_p(\omega) \approx \omega LQ \quad \Longrightarrow \quad G(\omega) = \frac{1}{R_p(\omega)} \approx \frac{1}{\omega LQ}. \tag{74}$$

Thus

$$G_1(\omega) \approx \frac{1}{\omega L_{11}Q_1}, \qquad G_2(\omega) \approx \frac{1}{\omega L_{12}Q_2}, \tag{75}$$

and these can be inserted directly into (68) or (72).

1.6 Lossless resonance check

In the lossless case $G_1 = G_2 = 0$, parallel resonance occurs when

$$Y_{11}(j\omega)Y_{22}(j\omega) - Y_{12}^2(j\omega) = 0. \tag{76}$$

Solving for the two resonant frequencies gives

$$\omega_{1,2}^2 = \frac{1 + \eta \mp \sqrt{1 + \eta^2 - 2\eta(1 - 2k^2)}}{2\,L_{12}C_{12}(1 - k^2)}, \qquad \eta \equiv \frac{L_{12}C_{12}}{L_{11}C_{11}}, \tag{77}$$

and

$$f_{1,2} = \frac{\omega_{1,2}}{2\pi}. \tag{78}$$

2 Approximate pole/zero model and numerical example

2.1 Choice of reference frequency f_{ref}

To obtain a rational approximation with *constant* conductances $\bar{G}_1, \bar{G}_2$ over a band $\omega \in [\omega_1, \omega_2]$, we use a minimax fractional–error criterion. For one inductor,

$$G(\omega) = \frac{1}{\omega LQ} \;\Rightarrow\; \bar{G} = \frac{1}{\omega_{\mathrm{ref}}LQ}, \qquad \varepsilon(\omega) = \frac{\bar{G} - G(\omega)}{G(\omega)} = \frac{\omega}{\omega_{\mathrm{ref}}} - 1. \tag{79}$$

Equal and opposite endpoint errors, $\varepsilon(\omega_1) = -\varepsilon(\omega_2)$, imply

$$\omega_{\text{ref}} = \frac{\omega_1 + \omega_2}{2} \quad \Longrightarrow \quad f_{\text{ref}} = \frac{f_1 + f_2}{2}. \tag{80}$$

For the design band $f_1 = 4.20\,\text{GHz}$ and $f_2 = 8.20\,\text{GHz}$,

$$f_{\text{ref}} = \frac{4.2 + 8.2}{2} = 6.20\,\text{GHz}. \tag{81}$$

2.2 Numerical constants and conductances

Using

$$L_{11} = 1.9\,\text{nH}, \quad L_{12} = 1.6\,\text{nH}, \quad C_{11} = C_{12} = 0.5\,\text{pF}, \quad k = 0.55, \tag{82}$$

we obtain

$$\Delta_L = (1 - k^2)L_{11}L_{12} = 2.1204 \times 10^{-18}\ \text{H}^2, \qquad \frac{L_{11}}{\Delta_L} = 8.9606 \times 10^{8}\ \text{s}^{-2}. \tag{83}$$

With $Q_1 = 8.2$, $Q_2 = 2$ and $f_{\text{ref}} = 6.20\,\text{GHz}$,

$$\bar{G}_1 = \frac{1}{2\pi f_{\text{ref}} L_{11} Q_1} = 1.6476 \times 10^{-3}\ \text{S}, \tag{84}$$

$$\bar{G}_2 = \frac{1}{2\pi f_{\text{ref}} L_{12} Q_2} = 8.0219 \times 10^{-3}\ \text{S}. \tag{85}$$

2.3 Zeros from the numerator

From (68) we see that

$$Z_L(s) = \frac{Y_{22}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)} \ \Rightarrow\ N(s) = s^2 Y_{22}(s), \tag{86}$$

so

$$Y_{22}(s) = \bar{G}_2 + sC_{12} + \frac{L_{11}}{\Delta_L}\frac{1}{s}, \qquad N(s) = C_{12}s^3 + \bar{G}_2 s^2 + \frac{L_{11}}{\Delta_L}s. \tag{87}$$

Thus $z_0 = 0$ and the finite zeros satisfy

$$C_{12}s^2 + \bar{G}_2 s + \frac{L_{11}}{\Delta_L} = 0, \tag{88}$$

with roots

$$s_{z1,2} = \frac{-\bar{G}_2 \pm \sqrt{\bar{G}_2^2 - 4C_{12}\frac{L_{11}}{\Delta_L}}}{2C_{12}} = -\,8.0219 \times 10^{9} \pm j\,4.1566 \times 10^{10}\ \text{rad/s}. \tag{89}$$

The corresponding zero frequency is

$$f_z = \frac{|\Im\{s_{z1,2}\}|}{2\pi} = 6.62\,\text{GHz}. \tag{90}$$

2.4 Poles from the denominator

The denominator is

$$D(s) = s^2\big(Y_{11}(s)Y_{22}(s) - Y_{12}^2(s)\big), \tag{91}$$

which expands to

$$D(s) = (C_{11}C_{12})s^4 + (\bar{G}_1C_{12} + \bar{G}_2C_{11})s^3 + \Big(\bar{G}_1\bar{G}_2 + \frac{C_{11}L_{11} + C_{12}L_{12}}{\Delta_L}\Big)s^2 + \frac{\bar{G}_1L_{11} + \bar{G}_2L_{12}}{\Delta_L}s + \frac{1}{\Delta_L}. \tag{92}$$

Numerically,

$$a_4 = 2.50 \times 10^{-25}, \quad a_3 = 4.8348 \times 10^{-15}, \quad a_2 = 8.3853 \times 10^{-4}, \quad a_1 = 7.5295 \times 10^{6}, \quad a_0 = 4.7161 \times 10^{17}, \tag{93}$$

and the four poles $D(s) = 0$ are

$$p_{1,2} = -\,5.2378 \times 10^9 \pm j\,4.9821 \times 10^{10}\ \text{rad/s}, \tag{94}$$

$$p_{3,4} = -\,4.4317 \times 10^9 \pm j\,2.7056 \times 10^{10}\ \text{rad/s}. \tag{95}$$

Their imaginary parts correspond to

$$f_1 = \frac{|\Im\{p_{1,2}\}|}{2\pi} = 7.93\,\text{GHz}, \qquad f_2 = \frac{|\Im\{p_{3,4}\}|}{2\pi} = 4.31\,\text{GHz}. \tag{96}$$

Derivation of Z_{in} and Z_D From the Small-Signal Model

$s = j\omega$.

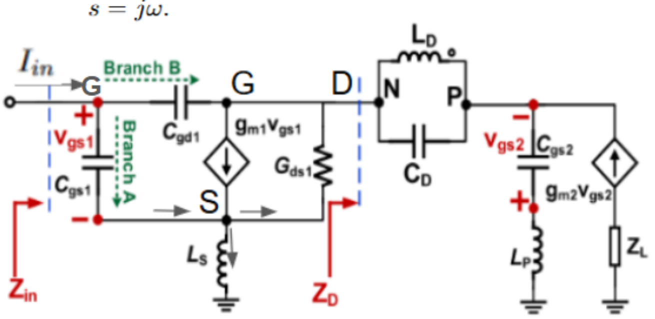

Figure 4: Small-signal equivalent circuit of the input stage for Zin derivation

Node voltages: V_g at G, V_s at S, and V_d at D (M1 drain, left of the tank). M1 has $C_{gs1}, C_{gd1}, g_{m1}, g_{ds1}, L_S$. The right-hand network as seen from D is denoted $Z_D(\omega)$ and will be derived at the end.

The input current is the sum of the two branches highlighted in the schematic:

- Branch A: through C_{gs1} (path $G \to S$); - Branch B: through C_{gd1} into the drain network Z_D (path $G \to D \to Z_D$).

Branch currents (all defined as leaving the node indicated):

$$i_{gs1} = sC_{gs1}(V_g - V_s), \qquad i_{gd1} = sC_{gd1}(V_g - V_d),$$

$$i_{ds1} = g_{ds1}(V_d - V_s), \qquad i_{gm1} = g_{m1}(V_g - V_s),$$

$$i_{Z_D} = \frac{V_d}{Z_D}, \qquad i_{L_S} = \frac{V_s}{sL_S}.$$

1) KCL at the drain node D: expression for V_d

Sum of currents leaving node D is zero:

$$\frac{V_d}{Z_D} + sC_{gd1}(V_d - V_g) + g_{ds1}(V_d - V_s) + g_{m1}(V_g - V_s) = 0. \tag{97}$$

Define

$$A \ \equiv \ \frac{1}{Z_D} + sC_{gd1} + g_{ds1}. \tag{98}$$

Solving (97) for V_d gives

$$V_d = \frac{(sC_{gd1} - g_{m1})V_g + (g_{ds1} + g_{m1})V_s}{A}. \tag{99}$$

2) KCL at the source node S: ratio V_s/V_g

Currents leaving node S:

$$\frac{V_s}{sL_S} + sC_{gs1}(V_s - V_g) + g_{ds1}(V_s - V_d) - g_{m1}(V_g - V_s) = 0. \tag{100}$$

Define

$$B \ \equiv \ \frac{1}{sL_S} + sC_{gs1} + g_{ds1} + g_{m1}, \tag{101}$$

so that (100) can be written as

$$B\,V_s = (sC_{gs1} + g_{m1})V_g + g_{ds1}V_d. \tag{102}$$

Substitute V_d from (99):

$$BV_s = (sC_{gs1} + g_{m1})V_g + \frac{g_{ds1}}{A}\Big[(sC_{gd1} - g_{m1})V_g + (g_{ds1} + g_{m1})V_s\Big]. \tag{103}$$

Rearranging (103) gives

$$\big[AB - g_{ds1}(g_{ds1} + g_{m1})\big]V_s = \big[A(sC_{gs1} + g_{m1}) + g_{ds1}(sC_{gd1} - g_{m1})\big]V_g. \tag{104}$$

Hence

$$\frac{V_s}{V_g} = \frac{A(sC_{gs1} + g_{m1}) + g_{ds1}(sC_{gd1} - g_{m1})}{AB - g_{ds1}(g_{ds1} + g_{m1})}. \tag{105}$$

For compactness, denote

$$r \ \equiv \ \frac{V_s}{V_g} = \frac{N_r}{D_r}, \qquad \begin{cases} N_r = A(sC_{gs1} + g_{m1}) + g_{ds1}(sC_{gd1} - g_{m1}), \\ D_r = AB - g_{ds1}(g_{ds1} + g_{m1}). \end{cases} \tag{106}$$

3) Drain-to-gate gain V_d/V_g

From (99),

$$\frac{V_d}{V_g} = \frac{(sC_{gd1} - g_{m1}) + (g_{ds1} + g_{m1})\,r}{A}. \tag{107}$$

4) Input current and input impedance Z_{in}

The input current is the sum of the branch currents:

$$I_{in} = sC_{gs1}(V_g - V_s) + sC_{gd1}(V_g - V_d), \tag{108}$$

i.e. Branch A (through C_{gs1}) in parallel with Branch B (through C_{gd1} into Z_D).

Dividing (108) by V_g, using $r = V_s/V_g$ from (105) and V_d/V_g from (107), and simplifying, one obtains after algebra

$$Z_{in} = \frac{V_g}{I_{in}} = \frac{\text{numerator}(s)}{\text{denominator}(s)}, \tag{109}$$

where both numerator and denominator are polynomials in s containing $C_{gs1}, C_{gd1}, g_{m1}, g_{ds1}, L_S$ and Z_D.

To write (109) in the compact form used in the paper, define

$$T = sL_S\big(g_{m1} + sC_{gs1} + g_{ds1}\big), \qquad C_T = C_{gs1} + \frac{g_{ds1}}{s}, \tag{110}$$

and use (98)–(101). After straightforward simplification,

$$Z_{in} = \frac{\big(1 + sC_{gd1}Z_D\big)(1+T) + Z_D g_{ds1}\big(1 - s^2 L_S C_{gs1}\big)}{s\Big\{(Z_D + sL_S)\big[g_{ds1}C_T + C_{gd1}(g_{m1} + sC_{gs1})\big] + C_T\Big\}}. \tag{111}$$

This matches the Z_{in} expression referred to as Eq. (5) in the paper.

5) Drain looking-in impedance Z_D

From node D, the right-hand network is:

- A series peaking inductor L_P feeding the common-gate M2 input with shunt admittance $g_{m2} + sC_{gs2}$, giving

$$Z_{\mathrm{CG}+L_P} = sL_P + \frac{1}{g_{m2} + sC_{gs2}} = \frac{1 + sL_P(g_{m2} + sC_{gs2})}{g_{m2} + sC_{gs2}}. \tag{112}$$

  In the operating regime $g_{m2} \gg \omega C_{gs2}$ and $\omega^2 L_P C_{gs2} \ll 1$, this can be approximated as

$$Z_{\mathrm{CG}+L_P} \approx \frac{1 - \omega^2 L_P C_{gs2}}{g_{m2} + j\omega C_{gs2}}. \tag{113}$$

- A parallel tank $L_D \parallel C_D$ with impedance

$$Z_{L_D \parallel C_D} = \frac{j\omega L_D}{1 - \omega^2 L_D C_D}. \tag{114}$$

Adding the two blocks yields

$$Z_D(\omega) = \frac{1 - \omega^2 L_P C_{gs2}}{g_{m2} + j\omega C_{gs2}} + \frac{j\omega L_D}{1 - \omega^2 L_D C_D}. \tag{115}$$

Under the simplifying limits $g_{m2} \gg \omega C_{gs2}$ and $\omega^2 L_P C_{gs2} \ll 1$,

$$Z_D(\omega) \approx \frac{1}{g_{m2}} + \frac{j\omega L_D}{1 - \omega^2 L_D C_D}, \tag{116}$$

which is the form used for the Fig. 4 calculations in the paper.

6) Intuition From the Branch-A / Branch-B View

If Branch B is switched off by setting $C_{gd1} = 0$ and g_{ds1} is neglected, only Branch A (through C_{gs1} and L_S) sets the input impedance:

$$Z_{in} \approx \frac{1}{sC_{gs1}} + \frac{g_{m1}L_S}{C_{gs1}} + sL_S. \tag{117}$$

The real part is then

$$\Re\{Z_{in}\} \approx \frac{g_{m1}L_S}{C_{gs1}}, \tag{118}$$

which explains the flat "plateau" versus frequency, while the imaginary part $\Im\{Z_{in}\} \approx \omega L_S - 1/(\omega C_{gs1})$ is cancelled by choosing L_G at the input.

Branch B (through C_{gd1} into Z_D) then:

- adds $\Re\{Z_{in}\}$ via the real part of Z_D (mainly $1/g_{m2}$ and tank losses), helping keep $\Re\{Z_{in}\} > 0$ and tune the plateau;
- contributes a frequency-dependent effective capacitance from the L_D–C_D tank, which shifts the frequency where $\Im\{Z_{in}\} \approx 0$ and thus the $|S_{11}|$ dip.

In this way the two branches of the schematic directly map to the structure of the final Z_{in} expression (111).

2.5 Reflection coefficient and the $|S_{11}|$ dip

From the input reflection coefficient,

$$S_{11}(\omega) = \Gamma_{in}(\omega) = \frac{Z_{in}(\omega) - Z_0}{Z_{in}(\omega) + Z_0}, \tag{119}$$

where $Z_0 = 50\ \Omega$ and

$$Z_{in}(\omega) = R(\omega) + jX(\omega), \qquad R(\omega) = \Re\{Z_{in}\},\ \ X(\omega) = \Im\{Z_{in}\}. \tag{120}$$

Taking the magnitude gives

$$|S_{11}(\omega)| = \left|\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right| = \frac{\sqrt{(R - Z_0)^2 + X^2}}{\sqrt{(R + Z_0)^2 + X^2}}. \tag{121}$$

Dip frequency. The $|S_{11}|$ dip (best match) occurs near the frequency where the input reactance crosses zero,

$$X(\omega_{\text{dip}}) = \Im\{Z_{in}(j\omega_{\text{dip}})\} = 0, \tag{122}$$

i.e., where Z_{in} becomes purely real.

How low the dip goes (depth). At the zero-crossing $X = 0$, (121) reduces to

$$|S_{11}|_{\text{dip}} = \left|\frac{R(\omega_{\text{dip}}) - Z_0}{R(\omega_{\text{dip}}) + Z_0}\right|. \tag{123}$$

Thus the dip depth in dB is

$$|S_{11}|_{\text{dip,dB}} = 20\log_{10}(|S_{11}|_{\text{dip}}) = 20\log_{10}\left|\frac{R(\omega_{\text{dip}}) - Z_0}{R(\omega_{\text{dip}}) + Z_0}\right|. \tag{124}$$

Therefore, the dip is deepest when $R(\omega_{\text{dip}}) \approx Z_0$ (perfect match).

Stage-2 Gain Derivation (M3–M4) With C_{B2}–R_{B2} and $L_2 \parallel C_2 \parallel R_{p3}$

Small–Signal Model and Definitions

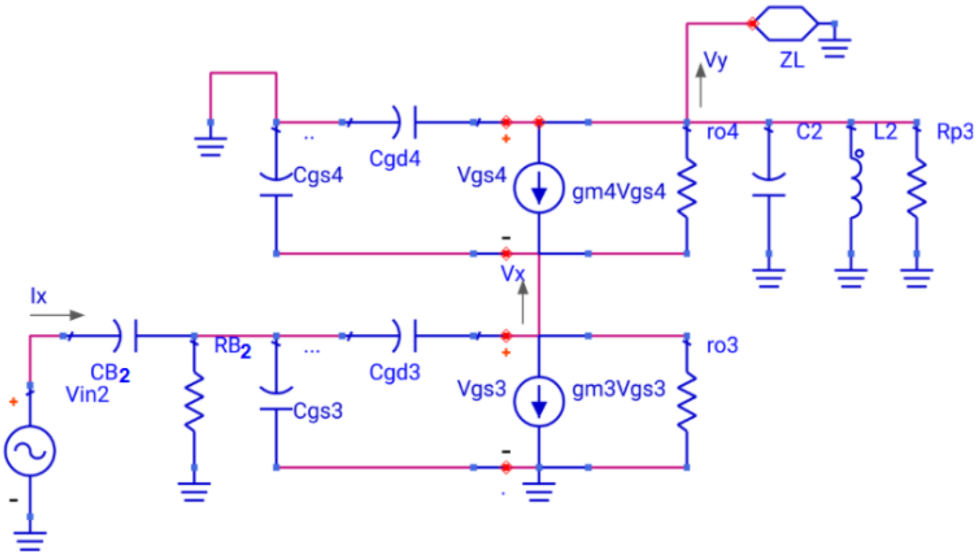

Figure 5: Small-signal model for 2nd and 3rd stage gain derivation

We use the midband small–signal model corresponding to the schematic:

- M3: common-source with transconductance g_{m3} and output resistance r_{o3}. Source at AC ground, so $v_{gs3} = v_{g3}$.
- M4: common-gate cascode with transconductance g_{m4} and output resistance r_{o4}. Gate is AC grounded. Its source/drain voltages are v_x and v_y.
- Output tank at node v_y: $L_2 \parallel C_2 \parallel R_{p3}$.
- Gate high-pass: series C_{B2} into R_{B2} to ground, feeding the M3 gate.

Node voltages:

v_{g3} (gate of M3), $\quad v_x$ (M3 drain/M4 source), $\quad v_y$ (M4 drain/output).

Load admittance and impedance at the output node:

$$Y_L(s) = \frac{1}{R_{p3}} + sC_2 + \frac{1}{sL_2}, \qquad Z_L(s) = \frac{1}{Y_L(s)}, \qquad s = j\omega. \tag{125}$$

Gate high-pass transfer:

$$H_{\text{HP2}}(s) \triangleq \frac{v_{g3}}{v_{\text{in},2}} = \frac{sR_{B2}C_{B2}}{1 + sR_{B2}C_{B2}}. \tag{126}$$

We first find the gate-referenced gain $A_{v2,\text{gate}}(s) \equiv v_y/v_{g3}$, then include (138) to get the input-referenced gain $A_{v2}(s) \equiv v_y/v_{\text{in},2}$.

KCL at the Output Node v_y

Currents *leaving* v_y go into the tank admittance Y_L, into r_{o4}, and through the M4 dependent source (current $g_{m4}v_{gs4} = -g_{m4}v_x$ from drain to source):

$$\left(Y_L(s) + \tfrac{1}{r_{o4}}\right)v_y + \left(g_{m4} - \tfrac{1}{r_{o4}}\right)v_x = 0. \tag{127}$$

Hence

$$v_y = -\frac{g_{m4} - \frac{1}{r_{o4}}}{Y_L(s) + \frac{1}{r_{o4}}}\, v_x. \tag{128}$$

KCL at the Cascode Source Node v_x

Currents *leaving* v_x go through r_{o3}, r_{o4}, and via the M4 source (current $+g_{m4}v_x$ leaving). M3 injects $g_{m3}v_{g3}$ from v_x to ground:

$$\Big(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\Big)v_x - \tfrac{1}{r_{o4}}v_y = g_{m3}v_{g3}. \tag{129}$$

Substitute (140) into (141):

$$\left[\frac{1}{r_{o3}} + \frac{1}{r_{o4}} - g_{m4} + \frac{\frac{1}{r_{o4}}\big(g_{m4} - \frac{1}{r_{o4}}\big)}{Y_L(s) + \frac{1}{r_{o4}}}\right] v_x = g_{m3}v_{g3}. \tag{130}$$

Therefore

$$\frac{v_x}{v_{g3}} = \frac{g_{m3}}{\dfrac{1}{r_{o3}} + \dfrac{1}{r_{o4}} - g_{m4} + \dfrac{\frac{1}{r_{o4}}\big(g_{m4} - \frac{1}{r_{o4}}\big)}{Y_L(s) + \frac{1}{r_{o4}}}}. \tag{131}$$

Gate-Referenced Stage Gain

From (140) and (143),

$$\frac{v_y}{v_{g3}} = -\frac{g_{m4} - \frac{1}{r_{o4}}}{Y_L(s) + \frac{1}{r_{o4}}}\,\frac{v_x}{v_{g3}}. \tag{132}$$

A convenient identity is

$$\Big(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\Big)\tfrac{1}{r_{o4}} + \tfrac{1}{r_{o4}}\Big(g_{m4} - \tfrac{1}{r_{o4}}\Big) = \frac{1}{r_{o3}r_{o4}}, \tag{133}$$

so the denominator of (143) can be written as

$$\Big(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\Big)Y_L(s) + \frac{1}{r_{o3}r_{o4}}. \tag{134}$$

Thus the gate-referenced gain is

$$A_{v2,\mathrm{gate}}(s) \equiv \frac{v_y}{v_{g3}} = -\,g_{m3}\,\frac{g_{m4} - \frac{1}{r_{o4}}}{\Big(\frac{1}{r_{o3}} + \frac{1}{r_{o4}} - g_{m4}\Big)Y_L(s) + \frac{1}{r_{o3}r_{o4}}}. \tag{135}$$

Input-Referenced Stage Gain

Using the input high-pass (138),

$$A_{v2}(s) \equiv \frac{v_y}{v_{\mathrm{in},2}} = H_{\mathrm{HP2}}(s)\,A_{v2,\mathrm{gate}}(s) = \frac{sR_{B2}C_{B2}}{1 + sR_{B2}C_{B2}}\,A_{v2,\mathrm{gate}}(s). \tag{136}$$

Stage-2 Gain Derivation (M3–M4) With C_{B2}–R_{B2} and $L_2 \parallel C_2 \parallel R_{p3}$

Small–Signal Model and Definitions

We use the midband small–signal model that matches the schematic:

- M3: common-source with transconductance g_{m3} and output resistance r_{o3}. Source is at AC ground, so $v_{gs3} = v_{g3}$.
- M4: common-gate cascode with transconductance g_{m4} and output resistance r_{o4}. Gate is AC grounded. Its source/drain voltages are v_x and v_y.
- Output tank at node v_y: $L_2 \parallel C_2 \parallel R_{p3}$.
- Gate high-pass: series C_{B2} into R_{B2} to ground, feeding the M3 gate.

Node voltages:

v_{g3} (M3 gate), $\quad v_x$ (M3 drain / M4 source), $\quad v_y$ (M4 drain / output).

Load admittance and impedance at the output node:

$$Y_{L2}(s) = \frac{1}{R_{p3}} + sC_2 + \frac{1}{sL_2}, \qquad Z_{L2}(s) = \frac{1}{Y_{L2}(s)}, \qquad s = j\omega. \tag{137}$$

Gate high-pass transfer:

$$H_{\mathrm{HP2}}(s) \triangleq \frac{v_{g3}}{v_{\mathrm{in},2}} = \frac{sR_{B2}C_{B2}}{1 + sR_{B2}C_{B2}}. \tag{138}$$

We first find the gate-referenced gain $A_{v2,\mathrm{gate}}(s) \equiv v_y/v_{g3}$, then include (138) for the input-referenced gain $A_{v2}(s) \equiv v_y/v_{\mathrm{in},2}$.

KCL at the Output Node v_y

Currents *leaving* v_y go into the tank admittance Y_{L2}, into r_{o4}, and through M4's dependent source (current $g_{m4}v_{gs4} = -g_{m4}v_x$ from drain to source):

$$\left(Y_{L2}(s) + \tfrac{1}{r_{o4}}\right)v_y + \left(g_{m4} - \tfrac{1}{r_{o4}}\right)v_x = 0. \tag{139}$$

Hence

$$v_y = -\frac{g_{m4} - \frac{1}{r_{o4}}}{Y_{L2}(s) + \frac{1}{r_{o4}}}\, v_x. \tag{140}$$

KCL at the Cascode Source Node v_x

Currents *leaving* v_x go through r_{o3}, r_{o4}, and via the M4 source (current $+g_{m4}v_x$ leaving). M3 injects $g_{m3}v_{g3}$ from v_x to ground:

$$\left(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\right)v_x - \tfrac{1}{r_{o4}}v_y = g_{m3}v_{g3}. \tag{141}$$

Substitute (140) into (141):

$$\left[\frac{1}{r_{o3}} + \frac{1}{r_{o4}} - g_{m4} + \frac{\frac{1}{r_{o4}}\left(g_{m4} - \frac{1}{r_{o4}}\right)}{Y_{L2}(s) + \frac{1}{r_{o4}}}\right] v_x = g_{m3}v_{g3}. \tag{142}$$

Therefore

$$\frac{v_x}{v_{g3}} = \frac{g_{m3}}{\dfrac{1}{r_{o3}} + \dfrac{1}{r_{o4}} - g_{m4} + \dfrac{\frac{1}{r_{o4}}\left(g_{m4} - \frac{1}{r_{o4}}\right)}{Y_{L2}(s) + \frac{1}{r_{o4}}}}. \tag{143}$$

Gate-Referenced Stage-2 Gain

From (140) and (143),

$$\frac{v_y}{v_{g3}} = -\frac{g_{m4} - \frac{1}{r_{o4}}}{Y_{L2}(s) + \frac{1}{r_{o4}}}\,\frac{v_x}{v_{g3}}. \tag{144}$$

A convenient identity is

$$\left(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\right)\tfrac{1}{r_{o4}} + \tfrac{1}{r_{o4}}\left(g_{m4} - \tfrac{1}{r_{o4}}\right) = \frac{1}{r_{o3} r_{o4}}, \tag{145}$$

so the denominator in (143) can be written as

$$\left(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\right) Y_{L2}(s) + \frac{1}{r_{o3} r_{o4}}. \tag{146}$$

Thus the gate-referenced Stage-2 gain is

$$A_{v2,\mathrm{gate}}(s) \equiv \frac{v_y}{v_{g3}} = -\,g_{m3}\,\frac{g_{m4} - \frac{1}{r_{o4}}}{\left(\tfrac{1}{r_{o3}} + \tfrac{1}{r_{o4}} - g_{m4}\right) Y_{L2}(s) + \frac{1}{r_{o3} r_{o4}}}. \tag{147}$$

Input-Referenced Stage-2 Gain

Using the input high-pass (138),

$$A_{v2}(s) \equiv \frac{v_y}{v_{\mathrm{in},2}} = H_{\mathrm{HP2}}(s)\, A_{v2,\mathrm{gate}}(s) = \frac{sR_{B2}C_{B2}}{1 + sR_{B2}C_{B2}}\, A_{v2,\mathrm{gate}}(s). \tag{148}$$

Note on Stage-3 Gain Derivation

Important note. Stage-3 (M5–M6) has the *same cascode + shunt-tank topology* as Stage-2:

- M5 plays the role of M3 with $(g_{m3}, r_{o3}) \to (g_{m5}, r_{o5})$, - M6 plays the role of M4 with $(g_{m4}, r_{o4}) \to (g_{m6}, r_{o6})$, - the load becomes $L_3 \parallel C_3 \parallel R_{p4}$ with admittance $Y_{L3}(s)$, - the input high-pass uses $(C_{B2}, R_{B2}) \to (C_{B3}, R_{B3})$.

Because the schematic and current directions are identical, the KCL steps for Stage-3 are the same as for Stage-2. Therefore we do *not* repeat the algebra and only state the corresponding results below (obtained by the substitutions above).

Compact g_{ds} Forms, Poles, Zeros, and Numerical Values

Common Load and High-Pass Definitions

$$Y_{L2}(s) = \frac{1}{R_{p3}} + sC_2 + \frac{1}{sL_2}, \qquad Y_{L3}(s) = \frac{1}{R_{p4}} + sC_3 + \frac{1}{sL_3}, \tag{149}$$

$$H_{\mathrm{HP2}}(s) = \frac{sR_{B2}C_{B2}}{1 + sR_{B2}C_{B2}}, \qquad H_{\mathrm{HP3}}(s) = \frac{sR_{B3}C_{B3}}{1 + sR_{B3}C_{B3}}. \tag{150}$$

Using $g_{ds3} = 1/r_{o3}$, $g_{ds4} = 1/r_{o4}$, etc. the gate-referenced gains can be written compactly.

Stage-2 (M3–M4, Load $L_2 \parallel C_2 \parallel R_{p3}$)

Gate-referenced gain (in g_{ds} notation).

$$A_{v2,\mathrm{gate}}(s) = -\,g_{m3}\,\frac{g_{m4} - g_{ds4}}{(g_{ds3} + g_{ds4} - g_{m4})\, Y_{L2}(s) + g_{ds3} g_{ds4}}. \tag{151}$$

Input-referenced gain.

$$A_{v2}(s) = H_{\mathrm{HP2}}(s)\, A_{v2,\mathrm{gate}}(s) = \frac{sR_{B2}C_{B2}}{1 + sR_{B2}C_{B2}}\, A_{v2,\mathrm{gate}}(s). \tag{152}$$

Poles and Zeros

Zeros.

- From the tank: as $s \to 0$, the term $1/(sL_2)$ dominates $Y_L(s)$ in (125), so $Y_L \to \infty$ and the denominator of (135) grows $\propto 1/s$, forcing $A_{v2,\mathrm{gate}}(s) \to 0$ linearly with s. Thus $A_{v2,\mathrm{gate}}(s)$ has a zero at $s = 0$.
- From the input high-pass: $H_{\mathrm{HP2}}(s)$ has a zero at $s = 0$.

Therefore the input-referenced stage has a *double zero at the origin*:

$$z_1 = 0, \qquad z_2 = 0.$$

Poles. The tank poles are the roots of

$$\big(g_{ds5} + g_{ds6} - g_{m6}\big)\, C_3\, s^2 + \left[\big(g_{ds5} + g_{ds6} - g_{m6}\big)\frac{1}{R_{p4}} + g_{ds5}g_{ds6}\right] s + \big(g_{ds5} + g_{ds6} - g_{m6}\big)\frac{1}{L_3} = 0, \tag{153}$$

with

$$s_{q1,2} = \frac{-\left[\big(g_{ds5} + g_{ds6} - g_{m6}\big)\frac{1}{R_{p4}} + g_{ds5}g_{ds6}\right] \pm \sqrt{\left(\big(g_{ds5} + g_{ds6} - g_{m6}\big)\frac{1}{R_{p4}} + g_{ds5}g_{ds6}\right)^2 - 4\big(g_{ds5} + g_{ds6} - g_{m6}\big)^2 \frac{C_3}{L_3}}}{2\big(g_{ds5} + g_{ds6} - g_{m6}\big)C_3}. \tag{154}$$

The input HPF adds a real pole at

$$p_{\mathrm{HP3}} = -\frac{1}{R_{B3}C_{B3}}. \tag{155}$$

High-r_o sanity check. If $r_{o3}, r_{o4} \to \infty$, $\frac{1}{r_{o3}} + \frac{1}{r_{o4}} - g_{m4} \approx -g_{m4}$ and $\frac{1}{r_{o3}r_{o4}} \approx 0$, so (135) reduces to

$$A_{v2,\mathrm{gate}}(s) \approx -g_{m3} Z_L(s), \tag{156}$$

i.e. a simple CS driving the shunt tank impedance Z_L; at resonance $Z_L(j\omega_0) \approx R_{p3}$, so $|A_{v2,\mathrm{gate}}| \approx g_{m3}R_{p3}$, matching the usual peak-gain rule of thumb.

Stage-3 (M5–M6, Load $L_3 \parallel C_3 \parallel R_{p4}$)

Because Stage-3 has the same topology, applying the same derivation with $(3,4) \to (5,6)$, $(L_2, C_2, R_{p2}) \to (L_3, C_3, R_{p4})$ gives:

Gate-referenced gain.

$$A_{v3,\mathrm{gate}}(s) = -\, g_{m5}\, \frac{g_{m6} - g_{ds6}}{\big(g_{ds5} + g_{ds6} - g_{m6}\big)\, Y_{L3}(s) + g_{ds5}g_{ds6}}. \tag{157}$$

Input-referenced gain.

$$A_{v3}(s) = H_{\mathrm{HP3}}(s)\, A_{v3,\mathrm{gate}}(s) = \frac{sR_{B3}C_{B3}}{1 + sR_{B3}C_{B3}}\, A_{v3,\mathrm{gate}}(s). \tag{158}$$

How the long Stage-2/3 gain collapses to Razavi's compact form

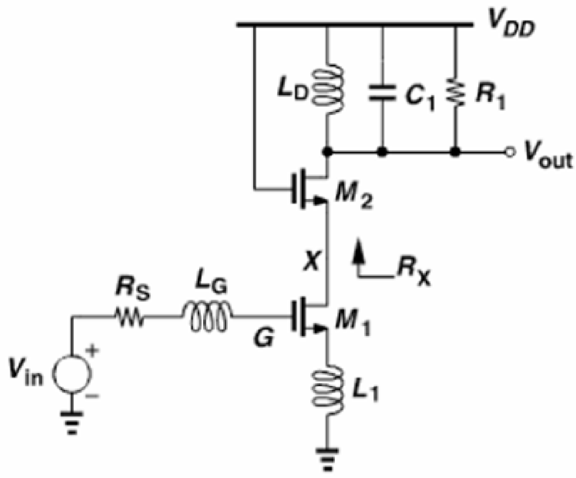

Figure 6: Inductively-degenerated Cascode CS LNA

The above figure from (Razavi, p. 290) is included as the textbook reference for a cascode CS LNA with inductive degeneration and a resonant load derivation. Stage-2 and 3 gain expressions are kept in full s-domain form (tank frequency response, finite loss, finite r_o, and the high-pass). To verify consistency with Razavi, I evaluate the result at resonance and apply the same simplifying assumptions (strong cascode and $|H_{HP}(j\omega_0)| \approx 1$), under which the gain collapses to the compact form $A_v \approx -g_m R_p$ (equivalently Razavi's $\omega_T R_1/(2\omega_0 R_S)$ under matching).

Start from the general midband transfer form of the stage ;

$$A_{v2}(s) \triangleq \frac{v_{out2}}{v_{in2}} = -g_{m3} \underbrace{H_{HP2}(s)}_{C_{B2}\text{–}R_B \text{ high-pass}} \underbrace{\Big(Z_{L2}(s) \parallel r_{o,\text{cas2}}\Big)}_{\text{load seen at drain}}. \tag{159}$$

Step 1: Evaluate at the tank resonance. Let $s = j\omega_{02}$ and choose ω_{02} as the Stage-2 load resonance. Then the reactive part cancels and the tank looks purely resistive:

$$Z_{L2}(j\omega_{02}) \approx R_{p3}. \tag{160}$$

Step 2: Strong cascode (large output resistance). If the cascode output resistance is much larger than the tank loss resistance,

$$r_{o,\text{cas2}} \gg R_{p3}, \tag{161}$$

then the parallel combination simplifies to

$$Z_{L2}(j\omega_{02}) \parallel r_{o,\text{cas2}} \approx R_{p3}. \tag{162}$$

Step 3: High-pass is flat at resonance. If the high-pass corner is well below the resonance,

$$\omega_{HP2} \ll \omega_{02}, \tag{163}$$

then

$$\big|H_{HP2}(j\omega_{02})\big| \approx 1. \tag{164}$$

Collapsed peak gain (Razavi-style). Substituting (160)–(164) into (159) gives the compact peak gain:

$$A_{v2,\max} \triangleq A_{v2}(j\omega_{02}) \approx -g_{m3} R_{p3}. \tag{165}$$

Connection to Razavi's compact expressions. Razavi's result at resonance under matching assumptions can be written as

$$A_v \approx \frac{\omega_T R_1}{2\omega_0 R_S} = \frac{R_1}{2\omega_0 L_1}. \tag{166}$$

Equation (165) has the same structure "(effective transconductance) × (effective load resistance)". In our notation, the correspondence at resonance is

$$R_1 \leftrightarrow R_{p3}, \qquad \omega_0 \leftrightarrow \omega_{02}, \qquad g_m \leftrightarrow g_{m3} \text{ (or } G_{m,\text{eff}} \text{ if degeneration is included)}, \tag{167}$$

so the long s-domain expression reduces to the same compact gain form when evaluated at resonance with the same simplifying assumptions.

Stage-3 note. Since Stage-2 and Stage-3 share the same schematic structure, the Stage-3 gain follows the same reduction steps:

$$A_{v3,\max} \approx -g_{m5} R_{p4}, \tag{168}$$

with $(\omega_{02}, R_{p3}, g_{m3})$ replaced by $(\omega_{03}, R_{p4}, g_{m5})$.

How the Stage-2 and Stage-3 Values Were Obtained

1) Tank from Target f_0 and Q_p

For a shunt (parallel) tank $L \parallel C \parallel R_p$,

$$\omega_0 = 2\pi f_0, \tag{169}$$

$$C = \frac{1}{\omega_0^2 L}, \tag{170}$$

$$R_p = \frac{Q_p}{\omega_0 C} = Q_p \omega_0 L. \tag{171}$$

Stage-2 target: $f_{0,2} = 5.50\,\text{GHz}$, $Q_{p2} \approx 5.0$, $L_2 = 1.00\,\text{nH}$.

$$\omega_{0,2} = 2\pi \cdot 5.50 \times 10^9/\text{s} \approx 3.46 \times 10^{10}/\text{s}, \tag{172}$$

$$C_2 = \frac{1}{\omega_{0,2}^2 L_2} \approx \frac{1}{(3.4558 \times 10^{10})^2 \cdot 1.0 \times 10^{-9}} \approx 0.837\,\text{pF}, \tag{173}$$

$$R_{p2} = Q_{p2}\,\omega_{0,2} L_2 \approx 5.0 \cdot (3.4558 \times 10^{10}) \cdot 1.0 \times 10^{-9} \approx 173\,\Omega. \tag{174}$$

Stage-3 target: $f_{0,3} = 7.20\,\text{GHz}$, $Q_{p3} = 8.2$, $L_3 = 1.00\,\text{nH}$.

$$\omega_{0,3} = 2\pi \cdot 7.20 \times 10^9/\text{s} \approx 4.52 \times 10^{10}/\text{s}, \tag{175}$$

$$C_3 = \frac{1}{\omega_{0,3}^2 L_3} \approx \frac{1}{(4.5239 \times 10^{10})^2 \cdot 1.0 \times 10^{-9}} \approx 0.489\,\text{pF}, \tag{176}$$

$$R_{p3} = Q_{p3}\,\omega_{0,3} L_3 \approx 8.2 \cdot (4.5239 \times 10^{10}) \cdot 1.0 \times 10^{-9} \approx 371\,\Omega. \tag{177}$$

2) Pick g_m From a Peak-Gain Target

For a CS driving a shunt tank, a useful resonance rule of thumb is

$$\big|A_v\big|_{\text{pk}} \approx g_m\, R_p \quad \Longrightarrow \quad g_m \approx \frac{\big|A_v\big|_{\text{pk}}}{R_p} \quad \text{(with } |A_v|_{\text{pk}} \text{ in linear, not dB).} \tag{178}$$

Stage-2: target peak $\approx 15.0\,\text{dB}$,

$$\big|A_v\big|_{\text{pk}} = 10^{15/20} \approx 5.623 \;\Rightarrow\; g_{m3} \approx \frac{5.623}{173} \approx 0.0330\,\text{S}. \tag{179}$$

Stage-3: choose g_{m5} strong but not overpowering,

$$g_{m5} = 0.0700\,\text{S} \quad \Rightarrow \quad \big|A_v\big|_{\text{pk}} \approx g_{m5} R_{p3} \approx 0.07 \times 371 \approx 26 \;\; (\approx 28.3\,\text{dB}). \tag{180}$$

3) Gate Caps From an f_T Guess

With a simple RF MOS estimate,

$$C_{\text{sum}} \approx \frac{g_m}{2\pi f_T}, \qquad C_{gs} \approx 0.8\, C_{\text{sum}}, \quad C_{gd} \approx 0.2\, C_{\text{sum}}. \tag{181}$$

Using $f_T = 100\,\text{GHz}$:

Stage-2 devices (from $g_{m3} \approx 0.0330\,\text{S}$, $g_{m4} = 0.0900\,\text{S}$):

$$C_{\text{sum},3} = \frac{0.033}{2\pi \cdot 10^{11}} \approx 52.0\,\text{fF}, \qquad C_{gs3} \approx 41.4\,\text{fF}, \quad C_{gd3} \approx 10.4\,\text{fF}, \tag{182}$$

$$C_{\text{sum},4} = \frac{0.090}{2\pi \cdot 10^{11}} \approx 143\,\text{fF}, \qquad C_{gs4} \approx 115\,\text{fF}, \quad C_{gd4} \approx 28.6\,\text{fF}. \tag{183}$$

Stage-3 devices (from $g_{m5} = 0.0700\,\text{S}$, $g_{m6} = 0.110\,\text{S}$):

$$C_{\text{sum},5} = \frac{0.070}{2\pi \cdot 10^{11}} \approx 112\,\text{fF}, \qquad C_{gs5} \approx 89.2\,\text{fF}, \quad C_{gd5} \approx 22.3\,\text{fF}, \tag{184}$$

$$C_{\text{sum},6} = \frac{0.110}{2\pi \cdot 10^{11}} \approx 175\,\text{fF}, \qquad C_{gs6} \approx 140\,\text{fF}, \quad C_{gd6} \approx 35.0\,\text{fF}. \tag{185}$$

Output-Node Effective C (Tank Shunt)

Because the cascode's C_{gd} is to ground at the output node, keep f_0 correct by

$$C_{2,\text{eff}} \approx C_2 - C_{gd4} \approx 0.837\,\text{pF} - 0.0286\,\text{pF} \approx 0.809\,\text{pF}, \tag{186}$$

$$C_{3,\text{eff}} \approx C_3 - C_{gd6} \approx 0.489\,\text{pF} - 0.0350\,\text{pF} \approx 0.454\,\text{pF}. \tag{187}$$

Cascode Output Resistance Check

With $r_o = 1/g_{ds}$ and the standard two-transistor cascode approximation

$$r_{o,\text{cas}} \approx r_{o,\text{CS}} + r_{o,\text{CG}} + g_{m,\text{CG}}\, r_{o,\text{CS}}\, r_{o,\text{CG}}, \tag{188}$$

we choose small but non-zero g_{ds} so $r_{o,\text{cas}} \gg R_p$ and the tank dominates.

Stage-2: $g_{ds3} = 3.00\,\text{mS} \Rightarrow r_{o3} \approx 333\,\Omega, \quad g_{ds4} = 2.00\,\text{mS} \Rightarrow r_{o4} \approx 500\,\Omega,$ (189)

$$r_{o,\text{cas2}} \approx 333 + 500 + 0.09 \cdot 333 \cdot 500 \approx 15.8\,\text{k}\Omega \gg R_{p2} \approx 173\,\Omega. \tag{190}$$

Stage-3: $g_{ds5} = 2.50\,\text{mS} \Rightarrow r_{o5} \approx 400\,\Omega, \quad g_{ds6} = 1.70\,\text{mS} \Rightarrow r_{o6} \approx 588\,\Omega,$ (191)

$$r_{o,\text{cas3}} \approx 400 + 588 + 0.11 \cdot 400 \cdot 588 \approx 26.9\,\text{k}\Omega \gg R_{p3} \approx 371\,\Omega. \tag{192}$$

Input High-Pass Corners (Series C_B into R_B)

Ignoring the small gate caps for the corner estimate,

$$f_{\mathrm{HP}} \approx \frac{1}{2\pi R_B C_B}. \tag{193}$$

$$\text{Stage-2: } R_{B2} = 50.0\,\mathrm{k\Omega},\ C_{B2} = 4.00\,\mathrm{pF} \Rightarrow f_{\mathrm{HP}} \approx \frac{1}{2\pi \cdot 50 \times 10^3 \cdot 4 \times 10^{-12}} \approx 0.800\,\mathrm{MHz}. \tag{194}$$

$$\text{Stage-3: } R_{B3} = 50.0\,\mathrm{k\Omega},\ C_{B3} = 1.00\,\mathrm{pF} \Rightarrow f_{\mathrm{HP}} \approx \frac{1}{2\pi \cdot 50 \times 10^3 \cdot 1 \times 10^{-12}} \approx 3.18\,\mathrm{MHz}. \tag{195}$$

Block	Parameter	Stage-2 (5.5 GHz)	Stage-3 (7.2 GHz)
Input HPF	R_B	50 k	50 k
	C_B	4 pF	1 pF
Tank	L	1.0 nH	1.0 nH
	f_0	5.5 GHz	7.2 GHz
	Q_p	≈ 5.0	8.2
	C (from f_0)	≈ 0.84 pF	≈ 0.489 pF
	$R_p = Q_p\omega_0 L$	$\approx 173\,\Omega$	$\approx 371\,\Omega$
CS (M3/M5)	g_m	≈ 0.033 S	0.070 S
	g_{ds}	3.0 mS	2.5 mS
	C_{gs}	≈ 42 fF	≈ 89 fF
	C_{gd}	≈ 10.5 fF	≈ 22 fF
CG (M4/M6)	g_m	0.090 S	0.110 S
	g_{ds}	2.0 mS	1.7 mS
	C_{gs}	≈ 115 fF	≈ 140 fF
	C_{gd}	≈ 28.7 fF	≈ 35 fF