Astronomy 401/Physics 903
Lecture 20
The Expansion of the Universe and Newtonian Cosmology

1 The expansion of the universe

In 1929 Hubble published the famous paper "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae," showing that galaxies had recessional velocities proportional to their distance, $v = H_0 d$ and that therefore the universe is expanding. What does this actually mean?

Suppose the Earth doubles in size in an hour. Right now, from Milwaukee, it's 500 miles to Pittsburgh, 1000 miles to Dallas, and 2000 miles to Seattle. After the expansion, it will be 1000 miles to Pittsburgh, 2000 miles to Dallas, and 4000 miles to Seattle, so the expansion velocities we will observe will be 500 miles an hour for Pittsburgh, 1000 miles an hour for Dallas, and 2000 miles an hour for Seattle. The velocity of expansion is directly proportional to the distance; this is a result of expansion that is isotropic and homogeneous. Note that we would observe the same thing no matter where we were, because every point is moving away from every other point. The expansion has no center.

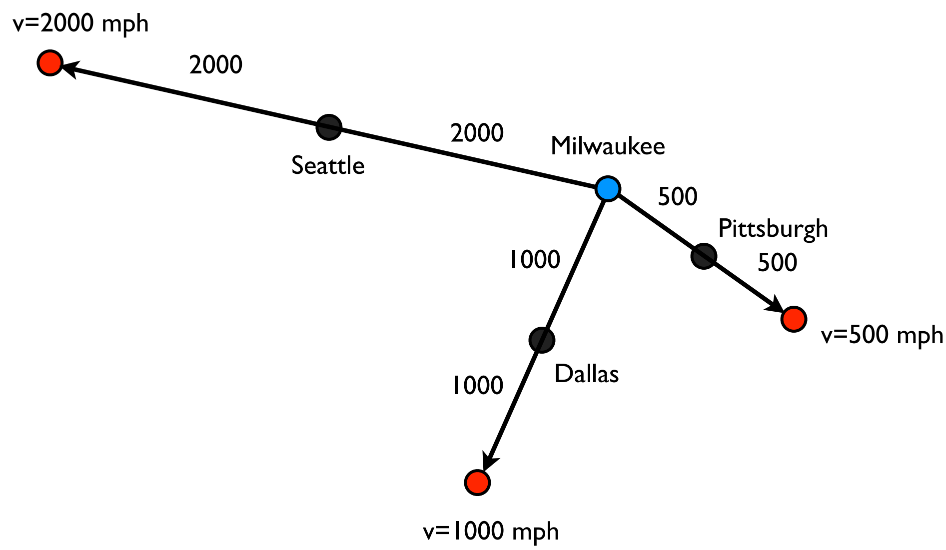

Figure 1: Measured velocity of recession is directly proportional to distance.

Galaxy redshifts are often described as Doppler shifts, but this isn't strictly correct. The redshift is due not to the galaxy moving through space, but because of the expansion of space itself. It is a **cosmological redshift**, not a Doppler shift; the redshift is produced by the expansion of the universe, as the wavelength of light is stretched along with space. The motion of galaxies due to the expansion of the universe is called the **Hubble flow**. Galaxies also have **peculiar velocities**, which are their velocities through space, independent of the Hubble flow. Also important to note that gravitationally bound structures (galaxies, clusters of galaxies) do not participate in the expansion.

1.1 The Hubble constant: review

The Hubble constant is measured using as many different independent distance indicators as possible, and through other tests of cosmological parameters we'll talk about later. For most of the 20th century, the Hubble constant was only known to be between 50 and 100 km s^{-1} Mpc^{-1} (and there were two distinct camps with very strong opinions about whether it was 50 or 100). For this reason, it's common to define the dimensionless parameter h:

$$H_0 = 100\,h \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{1}$$

This is incorporated into measurements of quantities that involve the Hubble constant. For example, the mass of a typical galaxy group is $2 \times 10^{13} h^{-1}$ M$_\odot$. We have better measurements of H_0 now, so $h \simeq 0.7$.

In conventional units, the Hubble constant is inverse time:

$$H_0 = 100\,h \text{ km s}^{-1} \text{ Mpc}^{-1} = 3.24 \times 10^{-18} h \text{ s}^{-1}, \tag{2}$$

so

$$H_0 = 2.27 \times 10^{-18} \text{ s}^{-1} \tag{3}$$

for $h = 0.70$.

To estimate how long ago the Big Bang occurred, we assume (incorrectly) that the recessional velocities of galaxies are constant. We call the time since the Big Bang t_H; this is the time required for a galaxy to travel to a distance d at speed v. So

$$d = vt_H = H_0 d t_H \tag{4}$$

and the **Hubble time** t_H is

$$t_H = \frac{1}{H_0} = 4.4 \times 10^{17} \text{ s} = 1.39 \times 10^{10} \text{ yr}. \tag{5}$$

So the universe is about 13.9 Gyr old, again assuming $H_0 = 70$ km s^{-1} Mpc^{-1}. In fact this is very close to what we estimate the real age of the universe to be.

2 Newtonian cosmology

Now we will describe the evolution of the universe mathematically. This will take some time, but it isn't too complicated when we consider only gravitational forces; the evolution of the universe is essentially a battle between expansion and gravity.

The universe we consider is homogeneous and isotropic, with constant density ρ. We will consider a spherical shell of radius r and mass m. The shell is expanding due to the expansion of the universe, with velocity $v = dr/dt = \dot{r}$.

As the universe expands, the mass inside the shell $M(< r)$ is **constant**—the expansion is uniform, so mass that starts within the shell will always be within the shell. So $\rho r^3 =$ constant.

Because the mass is uniformly distributed, mass outside the shell doesn't matter.

The total energy of the shell as it expands is the sum of its kinetic and potential energies:

$$\frac{1}{2}mv^2 - \frac{GM(< r)m}{r} = E \tag{6}$$

Now let's parameterize the total energy of the shell:

$$E = -\epsilon \, \frac{1}{2} m c^2, \tag{7}$$

where ϵ is just a dimensionless number which we'll rewrite and discuss in a moment. So

$$\frac{1}{2} m v^2 - \frac{G M(<r) m}{r} = -\epsilon \, \frac{1}{2} m c^2. \tag{8}$$

Next we substitute $M(<r) = (4\pi/3) r^3 \rho$, cancel the shell mass m, and multiply by 2:

$$v^2 - \frac{8\pi}{3} G \rho r^2 = -\epsilon c^2. \tag{9}$$

Now let's return to our dimensionless number ϵ, and rewrite it as

$$\epsilon \equiv K x^2. \tag{10}$$

x has units of length, and we will discuss it soon. K therefore has units of (length)$^{-2}$, and it contains information about whether the shell is **gravitationally bound or unbound**.

- Bound: $K > 0$. Total energy $E < 0$, so the shell can recollapse. We call this **closed**.
- Unbound: $K < 0$. Total energy $E > 0$, so kinetic energy wins and the shell keeps expanding. We call this **open**.
- $K = 0$: **Critical** or **flat**. Perfect balance between expansion and gravitational resistance.

So, we have

$$v^2 - \frac{8\pi}{3} G \rho r^2 = -K c^2 x^2. \tag{11}$$

Now we write the radius r in terms of a dimensionless global **scale factor** $a(t)$, which describes the overall expansion of the universe:

$$r(t) = a(t) x \tag{12}$$

$a(t)$ is called the "scale factor" of the universe, and x is called the **comoving coordinate**. The comoving coordinate describes the distance between objects, but it stays constant as the universe expands; this is a useful way to talk about distances when the whole universe is expanding.

For instance, at time t_1, two galaxies are separated by a distance $r(t_1) = a(t_1)x$, and at t_2 they are separated by a distance $r(t_2) = a(t_2)x$. Their comoving separation stays the same, but their physical separation r increases due to the increasing scale factor $a(t)$.

Now we rewrite the expansion velocity

$$v = \frac{dr}{dt} = x \frac{da}{dt} = x \dot{a} \tag{13}$$

and substitute this and $r = ax$ into Equation 11:

$$x^2 \dot{a}^2 - \frac{8\pi}{3} G \rho a^2 x^2 = -K c^2 x^2. \tag{14}$$

We cancel x:

$$\boxed{\dot{a}^2 - \frac{8\pi}{3}G\rho a^2 = -Kc^2} \tag{15}$$

This is an expression for the evolution of only the scale factor a — there are no more references to specific shells or coordinates.

Now we'll do a few more things. First pull out a factor of a^2 on the left side:

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2 = -Kc^2. \tag{16}$$

We can choose the normalization of $a(t)$ in any way we like. For convenience, let's set the scale factor today equal to 1: $a(t_0) = 1$ (t_0 = time today; the 0 subscript refers to the value of something at the present time, e.g. H_0 is the value of the Hubble parameter [which is not a constant] today, and ρ_0 is the density of the universe today). So

$$\left[\frac{\dot{a}(t_0)}{a(t_0)}\right]^2 - \frac{8\pi}{3}G\rho_0 = -Kc^2 \tag{17}$$

at the present time.

Now let's look at the first term. According to the Hubble law,

$$v(t) = H(t)r(t) = H(t)a(t)x. \tag{18}$$

Also,

$$v(t) = \frac{dr(t)}{dt} = x\frac{da(t)}{dt}. \tag{19}$$

Equating these two expressions for v, we find an expression for the Hubble parameter at any time t:

$$H(t) = \frac{da(t)/dt}{a}. \tag{20}$$

So in general,

$$\boxed{H(t) = \frac{\dot{a}(t)}{a(t)}} \tag{21}$$

and at the present day

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)}. \tag{22}$$

We recognize this as the left term in Equation 17 and substitute it in:

$$\boxed{H_0^2 - \frac{8\pi}{3}G\rho_0 = -Kc^2} \tag{23}$$

In order from left to right, the three terms in this equation describe the kinetic, potential and total energies.

Now let's return to that "critical" universe with $K = 0$, where expansion and gravity are perfectly balanced. We can calculate the density we need to make this happen. This is the **critical density** ρ_c. If $K = 0$,

$$H_0^2 - \frac{8\pi}{3}G\rho_c = 0 \tag{24}$$

and

$$\boxed{\rho_c = \frac{3H_0^2}{8\pi G}} \tag{25}$$

This is about 9.2×10^{-27} kg m^{-3}. For comparison, our best estimate of the density of baryonic[1] matter is 4.2×10^{-28} kg m^{-3}; this is about 4% of the critical density.

If $\rho_0 < \rho_c$, there isn't enough mass to reverse the expansion and the universe is open and will expand forever. If $\rho_0 > \rho_c$, gravity wins: the universe is closed and it will recollapse.

We parameterize the density of the universe in terms of a dimensionless number Ω:

$$\Omega_0 \equiv \frac{\rho_0}{\rho_c} \tag{26}$$

- $\Omega_0 = 1$: flat universe with $K = 0$
- $\Omega_0 < 1$: open universe with $K < 0$
- $\Omega_0 > 1$: closed universe with $K > 0$

We can then also rewrite Equation 23 as

$$H_0^2(\Omega_0 - 1) = Kc^2 \tag{27}$$

at the present day.

3 Evolution of the scale factor

We return to our equation for the evolution of the scale factor $a(t)$. This equation is known as the Friedmann equation; it's one of the most important equations in cosmology, and it is properly derived from the theory of general relativity. Our current version of the equation includes only matter, but we will broaden it later to include other components of the universe. The matter-only Friedmann equation is

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi}{3} G\rho a^2(t) = -Kc^2. \tag{28}$$

Remember that the mass within our original shell was constant,

$$\rho(t) r^3(t) = \text{ constant.} \tag{29}$$

Writing this in terms of the scale factor,

$$\rho(t) a^3(t) x^3 = \text{ constant,} \tag{30}$$

where the comoving coordinate x is also constant. So

$$\rho(t) a^3(t) = \rho(t_0) a^3(t_0) = \rho_0 \tag{31}$$

since $a(t_0) = 1$.

[1]Baryons are particles made of three quarks. Protons and neutrons are the most common. Baryons make up nearly all of the visible matter in the universe. (Electrons are leptons—not composed of quarks.) Most dark matter is probably non-baryonic.

Therefore

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi G}{3}\frac{\rho(t)a^3(t)}{a(t)} = -Kc^2 \tag{32}$$

which can be written

$$\boxed{\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi G\rho_0}{3a(t)} = -Kc^2.} \tag{33}$$

This is a differential equation for $a(t)$, and we can solve it to get an expression for the evolution of the size of the universe.