Astronomy 401/Physics 903
Lecture 27
Cosmological Distances II

1 Angular diameter distance

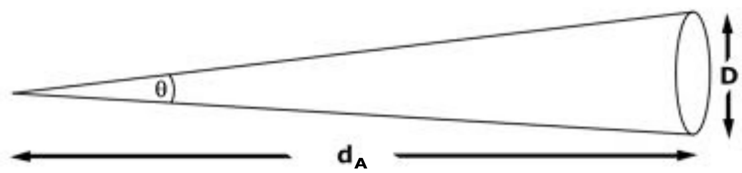

Figure 1: The physical diameter D is given by $D = d_A\theta$, when θ is in radians.

Astronomers are often interested in the physical size of a galaxy (i.e., its size in kpc), but what we measure is the angular size (in arcsec or radians). Converting from angular to physical size requires the distance, and this is not so simple when the universe is curved and expanding. We need to use the Robertson-Walker metric to find the distance, as we did for the proper distance. Now $dx = 0$ (the object is at comoving coordinate x), and we integrate over $d\theta$ since we are integrating over the angle subtended by the object:

$$ds = a(t)x\,d\theta \tag{1}$$

so

$$\begin{aligned} D &= \int ds &(2)\\ &= \int_0^\theta a(t)x\;d\theta &(3)\\ &= a(t)x\,\theta. &(4)\end{aligned}$$

In a flat and non-expanding universe, $D = d_A\theta$, where d_A is the distance to the object. In order to preserve this relationship between physical and angular size, we define by analogy a distance that works the same way, i.e. a distance we can use to translate between physical and angular size. This is called the **angular diameter distance**,

$$d_A = a(t)x, \tag{5}$$

or, in terms of redshift,

$$d_A = \frac{x(z)}{1+z}. \tag{6}$$

In order to evaluate this, we need to know the comoving coordinate as a function of redshift (the coordinate distance or comoving distance). To find this it is useful to define the integral

$$I(z) = H_0\int_0^z \frac{dz'}{H(z')}. \tag{7}$$

Using the expression for $H(z)$ defined by Equation 7 of Lecture 24,

$$H(z) = H_0(1+z)\left[\Omega_{m,0}(1+z) + \Omega_{\mathrm{rel},0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0\right]^{1/2}, \tag{8}$$

this is

$$I(z) \equiv \int_0^z \frac{dz'}{\sqrt{\Omega_{m,0}(1+z')^3 + \Omega_{\text{rel},0}(1+z')^4 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z')^2}}. \tag{9}$$

Then the present proper distance is

$$d_{p,0}(z) = \frac{c}{H_0} I(z) \tag{10}$$

and the comoving distance $x(z)$ is

$$x(z) = \frac{c}{H_0} I(z) \qquad (\Omega_0 = 1) \tag{11}$$

$$x(z) = \frac{c}{H_0\sqrt{\Omega_0 - 1}} \sin\left[I(z)\sqrt{\Omega_0 - 1}\right] \qquad (\Omega_0 > 1) \tag{12}$$

$$x(z) = \frac{c}{H_0\sqrt{1-\Omega_0}} \sinh\left[I(z)\sqrt{1-\Omega_0}\right] \qquad (\Omega_0 < 1). \tag{13}$$

We substitute this into Equation 6,

$$d_A = \frac{x(z)}{1+z}, \tag{14}$$

to calculate d_A. We won't write out the long result here; in practice we do these things numerically, and there are many online cosmological calculators (and an iPhone app) that will calculate all the relevant distances for you.

In a static and Euclidean universe the angular diameter distance is equal to the proper distance. But recall that we derived the proper distance from the Robertson-Walker metric, and for a flat universe $d_p = x$ (the proper distance is equal to the comoving or coordinate distance). This means that for a flat universe the angular diameter distance is equal to the proper distance at the time the light was emitted: $d_A = d_p(t_0)/(1+z) = d_p(t_e)$.

The angular diameter distance is weird! In most cosmological models, the angular size of an object decreases with distance as we would expect until about $z \sim 1.5$ (depending on the cosmological model), and beyond this distance, the angular size gets *bigger* again. This is shown in Figure 2.

2 Luminosity distance

We also want to be able to relate the flux we receive from an object to its intrinsic luminosity. In static, Euclidean space,

$$F = \frac{L}{4\pi d_L^2}, \tag{15}$$

where d_L is the distance to the object. But in our possibly curved and certainly expanding universe things are again more complicated:

- The energy of each photon from the object is reduced because of the cosmological redshift, $\lambda_{\text{obs}} = \lambda_{\text{em}}(1+z)$
- The arrival rate of photons is lower because of **cosmological time dilation**,

  $$\frac{\Delta t_{\text{obs}}}{\Delta t_{\text{em}}} = 1 + z. \tag{16}$$

  This is another consequence of the cosmological redshift. If we think of the light as a wave, the wavelength is stretched by a factor of $(1+z)$, but if we instead consider individual photons their arrival rate will be decreased by a factor of $(1+z)$.

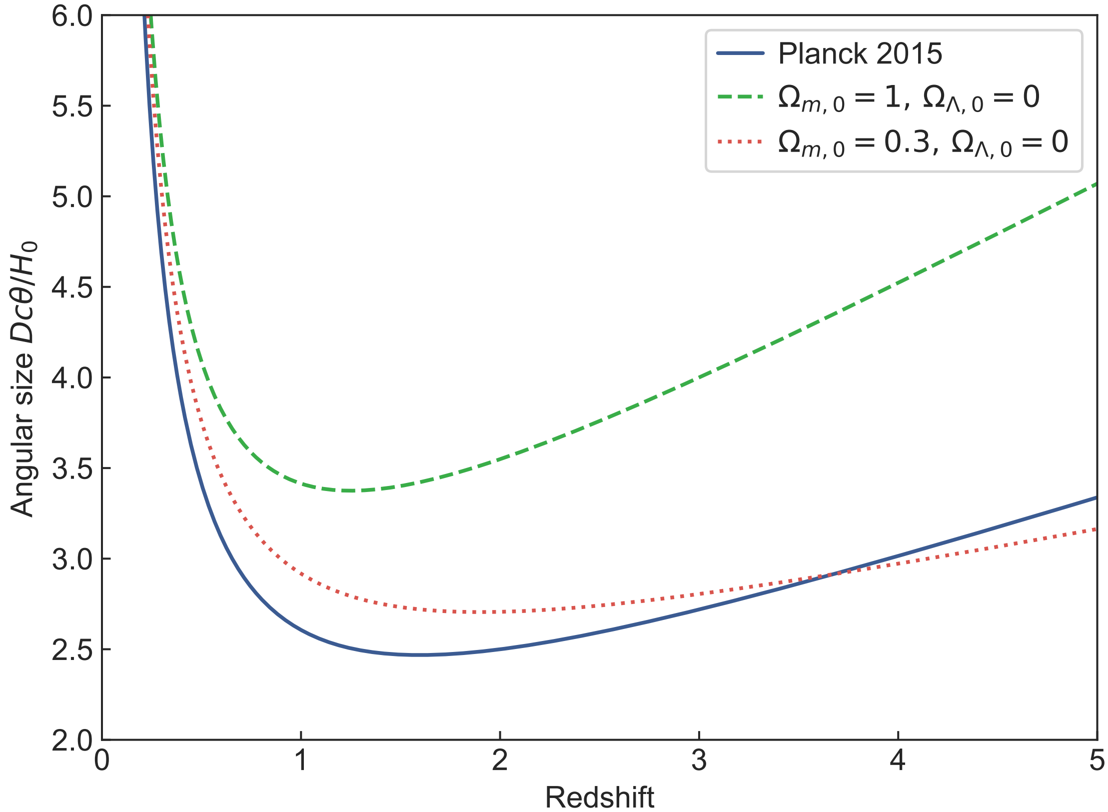

Figure 2: The angular diameter θ of a galaxy in units of $H_0 D/c$ as a function of redshift for several different cosmological models.

- From the Robertson-Walker metric, the object is at comoving coordinate x, and at the time of measurement (i.e. now), photons from the object are spread over a sphere with surface area $4\pi x^2$.

The result is

$$F = \frac{[L/(1+z)^2]}{4\pi x^2} \tag{17}$$

$$= \frac{L}{4\pi x^2 (1+z)^2} \tag{18}$$

where the two factors of $(1+z)$ account for the lower energy and reduced arrival rate of the photons. To retain the usual relationship between flux and luminosity, we define the **luminosity distance** d_L, such that

$$F = \frac{L}{4\pi d_L^2}. \tag{19}$$

So

$$d_L(z) = x(z)(1+z) \tag{20}$$

where we use our previously calculated $x(z)$.

We can use this to construct the distance modulus as usual:

$$m - M = 5 \log\left(\frac{d_L(z)}{10\ \text{pc}}\right). \tag{21}$$

This is called the **redshift-magnitude relation**.

So we have the luminosity distance

$$d_L(z) = x(z)(1+z) \tag{22}$$

and the angular diameter distance

$$d_A = \frac{x(z)}{1+z}. \tag{23}$$

The two distances are related by a factor of $(1+z)^2$:

$$d_A(z) = \frac{d_L(z)}{(1+z)^2}. \tag{24}$$

This means that at $z \sim 1$, the angular diameter distance is a factor of ~ 4 smaller than the luminosity distance, and the two distances diverge further with increasing redshift. The comoving, luminosity and angular diameter distances as a function of redshift are shown in Figure 3.

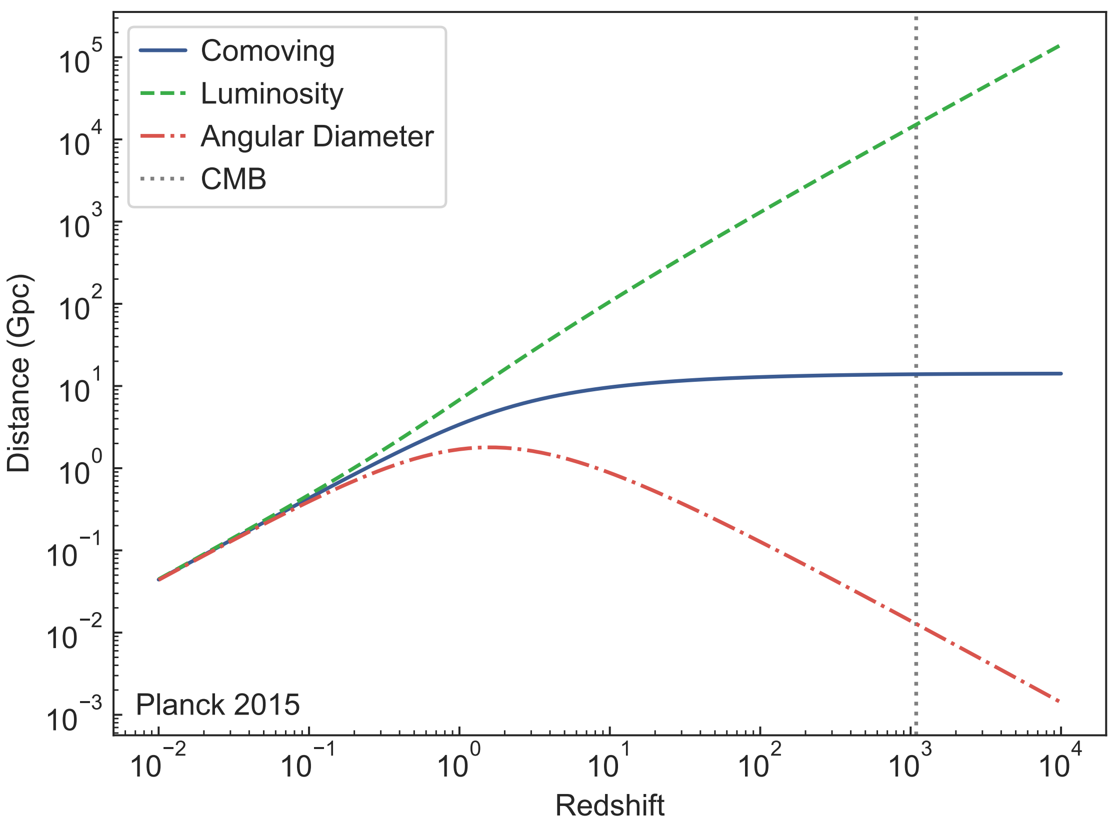

Figure 3: The comoving, luminosity and angular diameter distances as a function of redshift, for the Planck 2015 cosmological model. The CMB redshift $z = 1089$ is marked by the dotted vertical line for reference. The comoving distance to objects at this redshift is 13.9 Gpc.

3 Surface brightness

The difference between the luminosity and angular diameter distances has strong implications for the surface brightness of objects as a function of redshift. The apparent surface brightness is

$$\Sigma = \frac{\text{flux}}{\text{angular area}} = \frac{L/4\pi d_L^2}{(D/d_A)^2} = \left[\frac{L}{4\pi D^2}\right]\frac{d_A^2}{d_L^2} \tag{25}$$

where the term in square brackets does not depend on redshift because it contains only intrinsic properties of the galaxy.

So

$$\begin{aligned} \Sigma &\propto \left(\frac{d_A}{d_L}\right)^2 && (26) \\ &\propto \frac{1}{(1+z)^4} && (27) \end{aligned}$$

This is a very strong dependence! At $z \sim 1$, a galaxy's surface brightness has dropped by a factor of $\sim 2^{-4} = 1/16$. This is another reason why distant galaxies are difficult to observe.