Astronomy 401/Physics 903
Lecture 23
The Cosmological Constant

Consider again the now familiar equation

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2 = -Kc^2. \tag{1}$$

We derived this equation from Newtonian physics, by considering the kinetic and potential energy of the universe; recall that the first term refers to the kinetic energy of the universe, the second to the potential energy, and the term on the right to the total energy. A universe containing only matter must be either contracting or expanding: an initially static universe will begin to contract because of gravity, and an initially expanding universe will either continue to expand or reach a maximum and then collapse, depending on the balance of kinetic and potential energies. The only way to make a matter-only universe static is for it to be empty, with $\rho = 0$.[1]

This equation is also a solution to Einstein's field equations for an isotropic, homogeneous universe. In 1922 the Russian mathematician Aleksandr Friedmann solved the field equations and obtained this equation for a non-static universe. We've been calling it the Friedmann equation all along, but strictly speaking this refers to the equation as derived from general relativity. As we will soon see, the constant K refers to the curvature of the universe.

1 The cosmological constant

Einstein developed his field equations before Hubble's discovery of the expanding universe, and he believed that the universe was static. In their original form, his field equations couldn't produce a static, non-empty universe, so Einstein added an additional term (a constant of integration) in order to make the universe static. This term is the cosmological constant Λ, and with this addition the general solution to Einstein's field equations is

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho - \frac{1}{3}\Lambda c^2\right]a^2 = -Kc^2. \tag{2}$$

In our original Newtonian derivation, this would result from adding an additional potential energy term

$$U_\Lambda \equiv -\frac{1}{6}\Lambda m c^2 r^2 \tag{3}$$

to our equation for the energy balance of the universe. The result of this new potential is a force

$$\mathbf{F}_\mathbf{\Lambda} = -\frac{\partial U_\Lambda}{\partial r}\,\hat{\mathbf{r}} = \frac{1}{3}\Lambda m c^2 r\,\hat{\mathbf{r}} \tag{4}$$

which is radially outward for $\Lambda > 0$: a repulsive force on the mass shell countering gravity, which allowed Einstein to balance the universe in an (unstable) equilibrium.

[1] In our Newtonian analysis, if the mass density of the universe is ρ, the gravitational potential Φ is given by Poisson's equation $\nabla^2\Phi = 4\pi G\rho$. The gravitational acceleration $\vec{a}$ at any point in space is the gradient of the potential, $\vec{a} = -\vec{\nabla}\Phi$. If the universe is permanently static, $\vec{a}$ must be zero everywhere, meaning that the potential Φ must be constant in space. But if Φ is constant, then $\rho = 1/(4\pi G)\,\nabla^2\Phi = 0$, and the universe is empty.

2 Effects of the cosmological constant

After Hubble's discovery of the expanding universe, Einstein called the inclusion of this term the "biggest blunder" of his life. However, recent results have indicated that the universe is actually dominated by some sort of energy which behaves like the cosmological constant. We call this **dark energy**, and we'll now look at its effect on the dynamics of the universe.

We write the Friedmann equation in a form that makes it clear that we're now dealing with a three-component universe of mass, relativistic particles and dark energy:

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G(\rho_m + \rho_{\rm rel}) - \frac{1}{3}\Lambda c^2\right]a^2 = -Kc^2. \tag{5}$$

The fluid equation also comes from solving Einstein's field equations with the inclusion of the cosmological constant

$$\frac{d(a^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(a^3)}{dt} \tag{6}$$

where ρ and P are the density and pressure due to every component of the universe. (Λ does not appear in the fluid equation.)

As discussed for the two-component universe earlier, the Friedmann equation and the fluid equation can be combined to produce the acceleration equation:

$$\frac{d^2a}{dt^2} = \left[-\frac{4\pi G}{3}\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right]a \tag{7}$$

We now define the equivalent mass density of dark energy

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0} \tag{8}$$

so that the Friedmann equation becomes

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G(\rho_m + \rho_{\rm rel} + \rho_\Lambda)\right]a^2 = -Kc^2. \tag{9}$$

Note that because ρ_Λ remains constant as the universe expands, more and more dark energy must appear to fill the larger volume.

We can calculate the pressure due to dark energy from the fluid equation

$$\frac{d(a^3\rho_\Lambda)}{dt} = -\frac{P_\Lambda}{c^2}\frac{d(a^3)}{dt} \tag{10}$$

which is

$$3a^2\rho_\Lambda\frac{da}{dt} + a^3\frac{d\rho_\Lambda}{dt} = -\frac{P_\Lambda}{c^2}3a^2\frac{da}{dt} \tag{11}$$

Because ρ_Λ remains constant $d\rho_\Lambda/dt = 0$, so the second term is zero. Canceling $3a^2 da/dt$ from both sides, we find

$$P_\Lambda = -\rho_\Lambda c^2 \tag{12}$$

which is the equation of state for dark energy. In the general equation of state $P = w\rho c^2$, $w = -1$ for the cosmological constant. The pressure due to the cosmological constant is *negative*, while the equivalent mass density is positive. (Note that $w = -1$ for the simplest model of dark energy discussed here; other models with varying values of w are possible, and constraining w observationally is one of the primary goals in the study of dark energy.)

We can then substitute expressions for ρ_Λ and P_Λ into the acceleration equation (Equation 7):

$$\frac{d^2a}{dt^2} = \left[-\frac{4\pi G}{3}\left[\rho_m + \rho_{\rm rel} + \rho_\Lambda + \frac{3(P_m + P_{\rm rel} + P_\Lambda)}{c^2}\right]\right] a. \tag{13}$$