Astronomy 401/Physics 903
Problem Set 9
Due in class, **Thursday April 25, 2019**

1 Circles on the Earth

Suppose that the Earth is a perfectly smooth sphere, and that the fractional uncertainty in your ability to measure distances on this sphere is $\sigma_x/x = 10^{-6}$ (i.e. the uncertainty on your measurement of 1 meter is 10^{-6} m $= 1\,\mu$m). If you were to draw a circle of radius r on the surface of the Earth, how large would r have to be in order for you to be able to detect a difference between the expected ($C = 2\pi r$) and measured values of the circle's circumference? Remember that a measurement must have significance of at least 3σ in order to be a detection.

What if your fractional uncertainty is 10^{-3}, i.e. the uncertainty on your measurement of 1 meter is 1 mm?

2 Single component universes

The Friedmann equation governing the dynamics of a flat universe with a single component of (matter or equivalent mass) density ρ is

$$\left(\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}\rho\, a^2 = 0. \tag{1}$$

Use this equation to derive the dependence of the scale factor a on time t, for

a) a universe containing only matter with density ρ_m;

b) a universe containing only radiation with density ρ_r;

c) a universe containing only a cosmological with density ρ_Λ.

Recall that $\rho_m(a) = \rho_{m,0}\, a^{-3}$, $\rho_r(a) = \rho_{r,0}\, a^{-4}$, and $\rho_\Lambda(a) = \rho_{\Lambda,0}$.

3 Acceleration and the equation of state

a) The acceleration equation for a universe containing a single component with density ρ and pressure P is

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) a. \tag{2}$$

If the single component of the universe has equation of state $P = w\rho c^2$, find an expression for the deceleration parameter $q(t)$ in terms of w and Ω, the ratio of ρ to the critical density ρ_c. Recall the definition

$$q(t) \equiv -\frac{a(t)[d^2a(t)/dt^2]}{[da(t)/dt]^2}. \tag{3}$$

b) Suppose that this universe is flat, with $\Omega = 1$. What is the value of w that separates an accelerating and a decelerating universe? (Also recall that q is defined such that $q > 0$ if the universe is decelerating.)

4 The cosmological constant

Einstein originally introduced the cosmological constant Λ to stabilize his model of a pressureless matter-only universe against expansion or contraction.

a) Find an expression for Λ in terms of the density ρ_m of a static model of a pressureless matter-only universe with a cosmological constant.

b) Find an expression for the curvature K for this static model. Is this model universe closed, open or flat?

c) Explain why Einstein's static model is in an unstable equilibrium, so any departure from equilibrium (expansion or contraction) will tend to increase.

5 Cosmological calculator

This problem is required only for students enrolled in Physics 903.

Use the equations given in lecture and in the text to write your own cosmological calculator. This will be a program that allows you to specify values for the cosmological parameters H_0, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$, and then calculates the age of the universe, the angular diameter distance, and the luminosity distance for a given redshift z. You may assume that $\Omega_{\text{rel},0} = 0$ and that $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0}$.

Use your program to plot the age of the universe and the values of the luminosity and angular diameter distances vs. redshift for the redshift range $0 < z < 10$, for three different cosmological models:

1. The Planck 2015/2016 values of H_0, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$ (these results are called Planck 2015 sometimes and Planck 2016 sometimes because they were released in 2015 but the publication date of the paper is 2016);
2. A universe in which $H_0 = 70$ km s^{-1} Mpc^{-1}, $\Omega_{m,0} = 1$ and $\Omega_{\Lambda,0} = 0$ (this is called the Einstein-de Sitter model); and
3. An open universe with $H_0 = 70$ km s^{-1} Mpc^{-1}, $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0$.

Also make a table for each of these models giving the value of each parameter at redshifts $z = 0, 1, 2, 3 \ldots 9, 10$.

For which of these three cosmological models is the universe now the oldest, and for which is it the youngest? Which model has the largest distances to galaxies at $z = 10$, and which the smallest?

Note: There are many codes and online calculators that will do this, including the `astropy` cosmology package. You are encouraged to use them to check your results, but for this problem you must do your own calculations beginning with the original cosmological equations.