Astronomy 401/Physics 903
Lecture 21
Newtonian Cosmology II

1 Evolution of the scale factor

We have derived a differential equation for the scale factor $a(t)$,

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{8\pi G\rho_0}{3a(t)} = -Kc^2. \tag{1}$$

and we can solve it to get an expression for the evolution of the size of the universe.

1.1 Flat universe

We'll start with the case in which the universe is flat, with $K = 0$ and density equal to the critical density,

$$\rho_0 = \rho_{c,0} = \frac{3H_0^2}{8\pi G}. \tag{2}$$

Then our differential equation is

$$\left[\frac{da(t)}{dt}\right]^2 - \frac{H_0^2}{a(t)} = 0. \tag{3}$$

We solve:

$$\begin{aligned}
\frac{da}{dt} &= \frac{H_0}{\sqrt{a(t)}} && (4)\\
\sqrt{a(t)}\; da &= H_0\; dt && (5)\\
\int_0^a \sqrt{a}\; da &= H_0 \int_0^t dt && (6)\\
\frac{2}{3}a^{3/2}(t) &= H_0 t && (7)
\end{aligned}$$

and

$$\boxed{a(t) = \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3}} \tag{8}$$

where $t_H \equiv 1/H_0$ is the Hubble time.

So, for a critical, flat universe ($\Omega_0 = 1$), $a(t) \propto t^{2/3}$. Note that this expands forever. Also note that, since $a(t_0) = 1$,

$$t_0 = \frac{2}{3H_0}. \tag{9}$$

This is the current age of a flat universe containing the critical density of matter (which is not the universe we live in, as we will see later).

1.2 Closed universe

Things get more complicated if the universe isn't flat. For a closed universe with $\Omega_0 = \rho_0/\rho_c > 1$ we can express the solution to our differential equation in parametric form:

$$\begin{aligned} a &= \frac{4\pi G\rho_0}{3Kc^2}(1-\cos\theta) && (10) \\ &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos\theta) && (11) \end{aligned}$$

and

$$\begin{aligned} t &= \frac{4\pi G\rho_0}{3K^{3/2}c^3}(\theta-\sin\theta) && (12) \\ &= \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta-\sin\theta) && (13) \end{aligned}$$

We increase the value of θ, and see what happens to a and t:

$\theta = 0$:

$$\begin{aligned} a &= 0 && (14) \\ t &= 0 && (15) \end{aligned}$$

$\theta = \pi$:

$$\begin{aligned} a &= \frac{8\pi G\rho_0}{3Kc^2} = \frac{\Omega_0}{\Omega_o - 1} && (16) \\ t &= \frac{4\pi^2 G\rho_0}{3K^{3/2}c^3} = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} && (17) \end{aligned}$$

$\theta = 2\pi$:

$$\begin{aligned} a &= 0 && (18) \\ t &= 2\times t(\theta=\pi) && (19) \end{aligned}$$

So the universe recollapses! (These are the equations for a cycloid; see Figure 1.)

What's the current age of the universe in this model? This gets a bit messy. The scale factor today is $a_0 = 1$, so from Equation 11,

$$\frac{2(\Omega_0 - 1)}{\Omega_0} - 1 = -\cos\theta_0. \quad (20)$$

Solving for θ_0, we have

$$\theta_0 = \cos^{-1}\left[1 - \frac{2(\Omega_0 - 1)}{\Omega_0}\right] = \cos^{-1}\left[\frac{2}{\Omega_0} - 1\right]. \quad (21)$$

Using the identity

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}, \quad (22)$$

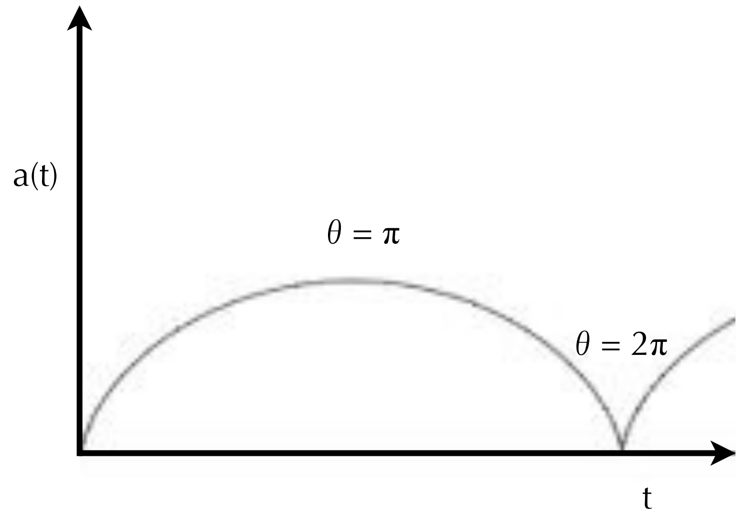

Figure 1: Evolution of the scale factor for a closed universe.

we see that

$$\begin{aligned}\sin\theta_0 &= \sin\left(\cos^{-1}\left[\frac{2}{\Omega_0}-1\right]\right) &(23)\\ &= \sin\left(\sin^{-1}\sqrt{1-\left[\frac{2}{\Omega_0}-1\right]^2}\right) &(24)\\ &= \sqrt{1-\left[\frac{2}{\Omega_0}-1\right]^2}. &(25)\end{aligned}$$

Inserting these expressions for θ_0 and $\sin\theta_0$ into Equation 13, we find the age of the universe

$$t_0 = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left[\frac{2}{\Omega_0}-1\right]-\sqrt{1-\left[\frac{2}{\Omega_0}-1\right]^2}\right]. \quad (26)$$

1.3 Open universe

For an open universe with $K < 0$ and $\Omega_0 < 1$, we again have parametric solutions for the evolution of the scale factor:

$$\begin{aligned}a &= \frac{4\pi G\rho_0}{3|K|c^2}(\cosh\theta-1) &(27)\\ &= \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh\theta-1) &(28)\end{aligned}$$

and

$$\begin{aligned}t &= \frac{4\pi G\rho_0}{3|K|^{3/2}c^3}(\sinh\theta-\theta) &(29)\\ &= \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\theta-\theta) &(30)\end{aligned}$$

where $\sinh\theta$ and $\cosh\theta$ are the hyperbolic sine and cosine functions:

$$\cosh\theta \equiv \frac{e^{\theta}+e^{-\theta}}{2} \geq 1 \tag{31}$$

$$\sinh\theta \equiv \frac{e^{\theta}-e^{-\theta}}{2} \geq \theta \tag{32}$$

In this case a is monotonically increasing, so the universe expands forever.

For all of these expressions, you can see that the age of the universe depends on Ω_0. More dense universes are younger.[1]

2 Cosmological redshift

The evolution of the scale factor affects more than galaxies. The expansion of the universe also changes the wavelength of light; light stretches as the universe expands.

Suppose light is emitted with wavelength $\lambda_1 = \lambda_{\rm em}$ at some time $t_1 \ll t_0$, when the universe was smaller than its current size—for example, suppose the scale factor $a_1(t_1) = 1/3$ (i.e. the universe was one-third of its current size when the light was emitted). We detect the light at time t_0, when the universe has tripled in size, so $a_0 = 1 = 3a_1$. The wavelength of the light has expanded along with the universe, so it is now $\lambda_0 = 3\lambda_{\rm em}$. So, recalling the definition of redshift,

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} \equiv (1+z) = \frac{a(t_0)}{a(t_1)}. \tag{33}$$

For this example, the redshift $z = 2$, since the observed wavelength of the light is $1 + z = 3$ times longer than the emitted wavelength.

So the redshift and scale factor are directly related, and when we talk about the history of the universe we usually just parameterize it in terms of the redshift we would measure today for photons which were emitted at some earlier time $t < t_0$ when the universe was more compact.

$$(1+z) = \frac{a(t_0)}{a(t_1)}, \tag{34}$$

so

$$a(t) = \frac{1}{(1+z)}, \tag{35}$$

independent of any of the cosmological parameters like H_0 or Ω_0. So, for example, at a redshift $z = 3$, the universe was 1/4 of its current size.

For a flat universe we derived

$$a(t) = \left(\frac{t}{\frac{2}{3}t_H}\right)^{2/3} \tag{36}$$

which can be rearranged to

$$t(z) = \left(\frac{2}{3}t_H\right)\left(\frac{1}{(1+z)^{3/2}}\right). \tag{37}$$

So as we found before, today at $z = 0$ $t_0 = (2/3)t_H$, and at $z = 3$, $t = t_0/8$. If $\Omega_0 = 1$ and we observe a galaxy at redshift $z = 3$, we are observing a galaxy at 1/8 the current age of the universe.

[1] For a given value of H_0; this makes sense, since for a given expansion rate H_0 a higher density universe will take less time to reach its current density.

3 Lookback time

It's also useful to define the "lookback time" to redshift z, the amount of time which has passed between when a redshifted photon was emitted and when we detect it today.

$$t_{\text{lookback}} \equiv t_0 - t(z) \tag{38}$$

In other words, this is just the difference in age between the universe today and the universe at time $t(z)$. The lookback time depends on Ω_0 and H_0.

4 The expansion rate of the universe and the flatness problem

The rate at which the universe is expanding is a function of time—the value of the Hubble "constant" isn't constant, $H(z) \neq H_0$ and

$$H(z) = \frac{\dot{a}(t)}{a(t)} \neq \frac{\dot{a}(t_0)}{a(t_0)}. \tag{39}$$

(Recall that the scale factor is related to redshift by $a(t) = 1/(1+z)$.)

We can derive an expression for the evolution of the Hubble parameter with redshift. We return to our equation for the evolution of the scale factor

$$\left[\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2(t) = -Kc^2. \tag{40}$$

Kc^2 is constant and $H(z) = \dot{a}(t)/a(t)$, so

$$\left[H^2(z) - \frac{8\pi}{3}G\rho\right]a^2(t) = \left[H_0^2 - \frac{8\pi}{3}G\rho_0\right]a^2(t_0). \tag{41}$$

Since $a(t_0) = 1$ and $a(t) = 1/(1+z)$,

$$H^2(z) - \frac{8\pi}{3}G\rho = \left[H_0^2 - \frac{8\pi}{3}G\rho_0\right](1+z)^2. \tag{42}$$

From the definition of the critical density

$$\Omega_o = \frac{\rho_0}{\rho_c} = \frac{8\pi G\rho_0}{3H_0^2}, \tag{43}$$

$$\frac{8\pi G\rho_0}{3} = H_0^2\Omega_0, \tag{44}$$

and because mass is conserved,

$$\rho(z)a^3(t) = \rho_0 a^3(t_0) \Rightarrow \rho(z) = \rho_0(1+z)^3. \tag{45}$$

The universe was denser at higher redshifts. So

$$\frac{8\pi G\rho(z)}{3} = H_0^2\Omega_0(1+z)^3 \tag{46}$$

and we have

$$H^2(z) - H_0^2\Omega_0(1+z)^3 = H_0^2[1-\Omega_0](1+z)^2 \tag{47}$$

and

$$H^2(z) = H_0^2(1+z)^2[\Omega_0(1+z) + 1 - \Omega_0] \tag{48}$$

which simplifies to

$$\boxed{H^2(z) = H_0^2(1+z)^2(1+\Omega_0 z)}. \tag{49}$$

This tells us about the evolution of the expansion rate of the universe.

It is also useful to calculate the evolution of the density of the universe. As noted above, the universe used to be smaller and denser, so $\Omega(z) \neq \Omega_0$. We can calculate $\Omega(z)$ starting from Equation 41 and using the definition of the density parameter Ω as a function of redshift,

$$\Omega(z) = \frac{\rho(z)}{\rho_c(z)} = \frac{8\pi G}{3H^2(z)}\rho(z). \tag{50}$$

Equation 41 then becomes

$$H^2(z) - H^2\Omega(z) = [H_0^2 - H_0^2\Omega](1+z)^2, \tag{51}$$

since $a = 1/(1+z)$. Rearranging, we have

$$1 - \Omega(z) = \frac{(1-\Omega_0)(1+z)^2}{H^2(z)/H_0^2}. \tag{52}$$

Finally, substituting $H^2(z)/H_0^2$ from Equation 49, the result is

$$\boxed{1 - \Omega(z) = \frac{1-\Omega_0}{1+\Omega_0 z}} \tag{53}$$

The general behavior of Equation 53 is shown in Figure 2, for an open universe with $\Omega_0 = 0.3$ (dashed blue line) and a closed universe with $\Omega_0 = 1.5$ (solid red line). In both cases, $\Omega(z) \to 1$ at high redshift. Flat universes are always flat, but open and closed universes also used to be very close to flat. This is a problem which we'll talk more about later. $\Omega = 1$ is unstable: if the universe was even slightly more dense than ρ_c, then at late times it's much denser than ρ_c, and if it were slightly less dense than ρ_c, it should now be much less dense than ρ_c. So the current matter density of the universe ($\Omega_m \sim 0.3$) is very unlikely! In order to have this density today, the universe would have to have formed with a density within a very tiny fraction of the critical density—one part in 10^{62} or less. This is called the **flatness problem**. This is one of the main motivations for inflation, the idea that the universe went through a phase of exponential expansion very (*very*) soon after the Big Bang. This is also one of the reasons why theorists favored a flat universe even when observations favored an open universe.

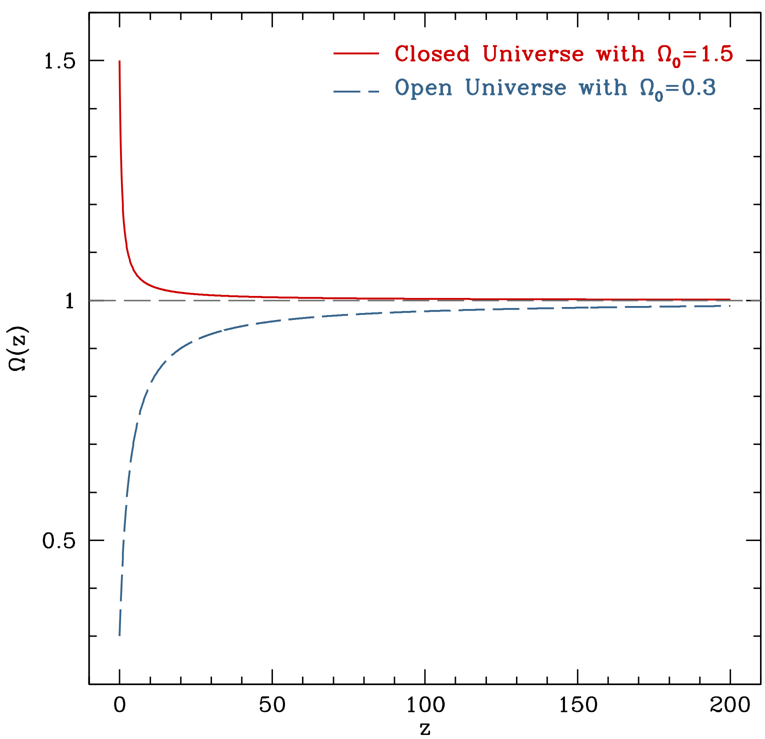

Figure 2: Evolution of the density parameter with redshift.