Astronomy 401/Physics 903
Lecture 24
The Cosmological Constant II

1 The cosmological constant and the expansion of the universe

The Friedmann equation can also be written in terms of the Hubble constant and the density parameter Ω (we showed this for the single component universe a while back) as

$$H^2[1-(\Omega_m+\Omega_{\rm rel}+\Omega_\Lambda)]a^2=-Kc^2 \tag{1}$$

where

$$\Omega_m = \frac{\rho_m}{\rho_c}=\frac{8\pi G\rho_m}{3H^2} \tag{2}$$

$$\Omega_{\rm rel} = \frac{\rho_{\rm rel}}{\rho_c}=\frac{8\pi G\rho_{\rm rel}}{3H^2} \tag{3}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}=\frac{\Lambda c^2}{3H^2}. \tag{4}$$

We define the **total density parameter**

$$\Omega\equiv\Omega_m+\Omega_{\rm rel}+\Omega_\Lambda. \tag{5}$$

Note that Ω without a subscript refers to the total density parameter for all of the components of the model under consideration. The Friedmann equation is then

$$H^2(1-\Omega)a^2=-Kc^2. \tag{6}$$

A flat universe with $K=0$ requires $\Omega(t)=1$.

We can also find the evolution of the Hubble parameter with redshift. This is given by

$$H(z)=H_0(1+z)\left[\Omega_{m,0}(1+z)+\Omega_{\rm rel,0}(1+z)^2+\frac{\Omega_{\Lambda,0}}{(1+z)^2}+1-\Omega_0\right]^{1/2}. \tag{7}$$

The Planck (2015) measurements of the cosmic microwave background in combination with other methods give us our best values for the three components of the universe today:

$$\Omega_{m,0} = 0.3089\pm0.0062 \tag{8}$$

$$\Omega_{\rm rel,0} = 9.161\times10^{-5} \tag{9}$$

$$\Omega_{\Lambda,0} = 0.6911\pm0.0062 \tag{10}$$

Adding these together, the total density parameter $\Omega_0=1.000\pm0.0088$. Within our ability to measure it, the universe is flat and currently dominated by dark energy.

We can also define the **deceleration parameter** $q(t)$:

$$q(t)\equiv-\frac{a(t)[d^2a(t)/dt^2]}{[da(t)/dt]^2} \tag{11}$$

The name and the minus sign, which gives a positive value for a decelerating universe, reflect the once-common belief that the universe had to be decelerating. The deceleration parameter can also be written in terms of the density parameters for the different components of the universe:

$$q(t) = \frac{1}{2}\Omega_m(t) + \Omega_{\rm rel}(t) - \Omega_\Lambda(t). \tag{12}$$

With current values,

$$q_0 = -0.54, \tag{13}$$

telling us (because of the minus sign) that the universe is currently accelerating.

2 The Λ era

We've already discussed the dependence of the density of radiation and matter on the scale factor: $\rho_m \propto a^{-3}$ and $\rho_{\rm rel} \propto a^{-4}$. Because the radiation density decreases more quickly as the universe expands, the universe was dominated by radiation at early times but then became matter-dominated at a redshift of $z_{r,m} = 3270$. We now have another component to consider, ρ_Λ, which is *constant*. This means that at some point, as the matter density decreases as the universe expands, the universe will become dominated by the cosmological constant. As it turns out, this has already happened. Since $a^3\rho_m = \rho_{m,0}$ and $\rho_\Lambda = \rho_{\Lambda,0} =$ constant, the scale factor at the equality of matter and dark energy is

$$a_{m,\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = 0.764, \tag{14}$$

which corresponds to a redshift $z_{m,\Lambda} = 0.308$.

We can also use the acceleration equation to find when the acceleration of the universe changed from negative to positive. The result is

$$a_{\rm accel} = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3} = 0.607, \tag{15}$$

corresponding to a redshift $z_{\rm accel} = 0.648$. So the universe began to accelerate *before* the equality of matter and dark energy; this is because dark energy has pressure as well as equivalent mass density which affects the dynamics of the universe, and (non-relativistic) matter does not. Note also that unlike all of the other cosmological times we've calculated, these are relatively recent. We routinely observe objects at these redshifts, and can use them to understand the dynamics of the universe. More on that when we discuss observational cosmology.

Another consequence of the fact that $\rho_m \propto a^{-3}$ and $\rho_{\rm rel} \propto a^{-4}$ while ρ_Λ is constant is that in the early universe, the densities of matter and radiation were much higher than the dark energy density. This means that we can neglect Λ in the early universe, and all the results we've already derived for the early universe are still valid.

Finally, we can derive the time dependence of the scale factor a for a flat universe by setting $K = 0$ in the Friedmann equation. An analytic solution requires neglecting the relativistic particle density $\rho_{\rm rel}$, and the eventual result is

$$t(a) = \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\ln\left[\sqrt{\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)a^3} + \sqrt{1 + \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)a^3}\,\right]. \tag{16}$$

Since the scale factor $a = 1/(1+z)$, this is also an expression for the age of the universe as a function of redshift:

$$t(z) = \frac{2}{3}\frac{1}{H_0\sqrt{\Omega_{\Lambda,0}}}\ln\left[\sqrt{\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)(1+z)^{-3}} + \sqrt{1+\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)(1+z)^{-3}}\,\right]. \tag{17}$$

We can find the current age of the universe by setting $z = 0$ or $a = 1$:

$$t_0 = 1.38 \times 10^{10} \text{ yr}, \tag{18}$$

in agreement with the best-fit age of the universe from Planck, 13.799 ± 0.021 Gyr. We can also use this expression to find the time at which the universe began to accelerate: $t_{\text{accel}} = 7.57$ Gyr, so the universe has been accelerating for somewhat more than the second half of its existence.

3 What is dark energy?

We don't know, but a possible candidate is **vacuum energy**. In classical physics there is no such thing, but quantum physics allows for energy in a vacuum. The Heisenberg uncertainty principle allows particle-antiparticle pairs to spontaneously appear and then annihilate in an otherwise empty vacuum. The total energy and lifetime of these particles must satisfy the relation

$$\Delta E \Delta t \geq h. \tag{19}$$

There is an energy density associated with these particle-antiparticle pairs, and this energy density is a quantum phenomenon that doesn't care at all about the expansion of the universe or the passage of time. However, calculating the actual value of this energy density is an exercise in quantum field theory that hasn't yet been completed. One suggestion is that the natural value for the vacuum energy density is the Planck energy density,

$$u_{\text{vac}} \sim u_P = \frac{E_P}{l_P^3}. \tag{20}$$

However, the Planck energy is large (by particle physics standards; $E_P = \sqrt{\hbar c^5/G} = 1.96 \times 10^9$ J) while the Planck length is very small ($l_P = 1.6 \times 10^{-35}$ m), and the resulting energy density $u_P \sim 4.8 \times 10^{113}$ J m^{-3} is 123 orders of magnitude larger than the current critical density of the universe, $u_c = 7.7 \times 10^{-10}$ J m^{-3} (!). Clearly we don't yet understand the energy density of the vacuum. The question of what dark energy is and why it has the value it does is very much still open, but observations of the expansion of the universe will help by constraining the value of u_{vac}.