Astronomy 401/Physics 903
Lecture 19
Error propagation and uncertainties

The distance ladder is interconnected—calibration at large distances depends on calibration locally (this is why it's called a ladder, since each higher step depends on the steps below), and errors propagate up the ladder.

There are two kinds of errors: *random errors*, which are imprecision due to a limited amount of data and are centered around the true value, and *systematic error*, which is a global offset so that all measurements will not converge to the true value (calibration problems, for example).

None of these distance measurements are perfect, and it's important to consider how uncertainties in our measurements and calibrations will affect derived quantities like distances.

For example: we measure flux F with uncertainty σ_F, and derive the distance

$$r = \sqrt{\frac{L}{4\pi F}}, \tag{1}$$

assuming that we are using a standard candle so L is known. What is the uncertainty in the distance?

We want to know how changes in F cause changes in r, so we look at derivatives in the relationship between flux and distance.

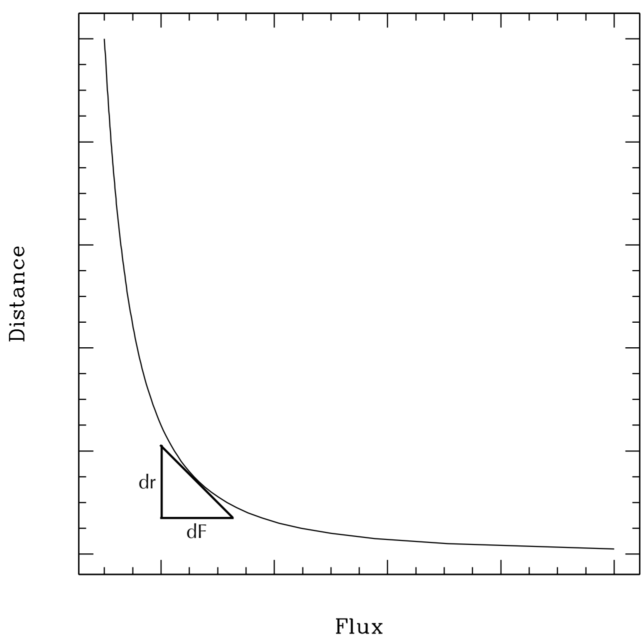

Figure 1: Uncertainties in flux and distance

$$r = \sqrt{\frac{L}{4\pi}} \times F^{-1/2} \tag{2}$$

$$dr = \sqrt{\frac{L}{4\pi}} \times \left[-\frac{1}{2} F^{-3/2} \right] dF \tag{3}$$

$$= \quad -\frac{1}{2}\sqrt{\frac{L}{4\pi F}}\,\frac{dF}{F} \tag{4}$$

So

$$\frac{dr}{r} = -\frac{1}{2}\frac{dF}{F} \tag{5}$$

and the fractional error in the distance is half the fractional error in the flux. A 20% error in the flux leads to a 10% error in the distance.

Formally, if $y = f(x)$, with uncertainty σ_x in x,

$$\sigma_y^2 = \left(\frac{dy}{dx}\right)^2 \sigma_x^2. \tag{6}$$

If there are multiple sources of error, $z = f(x, y)$ with uncertainties in x and y,

$$\sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2. \tag{7}$$

This is called the **propagation of errors**, and this relationship is only valid for random errors, not systematic!

We can use this to see the relationship between errors in fluxes and magnitudes:

$$\begin{aligned}
m &= C - 2.5\log F && (8)\\
&= C - 2.5\log e \ln F && (9)\\
&= C - 1.086\ln F && (10)\\
\frac{dm}{dF} &= -\frac{1.086}{F} && (11)
\end{aligned}$$

And

$$\begin{aligned}
\sigma_m^2 &= \left(\frac{dm}{dF}\right)^2 \sigma_F^2 && (12)\\
&= \left(-\frac{1.086}{F}\right)^2 \sigma_F^2 && (13)\\
&= 1.18\left(\frac{\sigma_F}{F}\right)^2 && (14)
\end{aligned}$$

So

$$\sigma_m \simeq 1.1\frac{\sigma_F}{F} \tag{15}$$

This means that an uncertainty in magnitude is a fractional uncertainty in flux! A magnitude error $\Delta m = 0.1$ is equivalent to a 10% uncertainty in flux.