Astronomy 401/Physics 903
Lecture 22
The Cosmic Microwave Background and the Thermal History of the Universe

1 Adding pressure to the model of the universe

So far we have only considered the gravitational effect of matter in our model of the universe. We will now broaden this a bit. We start with our now familiar differential equation for the evolution of the scale factor, the Friedmann equation:

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3}\right]a^2 = -Kc^2. \tag{1}$$

We will expand the definition of ρ to include relativistic particles like photons and neutrinos as well as normal matter. For normal matter, ρ is just the usual mass density. For relativistic particles, we make use of the equivalence of mass and energy: ρ is the energy density divided by c^2: $\rho = u/c^2$.

Remember that the Friedmann equation is essentially a statement of the conservation of energy; it says that the sum of the gravitational potential energy and the kinetic energy of expansion of the universe is constant. We will use another expression of conservation of energy to calculate the effects of components of the universe that produce pressure. Our universe is now filled with a fluid of density ρ (this is now the equivalent mass density), temperature T, and pressure P. We will apply the first law of thermodynamics to this fluid:

$$dU = dQ - PdV \tag{2}$$

(This says that the change in the internal energy of a system is equal to the amount of heat supplied to the system, minus the amount of work performed by the system on its surroundings. The work is equal to the pressure times the change in volume.) The universe has the same temperature everywhere, so there is no heat flow: $dQ = 0$, and therefore the expansion of the universe is adiabatic. We write this as a time derivative,

$$\frac{dU}{dt} = -P\frac{dV}{dt} \tag{3}$$

and substitute $V = \frac{4}{3}\pi r^3$ to obtain

$$\frac{dU}{dt} = -\frac{4}{3}\pi P\frac{d(r^3)}{dt}. \tag{4}$$

(Note that $d(r^3)/dt = 3r^2\dot{r}$.) We then define the internal energy per unit volume:

$$u = \frac{U}{\frac{4}{3}\pi r^3} \tag{5}$$

so that

$$\frac{dU}{dt} = \frac{4\pi}{3}\frac{d(r^3u)}{dt}. \tag{6}$$

Equating this with Equation 4 above, we find

$$\frac{d(r^3u)}{dt} = -P\frac{d(r^3)}{dt}. \tag{7}$$

We then write u in terms of the equivalent mass density ρ:

$$\rho = \frac{u}{c^2} \tag{8}$$

to get

$$\frac{d(r^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(r^3)}{dt}. \tag{9}$$

We then substitute $r = ax$ and cancel x because it's constant:

$$\boxed{\frac{d(a^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(a^3)}{dt}} \tag{10}$$

This is called the **fluid equation**, and is the second key equation describing the expansion of the universe. Note that for the pressureless model we considered earlier, $P = 0$ and $a^3\rho =$ constant, as we expect.

The fluid equation and the Friedmann equation are both statements of energy conservation. We can combine them to derive an equation that describes how the expansion of the universe behaves with time. We multiply the Friedmann equation by a:

$$a\left(\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho a^3 = -Kc^2 a \tag{11}$$

and take a time derivative:

$$\left(\frac{da}{dt}\right)^3 + 2a\left(\frac{da}{dt}\right)\left(\frac{d^2a}{dt^2}\right) - \frac{8\pi}{3}G\frac{d(\rho a^3)}{dt} = -Kc^2\frac{da}{dt}. \tag{12}$$

We replace Kc^2 using the Friedmann equation again, and use the fluid equation to replace the $d(\rho a^3)/dt$ term, which gives us, after a bunch of algebra,

$$\boxed{\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a} \tag{13}$$

This is the **acceleration equation** (the text refers to it as the second Friedmann equation). Note that the effect of pressure is to slow down the expansion: for positive matter density and pressure, the acceleration is negative, so the universe is slowing down. This may seem counterintuitive, but because there are no pressure gradients in an isotropic universe, there is no net force associated with the pressure. Instead, the kinetic energy of the particles has an equivalent mass density that exerts a gravitational attraction to slow down the expansion.

We'll write down the Friedmann equation again, so that we have all three important equations together:

$$\boxed{\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2 = -Kc^2} \tag{14}$$

These equations have three unknowns, a, ρ and P, but they are not independent: we can use any two to derive the third, as we just did for the acceleration equation. So to solve for a, ρ and P we need another equation: an **equation of state** that links the variables. We write the equation of state as

$$P = w\rho c^2, \tag{15}$$

where w is a constant, so the pressure is proportional to the energy density of the fluid. For matter with no pressure, as in our first model of the universe, $w = 0$. A fluid of photons or other massless particles is relativistic, and has the equation of state

$$P = \frac{1}{3}\rho c^2, \tag{16}$$

so $w = 1/3$.[1] We will return to this question of pressure and the equation of state when we discuss the cosmological constant.

[1] See Section 4.2.5 of the text for a derivation of this factor.

2 Cooling of the universe after the Big Bang

A key point of the Big Bang theory is that the early universe was very dense and hot. We expect that this hot, dense universe would have been in thermodynamic equilibrium, and that therefore the radiation field had a blackbody spectrum. We can compute the cooling of this radiation as the universe expands.

The energy density of blackbody radiation is

$$u = a_{\rm rad} T^4, \tag{17}$$

where $a_{\rm rad}$ is the radiation constant $a_{\rm rad} \equiv 4\sigma/c$. By replacing P in the fluid equation

$$\frac{d(a^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(a^3)}{dt} \tag{18}$$

with the equation of state $P = w\rho c^2$ we find

$$a^{3(1+w)}u = a^4 u = u_0, \tag{19}$$

since $w = 1/3$ for photons and $a_0 = 1$ (see Problem Set 7). This tells us that the energy density of the universe today is smaller by a factor of a^4 than it was at some earlier time with scale factor a. A factor of a^3 comes from the change in volume of the universe, and an additional factor of a comes from the lower energy of the longer wavelength photons we see today, because of the cosmological redshift. Therefore

$$a_{\rm rad} a^4 T^4 = a_{\rm rad} T_0^4 \tag{20}$$

and the current temperature of the blackbody radiation is related to the temperature at an earlier time by

$$aT = T_0. \tag{21}$$

When the universe was half as large it was twice as hot. Recalling that

$$a = \frac{1}{1+z} \tag{22}$$

we can also write

$$T = (1+z)T_0 \tag{23}$$

for the dependence of the temperature of the radiation on redshift.

We can make an order of magnitude estimate of the current temperature of the blackbody radiation by considering the conditions needed to produce helium in the early universe. The early universe was hot and dense enough for nuclear reactions to take place, and the heaviest element that was formed in these reactions was He (and a very small amount of Li). This fusion requires approximately $T \simeq 10^9$ K and $\rho_b \simeq 10^{-2}$ kg m^{-3}, where the b subscript refers to the baryon density. (If the temperature were higher the deuterium nuclei needed for the reaction would photodisassociate, and if the temperature were lower it would be too difficult to overcome the Coulomb barrier. The density is needed to produce the observed amount of He.) We can therefore estimate the value of the scale factor at the time of helium formation:

$$a \simeq \left(\frac{\rho_{b,0}}{\rho_b}\right)^{1/3} = 3.5 \times 10^{-9} \tag{24}$$

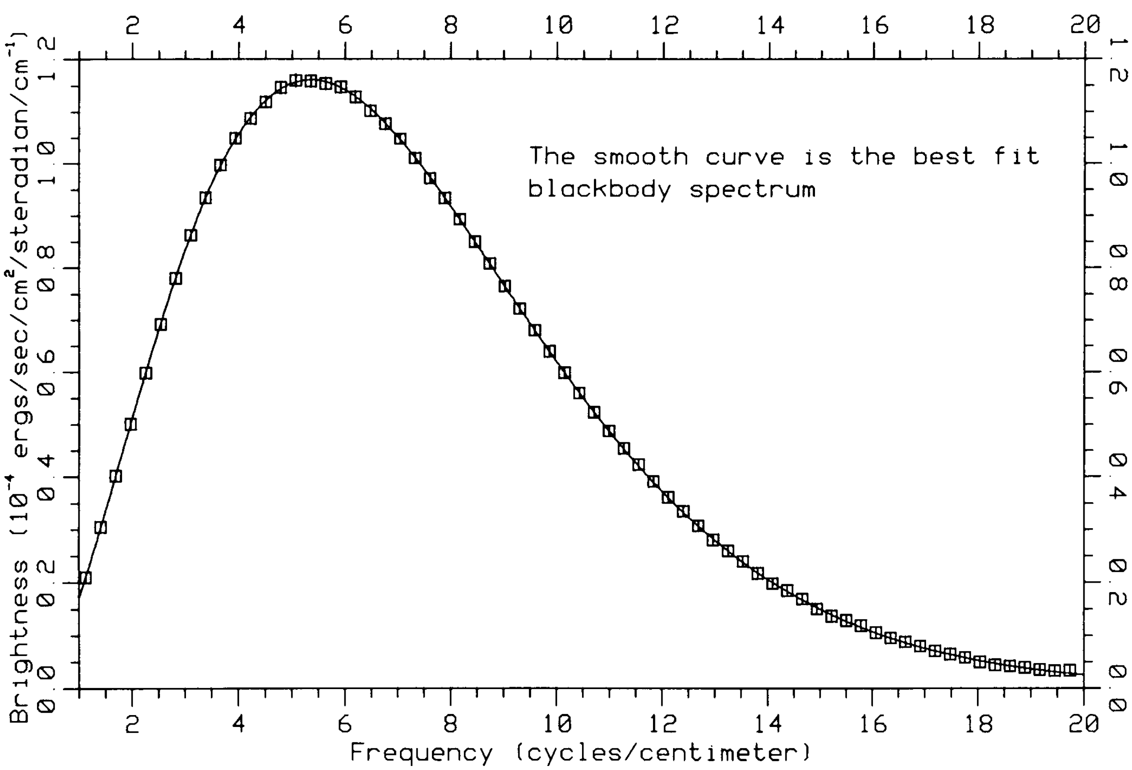
(since $\rho a^3 = \rho_0$). We can then combine this with the temperature required, $T(a) = 10^9$ K, to determine the current temperature of the radiation:

$$T_0 = aT(a) \simeq 3.5 \text{ K} \tag{25}$$

We will see that this simple calculation gives a very good estimate of the actual temperature of the radiation. Also note that this was predicted in 1948, well before it was discovered.

3 Discovery and measurement of the CMB

- Discovered in 1963 by Arno Penzias and Robert Wilson, working at Bell Labs
- Couldn't get rid of background hiss in their signal... even after removing the pigeons from the antenna
- Knew that a 3 K blackbody would produce the signal, but didn't know of any possible source until they heard about recent work at Princeton (Robert Dicke and Jim Peebles) calculating the temperature of relic radiation from the Big Bang
- Penzias and Wilson published the discovery with the title "A Measurement of Excess Antenna Temperature at 4080 Megacycles per Second," a very modest title for an extremely important discovery!
- The spectrum of the CMB was measured by the COBE satellite, which operated from 1989 to 1993, and proved to be a spectacular confirmation of the prediction—a nearly perfect blackbody with temperature 2.725 K



Mather et al 1990, Astrophysical Journal Letters, 354, L37

Figure 1: The spectrum of the CMB from the COBE satellite, and the best-fit blackbody model. The errors on the points are smaller than the points themselves.

- The CMB radiation fills the universe and is isotropic. An observer moving with respect to the Hubble flow (the general expansion of the universe) will see a Doppler shift in the CMB. This change in wavelength can be expressed as a change in temperature using Wien's law. The Sun's peculiar velocity produces a **dipole anisotropy** in the CMB: the temperature depends on the peculiar velocity of the Sun in the direction we're looking. This allows us to determine the peculiar velocity of the Sun with respect to the Hubble flow—it's 370.6 ± 0.4 km s^{-1} . We can decompose this into motions of the Sun around the Galaxy, the Milky Way within the Local Group, and the Local Group with respect to the Hubble flow.

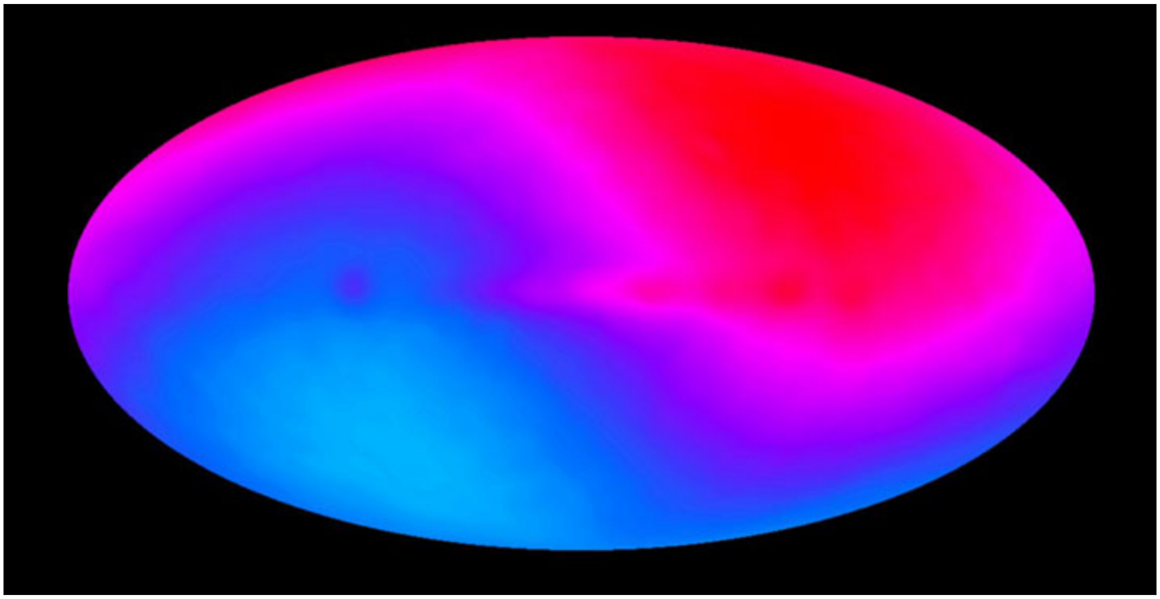


Figure 2: The CMB dipole, due to the peculiar velocity of the Sun

- Once we remove the dipole, the CMB is incredibly isotropic. However, it does have small variations in temperature: on scales of 1° or less, the temperature departs from the average value by about one part in 10^5. These anisotropies produce the famous map of the temperature variations in the CMB (most recently from the WMAP and Planck satellites), and measurements of these anisotropies are the primary source of our precise measurements of cosmological parameters. More on that later.

4 A two-component model of the universe

A complete model of the universe needs to include the effects of the CMB as well as matter. The CMB has a negligible effect on the dynamics of the universe now, but at very early times it was dominant. We will only talk about photons here, but the following applies to neutrinos and other relativistic particles as well.

We already saw that

$$a^{3(1+w)}u = a^4 u = u_0, \tag{26}$$

since $w = 1/3$ for photons and $a_0 = 1$. Converting the energy density into the equivalent mass density $\rho_{\rm rel} = u/c^2$, we see that

$$\rho_{\rm rel} = \rho_{\rm rel,0}\, a^{-4} = \rho_{\rm rel,0}\, (1+z)^4. \tag{27}$$

Compare this with the evolution of matter density with the scale factor,

$$\rho_m = \rho_{m,0}\, a^{-3} = \rho_{m,0}\, (1+z)^3. \tag{28}$$

As the scale factor becomes smaller with increasing redshift, $\rho_{\rm rel}$ increases more rapidly than ρ_m, which means that although matter dominates now, there must have been a time in the early universe as $a \to 0$ when

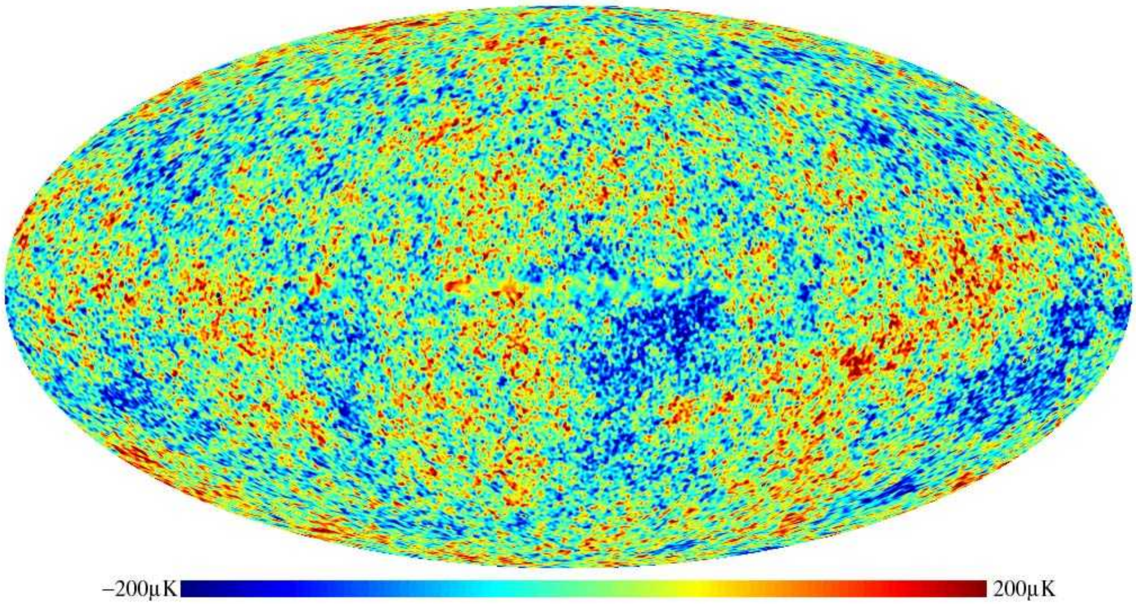


Figure 3: Anisotropies in the CMB, from WMAP

radiation (all relativistic particles) was dominant. We can find out when this was by setting $\rho_{\rm rel} = \rho_{\rm rel,0}/a^4$ equal to $\rho_m = \rho_{m,0}/a^3$. The value of the scale factor for matter-radiation equality is then

$$a_{r,m} = \frac{\rho_{\rm rel,0}}{\rho_{m,0}} = 3.05 \times 10^{-4}, \tag{29}$$

where we have inserted our best estimates of the current densities of radiation and matter (more on these later). The corresponding redshift is

$$z_{r,m} = \frac{1}{a_{r,m}} - 1 = 3270. \tag{30}$$

Since $aT = T_0$, the temperature at this time was

$$T_{r,m} = \frac{T_0}{a_{r,m}} = 8910 \text{ K}. \tag{31}$$

So the universe was **radiation dominated** until a redshift of $z = 3270$, at which time the temperature was $T = 8910$ K. After this, the universe was **matter dominated**.

We can also look at how the universe expands during the radiation era. Including the contributions of both matter and relativistic particles,

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}(\rho_m + \rho_{\rm rel})\right]a^2 = -Kc^2 \tag{32}$$

Writing ρ_m and $\rho_{\rm rel}$ in terms of their values today, we find

$$\left[\left(\frac{da}{dt}\right)^2 - \frac{8\pi G}{3}\left(\frac{\rho_{m,0}}{a} + \frac{\rho_{\rm rel,0}}{a^2}\right)\right] = -Kc^2 \tag{33}$$

We can set $K = 0$ because the early universe was extremely close to flat. We can then show that the time of matter and radiation equality was $t_{r,m} = 5.5 \times 10^4$ yrs, and that when $a \ll a_{r,m}$, $a \propto t^{1/2}$. This should be compared with the matter-dominated era, when $a \gg a_{r,m}$; in this case $a \propto t^{2/3}$, as we already showed for a flat universe containing only matter. So the universe expanded more slowly in the radiation era.

5 The origin of the CMB

What are we actually seeing when we look at the CMB radiation?

After the Big Bang, the universe was hot and dense, and filled with free electrons, free protons, and photons. The photons scattered off the free electrons, and could only travel short distances between scatterings. The frequent scatterings kept the electrons and photons in thermal equilibrium (they had the same temperature). As the universe expanded, the electrons became farther apart and the photons could travel longer between scatterings. This process, of the expansion diluting the density of a particle until it no longer interacts with other particles, is called **decoupling**, and it also occurred for other particles such as neutrinos—early in the history of the universe, neutrinos stopped interacting with other particles because the distances between them became too great.

In the case of photons, something else happened: the universe became cool enough for the free electrons and free protons to combine into neutral atoms, so the photons no longer had electrons to interact with, and matter and radiation began to evolve independently. This completed the decoupling of radiation and matter. The formation of neutral atoms is called **recombination**, even though that doesn't actually make sense since the electrons and protons were never combined to begin with.

Once there were no free electrons, the opacity of the universe was much lower, and the photons could stream freely without scattering. The CMB photons we see were last scattered during recombination.

For this reason we define the **surface of last scattering**. This is a spherical "surface," centered on the Earth, from which all the CMB photons come. Because the universe was opaque before recombination, this is the farthest redshift we can possibly observe. The surface of last scattering actually has a thickness Δz, because recombination didn't happen all at once.

The redshift at which recombination occurred, and the temperature of the universe at that time, can be calculated, and measured from the CMB observations. The result is

$$z_{\rm dec} = 1089 \pm 1 \tag{34}$$

and

$$T_{\rm dec} = T_0(1 + z_{\rm dec}) = 2970 \text{ K}. \tag{35}$$

This corresponds to an age of $t_{\rm dec} = 379,000$ years.