Astronomy 401/Physics 903
Lecture 26
Cosmological distances and the Robertson-Walker metric

1 Measuring distances and the Robertson-Walker metric

We have described how we can use measurements of distances made within a space to deduce things about the curvature of that space. These differences in the "big" distances we measure must reflect differences in the infinitesimal distances from which they are made up:

$$L = \int d\ell. \tag{1}$$

In Euclidean space, the distance between two points is

$$L^2 = x^2 + y^2 \tag{2}$$

or equivalently in polar coordinates,

$$L^2 = r^2. \tag{3}$$

We express infinitesimal separations similarly:

$$(d\ell)^2 = (dx)^2 + (dy)^2 \tag{4}$$

or in polar coordinates

$$(d\ell)^2 = (dr)^2 + (r\ d\phi)^2 \tag{5}$$

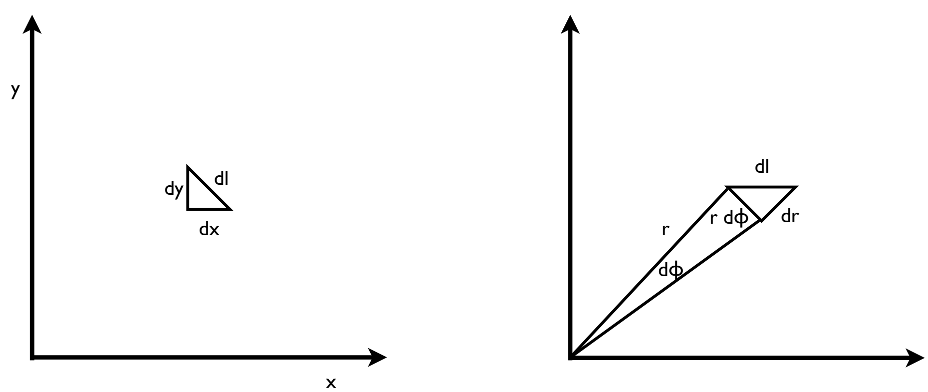

Figure 1: Infinitesimal distances.

We call the rule for how distances are measured in a space the **metric**, e.g. $d\ell^2 = dr^2 + r^2 d\phi^2$ is the metric for a flat Euclidean space.

We can define a different metric for a different geometry. For example, the distance between any two points on the surface of a sphere of radius R is given by

$$(d\ell)^2 = (R\ d\theta)^2 + (r\ d\phi)^2, \tag{6}$$

with radial coordinate r, polar angle θ and azimuthal angle ϕ. Since $r = R\sin\theta$, $dr = R\cos\theta\, d\theta$, and

$$R\, d\theta = \frac{dr}{\cos\theta} = \frac{dr}{\sqrt{1 - r^2/R^2}}, \tag{7}$$

since $r = R\sin\theta$ and $\sin^2\theta + \cos^2\theta = 1$, so $\cos^2\theta = 1 - r^2/R^2$. So our metric for distance on the surface of the sphere is

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\, d\phi)^2. \tag{8}$$

For a general two-dimensional surface with **curvature**

$$k \equiv \frac{1}{R^2}, \tag{9}$$

the metric is

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - kr^2}}\right)^2 + (r\, d\phi)^2. \tag{10}$$

We generalize this to three dimensions by changing from polar to spherical coordinates:

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - kr^2}}\right)^2 + (r\, d\theta)^2 + (r\sin\theta\, d\phi)^2. \tag{11}$$

k is now the curvature of the 3-d space. Note that as $k \to 0$ or $r \to 0$, the space acts Euclidean.

When considering the expanding universe, the curvature is a function of time. We therefore define

$$k(t) \equiv \frac{K}{a^2(t)}, \tag{12}$$

where K is a time-independent constant and a is the time-dependent scale factor. We can then rewrite the metric using K and substituting $r = a(t)x$, where $a(t)$ is again the scale factor and x is the comoving coordinate:

$$(dl)^2 = a^2(t)\left[\left(\frac{dx}{\sqrt{1 - Kx^2}}\right)^2 + (x\, d\theta)^2 + (x\sin\theta\, d\phi)^2\right] \tag{13}$$

Finally, we are measuring distances in *spacetime*, not just space. By distance, we mean the proper distance between two spacetime events that occur simultaneously, according to an observer. The spacetime metric is

$$(ds)^2 = (c\, dt)^2 - (d\ell)^2. \tag{14}$$

Therefore, we have

$$\boxed{(ds)^2 = (c\, dt)^2 - a^2(t)\left[\left(\frac{dx}{\sqrt{1 - Kx^2}}\right)^2 + (x\, d\theta)^2 + (x\sin\theta\, d\phi)^2\right]} \tag{15}$$

This is the **Robertson-Walker metric**, and it is used to determine the spacetime interval between two events in any homogeneous, isotropic space (both positively and negatively curved, though we only showed it for the simplest, spherical case here).

Also note that the curvature parameter K is the K that appears in the Friedmann equation,

$$\left[\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{8\pi}{3}G\rho\right]a^2(t) = -Kc^2. \tag{16}$$

In our Newtonian derivation of this equation, K referred to the total energy of the universe, but when this equation is derived from general relativity, K is the curvature constant defined above. Thus if $K = 0$, the space is Euclidean, i.e. flat. If $K > 0$, $1/\sqrt{K}$ can be interpreted as the radius of curvature of the three-dimensional space; the two-dimensional analogy is the surface of a sphere. If $K < 0$, the space is hyperbolic.

2 Cosmological distances

How do we measure distances in an expanding and potentially curved universe? This isn't an easy question, and the answer is different depending on exactly what we're trying to measure.

2.1 Hubble distance

An order of magnitude estimate for the size of the observable universe is easy: the speed of light multiplied by the age of the universe. The approximate age of the universe is the Hubble time t_H, and this distance is the Hubble distance,

$$d_H = c\,t_H = \frac{c}{H_0} \simeq 4.2 \text{ Gpc}. \tag{17}$$

2.2 Proper distance

Now we'll consider other distances. The **proper distance** is the distance between us and some object *now*—not the distance between us and the object when its light was emitted (note that this distance is not actually possible to measure). We can derive this using the Robertson-Walker metric

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\left(\frac{dx}{\sqrt{1-Kx^2}}\right)^2 + (x\,d\theta)^2 + (x\sin\theta\,d\phi)^2\right]. \tag{18}$$

Because this is the distance to the object now, $dt = 0$, and $d\theta = d\phi = 0$ along a radial line from us to the object, which is at comoving coordinate x. We set $a(t_0) = 1$ since we're interested in the distance right now. We can then find the proper distance to the object by integrating:

$$d_p = \int_0^x \frac{dx'}{\sqrt{1-Kx'^2}} \tag{19}$$

The solution to this depends on the value of K.

For a flat universe with $K = 0$, $d_p = x$, and the proper distance to the object is its comoving coordinate. The distance given by x is also called the **coordinate distance**.

For a closed universe with $K > 0$, the solution is

$$d_p = \frac{1}{\sqrt{K}}\sin^{-1}(x\sqrt{K}) \tag{20}$$

and for an open universe with $K < 0$, the proper distance is

$$d_p = \frac{1}{\sqrt{|K|}} \sinh^{-1}(x\sqrt{|K|}). \tag{21}$$

In a closed universe the proper distance to an object is greater than its coordinate distance, while in an open universe it is less. (This is an effect of the curvature, like the fact that for a closed universe the circumference of a circle is less than 2π times its radius.)

2.3 Horizon distance

A real calculation of the size of the observable universe must account for the fact that as the universe ages, photons from increasingly distant objects have more time to reach us. In other words, more of the universe may come into causal contact with us over time. The farthest observable point is called the **particle horizon**, and the proper distance to that point is the **horizon distance** d_h. This is the diameter of the largest causally connected region. The horizon distance at time t is

$$d_h(t) = a(t) \int_0^t \frac{c\ dt'}{a(t')} \tag{22}$$

As a photon moves toward us it travels a small distance $c\ dt$ in each interval of time dt. We can't just add these distances together because the universe expands as the photon travels, so we divide the small distance $c\ dt$ by the scale factor at the time t to account for the expansion. The integral portion of this expression is the comoving horizon distance, and we multiply by the scale factor $a(t)$ to obtain the proper distance to the horizon.

During the radiation era, the scale factor was $a(t) = Ct^{1/2}$, where C is a constant. Substituting this into Equation 22, we find the time dependence of the horizon distance during the radiation era,

$$d_h(t) = 2ct. \tag{23}$$

During the matter era (assuming a flat universe with $K = 0$), $a(t) = Ct^{2/3}$, which gives

$$d_h(t) = 3ct. \tag{24}$$

Notice that these distances are larger than ct, the distance travelled by a photon in time t. This is possible because of our definition of proper distance, as the distance between two events measured in a frame of reference where those two events happen at the same time. In order to actually measure the size of the visible universe at a particular time, we would have to devise some contrived scenario such as adding up distances measured by observers spread throughout the universe all making measurements at the same time. However, the photon emitted at time $t = 0$ traversed these same regions at earlier epochs, when the universe was smaller.

Also note that in the radiation and matter eras the horizon distance is proportional to t, while the scale factor is proportional to $t^{1/2}$ and $t^{2/3}$ respectively. This means that during those eras the size of the observable universe increased more rapidly that the universe expanded, and therefore the universe became increasingly causally connected as it aged.

The horizon distance in the matter-dominated era can be rewritten in terms of redshift as

$$d_h(z) = \frac{2c}{H_0\sqrt{\Omega_{m,0}}} \frac{1}{(1+z)^{3/2}} \tag{25}$$

We can then roughly estimate the present horizon distance by setting $z = 0$:

$$d_{h,0} \approx \frac{2c}{H_0\sqrt{\Omega_{m,0}}} = 16.3 \text{ Gpc}. \tag{26}$$

In the current, Λ-dominated era, the expression for the scale factor as a function of time is

$$a(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t\sqrt{\Omega_{\Lambda,0}}\right) \tag{27}$$

(this is the inversion of the expression for $t(a)$ we saw earlier). For the horizon distance, this gives

$$d_h(t) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t\sqrt{\Omega_{\Lambda,0}}\right) \int_0^t \frac{c\,dt'}{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_0 t'\sqrt{\Omega_{\Lambda,0}}\right)} \tag{28}$$

This has no simple analytic solution and must be integrated numerically. The result is that at the present time, the distance to the particle horizon in a flat universe dominated by the cosmological constant is

$$d_{h,0} = 14.6 \text{ Gpc}. \tag{29}$$

In the Λ era, the integral portion of Equation 28, without the term in front, is the present distance to the point that will be at the particle horizon at time t. As $t \to \infty$, this integral converges to 19.3 Gpc. This means that the proper distance today to the farthest object that will ever be observable in the future is 19.3 Gpc. Everything within a sphere of radius 19.3 Gpc will eventually become visible, and everything beyond is hidden forever.

In the future both the particle horizon and the scale factor will grow exponentially, as can be shown for a universe dominated by the cosmological constant. But the particle horizon will never catch up to an object that is more than 19.3 Gpc away, so its light will never reach us. If we observe an object at the particle horizon over time, its photons will be increasingly redshifted and their arrival rate will decline toward zero due to cosmological time dilation (more on that later), so the object will fade from view.