Astronomy 401/Physics 903
Lecture 25
Non-Euclidean Geometries

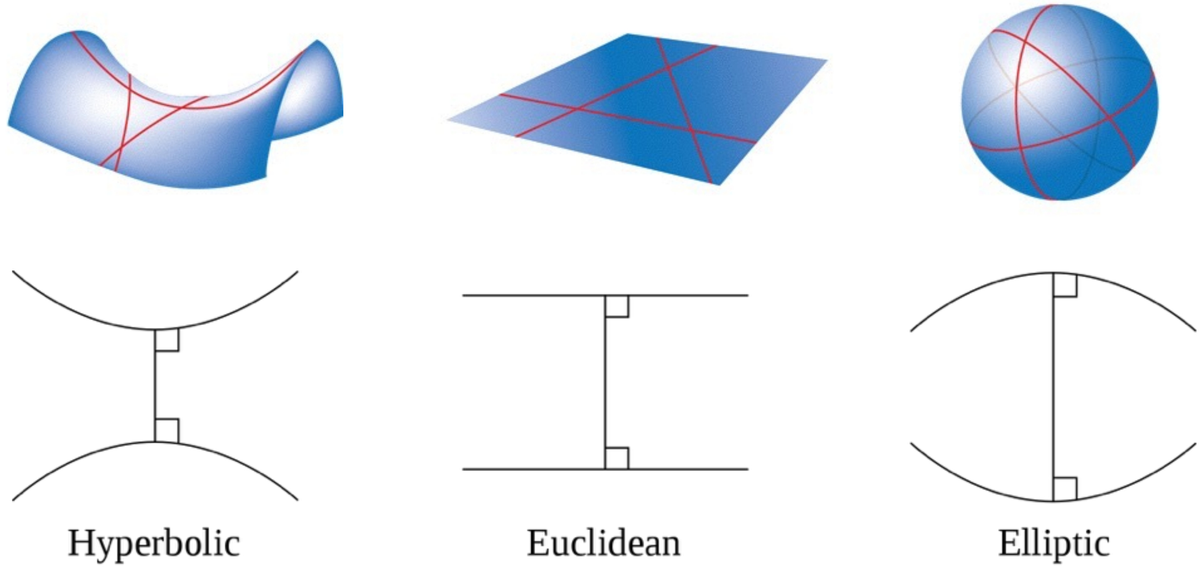
We've been talking about different scenarios for the evolution of the universe: open, flat or closed, depending on the density of mass and energy. This is very closely related to the geometry (i.e. the curvature) of the universe, which is described by general relativity. To understand the curvature of the universe, we need to consider some of the principles of non-Euclidean geometry.

1 Euclidean, Elliptic and Hyperbolic Geometries

In about 300 BC, Euclid worked out 5 postulates from which all the rules of geometry could be derived—these rules lay out the basic behaviors of straight lines, right angles, etc.

Modern statement of the 5th postulate: Given a line and a point not on the line, there is exactly one parallel line which passes through the point. This is also called the "parallel postulate."

In the18th century, mathematicians realized it was possible to make fully consistent definitions of geometry using Euclid's first 4 postulates, but modifying the 5th as shown in the figure below. These correspond to **curved spaces**.



hyperbolic geometry
- open, negatively curved
- at least two parallel lines
- angles of triangle add to less than 180°

normal Euclidean geometry
- flat
- one parallel line
- angles of triangle add to 180°

elliptic geometry
- closed, positively curved
- no parallel lines
- angles of triangle add to more than 180°

Figure 1: Modifications of the 5th postulate involving parallel lines lead to hyberbolic and elliptic (spherical) geometries. Two lines are parallel if they never intersect.

The examples shown in Figure 1 are curved 2-dimensional surfaces embedded in 3-d spaces—it's not too hard to understand the geometry of something when you can leave the space to look at it. But we can't do that with the universe—we can't pop off into a 4th spatial dimension to look at the curvature. Instead we have to try to understand it using measurements entirely within the space itself. These are called "inner properties."

The inner properties of a curved space are closely related to how distances are measured within that space:

- On a flat Euclidean plane, a circle has circumference $C = 2\pi r$.
- On a sphere of radius R, the radius of the circle is $r = R\theta$, but the circumference is

$$\begin{aligned} C &= 2\pi(R\sin\theta) && (1)\\ &= 2\pi r\left(\frac{\sin\theta}{\theta}\right) && (2)\\ &< 2\pi r && (3)\end{aligned}$$

  So, an observer on the surface of a sphere could measure the radius and circumference of a circle, and deduce that they were living in a positively curved world.

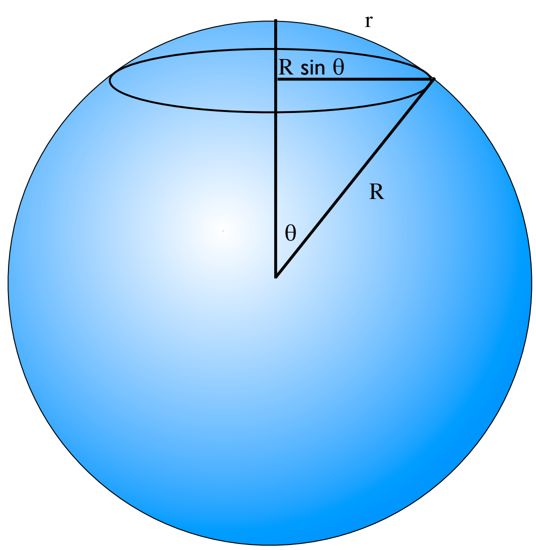


Figure 2: A circle on the surface of a sphere.

- Similarly, in a negatively curved space, $C > 2\pi r$.