Spherical Harmonics

Ying Xiong
School of Engineering and Applied Sciences
Harvard University
yxiong@seas.harvard.edu

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1 Definitions

Note that different conventions on this topic exist. In this document, we follow the one in Wikipedia [1], which is also consistent with that of Matlab (R2013b).

1.1 Associated Legendre polynomials [2]

The associated Legendre polynomials are defined as

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}\left(1-x^2\right)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}\left(x^2-1\right)^\ell, \quad \text{for } -1 \le x \le 1,\ \ell \ge 0,\ -\ell \le m \le \ell. \tag{1}$$

This definition has the property that

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x). \tag{2}$$

Matlab also provides implementation for "fully normalized associated Legendre functions", defined as

$$N_\ell^m = (-1)^m \sqrt{\frac{\left(\ell+\frac{1}{2}\right)!\,(\ell-m)!}{(\ell+m)!}} P_\ell^m \tag{3}$$

1.1.1 Low-order examples

Associated Legendre polynomials with $\ell \le 2$.

$$P_0^0(x) = 1,$$
$$P_1^{-1}(x) = -\tfrac{1}{2}P_1^1(x), \qquad P_1^0(x) = x, \qquad P_1^1(x) = -\left(1-x^2\right)^{1/2},$$
$$P_2^{-2}(x) = \tfrac{1}{24}P_2^2(x), \quad P_2^{-1}(x) = -\tfrac{1}{6}P_2^1(x), \quad P_2^0(x) = \tfrac{1}{2}(3x^2-1), \quad P_2^1(x) = -3x(1-x^2)^{1/2}, \quad P_2^2(x) = 3(1-x^2).$$

1.2 Spherical harmonic functions [1]

The spherical harmonic functions are defined as

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)\, e^{im\phi}, \tag{4}$$

or equivalently by fully normalized associate Legendre functions

$$Y_\ell^m(\theta,\phi) = (-1)^m \sqrt{\frac{1}{2\pi}} N_\ell^m(\cos\theta)\, e^{im\phi}. \tag{5}$$

A real basis of spherical harmonic functions can be defined as below

$$Y_{\ell,m} = \begin{cases} \frac{1}{\sqrt{2}}\left(Y_\ell^{-m} + (-1)^m Y_\ell^m\right) & \text{if } m > 0 \\ Y_\ell^0 & \text{if } m = 0 \\ \frac{i}{\sqrt{2}}\left(Y_\ell^m - (-1)^m Y_\ell^{-m}\right) & \text{if } m < 0. \end{cases} \tag{6}$$

Note we use two subscripts for real basis, as opposed to one subscript and one super-script for complex one. The real basis can usually be more conveniently expressed in terms of $(x,y,z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

1.2.1 Low-order examples [3]

Spherical harmonic functions with $\ell \le 2$.

$$\begin{gathered} Y_0^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}, \\ Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi}, \quad Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta, \quad Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi}, \\ Y_2^{-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi}, \quad Y_2^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi}, \quad Y_2^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right), \quad Y_2^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi}, \quad Y_2^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi}. \end{gathered}$$

Real spherical harmonic functions with $\ell \le 2$.

$$\begin{gathered} Y_{0,0} = \frac{1}{2}\sqrt{\frac{1}{\pi}}, \\ Y_{1,-1} = \frac{1}{2}\sqrt{\frac{3}{\pi}}y, \qquad Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}z, \qquad Y_{1,1} = \frac{1}{2}\sqrt{\frac{3}{\pi}}x, \\ Y_{2,-2} = \frac{1}{2}\sqrt{\frac{15}{\pi}}xy, \quad Y_{2,-1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}yz, \quad Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3z^2 - 1\right), \quad Y_{2,1} = \frac{1}{2}\sqrt{\frac{15}{\pi}}xz, \quad Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{\pi}}\left(x^2 - y^2\right). \end{gathered}$$

1.3 Inner product in spherical space

The inner product in the spherical space is defined as

$$\langle f, g\rangle = \int_{\mathbb{S}^2} f(\omega)\bar{g}(\omega)d\omega = \int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi} f(\theta,\phi)\bar{g}(\theta,\phi)\sin\theta\, d\phi\, d\theta. \tag{7}$$

By our convention, $\langle Y_\ell^m, Y_{\ell'}^{m'}\rangle = \delta_{\ell,\ell'}\delta_{m,m'}$.

1.4 Decomposition of a spherical function

Any well-behaved spherical function f can be decomposed as

$$f = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell,m} Y_{\ell,m}, \tag{8}$$

where

$$f_\ell^m = \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\omega)\bar{Y}_\ell^m(\omega)d\omega, \tag{9}$$

$$f_{\ell,m} = \langle f, Y_{\ell,m}\rangle = \int_{\mathbb{S}^2} f(\omega)Y_{\ell,m}(\omega)d\omega. \tag{10}$$

2 Applications in Lambertian Reflectance [4, 5, 6]

We use the real form spherical harmonics in this application.

2.1 Convolution and Funke-Hecke theorem [4]

Let $L(\omega)$ be a spherical function and $k(x)$ a scalar function, we define "convolution" as

$$E\left(\omega_1\right) = \left(k * L\right)\left(\omega_1\right) \stackrel{\text{def}}{=} \int_{\mathbb{S}^2} k(\omega_1 \cdot \omega_2) L(\omega_2) d\omega_2, \tag{11}$$

where the result $E(\omega_1)$ is another spherical function.

Theorem 1 (Funk-Hecke)**.** *Let $k(x)$ be a bounded integrable function on $[-1,1]$, whose Harmonic expansion*

$$k(\cos\theta) = \sum_{\ell=0}^{\infty} k_\ell Y_{\ell,0}(\theta,\phi) = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} k_\ell P_\ell^0(\cos\theta). \tag{12}$$

Then we have

$$k * Y_{\ell,m} = \sqrt{\frac{4\pi}{2\ell+1}} k_l Y_{\ell,m}. \tag{13}$$

The theorem implies that

$$E = k * L = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\sqrt{\frac{4\pi}{2\ell+1}} k_\ell L_{\ell,m} \right) Y_{\ell,m}. \tag{14}$$

2.2 Lambertian Reflectance [5]

Let L denote distant lighting distribution, and $E(\mathbf{n})$ the irradiance of the surface normal $\mathbf{n}$. Then we have

$$E\left(\mathbf{n}\right) = \int_{\Omega(\mathbf{n})} L(\omega) \left(\mathbf{n} \cdot \omega\right) d\omega \tag{15}$$

where $\mathbf{n}$ and ω are unit direction vectors and $\Omega\left(\mathbf{n}\right)$ is the upper hemisphere of $\mathbf{n}$. Define

$$A\left(\theta\right) = \max\left[\cos\theta, 0\right] = \sum_{\ell=0}^{\infty} A_\ell Y_{\ell,0}(\theta,\phi). \tag{16}$$

Then we have

$$E(\mathbf{n}) = \int_{\mathbb{S}^2} L(\omega) A(\mathbf{n}\cdot\omega) d\omega = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} A_\ell L_{\ell,m} Y_{\ell.m}(\mathbf{n}), \tag{17}$$

or equivalently,

$$E_{\ell,m} = \sqrt{\frac{4\pi}{2\ell+1}} A_\ell L_{\ell,m} = \widehat{A}_\ell L_{\ell,m}, \quad \text{with } \widehat{A}_\ell = \sqrt{\frac{4\pi}{2\ell+1}} A_\ell. \tag{18}$$

The observation is that $\widehat{A}_\ell$ decays fast with respect to ℓ, and therefore the first several spherical harmonics (*e.g.* $\ell \leq 2$) captures most of the energy in $E\left(\mathbf{n}\right)$.

The lighting coefficients can be found by an integration (pre-filtering)

$$L_{\ell,m} = \int_{\theta=0}^{\pi} \int_{\phi=-\pi}^{\pi} L(\theta,\phi) Y_{\ell,m}(\theta,\phi)\ \sin\theta\ d\theta\ d\phi. \tag{19}$$

2.3 Some Heuristics

2.3.1 $A(\theta)$ as a low-pass filter

The irradiance E is the convolution of lighting L and the function $A(\theta)$, which serves as a low-pass filter (because $\widehat{A}_\ell$ decays fast with respect to ℓ).

If the $A(\theta)$ were simply $\cos\theta$, *i.e.* without the zero clipping, it will be a perfect low-pass filter, because only the first order component would be non-zero.

2.3.2 Distant point light source

Now we consider a special case when lighting L is a distant point light source, which is equivalent to a directional light source. It can be written as a delta function

$$L(\theta, \phi) = \delta_{\theta_0, \phi_0}(\theta, \phi). \tag{20}$$

If the function $A(\theta)$ were simply $\cos\theta$, the rendering function is exactly captured by the first-order spherical harmonics, because $\hat{A}_\ell = 0$ for $\ell = 0$ and $\ell \geq 2$. In this case, the delta function will produce the same response $E(\mathbf{n})$ as the corresponding combination of first-order spherical harmonics.

In reality, when $A(\theta)$ has zero-clipping, the higher-order component will not be entirely cut-off and will need to be taken into consideration. The same first-order spherical harmonics will only produce exact $E\left(\mathbf{n}\right)$ on the hemispher $\Omega\left(\mathbf{n}\right)$, but not the whole sphere $\mathbb{S}^2$; and the best decomposition in the whole sphere can be different from that of the hemisphere.

References

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