Trig approximation

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Warmup

Given $f(x)$ & an orthogonal polynomial basis $\{\phi_j\}_{j=0}^{n}$, wrt $w(x) \geq 0$
construct a n^{th} degree polynomial approx of f.

Soln: Least squares approx.

$$f(x) = \sum_{j=0}^{\infty} a_j \phi_j(x) \quad \text{- in dreams}$$

$$f(x) \sim \sum_{j=0}^{n} a_j \phi_j(x) = P_n(x)$$

let's take inner product w/ ϕ_i and force equality.

$$\Rightarrow \langle f, \phi_i \rangle_w = \langle \sum_{j=0}^{n} a_j \phi_j(x), \phi_i(x) \rangle_w$$

$$= \sum_{j=0}^{n} a_j \underbrace{\langle \phi_j, \phi_i(x) \rangle_w}_{=0 \text{ for } j \neq i}$$

$$= a_i \langle \phi_i, \phi_i \rangle_w$$

$$\rightarrow a_i = \frac{\langle f, \phi_i \rangle_w}{\langle \phi_i, \phi_i \rangle_w}$$

Before $g(x) = \left(f(x) - \sum_{j=0}^{n} a_j \phi_j\right)^2$ minimize g.

Fourier for f defined on $[-\pi, \pi]$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$

We know $\{1, \cos(x), \cos(2x), \dots, \sin x, \sin(2x), \dots\}$ are mutually orthogonal on $[-\pi, \pi]$

So we can find the coefficient via innerproducts. $\ell \in \mathbb{Z}^+$

$$\int_{-\pi}^{\pi} f(x)\cos(\ell x)\,dx = \int_{-\pi}^{\pi} a_0 \cos(\ell x)\,dx + \int_{-\pi}^{\pi} \Big(\sum_{k=1}^{\infty} a_k \cos(kx)\cos(\ell x)$$

$$+ b_k \sin(kx)\cos(\ell x) \Big)\,dx$$

$$= a_0 \underbrace{\int_{-\pi}^{\pi} \cos(\ell x)\,dx}_{=0} + \sum_{k=1}^{\infty} a_k \underbrace{\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\,dx}_{=0 \text{ if } \ell \neq k}$$

$$+ b_k \sum_{k=1}^{\infty} \underbrace{\int_{-\pi}^{\pi} \sin(kx)\cos(\ell x)\,dx}_{=0}$$

$$= a_\ell \int_{-\pi}^{\pi} \cos^2(\ell x)\,dx = a_\ell \pi$$

$$a_\ell = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\ell x)\,dx \qquad \ell = 1, 2, \dots$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad \int_{-\pi}^{\pi} 1\,dx$$

$$b_\ell = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(\ell x)\,dx$$

The trig approx. is the partial sum.

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$$f(x) \sim S_n(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$