Splines

Monday, October 4, 2021 2:00 PM

Warm-up

So far interpolation ... $\{x_j\}_{j=0}^n$ - interpolation nodes

$\{f(x_j)\}_{j=0}^n$ = data at nodes

How can we build a poly. to approx. f?

Lagrange $\quad P_n(x) = \sum_{j=0}^n f(x_j) L_j(x)$

L_j ← Lagrange polynomial.

Newton Divided-Differences.

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_n)$$

Monomials: $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdot + a_n x^n$

Barycentric $\quad P_n(x) = \ell(x) \sum_{j=0}^n \frac{f(x_j) w_j}{(x-x_j)}$

in exact arthimetic they are same.

Difference.

Lagrange - stable · cost $O(n^2)$ to evaloate.

Newton divided differences: not very stable. $O(n)$ cost

Monomials: not stable.

Barycentric: stable cost $O(n)$ to evaluate.

Error: $f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0) \cdots (x-x_n)$

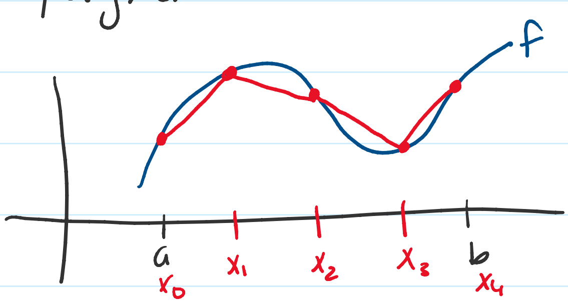
Hermite: $\{x_j\}_{j=0}^n$ $\{f(x_j)\}_{j=0}^n$ $\{f'(x_j)\}_{j=0}^n$

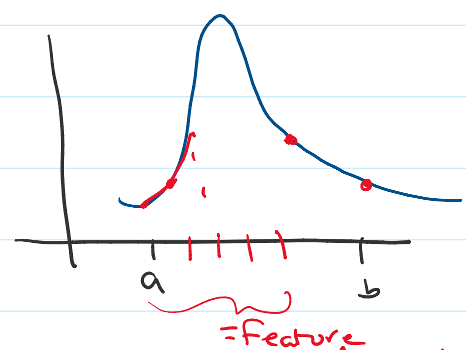
- approximates polynomials up to order $2n+1$ exactly

§3.5 splines

Instead of a "global" approximation on an interval $[a,b]$ w/ 1 polynomial, We will create piecewise approximations.

Ex Approximate $f(x)$ with piecewise linear polynomials

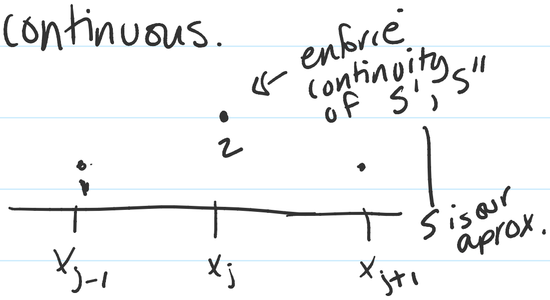




NODE placement depends on where feature s of f are.

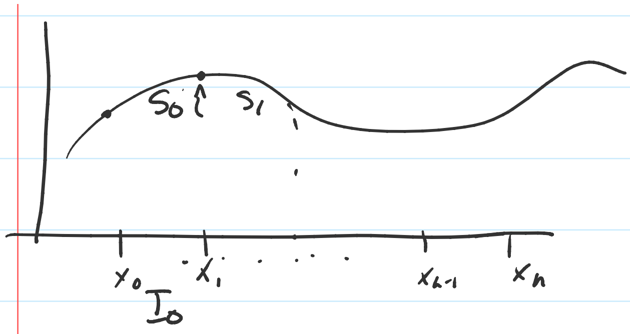
Piecewise linear is easy to understand but it not great for capturing functions w/ features.
-place more nodes or go higher order.

let's try going higher order. we need more info.
we will force the 1^{st} & 2^{nd} derivatives to be continuous.



We are given $\{x_j\}_{j=0}^{n}$ $\{f(x_j)\}_{j=0}^{n}$
we want our piecewise approx to have continuous 1^{st} & second derivatives.

let's build these approximations.



let S_i denote the polynomial approx. on interval $[x_i, x_{i+1}] = I_i$

Then $S_i(x_i) = f(x_i) \quad S_i(x_{i+1}) = f(x_{i+1})$

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$$
$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$$

Note: At the endpts, there is not enough info. to define a cubic polynomial

There are two standard options for dealing with the endpts.

1- Natural (or free) boundary

$$S_0''(x_0) = 0 = S_{n-1}''(x_n)$$

2- Clamped boundary

$$S_0'(x_0) = f'(x_0) \qquad S_{n-1}'(x_n) = f'(x_n)$$