Chebychev

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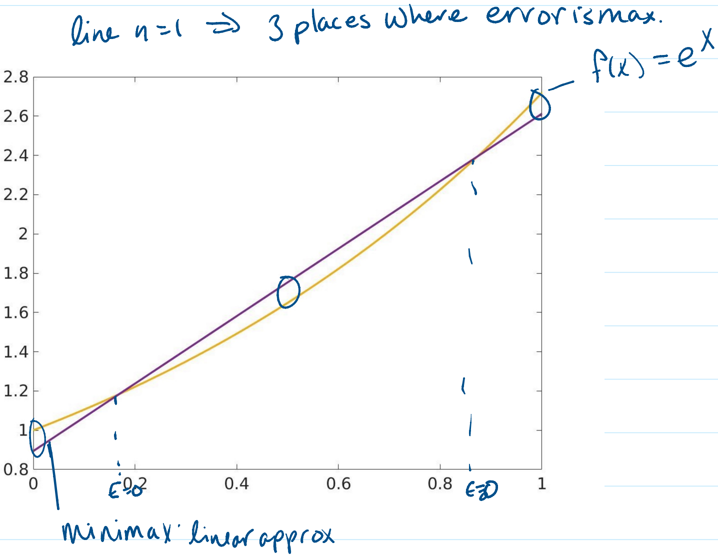
Why are Chebychev polynomials important?

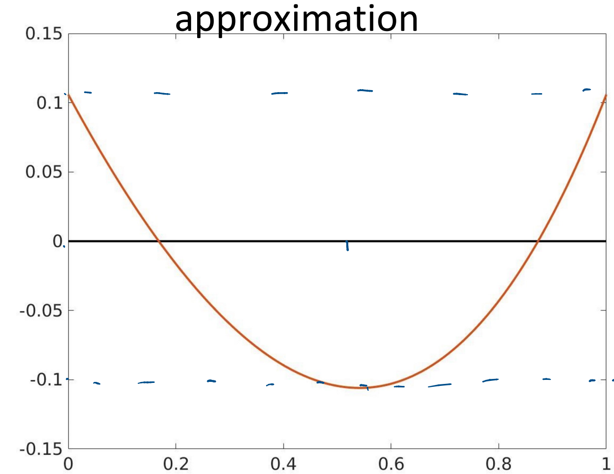
One can show that if $p_n(x)$ is the optimal degree n polynomial approx. of $f(x)$ in the max norm on $[a,b]$ then the chebychev nodes are nearly optimal.

The optimal polynomial has an error $E(x) = f(x) - p_n(x)$ with at least $n+2$ equally large alternating sign. (The equi-oscillation thm)

Linear minimax approximation of f(x) = exp(x)



Plot of the error in the approximation



It turns out that the least squares approx w/ $w(x) = 1/\sqrt{1-x^2}$ on $[-1,1]$ is very close to the minimax approx.

The Chebychev polynomials are orthogonal wrt $w(x) = 1/\sqrt{1-x^2}$ on $[-1,1]$.

The Chebychev polynomials are given by

$$T_n(x) = \cos(n \cos^{-1} x)$$

We need to verify 3 things

1- $T_n(x)$ is in fact a polynomial of degree n.

2- That they are orthogonal wrt $w(x) = 1/\sqrt{1-x^2}$ on $[-1,1]$

3- Satisfy a 3 term recursion

Question 1: $T_0 = \cos(0\cos^{-1}(x)) = 1$

$T_1 = \cos(\cos^{-1}x) = x$

$T_n(x) = \cos(n\cos^{-1}x)$ let $\theta = \cos^{-1}x \to x = \cos\theta$

$T_{n+1}(x) = \cos((n+1)\theta) = \cos n\theta\cos\theta - \sin n\theta\sin\theta$

$T_{n-1}(x) = \cos((n-1)\theta) = \cos n\theta\cos\theta + \sin n\theta\sin\theta$

$T_{n+1}(x) + T_{n-1}(x) = 2T_n(x)\cos\theta$

$= 2xT_n(x)$

$\Rightarrow T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ ✓ 3 term recursion

We need to check orthogonality?

we need to verify $\int_{-1}^{1} T_n(x)T_m(x)w(x)\,dx = 0$ for $m \neq n$

$$\int_{-1}^{1}\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)\frac{1}{\sqrt{1-x^2}}\,dx$$

Sub $\cos^{-1}x = \theta$, $d\theta = \frac{-dx}{\sqrt{1-x^2}}$

$$= -\int_{\pi}^{0}\cos(n\theta)\cos(m\theta)\,d\theta$$

$$= \int_0^{\pi}\cos(n\theta)\cos(m\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left(\cos((m+n)\theta) + \cos((m-n)\theta)\right)d\theta$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n}\Bigg|_0^{\pi} + \frac{\sin((m-n)\theta)}{m-n}\Bigg|_0^{\pi}\right] = 0$$