APPM 4650 — Error analysis of iterative methods

Definition 0.1. *Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If there exists postive constants λ and α such that*

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=1}^{\infty}$ converges to p with an order α and asympotic error constant λ. If $\alpha = 1$ and $\lambda < 1$ then the sequence converges linearly. *If $\alpha = 2$, the sequence is* quadratically *convergent.*
Note: In general people do not worry about λ much.

Example: Show that the fixed point iteration converges linearly when $g \in C^1[a,b]$ and $|g'(x)| < k < 1$ near the fixed point p.
Solution: We need to look at $|p_{n+1}-p|$ and create a relation with $|p_n - p|$.

$$\begin{aligned}
|p_{n+1}-p| &= |g(p_n) - g(p)| \text{ by definition of the fixed point iteration} \\
&\leq |g'(\eta)||p_n - p| \text{ by the mean value theorem} \\
&< k|p_n - p| \text{ since } |g'(x)| < k
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \left|\frac{p_{n+1}-p}{p_n-p}\right| < k$$

and the fixed point iteration converges linearly.

Theorem 2.8 from textbook does this proof differently. Theorem 2.9 says that if $g'(p) = 0$, the fixed point iteration converges quadratically. This is what the textbook uses to discuss the order of convergence for Newton's method. We will do something more savy here.

Theorem 0.1. *Assume that f is twice continuously differentiable on an open interval (a,b) and that there exists $p \in [a,b]$ with $f'(p) \neq 0$. Newton's iteration is defined by*

$$p_{n+1} = p_n + \frac{f(p_n)}{f'(p_n)} \quad n = 1, 2, \ldots$$

Assume also that $p_n \to p$ as $n \to \infty$. Then for n sufficiently large

$$|p_{n+1} - p| \leq M|p_n - p|^2$$

where $M > \left|\frac{f''(x)}{2f'(x)}\right|$ for all x near p. Thus Newton's method is quadratically convergent.

Proof. Let $e_n = p_n - p$ denote the error at step n.
Using the Taylor expansion we can write $f(p)$ in terms of p_n and e_n.

$$f(p) = f(p_n - e_n) = f(p_n) - e_n f'(p_n) + \frac{e_n^2}{2} f''(\eta_n)$$

for some η_n between p_n and p.
We know that p is the root of $f(x)$. So

$$0 = f(p_n - e_n) = f(p_n) - e_n f'(p_n) + \frac{e_n^2}{2} f''(\eta_n)$$

Since $f'(p) \neq 0$, we have $f'(p_n) \neq 0$ as long as it is close enough to p.
Dividing by $f'(p_n)$, we get

$$0 = \underbrace{\frac{f(p_n)}{f'(p_n)} - (p_n - p)}_{e_{n+1}} + \frac{e_n^2 f''(\eta_n)}{2f'(p_n)}$$

Thus

$$e_{n+1} \leq \frac{e_n^2 f''(\eta_n)}{2f'(p_n)} \leq M e_n^2$$

□

What happens if the root has multiplicity 2 or greater?

Definition 0.2. *A solution p of $f(x) = 0$ is a* zero of multiplicity m *if for $x \neq p$, we can write $f(x) = (x-p)^m q(x)$ where $\lim_{x\to p} q(x) \neq 0$.*

Theorem 0.2. *(Theorem 2.11) The function $f \in C^1[a,b]$ has a simple zero at $p \in [a,b]$ if and only if $f(p) = 0$ but $f'(p) \neq 0$.*

The proof of this theorem is easy. Please look at on your own.

Theorem 0.3. *(Theorem 2.12) The function $f \in C^m[a,b]$ has a root at $p \in (a,b)$ of multiplicy m if and only if $0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p)$ and $\lim_{x\to p} f^{(m)}(x) \neq 0$.*

What happens to the convergence of Newton's method when a root has multiplicty greater than 1?

See explore_newton.m or explore_newton.py
Sadly the convergence drops from 2nd order to first order.

One option to restore second order convergence is to apply Newton's method to

$$\mu(x) = \frac{f(x)}{f'(x)}$$

It is reasonable to ask "Why does this work?"
Suppose α is root of multiplicity m. Then we know that we can write $f(x)$ as $f(x) = (x-\alpha)^m q(x)$ where $q(\alpha) \neq 0$. Also,

$$f'(x) = (x-\alpha)^m q'(x) - m(x-a)^{m-1} q(x).$$

Then

$$\begin{aligned}
\mu(x)* &= \frac{f(x)}{f'(x)} \\
&= \frac{(x-\alpha)^m q(x)}{(x-\alpha)^m q'(x) - m(x-a)^{m-1} q(x)} \\
&= \frac{(x-\alpha)^m q(x)}{(x-\alpha)^{m-1}((x-\alpha)q'(x) - mq(x))} \\
&= \frac{(x-\alpha) q(x)}{((x-\alpha)q'(x) - mq(x))}
\end{aligned}$$

Thus α is a root of $\mu(x)$ with multiplicity 1.
There is another way to fix this problem. You may see it soon.
Of course it is possible for Newton to fail. The most obvious ways is for the root to not have a basin of convergence at all. Another way is for an iterate to land on a point where the tangent line is horizontal. Another possible failure is a cyclic set of iterations.
Example: Let $f(x) = x(x-1)(x-2)$. Then there exist two points $x_0 = \alpha$ and $x_1 = \beta$ such that Newton will iterate between the two forever. Below is a drawing illustrating this.