APPM 4650 — Fixed point iteration

The fixed point iteration is our root finding technique. It is actually a generalization of the root finding techniques we will learn next.

Definition 0.1. *The number p is a* fixed point *of a given function $g(x)$ if $g(p) = p$.*

Graphically, the fixed point is the value of x such that the function $g(x)$ and the line $y = x$ cross. Insert figure

Question: How does this relate to root finding?
Answer: In root finding, our goal is to find c such that $f(c) = 0$. We can define a function $g(x)$ so that c is the fixed point.
Possible options include: $g(x) = x - f(x)$ and $g(x) = x - 10f(x)$. We will determine how to pick a good $g(x)$ shortly.

Conversely you can turn a fixed point problem into a root finding problem. This tends to be easier. For example, the fixed point problem is given $g(x)$, find p such that $g(p) = p$. This is the same as finding the root of $f(x) = x - g(x)$.

Another reasonable question is given a function $g(x)$ and an interval $[a, b]$, how doe we if there a fixed point in $[a, b]$? Fortunately there is a nice theorem that tells us when this is true. It is built off of intuition and calculus.

Theorem 0.1 (2.3 in text)**.** *(i) If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed point in $[a, b]$.*

(ii) If, in addition, $g'(x)$ exists on $[a, b]$ and there exists a positive constant $k < 1$ with $|g'(x)| < k$ for all $x \in [a, b]$ then there is exactly one fixed point in $[a, b]$.

Proof. (i) If $g(a) = a$ or $g(b) = b$, the endpont is the fixed point.
If not $g(a) > a$ and $g(b) < b$.
Let's define $f(x) = g(x) - x$. Then we know that $f(a) > 0$ and $f(b) < 0$. The intermediate value theorem tells us that there exists a $p \in [a, b]$ such that $f(p) = 0$; i.e. $p = g(p)$.

(ii) Our goal for this part of the proof is to show that the fixed point is unique. The technique for doing this (almost) always the same. It is a proof by contradiction where you assume that there are two. Here we assume that there are two fixed points.

Suppose that (i) is satisfied; i.e. $g(x)$ is onto in $[a, b]$ and $|g'(x)| < k < 1$ for all $x \in [a, b]$.
Now we assume that we have two distinct fixed points p and q in $[a, b]$ ($p \neq q$).
The mean value theorem says that there exists a $\xi \in [a, b]$ such that

$$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

Now lets look at the difference of p and q.

$$
\begin{aligned}
|p-q| &= |g(p)-g(q)| \text{ since fixed points} \\
&= |g'(\xi)||p-q| \text{ by the mean value theorem} \\
&< k|p-q| \text{ since } |g'(x)| < k \text{ forall } x \in [a,b] \\
&< |p-q| \text{ since } k < 1
\end{aligned}
$$

This is a contradiction since $|p-q|$ cannot be less than itself. Thus there is only one fixed point in the interval.

□

What is the fixed point iteration?
The idea is to march along the line $y = x$ in hopes of finding $x = g(x)$.

More specifically, given a function $g(x)$ and intiial approximation p_0, we will construct a sequence $\{p_n\}_{n=0}^{\infty}$ where $p_n = g(p_{n-1})$ for $n > 0$. If the sequence p_n converges to p, and g is continous then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p)$$

PSEUDOCODE: Fixed point iteration

Input: $g(x)$, p_0, ϵ= stopping tolerance, $N_{\max}$=max number of iterations
Output: p^*=the approximate fixed point and *ier*=the error message

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(1) Set $i = 1$
(2) **while** $i < N_{\max}$
(3) Set $p = g(p_0)$ (calculate p_i)
(4) If $|p - p_0| < \epsilon$, set $p^* = p$, $ier = 0$ and exit
(5) Set $i = i + 1$
(6) Set $p_0 = p$ (get ready for next iteration)
end Set $ier = 1$ and output "Fixed point ran out of iterations"

Example: Apply the fixed point iteration to

$$
\begin{aligned}
f_1(x) &= 1 + 0.5\sin(x) \\
f_2(x) &= 3 + 2\sin(x)
\end{aligned}
$$

The fixed points are $\alpha_1 = 1.49870113351785$ and $\alpha_2 = 3.09438341304928$.
Do the fixed point iterations converge to the fixed points? If it does not work, do you have an idea why? We will explore this next.

Theorem 0.2. *(2.4 from text) Let $g \in C[a,b]$ such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that $g'(x)$ exists on (a,b) and there exists a constant k such $0 < k < 1$ with $|g'(x)| < k$ for all $x \in [a,b]$. Then for any initial guess $p_0 \in [a,b]$, the sequence defined by $p_n = g(p_{n-1})$ $n \geq 1$ converges to a unique fixed point in $[a,b]$.*

Proof. By Theorem 2.3, we know that there exists a unique fixed point in $[a,b]$. We also know that $|g'(x)| < k < 1$ for all $x \in [a,b]$. Let p denote the fixed point.

By the mean value theorem, there exists an η_n in the interval containing p and p_n such that

$$|p_n - p| = |g(p_{n-1}) - g(p)| \leq |g'(\eta_n)||p_{n-1} - p| < k|p_{n-1} - p|$$

Thus we can continue applying the mean value theorem to find

$$\begin{aligned} |p_n - p| &< k|p_{n-1} - p| \\ &< k^2|p_{n-2} - p| \\ &\vdots \\ &< k^n|p_0 - p| \end{aligned}$$

Since $0 < k < 1$, $\lim_{n\to\infty} k^n = 0$. Therefore

$$\lim_{n\to\infty} |p_n - p| < \lim_{n\to\infty} k^n|p_0 - p| = 0$$

i.e. $\{p_n\}_{n=1}^{\infty}$ converges to p.

□

We get something nice for free with the proof of this theorem. What is the order of convergence of the fixed point interation?
It is first order since

$$\frac{|p_n - p|}{|p_{n-1} - p|} < k$$

for large n.

Example: Use a fixed point iteration to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = -0$ on $[1, 2]$. Use $p_0 = 1$.
Solution: The first thing that we need to do is find a function $g(x)$ such that the fixed point iteration will converge.
We want to pick $g(x)$ such that

- g is onto $[1, 2]$
- $|g'(x)| < 1$ for all $x \in [1, 2]$

We have choices:

Option 1: $x = x^3 - 1 = g_1(x)$
$g_1(x)$ is continuous but we must make sure it is onto.

$$\min_{x\in[1,2]} x^3 - 1 = 0 \text{ at } x = 1$$

$$\max_{x\in[1,2]} x^3 - 1 = 7 \text{ at } x = 2$$

Thus $g_1(x)$ is not onto so this is not a good canditate for the fixed point iteration.

Let's check the second condition just for practice.

$$|g_1'(x)| = 3x^2$$

The bound of $|g_1'(x)|$ for $x \in [1, 2]$ is 12 which is much greater than 1. Thus g_1 fails to satisfy any of the the conditions needed for the fixed point iteration to work.

Option 2: $x = (x+1)^{1/3} = g_2(x)$

$g_2(x)$ is continuous but we must make sure it is onto.

$$\min_{x\in[1,2]} (x+1)^{1/3} = 2^{1/3} > 1 \text{ at } x = 1$$

$$\max_{x\in[1,2]} (x+1)^{1/3} = 3^{1/3} < 2 \text{ at } x = 2$$

Thus g_2 is onto.

Let's verify the second condition.

$$g_2'(x) = 1/3(x+1)^{-2/3}$$

We need to bound this

$$|g_2'(x)| = 1/3|(x+1)^{-2/3}| \leq 1/3 < 1$$

for all $x \in [1,2]$. Thus this is a good choice of function to apply the fixed point iteration to.