Homework #3

1.) a.) find upper bound on number of iterations in the bisection needed to approximate the solution of $x^3+x-4=0$ lying in the interval $[1,4]$ w/ an accuracy of 10^{-3}

$$\boxed{|b_n-a_n| < \varepsilon \iff 2^n > \frac{|b_0-a_0|}{\varepsilon} \iff n > \log_2\left(\frac{|b_0-a_0|}{\varepsilon}\right)}$$

$$n > \log_2\left(\frac{|4-1|}{10^{-3}}\right)$$

$$n > \log_2(3\cdot 10^3) = 11.5507$$

$n = 12$ ~~or greater~~

b.) They were the same, my program had 12 iterations

2.) a. $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$ $\quad x^0 = 2$

$$g(x) = -16 + 6x + \frac{12}{x}$$

$$g'(x) = 6 - \frac{12}{x^2}$$

$$g'(2) = 6 - \frac{12}{4} = 3 > 1$$

won't converge because not $|g'(x)| < 1$

b.) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$ $\quad x_0 = 3^{1/3}$

$$g'(x) = \frac{2}{3} - \frac{2}{x^3}$$

$$\left|\frac{2}{3} - \frac{2}{(\sqrt[3]{3})^3}\right| = 0 < 1 \implies \text{converges}$$

$g'(x_0) = 0 \implies$ converges atleast quadratically

$\longrightarrow$

Use taylor expansion at x_0

$$g(x) = g(x_0) + g'(x_0)(x_n - x_0) + \frac{g''(c)(x_n - x_0)^2}{2}$$

C is betwee x and x_0

$$g'(x_0) = 0 \Rightarrow g(x) = g(x_0) + \frac{g''(c)(x_n - x_0)^2}{2}$$

sub ↓ sub ↓

$$x_{n+1} = x_0 + \frac{g''(c)(x - x_0)^2}{2}$$

↓ chang x_0 to x_n

$$x_{n+1} - x_0 = \frac{g''(c)(x_n - x_0)^2}{2}$$

$$\lim_{n\to\infty}\left(\frac{x_{n+1} - x_0}{(x_n - x_0)^2} = \frac{g''(c)}{2}\right)$$

$$\lim_{n\to\infty}\frac{|x_{n+1} - x_0|}{|x_n - x_0|^2} = \lim_{n\to\infty}\frac{g''(c)}{2}$$

order of converge = 2

c) $x_{n+1} = \frac{12}{1+x_n}$ $\quad x_0 = 3$

$$g'(x) = \frac{-12}{(1+x_n)^2}$$

$$|g'(3)| = \left|\frac{-12}{(1+3)^2}\right| = \frac{3}{4} \leq 1 \Rightarrow \text{converges}$$

$\Rightarrow$ converses linearly $0 \leq 3/4 \leq 1$

$$\lim_{n\to\infty}\frac{|x_{n+1} - x_0|}{|x_n - x_0|} < \frac{3}{4}$$

3.) Roots to be determined w atleast 10 accurate digits

n accurate digits is equal to a relative error smaller than 0.5×10^{-n}

relative error $< 0.5 \times 10^{-10}$

$x - 4\sin(2x) - 3 = 0$

a) 3 zero crossings

b.) $x_{n+1} = -\sin(2x_n) + \frac{5x_n}{4} - 3/4$

4i) Frozen when $T(x,t)=0$ at x distance and t time

$$T(x,t) = T_s + (T_i - T_s)\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

T_s = constant temp during cold period
T_i = initial soil temp before cold snap
α = thermal conductivity (m^2 per sec)

a.) Apply fixed point

$$f(x) = T(x,t) = 0$$

$$f(x) = T_s + (T_i - T_s)\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$f'(x) = (T_i - T_s)\frac{d}{dx}\left(\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right)$$

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}\,ds$$

$$\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\alpha t}}} e^{-s^2}\,ds$$

$$\frac{d}{dt}(\text{erf}(t)) = \frac{2}{\sqrt{\pi}}e^{-t^2}$$

$$\frac{d}{dx}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{1}{2\sqrt{\alpha t}}$$ u-sub

$$f'(x) = (T_i - T_s)\left(\frac{1}{2\sqrt{\alpha t}}\cdot\frac{2}{\sqrt{\pi}}e^{-\left(\frac{x}{2\sqrt{\alpha t}}\right)^2}\right)$$

$$t = \underset{\text{sec}}{60}\cdot\underset{\text{min}}{60}\cdot\underset{\text{hour}}{24}\cdot\underset{\text{day}}{60}$$

5.) Let $f(x)$ denote a function w/ root a of multiplicity m. That means we can write $f(x) = (x-a)^m h(x)$ w/ $h(a) \neq$ and $h(x)$ is at least twice continuous differentiable

a.) Newton is a fixed point iteration define by

$$p_{n+1} = p_n + \frac{f(p_n)}{f'(p_n)} \quad \text{where } f'(p_n) \neq 0$$

$$\text{so } g(x) = x + \frac{f(x)}{f'(x)}$$ plug in

6.) $f(x) = x^6 - x - 1 \qquad x_0 = 2, \; x_1 = 1$

$\cancel{x = x^6 - 1}$

$f'(x) = 6x^5 - 1$

$\cancel{g(x) = x^6 - 1}$

$\cancel{g'(x) =}$

$$g(x_{n+1}) = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1}$$

Newton

$x_1 =$