APPM 4600 — HOMEWORK # 2

1. (a) Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$.
   (b) Show that $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$.
   (c) Show that $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$.
   (d) Show that $\int_0^{\varepsilon} e^{-x^2} dx = O(\varepsilon)$ as $\varepsilon \to 0$.

2. Consider solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the inverse of $\mathbf{A}$ is $\begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$. In this problem we will investigate a perturbation in $\mathbf{b}$ of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ and the numerical effects of the condition number.

   (a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.
   (b) What is the condition number of $\mathbf{A}$?
   (c) Let Δb_1 and Δb_2 be of magnitude 10^{-5}; not necessarily the same value. What is the relative error in the solution? What is the relationship between the relative error, the condition number, and the perturbation. Is the behavior different if the perturbations are the same? Which is more realistic: same value of perturbation or different value of perturbation?

3. Let $f(x) = e^x - 1$

   (a) What is the relative condition number $\kappa(f(x))$? Are there any values of x for which this is ill-conditioned?
   (b) Consider computing $f(x)$ via the following algorithm:

```
1: y = math.e^x
2: return y -1
```

   Is this algorithm stable? Justify your answer
   (c) Let x have the value $9.999999995000000 \times 10^{-10}$, in which case $f(x)$ is equal to 10^{-9} up to 16 decimal places. How many correct digits does the algorithm listed above give you? Is this expected?
   (d) Find a polynomial approximation of $f(x)$ that is accurate to 16 digits for $x = 9.999999995000000 \times 10^{-10}$. Hint: use Taylor series, and remember that 16 digits of accuracy is a relative error, not an absolute one.
   (e) Verify that your answer from part (d) is correct.
   (f) [Optional] How many digits of precision do you have if you do a simpler Taylor series?
   (g) [Fact; no work required] Matlab provides `expm1` and Python provides `numpy.expm1` which are special-purpose algorithms to compute $e^x - 1$ for $x \approx 0$. You could compare your Taylor series approximation with `expm1`.

4. Consider the equation $2x - 1 = \sin x$.

   (a) Find a closed interval $[a, b]$ on which the equation has a root r, and use the Intermediate Value Theorem to prove that r exists.

   (b) Prove that r from (a) is the only root of the equation (on all of $\mathbb{R}$).

   (c) Use the bisection code from class (or your own) to approximate r to eight correct decimal places. Include the calling script, the resulting final approximation, and the total number of iterations used.

5. The function $f(x) = (x - 5)^9$ has a root (with multiplicity 9) at $x = 5$ and is monotonically increasing (decreasing) for $x > 5$ ($x < 5$) and should thus be a suitable candidate for your function above. Use `a=4.82` and `b=5.2` and `tol = 1e-4` and use `bisection` with:

   (a) $f(x) = (x - 5)^9$.

   (b) The expanded expanded version of $(x - 5)^9$, that is, $f(x) = x^9 - 45x^8 + \ldots - 1953125$.

   (c) Explain what is happening.