APPM 4650 — Intro to interpolation

What is interpolation?

Given a set of point $\{x_j\}_{j=0}^n$ and funciton evaluations $\{f(x_j)\}_{j=0}^n$ create an approximation to the function $f(x)$.

Sometimes you have more information than just function evaluations. For example, you may have derivative information too.

Definition 0.1. *We call the points* $\{x_j\}_{0=1}^n$ *the intepolation* nodes*.*

You can use any set of functions to create the approximation. The most common choice for the set of functions to approximate with are polynomials.

Theorem 0.1. *(Weierstrass approximation Theorem) Suppose f is continuous on $[a,b]$. For each $\epsilon > 0$ there exists a polynomial $p(x)$ such taht*

$$|f(x) - p(x)| < \epsilon \text{ for all } x \in [a,b].$$

This does not say anything about how to construct the polynomial or the order of the polynomial needed.

What do we know? Taylor approximations from calculus but his is not useful here.

It would be great if we could make a polynomial that went through all the points.

Let suppose we only have data at two points x_0 and x_1. We know that we can fit this data with a line. Instead of going back to algebra, we will try to come up with something that extends naturally to higher order approximations (i.e. situations where we have more data).

Let's start by making a line that goes through 1 at x_0 and 0 at x_1.

Draw the line and label everything

The formula for the line is $L_0(x) = \frac{x-x_1}{x_0-x_1}$. Likewise we can construct a line that is O at x_0 and 1 at x_1. $L_1(x) = \frac{x-x_0}{x_1-x_0}$

Then the linear approximation of $f(x)$ is $p_1(x) = f(x_0)L_0(x) + f(x_1)L_1(x)$.

Example: Create at linear approximation to $f(x) = e^x$ through the nodes $x_0 = 0$ and $x_1 = 1$.
Soln: $p_1(x) = \frac{x-1}{0-1} + e\frac{x-0}{1-0}$

What if we want or need a higher approximation?
We need to sample more points and build high order basis polynomials that are 1 at a single node and zero at all the other points. These basis polynomials are called *Lagrange polynomials.*

Bask to the original problem... Suppose we are given a set of point $\{x_j\}_{j=0}^n$ and funciton evaluations $\{f(x_j)\}_{j=0}^n$ create an approximation to the function $f(x)$.

The j^{th} Lagrange polynomial of order n is

$$L_j(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

Example: Create at cubic approximation to $f(x) = e^x$ through the nodes $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, and $x_3 = 11/3$.
Soln: $p_3(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_2)L_3(x)$ where L_j is the j^{th} cubic Lagrange polynomial.

See {interpolation.m} to see performance of the linear and cubic approximations.

How accurate is this approxiation?

Theorem 0.2. *(3.3) Suppose* $x_0, x_1, \ldots, x_n$ *are distinct numbers in the interval* $[a, b]$ *and* $f \in C^{n+1}[a, b]$. *Then, for each* $x \in [a, b]$, *there exists an* η *(not generally known) between* $x_0, x_1, \ldots, x_n$ *in* $[a, b]$ *such that*

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x - x_0) \cdots (x - x_n)$$

where $p_n)$ *is the Lagrange approxiation of degree* n *through* $\{x_j\}_{j=0}^{n}$.