APPM 4600 — Exam 2 Review Sheet

- **Systems of Nonlinear equations and root finding methods (nD):**
  (a) Consider the fixed point iteration applied to $F : R^n \to R^n$. How would you apply this to a root finding problem? Under what conditions on its Jacobian G is F a contractive map?
  (b) Derive Newton's method for systems of nonlinear equations. Under what assumptions does this method converge quadratically? How does the Newton's method for nonlinear systems of equations differ from Newton's method for scalar equations?

- **Quasi-Newton methods:**
  (a) What are the different quasi-Newton methods we talked about?
  (b) What is the motivation for creating these techniques?
  (c) How do these techniques address the difficulty that motivated their development? Can you, for instance, compare the cost per step with Newton?

- **Steepest Descent**
  (a) What is the basic idea behind gradient descent? How can we use it to solve a rootfinding problem $F(x) = 0$?
  (b) What is the basic idea behind using a line-search to decide how big of a step to take in steepest descent?

- **Interpolation:**
  (a) What are the different interpolation techniques? (Lagrange, Newton, Hermite, splines)
  (b) What data is needed for each one?
  (c) What is the form for the representation? How is it constructed? How stable is it?
  (d) What is the error for each technique? Given fixed nodes, how would you create an upper bound on the error? Given a function that you can evaluate easily, what can you control to minimize the error?
  (e) Why are some sets of points better than others for polynomial interpolation?

- **Approximation:**
  (a) What is the difference between discrete approximation, L^2 approximation and interpolation?
  (b) How do we construct each approximation technique?
  (c) When do you use each approximation technique?
  (d) How do we build a family of orthogonal polynomials using Gram-Schmidt? What form does the 3-term recurrence have, and how do we use it? Once I have a family of orthogonal polynomials (e.g. Legendre, Chebyshev), how do I find the coefficients for the L2 approximant?

Note: Below are some practice problems. This list of problems is not exhaustive. You should not expect the exam to look exactly like this. These are simply some problems that you should be able to solve using the material from class. Making sure you understand all the concepts and algorithms introduced in class, homeworks and labs is the best way to prepare you for the exam. Be sure to understand why, when and how an algorithm works in addition to the limitations.

1. Consider the problem of finding the intersections between two circles of radius 1, with centers at $(1,0)$ and $(2,1)$, respectively. Upon ploting both circles, we clearly see two roots: $(1,1)$ and $(2,0)$. We can write this as a system of two nonlinear equations for coordinates (x,y):

$$\begin{aligned} F_1(x) &= (x-1)^2 + y^2 - 1 = 0 \\ F_2(x) &= (x-2)^2 + (y-1)^2 - 1 = 0 \end{aligned}$$

   Where $F(x) = [F_1(x) \;\; F_2(x)]$. We find the Jacobian of $F(x)$:

$$J_F(x) = \begin{bmatrix} 2(x-1) & 2y \\ 2(x-2) & 2(y-1) \end{bmatrix}$$

   (a) We use the Newton method starting at initial point $(x0, y0) = (1,0)$. We immediately get a complaint from our software that reads 'LinAlgError: Singular Matrix'. Compute the Jacobian at this initial point and explain why we got this message.

   (b) After running the Newton method for a good amount of initial points $(x0, y0)$, we notice a pattern. Iterations starting above the line $y = x - 1$ converge to $(1,1)$ and iterations starting below it converge to $(2,0)$. If we start exactly at this line, we get the same error message we got in (a). Plug in $y = x - 1$ in the formula for the Jacobian and explain why that might be the case.

2. Consider the Broyden Quasi-Newton method to find roots for a system of $n = 1000$ equations of the form $F(x) = 0$. Given $B_0 = J_F(x_0)$ (Jacobian at initial guess).

   - Write down the first step of the Broyden method.
   - Assume you have access to the matrix B_0^{-1}, or that given vectors v, you can compute the product $B_0^{-1}v$. Briefly explain how the Broyden method avoids having an expensive $O(n^3)$ step while retaining superlinear convergence.

3. Consider the method of steepest descent for root finding of the system $F(x) = 0$. We use it to minimize the function:

$$g(x) = ||F(x)||^2 = \sum_{j=1}^{n} F_j(x)^2$$

   (a) Derive an expression for the gradient of $g(x)$ involving $F(x)$ and its derivatives (Jacobian matrix $J_F(x)$).

(b) The method of steepest descent takes steps of the form:

$$x_{n+1} = x_n - \alpha_n \nabla g(x_n)$$

Say you decide to pick $\alpha_n = 1$ for all n. Explain: What can go wrong with this "lazy steepest descent"?

4. Let $f(x) = \sqrt{1+x}$ with interpolation nodes $x_0 = 0$, $x_1 = 1/2$ and $x_2 = 1$.

   (a) Create the highest order approximation possible using these interpolation nodes (hint: you can use derivatives of $f(x)$).

   (b) Derive a bound on the error when approximation $f(x)$ for $x \in [0,1]$ using the polynomial from part (a).

5. Using Lagrange interpolation as your inspiration, create a third degree polynomial such that $p(x_0) = y_0$, $p'(x_1) = y_2$ and $p''(x_1) = y_3$.

6. Given the following data for a smooth function $f(x)$

x	-2	-1	0	1	2	3
$f(x)$	-5	1	1	1	7	25

   (a) A routine that computes the barycentric weights w produces the following weights:

$$w = \frac{1}{120}\begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

   Write down a barycentric formula for the Lagrange interpolant.

   (b) Use divided differences to determine the interpolating polynomial (represented in the Newton polynomial basis) given the following data in the interval $[-2, 3]$.

7. What does it mean for a cubic spline to be natural, clamped or periodic? How do you impose these conditions, and how do they change the linear system (matrix and right-hand-side) to find the coefficients of the cubic "pieces" of these splines? Write down one example (e.g. periodic).

8. Given the data in the previous exercise:

x	-2	-1	0	1	2	3
$f(x)$	-5	1	1	1	7	25

   Write down the matrix M and vector y, and using them, the system of normal equations (don't solve it) to find the least squares fit for this data using (a) a line (b) a quadratic polynomial.

9. Find an orthogonal basis of polynomials for the interval $[0,1]$ and weight function $w(x) = 1$. You can use Gram-Schmidt, the 3-term recurrence or the Legendre basis.

10. Using the basis from the previous exercise, find the coefficients for the continuous quadratic least squares approximation of $f(x) = \sin(x)$.