APPM 4650 — Bisection

Goal of root finding techniques: Given a function $f(x)$ that is continuous in an area of interest, find one root.

Sometimes we are given an interval in which the root lives or a rough guess. Sometimes the derivative may be available but not always.
The root finding techniques we will talk about in this class can be used to find eigenvalues, in optimization problems, etc.
The first technique we will talk about is bissection.

Goal: Given a function $f(x)$ that is continuous on $[a, b]$ and satisfies the condition that $f(a)$ and $f(b)$ have opposite sign, find the value $x^* \in [a, b]$ such that $f(x^*)$ is approximately 0.

Question: How do we even know there is a root in the interval $[a, b]$?
Answer: The intermediiat value theorem says that there exist a $p \in [a, b]$ such that $f(p) = 0$.

Question: Will $x^* = p$?
Answer: Maybe not. It could be that p is not representable by the computer.

Question: What do mean by $f(x^*)$ is approximately 0?
Answer: It is possible that we cannot evaluate $f(x)$ to be 0 for any x. Thus we typically use a stopping tolerance which decides when the value of $f(x)$ is small enough. i.e. $|f(x^*)| \leq 100\epsilon_M$.

Idea behind bissection:

- Find the midpoint of the interval
- Check sign of the function at the endpoints and the midpoint to determine which half of the interval does the root live.
- Pick that interval as the new interval to cut in half and repeat.

Question: When should we stop?
Answer: If f evaluated at the midpoint is small enough or the intervals get too small.

Question: Let $c = 1/2(a + b)$ denote the midpoint of the interval $[a, b]$. How can we determine which interval the root is in $[a, c]$ or $[c, b]$?
Answer: Look at the sign of $f(a)f(c)$ and $f(b)f(c)$, which ever is less than zero contains the root.

We are now in a position to write the pseudocode. ‘

PSEUDOCODE: Bisection

Input: $a, b, f(x)$, tol and $N_{\max}$
Output: x^* the approximate root and ier the error message
(I like to say $ier = 0$ is success and $ier \neq 0$ is failure. Different ier values can mean different sources of failure.)

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(1) if $|f(a)| = 0$, $x^* = a$ and exit
    if $|f(b)| = 0$, $x^* = b$ and exit
(2) Set $fa = f(a)$
(3) **for** $i = 1, \ldots, N_{\max}$
(4)   Set $c = (b + a)/2$ (calculate midpoint)
(5)   Set $fc = f(c)$
(6)   If $|fc| = 0$ or $(b - a)/2 < tol$
        Set $x^* = c$ and $ier = 0$ then exit
(7)   If $(fc) * (fa) > 0$ (find the next interval)
        Set $a = c$ and $fa = fc$.
    else
        Set $b = c$.
    end if
  end
(8) If $i = N_{\max}$, $ier = 0$ and output "bissection failed"

Convergence: How fast will the bissection method converge to the root?
Let c_k denote the midpoint at iteration k. Then $|c_k - p| \leq 1/2|c_{k-1} - p|$. Thus

$$|c_k - p| \leq (1/2)^k |c_1 - p|.$$

What type of convergence is this?