APPM 4600 — HOMEWORK # 3

1. (a) Use a theorem from class (Theorem 2.1 from text) to find an upper bound on the number of iterations in the bisection needed to approximate the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$ with an accuracy of 10^{-3}.

   (b) Find an approximation of the root using the bisection code from class to this degree of accuracy. How does the number of iterations compare with the upper bound you found in part (a)?

2. **Definition 1** *Suppose* $\{p_n\}_{n=0}^{\infty}$ *is a sequence that converges to* p *with* $p_n \neq p$ *for all* n*. If there exists postive constants* λ *and* q *such that*

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^q} = \lambda$$

then $\{p_n\}_{n=1}^{\infty}$ *converges to* p *with an order* q *and asympotic error constant* λ*. If* $q = 1$ *and* $\lambda < 1$ *then the sequence converges* linearly*. If* $q = 2$*, the sequence is* quadratically *convergent.*

Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence. (**Hint: what condition can you evaluate for fixed point function** $g(x)$ **at the fixed point to determine this?**)

   (a) (10 points) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $x_* = 2$

   (b) (10 points) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{1/3}$

   (c) (10 points) $x_{n+1} = \frac{12}{1+x_n}$, $x_* = 3$

3. All the roots of the scalar equation

$$x - 4\sin(2x) - 3 = 0,$$

are to be determined with at least 10 accurate digits[1].

   (a) Plot $f(x) = x - 4\sin(2x) - 3$. All the zero crossings should be in the plot. How many are there?

   (b) Write a program or use the code from class to compute the roots using the fixed point iteration

$$x_{n+1} = -\sin(2x_n) + 5x_n/4 - 3/4.$$

   Use a stopping criterium that gives an answer with ten correct digits. (*Hint: you may have to change the error used in determing the stopping criterion.*) Find, empirically which of the roots that can be found with the above iteration. Give a theoretical explanation.

[1] n accurate digits is equivalent to a relative error smaller than 0.5×10^{-n}.

4. In laying water mains, utilities must be concerned with the possibility of freezing. Although soil and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x,t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

where T_s is the constant temperature during a cold period, T_i is the initial soil temperature before the cold snap, α is the thermal conductivity (in meters^2 per second), and

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2) ds$$

Assume that $T_i = 20$ [degrees C], $T_s = -15$[degrees C], $\alpha = 0.138 \cdot 10^{-6}$ [meters^2 per second]. *It is convenient to use* `scipy` *to evaluate the erf function.*

For parts (b) and (c), run your experiments with a tolerance of $\epsilon = 10^{-13}$.

   (a) We want to determine how deep a water main should be buried so that it will only freeze after 60 *days* exposure at this constant surface temperature.
   Formulate the problem as a root finding problem $f(x) = 0$. What is f and what is f'?
   Plot the function f on $[0, \bar{x}]$, where $\bar{x}$ is chosen so that $f(\bar{x}) > 0$.

   (b) Compute an approximate depth using the Bisection Method with starting values $a_0 =$ 0[meters] and $b_0 = \bar{x}$ [meters].

   (c) Compute an approximate depth using Newton's Method with starting value $x_0 = 0.01$[meters]. What happens if you start with $x_0 = \bar{x}$?

5. Let $f(x)$ denote a function with root α of multiplicity m. That means we can write $f(x) = (x - \alpha)^m h(x)$ with $h(\alpha) \neq 0$ and $h(x)$ is at least twice continuously differentiable.

   (a) Using fixed point theory or otherwise, show that Newton's method applied to $f(x)$ only converges linearly to the root α.

   (b) Show that the fixed point iteration applied to $g(x) = x - m\frac{f(x)}{f'(x)}$ is second order convergent.

   (c) What does part (b) provide for Newton's method in the case of roots with multiplicity greater than 1?

6. Use Newton and Secant method to approximate the largest root of

$$f(x) = x^6 - x - 1.$$

   Start Newton's method with $x_0 = 2$. Start Secant method with $x_0 = 2$ and $x_1 = 1$.

   (a) Create at table of the error for each step in the iteration. Does the error decrease as you expect?

   (b) Plot $|x_{k+1} - \alpha|$ vs $|x_k - \alpha|$ on log-log axes where α is the exact root for both methods. What are the slopes of the lines that result from this plot? How does this relate to the order?