Least squares continuous

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Warm-up

What was a good choice of interpolation nodes?

Chebychev was better than Equispaced.

- Clustering near endpts killed the wild ossillations.

We are going to build least squares approx. of functions.

Chebychev polynomials

- clustering of nodes
- orthogonal wrt a specific weight function. on $[-1,1]$.

Def:
- We say $f(x)$ is <u>orthogonal</u> to $g(x)$ wrt the weight function $w(x)>0$ on (a,b)

$$\Rightarrow \int_a^b f(x)\,g(x)\,w(x)\,dx = 0$$

- $\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx$ is the $\mathcal{L}^2$-innerproduct.

Ex:

Weight functions	orthogonal polynomials
$w(x) = \frac{1}{\sqrt{1-x^2}}$	1^{st} kind chebychev polynomials
$w(x) = 1$	Legendre
$w(x) = \sqrt{1-x^2}$	2^{nd} kind Chebychev polynomials

defined on $(-1,1)$

Recall: ℓ^2-approx. $\{x_j\}_{j=0}^n$ $\{y_j\}_{j=0}^n$

$$p(x) = \sum_{j=0}^{n} a_j x^j$$

$$\min \left| y_i - \sum_{j=0}^{n} a_j x^j \right|^2 \rightarrow \text{have to solve the normal eqn.}$$

$\min \sum (y_i - \sum_{j=0} a_j x^j)^2$ → have to solve the normal eqn.

Now we do a continuous approx.
Given $f(x)$ & a set of orthogonal polynomials $\phi_j(x)$ + $w(x) \geq 0$ on (a,b)

Write my approx. $p_n(x) = \sum_{j=0}^{n} a_j \phi_j(x)$

corresponding minimization problem

$$\min_{a_0, \dots a_n} \underbrace{\int_a^b w(x) \left(f(x) - p_n(x) \right)^2 dx}_{E(a_0, \dots a_n)}$$

min will happen when $\frac{dE}{da_j} = 0$ for $j = 0, \dots n$

$$\frac{dE}{da_j} = \frac{d}{da_j} \left(\int_a^b w(x) \left(f(x) - \sum_{i=0}^{n} a_i \phi_i(x) \right)^2 dx \right)$$

$$= -2 \int_a^b w(x) \left(f(x) - \sum_{i=0}^{n} a_i \phi_i(x) \right) \phi_j(x)\, dx = 0$$

Since this holds for all j, we get a linear system

$$\underbrace{\begin{bmatrix} \int w \phi_0 \phi_0 dx & \cdots & \int w \phi_0 \phi_n dx \\ \vdots & & \vdots \\ \int w \phi_n \phi_0 dx & \cdots & \int w \phi_n \phi_n dx \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}}_{\bar{a}} = \underbrace{\begin{bmatrix} \int w f(x) \phi_0 dx \\ \vdots \\ \int w f(x) \phi_n dx \end{bmatrix}}_{\bar{b}}$$

$$M \bar{a} = \bar{b}$$

where $M_{ij} = \int_a^b w(x) \phi_i(x) \phi_j(x)\, dx$

$$M \quad a = b$$

where $M_{ij} = \int_a^b w(x)\, \phi_i(x)\, \phi_j(x)\, dx$

Since $\{\phi_j\}_{j=0}^{n}$ are orthogonal

$$M_{ij} = 0 \text{ if } i \neq j$$

So M is now diagonal

$$M = \begin{bmatrix} \alpha_0 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix} \quad \text{where} \quad \alpha_j = \int_a^b (\phi_j(x))^2\, w(x)\, dx$$

So our constants are $a_j = \dfrac{\int_a^b f(x)\, \phi_j(x)\, w(x)\, dx}{\int_a^b (\phi_j(x))^2\, w(x)\, dx}$

$\Rightarrow P_n(x) = \sum_{j=0}^{n} a_j\, \phi_j(x)$ This is lovely!

There are two ways of constructing orthogonal polynomials.

1- Gram-Schmidt. - General & matches linear algebra.

2 - three term recursion - trickier to develop

- faster for construction of the polynomials

let $\langle \phi_i, \phi_j \rangle_w = \int_a^b \phi_i(x)\, \phi_j(x)\, w(x)\, dx$

<u>Gram-Schmidt</u>

Always start with $\phi_0 = 1$

$\phi_1(x) = x - b_0 \phi_0$ pick b_0 so that $\langle \phi_0, \phi_1 \rangle_w = 0$

unknown

$\phi_1(x) = x - \underset{\uparrow \text{unknown}}{b_0}\, \phi_0$ pick b_0 so that $\langle \phi_0, \phi_1 \rangle_\omega = 0$

$$\langle \phi_1(x), \phi_0(x) \rangle_\omega = \langle x - b_0 \phi_0, \phi_0 \rangle_\omega = \langle x, \phi_0 \rangle_\omega - b_0 \langle \phi_0, \phi_0 \rangle_\omega = 0$$

$$\rightarrow \quad b_0 = \frac{\langle x, \phi_0 \rangle_\omega}{\langle \phi_0, \phi_0 \rangle_\omega}$$

Next $\phi_2(x) = x^2 - c_0 \phi_0 - c_1 \phi_1$

want $\langle \phi_2, \phi_0 \rangle_\omega = 0$ & $\langle \phi_2, \phi_1 \rangle_\omega = 0$

This will give us c_0 & c_1

$$\langle \phi_2, \phi_0 \rangle_\omega = \langle x^2 - c_0 \phi_0 - c_1 \phi_1, \phi_0 \rangle_\omega = \langle x^2, \phi_0 \rangle - c_0 \langle \phi_0, \phi_0 \rangle_\omega - c_1 \overset{=0}{\cancel{\langle \phi_0, \phi_1 \rangle}}$$

$$\Rightarrow c_0 = \frac{\langle x^2, \phi_0 \rangle_\omega}{\langle \phi_0, \phi_0 \rangle_\omega} \qquad \text{likewise} \qquad c_1 = \frac{\langle x^2, \phi_1 \rangle_\omega}{\langle \phi_1, \phi_1 \rangle_\omega}$$

Warm-up

What does it mean for a set of polynomials $\{\phi_j\}$ to be orthogonal wrt $w(x) \geq 0 \ \forall x \in I$?

Soln: $\langle \phi_j, \phi_i \rangle_\omega = \int_I \phi_j(x)\, \phi_i(x)\, w(x)\, dx = 0$ if $i \neq j$

What if you want your basis to be orthonormal?

Soln:
If $\{\phi_j\}$ are orthogonal. then

$\{\psi_j\}$ ... $\psi_j = \phi$ ... $\|\phi_j\| = \sqrt{\langle \phi_j, \phi_j \rangle}$

$\{\psi_j\}$ where $\psi_j = \dfrac{\phi_j}{\|\phi_j\|_w}$, $\|\phi_j\|_w = \sqrt{\langle \phi_j, \phi_j \rangle_w}$

Another way to get orthogonal polynomials is via the 3 term recursion. (Text §8.7 p. 515)

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k \phi_{k-2}(x)$$

where $\phi_{-1} = 0$, $\phi_0 = 1$ $\Rightarrow$ $\phi_1 = (x - b_1)\phi_0$

The constants are given by

$$b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1} \rangle_w}{\langle \phi_{k-1}, \phi_{k-1} \rangle_w} \quad \text{for } k \geq 1$$

& $$c_k = \frac{\langle \phi_{k-2}, x\phi_{k-1} \rangle}{\langle \phi_{k-2}, \phi_{k-2} \rangle_w} \quad \text{for } k \geq 2$$