Numerical methods for solving the class-based mean-field equations

August 19, 2021

To solve the MF equation numerically we discretise the opinion space into n equally spaced states. Any individuals with opinion $x \in [\frac{i-1}{n}, \frac{i}{n}]$ we say is in state i, $i \in \{1, \cdots, n\}$. We write $P_i(t)$ as the proportion of the population in state i at time t. Then

$$\begin{aligned} P_i(t+dt) = P_i(t) &+ \text{proportion of nodes that move to state i in dt} \\ &- \text{proportion of nodes that move out of state i in dt.} \end{aligned} \tag{1}$$

The proportion of nodes that move to state i in dt is

$$\sum_{k \neq i} P_k(t) B_{ki}(dt),$$

where $B_{ki}(dt)$ is the probability that a node transitions from state k to state i in dt. Similarly, the proportion of nodes that move away from state i in dt is

$$\sum_{k \neq i} P_i(t) B_{ik}(dt)$$

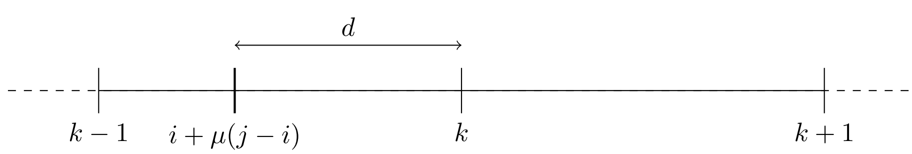
and so Eqn. 1 becomes

$$P_i(t+dt) = P_i(t) + \sum_{k \neq i} P_k(t) B_{ki} dt - \sum_{k \neq i} P_i(t) B_{ik}(dt). \tag{2}$$

Before deriving the transition probabilities, we need to decide what happens if interaction leads to a new opinion which lies between states. Suppose two individuals I_i and I_j, with opinions i/n and j/n (states i and j) respectively, interact. The new opinion of I_i is

$$\frac{1}{n}\left[i + \mu(j-i)\right]. \tag{3}$$

If this falls between two states $k-1$ and k then we can assign a probability that the new opinion will go to either of these states which is proportional to the distance from each state. More specifically, if the distance from $i + \mu(j-i)$ to k is d then we can say the the new state of I_i will be k with probability $1-d$ and $k-1$ with probability d. (Here we are assuming the distance between states is 1).



Now an individual can move from state i to state k if they interact with someone in state j such that $|\frac{i}{n} - \frac{j}{n}| < \epsilon$ and $\frac{k-1}{n} < \frac{i}{n} + \mu\left(\frac{j}{n} - \frac{i}{n}\right) < \frac{k+1}{n}$. The probability that they interact with someone in state j in time dt is $P_j(t)dt$ and this will result in an opinion change if $|i-j| < \epsilon$. Summing over possible values of j,

$$B_{i,k}(dt) = \sum_{\substack{j: k-1 < i+\mu(j-i) < k+1 \\ |j-i| < \lceil \epsilon n \rceil}} P_j(t) dt (1 - |k - (i + \mu(j-i))|)$$

An individual will stay is state i if (1) interaction leads to a new opinion in $\left(\frac{i-1}{n}, \frac{i+1}{n}\right)$ or (2) they interact with someone whose opinion is too far away to cause an opinion change, $|i-j| \geq \epsilon$. Thus we have

$$B_{i,i}(dt) = \sum_{\substack{j:i-1<i+\mu(j-i)<i+1 \\ |j-i|<\lceil \epsilon n \rceil}} P_j(t)dt(1-|i-(i+\mu(j-i))|) + \sum_{j:|j-i|\geq\lceil \epsilon n \rceil} P_j(t)dt$$

We can show (?) that $B_{i,i}(dt) = \sum_{k\neq i} B_{i,k}$, and so Eqn. 2 becomes

$$P_i(t+dt) = P_i(t) + \sum_{k\neq i} P_k(t)B_{ki}dt - P_i(t)B_{ii}(dt). \tag{4}$$

We construct a matrix of transition rates Q from the matrix of transition probabilities $B(dt)$ by setting

$$\begin{cases} Q_{ik} = \lim_{dt\to 0} \frac{B_{ik}(dt)}{dt}, & k \neq i \\ Q_{ii} = -\lim_{dt\to 0} \frac{B_{ii}(dt)}{dt}. \end{cases} \tag{5}$$

Dividing by dt in Eqn. 4 and taking the limit $dt \to 0$ we have

$$\lim_{dt\to 0} \frac{P_i(t+dt) - P_i(t)}{dt} = \sum_{k\neq i} P_k(t) \lim_{dt\to 0} \frac{B_{ki}(dt)}{dt} - P_i(t) \lim_{dt\to 0} \frac{B_{ii}(dt)}{dt},$$

which becomes

$$\frac{\partial P_i}{\partial t} = \sum_{k\neq i} P_k(t)Q_{ki},$$

or

$$\frac{\partial P}{\partial t} = P(t)Q,$$

where $P(t) = (P_1(t), \dots, P_n(t))$.

Note that the entries of Q are

$$Q_{ik} = \sum_{\substack{j:k-1<i+\mu(j-i)<k+1 \\ |j-i|<\lceil \epsilon n \rceil}} P_j(t)(1-|k-(i+\mu(j-i))|), \qquad k \neq i$$

$$Q_{ii} = -\sum_{k\neq i} Q_{ik}$$

We can extend this to multiple classes,

$$\frac{\partial P_l}{\partial t} = \sum_{m=1}^{N_C} C_{lm}P_l(t)Q_m,$$

where C_{lm} are the coefficients in the CBMF equations and Q_m is the transition matrix for the distribution of class m.