Supplementary Material for

Decentralized Pose Graph Optimization on Manifolds for Multi-Robot Object-based Collaborative Mapping

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[Theoretical analysis under a simplified setting]

To show that the update directions v_i will eventually align and thus mimic Newton tracking, we conduct a rigorous analysis that constitutes the following two stages. By examining the primal-dual dynamics under standard convexity assumptions, we first prove that both the public variables y_i and the dual variables λ_i will converge to certain limit points. Then, we perform a standard limit point analysis, which shows that both quantities indeed converge to a limit point that satisfies the global optimality conditions, i.e., y_i converging to a consensus value and v_i vanishing to zero. The detailed derivation are given as below.

Proof: Fist of all, for notational convenience, let $z_i := [x_i, y_i]$ denote the concatenation of private and public variables of robot i. The consistency constraint for public variables between any two robots $i \neq j$ can be expressed as $\varphi(A_i z_i, A_j z_j)$, where matrices $\{A_i\}_{i\in[N]}$ extract the public components (Without loss of generality, we omit these matrices in subsequent analysis for simplicity). Let $\boldsymbol{z} := [z_1, \ldots, z_N]$ denote the stacked vector of all robot variables, and $\boldsymbol{\lambda} := [\lambda_{ij}]_{i\neq j\in[N]}$ represent all dual variables. As a result, with $f(\boldsymbol{z}) = \sum_{i=1}^{N} f_i(\boldsymbol{z}_i)$, the global augmented Lagrangian becomes

$$L\left(\boldsymbol{z}, \lambda\right) = f(\boldsymbol{z}) + \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \left\langle \lambda_{ij}, w_{ij}\varphi\left(z_i, z_j\right)\right\rangle + \frac{\beta}{2}\left\| w_{ij}\varphi\left(z_i, z_j\right)\right\|^2 \right). \tag{0.1}$$

STEP 1: Convergence to certain cluster point

For analytical simplicity, let the consistency constraint $\varphi(\cdot)$ be defined as $\varphi(z_i, z_j) := z_i - z_j$. Then, the primal-dual update process becomes

$$\boldsymbol{z}^{k+1} = \underset{\boldsymbol{z}_i, x_i}{\arg\min}\, L\left(\boldsymbol{z}^k, \boldsymbol{\lambda}^k\right), \tag{0.2}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \beta D_W B \boldsymbol{z}^{k+1}, \tag{0.3}$$

with $L\left(\boldsymbol{z}^k, \boldsymbol{\lambda}^k\right) = f\left(\boldsymbol{z}^k\right) + \left\langle \boldsymbol{\lambda}^k, D_W B \boldsymbol{z}^k \right\rangle + \frac{\beta}{2}\boldsymbol{z}^k (D_W B)^\top D_W B \boldsymbol{z}^k$ where B is the incidence matrix and D_W is a diagonal matrix with weights w_{ij} as the diagonal entries.

According to the first-order optimality condition, from (2.2) and (2.4) we have

$$\begin{aligned} &\nabla f\left(\boldsymbol{z}^{k+1}\right) + (D_W B)^\top \boldsymbol{\lambda}^k + \beta (D_W B)^\top D_W B \boldsymbol{z}^{k+1} = 0 \\ &\Leftrightarrow \nabla f\left(\boldsymbol{z}^{k+1}\right) + (D_W B)^\top \left(\boldsymbol{\lambda}^k + \beta D_W B \boldsymbol{z}^{k+1}\right) = 0 \\ &\Leftrightarrow \nabla f\left(\boldsymbol{z}^{k+1}\right) + (D_W B)^\top \boldsymbol{\lambda}^{k+1} = 0. \end{aligned} \tag{0.4}$$

where we have used the dual update in Eq. (0.3) to obtain the last relation.

Now, let $(\boldsymbol{z}^*, \lambda^*)$ with $\boldsymbol{z}^* = 1 \cdot z^*$ be a saddle point. Then, we have

$$\nabla f\left(\boldsymbol{z}^*\right) + (D_W B)^\top \boldsymbol{\lambda}^* + \beta (D_W B)^\top D_W B \boldsymbol{z}^* = 0. \tag{0.5}$$

Since $B\boldsymbol{z}^* = 0$, we further have

$$\nabla f\left(\boldsymbol{z}^*\right) + (D_W B)^\top \boldsymbol{\lambda}^* = 0. \tag{0.6}$$

Besides, due to the convexity of $f(\boldsymbol{z})$, we obtain

$$\left\langle \nabla f\left(\boldsymbol{z}^{k+1}\right) - \nabla f\left(\boldsymbol{z}^*\right), \boldsymbol{z}^{k+1} - \boldsymbol{z}^* \right\rangle \geq 0. \tag{0.7}$$

Then, substituting (0.4) and (0.6) into (0.7), we have

$$\begin{aligned} &\left\langle (D_W B)^\top \left(\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*\right), \boldsymbol{z}^{k+1} - \boldsymbol{z}^* \right\rangle \\ &\quad = \left\langle \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*, D_W B \left(\boldsymbol{z}^{k+1} - \boldsymbol{z}^*\right) \right\rangle \\ &\quad = \left\langle \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*, D_W B \boldsymbol{z}^{k+1} \right\rangle \\ &\quad = \left\langle \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*, \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k \right\rangle \leq 0, \end{aligned} \tag{0.8}$$

where we have used the dual update in Eq. (0.3) to obtain the last relation.

Knowing the fact that $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$, letting $a := \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*$, $b := \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k$ we have

$$\left\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*\right\|^2 - \left\|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*\right\|^2 \leq -\left\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k\right\|^2. \tag{0.9}$$

The rest of the proof follows the standard limit point analysis [A1] which shows that the sequence of variables $(\boldsymbol{z}^k, \boldsymbol{\lambda}^k)$ will gradually converge to certain limit point $(\boldsymbol{z}^\infty, \boldsymbol{\lambda}^\infty)$.

[A1] *Chambolle, Antonin, and Thomas Pock. "A first-order primal-dual algorithm for convex problems with applications to imaging." Journal of mathematical imaging and vision 40 (2011): 120-145.*

[A2] *Zhang, J., Ma, S. Zhang, S. Primal-dual optimization algorithms over Riemannian manifolds: an iteration complexity analysis. Math. Program. 184, 445–490 (2020).*

STEP 2: Limit point analysis

Having established the convergence to certain limit point, we are now ready to show the update directions v_i indeed eventually reach consensus. Let $k \to \infty$ and then we have for each robot i,

$$y_i^\infty = Retr_{y_i^\infty}(-\alpha v_i^\infty), \tag{0.10}$$

$$x_i^\infty = Retr_{x_i^\infty}(-\gamma u_i^\infty), \tag{0.11}$$

$$\lambda_{ij}^\infty = \lambda_{ij}^\infty + \beta w_{ij} \varphi\left(y_i^\infty, y_j^\infty\right). \tag{0.12}$$

Since λ_{ij}^k converges to certain limit point, it follows from Eq. (0.12) that

$$\varphi\left(y_i^\infty, y_j^\infty\right) = 0. \tag{0.13}$$

which implies that the public variables of all robots converge to a consensus value.

Likewise, according to Eq. (0.10), we obtain

$$v_i^\infty = -\frac{1}{\alpha} Retr_{y_i^\infty}^{-1}(y_i^\infty) = \mathbf{0}, \tag{0.14}$$

which implies that the update directions v_i eventually converge to zero. Thus, we complete the proof.

Remark: It should be noted that the above convergence analysis is conducted under a simplified setting, where we assume the convexity of the cost functions and adopt a specific form of the residual function. While practical applications (e.g., PGO) often involve non-convex problems on Riemannian manifolds, empirical evidence shows effective convergence under proper initialization. Recent advances in Riemannian non-convex optimization [A2] suggest potential extensions of our framework but a formal analysis under non-convex settings with general residual function φ remains challenging due to the presence of non-linear geometry (e.g., curvature, retractions), which is left for our ongoing work.