Supplementary Material for

Decentralized Pose Graph Optimization on Manifolds for Multi-Robot Object-based Collaborative Mapping

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[Theoretical analysis under a simplified setting]

To show that the update directions v_i will eventually align and thus mimic Newton tracking, we conduct a rigorous analysis that constitutes two stages as follows. First, by examining the primal-dual dynamics under standard convexity assumptions, we prove asymptotic convergence to the limit points. Second, we perform a thorough limit point analysis, which shows that both quantities converge to a consistent equilibrium point that satisfies the global optimality conditions, i.e., both y_i and the update directions v_i converging to certain consensus value eventually (note that $\boldsymbol{v}$ vanishes as well). The detailed derivation are given as below.

Fist of all, for notational convenience, let $z_i := [x_i, y_i]$ denote the concatenation of private and public variables of robot i. The consistency constraint for public variables between any two robots $i \neq j$ can be expressed as $\varphi(A_i z_i, A_j z_j)$, where matrices $\{A_i\}_{i\in[N]}$ extract the public components (Without loss of generality, we omit these matrices in subsequent analysis for simplicity). Let $\boldsymbol{z} := [z_1, \ldots, z_N]$ denote the stacked vector of all robot variables, and $\boldsymbol{\lambda} := [\lambda_{ij}]_{i\neq j\in[N]}$ represent all dual variables. As a result, with $f(\boldsymbol{z}) = \sum_{i=1}^{N} f_i(\boldsymbol{z}_i)$, the global Lagrange function becomes

$$L(\boldsymbol{z}, \lambda) = f(\boldsymbol{z}) + \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} \langle \lambda_{ij}, w_{ij}\varphi(z_i, z_j)\rangle + \frac{\beta}{2}\|w_{ij}\varphi(z_i, z_j)\|^2 \right). \tag{0.1}$$

STEP 1: Convergence to certain cluster point

For analytical simplicity, let the consistency constraint $\varphi(\cdot)$ be defined with $\varphi(z_i, z_j) := z_i - z_j$. Then, the primal-dual update process becomes

$$\boldsymbol{z}^{k+1} = \underset{\boldsymbol{z}_i, x_i}{\arg\min}\, L\left(\boldsymbol{z}^k, \boldsymbol{\lambda}^k\right), \tag{0.2}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \beta D_W B \boldsymbol{z}^{k+1}. \tag{0.3}$$

with

$$L\left(\boldsymbol{z}^k, \boldsymbol{\lambda}^k\right) = f\left(\boldsymbol{z}^k\right) + \left\langle \boldsymbol{\lambda}^k, D_W B \boldsymbol{z}^k \right\rangle + \frac{\beta}{2}\boldsymbol{z}^k (D_W B)^\top D_W B \boldsymbol{z}^k. \tag{0.4}$$

where B is the incidence matrix and D_W is a diagonal matrix with weights w_{ij} as the diagonal entries.

According to the first-order optimality condition, we have

$$\begin{aligned} &\nabla f\left(\boldsymbol{z}^{k+1}\right) + (D_W B)^\top \boldsymbol{\lambda}^k + \beta (D_W B)^\top D_W B \boldsymbol{z}^{k+1} = 0 \\ &\Leftrightarrow \nabla f\left(\boldsymbol{z}^{k+1}\right) + (D_W B)^\top \left(\boldsymbol{\lambda}^k + \beta D_W B \boldsymbol{z}^{k+1}\right) = 0 \\ &\Leftrightarrow \nabla f\left(\boldsymbol{z}^{k+1}\right) + (D_W B)^\top \boldsymbol{\lambda}^{k+1} = 0. \end{aligned} \tag{0.5}$$

where we have used the dual update in Eq. (0.3).

Let $(\boldsymbol{z}^*, \lambda^*)$ with $\boldsymbol{z}^* = 1 \cdot z^*$ be a saddle point. Then, we have

$$\nabla f(\boldsymbol{z}^*) + (D_W B)^\top \boldsymbol{\lambda}^* + \beta (D_W B)^\top D_W B \boldsymbol{z}^* = 0. \tag{0.6}$$

Since $B\boldsymbol{z}^* = 0$, we further have

$$\nabla f(\boldsymbol{z}^*) + (D_W B)^\top \boldsymbol{\lambda}^* = 0. \tag{0.7}$$

Besides, due to the convexity of $f(\boldsymbol{z})$, we obtain

$$\left\langle \nabla f\left(\boldsymbol{z}^{k+1}\right) - \nabla f(\boldsymbol{z}^*), \boldsymbol{z}^{k+1} - \boldsymbol{z}^* \right\rangle \geq 0. \tag{0.8}$$

Then, substituting (0.5) and (0.7) into (0.8), we have

$$\begin{aligned} &\left\langle (D_W B)^\top \left(\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*\right), \boldsymbol{z}^{k+1} - \boldsymbol{z}^* \right\rangle \\ &= \left\langle \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*, D_W B\left(\boldsymbol{z}^{k+1} - \boldsymbol{z}^*\right) \right\rangle \\ &= \left\langle \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*, D_W B \boldsymbol{z}^{k+1} \right\rangle \\ &= \left\langle \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*, \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k \right\rangle \leq 0. \end{aligned} \tag{0.9}$$

where we have used the dual update in Eq. (0.3).

Knowing the fact that $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$, letting $a := \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*$, $b := \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k$ we have

$$\left\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*\right\|^2 - \left\|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*\right\|^2 \leq -\left\|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k\right\|^2. \tag{0.10}$$

According to the standard limit point analysis [A1], the sequence of variables$(\boldsymbol{z}^k, \boldsymbol{\lambda}^k)$ will gradually converge to a certain limit point $(\boldsymbol{z}^*, \boldsymbol{\lambda}^*)$. It follows that the limit $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{\lambda}^*)$ is a fixed point of the iteration. Our current convergence analysis assumes the convexity of the local cost functions to simplify theoretical derivations. While practical applications like PGO often involve non-convex problems, empirical evidence shows effective convergence under proper initialization. Recent advances in Riemannian non-convex optimization [A2] suggest potential extensions of our framework. Formal analysis under general non-convex settings remains a key focus of our ongoing research.

[A1] *Chambolle, Antonin, and Thomas Pock. "A first-order primal-dual algorithm for convex problems with applications to imaging." Journal of mathematical imaging and vision 40 (2011): 120-145.*

[A2] *Zhang, J., Ma, S. Zhang, S. Primal-dual optimization algorithms over Riemannian manifolds: an iteration complexity analysis. Math. Program. 184, 445–490 (2020).*

STEP 2: Limit point analysis

Having established convergence to the limit point, we now analyze the equilibrium behavior to validate the directional consensus of v_i at convergence. As $k \to \infty$, we analyze the algorithm update procedure of robot i as follows,

$$y_i^\infty = Retr_{y_i^\infty}(-\alpha v_i^\infty), \tag{0.11}$$

$$x_i^\infty = Retr_{x_i^\infty}(-\gamma u_i^\infty), \tag{0.12}$$

$$\lambda_{ij}^\infty = \lambda_{ij}^\infty + \beta w_{ij} \varphi\left(y_i^\infty, y_j^\infty\right). \tag{0.13}$$

From Eq. (0.13), we have

$$\varphi\left(y_i^{\infty}, y_j^{\infty}\right) = 0. \tag{0.14}$$

It suggests that the public variables of all robots can converge on the consensus.

Similarly, for Eq. (0.11) when $k \to \infty$, we have

$$v_i^{\infty} = -\frac{1}{\alpha} Retr_{y_i^{\infty}}^{-1}(y_i^{\infty}) = \mathbf{0}, \tag{0.15}$$

which implies that the update directinos v_i eventually converge to zero. Thus, we complete the proof.

In Euclidean settings, Zhang et al.[25] demonstrated that distributed Newton tracking achieves linear convergence by approximating the global Newton direction through local updates despite heterogeneity in H_i and g_i. However, formal proof for the Riemannian manifold case with general residual function φ is more challenging due to non-linear geometry (e.g., curvature, retractions), which is left for our ongoing work.