Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 3

Second-Order Linear Differential Equations

Section 3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Solutions of Linear Homogeneous Equations: Differential Operator Notation

- Let p, q be continuous functions on an interval $I = (\alpha, \beta)$ which could be infinite. For any function y that is twice differentiable on I, define the differential operator L by

$$L[y] = y'' + p\,y' + q\,y$$

- Note that $L[y]$ is a function on I, with output value

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

- For example,

$$p(t) = t^2, q(t) = e^{2t}, y(t) = \sin t, I = (0,\ 2\pi)$$
$$L[y](t) = -\sin t + t^2 \cos t + e^{2t} \sin t$$

Solutions for Initial Value Second Order Linear Homogeneous Equations

- In this section we will discuss the second order linear homogeneous equation $L[y](t) = 0$, along with initial conditions as indicated below:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

- Are there solutions to this initial value problem, and if so, are they unique?
- What can be said about the form and structure of solutions that might be helpful in finding solutions to particular problems?
- The above two questions are addressed in the theorems of this section.

Theorem 3.2.1 (Existence and Uniqueness)

- Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t_0) = y_0,\; y'(t_0) = y'_0$$

- where p, q, and g are continuous on an open interval I that contains t_0. Then there exists a unique solution $y = \phi(t)$ on I.
- While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equations.

Example 3.2.1

- Find the longest interval in which the solution of the initial value problem is certain to exist:

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0,\ y(1) = 2,\ y'(1) = 1$$

- Writing the differential equation in the form :

$$y'' + p(t)y' + q(t)y = g(t)$$

where: $$p(t) = \frac{1}{t-3},\ q(t) = -\frac{t+3}{t(t-3)} \text{ and } g(t) = 0$$

- The only points of discontinuity for these coefficients are $t = 0$ and $t = 3$. So the longest open interval containing the initial point $t = 1$ in which all the coefficients are continuous is $0 < t < 3$
- Therefore, the longest interval in which Theorem 3.2.1 guarantees the existence of the solution is $0 < t < 3$

Example 3.2.1 Cont.

- Find the unique solution of the initial value problem:

$$y'' + p(t)y' + q(t)y = 0, y(t_0) = 0, y'(t_0) = 0$$

where p, q are continuous on an open interval I containing t_0.

- In light of the initial conditions, note that $y = 0$ is a solution to this homogeneous initial value problem.
- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that $y = 0$ is the only solution of this problem.

Theorem 3.2.2: Principle of Superposition

- If y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + y_2c_2$ is also a solution, for all constants c_1 and c_2.

- To prove this theorem, substitute $c_1y_1 + c_2y_2$ in for y in the equation above, and use the fact that y_1 and y_2 are solutions.
- Thus for any two solutions y_1 and y_2, we can construct an infinite family of solutions, each of the form $y = c_1y_1 + c_2\,y_2$.
- Can all solutions can be written this way, or do some solutions have a different form altogether? To answer this question, we use the Wronskian determinant.

The Wronskian Determinant (part one)

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)y' + q(t)y = 0$$
- From Theorem 3.2.2, we know that $y = c_1 y_1 + c_2 y_2$ is a solution to this equation.
- Next, find coefficients such that $y = c_1 y_1 + c_2 y_2$ satisfies the initial conditions
$$y(t_0) = y_0,\ y'(t_0) = y'_0$$
- To do so, we need to solve the following equations:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y'_1(t_0) + c_2 y'_2(t_0) &= y'_0 \end{aligned}$$

The Wronskian Determinant (part two)

- The determinant of coefficients of the equations for initial conditions c_1 and c_2 is known as the Wronskian:

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2(t_0)$$

- If $W \neq 0$, then unique solution for c_1 and c_2 can be obtained for all values of y_0 and y_0':

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2(t_0)}, \qquad c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2(t_0)},$$

- In terms of determinants: $c_1 = \dfrac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \dfrac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$

The Wronskian Determinant (part three)

- In order for these formulas to be valid, the determinant W in the denominator cannot be zero:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

- The **Wronskian determinant**, or more simply, the Wronskian of the solutions y_1 and y_2 is sometimes referred to with the notation:

$$W[y_1, y_2](t_0)$$

Theorem 3.2.3

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

with the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0'$$

Then it is always possible to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W = y_1 y_2' - y_1' y_2$$

is not zero at the point t_0

Example 3.2.3

- In Example 2 of Section 3.1, we found that

$$y_1(t) = e^{-2t} \text{and } y_2(t) = e^{-3t}$$

were solutions to the differential equation

$$y'' + 5y' + 6y = 0$$

- Find the Wronskian of y_1 and y_2.
- The Wronskian of these two functions is

$$W\left[e^{-2t}, e^{-3t}\right] = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

- Since W is nonzero for all values of t, the functions can be used to construct solutions of the differential y_1 and y_2 equation with initial conditions at any value of t.

Theorem 3.2.4: Fundamental Solutions

- Suppose y_1 and y_2 are solutions to the equation

  $$L[y] = y'' + p(t)y' + q(t)y = 0.$$

  Then the family of solutions

  $$y = c_1 y_1 + c_2 y_2$$

  with arbitrary coefficients c_1, c_2 includes every solution to the differential equation if and only if there is a point t_0 such that $W[y_1, y_2](t_0) \neq 0$.
- The expression $y = c_1 y_1 + c_2 y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

Example 3.2.4

- Consider the general second order linear equation below, with the two solutions indicated:

$$y'' + p(t)\,y' + q(t)\,y = 0$$

- Suppose the functions below are solutions to this equation:

$$y_1 = e^{r_1 t},\, y_2 = e^{r_2 t},\quad r_1 \neq r_2$$

- The Wronskian of y_1 and y_2 for all t is given by:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = \left(r_2 - r_1\right)\exp\left[\left(r_1 + r_2\right)t\right]$$

- Thus y_1 and y_2 form a fundamental set of solutions to the equation, and can be used to construct all of its solutions.
- The general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Example 3.2.5: Fundamental Solutions (part one)

- Consider the following differential equation:

$$2t^2 y'' + 3t\, y' - y = 0,\ \ t > 0$$

- Show that the functions below are fundamental solutions:

$$y_1 = t^{1/2},\ y_2 = t^{-1}$$

- To show this, first substitute y_1 into the equation:

$$2t^2\left(\frac{-t^{-3/2}}{4}\right) + 3t\left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0$$

- Thus y_1 is a indeed a solution of the differential equation.
- Similarly, y_2 is also a solution:

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0$$

Example 3.2.5: Fundamental Solutions (part two)

- Recall that

$$y_1 = t^{1/2}, y_2 = t^{-1}$$

- To show that y_1 and y_2 form a fundamental set of solutions, we evaluate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2}$$

- Since $W \neq 0$ for $t > 0$, y_1 and y_2 form a fundamental set of solutions for the differential equation

$$2t^2 y'' + 3t\,y' - y = 0, \;\; t > 0$$

Theorem 3.2.5: Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients p and q are continuous on some open interval I:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

- Let t_0 be a point in I, and y_1 and y_2 solutions of the equation with y_1 satisfying initial conditions

$$y_1(t_0) = 1,\ y'_1(t_0) = 0$$

and y_2 satisfying initial conditions

$$y_2(t_0) = 0,\ y'_2(t_0) = 1$$

- Then y_1 and y_2 form a fundamental set of solutions to the given differential equation.

Example 3.2.6: Apply Theorem 3.2.5

- Find the fundamental solution set specified by Theorem 3.2.5 for the differential equation and initial point

$$y'' - y = 0, \quad t_0 = 0$$

- In Section 3.1, we found two solutions of this equation:

$$y_1 = e^t, y_2 = e^{-t}$$

  The Wronskian of these solutions is $W[y_1, y_2](t_0) = -2 \neq 0$ so they form a fundamental set of solutions.

- But these two solutions do not satisfy the initial conditions stated in Theorem 3.2.5, and thus they do not form the fundamental set of solutions mentioned in that theorem.

- Let y_3 and y_4 be the fundamental solutions of Theorem 3.2.5.

$$y_3(0) = 1, \; y_3'(0) = 0; \quad y_4(0) = 0, \; y_4'(0) = 1$$

Example 3.2.6: General Solution

- Since y_1 and y_2 form a fundamental set of solutions,

$$y_3 = c_1 e^t + c_2 e^{-t}, \quad y_3(0) = 1, y_3'(0) = 0$$

$$y_4 = d_1 e^t + d_2 e^{-t}, \quad y_4(0) = 0, y_4'(0) = 1$$

- Solving each equation, we obtain

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t), \quad y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$$

- The Wronskian of y_3 and y_4 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0$$

- Thus y_3, y_4 form the fundamental set of solutions indicated in Theorem 3.2.5, with general solution in this case

$$y(t) = k_1 \cosh(t) + k_2 \sinh(t)$$

Example 3.2.6: Many Fundamental Solution Sets

- Thus
$$S_1 = \left\{ e^t, e^{-t} \right\}, \quad S_2 = \left\{ \cosh t, \sinh t \right\}$$
  both form fundamental solution sets to the differential equation and initial point
$$y'' - y = 0, \quad t_0 = 0$$
- In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.

Theorem 3.2.6

Consider again the equation (2):

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous real-valued functions.

If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part u and its imaginary part v are also solutions of this equation.

Theorem 3.2.7 (Abel's Theorem)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on some open interval I, then the $W[y_1,y_2](t)$ is given by:

$$W[y_1, y_2](t) = ce^{-\int p(t)\,dt}$$

where c is a constant that depends on y_1 and y_2 but not on t.

- Note that $W[y_1,y_2](t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Example 3.2.7 Applying Abel's Theorem

- Recall the following differential equation and its solutions:

  $2t^2 y'' + 3t\,y' - y = 0,\ t > 0$ with solutions $y_1 = t^{1/2}, y_2 = t^{-1}$

- We computed the Wronskian for these solutions to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{3}{2}t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

- Writing the differential equation in the standard form

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,\ t > 0$$

- So $p(t) = \dfrac{3}{2t}$ and the Wronskian given by Theorem 3.2.6 is

$$W[y_1, y_2](t) = ce^{-\int \frac{3}{2t}dt} = ce^{-\frac{3}{2}\ln t} = ct^{-3/2}$$

- This is the Wronskian for any pair of fundamental solutions. For the solutions given above, we must let $c = -\dfrac{3}{2}$.

Summary

- To find a general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \ \alpha < t < \beta$$

  we first find two solutions y_1 and y_2.
- Then make sure there is a point t_0 in the interval such that $W[y_1, y_2](t_0) \neq 0$.
- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1y_1 + c_2 y_2$.
- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.

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