Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 3

Second Order Linear Differential Equations

Section 3.6 Variation of Parameters

Variation of Parameters

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are continuous functions on an open interval I.

- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the **variation of parameters** method to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to the homogeneous equation.

- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals must be evaluated, and this can present difficulties.

Example 3.6.1: Variation of Parameters

- Find the general solution to:

$$y'' + 4y = 8\tan t, \quad -\pi/2 < t < \pi/2$$

- We cannot use the undetermined coefficients method since $g(t)$ is a quotient of sin t or cos t, instead of a sum or product.
- Recall that the solution to the homogeneous equation is

$$y_C(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

- To find a particular solution to the nonhomogeneous equation, we begin with the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

- Then

$$y'(t) = u_1'(t)\cos(2t) - 2u_1(t)\sin(2t) + u_2'(t)\sin(2t) + 2u_2(t)\cos(2t)$$

- or $\quad y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$

Example 3.6.1: Derivatives and 2nd Equation

- From the previous slide,

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$$

- We need two equations to solve for u_1 and u_2. The first equation is the differential equation. To get a second equation, we will require:

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

- Then

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t)$$

- Next,

$$y''(t) = -2u_1'(t)\sin(2t) - 4u_1(t)\cos(2t) + 2u_2'(t)\cos(2t) - 4u_2(t)\sin(2t)$$

Example 3.6.1: Two Equations for and

- Recall that our differential equation is

$$y'' + 4y = 8\tan t$$

- Substituting y'' and y into this equation, we obtain

$$-2u_1'(t)\sin(2t) - 4u_1(t)\cos(2t) + 2u_2'(t)\cos(2t) - 4u_2(t)\sin(2t)$$
$$+4\big(u_1(t)\cos(2t) + u_2(t)\sin(2t)\big) = 8\tan t$$

- This equation simplifies to

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$

- Thus, to solve for u_1 and u_2, we have the two equations:

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$
$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

Example 3.6.1: Solve for u_1'

- To find u_1 and u_2 , we first need to solve for u_1' and u_2'

$$-2u_1'(t)\sin(2t)+2u_2'(t)\cos(2t)=8\tan t$$
$$u_1'(t)\cos(2t)+u_2'(t)\sin(2t)=0$$

- From second equation, $u_2'(t)=-u_1'(t)\dfrac{\cos 2t}{\sin 2t}$

- Substituting this into the first equation,

$$-2u_1'(t)\sin(2t)+2\left[-u_1'(t)\frac{\cos(2t)}{\sin(2t)}\right]\cos(2t)=8\tan t$$
$$-2u_1'(t)\sin^2(2t)-2u_1'(t)\cos^2(2t)=8\tan t\sin(2t)$$
$$-2u_1'(t)\left[\sin^2(2t)+\cos^2(2t)\right]=8\left[\frac{2\sin^2 t\cos t}{\cos t}\right]$$
$$u_1'(t)=-8\sin^2 t$$

Example 3.6.1: Solve for u_1 and u_2

- From the previous slide,

$$u_1'(t) = -8\sin^2 t, \quad u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$

- Then

$$u_2'(t) = 8\sin^2 t\,\frac{\cos(2t)}{\sin(2t)} = 4\,\frac{\sin t(2\cos^2 t - 1)}{\cos t} = 4\sin t\left(2\cos t - \frac{1}{\cos t}\right)$$

- Thus

$$u_1(t) = \int u_1'(t)\,dt = 4\sin t\cos t - 4t + c_1$$

$$u_2(t) = \int u_2'(t)\,dt = 4\ln(\cos t) - 4\cos^2 t + c_2$$

Example 3.6.1: General Solution

- Recall our equation and homogeneous solution y_C:

$$y'' + 4y = 8\tan t, \quad y_C(t) = c_1\cos(2t) + c_2\sin(2t)$$

- Using the expressions for u_1 and u_2 on the previous slide, the general solution to the differential equation is

$$\begin{aligned} y(t) &= u_1(t)\cos 2t + u_2(t)\sin 2t + y_C(t) \\ &= (4\sin t\cos t)\cos(2t) + (4\ln(\cos t) - 4\cos^2 t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \\ &= -2\sin(2t) - 4t\cos(2t) + 4\ln(\cos t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \end{aligned}$$

Variation of Parameters Summary

- Suppose y_1, y_2 are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation $y'' + p(t)y' + q(t)y = g(t)$, where we note that the coefficient on y'' is 1.
- To find u_1 and u_2, we need to solve the equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

- Doing so, and using the Wronskian, we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W[y_1, y_2](t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W[y_1, y_2](t)}$$

- Thus

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\,dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\,dt + c_2$$

Theorem 3.6.1

- Consider the equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$
$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

- If the functions p, q and g are continuous on an open interval I, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\,ds$$

and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

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