Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

Chapter 2

First-Order Differential Equations

Section 2.6
Exact Differential Equations and Integrating Factors

Exact Differential Equations Definition

- Consider a first order ODE of the form

$$M(x, y) + N(x, y)y' = 0$$

- Suppose there is a function $\Psi(x, y)$ such that

$$\Psi_x(x, y) = M(x, y),\ \Psi_y(x, y) = N(x, y)$$

and such that $\Psi(x, y) = c$ defines $y = \phi(x)$ implicitly. Then

$$M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x,\phi(x))$$

and hence the original ODE becomes

$$\frac{d}{dx}\psi(x,\phi(x)) = 0$$

- Thus $\Psi(x, y) = c$ defines a solution implicitly.
- In this case, the ODE is said to be an **exact differential equation.**

Example 2.6.1: Exact Equation

- Consider the equation:

$$2x + y^2 + 2xyy' = 0$$

- It is neither linear nor separable, but there is a function φ such that

$$2x + y^2 = \frac{\partial \psi}{\partial y} \text{ and } 2xy = \frac{\partial \psi}{\partial x}$$

- The function that works is $\Psi(x, y) = x^2 + xy^2$
- Thinking of y as a function of x and calling upon the chain rule, the differential equation and its solution become

$$\frac{d\psi}{dx} = \frac{d}{dx}\left(x^2 + xy^2\right) = 0 \Rightarrow \psi\left(x, y\right) = x^2 + xy^2 = c$$

Theorem 2.6.1

- Suppose an ODE can be written in the form

$$M(x,y) + N(x,y)y' = 0 \qquad (1)$$

where the functions M, N, M_y and N_x are all continuous in the rectangular region R: $\alpha < x < \beta$, $\gamma < y < \delta$ Then Eq. (1) is an **exact** differential equation in R if and only if

$$M_y(x,y) = N_x(x,y) \qquad (2)$$

- That is, there exists a function ψ satisfying the conditions

$$\psi_x(x,y) = M(x,y),\ \psi_y(x,y) = N(x,y) \qquad (3)$$

if and only if M and N satisfy Equation (2).

Example 2.6.2: Exact Equation

- Consider the following differential equation.

$$\left(y\cos x+2xe^{y}\right)+\left(\sin x+x^{2}e^{y}-1\right)y'=0$$

- Then

$$M\left(x,y\right)=y\cos x+2xe^{y},N\left(x,y\right)=\sin x+x^{2}e^{y}-1$$

and hence

$$M_{y}\left(x,y\right)=\cos x+2xe^{y}=N_{x}\left(x,y\right)\Rightarrow \text{ODE is exact}$$

- From Theorem 2.6.1,

$$\psi_{x}\left(x,y\right)=M=y\cos x+2xe^{y},\psi_{y}\left(x,y\right)=N=\sin x+x^{2}e^{y}-1$$

- Thus

$$\psi\left(x,y\right)=\int\psi_{x}\left(x,y\right)dx=\int\left(y\cos x+2xe^{y}\right)dx=y\sin x+x^{2}e^{y}+h\left(y\right)$$

Example 2.6.2: Exact Equation Solution

- We have

$$\psi_x(x,y) = M = y\cos x + 2xe^y, \psi_y(x,y) = N = \sin x + x^2e^y - 1$$

and

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int \left(y\cos x + 2xe^y\right)dx = y\sin x + x^2e^y + h(y)$$

- It follows that $\psi_y(x,y) = \sin x + x^2e^y - 1 = \sin x + x^2e^y + h'(y)$

$$\Rightarrow h'(y) = -1 \Rightarrow h(y) = -y + k$$

- Thus

$$\psi(x,y) = y\sin x + x^2e^y - y + k$$

- By Theorem 2.6.1, the solution is given implicitly by

$$y\sin x + x^2e^y - y = c$$

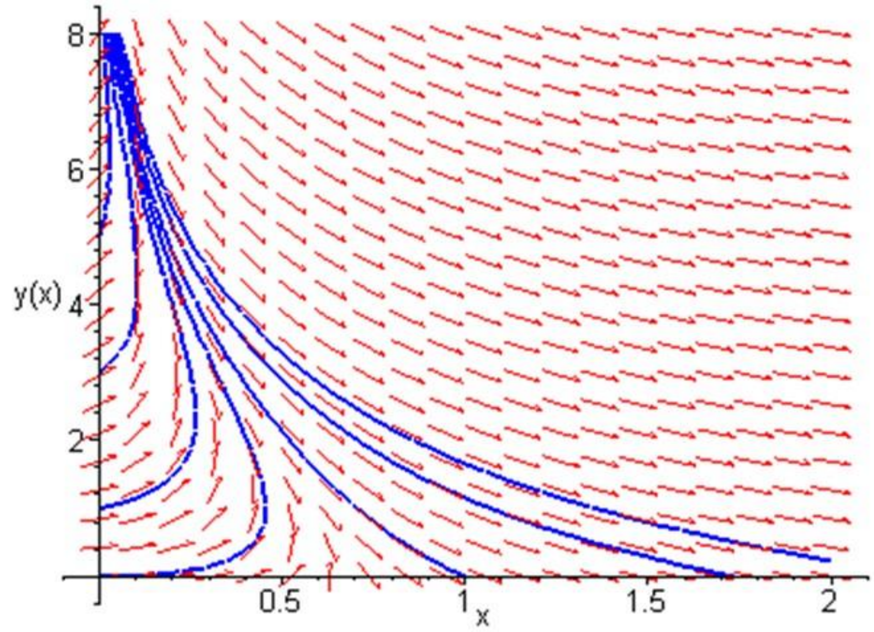
Example 2.6.2: Direction Field and Solution Curves

Our differential equation and solutions are given by

$$\left(y\cos x + 2xe^{y}\right) + \left(\sin x + x^{2}e^{y} - 1\right)y' = 0,$$

$$y\sin x + x^{2}e^{y} - y = c$$

- A graph of the direction field for this differential equation, along with several solution curves, is given at right:



Example 2.6.3: Non-Exact Equation Presentation

- Consider the following differential equation.

$$\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$$

- Then

$$M(x,y) = 3xy + y^2, N(x,y) = x^2 + xy$$

and hence

$$M_y(x,y) = 3x + 2y \neq 2x + y = N_x(x,y) \Rightarrow \text{ODE is not exact}$$

- To show that our differential equation cannot be solved by this method, let us seek a function ψ such that

$$\psi_x(x,y) = M = 3xy + y^2, \psi_y(x,y) = N = x^2 + xy$$

- Thus

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int \left(3xy + y^2\right)dx = \frac{3}{2}x^2 y + xy^2 + h(y)$$

Example 2.6.3: Non-Exact Equation Confirmation

- We seek ψ such that

$$\psi_x(x,y) = M = 3xy + y^2, \ \psi_y(x,y) = N = x^2 + xy$$

and $\psi(x,y) = \int \psi_x(x,y)\,dx = \int \left(3xy + y^2\right)dx = 3x^2y/2 + xy^2 + C(y)$

- Then

$$\psi_y(x,y) = x^2 + xy = \frac{3}{2}x^2 + 2xy + h'(y)$$

$$\Rightarrow h'(y) \stackrel{?}{=} -\frac{1}{2}x^2 - xy$$

- Because $h'(y)$ depends on x as well as y, there is no function $\psi(x, y)$ such that

$$\frac{d\psi}{dx} = \left(3xy + y^2\right) + \left(x^2 + xy\right)y'$$

Integrating Factors

- It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$:

$$M(x,y) + N(x,y)y' = 0$$

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

- For this equation to be exact, we need

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

- This partial differential equation may be difficult to solve. If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

provided right side is a function of x only. Similarly if μ is a function of y alone. See text for more details.

Example 2.6.4: Non-Exact Equation

- Consider the following non-exact differential equation.

$$\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$$

- Seeking an integrating factor, we solve the linear equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \Leftrightarrow \frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \mu(x) = x$$

- Multiplying our differential equation by μ, we obtain the exact equation

$$\left(3x^2y + xy^2\right) + \left(x^3 + x^2y\right)y' = 0,$$

which has its solutions given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = c$$

Copyright