Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 5

Series Solutions of Second-Order Linear Equations

Section 5.1 Review of Power Series

Review of Power Series

- Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation.
- So far, we have a systematic procedure for constructing fundamental solutions if the equation has constant coefficients.
- For a larger class of equations with variable coefficients, we must search for solutions beyond the familiar elementary functions of calculus.
- The principal tool we need is the representation of a given function by a power series.
- Then, similar to the undetermined coefficients method, we assume the solutions have power series representations, and then determine the coefficients so as to satisfy the equation.

Convergent Power Series

- A **power series** about the point x_0 has the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

and is said to **converge** at a point x if

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x - x_0)^n$$

exists for that x.

- Note that the series converges for $x = x_0$. It may converge for all x, or it may converge for some values of x and not others.

Absolute Convergence

- A power series about the point x_0

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to **converge absolutely** at a point x if the series

$$\sum_{n=0}^{\infty} \left| a_n (x - x_0)^n \right| = \sum_{n=0}^{\infty} |a_n| |x - x_0|^n$$

converges.

- If a series converges absolutely, then the series also converges. The converse, however, is not necessarily true.

Ratio Test

- One of the most useful tests for the absolute convergence of a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is the ratio test. If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L,$$

then the power series converges absolutely at that value of x if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. The test is inconclusive if $|x - x_0|L = 1$.

Example 5.1.1

- For which values of x does power series below converge.

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(x-2)^n=(x-2)-2(x-2)^2+3(x-2)^3-\boxed{?}$$

- Using the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n}\right|=|x-2|\lim_{n\to\infty}\frac{n+1}{n}=|x-2|$$

converges absolutely for $1 < x < 3$

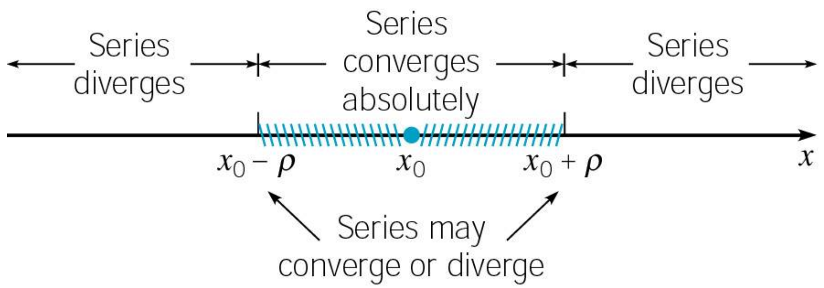
- At $x = 1$ and $x = 3$, the corresponding series are, respectively,

$$\sum_{n=1}^{\infty}(1-2)^n=\sum_{n=1}^{\infty}(-1)^n,\qquad \sum_{n=1}^{\infty}(3-2)^n=\sum_{n=1}^{\infty}(1)^n$$

- Both series diverge, since the n^{th} terms do not approach zero.
- Therefore, the interval of convergence is (1, 3).

Radius of Convergence

- There is a nonnegative number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for all x satisfying $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$.
- For a series that converges only at x_0, we define ρ to be zero.
- For a series that converges for all x, we say that ρ is infinite.
- If $\rho > 0$, then $|x-x_0| < \rho$ is called the **interval of convergence**.
- The series may either converge or diverge when $|x-x_0| = \rho$.



Example 5.1.2

- Find the radius of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$$

- Using the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)2^{n+1}}\frac{n2^n}{(x+1)^n}\right| = \frac{|x+1|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x+1|}{2}$$

converges absolutely for

$$-3 < x < 1$$

- At $x = -3$ and $x = 1$, the corresponding series are, respectively,

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty}\frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

- The alternating series on the left is convergent but not absolutely convergent. The series on the right, called the harmonic series, is divergent. Therefore the interval of convergence is $[-3, 1)$, and hence the radius of convergence is $\rho = 2$.

Taylor Series

- Suppose that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges to $f(x)$ for $|x - x_0| < \rho$.
- Then the value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

and the series is called the **Taylor series** for f about $x = x_0$.

- Also, if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

then f is continuous and has derivatives of all orders on the interval of convergence. Further, the derivatives of f can be computed by differentiating the relevant series term by term.

Analytic Functions

- A function f that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

with a radius of convergence $\rho > 0$, is said to be **analytic** at x_0.

- All of the familiar functions of calculus are analytic.
- For example, $\sin x$ and e^x are analytic everywhere, while $1/x$ is analytic except at $x = 0$, and $\tan x$ is analytic except at odd multiples of $\pi/2$.
- If f and g are analytic at x_0, then so are $f \pm g$, fg, and f/g ; see text for details on these arithmetic combinations of series.

Series Equality

- If two power series are equal, that is,

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n = \sum_{n=1}^{\infty} b_n (x - x_0)^n$$

for each x in some open interval with center x_0, then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$

- In particular, if, for each x:

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n = 0$$

then $a_n = 0$ for $n = 0, 1, 2, 3, \ldots$

Shifting Index of Summation

- The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable.
- Thus it is immaterial which letter is used for the index of summation:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

- Just as we make changes in the variable of integration in a definite integral, we find it convenient to make changes of summation in calculating series solutions of differential equations.

Example 5.1.3: Shifting Index of Summation

- Rewrite the series below as one starting with the index $n = 0$.

$$\sum_{n=2}^{\infty} a_n (x)^n$$

By letting $m = n - 2$ in this series, then $n = 2$ corresponds to $m = 0$, and hence

$$\sum_{n=2}^{\infty} a_n (x)^n = \sum_{m=0}^{\infty} a_{m+2} (x)^{m+2}$$

- Replacing the dummy index m with n, we obtain

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}$$

as desired.

Example 5.1.4: Rewriting a Generic Term

- Write the following series as a series whose generic term involves $(x - x_0)^n$ rather than $(x - x_0)^{(n-2)}$:

$$\sum_{n=2}^{\infty} (n+2)(n+1)a_n (x - x_0)^{n-2}$$

- If $m = n - 2$, then $n = 2$ corresponds to $m = 0$, so:

$$\sum_{n=2}^{\infty} (n+2)(n+1)a_n (x - x_0)^{n-2} = \sum_{m=0}^{\infty} (m+4)(m+3)a_{m+2} (x - x_0)^{m}$$

- Replacing the dummy index m with n, we obtain

$$\sum_{n=0}^{\infty} (n+4)(n+3)a_{n+2} (x - x_0)^{n}$$

as desired.

Example 5.1.5: Reindexing a Series

- Write the following series as a series whose generic term involves x^{r+n}

$$x^2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

- Begin by taking x^2 inside the summation and letting $m = n+1$

$$x^2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n+1} = \sum_{m=1}^{\infty} (r+m-1) a_{m-1} x^{r+m}$$

- Replacing the dummy index m with n, we obtain the desired result

$$\sum_{n=1}^{\infty} (r+n-1) a_{n-1} x^{r+n}$$

Example 5.1.6: Determining Coefficients (part one)

- Assume that for all x, and determine what this implies about the coefficients a_n.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

- Begin by writing both series with the same powers of x. As before, for the series on the left, replace n by $n+1$ and start counting 1 lower. The above equality becomes:

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n \Rightarrow (n+1) a_{n+1} = a_n \Rightarrow a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, 3, \ldots$

Example 5.1.6: Determining Coefficients (part two)

- Using the recurrence relationship just derived:

$$a_{n+1} = \frac{a_n}{n+1}$$

- We can solve for the coefficients successively by letting $n = 0, 1, 2, \ldots$

$$a_1 = a_0,\ a_2 = \frac{a_1}{2} = \frac{a_0}{2},\ a_3 = \frac{a_2}{3} = \frac{a_0}{3!},\ \cdots,\ a_n = \frac{a_0}{n!}$$

- Using these coefficients in the original series, we get a recognizable Taylor series:

$$a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$$

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