Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 3

Second-Order Linear Differential Equations

Section 3.4 Repeated Roots; Reduction of Order

Repeated Roots; Reduction of Order

- Recall our 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

where a, b and c are constants

- Assuming an exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, then $r_1 = r_2 = \dfrac{-b}{2a}$, with both roots yielding the same solution:

$$y_1(t) = ce^{-bt/(2a)}$$

Finding the Second Solution Using Multiplying Factor $v(t)$

- We know that if $y_1(t)$ is a solution, then $y_2(\text{t}) = cy_1(t)$ is a solution.
- Since y_1 and y_2 are linearly dependent, we can generalize this approach and multiply by a function $v(t)$, and determine conditions for which y_2 is a solution:

$$y_1(t) = e^{-bt/(2a)} \text{ a solution } \square \quad \text{try } y_2(t) = v(t)e^{-bt/(2a)}$$

$$y_2(t) = v(t)e^{-bt/(2a)}$$

$$y_2'(t) = v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)}$$

$$y_2''(t) = v''(t)e^{-bt/(2a)} - \frac{b}{2a}v'(t)e^{-bt/(2a)} - \frac{b}{2a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/(2a)}$$

$$y_2'' = \upsilon''(t)e^{-bt/(2a)} - \frac{b}{a}\upsilon'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}\upsilon(t)e^{-bt/(2a)}$$

Finding the Multiplying Factor $v(t)$

- Substituting derivatives into the original ODE, we can find and expression for v:

$$e^{-bt/(2a)}\left\{a\left[v''(t)-\frac{b}{a}v'(t)+\frac{b^2}{4a^2}v(t)\right]+b\left[v'(t)-\frac{b}{2a}v(t)\right]+cv(t)\right\}=0$$

$$av''(t)-bv'(t)+\frac{b^2}{4a}v(t)+bv'(t)-\frac{b^2}{2a}v(t)+cv(t)=0$$

$$av''(t)+\left(\frac{b^2}{4a}-\frac{b^2}{2a}+c\right)v(t)=0$$

$$av''(t)+\left(\frac{b^2}{4a}-\frac{2b^2}{4a}+\frac{4ac}{4a}\right)v(t)=0 \iff av''(t)+\left(\frac{-b^2}{4a}+\frac{4ac}{4a}\right)v(t)=0$$

$$av''(t)-\left(\frac{b^2-4ac}{4a}\right)v(t)=0$$

$$v''(t)=0 \quad\Rightarrow\quad v(t)=k_3t+k_4$$

General Solution

- To find our general solution, we have:

$$\begin{aligned} y(t) &= k_1 e^{-bt/(2a)} + k_2 v(t) e^{-bt/(2a)} \\ &= k_1 e^{-bt/(2a)} + \left(k_3 t + k_4\right) e^{-bt/(2a)} \\ &= c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)} \end{aligned}$$

- Thus the general solution for repeated roots is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

Wronskian

- The general solution is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

- Thus every solution is a linear combination of

$$y_1(t) = e^{-bt/(2a)},\ y_2(t) = t e^{-bt/(2a)}$$

- The Wronskian of the two solutions is

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{-bt/(2a)} & t e^{-bt/(2a)} \\ -\dfrac{b}{2a} e^{-bt/(2a)} & \left(1 - \dfrac{bt}{2a}\right) e^{-bt/(2a)} \end{vmatrix} \\ &= e^{-bt/a}\left(1 - \frac{bt}{2a}\right) + e^{-bt/a}\left(\frac{bt}{2a}\right) \\ &= e^{-bt/a} \neq 0 \quad \text{for all } t \end{aligned}$$

- Thus y_1 and y_2 form a fundamental solution set for equation.

Example 3.4.1 (part one)

- Solve the differential equation:

$$y'' + 4y' + 4y = 0$$

- Assuming an exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 4r + 4 = 0 \Leftrightarrow (r+2)^2 = 0 \Leftrightarrow r = -2$$

- So one solution is $y_1(t) = e^{-2t}$ and a second solution is found:

$$\begin{aligned} y_2(t) &= v(t)e^{-2t} \\ y_2'(t) &= v'(t)e^{-2t} - 2v(t)e^{-2t} \\ y_2''(t) &= v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t} \end{aligned}$$

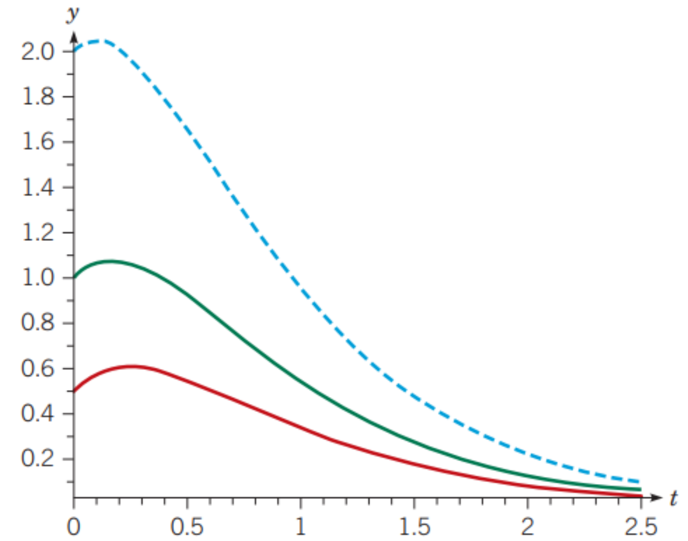
- Substituting these into the differential equation and simplifying yields $v''(t) = 0,\ v'(t) = k_1,\ v(t) = k_1 t + k_2$ where k_1 and k_2 are arbitrary constants.

Example 3.4.1 (part two)

- Letting $k_1 = 1$ and $k_2 = 0$, $v(t) = t$ and $y_2(t) = te^{-2t}$
- So the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

- Note that both y_1 and y_2 tend to 0 as $t \to \infty$ regardless of the values of c_1 and c_2
- The figure shows three solutions of this equation with different sets of initial conditions.
  - $y(0) = 2$, $y'(0) = 1$ (top)
  - $y(0) = 1$, $y'(0) = 1$ (middle)
  - $y(0) = ½$, $y'(0) = 1$ (bottom)



Example 3.4.2 (part one)

- Find the solution of the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \;\; y(0) = 2, \;\; y'(0) = \frac{1}{3}$$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \;\Rightarrow\; r^2 - r + \frac{1}{4} = 0 \;\Leftrightarrow\; (r - \frac{1}{2})^2 = 0 \;\Leftrightarrow r = \frac{1}{2}$$

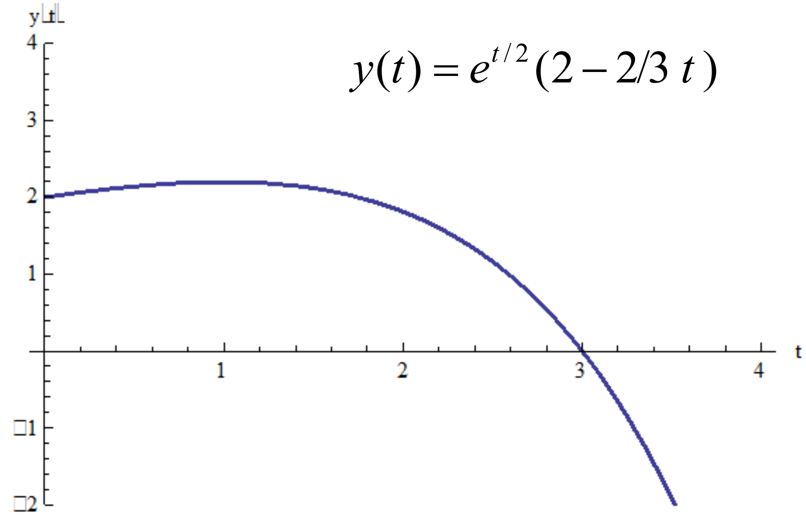
- Thus the general solution is

$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

- Using the initial conditions:

$$\left.\begin{array}{rcl} c_1 & = & 2 \\ \frac{1}{2}c_1 + c_2 & = & \frac{1}{3} \end{array}\right\} \Rightarrow c_1 = 2, \; c_2 = -\frac{2}{3}$$

- Thus $y(t) = 2e^{t/2} - \dfrac{2}{3} t e^{t/2}$



Example 3.4.2 (part two)

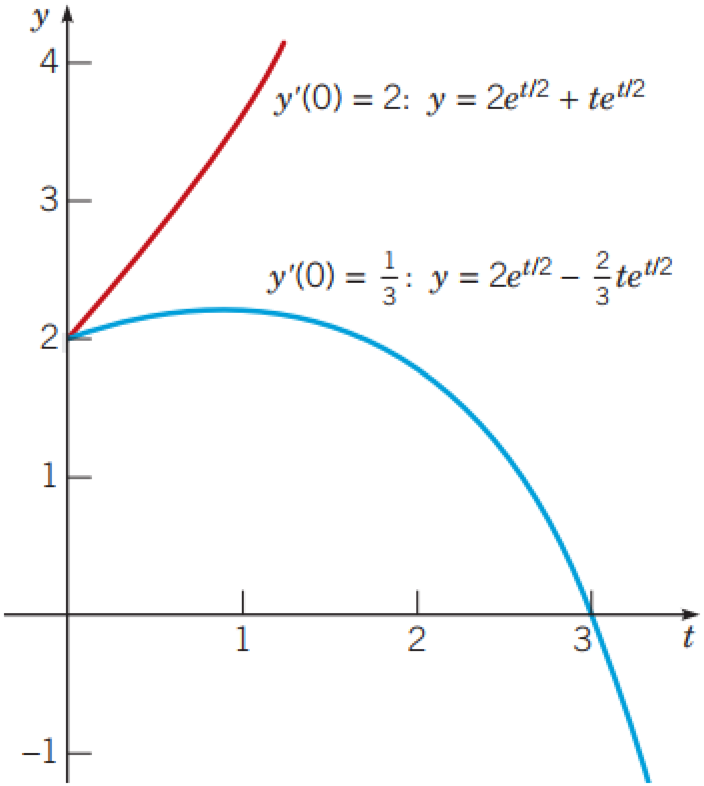
- Suppose that the initial slope in the previous problem was increased

$$y(0) = 2, \;\; y'(0) = 2$$

- The solution of this modified problem is

$$y(t) = 2e^{t/2} + te^{t/2}$$

- Notice that the coefficient of the second term is now positive. This makes a difference in the graph, since the exponential function is raised to a positive power.



Reduction of Order

- The method used so far in this section also works for equations with nonconstant coefficients:

$$y'' + p(t)\,y' + q(t)\,y = 0$$

- That is, given that y_1 is solution, try $y_2 = v(t)y_1$:

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2'(t) &= v'(t)y_1(t) + v(t)y_1'(t) \\ y_2''(t) &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \end{aligned}$$

- Substituting these into the original ODE and collecting terms,

$$y_1 v'' + \left(2y_1' + py_1\right)v' + \left(y_1'' + py_1' + qy_1\right)v = 0$$

- Since y_1 is a solution to the differential equation, this last equation reduces to a first order equation in v:

$$y_1 v'' + \left(2y_1' + py_1\right)v' = 0$$

Example 3.4.3: Reduction of Order

- Given the variable coefficient equation and solution y_1,

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution:

$$y_2(t) = v(t)\, t^{-1}$$

$$y_2'(t) = v'(t)\, t^{-1} - v(t)\, t^{-2}$$

$$y_2''(t) = v''(t)\, t^{-1} - 2v'(t)\, t^{-2} + 2v(t)\, t^{-3}$$

- Substituting these into the original ODE and collecting terms,

$$2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} = 0$$

$$\Leftrightarrow 2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$$

$$\Leftrightarrow \; 2tv'' - v' = 0$$

$$\Leftrightarrow \; 2tu' - u = 0, \;\; \text{where } u(t) = v'(t)$$

Example 3.4.3: Finding $v(t)$

- To solve

  $$2tu' - u = 0, \quad u(t) = v'(t)$$

  for u, we can use the separation of variables method:

  $$2t\frac{du}{dt} - u = 0 \quad \Leftrightarrow \quad \int \frac{du}{u} = \int \frac{1}{2t}\,dt \quad \Leftrightarrow \quad \ln|u| = \frac{1}{2}\ln|t| + C$$

  $$\Leftrightarrow \quad |u| = |t|^{1/2} e^{C} \quad \Leftrightarrow \quad u = ct^{1/2}, \quad \text{since } t > 0.$$

- Thus $v' = ct^{1/2}$

  and hence

  $$v(t) = \frac{2}{3}c\,t^{3/2} + k$$

Example 3.4.3: General Solution

- Since $v(t) = \frac{2}{3} c t^{3/2} + k$

$$y_2(t) = \left(\frac{2}{3} c t^{3/2} + k \right) t^{-1} = \frac{2}{3} c t^{1/2} + k\, t^{-1}$$

- Recall $y_1(t) = t^{-1}$
- So we can neglect the second term of y_2 to obtain

$$y_2(t) = t^{1/2}$$

- The Wronskian of $y_1(t)$ and $y_2(t)$ can be computed

$$W\left[y_1, y_2 \right](t) = \frac{3}{2} t^{-3/2} \neq 0 \text{ for } t > 0$$

- Hence the general solution to the differential equation is

$$y(t) = c_1 t^{-1} + c_2 t^{1/2}$$

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