Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 3

Second-Order Linear Differential Equations

Section 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Nonhomogeneous Equations; Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, and g are continuous functions on an open interval I.

- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the method of undetermined coefficients to solve the nonhomogeneous equation, which relies on knowing solutions to the homogeneous equation.

Theorem 3.5.1

- If Y_1 and Y_2 are solutions of the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_1 - Y_2$ is a solution of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

- If, in addition, y_1 and y_2 form a fundamental solution set of the homogeneous equation, then there exist constants c_1 and c_2 such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem 3.5.2 (General Solution)

- To solve the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

we need to do three things:

1. Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation. This is called the **complementary solution** and may be denoted by $y_c(t)$.
2. Find any solution $Y(t)$ of the nonhomogeneous equation. This is often referred to as a **particular solution.**
3. Form the sum of the functions found in steps 1 and 2.

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

- In this section we use the method of **undetermined coefficients** to find a particular solution Y to the nonhomogeneous equation, assuming we can find solutions y_1, y_2 for the homogeneous case.
- The method of undetermined coefficients is usually limited to equations in which p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

Example 3.5.1: Exponential $g(t)$

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}$$

- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:

$$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t},\ Y''(t) = 4Ae^{2t}$$

- Substituting these derivatives into the differential equation,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Leftrightarrow \quad -6Ae^{2t} = 3e^{2t} \quad \Leftrightarrow \quad A = -\frac{1}{2}$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = -\frac{1}{2}e^{2t}$$

Example 3.5.2: $g(t) = A \sin t$, First Attempt

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin t$$

- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$

- Substituting these derivatives into the differential equation,

$$\begin{aligned}
&-A\sin t - 3A\cos t - 4A\sin t = 2\sin t \\
\Leftrightarrow\ & (2+5A)\sin t + 3A\cos t = 0 \\
\Leftrightarrow\ & c_1 \sin t + c_2 \cos t = 0
\end{aligned}$$

- Since sin(x) and cos(x) are not multiples of each other, we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 0$ and $3A = 0$, which is impossible.

Example 3.5.2: $g(t) = A \sin t$, Particular Solution

- Our next attempt at finding a Y is

$$Y(t) = A \sin t + B \cos t$$
$$\Rightarrow Y'(t) = A \cos t - B \sin t,\ Y''(t) = -A \sin t - B \cos t$$

- Substituting these derivatives into the ODE, we obtain

$$\left(-A \sin t - B \cos t\right) - 3\left(A \cos t - B \sin t\right) - 4\left(A \sin t + B \cos t\right) = 2 \sin t$$
$$\Leftrightarrow \left(-5A + 3B\right) \sin t + \left(-3A - 5B\right) \cos t = 2 \sin t$$
$$\Leftrightarrow -5A + 3B = 2,\ -3A - 5B = 0$$
$$\Leftrightarrow A = -\frac{5}{17},\ B = \frac{3}{17}$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{-5}{17} \sin t + \frac{3}{17} \cos t$$

Example 3.5.3: $g(t)$ is a product of e^t and cos $2t$

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = -8e^t \cos(2t)$$

- We seek Y satisfying this equation, as follows:

$$\begin{aligned}
Y(t) &= Ae^t \cos(2t) + Be^t \sin(2t) \\
Y'(t) &= Ae^t \cos(2t) - 2Ae^t \sin(2t) + Be^t \sin(2t) + 2Be^t \cos(2t) \\
&= (A + 2B)e^t \cos(2t) + (-2A + B)e^t \sin(2t) \\
Y''(t) &= (A + 2B)e^t \cos(2t) - 2(A + 2B)e^t \sin(2t) + (-2A + B)e^t \sin(2t) \\
&\quad + 2(-2A + B)e^t \cos(2t) \\
&= (-3A + 4B)e^t \cos(2t) + (-4A - 3B)e^t \sin(2t)
\end{aligned}$$

- Substituting these into the ODE and solving for A and B:

$$A = \frac{10}{13},\ B = \frac{2}{13} \quad \square \quad Y(t) = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t)$$

Discussion: Sum $g(t)$

- Consider again our general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Suppose that $g(t)$ is sum of functions:

$$g(t) = g_1(t) + g_2(t)$$

- If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$
$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

Example 3.5.4: Sum $g(t)$

- Consider the equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

- Our equations to solve individually are

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$

- Our particular solution is then

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

Example 3.5.5: First Attempt

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2e^{-t}$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = Ae^{-t} \Rightarrow Y'(t) = -Ae^{-t}, \; Y''(t) = Ae^{-t}$$

- Substituting these derivatives into differential equation,

$$(A + 3A - 4A)e^{-t} = 2e^{-t}$$

- Since the left side of the above equation is always 0, no value of A can be found to make $Y(t) = Ae^{-t}$ a solution to the nonhomogeneous equation.
- To understand why this happens, we will look at the solution of the corresponding homogeneous differential equation.

Example 3.5.5: Homogeneous Solution

- To solve the corresponding homogeneous equation:

$$y'' - 3y' - 4y = 0$$

- We use the techniques from Section 3.1 and get

$$y_1(t) = e^{-t} \text{and } y_2(t) = e^{4t}$$

- Thus our assumed particular solution $Y(t) = Ae^{-t}$ solves the homogeneous equation instead of the nonhomogeneous equation.

- So we need another form for $Y(t)$ to arrive at the general solution of the form:

$$y(t) = c_1 e^{-t} + c_2 e^{4t} + Y(t)$$

Example 3.5.5: Particular Solution

- Our next attempt at finding a $Y(t)$ is:

$$Y(t) = Ate^{-t}$$

$$Y'(t) = Ae^{-t} - Ate^{-t}$$

$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = Ate^{-t} - 2Ae^{-t}$$

- Substituting these into the ODE,

$$Y'' - 3Y' - 4Y = \left(-2A - 3A\right)e^{-t} + \left(A + 3A - 4A\right)te^{-t} = 2e^{-t}$$

Solve for A: $\quad -5A = 2, \text{so } A = -\frac{2}{5}$

$$Y(t) = -\frac{2}{5}te^{-t}$$

- So the general solution to the IVP is $y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{2}{5} t e^{-t}$

Summary: Undetermined Coefficients Part 1

- For the differential equation

$$ay'' + by' + cy = g(t)$$

  where a, b, and c are constants, if $g(t)$ belongs to the class of functions discussed in this section (involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of these), the method of undetermined coefficients may be used to find a particular solution to the nonhomogeneous equation.

- The first step is to find the general solution for the corresponding homogeneous equation with $g(t) = 0$.

$$y_C(t) = c_1 y_1(t) + c_2 y_2(t)$$

Summary: Undetermined Coefficients Part 2

- The second step is to select an appropriate form for the particular solution, $Y(t)$, to the nonhomogeneous equation and determine the derivatives of that function.
- After substituting $Y(t)$, $Y'(t)$, and $Y''(t)$ into the nonhomogeneous differential equation, if the form for $Y(t)$ is correct, all the coefficients in $Y(t)$ can be determined.
- Finally, the general solution to the nonhomogeneous differential equation can be written as

$$y_{gen}(t) = y_C(t) + Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

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