Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 1

Introduction

Section 1.3
Classification of Differential Equations

Methods for Classifying Differential Equations

- The main purpose of this course is to discuss properties of solutions of differential equations, and to present methods of finding solutions or approximating them.
- To provide a framework for this discussion, in this section we give several ways of classifying differential equations.

Ordinary Differential Equations

- When the unknown function depends on a single independent variable, only ordinary derivatives appear in the equation.
- In this case the equation is said to be an ordinary differential equations (ODE).
- The equations discussed in the preceding two sections are ordinary differential equations. For example,

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad \frac{dp}{dt} = \frac{p}{2} - 450$$

- Another example of an ordinary differential equation, for charge $Q(t)$ on a capacitor in a circuit:

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Partial Differential Equations

- When the unknown function depends on several independent variables, partial derivatives appear in the equation.
- In this case the equation is said to be a partial differential equation (PDE).
- Examples:

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \qquad \text{heat conduction}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \qquad \text{wave equation}$$

Systems of Differential Equations

- Another classification of differential equations depends on the number of unknown functions that are involved.
- If there is a single unknown function to be found, then one equation is sufficient. If there are two or more unknown functions, then a system of equations is required.
- For example, predator-prey equations have the form

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy$$

where $x(t)$ and $y(t)$ are the respective populations of prey and predator species. The constants a, c, α, and γ depend on the particular species being studied.

- Systems of equations are discussed in Chapter 7 and 9.

Order of Differential Equations

- The order of a differential equation is the order of the highest derivative that appears in the equation.
- More generally, the equation

$$F\left(t, y, y', \boxed{?}, y^{(n)}\right) = 0 \text{ where } y = u(t)$$

is an ordinary differential equation of the n^{th} order.

- Example: $y''' + 2e^t y'' + yy' = t^4$ is a third order differential equation, for $y = u(t)$.
- We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

We will be studying differential equations for which the highest derivative can be isolated: $y^{(n)} = f\left(t, y, y', y'', \boxed{?}, y^{(n-1)}\right)$

Linear & Nonlinear Differential Equations

- An ordinary differential equation

$$F\left(t, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$$

is **linear** if F is linear in the variables $y, y', \ldots, y^{(n)}$

- Thus the general linear ODE has the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t)$$

- Example: Determine whether the equations below are linear or nonlinear.

$$(1)\; y' + 3y = 0 \qquad (2)\; y'' + 3e^y y' - 2t = 0 \qquad (3)\; y'' + 3y' - 2t^2 = 0$$

$$(4)\; \frac{d^4 y}{dt^4} - t\frac{d^2 y}{dt^2} + 1 = t^2 \qquad (5)\; u_{xx} + uu_{yy} = \sin t \qquad (6)\; u_{xx} + \sin(u)u_{yy} = \cos t$$

Solutions to Differential Equations

- A solution $\phi(t)$ to an ordinary differential equation

$$y^{(n)}(t) = f\left(t, y, y', y'', y''', \ldots, y^{(n-1)}\right)$$

satisfies the equation:

$$\phi^{(n)}(t) = f\left(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)}\right)$$

- It is not always easy to find solutions of differential equations. However, it is relatively easy to determine whether a function is actually a solution. Just substitute the function into the equation.
- Example: Verify the following solutions of the ODE

$$y'' + y = 0; y_1(t) = \sin t, y_2(t) = -\cos t, y_3(t) = 2\sin t$$

Solutions to Differential Equations Cont.

There are three important questions in the study of differential equations:

- Is there a solution? (Existence)
- If there is a solution, is it unique? (Uniqueness)
- If there is a solution, how do we find it?
  (Analytical Solution, Numerical Approximation, etc.)

Historical Background

- The development of differential equations is a significant part of the general development of mathematics.
- Isaac Newton (1642–1727) was born in England, and is known for his development of calculus and laws of physics (mechanics), as well as a series solution to ODE's, 1665 –1690.
- Gottfried Leibniz (1646–1716) was born in Leipzig, Germany. He is known for his development of calculus (1684), notation for the derivative (dy/dx), method separation of variables (1691), and first order ODE methods (1694).

The Bernoullis

- Jakob Bernoulli (1654–1705) & Johann Bernoulli (1667–1748) were both raised in Basel, Switzerland.
- They used calculus and integrals in the form of differential equations to solve mechanics problems.
- Daniel Bernoulli (1700–1782), son of Johann, is known for his work on partial differential equations and applications, and Bessel functions.

Leonhard Euler (1707–1783)

- Leonhard Euler (pronounced "oiler"), was raised in Basel, Switzerland, and was the most prolific mathematician of all time. His collected works fill more than 70 volumes.
- He formulated problems in mechanics into mathematical language and developed methods of solution. "First great work in which analysis is applied to science of movement" (Lagrange).
- He is known also for his work on exactness of first order ODEs (1734), integrating factors (1734), linear equations with constant coefficients (1743), series solutions to differential equations (1750), numerical procedures (1768), PDEs (1768), and calculus of variations (1768).

Joseph-Louis Lagrange (1736–1813).

- Lagrange was raised in Turin, Italy. He was mostly self-taught in beginning, and became math professor at age 19.
- Lagrange most famous for his *Analytical Mechanics* (1788) work on Newtonian mechanics.
- Lagrange showed that the general solution of a nth order linear homogeneous ODE is a linear combination of n independent solutions (1762–1765). He also gave a complete development of variation of parameters (1774–1775), and studied PDEs and calculus of variations.

Pierre-Simon de Laplace (1749–1827).

- Laplace was raised in Normandy, France, and was preeminent in celestial mechanics (1799–1825).
- Laplace's equation in PDEs is well known, and he studied it extensively in connection with gravitational attraction.
- The Laplace transform is named after him.

The 1800s

- By the end of the 1700s, many elementary methods of solving ordinary differential equations had been discovered.
- In the 1800s, interest turned to theoretical questions of existence and uniqueness, and series expansions (Ch 5).
- Partial differential equations also became a focus of study, as their role in mathematical physics became clear.
- A classification of certain useful functions arose, called Special Functions. This group included Bessel functions, Chebyshev polynomials, Legendre polynomials, Hermite polynomials, Hankel polynomials.

The 1900s – Present

- Computational results were an essential element in the discovery of “chaos theory”.
- In 1961, Edward Lorenz (1917 – 2008), was developing weather prediction models when he observed different results upon restarting a simulation in the middle of the time period using previously computed results.
- He used three-digit approximations rather than restart the computation with the six-digit values that were stored in the computer.
- The common trait that small changes in a problem produce large changes in the solution is one of the defining characteristics of chaos.

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