Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 5

Series Solutions of Second Order Linear Equations

Section 5.2 Series Solutions Near an Ordinary Point, Part I

Second Order Linear Equations with Variable Coefficients

- In Chapter 3, we examined methods of solving second order linear differential equations with constant coefficients.
- We now consider the case where the coefficients are functions of the independent variable, which we will denote by x.
- It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

  since the method for the nonhomogeneous case is similar.
- We primarily consider the case when P, Q, R are polynomials, and hence also continuous.
- We will see that the method of solution is also applicable when P, Q and R are general analytic functions.

Ordinary Points

- Assume P, Q, R are polynomials with no common factors, and that we want to solve the equation below in a neighborhood of a point of interest x_0:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

- The point x_0 is called an **ordinary point** if $P(x_0) \neq 0$. Since P is continuous, $P(x) \neq 0$ for all x in some interval about x_0. For x in this interval, divide the differential equation by P to get

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

- Since p and q are continuous, Theorem 3.2.1 says there is a unique solution, given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$

Singular Points

- Suppose we want to solve the equation below in some neighborhood of a point of interest x_0:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

- The point x_0 is called a **singular point** if $P(x_0) = 0$.
- Since P, Q, and R are polynomials with no common factors such as $(x - x_0)$, it follows that $Q(x_0) \neq 0$ or $R(x_0) \neq 0$.
- Then at least one of p or q becomes unbounded as $x \to x_0$, and therefore Theorem 3.2.1 does not apply in this situation.
- Sections 5.4 through 5.7 deal with finding solutions in the neighborhood of a singular point.

Series Solutions Near Ordinary Points

- In order to solve our equation near an ordinary point x_0,

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$$

  we will assume a series representation of the unknown solution function y:

$$y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

- As long as we are within the interval of convergence, this representation of y is continuous and has derivatives of all orders.

Example 5.2.1: Series Solution to a Homogeneous Equation

- Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = 1$. Thus every point x is an ordinary point. We will take $x_0 = 0$.
- Assume a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

- Differentiate term by term to obtain

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Example 5.2.1: Combining the Series

- Our equation is

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

- Shifting indices by replacing n with $n + 2$, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + a_n\right)x^n = 0$$

Example 5.2.1: Recurrence Relation

- Our equation is $\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2}+a_n\right)x^n=0$
- For this equation to be valid for all x, the coefficient of each power of x must be zero, and hence

$$(n+2)(n+1)a_{n+2}+a_n=0, \quad n=0,1,2,\ldots$$

or

$$a_{n+2}=\frac{-a_n}{(n+2)(n+1)}, \quad n=0,1,2,\ldots$$

- This type of equation is called a **recurrence relation**.
- Next, we find the individual coefficients a_0, a_1, a_2, …

Example 5.2.1: Even Coefficients

Starting with the recurrence relationship: $a_{n+2} = \dfrac{-a_n}{(n+2)(n+1)}$

To find a_2, a_4, a_6, …., we proceed as follows:

$$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}, \qquad a_4 = -\frac{a_2}{4\cdot 3} = +\frac{a_0}{4!}, \quad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}, \ldots .$$

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!} a_0, \quad k = 1, 2, 3, \ldots . \qquad \text{for } k = n/2$$

Example 5.2.1: Odd Coefficients

Starting with the recurrence relationship: $a_{n+2} = \dfrac{-a_n}{(n+2)(n+1)}$

To find a_3, a_5, a_7, …., we proceed as follows:

$$a_3 = -\frac{a_1}{2\cdot 3} = -\frac{a_1}{3!}, \qquad a_5 = -\frac{a_3}{5\cdot 4} = +\frac{a_1}{5!}, \quad a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7!}, \ldots,$$

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1, \quad k = 1, 2, 3, \ldots . \qquad \text{for } k = n/2$$

Example 5.2.1: Solution

- We now have the following information:

$$y(x)=\sum_{n=0}^{\infty}a_n x^n, \text{ where } a_{2k}=\frac{(-1)^k}{(2k)!}a_0, \quad a_{2k+1}=\frac{(-1)^k}{(2k+1)!}a_1$$

- Thus

$$y(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

the first and second terms represent two series solutions y_1 and y_2

- Note: a_0 and a_1 are determined by the initial conditions. (Expand series a few terms to see this.)
- Also, by the ratio test it can be shown that these two series converge absolutely on $(-\infty, \infty)$ and hence the manipulations we performed on the series at each step are valid.

Example 5.2.1: Functions Defined by IVP

- Our solution is

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

- From Calculus, we know this solution is equivalent to

$$y(x) = a_0 \cos x + a_1 \sin x$$

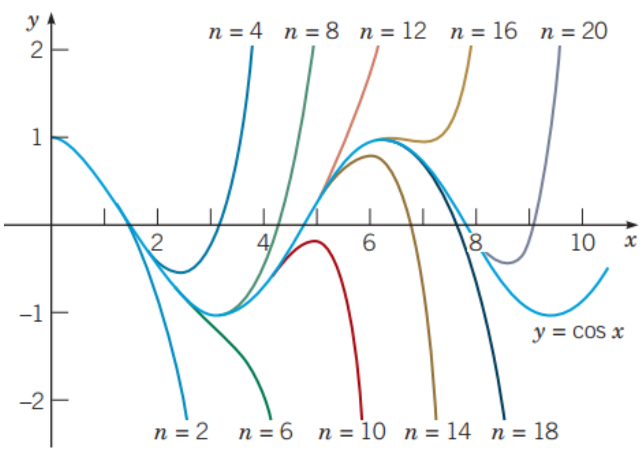
- In hindsight, we see that cos x and sin x are indeed fundamental solutions to our original differential equation

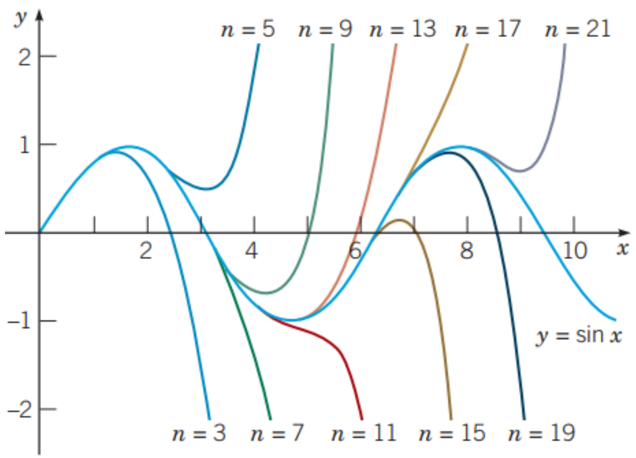
$$y'' + y = 0, \quad -\infty < x < \infty$$

- Many important functions are defined by the initial value problems solved by cos x and sin x.

Example 5.2.1: Graphs

- The graphs below show the partial sum approximations of cos x and sin x.
- As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each x in this interval the accuracy improves.
- The truncated power series provides only a local approximation in the neighborhood of $x = 0$.





Example 5.2.2: Airy's Equation

- Find a series solution of Airy's equation:

$$y'' - xy = 0, \ -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$. Thus every point x is an ordinary point. We will take $x_0 = 0$.
- Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ \ y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \ \ y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Example 5.2.2: Combine the Series

- Starting with equation: $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$
- Shift the index in the first term by replacing *n* with *n* + 2:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

- Shift the index of the second term by replacing *n* with *n* – 1 and starting the summation at 1 rather than zero:

$$2 \cdot 1 a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Example 5.2.2: Recurrence Relation

- Start with equation: $2 \cdot 1 a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$
- For this equation to be valid for all x, the coefficient of each power of x must be zero; hence $a_2 = 0$ and

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

or

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \ldots$$

Example 5.2.2: Determine the Coefficients

- We have $a_2 = 0$ and

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \quad n = 0, 1, 2, \ldots$$

- For this recurrence relation, note that $a_2 = a_5 = a_8 = \ldots = 0$.
- Next, we find the coefficients a_0, a_3, a_6, ….
- We do this by finding a formula for a_{3n}, $n = 1, 2, 3, \ldots$
- After that, we find a_1, a_4, a_7, …, by finding a formula for a_{3n+1}, $n = 1, 2, 3, \ldots$

Example 5.2.2: Find a_{3n}

- Given $a_{n+3} = \dfrac{a_n}{(n+2)(n+3)}$
- Find a_3, a_6, a_9, ….

$$a_3 = \frac{a_0}{2\cdot 3},\quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6},\quad a_9 = \frac{a_6}{8\cdot 9} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9},\ \cdots$$

- The general formula for this sequence is

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)},\quad n \ge 1$$

Example 5.2.2: Find a_{3n+1}

- Given $a_{n+3} = \dfrac{a_n}{(n+2)(n+3)}$
- Find a_4, a_7, a_{10}, ….

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}, \quad a_{10} = \frac{a_7}{9\cdot 10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}, \cdots$$

- The general formula for this sequence is

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}, \quad n \geq 1$$

Example 5.2.2: Solution

- Thus our solution is

$$y(x) = a_0\left[1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right]+a_1\left[x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right]$$

where a_0, a_1 are arbitrary (determined by initial conditions).

- Consider the two cases

$$(1)\ a_0 = 1,\ \ a_1 = 0 \ \text{ and satisfying } y(0) = 1,\ y'(0) = 0$$

$$(2)\ a_0 = 0,\ \ a_1 = 1 \text{ and satisfying } y(0) = 0,\ y'(0) = 1$$

- The corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, since $W[y_1, y_2]\ (0) = 1 \neq 0$, where

$$W(y_1,y_2)(0)=\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0)\end{vmatrix} = y_1(0)y_2'(0)-y_1'(0)y_2(0)$$

Example 5.2.2: Fundamental Solutions

- Our solution:

$$y(x) = a_0\left[1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right]+a_1\left[x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right]$$

- For the cases

  (1) $a_0 = 1,\ \ a_1 = 0$ and where $y(0) = 1, y'(0) = 0$

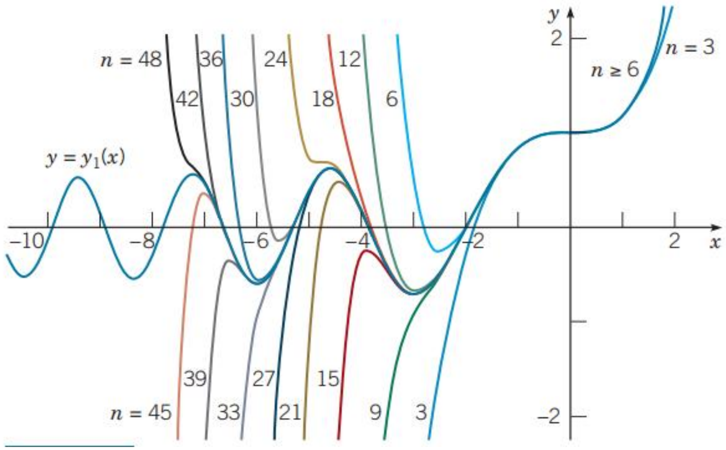
  (2) $a_0 = 0,\ \ a_1 = 1$ and where $y(0) = 0, y'(0) = 1,$

  the corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with the general solution

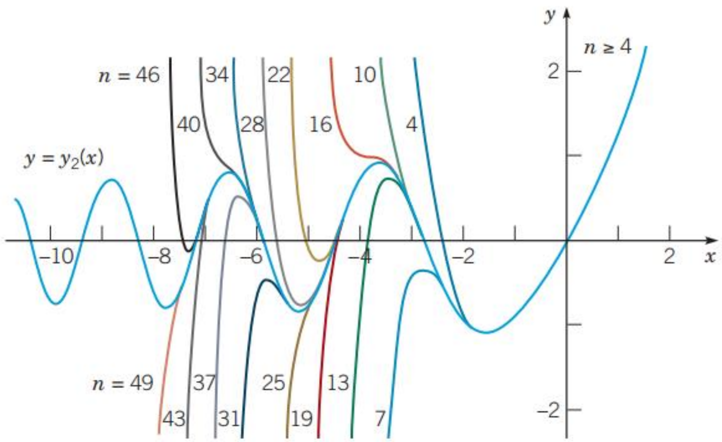
$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

Example 5.2.2: Solution Graphs

The graphs of the solutions y_1 and y_2 are given below. The accuracy interval for each approximation increases as n increases.



Polynomial approximations to $y = y_1(x)$ where the value of n is the degree of the approximating polynomial.



Polynomial approximations to $y = y_2(x)$ where the value of n is the degree of the approximating polynomial.

Example 5.2.3: Airy's Equation where $x = 1$ is an ordinary point

- Find a series solution of Airy's equation in powers of $x - 1$:

$$y'' - xy = 0, \ -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$. Thus every point x is an ordinary point. We will take $x_0 = 1$.
- Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n, \ \ y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \ \ y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

- Substituting these into ODE & shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} a_n (x-1)^n$$

Example 5.2.3: Rewriting the Series Equation

- Our equation is

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty}a_n(x-1)^n$$

- The x on right side can be written as $1 + (x - 1)$; and thus

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n &= [1+(x-1)]\sum_{n=0}^{\infty}a_n(x-1)^n \\ &= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1} \\ &= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n\end{aligned}$$

Example 5.2.3: Solving the Recurrence Relation

- Thus our equation becomes

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$$

- The corresponding recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1} \quad \text{for n} \geq 1$$

- Equating like powers of $x - 1$, we obtain

$$\begin{aligned} 2a_2 &= a_0 && \Rightarrow a_2 = \frac{a_0}{2}, \\ (3\cdot 2)a_3 &= a_1 + a_0 && \Rightarrow a_3 = \frac{a_0}{6} + \frac{a_1}{6}, \\ (4\cdot 3)a_4 &= a_2 + a_1 && \Rightarrow a_4 = \frac{a_0}{24} + \frac{a_1}{12}, \\ &\vdots \end{aligned}$$

Example 5.2.3: The Solution

- We now have the following information:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

and

$$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right]$$
$$+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

Example 5.2.3: Recursion with More Than Two Terms

- Our solution: $$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right]$$ $$+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

- The recursion has three terms,

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1},\ (n \geq 1)$$

and determining a general formula for the coefficients a_n can be difficult or impossible.

- However, we can generate as many coefficients as we like, preferably with the help of a computer algebra system.

Example 5.2.3: Solution and Convergence

- Since we don't have a general formula for the a_n, we cannot use a convergence test (i.e., the ratio test) on our power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

- This means our manipulations of the power series to arrive at our solution are suspect. However, the results of Section 5.3 will confirm the convergence of our solution.
- It can be shown that the solutions $y_3(x)$, $y_4(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with general solution

$$y(x) = a_0 y_3(x) + a_1 y_4(x)$$

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