Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 2

First-Order Differential Equations

Section 2.7
Numerical Approximations: Euler's Method

Analytical Solutions to First Order ODE's not Always Available

- Recall that a first order initial value problem has the form

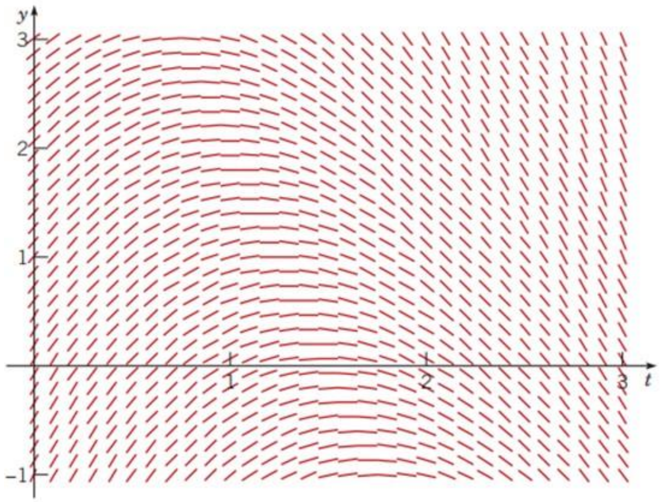
$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

- If f and $\partial f/\partial y$ are continuous, then this IVP has a unique solution $y = \phi(t)$ in some interval about t_0.
- When the differential equation is linear, separable or exact, we can find the solution by symbolic manipulations.
- However, the solutions for most differential equations of this form cannot be found by analytical means.
- Therefore it is important to be able to approach the problem in other ways.

Direction Fields

- For the first order initial value problem $y' = f(t, y), \quad y(t_0) = y_0,$ we can sketch a direction field and visualize the behavior of solutions. This has the advantage of being a relatively simple process, even for complicated equations. However, direction fields do not lend themselves to quantitative computations or comparisons.

direction field solution
to $dy/dt = 3 - 2t - 0.5y$



Numerical Methods

- For our first order initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

  an alternative is to compute approximate values of the solution $y = \phi(t)$ at a selected set of t-values.
- Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.
- There are many numerical methods that produce numerical approximations to solutions of differential equations, some of which are discussed in Chapter 8.
- In this section, we examine the **tangent line method**, which is also called **Euler's Method.**

Euler's Method: Tangent Line Approximation

- For the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

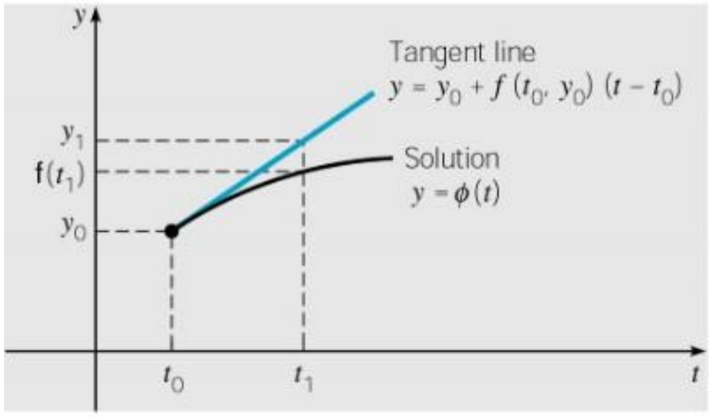
we begin by approximating solution $y = \phi(t)$ at initial point t_0.

- The solution passes through initial point (t_0, y_0) with slope $f(t_0, y_0)$. The line tangent to the solution at this initial point is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

- The tangent line is a good approximation to solution curve on an interval short enough.
- Thus if t_1 is close enough to t_0, we can approximate $y = \phi(t_1)$ by

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$



Euler's Formula

- For a point t_2 close to t_1, we approximate $y = \phi(t_2)$ using the line passing through (t_1, y_1) with slope $f(t_1, y_1)$:

$$y_2 = y_1 + f\left(t_1, y_1\right)\left(t_2 - t_1\right)$$

- Thus we create a sequence y_n of approximations $y = \phi(t_n)$

$$y_1 = y_0 + f_0 \cdot \left(t_1 - t_0\right)$$
$$y_2 = y_1 + f_1 \cdot \left(t_2 - t_1\right)$$
$$\Downarrow$$
$$y_{n+1} = y_n + f_n \cdot \left(t_{n+1} - t_n\right)$$

where $f_n = f(t_n, y_n)$.

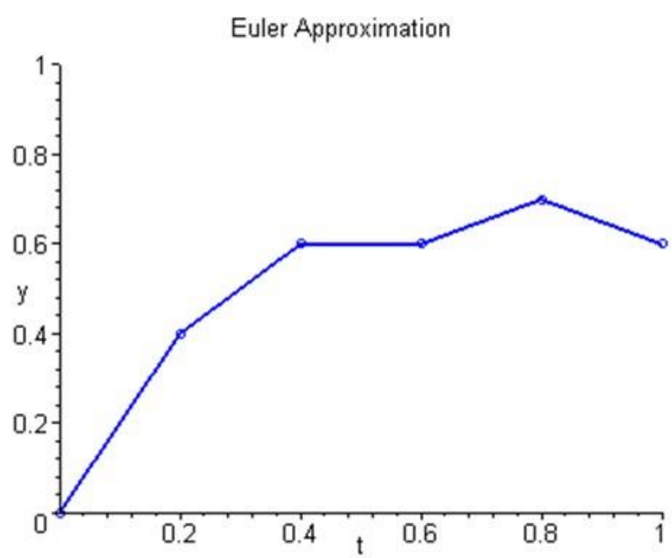
- For a uniform step size $t_{n+1} = -t_n + h$, Euler's formula becomes

$$y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \ldots$$

Euler Approximation

- To graph an Euler approximation, we plot the points (t_0, y_0), $(t_1, y_1), \ldots, (t_n, y_n)$, and then connect these points with line segments.

$$y_{n+1} + y_n + f_n \cdot \left(t_{n+1} - t_n\right), \text{where } f_n = f\left(t_n, y_n\right)$$



Example 2.7.1: Euler's Method: Approximate Solutions

- For the initial value problem $\frac{dy}{dt} = 3 - 2t - 0.5y, \quad y(0) = 1$

  we can use Euler's method with $h = 0.2$ to approximate the solution at $t = 0.2, 0.4, 0.6, 0.8$, and 1.0 as shown below.

n	t_n	y_n	$f_n = f(t_n, y_n)$	Tangent Line	Exact $y(t_n)$
0	0.0	1.00000	2.5	$y = 1 + 2.5(t - 0)$	1.00000
1	0.2	1.50000	1.85	$y = 1.5 + 1.85(t - 0.2)$	1.43711
2	0.4	1.87000	1.265	$y = 1.87 + 1.265(t - 0.4)$	1.75650
3	0.6	2.12300	0.7385	$y = 2.123 + 0.7385(t - 0.6)$	1.96936
4	0.8	2.27070	0.26465	$y = 2.2707 + 0.26465(t - 0.8)$	2.08584
5	1.0	2.32363			2.11510

Example 2.7.1: Exact Solution

- We can find the exact solution to our IVP, as in Chapter 2.1:

$$y' = 3 - 2t - 0.5y, \quad y(0) = 1$$

$$y' + 0.5y = 3 - 2t$$

$$e^{0.5t}y' + 0.5e^{0.5t}y = 3e^{0.5t} - 2te^{0.5t}$$

$$e^{0.5t}y = 14e^{0.5t} - 4te^{0.5t} + k$$

$$y = 14 - 4t + ke^{-0.5t}$$

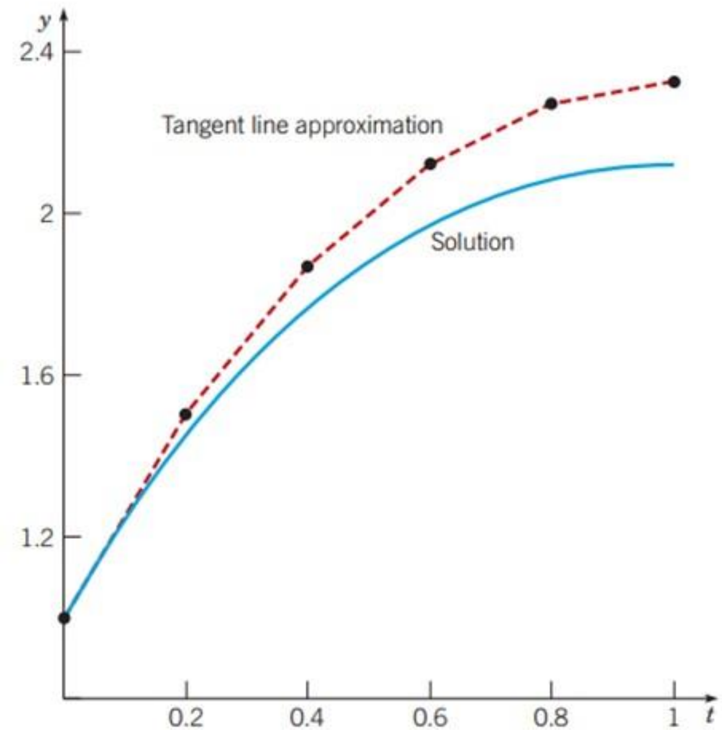
$$y(0) = 1 \Rightarrow k = -13$$

$$\Rightarrow y = 14 - 4t - 13e^{-0.5t}$$

Example 2.7.1: Error Analysis

From the figure below, we see that the errors start small, but get larger as shown by the divergence of the tangent line approximation curve (dotted red) from the exact solution (blue).

This divergence is due to the fact that the exact solution is not linear on [0, 1].



$$\text{Percent Relative Error} = \frac{y_{exact} - y_{approx}}{y_{exact}} \times 100$$

Example 2.7.2: Euler's Method - Defining Domain of Interest

- For the initial value problem

$$\frac{dy}{dt} = 3 - 2t - 0.5y, \quad y(0) = 1$$

we can use Euler's method with various step sizes to approximate the solution at t = 1.0, 2.0, 3.0, 4.0, and 5.0 and compare our results to the exact solution

$$y = 14 - 4t - 13e^{-0.5t}$$

at those values of t.

Example 2.7.2: Euler's Method - Choosing Step Size

- Comparison of exact solution with Euler's Method for $h = 0.1$, 0.05, 0.025, 0.01

t	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$	Exact
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	2.2164	2.1651	2.1399	2.1250	2.1151
2.0	1.3397	1.2780	1.2476	1.2295	1.2176
3.0	−0.7903	−0.8459	−0.8734	−0.8898	−0.9007
4.0	−3.6707	−3.7152	−3.7373	−3.7506	−3.7594
5.0	−7.0003	−7.0337	−7.0504	−7.0604	−7.0671

Example 2.7.3: Euler's Method - Approximate Solution over Domain of Interest

- For the initial value problem

$$\frac{dy}{dt} = 4 - t + 2y, \quad y(0) = 1$$

we can use Euler's method with $h = 0.1$ to approximate the solution at $t = 1, 2, 3,$ and 4, as shown below.

$$y_1 = y_0 + f_0 \cdot h = 1 + \big(4 - 0 + (2)(1)\big)(0.1) = 1.6$$

$$y_2 = y_1 + f_1 \cdot h = 1.6 + \big(4 - 0.1 + (2)(1.6)\big)(0.1) = 2.31$$

$$y_3 = y_2 + f_2 \cdot h = 2.31 + \big(4 - 0.2 + (2)(2.31)\big)(0.1) \approx 3.15$$

$$y_4 = y_3 + f_3 \cdot h = 3.15 + \big(4 - 0.3 + (2)(3.15)\big)(0.1) \approx 4.15$$

- Exact solution (see Chapter 2.1):

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

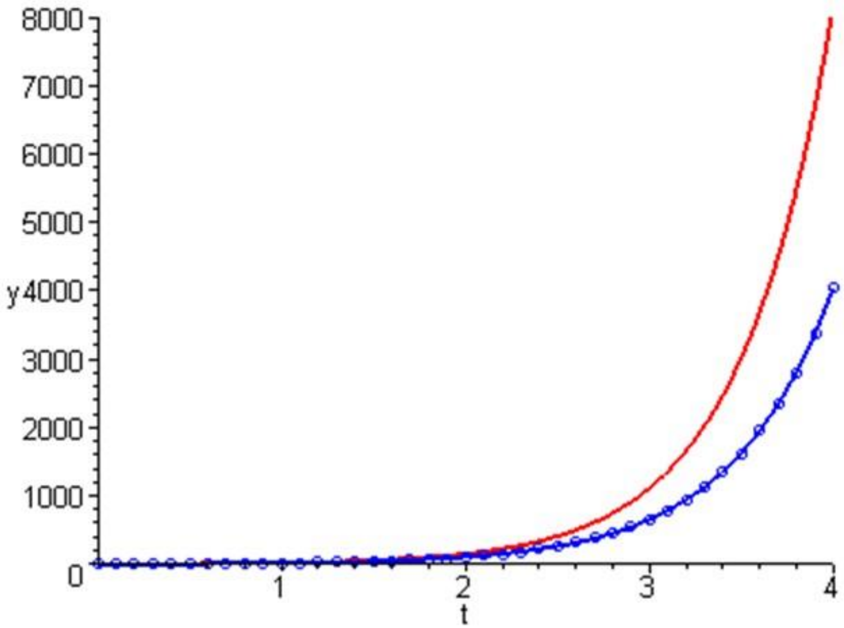
Example 2.7.3: Error Analysis

- The first five Euler approximations are given in table below for $h = 0.1$, 0.05, 0.025, 0.01.
- With increasing number of approximations, the difference between the Euler approximated solutions and exact solution increase.
- The errors are small initially, but quickly reach an unacceptable level. This suggests a nonlinear solution.

t	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	15.77728	17.25062	18.10997	18.67278	19.06990
2.0	104.6784	123.7130	135.5440	143.5835	149.3949
3.0	652.5349	837.0745	959.2580	1045.395	1109.179
4.0	4042.122	5633.351	6755.175	7575.577	8197.884
5.0	25026.95	37897.43	47555.35	54881.32	60573.53

Example 2.7.3: Error Analysis & Graphs

- The graphs below show the exact solution (red) plotted together with the Euler approximation (blue) for $h = 0.1$



t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution:

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

Euler Method Summary: Approximate Applies to a Particular Initial Condition

- Recall that if f and $\frac{\partial f}{\partial y}$ are continuous, then our first order initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

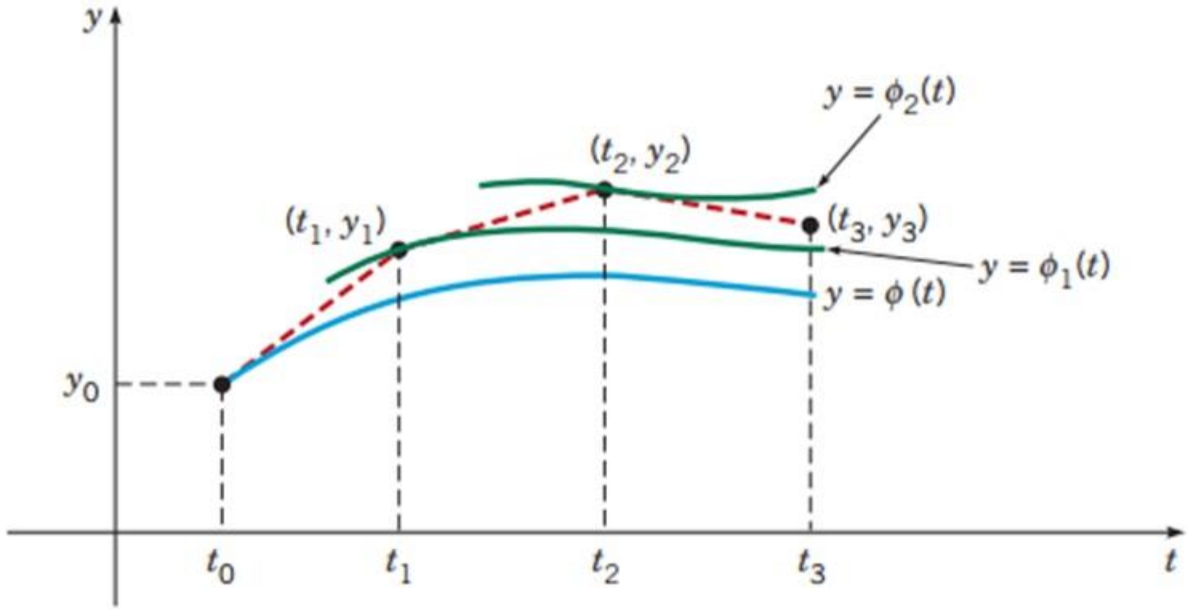
has a solution $y = \phi(t)$ in some interval about t_0.

- In fact, the equation has infinitely many solutions, each one indexed by a constant c determined by the initial condition.
- Thus $\phi(t)$ is the member of an infinite family of solutions that satisfies $\phi(t_0) = y_0$
- The Euler approximation will use only one of these solutions as input for the first approximation step.

Euler Method Summary: Iterative Steps

- The first step of Euler's method uses the tangent line to ϕ at the point (t_0, y_0) in order to estimate y_1.
- The point (t_1, y_1) is typically not on the graph of ϕ, because y_1 is an approximation of $\phi(t_1)$.
- Thus the next iteration of Euler's method does not use a tangent line approximation to ϕ, but rather to a nearby solution ϕ_1 that passes through the point (t_1, y_1).
- Thus Euler's method uses a succession of tangent lines to a sequence of different solutions $\phi(t), \phi_1(t), \phi_2(t), \ldots$ of the differential equation.

Euler Method Summary: Successive Tangents



The Euler Method uses tangents and function values from prior steps as inputs to subsequent steps.

Error Bounds and Numerical Methods

- In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- In our examples, we compared approximations with exact solutions. However, numerical procedures are usually used when an exact solution is not available. What is needed are bounds for (or estimates of) errors, which do not require knowledge of exact solution. More discussion on these issues and other numerical methods is given in Chapter 8.
- Since numerical approximations ideally reflect behavior of solution, a member of a diverging family of solutions is harder to approximate than a member of a converging family.
- Also, direction fields are often a relatively easy first step in understanding the behavior of solutions.

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