Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 10

Partial Differential Equations and Fourier Series

Section 10.1 Two-Point Boundary Value Problems

Two-Point Boundary Value Problems

- In many important physical problems there are two or more independent variables, so the corresponding mathematical models involve partial differential equations.
- This chapter treats one important method for solving partial differential equations, known as **separation of variables**.
- An essential feature of the separation of variables is the replacement of a partial differential equation by a set of ordinary differential equations, which must be solved subject to given initial or boundary conditions.
- Section 10.1 deals with some basic properties of boundary value problems for ordinary differential equations.
- The solution of the partial differential equation is then a sum, usually an infinite series, formed from the solutions to the ordinary differential equations, as we see later in the chapter.

Boundary Conditions

- Up to this point we have dealt with initial value problems, consisting of a differential equation together with suitable initial conditions at a given point.
- A typical example, as discussed in Chapter 3, is

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0,\ y'(t_0) = y_0'$$

- Physical applications often require the dependent variable y or its derivative y' to be specified at two different points.
- Such conditions are called **boundary conditions**.
- The differential equation and suitable boundary conditions form a **two-point boundary value problem**.
- A typical example is

$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0,\ y(\beta) = y_1$$

Homogeneous Boundary Value Problems

- The natural occurrence of boundary value problems usually involves a space coordinate as the independent variable, so we use x instead of t in the boundary value problem

$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0,\ y(\beta) = y_1$$

- Boundary value problems for nonlinear equations can be posed, but we restrict ourselves to linear equations only.
- If the above boundary value problem has the form

$$y'' + p(x)y' + q(x)y = 0, \quad y(\alpha) = 0,\ y(\beta) = 0$$

then it is said to be **homogeneous**. Otherwise, the problem is **nonhomogeneous**.

Solutions to Boundary Value Problems

- To solve the boundary value problem,

$$y'' + p(x)y' + q(x)y = g(x), \quad y(\alpha) = y_0,\ y(\beta) = y_1$$

  we need to find a function $y = y(x)$ that satisfies the differential equation on the interval $\alpha < x < \beta$ and that takes on the specified values y_0 and y_1 at the endpoints.
- Initial value and boundary value problems may superficially appear similar, but their solutions differ in important ways.
- Under mild conditions on the coefficients, an initial value problem is certain to have a unique solution.
- Yet for similar conditions, boundary value problems may have a unique solution, no solution, or infinitely many solutions.
- In this respect, linear boundary value problems resemble systems of linear algebraic equations.

Linear Systems

- Consider the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an n x n matrix, $\mathbf{b}$ is a given n x 1 vector, and $\mathbf{x}$ is an n x 1 vector to be determined.
- Recall the following facts (see Section 7.3):
  - If $\mathbf{A}$ is nonsingular, then $\mathbf{Ax} = \mathbf{b}$ has unique solution for any $\mathbf{b}$.
  - If $\mathbf{A}$ is singular, then $\mathbf{Ax} = \mathbf{b}$ has no solution unless $\mathbf{b}$ satisfies a certain additional condition, in which case there are infinitely many solutions.
  - The homogeneous system $\mathbf{Ax} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$.
  - If $\mathbf{A}$ is nonsingular, then this is the only solution, but if $\mathbf{A}$ is singular, then there are infinitely many (nonzero) solutions.
- Thus the nonhomogeneous system has a unique solution if the homogeneous system has only the solution $\mathbf{x} = \mathbf{0}$, and the nonhomogeneous system has either no solution or infinitely many solutions if homogeneous system has nonzero solutions.

Example 10.1.1

Solve the boundary value problem

$$y'' + 2y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

- The general solution of the differential equation is

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right)$$

- The first boundary condition requires that $c_1 = 1$.
- The second boundary condition implies that

$$c_1 \cos\left(\sqrt{2}\,\pi\right) + c_2 \sin\left(\sqrt{2}\,\pi\right) = 0, \quad \text{so} \quad c_2 = -\cot\left(\sqrt{2}\,\pi\right) \cong -0.2762.$$

- Thus the solution to the boundary value problem is

$$y = \cos\left(\sqrt{2}\,x\right) - \cot\left(\sqrt{2}\,\pi\right)\sin\left(\sqrt{2}\,x\right)$$

- This is an example of a nonhomogeneous boundary value problem with a unique solution.

Example 10.1.2

Solve the boundary value problem

$$y'' + y = 0, \quad y(0) = 1, \; y(\pi) = a, \qquad a \text{ arbitrary.}$$

- The general solution of the differential equation is
$$y = c_1 \cos(x) + c_2 \sin(x)$$
- The first boundary condition requires that $c_1 = 1$, while the second requires $c_1 = -a$. These two conditions are incompatible if $a \neq -1$, so there is no solution.
- However, if $a = -1$, then there are infinitely many solutions:
$$y = \cos(x) + c_2 \sin(x)$$
- This example illustrates that a nonhomogeneous boundary value problem may have no solution, and also that under special circumstances it may have infinitely many solutions.

Nonhomogeneous Boundary Value Problem and Corresponding Homogeneous Problem

- The nonhomogeneous boundary value problem has a corresponding homogeneous problem and boundary conditions:

$$y'' + p(x)y' + q(x)y = 0, \quad y(\alpha) = 0, \ y(\beta) = 0$$

- Observe that this problem has the solution $y = 0$ for all x, regardless of the coefficients $p(x)$ and $q(x)$.
- This solution is often called the trivial solution and is rarely of interest.
- What we would like to know is whether the problem has other, nonzero solutions.

Example 10.1.3

Solve the boundary value problem

$$y'' + 2y = 0, \quad y(0) = 0, \ y(\pi) = 0$$

- The general solution of the differential equation is given by:

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

- The first boundary condition requires that $c_1 = 0$.
- From the second boundary condition, we have $c_2 = 0$.
- Thus the only solution to the boundary value problem is $y = 0$.
- This example illustrates that a homogeneous boundary value problem may have only the trivial solution $y = 0$.

Example 10.1.4

Solve the boundary value problem

$$y'' + y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- The general solution is given by

$$y = c_1 \cos(x) + c_2 \sin(x)$$

- The first boundary condition requires $c_1 = 0$, while the second boundary condition is satisfied regardless of the value of c_2.
- Thus there are infinitely many solutions of the form

$$y = c_2 \sin(x), \text{ where } c_2 \text{ remains arbitrary}$$

- This example illustrates that a homogeneous boundary value problem may have infinitely many (nontrivial) solutions.

Linear Boundary Value Problems

- Thus examples 10.1.1 through 10.1.4 illustrate (but do not prove) that there is the same relationship between homogeneous and nonhomogeneous linear boundary value problems as there is between homogeneous and nonhomogeneous linear algebraic systems.
- A nonhomogeneous boundary value problem (Example 10.1.1) has a unique solution, and the corresponding homogeneous problem (Example 10.1.3) has only the trivial solution.
- Further, a nonhomogeneous problem (Example 10.1.2) has either no solution or infinitely many solutions, and the corresponding homogeneous problem (Example 10.1.4) has nontrivial solutions.

Eigenvalue Problems

- Recall from Section 7.3 the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$.
- Note that $\mathbf{x} = \mathbf{0}$ is a solution for all λ, but for certain λ, called eigenvalues, there are nonzero solutions, called eigenvectors.
- The situation is similar for boundary value problems.
- Consider the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- This is the same problem as in Example 10.1.3 if $\lambda = 2$, and is the same problem as in Example 10.1.4 if $\lambda = 1$.
- Thus the above boundary value problem has only the trivial solution ($y = 0$) for $\lambda = 2$, and has other, nontrivial, solutions for $\lambda = 1$.

Eigenvalues and Eigenfunctions

- Thus our boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

  has only the trivial solution for $\lambda = 2$, and has other, nontrivial solutions for $\lambda = 1$.

- By extension of the terminology for linear algebraic systems, the values of λ for which nontrivial solutions occur are called **eigenvalues**, and the nontrivial solutions themselves are called **eigenfunctions**.
- Thus $\lambda = 1$ is an eigenvalue of the boundary value problem and $\lambda = 2$ is not.
- Further, any nonzero multiple of $\sin(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda = 1$.

A Boundary Value Problem for $\lambda > 0$

- We now seek other eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- We consider separately the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.
- Suppose first that $\lambda > 0$. To avoid the frequent appearance of radical signs, let $\lambda = \mu^2$, where $\lambda > 0$.
- Our boundary value problem is then

$$y'' + \mu^2 y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- The general solution is

$$y = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

- The first boundary condition requires $c_1 = 0$, while the second is satisfied regardless of c_2 as long as $\mu = n$, $n = 1, 2, 3, \ldots$.

Eigenvalues, Eigenfunctions for $\lambda > 0$

- We have $\lambda = \mu^2$ and μ restricted to positive integer values. Thus the eigenvalues of

$$y'' + \lambda y = 0, \quad y(0) = 0,\ y(\pi) = 0$$

are

$$\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 9,\ \ldots,\ \lambda_n = n^2,\ \ldots$$

with corresponding eigenfunctions

$$y_1 = a_1 \sin(x),\ y_2 = a_2 \sin(2x), \ldots,\ y_n = a_n \sin(nx), \ldots$$

where $a_1, a_2, \ldots, a_n, \ldots$ are arbitrary constants. Choosing each constant to be 1, we have

$$y_1(x) = \sin(x), \quad y_2(x) = \sin(2x), \ldots, \quad y_n(x) = \sin(nx), \ldots,$$

Boundary Value Problem for $\lambda < 0$

- Now suppose $\lambda < 0$, and let $\lambda = -\mu^2$, where $\mu > 0$.
- Then our boundary value problem becomes

$$y'' - \mu^2 y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- The general solution is

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

- We have chosen $\cosh(\mu x)$ and $\sinh(\mu x)$ instead of $\exp(\mu x)$ and $\exp(-\mu x)$ for convenience in applying the boundary conditions.
- The first boundary condition requires that $c_1 = 0$, and from the second boundary condition, we have $c_2 = 0$.
- Thus the only solution is $y = 0$, and hence there are no negative eigenvalues for this problem.

Boundary Value Problem for $\lambda = 0$

- Now suppose $\lambda = 0$. Then our problem becomes

$$y'' = 0, \quad y(0) = 0, \; y(\pi) = 0$$

- The general solution is

$$y = c_1 x + c_2$$

- The first boundary condition requires that $c_2 = 0$, and from the second boundary condition, we have $c_1 \pi = 0 \Rightarrow c_1 = 0$.
- Thus the only solution is $y = 0$, and $\lambda = 0$ is not an eigenvalue for this problem.

Real Eigenvalues

- Thus the only real eigenvalues of

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(\pi) = 0$$

  are $\lambda_n = n^2$ for $n = 1, 2, 3, \cdots$ and that the corresponding eigenfunctions proportional to $\sin(nx)$.
- There is a possibility of complex eigenvalues, but for this particular boundary value problem it can be shown that there are no complex eigenvalues.
- Later, in Section 11.2, we discuss an important class of boundary problems that includes the one above.
- One of the useful properties of this class of problems is that all their eigenvalues are real.

Boundary Value Problem on [0, L]

- In later sections of this chapter, we will often encounter this boundary value problem on a more general interval [0, L]:

$$y'' + \lambda y = 0, \quad y(0) = 0, \; y(L) = 0$$

- If we let $\lambda = \mu_2$, $\mu > 0$ as before, the general solution is

$$y = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

- The first boundary condition requires $c_1 = 0$, and the second requires $\mu = n\pi/L$, regardless of the value of c_2.
- Thus, as before, the eigenvalues and eigenvectors are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \mathrm{K}$$

Where the eigenfunctions $y_n(x)$ are determined only up to an arbitrary multiplicative constant.

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