Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

Chapter 2

First-Order Differential Equations

Section 2.2
Separable Differential Equations

Definition of Separable Differential Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order equation

$$\frac{dy}{dx} = f(x, y)$$

- We can rewrite this in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

- For example, we can let $M(x,y) = -f(x,y)$ and $N(x,y) = 1$ to do this, but there may be other ways as well. In differential form,

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

- If M is a function of x only and N is a function of y only, then

$$M(x)\,dx + N(y)\,dy = 0$$

- In this case, the equation is called **separable**.

Example 2.2.1: Finding a General Solution for a Separable Equation

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

- Separating variables, and using calculus, we obtain

$$\left(1-y^2\right)dy = \left(x^2\right)dx$$

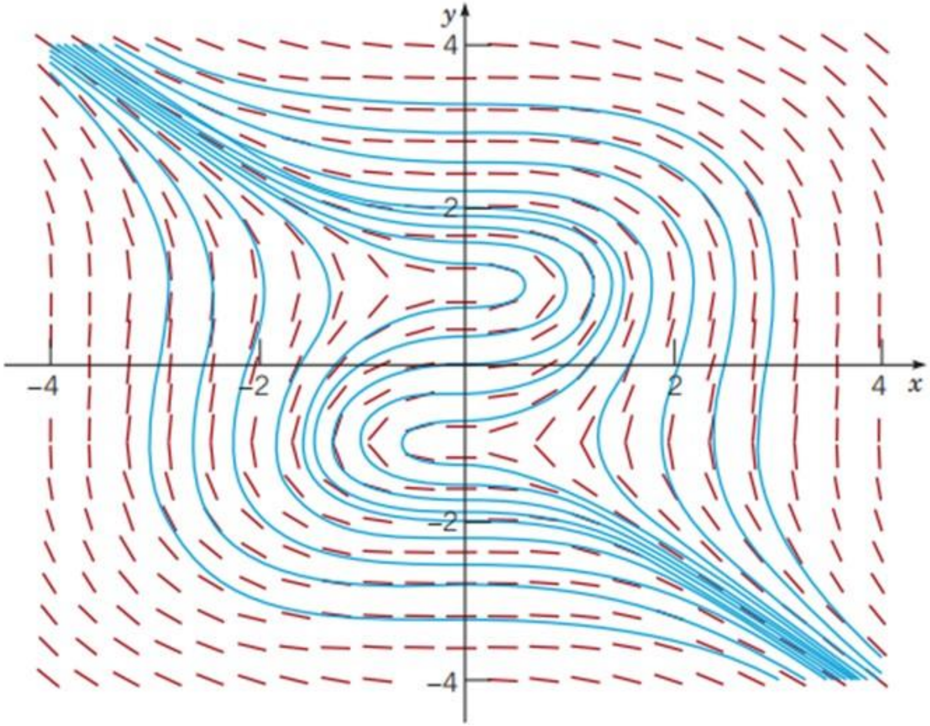
$$\int \left(1-y^2\right)dy = \int \left(x^2\right)dx$$

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + c$$

$$3y - y^3 = x^3 + c$$

Example 2.2.1: Graphing Solutions for a Separable Equation

The general solution $3y - y^3 = x^3 + c$ defines the solution y implicitly. A graph showing the direction field and implicit plots of several solution curves for the differential equation is shown below:



Example 2.2.2: Finding an Implicit General Solution

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

- Separating variables and using calculus, we obtain

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx$$
$$2\int (y-1)\,dy = \int (3x^2 + 4x + 2)\,dx$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

The equation above defines the solution y implicitly. An explicit expression for the solution can be found (next slide)

Example 2.2.2: Rearrange to create an explicit general solution in terms of "y"

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

The general equation above defines the solution y implicitly. An explicit expression for the solution can be found in this case:

$$y^2 - 2y - \left(x^3 + 2x^2 + 2x + c\right) = 0 \Rightarrow y = \frac{2 \pm \sqrt{4 + 4\left(x^3 + 2x^2 + 2x + c\right)}}{2}$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

Example 2.2.2: Initial Value Condition (0, −1)

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$(-1)^2 - 2(-1) = C \Rightarrow C = 3$$

- The implicit particular solution is $y^2 - 2y = x^3 + 2x^2 + 2x + 3$
- An explicit expression of y is:

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

- Of which only $y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$ satisfies the $y(0) = -1$ initial condition.

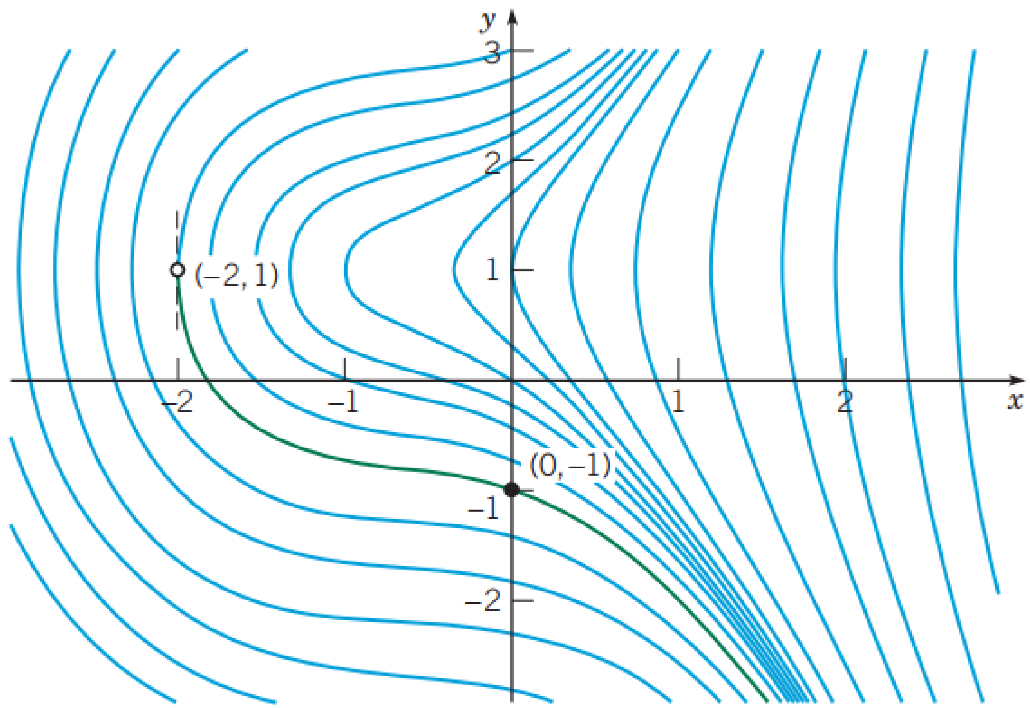
Example 2.2.2: Solution Domain

If initial condition is $y(0) = -1$, then we choose the positive sign, instead of negative sign, on the square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

The solution is valid over x-values for which the quantity under the radical is positive: $x > -2$

- domain shown in green in the graph
- vertical tangent at domain boundary point $(-2,1)$



Example 2.2.3: Implicit Solution of an Initial Value Problem

- Consider the following initial value problem:

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \; y(0) = 1$$

- Separating variables and using calculus, we obtain

$$\left(4 + y^3\right) dy = \left(4x - x^3\right) dx$$
$$\int \left(4 + y^3\right) dy = \int \left(4x - x^3\right) dx$$
$$4y + \frac{1}{4} y^4 = 2x^2 - \frac{1}{4} x^4 + c$$
$$y^4 + 16y + x^4 - 8x^2 = C \text{ where } C = 4c$$

- Using the initial condition, $y(0) = 1$, it follows that $C = 17$.

$$y^4 + 16y + x^4 - 8x^2 = 17$$

Example 2.2.3: General Solution vs. Particular Solution Through Point (0,1)

Thus the general solution is

$$y^4 + 16y + x^4 - 8x^2 = c$$

and the solution through (0,1) is

$$y^4 + 16y + x^4 - 8x^2 = 17$$

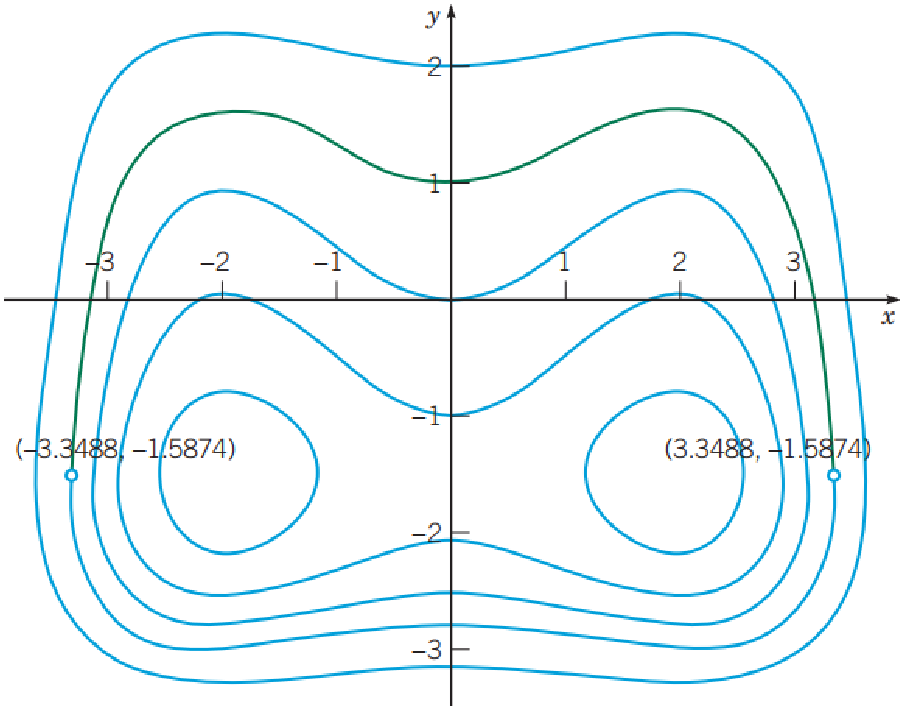
Vertical asymptotes for this particular solution will be found where:

$$y = \sqrt[3]{-4} \approx -1.5874$$

$$x \approx \pm 3.3488$$

Example 2.2.3: Graph of Solutions

- The graph of the solution through (0, 1) is shown in green
- The points identified with open dots correspond to the solution domain boundaries where the tangent lines are vertical.



Parametric Equations

- The differential equation: $\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}$

  is sometimes easier to solve if x and y are thought of as dependent variables of the independent variable t and rewriting the single differential equation as the system of differential equations:

$$\frac{dy}{dt} = F(x,y) \text{ and } \frac{dx}{dt} = G(x,y)$$

Chapter 9 is devoted to the solution of systems such as these.

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