Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

Chapter 2

First-Order Differential Equations

Section 2.9
First-Order Difference Equations

Introduction to Difference Equations

- Although a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more appropriate. Examples of this include accounts where interest is paid or charged monthly rather than continuously, applications involving drug dosages, and certain population growth problems where the population one year depends on the population in the previous year. For example,

$$y_{n+1} = f\left(n, y_n\right), \quad n = 0, 1, 2, \ldots$$

- Notice here that the independent variable n is discrete. Such equations are classified according to order, as linear or nonlinear, as homogeneous or nonhomogeneous. There is frequently an initial condition describing the first term y_0.

Difference Equation and Equilibrium Solution

- Assume for now that the state at year $n+1$ depends only on the state at year n, and not on the value of n itself

$$y_{n+1} = f\left(n, y_n\right), \quad n = 0, 1, 2, \ldots$$

- Then $y_1 = f\left(y_0\right), y_2 = f\left(y_1\right) = f\left(f\left(y_0\right)\right),\ y_3 = f\left(y_2\right) = f^3\left(y_0\right), \ldots, y_n = f^n\left(y_0\right)$
- This procedure is referred to as iterating the difference and it is often of interest to determine the behavior of y_n as n $\rightarrow \infty$.
- An **equilibrium solution** exists when

$$y_n = f\left(y_n\right)$$

and this is often of special interest, just as it is in differential equations.

Linear Homogeneous Difference Equations

- Suppose that the population of a certain species in a region in year $n+1$ is a positive multiple of the population in year n:

$$y_{n+1} = \rho_n y_n, \quad n = 0, 1, 2, ..$$

- Notice that the reproduction rate may differ from year to year.

$$y_n = \rho_{n-1} \cdots \rho_0 y_0, \qquad n = 1, 2, ...$$

- If the reproduction rate has the same value ρ for all n:

$$y_n = \rho^n y_0$$

- If the initial value y_0 is zero, then the equilibrium solution = 0
- Otherwise

$$\lim_{n\to\infty} y_n = \begin{cases} 0, & \text{if } |\rho| < 1; \\ y_0, & \text{if } \rho = 1; \\ \text{does not exist}, & \text{otherwise.} \end{cases}$$

Adding and Subtracting a Term to the Equation

- Suppose we have a net increase in population each year due to migration: $y_{n+1} = \rho y_n + b_n, \quad n = 0, 1, 2, \ldots$

- Then iterating this: $$y_1 = \rho y_0 + b_0,$$
$$y_2 = \rho\left(\rho y_0 + b_0\right) + b_1 = \rho^2 y_0 + \rho b_0 + b_1,$$
$$y_3 = \rho\left(\rho^2 y_0 + \rho b_0 + b_1\right) + b_2 = \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2,$$
$$y_n = \rho^n y_0 + \rho^{n-1} b_0 + \cdots + \rho b_{n-2} + b_{n-1} = \rho^n y_0 + \sum_{j=0}^{n-1} \rho^{n-1-j} b_j$$

- If the migration is constant (b) each year:
$$y_n = \rho^n y_0 + \left(1 + \rho + \rho^2 + \cdots + \rho^{n-1}\right) b$$

- And as long as $\rho \neq 1$, we can use the geometric series formula to get:
$$y_n = \rho^n y_0 + \frac{1-\rho^n}{1-\rho} b \quad = \rho^n\left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}$$

Conditions for an Equilibrium

- Letting $n \to \infty$ in the equation for y_n we get:

$$\lim_{n\to\infty} y_n = \left[\lim_{n\to\infty} \rho^n\right]\left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}$$

- Recall that $\rho \neq 1$. If it were, the sequence would become:

$$y_n = y_0 + nb \to \infty \text{ as } n \to \infty$$

- If $|\rho| < 1, \lim_{n\to\infty} \rho^n = 0$, so $y_n \to \dfrac{b}{1-\rho}$, an equilibrium solution.
- If $|\rho| > 1$ or if $\rho = -1$, $\lim_{n\to\infty} \rho^n$ does not exist, so the $\lim_{n\to\infty} y_n$ fails to exist unless

$$y_0 = \frac{b}{(1-\rho)} \Rightarrow \text{the solution starts at its equilibrium and stays there.}$$

Example 2.9.1: Extending the Model

- If we have a \$10,000 car loan at an annual interest rate of 12%, and we wish to pay it off in four years by making **monthly payments (–*b*),** we can adapt the previous result as follows:

$$y_n = \text{loan balance}(\$) \text{ in the } n^{\text{th}} \text{ month, } y_0 = 10{,}000$$

$$\rho = 1 + \frac{0.12}{12} = 1.01, 1 - \rho = -0.01, \frac{b}{1-\rho} = -100b$$

$$y_n = \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho} = 1.01^n \left(10{,}000 + 100b\right) - 100b$$

- To pay the loan off in four years, we set $y_{48} = 0$ and solve for ***b*:**

$$y_{48} = 1.01^{48} \left(10{,}000 + 100b\right) - 100b = 0 \Rightarrow$$

$$b = -100 \frac{1.01^{48}}{1.01^{48} - 1} \approx -263.34$$

- The total amount paid on the loan is 48(263.34)=\$12,640.32, so the amount of interest paid is \$2640.32.

Nonlinear Difference Equations

- As is the case with differential equations, nonlinear difference equations are much more complicated and have much more varied solutions than linear equations.
- We will analyze only the logistic equation, which is similar to the logistic differential equation discussed in 2.5.

$$y_{n+1} = \rho y_n \left(1 - \frac{y_n}{k}\right), \quad n = 0, 1, 2, \ldots$$

$$\text{Letting } u_n = y_n / k \quad , u_{n+1} = \rho u_n (1 - u_n)$$

- Seeking the equilibrium solution yields:

$$u_{n+1} = u_n = \rho u_n - \rho u_n^2 \Rightarrow u_n = 0, u_n = \frac{\rho - 1}{\rho}$$

- Are either of these equilibrium solutions asymptotically stable?

Examining Points Near Equilibrium Solutions

- For the first equilibrium solution of zero, the quadratic term ≈ 0:

$$\text{Near } u_n = 0,\ u_{n+1} = \rho u_n - \rho u_n{}^2 \approx \rho u_n \Rightarrow u_{n+1} = \rho u_n$$

- We have already examined this equation and concluded that for $|\rho| < 1$, the solution is asymptotically stable.
- We will now consider solutions near the second equilibrium:

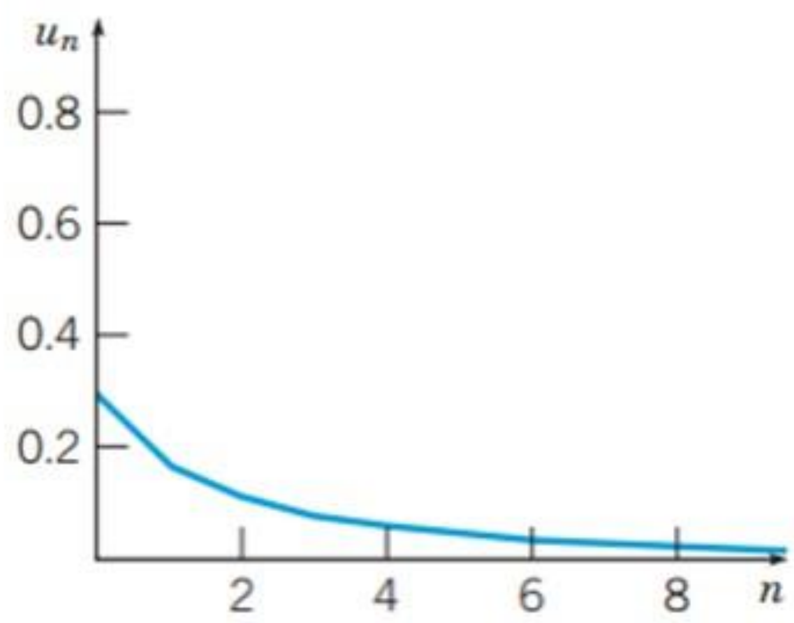
$$\text{Let } u_n = \frac{\rho - 1}{\rho} + v_n \text{ where } v_n \text{ is assumed to be small so quadratic term } \approx 0,$$

$$\Rightarrow v_{n+1} = (2-\rho)v_n - \rho v_n^2 \approx (2-\rho)v_n \ \left(\text{after simplifying } \mathrm{u}_{\mathrm{n+1}} \text{ expression}\right)$$

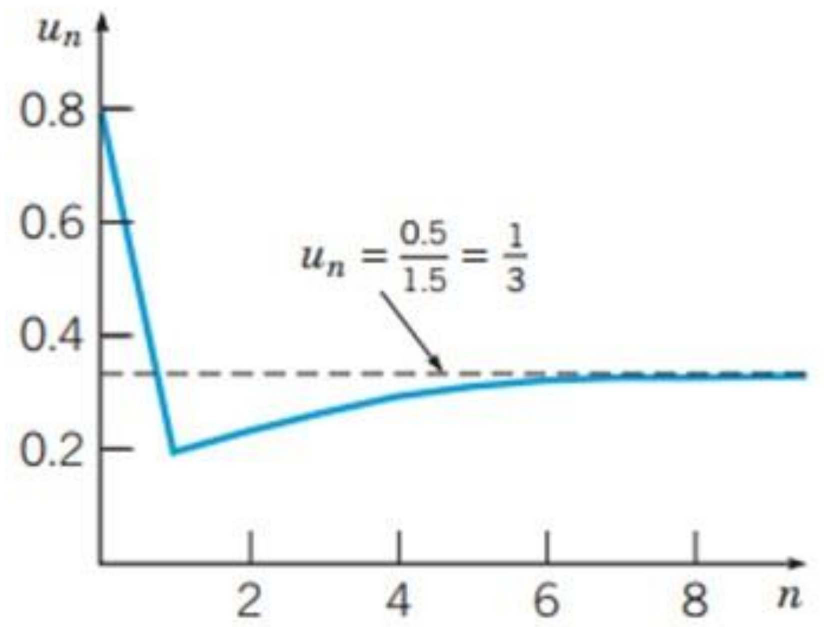
$$\Rightarrow v_{n+1} = (2-\rho)v_n$$

- From our previous discussion, we can conclude that $v_n \rightarrow 0$ provided $|2 - \rho| < 1$ or $1 < \rho < 3$. So, for these values of ρ, we can conclude that the solution is asymptotically stable.

Solutions for Varying Initial States and Varying Parameter Values Between 0 and 3 (part one)

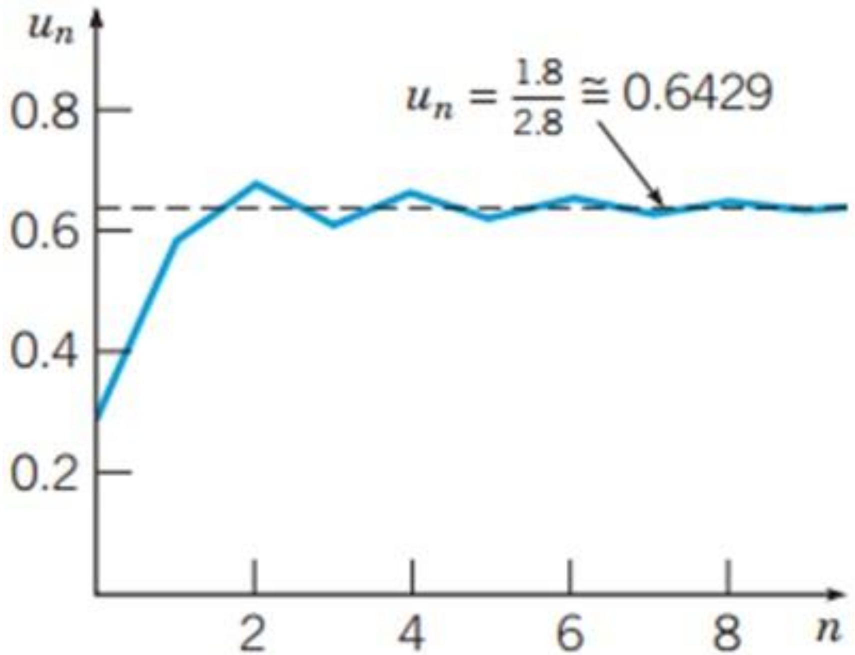


$y_0 = 0.3, \rho = 0.8$



$y_0 = 0.8, \rho = 1.5$

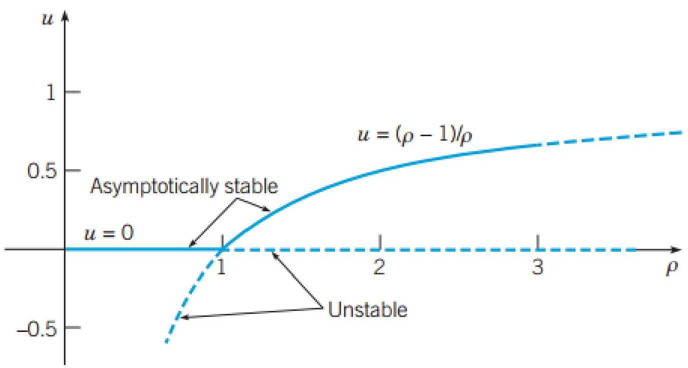
Solutions for Varying Initial States and Parameter Values Between 0 and 3 (part two)



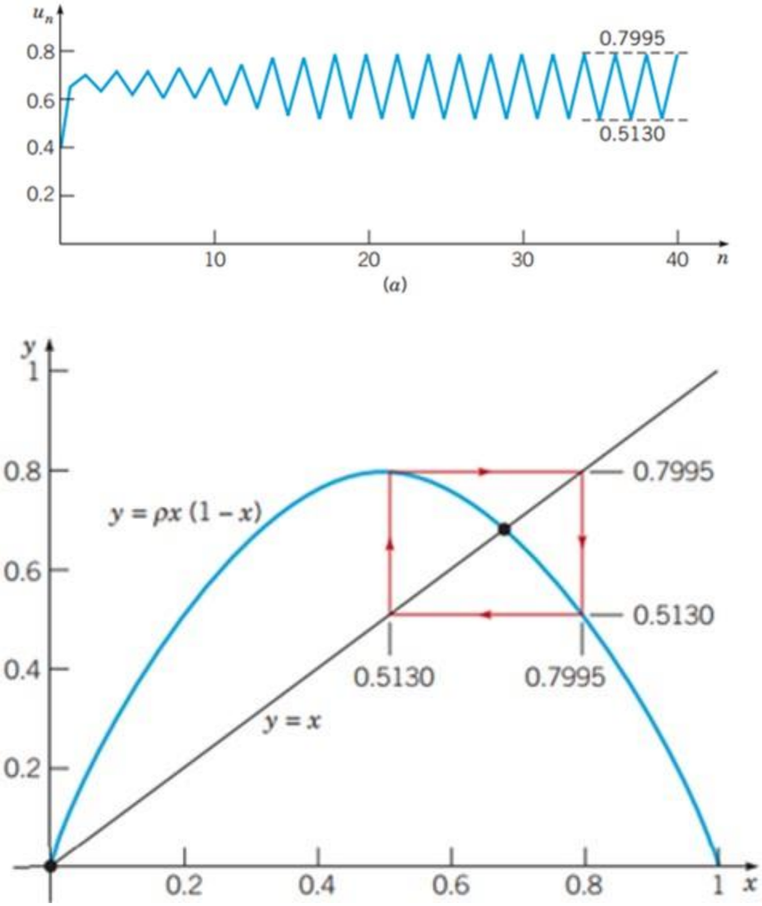
$y_0 = 0.3, \rho = 2.8$

Summary of Asymptotic Stability Intervals

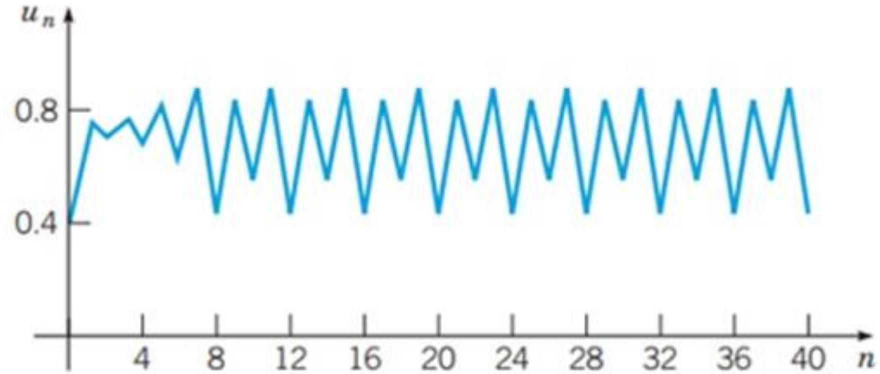
- We found that the difference equation $u_{n+1} = \rho u_n - \rho u_n^2$ has two equilibrium solutions: $u_n = 0,\ u_n = \dfrac{\rho - 1}{\rho}$
- Considering nonnegative values of the parameter ρ, the first equilibrium solution required that $0 \le \rho < 1$, while the second equilibrium solution required that $1 < \rho < 3$. there is an **exchange of stability** from one equilibrium solution to the other at $\rho = 1$. This is demonstrated in the chart below:

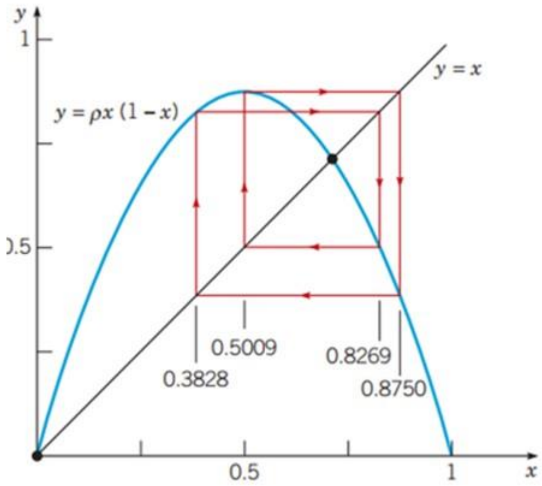


Solutions of the Difference Equation That Do Not Approach an Equilibrium - 2 Cycle



Solutions of the Difference Equation That Do Not Approach an Equilibrium - 4 Cycles



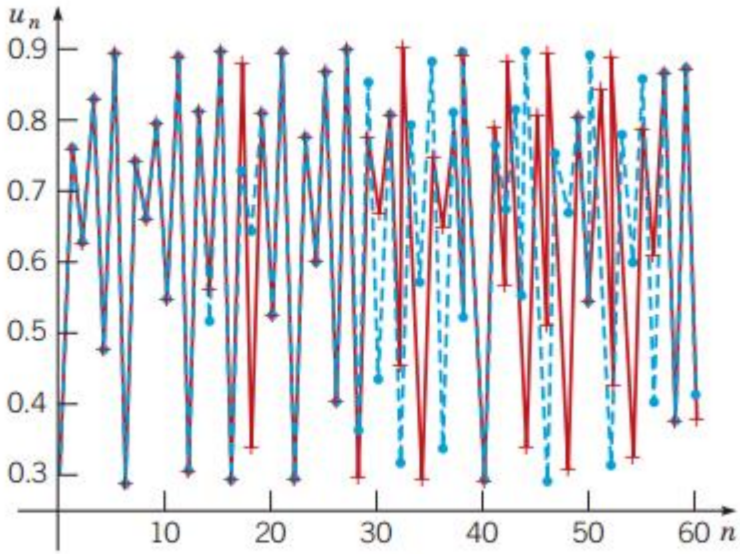


Solutions of the Difference Equation That Do Not Approach an Equilibrium: General Cyclic and Chaotic Behavior

- Notice from the preceding graphs how the behavior of the solution to the difference equation $u_{n+1} = \rho u_n(1-u_n)$ behaves rather unpredictably when $\rho > 3$. First, at $\rho = 3.2$, we saw the sequence oscillate between two values, creating a period of two. Then, at $\rho = 3.5$, the terms in the sequence were oscillating among four values, creating a period of 4. It is actually around $\rho = 3.449$ that this doubling of the period occurs and this is called a point of **bifurcation**. As ρ increases slightly further, periodic solutions of period 8, 16, … occur.
- By the time we reach $\rho > 3.57$, the solutions possess some regularity, but no discernible detailed pattern is present for most values of ρ. The term **chaotic** is used to describe this situation. One of the features of chaotic solutions is extreme sensitivity to the initial conditions. This is demonstrated on the following slide.

Chaotic Solutions Graphical Example

- Below are two solutions to $u_{n+1} = 3.65u_n(1 - u_n)$
- The blue solution corresponds to the initial state $u_0 = 0.300$
- The red solution corresponds to the initial state $u_0 = 0.305$



What Chaotic Solutions May Suggest

- On the basis of Robert May's analysis of the nonlinear equation we have considered

$$u_{n+1} = \rho u_n (1 - u_n) \text{ and similarly } y' = ry(1 - y)$$

  as a model for the population of certain insect species, we might conclude that if the growth rate ρ is too large, it will be impossible to make effective long-range predictions about these insect populations.
- It is increasingly clear that chaotic solutions are much more common than was suspected at first, and that they may be part of the investigation of a wide range of phenomena.

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