Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 5

Series Solutions of Second Order Linear Equations

Section 5.3 Series Solutions Near an Ordinary Point, Part II

Analytic Functions and Series Solutions Near Ordinary Points

- A function p is **analytic** at x_0 if it has a Taylor series expansion that converges to p in some interval about x_0

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$

- The point x_0 is an **ordinary point** of the equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at x_0. Otherwise x_0 is a **singular point**.

- If x_0 is an ordinary point, then p and q are analytic and have derivatives of all orders at x_0, and this enables us to solve for a_n in the solution expansion $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ See text.

Theorem 5.3.1

- If x_0 is an ordinary point of the differential equation

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$$

then the general solution for this equation is

$$y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0y_1(x)+a_1y_2(x)$$

where a_0 and a_1 are arbitrary, and y_1, y_2 are linearly independent series solutions that are analytic at x_0.

- Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

Radius of Convergence

- Thus if x_0 is an ordinary point of the differential equation, then there exists a series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- Further, the radius of convergence of the series solution is at least as large as the minimum of the radii of convergence of the series for p and q.
- These radii of convergence can be found in two ways:
  1. Find the series for p and q, and then determine their radii of convergence using a convergence test.
  2. If P, Q and R are polynomials with no common factors, then it can be shown that Q/P and R/P are analytic at x_0 if $P(x_0) \neq 0$, and the radius of convergence of the power series for Q/P and R/P about x_0 is the distance to the nearest zero of P (including complex zeros).

Example 5.3.1 (part one)

Let $y = \phi(x)$ be a solution of the initial value problem

$$(1+x^2)y''+2xy'+4x^2y=0,\ y'(0)=1$$

Determine $\phi''(0)$, $\phi'''(0)$, $\phi^{(4)}(0)$.

- To find $\phi''(0)$, evaluate the equation when $x = 0$:

$$(1+0^2)y''+2(0)y'+4(0)^2y=0$$

so $\phi''(0) = 0$.

Example 5.3.1 (part two)

- To find $\phi'''(0)$, differentiate the equation with respect to x:

$$(1+x^2)\phi'''(x)+2x\phi''(x)+2x\phi''(x)+2\phi'(x)+4x^2\phi'(x)+8x\phi(x)=0$$

- Then evaluate at $x = 0$: $\phi'''(0) + 2\phi'(0) = 0$

  Thus $\phi'''(0) = -2\phi'^{(0)} = -2$

- Differentiating $\phi'''(x)$ above with respect to x:

$$(1+x^2)\phi^{(4)}(x)+2x\phi'''(x)+4x\phi'''(x)+4\phi''(x)+(4x^2+2)\phi''(x)$$
$$+8x\phi'(x)+8x\phi'(x)+8\phi(x)=0$$

- Evaluate at $x = 0$: $\phi^{(4)}(0)+6\phi''(0)+8\phi(0)=0$

  Use $\phi(0) = 0$ and $\phi''(0) = 0$ to give $\phi^{(4)} = 0$

Example 5.3.2

Let $f(x) = (1 + x^2)^{-1}$. Find the radius of convergence of the Taylor series of f about $x = 0$.

- The Taylor series of f about $x_0 = 0$ is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

- Using the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} x^2 < 1, \text{ for } |x| < 1$$

- Thus the radius of convergence is $\rho = 1$.
- Alternatively, note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance in the complex plane from 0 to i or $-i$ is 1, we see again that $\rho = 1$.

Example 5.3.3

- Find the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$ and about $x = 1$. First observe:

$$(x^2 - 2x + 2) = 0 \Rightarrow x = 1 \pm i$$

- Since the denominator cannot be zero, this establishes the bounds over which the function can be defined.
- In the complex plane, the distance from $x_0 = 0$ to $x = 1 \pm i$ is $\sqrt{2}$. So, the radius of convergence of the Taylor series expansion about $x = 0$ is $\sqrt{2}$.
- In the complex plane, the distance from $x_0 = 1$ to $1 \pm \mathrm{i}$ is 1 , so the radius of convergence for the Taylor series expansion about $x_0 = 0$ is $\rho = 1$.

Example 5.3.4: Legendre Equation (part one)

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad \alpha \text{ a constant.}$$

- Here, $P(x) = 1 - x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha + 1)$; all of which are polynomials.
- Thus $x_0 = 0$ is an ordinary point, since $p(x) = -2x/(1 - x^2)$ and $q(x) = \alpha(\alpha + 1)/(1 - x^2)$ are analytic at $x_0 = 0$.
- Also, p and q have singular points at $x = \pm 1$.
- Thus the radius of convergence for the Taylor series expansions of p and q about $x_0 = 0$ is $\rho = 1$.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$.

Example 5.3.4: Legendre Equation (part two)

- Thus, for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$.

- It can be shown that if ρ is a positive integer, then one of the series solutions terminates after a finite number of terms, and hence converges for all x, not just for $|x| < 1$.

Example 5.3.5: Radius of Convergence

- Determine a lower bound for the radius of convergence of the series solution about both $x = 0$ and $x = -1/2$ for the equation
$$(1+x^2)y'' + 2xy' + 4x^2 y = 0$$
- Here, $P(x) = 1 + x^2$, $Q(x) = 2x$, $R(x) = 4x^2$.
- Thus both $x = 0$ and $x = -\frac{1}{2}$ are ordinary points, since $p(x) = 2x/(1 + x^2)$ and $q(x) = 4x^2 /(1 + x^2)$ are analytic at both points.
- Also, p and q have singular points at $x = \pm i$, so the complex plane distances from 0 to $\pm i$ and from $-\frac{1}{2}$ to $\pm i$ are 1 and $\frac{\sqrt{5}}{2}$, respectively.
- Thus the radius of convergence for the Taylor series expansions of p and q about $x_0 = 0$ is $\rho = 1$ and about $x_0 = -\frac{1}{2}$ is $\rho = \sqrt{5}/2$.

Example 5.3.5: Solution Theory

- Thus for the equation

$$(1+x^2)y'' + 2xy' + 4x^2 y = 0,$$

  the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = -1/2$ are $\rho = 1$ and $\sqrt{5}/2$, respectively, by Theorem 5.3.1.
- Suppose that initial conditions $y(0) = y_0$ and $y(0) = y_0'$ are given. Since $1 + x^2 \neq 0$ for all x, there exists a unique solution of the initial value problem on $(-\infty, \infty)$ by Theorem 3.2.1.
- On the other hand, Theorem 5.3.1 only guarantees a solution of the form

$$\sum_{n=0}^{\infty} a_n x^n \text{ for } -1 < x < 1, \text{ where } a_0 = y_0 \text{ and } a_1 = y_0'.$$

- Thus the unique solution on $(-\infty, \infty)$ may not have a power series about $x_0 = 0$ that converges for all x.

Example 5.3.6

- Can we determine a series solution and about $x = 0$? If a solution exists, what is the radius of convergence?

$$y'' + (\sin x)y' + (1 + x^2)y = 0$$

- Here, $P(x) = 1$, $Q(x) = \sin x$, $R(x) = 1 + x^2$.
- Note that $p(x) = \sin x$ is not a polynomial, but recall that it does have a Taylor series about $x_0 = 0$ that converges for all x.
- Similarly, $q(x) = 1 + x^2$ has a Taylor series about $x_0 = 0$, namely $1 + x^2$, which converges for all x.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is infinite.

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