Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 3

Second-Order Linear Differential Equations

Section 3.1 Homogeneous Differential Equations with Constant Coefficients

2^{nd} Order Linear Homogeneous Equations-Constant Coefficients

- A **second order ordinary differential equation** has the general form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

where f is some given function.

- This equation is said to be **linear** if f is linear in y and y':

$$y'' + p(t)y' + q(t)y = g(t)$$

Otherwise the equation is said to be **nonlinear.**

- A second order linear equation often appears as

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

- If $g(t)$ or $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Homogeneous Equations, Initial Values

- In Sections 3.5 and 3.6, we will see that once a solution to a homogeneous equation is found, then it is possible to solve the corresponding nonhomogeneous equation, or at least express the solution in terms of an integral.
- The focus of this chapter is thus on homogeneous equations; and in particular, those with constant coefficients:

$$ay'' + by' + cy = 0$$

  We will examine the variable coefficient case in Chapter 5.
- Initial conditions typically take the form

$$y(t_0) = y_0, \quad y'(t_0) = y_0',$$

- Thus solution passes through (t_0, y_0), and the slope of solution at (t_0, y_0) is equal to y_0'.

Example 3.1.1: General Solution

- Find the general solution for the second order linear differential equation:
$$y'' - y = 0$$
- Two solutions of this equation are
$$y_1(t) = e^t, \quad y_2(t) = e^{-t}$$
- Other solutions include
$$y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$$
- Based on these observations, we see that there are infinitely many solutions of the form
$$y(t) = c_1 e^t + c_2 e^{-t}$$
- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form.

Example 3.1.1: Initial Conditions

- Now consider the following initial value problem for our equation:

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

- We have found a general solution of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- Using the initial equations,

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= c_1 - c_2 = -1 \end{aligned}\right\} \Rightarrow c_1 = \frac{1}{2},\ c_2 = \frac{3}{2}$$

- Thus

$$y(t) = \frac{1}{2} e^t + \frac{3}{2} e^{-t}$$

The Characteristic Equation

- To solve the 2nd order equation with constant coefficients,

$$ay'' + by' + cy = 0$$

we begin by assuming a solution of the form $y = e^{rt}$.

- Substituting this into the differential equation, we obtain

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

- Simplifying,

$$e^{rt}(ar^2 + br + c) = 0$$

and hence

$$ar^2 + br + c = 0$$

- This last equation is called the **characteristic equation** of the differential equation.
- We then solve for r by factoring or using quadratic formula.

The General Solution

- Using the quadratic formula on the characteristic equation

$$ar^2 + br + c = 0,$$

  we obtain two solutions, r_1 and r_2.
- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.
- In this case, the general solution has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Initial Conditions

- For the initial value problem
  $ay'' + by' + cy = 0$ where $y(t_0) = y_0$ and $y'(t_0) = y_0'$

  we use the general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  together with the initial conditions to find c_1 and c_2. That is,

$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y'_0 \end{aligned}\right\} \Rightarrow c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0},\ c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0}$$

- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 3.1.2 General Solution

- Consider the linear differential equation

$$y'' + 5y' + 6y = 0$$

- Assuming an exponential solution leads to the characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 6 = 0 \; \Leftrightarrow \; (r+2)(r+3) = 0$$

- Factoring the characteristic equation yields two solutions: $r_1 = -2$ and $r_2 = -3$
- Therefore, the general solution to this differential equation has the form

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Example 3.1.3 Particular Solution

- Consider the initial value problem

$$y''+5y'+6y=0,\ y(0)=2,\ y'(0)=3$$

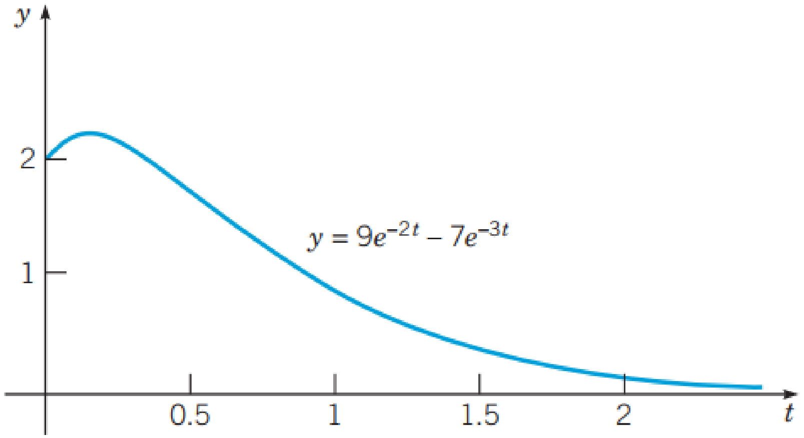
- From the preceding example, we know the general solution has the form: $$y(t)=c_1e^{-2t}+c_2e^{-3t}$$

- With derivative: $y'(t)=-2c_1e^{-2t}-3c_2e^{-3t}$

- Using the initial conditions:

$$\left.\begin{aligned} c_1+c_2&=2 \\ -2c_1-3c_2&=3 \end{aligned}\right\} \Rightarrow c_1=9,\ c_2=-7$$

- Thus $y(t)=9e^{-2t}-7e^{-3t}$



Example 3.1.4: Initial Value Problem

- Consider the initial value problem

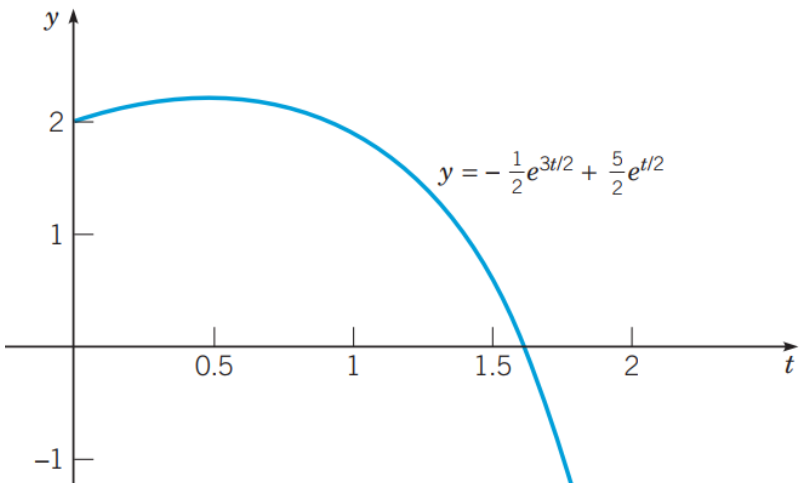
$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \; y'(0) = \frac{1}{2}$$

- Then $y(t) = e^{rt} \;\Rightarrow\; 4r^2 - 8r + 3 = 0 \;\Leftrightarrow\; (2r-3)(2r-1) = 0$
- Factoring yields two solutions, $r_1 = \frac{3}{2}$ and $r_2 = \frac{1}{2}$ $\quad y(t) = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}$
- The general solution has the form

$$y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 &= \frac{1}{2} \end{aligned}\right\} \Rightarrow c_1 = -\frac{1}{2}, \; c_2 = \frac{5}{2}$$



Example 3.1.5: Find Maximum Value

For the initial value problem in Example 3.3, to find the maximum value attained by the solution, we set $y'(t) = 0$ and solve for t:

Set $y' = 0$ and multiply by e^{3t} to find t_{max} which satisfies

$e^t = \frac{7}{6}$, hence:

$$t_{max} = ln\left(\frac{7}{6}\right) \cong 0.15415$$

The corresponding maximum value y_{max} is given by:

$$y_{max} = 9e^{-2t_m} - 7e^{-3t_m} = \frac{108}{49} \cong 2.20408$$

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