Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 3

Second-Order Linear Differential Equations

Section 3.3 Complex Roots of the Characteristic Equation

Complex Roots of Characteristic Equation

- Recall our discussion of the equation

$$ay'' + by' + cy = 0$$

where a, b and c are constants

- Assuming an exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Application of the quadratic formula (or factoring) yields two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then the roots are conjugate complex numbers:

$$r_1 = \lambda + i\mu \ \text{ and } \ r_2 = \lambda - i\mu$$

Thus

$$y_1(t) = e^{(\lambda + i\mu)t}, \quad y_2(t) = e^{(\lambda - i\mu)t}$$

Euler's Formula; Complex Valued Solutions

- Substituting *it* into the Taylor series for e^t, we obtain **Euler's formula**:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$$

- Generalizing Euler's formula, we obtain

$$e^{i\mu t} = \cos(\mu t) + i \sin(\mu t)$$

- Then

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right) = e^{\lambda t}\cos(\mu t) + ie^{\lambda t}\sin(\mu t)$$

- Therefore

$$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t}\cos(\mu t) + ie^{\lambda t}\sin(\mu t)$$
$$y_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t}\cos(\mu t) - ie^{\lambda t}\sin(\mu t)$$

Real-Valued Solutions

- Our two solutions thus far are complex-valued functions:

$$y_1(t) = e^{\lambda t}\cos(\mu t) + ie^{\lambda t}\sin(\mu t)$$
$$y_2(t) = e^{\lambda t}\cos(\mu t) - ie^{\lambda t}\sin(\mu t)$$

- We would prefer to have real-valued solutions, since our differential equation has real coefficients.
- To achieve this, recall that linear combinations of solutions are themselves solutions:

$$y_1(t) + y_2(t) = 2e^{\lambda t}\cos(\mu t)$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t}\sin(\mu t)$$

- Ignoring constants, we obtain the two solutions

$$y_3(t) = e^{\lambda t}\cos(\mu t), \quad y_4(t) = e^{\lambda t}\sin(\mu t)$$

Real-Valued Solutions: The Wronskian

- Thus we have the following real-valued functions:

$$u(t) = e^{\lambda t}\cos(\mu t), \quad \mathrm{v}(t) = e^{\lambda t}\sin(\mu t)$$

- Computation of the Wronskian gives:

$$W[u, \upsilon](t) = \mu e^{2\lambda t}$$

- Thus y_3 and y_4 form a fundamental solution set for our ODE, and the general solution can be expressed as

$$y = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$$

Example 3.3.1 (part one)

- Find the general solution of the differential equation:

$$y'' + y' + 9.25y = 0$$

- The characteristic equation is $r^2 + r + 9.25 = 0$ with roots given by:

$$r_1 = -\frac{1}{2} + 3i, \quad r_2 = -\frac{1}{2} - 3i$$

- Therefore, separating the real and imaginary components,

$$\lambda = -\frac{1}{2}, \mu = 3$$

and thus the general solution is

$$y = e^{-t/2}\left(c_1 \cos\left(3t\right) + c_2 \sin\left(3t\right)\right)$$

Example 3.3.1 (part two)

- Using the general solution just determined

$$y = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

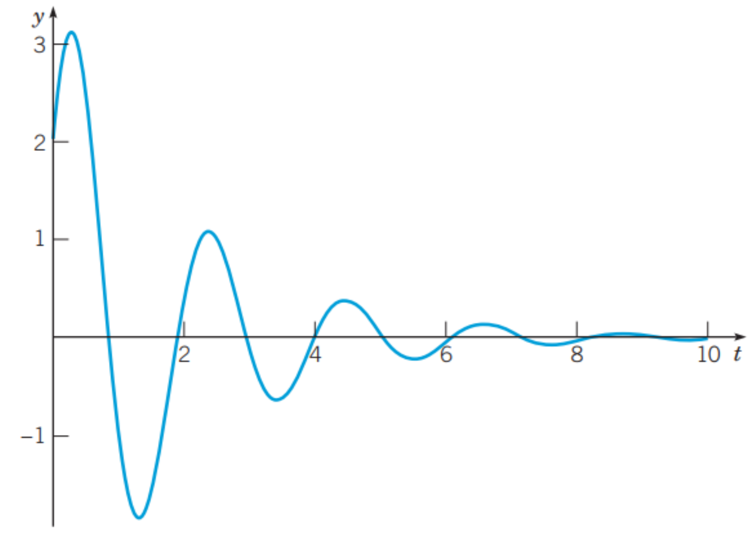
- We can determine the particular solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 8$

- So $\left.\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= -\tfrac{1}{2}c_1 + 3c_2 = 8 \end{aligned}\right\} \Rightarrow c_1 = 2,\ c_2 = 3$

- Thus the solution of this IVP is

$$y = e^{-t/2}\left(2\cos(3t) + 3\sin(3t)\right)$$

- The solution is a decaying oscillation



Example 3.3.2

- Find the solution to the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1$$

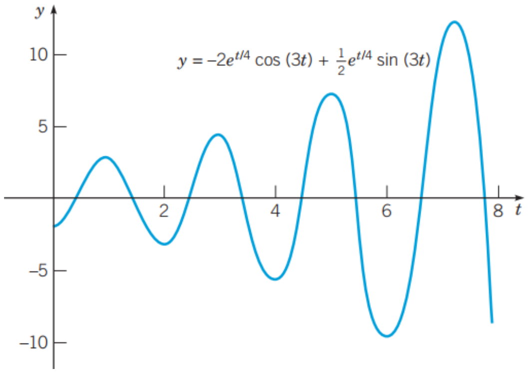
- Then $y(t) = e^{rt} \Rightarrow 16r^2 - 8r + 145 = 0 \Leftrightarrow r = \frac{1}{4} \pm 3i$
- Thus the general solution is $y(t) = c_1 e^{t/4}\cos(3t) + c_2 e^{t/4}\sin(3t)$
- And $y(0) = c_1 = -2$

  $y'(0) = \frac{1}{4}c_1 + 3c_2 = 1$ so, $c_1 = -2; c_2 = \frac{1}{2}$

- The solution of the IVP is

$$y(t) = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t)$$

- The solution is a growing oscillation



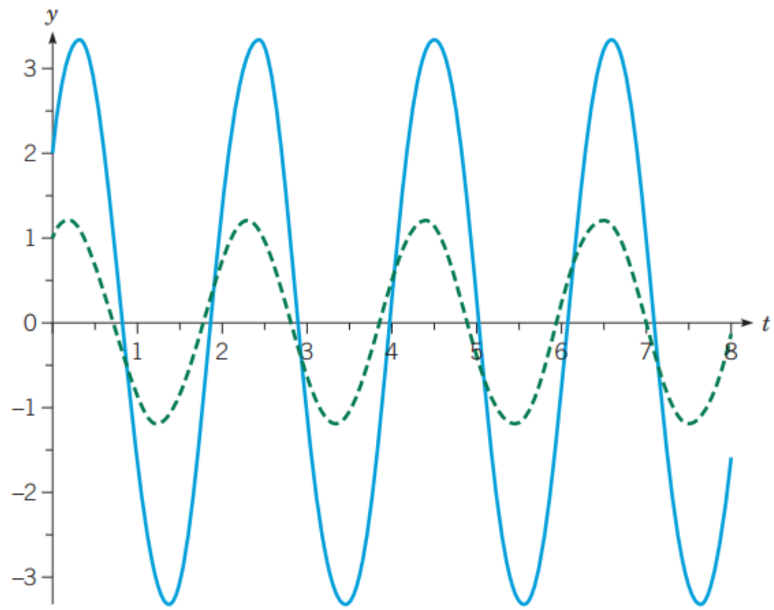
Example 3.3.3

- Find the general solution of $y'' + 9y = 0$
- Then $y(t) = e^{rt} \Rightarrow r^2 + 9 = 0 \Leftrightarrow r = \pm 3i$
- Therefore $\lambda = 0,\ \mu = 3$

  and thus the general solution is

  $$y(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

- Because $\lambda = 0$, there is no exponential factor in the solution, so the amplitude of each oscillation is constant. The figure shows the graph with two sets of initial conditions.



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