Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 1

Introduction

Section 1.2
Solutions of Some Differential Equations

Solutions for Free Fall and Population Prediction Mathematical Models

- Recall the differential equations:

$$m\frac{dv}{dt} = mg - \gamma v \qquad \frac{dp}{dt} = rp - k$$

- These equations have the general form $\dfrac{dy}{dt} = ay - b$
- We can use methods of calculus to solve differential equations of this form.

Example 1.2.1: Mice and Owls Population Prediction Solution

- To solve the differential equation

$$\frac{dp}{dt} = 0.5p - 450$$

we use methods of calculus, as follows.

$$\frac{dp}{dt} = \frac{p-900}{2} \Rightarrow \frac{dp/dt}{p-900} = \frac{1}{2} \Rightarrow \int \frac{dp}{p-900} = \int \frac{1}{2}dt$$

$$\Rightarrow \ln|p-900| = \frac{1}{2}t + C \Rightarrow |p-900| e^{\frac{1}{2}t+C}$$

$$\Rightarrow p - 900 = \pm e^{\frac{1}{2}t} e^{C} \Rightarrow p = 900 + ce^{\frac{1}{2}t}, \; c = \pm e^{C}$$

- Thus the solution is

$$p = 900 + ce^{\frac{1}{2}t}$$

where c is a constant.

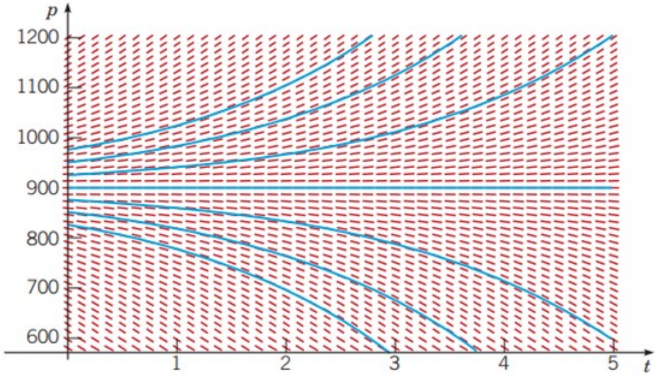
Example 1.2.1: Integral Curves

- Thus we have infinitely many solutions to our equation,

$$\frac{dp}{dt} = 0.5p - 450 \Rightarrow p = 900 + ce^{\frac{1}{2}t},$$

since k is an arbitrary constant.

- Graphs of solutions (**integral curves**) for several values of c, and direction field for differential equation, are given below.
- Choosing $c = 0$, we obtain the equilibrium solution, while for $c > 0$, the solutions diverge from equilibrium solution.



Example 1.2.1: Initial Conditions

- A differential equation often has infinitely many solutions. If a point on the solution curve is known, such as an initial condition, then this determines a unique solution.
- In the mice/owl differential equation, suppose we know that the mice population starts out at 850. Then $p(0) = 850$, and

$$p(t) = 900 + ce^{0.5t}$$

$$p(0) = 850 = 900 + ce^{0}$$

$$c = -50$$

Solution:

$$p(t) = 900 - 50e^{\frac{1}{2}t}$$

Solution to a General Equation

- To solve the general equation $\dfrac{dy}{dt} = ay - b$

  we use methods of calculus, as follows.

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right) \Rightarrow \frac{dy/dt}{y - b/a} = a \Rightarrow \int \frac{dy}{y - b/a} \int a\,dt$$

$$\Rightarrow \ln\left|y - b/a\right| = at + C \Rightarrow \left|y - b/a\right| = e^{at+C}$$

$$\Rightarrow y - b/a = \pm e^{at}e^{C} \Rightarrow y = b/a + ce^{at}, c = \pm e^{C}$$

- Thus the general solution is

$$y = \frac{b}{a} + ce^{at},$$

  where c is a constant.

Initial Value Problem

- Next, we solve the initial value problem

$$y' = ay - b, \quad y(0) = y_0$$

- From previous slide, the solution to differential equation is

$$y = \frac{b}{a} + ce^{at}$$

- Using the initial condition to solve for c, we obtain

$$y(0) = y_0 = \frac{b}{a} + ce^{0} \quad \Rightarrow \quad c = y_0 - \frac{b}{a}$$

and hence the solution to the initial value problem is

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

Equilibrium Solution

- To find the equilibrium solution, set $y' = 0$ & solve for y:

$$y' = ay - b, \text{ set equal to zero} \Rightarrow y(t) = \frac{b}{a}$$

- From the previous slide, our solution to the initial value problem is:

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

- Note the following solution behavior:
  - If $y_0 = b/a$, then y is constant, with $y(t) = b/a$
  - If $y_0 > b/a$ and $a > 0$, then y increases exponentially without bound
  - If $y_0 > b/a$ and $a < 0$, then y decays asymptotically to b/a
  - If $y_0 < b/a$ and $a > 0$, then y decreases exponentially without bound
  - If $y_0 < b/a$ and $a < 0$, then y increases asymptotically to b/a

Example 1.2.2: Free Fall Equation General Solution

- Recall equation modeling free fall descent of 10 kg object, assuming an air resistance coefficient $\gamma = 2$ kg/s:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

- Suppose object is dropped from 300 m above ground.
  a) Find the velocity at any time t.
  b) How long until it hits ground and how fast will it be moving then?
- For part (a), we need to solve the initial value problem

$$v' = 9.8 - 0.2v, \quad v(0) = 0$$

- Using result from previous slide, we have

$$\frac{d\upsilon/dt}{\upsilon - 49} = -\frac{1}{5} \Rightarrow \quad \ln\left|\upsilon(t) - 49\right| = -\frac{t}{5} + C \quad \Rightarrow \upsilon(t) = 49 + ce^{-t/5}$$

Example 1.2.2: Free Fall Equation Initial Value Solution

- To determine the particular value of c that corresponds to the initial condition, we substitute $t = 0$ and $v = 0$.
- This gives us

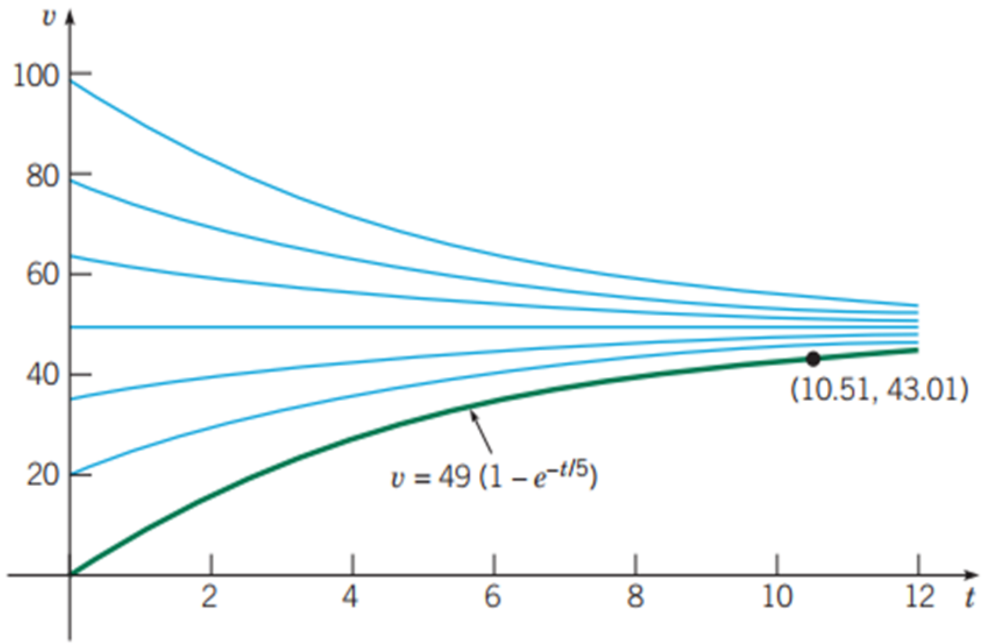
$$v(t) = 49 + ce^{-t/5} \Rightarrow 0 = 49 + ce^{0} \Rightarrow c = -49$$

- With the result that $c = -49$. And the solution of the initial value problem is

$$v(t) = 49\left(1 - e^{-t/5}\right)$$

Example 1.2.2: Graphs of Different Initial Value Solutions

- The graph of the solution found in part (a), along with the direction field for the differential equation, is given below.



Example 1.2.2: Time and Speed of Impact

- Next, given that the object is dropped from 300 m above ground, how long will it take to hit ground, and how fast will it be moving at impact?
- Solution: Let $x(t)$ = distance object has fallen at time t.

  It follows from our solution $v(t)$ that

$$\frac{dx}{dt} = 49 - 49e^{-t/5} \qquad \Rightarrow x(t) = 49t + 245e^{-t/5} + k$$

$$x(0) = 0 \Rightarrow k = -245 \quad \Rightarrow x(t) = 49t + 245e^{-t/5} - 245$$

- Let T be the time of impact:

$$49T + 245e^{-T/5} - 245 = 300.$$

- Using a solver, $T \cong 10.51$ seconds, hence

$$\upsilon(t) = 49\left(1 - e^{-0.2(10.51)}\right) \approx 43.01\,{}^{m}\!/\!{}_{s}$$

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