Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

Chapter 2

First-Order Differential Equations

Section 2.5
Autonomous Differential Equations and Population Dynamics

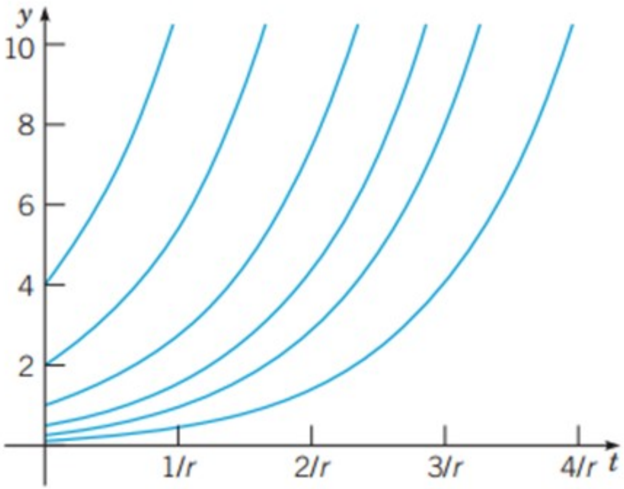
Autonomous Equations

- In this section we examine equations of the form $\frac{dy}{dt} = f(y)$, called **autonomous** equations, where the independent variable *t* does not appear explicitly.
- The main purpose of this section is to learn how geometric methods can be used to obtain qualitative information directly from a differential equation without solving it.
- Exponential Growth:

$$\frac{dy}{dt} = ry, \quad r > 0$$

- Solution: $y = y_0 e^{rt}$

graphs shows solution for different initial conditions



Logistic Growth

- An exponential model $y' = ry$, with solution $y = e^{rt}$, predicts unlimited growth, with rate $r > 0$ independent of population.
- If growth rate depends on population size, replace r by a function $h(y)$ to obtain:

$$\frac{dy}{dt} = h(y)y$$

- Choose growth rate $h(y)$ so that
  - $h(y) \cong r > 0$ when y is small,
  - $h(y)$ decreases as y grows larger, and
  - $h(y) < 0$ when y is sufficiently large.

  The simplest such function is $h(y) = r - ay$, where $a > 0$.
- Our differential equation then becomes $\frac{dy}{dt} = (r - ay)y, \quad r, a > 0$
- This equation is known as the Verhulst, or **logistic**, equation.

Logistic Equation

The logistic equation: $\dfrac{dy}{dt} = (r - ay)y, \quad r, a > 0$

can be rewritten as: $\dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right)y, \quad \text{where } K = \dfrac{r}{a}$

- The constant r is called the **intrinsic growth rate**
- K represents the **carrying capacity** of the population.

Logistic Equation: Equilibrium Solutions

- Our logistic equation is

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r, K > 0$$

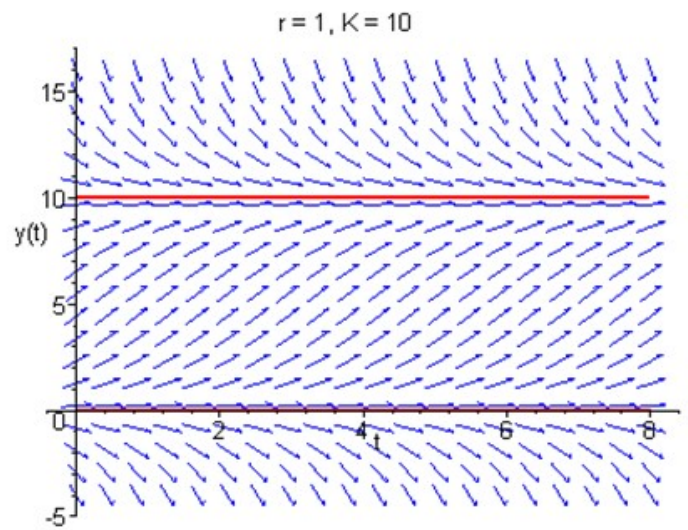
- Two **equilibrium solutions** are clearly present:

$$y = \phi_1(t) = 0, \quad y = \phi_2(t) = K$$

- In direction field below, with $r = 1$, $K = 10$, note the behavior of solutions near equilibrium solutions:

$y = 0$ is **unstable,**

$y = 10$ is **asymptotically stable.**



Autonomous Equations: Equilibrium Solutions

- Equilibrium solutions of a general first order autonomous equation $y' = f(y)$ can be found by locating roots of $f(y) = 0$.
- These roots of $f(y)$ are called **critical points.**
- For example, the critical points of the logistic equation

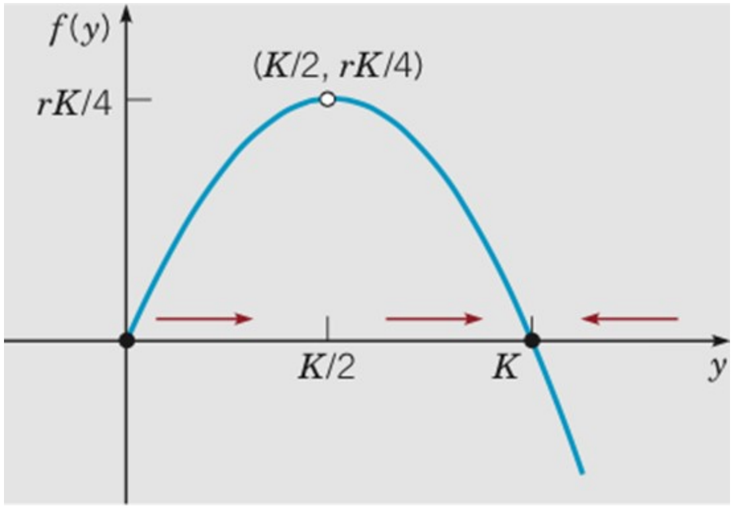
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

- are $y = 0$ and $y = K$.
- Thus critical points are constant functions (equilibrium solutions) in this setting.

Logistic Equation: Qualitative Analysis and Curve Sketching

- To better understand the nature of solutions to autonomous equations, we start by graphing $f(y)$ *vs*. y.
- In the case of logistic growth, that means graphing the following function and analyzing its graph using calculus.

$$f(y) = r\left(1 - \frac{y}{K}\right)y$$



Logistic Equation: Critical Points

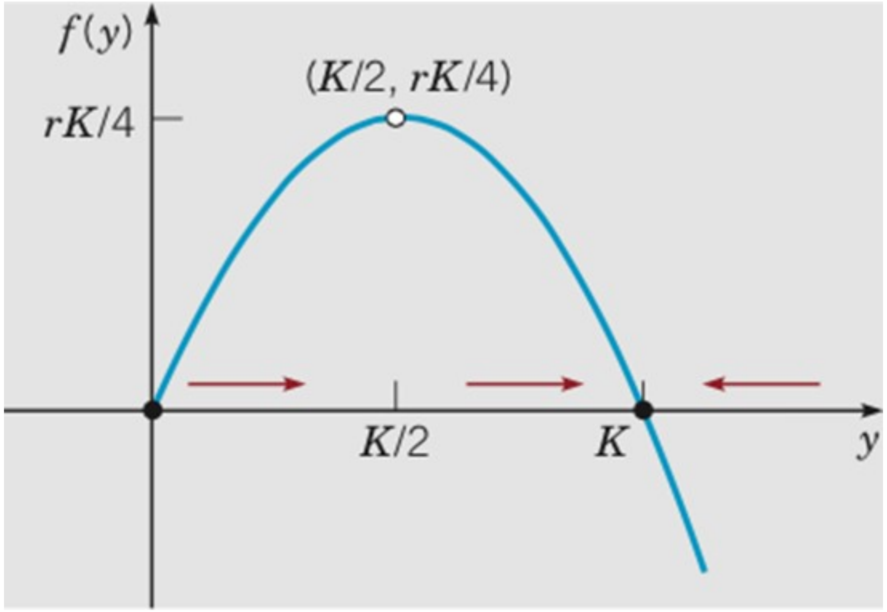
- The intercepts of f occur at $y = 0$ and $y = K$, corresponding to the critical points of logistic equation.
- The vertex of the parabola is $\left(\frac{K}{2}, \frac{rK}{4}\right)$, as shown below.

$$f(y) = r\left(1 - \frac{y}{K}\right)y$$

$$f'(y) = r\left[\left(-\frac{1}{K}\right)y + \left(1 - \frac{y}{K}\right)\right]$$

$$= -\frac{r}{K}\left[2y - K\right] \overset{set}{=} 0 \Rightarrow y = \frac{K}{2}$$

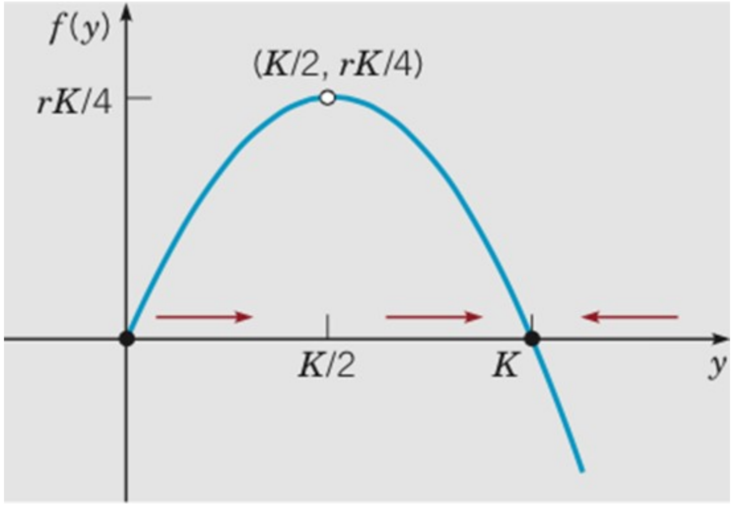
$$f\left(\frac{K}{2}\right) = r\left(1 - \frac{K}{2K}\right)\left(\frac{K}{2}\right) = \frac{rk}{4}$$



Logistic Solution: Increasing, Decreasing

- Note $\frac{dy}{dt} > 0$ for $0 < y < K$, so y is an increasing function of t there (indicate with right arrows along y-axis on $0 < y < K$).
- Similarly, y is a decreasing function of t for $y > K$ (indicate with left arrows along y-axis on $y > K$).
- In this context the y-axis is often called the **phase line**.

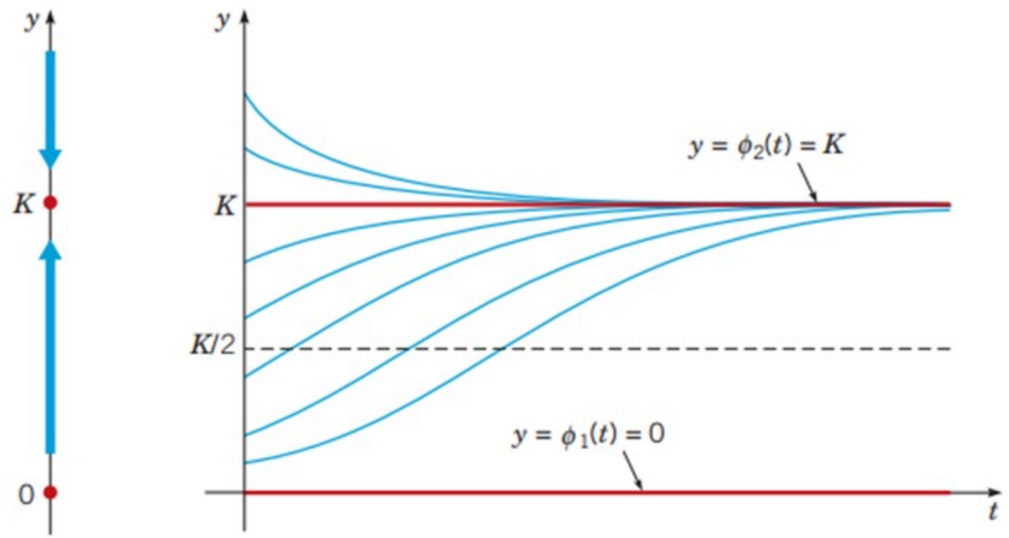
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r > 0$$



Logistic Solution: Steepness, Flatness

- Note $\frac{dy}{dt} \cong 0$ when $y \cong 0$ or $y \cong K$, so y is relatively flat there, and y gets steep as y moves away from 0 or K.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

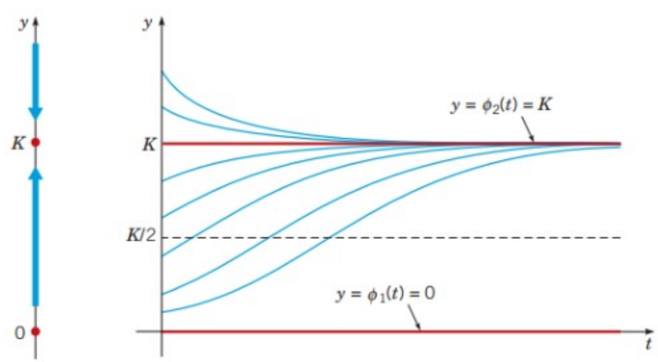


Logistic Solution: Concavity

- Examine the concavity of $y(t)$, using y'':

$$\frac{dy}{dt} = f(y) \Rightarrow \frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)f(y)$$

- The graph of y is concave up when f and f' have same sign, which occurs when $0 < y < \frac{K}{2}$ and $y > K$.
- The graph of y is concave down when f and f' have opposite signs, which occurs when $\frac{K}{2} < y < K$.
- The inflection point occurs at intersection of y and line $y = \frac{K}{2}$.



Logistic Solution: Curve Sketching

- Combining the information on the previous slides, we have:
  - The graph of y is increasing when $0 < y < K$.
  - The graph of y is decreasing when $y > K$.
  - The slope of y is approximately zero when $y \cong 0$ or $y \cong K$.
  - The graph of y is concave up when $0 < y < \frac{K}{2}$ and $y > K$.
  - The graph of y is concave down when $\frac{K}{2} < y < K$.
  - The inflection point is at $y = \frac{K}{2}$.
- Using this information, we can sketch solution curves y for different initial conditions.

Logistic Solution: Summary

- Using only the information present in the differential equation and without solving it, we obtained qualitative information about the solution y.
- For example, we know where the graph of y is the steepest, and hence where y changes most rapidly. Also, y tends asymptotically to the line $y = K$, for large t.
- The value of K is known as the **environmental carrying capacity**, or **saturation level**, for the species.
- Note how solution behavior differs from that of exponential equation, and thus the decisive effect of nonlinear term in logistic equation.

Solving the Logistic Equation - Integration of Separable ODE

- Provided $y \neq 0$ and $y \neq$ K, we can rewrite the logistic ODE:

$$\frac{dy}{(1 - y/K)y} = r\,dt$$

- Expanding the left side using partial fractions,

$$\frac{1}{(1 - y/K)y} = \frac{A}{1 - y/K} + \frac{B}{y} \Rightarrow 1 = Ay + B(1 - y/K) \Rightarrow B = 1, A = y/K$$

- Thus the logistic equation can be rewritten as

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right)dy = r\,dt$$

- Integrating the above result, we obtain

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

Solving the Logistic Equation: Explicit Solution Graph

- We have:

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

- If $0 < y_0 < K$, then $0 < y < K$ and hence

$$\ln y - \ln\left(1 - \frac{y}{K}\right) = rt + C$$

- Rewriting, using properties of logs and taking the exponential of both sides, we find that:

$$\frac{y}{1-(y/K)} = Ce^{rt}, \qquad y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \qquad \text{for } y(0) = y_0$$

Solution of the Logistic Equation: Effect of Initial Condition y_0

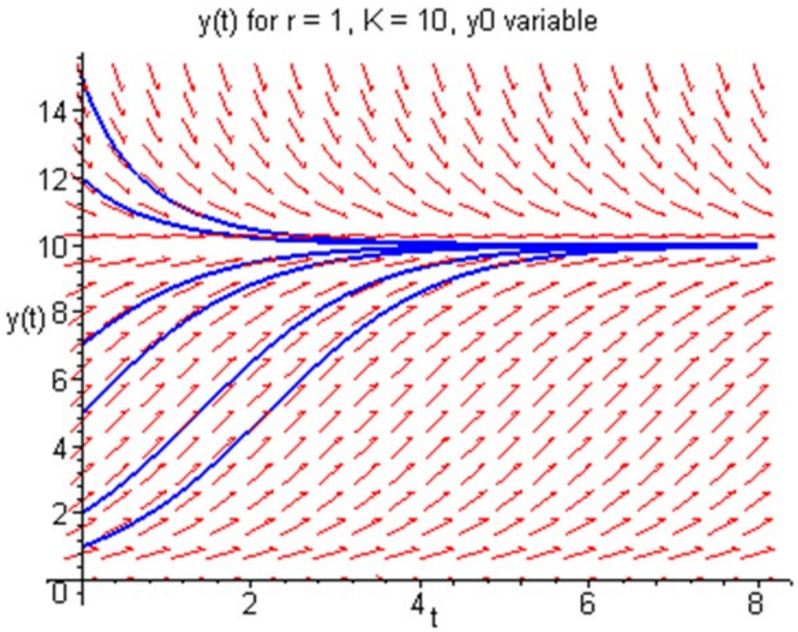
- We have: $y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

  for $0 < y_0 < K$.

- It can be shown that solution is also valid for $y_0 > K$. Also, this solution contains equilibrium solutions $y = 0$ and $y = K$.

- Hence the solution to this logistic equation is:

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$



Logistic Solution: Asymptotic Behavior

- The solution the to logistic ODE is

$$y = \frac{y_0 K}{y_0 + \left(K - y_0\right)e^{-rt}}$$

- We use limits to confirm the asymptotic behavior of the solution:

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{y_0 K}{y_0 + \left(K - y_0\right)e^{-rt}} = \lim_{t\to\infty} \frac{y_0 K}{y_0} = K$$

- Thus we can conclude that the equilibrium solution $y(t) = K$ is **asymptotically stable**, while equilibrium solution $y(t) = 0$ is **unstable**.
- The only way to guarantee that the solution remains near zero is to make $y_0 = 0$.

Example 2.5.1: Pacific Halibut - predicted biomass for a given time

Let y be the biomass (in kg) of halibut population at time t, with $r = 0.71$/year and $K = 80.5 \times 10^6$ kg. If $y_0 = 0.25K$, find

a) biomass 2 years later

b) the time τ such that $y(t) = 0.75$K.

For convenience, scale equation:

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + \left[1 - y_0/K\right]e^{-rt}}$$

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75e^{-(0.71)(2)}} \cong 0.5797$$

and hence: $y(2) \approx 0.5797K \approx 46.7 \times 10^6$ kg

Example 2.5.1: Pacific Halibut - time to reach 75% of carrying capacity

b) Find time τ, the time when for which $y(\tau) = 0.75K$.

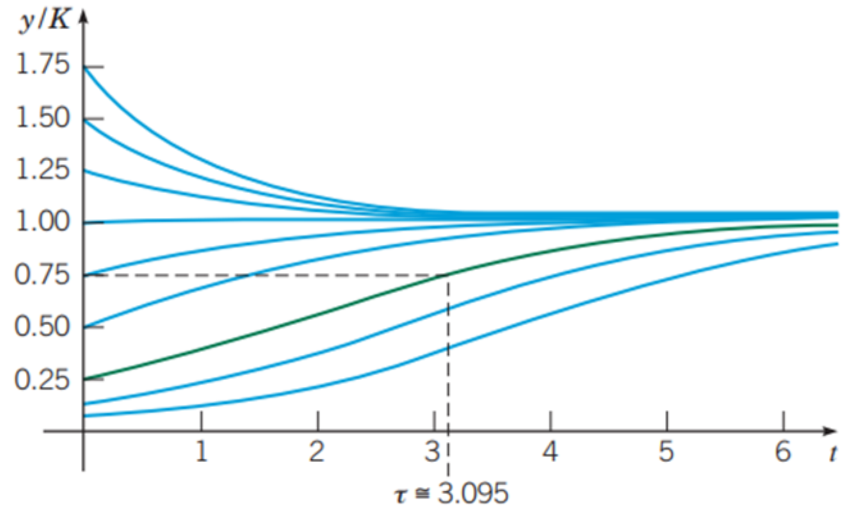
$$e^{-rt} = \frac{(y_0/K)(1-y/K)}{(y/K)(1-y_0/K)}$$, hence:

$$t = -\frac{1}{r}\ln\left(\frac{(y_0/K)(1-y/K)}{(y/K)(1-y_0/K)}\right)$$

Use given values of r and y_0/K:

$$\tau = -\frac{1}{0.71}\ln\frac{(0.25)(0.25)}{(0.75)(0.75)}$$

$$= \frac{1}{0.71}\ln 9 \approx 3.095 \text{ years}$$



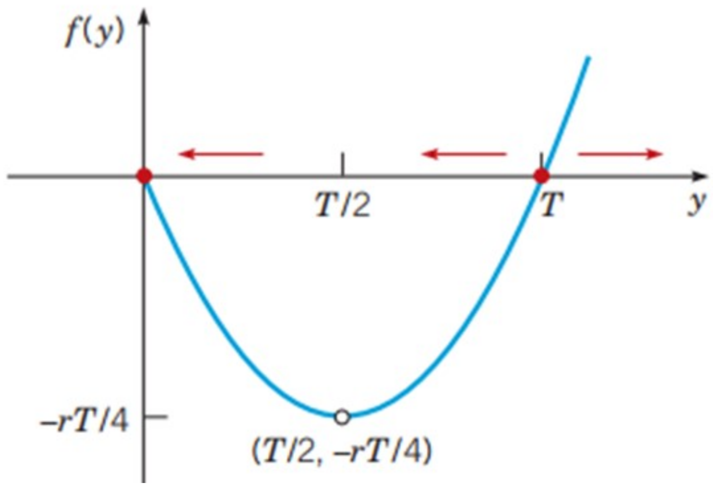
round to 3.1 years consistent with input data significant figures

Critical Threshold Equation

- Consider the following modification of the logistic Ordinary Differential Equation:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0$$

- The graph of the right hand side $f(y)$ is given below.



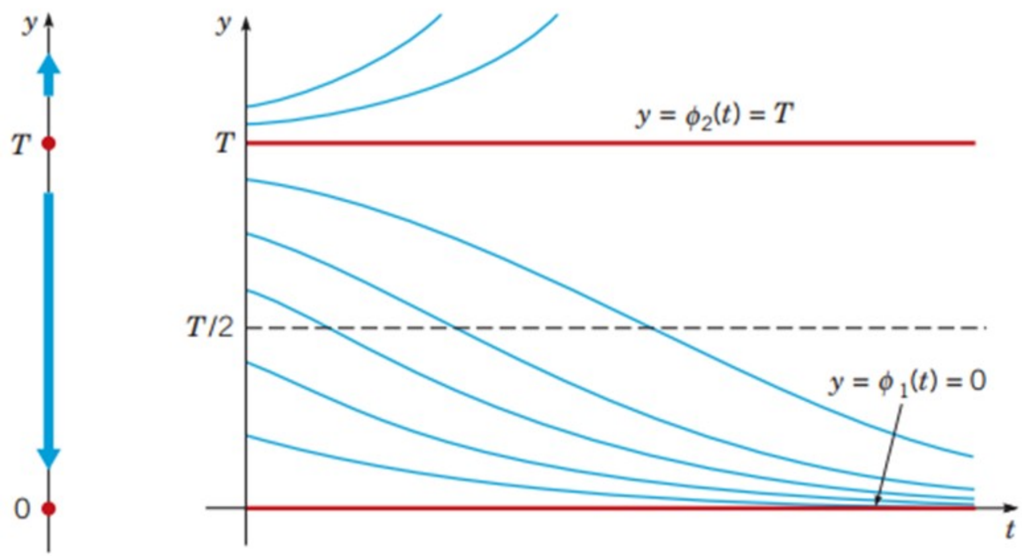
Critical Threshold Equation: Qualitative Analysis and Solution

- Performing an analysis similar to that of the logistic case, we obtain a graph of solution curves shown below.
- T is a **threshold level** for y_0, in that the population dies off or grows unbounded, depending on which side of T the initial value y_0 is located.
- Laminar fluid flow experiences similar threshold behavior (see text discussion).

Critical Threshold Equation: Graphical Solution

- It can be shown that the solution to the threshold equation

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y, \quad r > 0 \quad \textbf{which equals} \quad y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

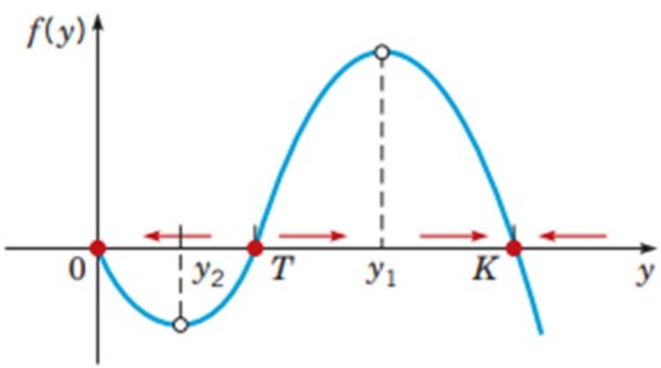


Logistic Growth with a Threshold

- In order to avoid unbounded growth for $y > T$ as in previous setting, consider the following modification of the logistic equation:

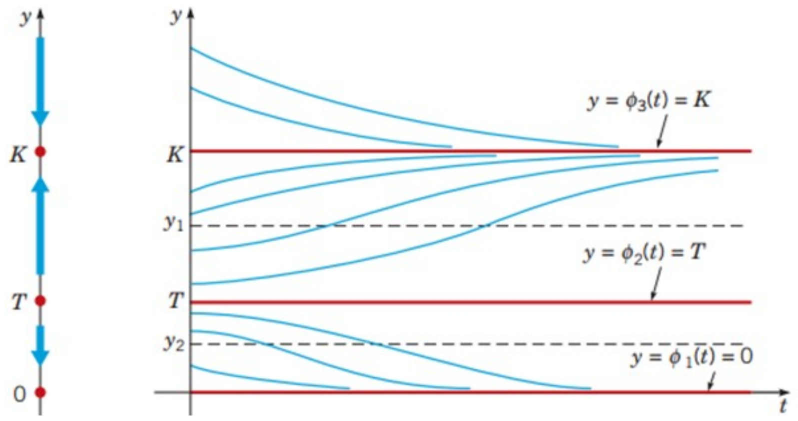
$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y, \quad r > 0 \text{ and } 0 < T < K$$

- The graph of the right hand side $f(y)$ is given below.



Logistic Growth with a Threshold: time-dependent trends

- An analysis similar to that of the logistic case, gives a graph of solution curves shown below right.
- T is the threshold value for y_0, in that the population dies off or grows towards K, depending on which side of T y_0 is.
- K is the carrying capacity level.
- $y = 0$ and $y = K$ are stable equilibrium solutions, and $y = T$ is an unstable equilibrium solution.



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