Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

Chapter 3

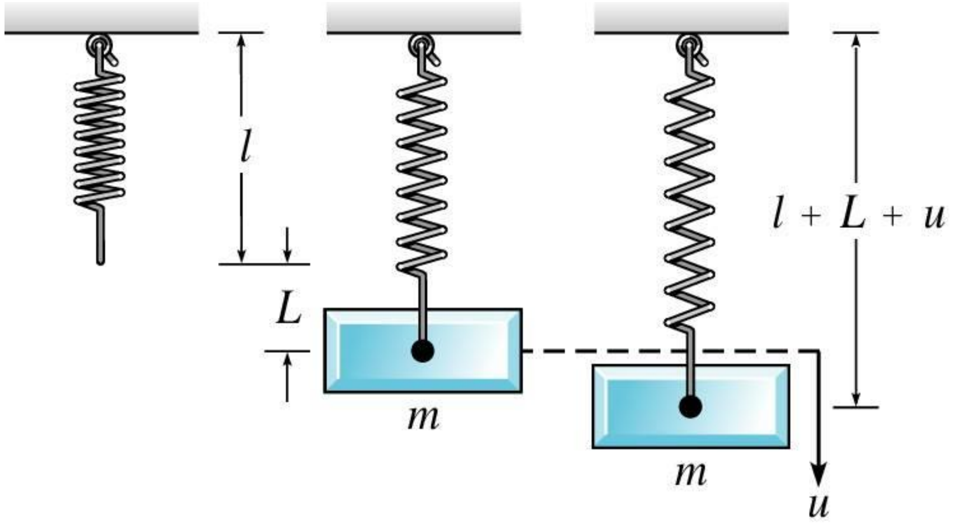
Second Order Linear Differential Equations

Section 3.8 Forced Periodic Vibrations

Forced Periodic Vibrations

- We continue the discussion of section 3.7, adding the presence of a periodic external force:

$$mu''(t)+\gamma u'(t)+ku(t)=F(t)$$



Forced Vibrations with Damping

- Consider the equation below for damped motion and external forcing function $F_0 \cos(\omega t)$

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

- The general solution of this equation has the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + A\cos(\omega t) + B\sin(\omega t) = u_C(t) + U(t)$$

where the general solution of the homogeneous equation is

$$u_C(t) = c_1 u_1(t) + c_2 u_2(t)$$

and the particular solution of the nonhomogeneous equation is

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

Homogeneous Solution

- The homogeneous solutions u_1 and u_2 depend on the roots r_1 and r_2 of the characteristic equation:

$$mr^2 + \gamma r + kr = 0 \;\Rightarrow\; r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- Since m, γ, and k are all positive constants, it follows that r_1 and r_2 are either real and negative, or complex conjugates with negative real part. In the first case,

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0,$$

while in the second case

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t\right) = 0.$$

- Thus in either case,

$$\lim_{t\to\infty} u_C(t) = 0$$

Transient and Steady-State Solutions

- Thus for the following equation and its general solution,

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_C(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)},$$

we have
$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 u_1(t) + c_2 u_2(t)\right) = 0$$

- Thus $u_C(t)$ is called the **transient solution**. Note however that

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

is a steady oscillation with same frequency as forcing function.

- For this reason, $U(t)$ is called the **steady-state solution**, or **forced response**.

Transient Solution and Initial Conditions

- For the following equation and its general solution,

$$u = c_1 u_1(t) + c_2 u_2(t) + A\cos(\omega t) + B\sin(\omega t) = u_c(t) + U(t)$$

  the transient solution $u_C(t)$ enables us to satisfy whatever initial conditions might be imposed.

- With increasing time, the energy put into system by initial displacement and velocity is dissipated through damping force. The motion then becomes the response $U(t)$ of the system to the external force $F_0 \cos(\omega t)$.

- Without damping, the effect of the initial conditions would persist for all time.

Example 3.8.1 (part one)

Consider a spring-mass system satisfying the differential equation and initial condition

$$u'' + u' + \frac{5}{4}u = 3\cos t, \;\; u(0) = 2, \; u'(0) = 3$$

Find the solution of this initial value problem and describe the behavior of the solution for large *t*.

- The homogeneous equation is given by:

$$u_C(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t$$

- A particular solution to the nonhomogeneous equation will have the form $U(t)$ = A cos t + B sin t and substitution gives $A = 12/17$ and $B = 48/17$.

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

Example 3.8.1 (part two)

- The general solution for the nonhomogeneous equation is

$$u = u_c(t) + U(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

- Applying the initial conditions yields

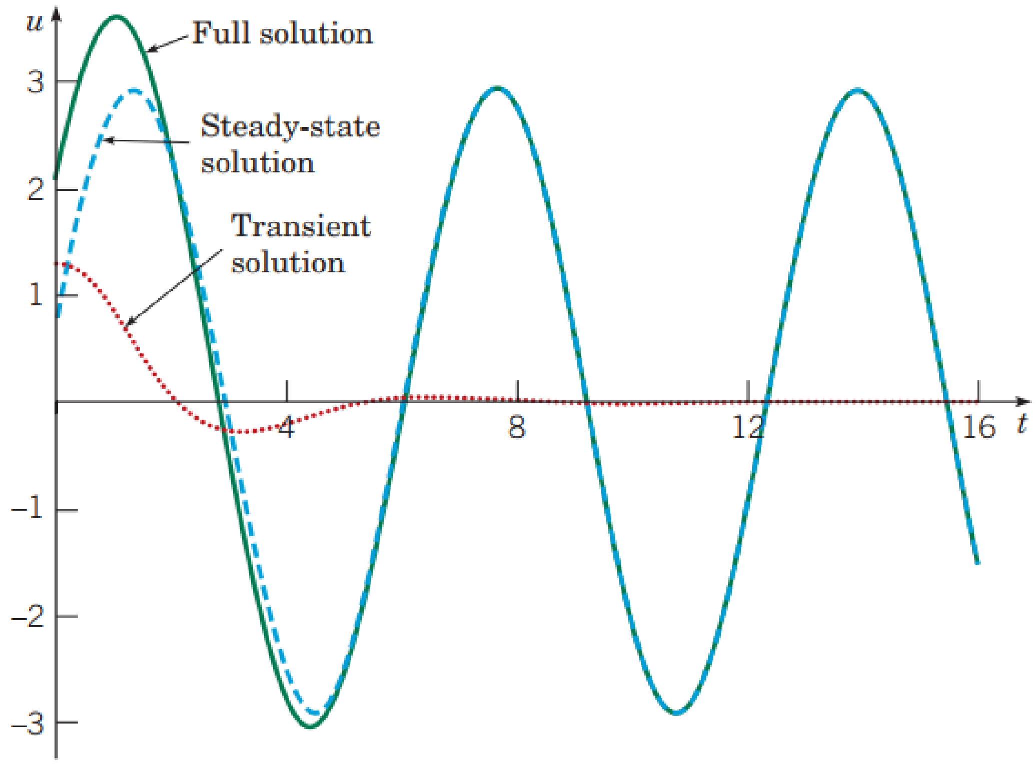
$$\left.\begin{aligned} u(0) &= c_1 + \frac{12}{17} = 2 \\ u'(0) &= -\frac{1}{2}c_1 + c_2 + \frac{48}{17} = 3 \end{aligned}\right\} \Rightarrow c_1 = \frac{22}{17},\ c_2 = \frac{14}{17}$$

- Therefore, the solution to the IVP is

$$u = \frac{22}{17}e^{-t/2}\cos t + \frac{14}{17}e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

Example 3.8.1 (part three)

The graph breaks the solution into its steady state ($U(t)$) and transient ($u_c(t)$)) components:



Rewriting the Forced Response

- Using trigonometric identities, it can be shown that the steady state response:

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

can be rewritten as $U(t) = R\cos(\omega t - \delta)$

- It can also be shown that amplitude and phase can be written as:

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \sin\delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

where $\omega_0^2 = \dfrac{k}{m}$

Forced Response Amplitude vs. Frequency

- The amplitude R of the steady state solution

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

depends on the driving frequency ω. For low-frequency excitation we have

$$\lim_{\omega \to 0} R = \lim_{\omega \to 0} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

where we recall $(\omega_0)^2 = k/m$. Note that F_0/k is the static displacement of the spring produced by force F_0.

- For high frequency excitation,

$$\lim_{\omega \to \infty} R = \lim_{\omega \to \infty} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = 0$$

Maximum Amplitude of Forced Response

- Summary of frequency effects on amplitude:

$$\lim_{\omega \to 0} R = F_0/k, \quad \lim_{\omega \to \infty} R = 0$$

- At an intermediate value of ω, the amplitude R may have a maximum value. To find this frequency ω, differentiate R and set the result equal to zero. Solving for $\omega_{\max}$, we obtain

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk}\right)$$

where $(\omega_0)^2 = k/m$. Note $\omega_{\max} < \omega_0$, and $\omega_{\max}$ is close to ω_0 for small γ. The maximum value of R is

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - \left(\gamma^2/4mk\right)}} \cong \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$$

approximation valid for small γ

Maximum Amplitude for Imaginary $\omega_{\max}$

- We have

$$\omega_{\max}^2 = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right)$$

and
$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$$

where the last expression is an approximation for small γ.

- If $\gamma^2/(mk) > 2$, then $\omega_{\max}$ is imaginary.
- In this case the maximum value of R occurs for $\omega = 0$, and R is a monotone decreasing function of ω.
- Recall that critical damping occurs when $\dfrac{\gamma^2}{mk} = 4$.

Resonance

- From the expression

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

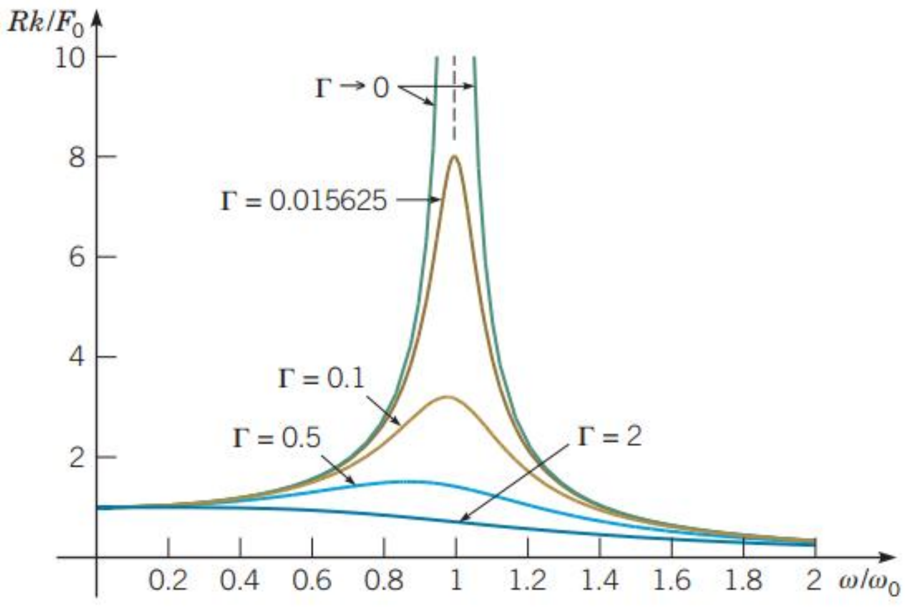
we see that $R_{\max} \cong \dfrac{F_0}{\gamma\omega_0}$ for small γ.

- Thus for lightly damped systems, the amplitude R of the forced response is large for ω near ω_0.
- This is true even for relatively small external forces, and the smaller the γ the greater the effect.
- This phenomenon is known as **resonance**. Resonance can be either good or bad, depending on circumstances. It must be taken into account when designing building structures.

Graphical Analysis of Quantities

To get a better understanding of the quantities we have been examining, we graph the ratios $\dfrac{Rk}{F_0}$ versus $\dfrac{\omega}{\omega_0}$ for several values of. $\Gamma = \dfrac{\gamma^2}{mk}$.

- Note that the peaks tend to get higher as damping decreases.
- As damping decreases to zero, the values of Rk/F_0 become asymptotic to $\omega = \omega_0$.
- The graph corresponding to $\Gamma = 0.015625$ is included because it appears in the next example.



Analysis of Phase Angle

- Recall that the phase angle δ given in the forced response

$$U(t) = R\cos(\omega t - \delta)$$

is characterized by the equations

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \ \sin\delta = \frac{\gamma\,\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

- For ω near zero, $\cos\delta \cong 1$ and $\sin\delta \cong 0$ and they rise and fall together.
- For $\omega = \omega_0$, $\cos\delta = 0$ and $\sin\delta = 1$ so $\delta = \dfrac{\pi}{2}$ and response lags behind the excitation.
- For very large ω, $\delta \cong \pi$, and the response is out of phase. That is the response is a minimum when excitation is a maximum.

Example 3.8.2: Forced Vibrations with Damping (part one)

- Consider the initial value problem

$$u'' + \frac{1}{8}u' + u = 3\cos(\omega t), \qquad u(0) = 2, \quad u'(0) = 0$$

  Show plots of the solution for different values of the forcing frequency ω, and compare them with corresponding plots of the forcing function.

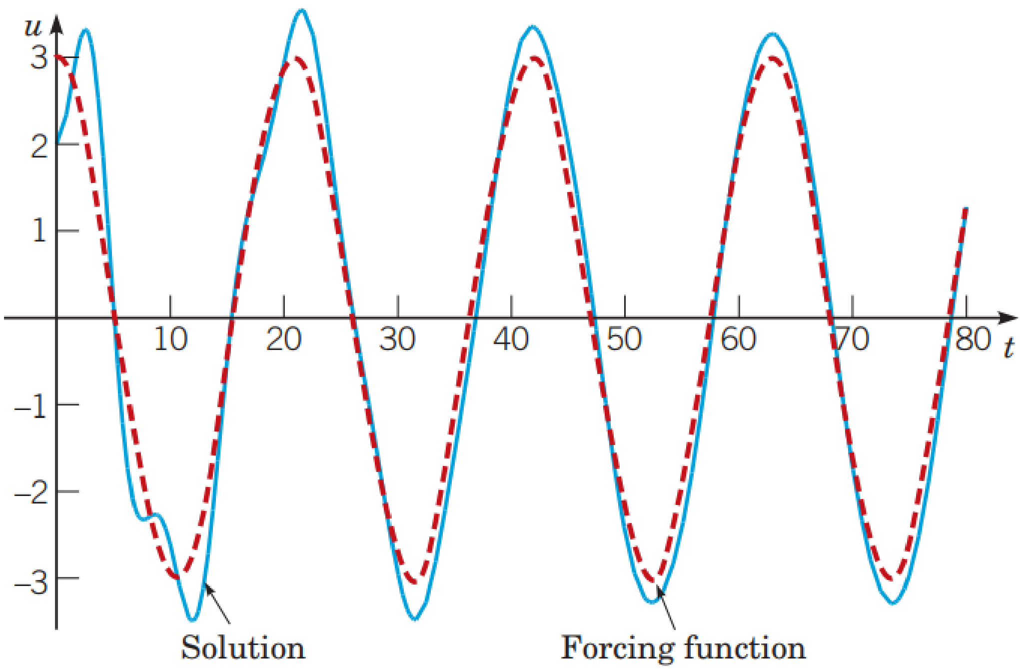
- For this systems $\omega_0 = 1$, $F_0/k = 3$, and $\Gamma = 1/64 = 0.015625$

- The unforced motion of this system was discussed in Ch 3.7, with the graph of the solution on the next slide, along with the graph of the ratios Rk/F vs. ω/ω_0 for different values of ω.

Example 3.8.2: Forced Vibrations with Damping (part two)

Graphs of the solution (solid blue), along with the external forcing function $F(t) = 3\cos(0.3t)$ (dashed red).

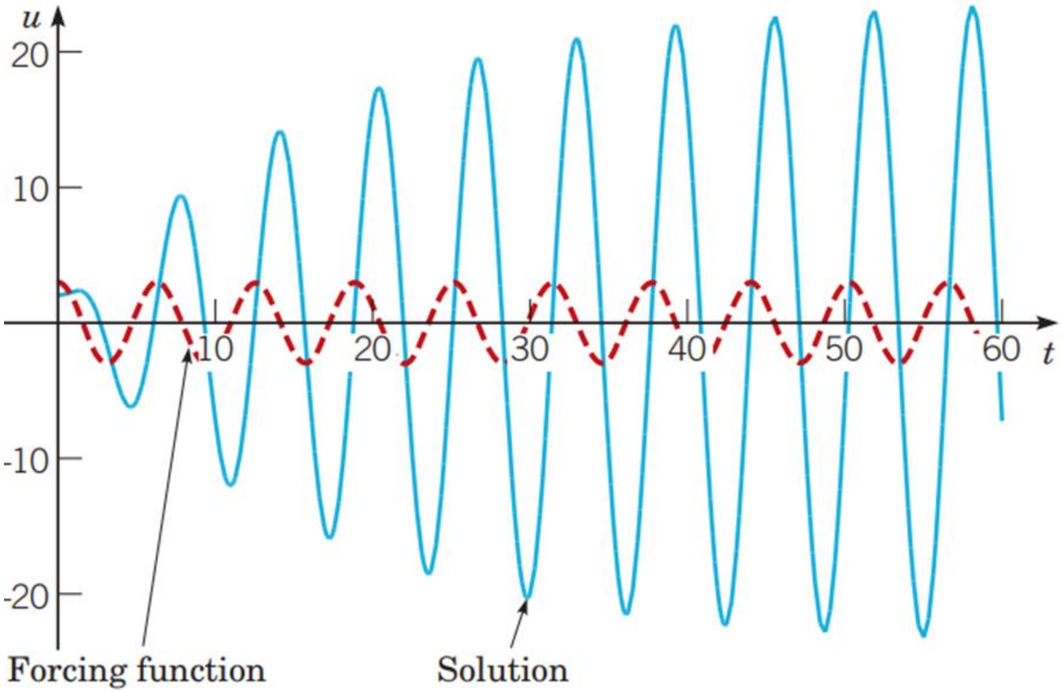
$\omega = 0.3$



Example 3.8.2: Forced Vibrations with Damping (part three)

Graphs of the solution (solid blue), along with the external forcing function $F(t) = 3\cos(t)$ (dashed red).

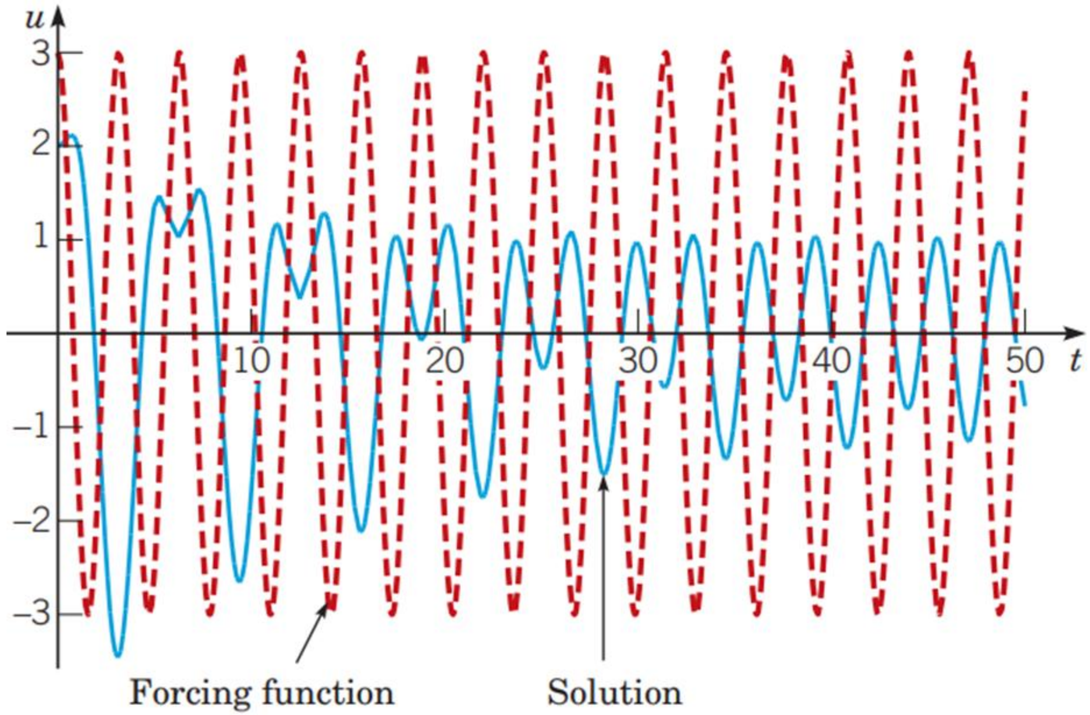
$\omega = 1.0$



Example 3.8.2: Forced Vibrations with Damping (part four)

Graphs of the solution (solid blue), along with the external forcing function $F(t) = 3\cos(2t)$ (dashed red).

$\omega = 2.0$



Undamped Equation: General Solution for the Case $\gamma = 0$

- Suppose there is no damping term. Then our equation is

$$mu''(t) + ku(t) = F_0 \cos(\omega t)$$

- Assuming $\omega \neq \omega_0$, the method of undetermined coefficients can be used to show that the general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Undamped Equation: Mass Initially at Rest

- If the mass is initially at rest, then the corresponding initial value problem is

$$mu''(t) + ku(t) = F_0 \cos \omega t, \;\; u(0) = 0, \; u'(0) = 0$$

- Recall that the general solution to the differential equation is

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

- Using the initial conditions to solve for c_1 and c_2, we obtain

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0$$

- Hence

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t) \right)$$

Undamped Equation: Solution to Initial Value Problem

- To simplify the solution even further, let $A = \frac{1}{2}(\omega_0 + \omega)t$ and $B = \frac{1}{2}(\omega_0 - \omega)t.$

  Then $A + B = \omega_0 t$ and $A - B = \omega t.$ Using the trigonometric identity

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

it follows that

$$\cos \omega t = \cos A \cos B + \sin A \sin B$$
$$\cos \omega_0 t = \cos A \cos B - \sin A \sin B$$

and hence

$$\cos \omega t - \cos \omega_0 t = 2 \sin A \sin B$$

Undamped Equation: Beats

- Using the results of the previous slide, the amplitude of motion can be expressed as:

$$\frac{2F_0}{m\left|\omega_0^2-\omega^2\right|}\left|\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)\right|$$

When $\left|\omega_0-\omega\right|\cong 0$, then $\omega_0+\omega$ is much greater than $|\omega_0-\omega|$, and $\sin\left(\frac{1}{2}(\omega_0+\omega)t\right)$ is oscillating more rapidly than $\sin\left(\frac{1}{2}(\omega_0-\omega)t\right)$

- Thus motion is a rapid oscillation with frequency $\frac{\omega_0+\omega}{2}$, but with slowly varying sinusoidal amplitude given by

$$\frac{2F_0}{m\left|\omega_0^2-\omega^2\right|}\left|\sin\frac{(\omega_0-\omega)t}{2}\right|$$

- This phenomenon is called a **beat**.

Example 3.8.3: Undamped, Mass Initially at Rest

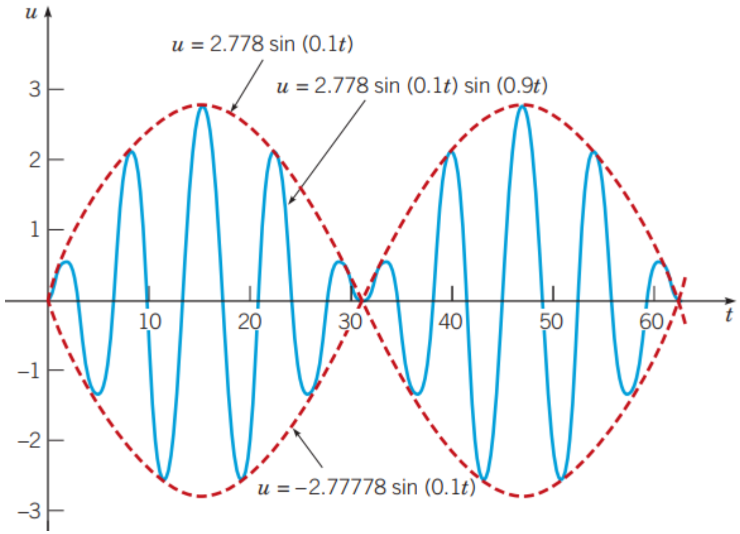
- Solve the initial value problem

$$u'' + u = \frac{1}{2}\cos(0.8t), \quad u(0) = 0, \quad u'(0) = 0,$$

- In this case, $\omega_0 = 1,\ \omega = 0.8,$ and $F_0 = \dfrac{1}{2}$, and hence the solution is

$$u = 2.778 \sin(0.1t) \sin(0.9t)$$

- The spring–mass system displacement oscillates with a frequency of 0.9, slightly less than the natural frequency $\omega_0 = 1$.
- The amplitude variation has a slow frequency of 0.1 and period of 20 π.
- A half-period of 10 π corresponds to a single cycle of increasing and then decreasing amplitude.



Undamped Equation: General Solution for the Case $\omega_0 = \omega$

- Recall our equation for the undamped case:

$$mu''(t) + ku(t) = F_0 \cos \omega t$$

- If the forcing frequency equals the natural frequency of the system, then the nonhomogeneous term $F_0 \cos \omega t$ is the solution of homogeneous equation. In this case, the solution is:

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

Example 3.8.4 (part one)

- Solve the initial value problem

$$u'' + u = \frac{1}{2}\cos t,\ \ u(0) = 0,\ u'(0) = 0$$

And plot the graph of the solution.

The general solution of the differential equation is

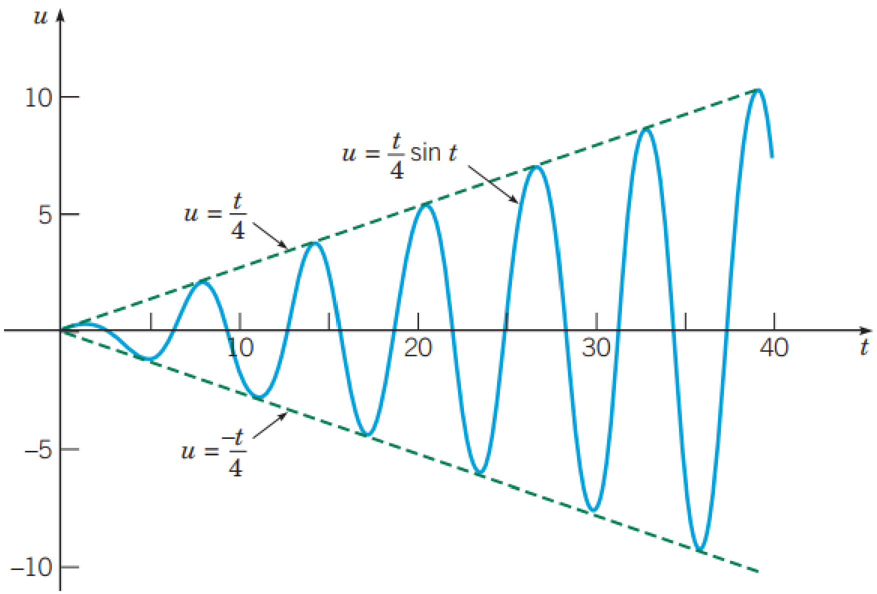
$$u = c_1 \cos t + c_2 \sin t + \frac{1}{4} t \sin t$$

And the initial conditions require that $c_1 = c_2 = 0$. Thus the solution of the given initial value problem is

$$u = \frac{t}{4}\sin t$$

Example 3.8.4 (part two)

A graph of the solution is shown below:



In general, motion u remains bounded if damping is present. However, the response of u to input function $F_0 \cos \omega t$ may be large if damping is small and $\omega \cong \omega_0$ in which case we have resonance.

Copyright