Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 2

First-Order Differential Equations

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Section 2.4
Differences Between Linear and Nonlinear Differential Equations

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Linear vs. Nonlinear Ordinary Differential Equations

- Recall that a first-order ODE has the form $y' = f(t, y)$, and is linear if f is linear in y, and nonlinear if f is nonlinear in y.
- Examples: $y' = ty - e^t \left(\text{linear}\right), y' = ty^2 \left(\text{non}-\text{linear}\right)$
- In this section, we will see that first-order linear and nonlinear equations differ in a number of ways, including:
  - The theory describing existence and uniqueness of solutions, and corresponding domains, are different.
  - Solutions to linear equations can be expressed in terms of a general solution, which is not usually the case for nonlinear equations.
  - Linear equations have explicitly defined solutions while nonlinear equations typically do not, and nonlinear equations may or may not have implicitly defined solutions.
- For both types of equations, numerical and graphical construction of solutions are important.

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Existence and Uniqueness Theorem for First-Order Linear Equations

- Consider the linear first order initial value problem:

$$y' + p(t)y = g(t),\ y(0) = y_0$$

If the functions p and g are continuous on an open interval containing the point $t = t_0$, then there exists a unique function that satisfies the initial value problem for each t in I.

- **Proof outline:** Use Ch 2.1 discussion and results:

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s) g(s) ds + y_0\right), \quad \mu(t) = \exp \int_{t_0}^{t} p(s) ds$$

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Existence and Uniqueness Theorem for First-Order Nonlinear Equations

- Consider the nonlinear first order initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

- Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ in the rectangle there is a unique solution $y = \phi(t)$ of the initial value problem.
- **Proof discussion:** Since there is no general formula for the solution of arbitrary nonlinear first order initial value problems, this proof is difficult, and beyond the scope of this course.
- The conditions stated in the theorem are sufficient but not necessary to guarantee existence of a solution, and continuity of f ensures existence but not uniqueness of $y = \phi(t)$

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Example 2.4.1: Linear Initial Value Problems

Use Theorem 2.4.1 to find an interval in which the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

has a unique solution. Next, do the same when the initial condition is changed to $y(-1) = 2$.

- Rewrite the original equation in standard form:

$$y' + \left(\frac{2}{t}\right)y = 4t \text{ where } p(t) = 2/t \text{ and } g(t) = 4t$$

- The solution to this initial value problem is defined for $t > 0$, the interval on which $p(t) = 2/t$ is continuous.

- If the initial condition is $y(-1) = 2$, then the solution is given by same expression as above, but is defined on $t < 0$.

- In either case, the Existence and Uniqueness Theorem guarantees that solution is unique on corresponding interval.

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Example 2.4.2: Nonlinear Initial Value Problem

Apply Theorem 2.4.2 to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

- The functions f and $\partial f / \partial y$ are given by

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

  and are continuous except on line $y = 1$.
- Thus we can draw an open rectangle about $(0, -1)$ in which f and $\partial f / \partial y$ are continuous, as long as it doesn't cover $y = 1$.
- How wide is the rectangle? Recall the solution defined for $x > -2$, with

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

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Example 2.4.2: Change Initial Condition

- Our nonlinear initial value problem is

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

with

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

which are continuous except on line $y = 1$.

- If we change the initial condition to $y(0) = 1$, then the Existence and Uniqueness Theorem is not satisfied. Solving this new initial value problem, we obtain:

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}, \quad x > 0$$

- Thus a solution exists but is not unique.

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Example 2.4.3: Nonlinear Initial Value Problem

Apply Theorem 2.4.2 to the following initial value problem and then solve the problem.

$$y' = y^{1/3}, y(0) = 0 \quad (t \geq 0)$$

- The functions f and $\partial f / \partial y$ are given by

$$f(t, y) = y^{1/3}, \frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-2/3}$$

- Thus f is continuous everywhere, but $\partial f / \partial y$ doesn't exist at $y = 0$, and hence the Existence and Uniqueness Theorem does not apply.
- Solutions exist, but are not unique.

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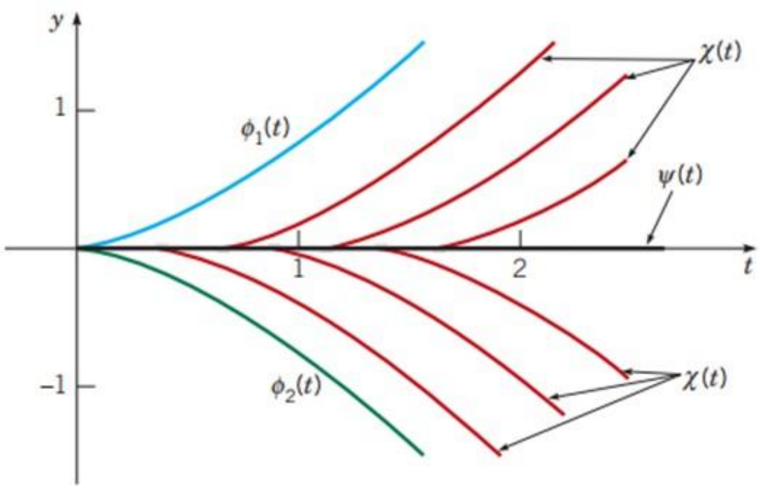
Example 2.4.3: Multiple Solutions

Separating variables, solving, and including the initial conditions, we obtain not one, but two solutions:

$$y^{-1/3}\,dy = dt \Rightarrow \frac{3}{2}y^{2/3} = t + c \Rightarrow y = \pm\left(\frac{2}{3}t\right)^{3/2}, \quad t \geq 0$$



The solution family for different values of t_0

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Example 2.4.4: Nonlinear Initial Value Problem

Solve the initial value problem:

$$y' = y^2, y(0) = 1$$

- The functions f and $\partial f / \partial y$ are given by

$$f(t, y) = y^2, \frac{\partial f}{\partial y}(t, y) = 2y$$

- Thus f and $\partial f / \partial y$ are continuous at $t = 0$, so the Existence and Uniqueness Theorem guarantees that solutions exist and are unique.

- Separating variables and solving, we obtain

$$y^{-2} dy = dt \Rightarrow -y^{-1} = t + c \Rightarrow y = -\frac{1}{t+c} \Rightarrow y = \frac{1}{1-t}$$

- The solution $y(t)$ is defined on $(-\infty, 1)$. Note that the singularity at $t = 1$ is not obvious from original IVP statement.

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Interval of Existence: Linear Equations

- Per the Existence and Uniqueness Theorem, the solution of a linear initial value problem $y' + p(t)y = g(t),\ y(t_0) = y_0$ exists throughout any interval about $t = t_0$ on which p and g are continuous.
- Vertical asymptotes or other discontinuities of a solution can only occur at points of discontinuity of p or g.
- However, a solution may be differentiable at points of discontinuity of p or g. See Chapter 2.1: Example 3 of text.
- Compare these comments with Example 1 and with previous linear equations in Chapter 1 and Chapter 2.

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Interval of Existence: Nonlinear Equations

- In the nonlinear case, the interval on which a solution exists may be difficult to determine.
- The solution $y = \phi(t)$ exists as long as $[t, \phi(t)]$ remains within a rectangular region indicated in Theorem 2.4.2. This is what determines the value of h in that theorem. Since $\phi(t)$ is usually not known, it may be impossible to determine this region.
- In any case, the interval on which a solution exists may have no simple relationship to the function f in the differential equation $y' = f(t, y)$, in contrast with linear equations.
- Furthermore, any singularities in the solution may depend on the initial condition as well as the equation.
- Compare these comments to the preceding examples.

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General Solutions

- For a first order linear equation, it is possible to obtain a solution containing one arbitrary constant, from which all solutions follow by specifying values for this constant.
- For nonlinear equations, such general solutions may not exist. That is, even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by specifying values for this constant.
- Consider $y' = y^2$ in example 4 where the solution $y = -\dfrac{1}{t+c}$

  contains an arbitrary constant, but does not include all solutions to the differential equation since the function $y = 0$ is a solution of the differential equation which cannot be obtained by specifying a value for c in solution above.

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Explicit Solutions: Linear Equations

- Per the Existence and Uniqueness Theorem, a solution of a linear initial value problem

$$y' + p(t)y = g(t),\ y(t_0) = y_0$$

exists throughout any interval about $t = t_0$ on which p and g are continuous, and this solution is unique.

- The solution has an explicit representation,

$$y = \frac{\int_{t_0}^{t} \mu(s) g(s)\, ds + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s) ds},$$

and can be evaluated at any appropriate value of t, as long as the necessary integrals can be computed.

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Explicit Solution Approximation

- For linear first order equations, an explicit representation for the solution can be found, as long as necessary integrals can be solved.
- If integrals can't be solved, then numerical methods are often used to approximate the integrals.

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Implicit Solutions: Nonlinear Equations

- For nonlinear equations, explicit representations of solutions may not exist.
- As we have seen, it may be possible to obtain an equation which implicitly defines the solution. If equation is simple enough, an explicit representation can sometimes be found.
- Otherwise, numerical calculations are necessary in order to determine values of y for given values of t. These values can then be plotted in a sketch of the integral curve.
- Recall the examples from earlier in the chapter and consider the following example

$$y' = \frac{y\cos x}{1+3y^3},\ y(0)=1 \quad \Rightarrow \quad \ln y + y^3 = \sin x + 1$$

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