Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

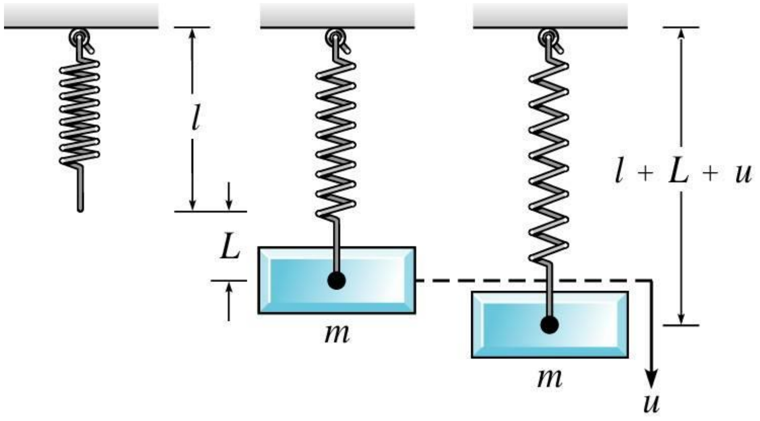
Chapter 3

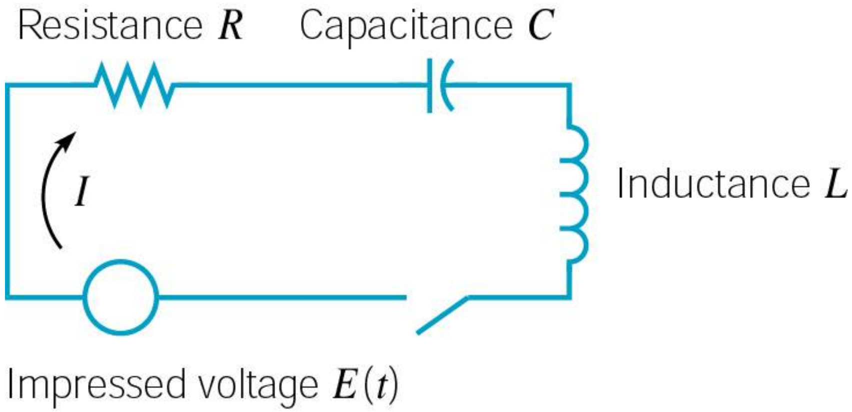
Second Order Linear Differential Equations

Section 3.7 Mechanical and Electrical Vibrations

Mechanical & Electrical Vibrations

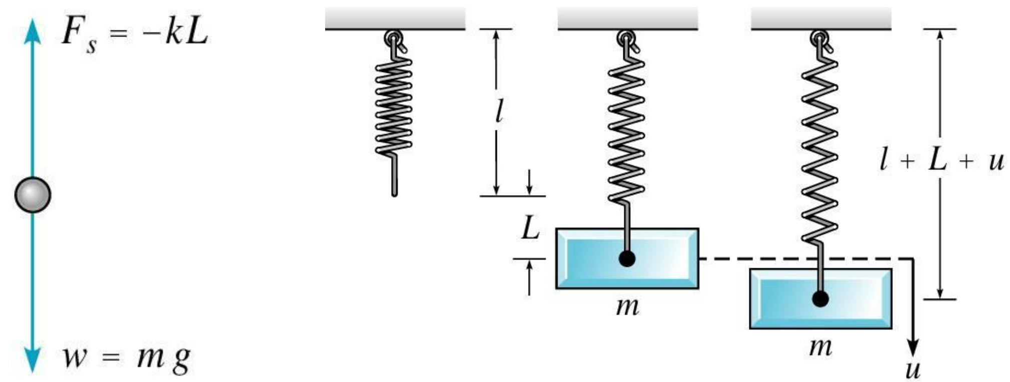
- Two important areas of application for second order linear equations with constant coefficients are in modeling mechanical and electrical oscillations.
- We will study the motion of a mass on a spring in detail.
- An understanding of the behavior of this simple system is the first step in investigation of more complex vibrating systems.





Spring – Mass System

- Suppose a mass m hangs from a vertical spring of original length l. The mass causes an elongation L of the spring.
- The force F_G of gravity pulls the mass down. This force has magnitude mg, where g is acceleration due to gravity.
- The force F_S of the spring stiffness pulls the mass up. For small elongations L, this force is proportional to L. This proportional relationship is known as Hooke's Law: $F_S = kL$
- When the mass is in equilibrium, the forces balance each other: $mg = kL$



Spring Model

- We will study the motion of a mass when it is acted on by an external force (forcing function) and/or is initially displaced.
- Let $u(t)$ denote the displacement of the mass from its equilibrium position at time t, measured downward.
- Let f be the net force acting on the mass. We will use Newton's 2nd Law: $mu''(t) = f(t)$ where $f(t)$ is the net force acting on mass m.
- In determining f, there are four separate forces to consider:
  - Weight: $w = mg$ (downward force)
  - Spring force: $F_s = -k(L + u)$ (up or down force, see next slide)
  - Damping force: $F_d(t) = -\gamma u'(t)$ (up or down, see following slide)
  - External force: $F(t)$ (up or down force, see text)

Spring Model: Spring Force Details

- The spring force F_s acts to restore a spring to the natural position, and is proportional to $L + u$. If $L + u > 0$, then the spring is extended and the spring force acts upward. In this case

$$F_s = -k(L+u)$$

- If $L + u < 0$, then spring is compressed a distance of $|L + u|$, and the spring force acts downward. In this case

$$F_s = k|L+u| = k[-(L+u)] = -k(L+u)$$

- In either case,

$$F_s = -k(L+u)$$

Spring Model: Damping Force Details

- The damping or resistive force F_d acts in the opposite direction as the motion of the mass. This can be complicated to model. F_d may be due to air resistance, internal energy dissipation due to action of spring, friction between the mass and guides, or a mechanical device (dashpot) imparting a resistive force to the mass.
- We simplify this and assume F_d is proportional to the velocity.
- In particular, we find that
  - If $u' > 0$, then u is increasing, so the mass is moving downward. Thus F_d acts upward and hence $F_d(t) = -\gamma u'(t)$.
  - If $u' < 0$, then u is decreasing, so the mass is moving upward. Thus F_d acts downward and hence $F_d(t) = -\gamma u'(t)$
  - In either case,

$$F_d(t) = -\gamma u'(t), \quad \gamma > 0$$

Spring Model: Differential Equation

- Taking into account these forces, Newton's Law becomes:

$$\begin{aligned} mu''(t) &= mg + F_s(t) + F_d(t) + F(t) \\ &= mg - k\left[L + u(t)\right] - \gamma u'(t) + F(t) \end{aligned}$$

- Recalling that $mg = kL$, this equation reduces to

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

  where the constants m, γ, and k are positive.
- We can prescribe initial conditions $u(0) = u_0$, $u'(0) = v_0$
- It follows from Theorem 3.2.1 that there is a unique solution to this initial value problem. Physically, if the mass is set in motion with a given initial displacement and velocity, then its position is uniquely determined at all future times.

Example 3.7.1: Identify Coefficients

- A 4 lb mass stretches a spring 2 in. The mass is displaced an additional 6 in and then released, and is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Formulate the IVP that governs the motion of this mass:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t), \;\; u(0) = u_0, \;\; u'(0) = v_0$$

- Find m: $w = mg \Rightarrow m = \dfrac{w}{g} \Rightarrow m = \dfrac{4\,\text{lb}}{32\,\text{ft/s}^2} \Rightarrow m = \dfrac{1}{8}\,\dfrac{\text{lb}\cdot\text{s}^2}{\text{ft}}$
- Find γ: $\gamma u' = 6\,\text{lb} \Rightarrow \gamma = \dfrac{6\,\text{lb}}{3\,\text{ft}/s} \Rightarrow \gamma = 2\,\dfrac{\text{lb}\cdot\text{s}}{\text{ft}}$
- Find k: $F_S = -kL \Rightarrow k = \dfrac{4\,\text{lb}}{2\,\text{in}} \Rightarrow k = 24\,\dfrac{\text{lb}}{\text{ft}}$

Example 3.7.1: Express the IVP

- Thus our differential equation becomes

$$\frac{1}{8}u''(t) + 2u'(t) + 24u(t) = 0$$

and hence the initial value problem can be written as

$$u''(t) + 16u'(t) + 192u(t) = 0$$
$$u(0) = \frac{1}{2}, \quad u'(0) = 0$$

- This problem can be solved using the methods of Chapter 3.3 and yields the solution

Spring Model: Undamped Free Vibrations (part one)

- Recall our differential equation for spring motion:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

- Suppose there is no external driving force and no damping. Then $F(t) = 0$ and $\gamma = 0$, and our equation becomes

$$mu''(t) + ku(t) = 0$$

- The general solution to this equation is

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where:

$$\omega_0^2 = \frac{k}{m}.$$

Spring Model: Undamped Free Vibrations (part two)

- Using trigonometric identities, the solution can be rewritten as:

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \Leftrightarrow R\cos(\omega_0 t - \delta)$$
$$u(t) = R\cos\delta\cos(\omega_0 t) + R\sin\delta\sin(\omega_0 t)$$

where

$$A = R\cos\delta,\ \ B = R\sin\delta \ \ \Rightarrow \ R = \sqrt{A^2 + B^2},\ \ \tan\delta = \frac{B}{A}$$

- Note that in finding δ, we must be careful to choose the correct quadrant. This is done using the signs of $\cos\delta$ and $\sin\delta$.

Spring Model: Undamped Free Vibrations (part three)

- Thus our solution is

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$$

where

$$\omega_0 = \sqrt{k/m}$$

- The solution is a shifted cosine (or sine) curve, that describes simple harmonic motion, with period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

- The circular frequency ω_0 (radians/time) is the **natural frequency** of the vibration, R is the **amplitude** of the maximum displacement of mass from equilibrium, and δ is the **phase** or phase angle (dimensionless).

Spring Model: Undamped Free Vibrations (part four)

- Note that our solution

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta), \quad \omega_0 = \sqrt{k/m}$$

is a shifted cosine (or sine) curve with period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Initial conditions determine A & B, hence also the amplitude R.
- The system always vibrates with the same frequency ω_0, regardless of the initial conditions.
- The period T increases as m increases, so larger masses vibrate more slowly. The period T decreases as k increases, so stiffer springs cause a system to vibrate more rapidly.

Example 3.7.2: Define the IVP

A 10 lb mass stretches a spring 2 in. The mass is displaced an additional 2 in and then set in motion with an initial upward velocity of 1 ft/s.

Determine the position of the mass at any later time, and find the period, amplitude, and phase of the motion.

$$mu''(t) + ku(t) = 0, \;\; u(0) = u_0, \;\; u'(0) = v_0$$

- Find m: $w = mg \Rightarrow m = \dfrac{w}{g} \Rightarrow m = \dfrac{10\,\text{lb}}{32\,\text{ft/s}^2} \Rightarrow m = \dfrac{5}{16}\,\dfrac{\text{lb}\cdot\text{s}^2}{\text{ft}}$
- Find k: $F_S = -kL \Rightarrow k = \dfrac{10\text{ lb}}{2\text{ in}} \Rightarrow k = 60\,\dfrac{\text{lb}}{\text{ft}}$
- Thus our IVP is $\dfrac{5}{16}u''(t) + 60u(t) = 0, \;\; u(0) = \dfrac{1}{6}, \;\; u'(t) = -1$

Example 3.7.2: Find the Solution

- Simplifying, we obtain

$$u''(t) + 192u(t) = 0, \quad u(0) = 1/6, \quad u'(0) = -1$$

- To solve, use methods of Ch 3.3 to obtain

$$u = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t)$$

Example 3.7.2: Find Period, Amplitude, and Phase

- The natural frequency is

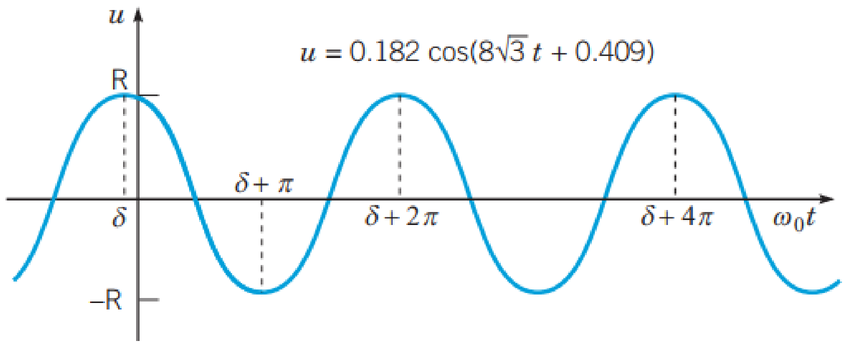
$$\omega_0 = \sqrt{k/m} = \sqrt{192} = 8\sqrt{3} \cong 13.856 \text{ rad/s}$$

- The period is

$$T = 2\pi/\omega_0 \cong 0.45345 \text{ s}$$

- The amplitude is

$$R = \sqrt{A^2 + B^2} \cong 0.18162 \text{ft}$$



- Next, determine the phase δ:

$$\tan\delta = \frac{B}{A} \Rightarrow \tan\delta = \frac{-\sqrt{3}}{4} \Rightarrow \delta = \tan^{-1}\left(\frac{-\sqrt{3}}{4}\right) \cong -0.40864 \text{ rad}$$

Note: numerical values for ω_0, T, and δ should be rounded to the appropriate significant figure based on the input data

Spring Model: Damped Free Vibrations

Suppose there is damping but no external driving force $F(t)$:

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

What is effect of the damping coefficient γ on the system?

- The characteristic equation is $mr^2 + \gamma r + k = 0$

  with roots: $r_1, r_2 = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \dfrac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \dfrac{4km}{\gamma^2}}\right)$

- Three cases for the solution:

$$\gamma^2 - 4km > 0, \quad u = Ae^{r_1 t} + Be^{r_2 t};$$

$$\gamma^2 - 4km = 0, \quad u = (A + Bt)e^{-\gamma t/(2m)}$$

$$\gamma^2 - 4km < 0, \quad u = e^{-\gamma t/(2m)}\left(A\cos(\mu t) + B\sin(\mu t)\right),$$

$$\mu = \frac{1}{2m}\left(4km - \gamma^2\right)^{1/2} > 0$$

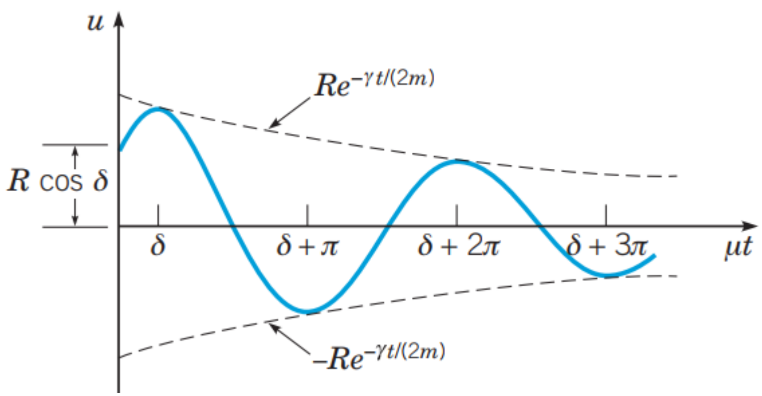
In all cases, the solution u tends to zero as $t \to \infty$

Damped Free Vibrations: Small Damping

- Of the cases for the solution form, the last is most important, which occurs when the damping is small. If we let $A = R\cos\delta$ and $B = \sin\delta$, displacement can be described by:

$$u = Re^{-\gamma t/(2m)}\cos\left(\mu t - \delta\right)$$

- The displacement u lies between the curves $u = \pm Re^{-\gamma t/(2m)}$, so the displacement vs. time curve resembles a cosine wave whose amplitude decreases as t increases. This motion is called a **dampened oscillation** or **dampened vibration**.



Damped Free Vibrations: Quasi-frequency

- Thus we have damped oscillations:

$$u = Re^{-\gamma t/(2m)} \cos(\mu t - \delta) \quad \text{where} \quad |u(t)| \le Re^{-\gamma t/(2m)}$$

- The amplitude R depends on the initial conditions, since

$$u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \;\; A = R\cos\delta, \;\; B = R\sin\delta$$

- Although the motion is not periodic, the parameter μ determines the mass oscillation frequency.
- Thus μ is called the **quasi-frequency**.
- Recall

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$$

Damped Free Vibrations: Quasi Period

- Comparing μ with the frequency ω_0 of undamped motion, we find:

$$\frac{\mu}{\omega_0} = \frac{\left(4km-\gamma^2\right)^{1/2} \big/ (2m)}{\sqrt{k/m}} = \left(1-\frac{\gamma^2}{4km}\right)^{1/2} \cong 1-\frac{\gamma^2}{8km}$$

approximation valid when $\gamma^2/4km$ is small $\Rightarrow$ "small damping"

- Similarly, the **quasi-period** is defined as $T_d = 2\pi/\mu$, where the relationship to period T is given by:

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1-\frac{\gamma^2}{4km}\right)^{-1/2} \cong 1+\frac{\gamma^2}{8km}$$

small damping increases quasi-period and decreases oscillation frequency

Damped Free Vibrations: Neglecting Damping for Small $\gamma^2/4km$

- Consider again the comparisons between damped and undamped frequency and period:

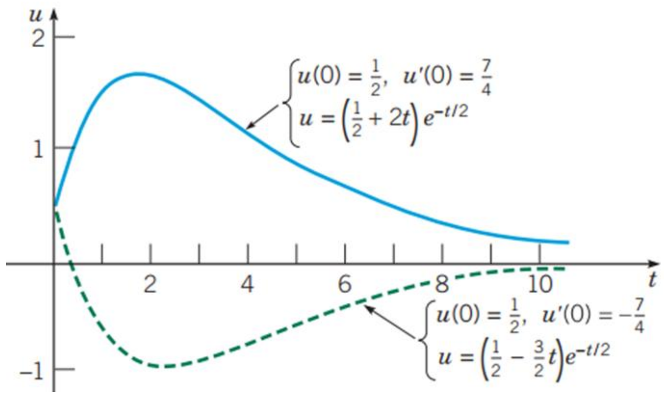
$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \quad \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

- The magnitude of γ alone is not as significant as the dimensionless ratio $\gamma^2/4km$. If $\gamma^2/4km$ is small, then damping has minimal effect on quasi-frequency and quasi-period.
- The mass oscillates about its equilibrium position.

Damped Free Vibrations: Critical and Overdamping

- Thus the nature of the solution changes as γ changes relative to $2\sqrt{km}$.
- If $\gamma = 2\sqrt{km}$, motion is **critically damped**.
- If $\gamma > 2\sqrt{km}$, motion is **overdamped**.

- For both critically and overdamped conditions, the mass passes through its equilibrium position once, then creeps back to it.



Critically dampened motion for two different initial conditions.

Example 3.7.3: Initial Value Problem

Suppose that the motion of a spring-mass system is governed by the initial value problem

$$u'' + \frac{1}{8}u' + u = 0, \;\; u(0) = 2, \;\; u'(0) = 0$$

Find the following:

a) quasi-frequency and quasi-period;
b) time at which mass passes through equilibrium position;
c) time τ such that $|u(t)| < 0.1$ for all $t > \tau$.

For Part (a), using methods of this chapter we obtain:

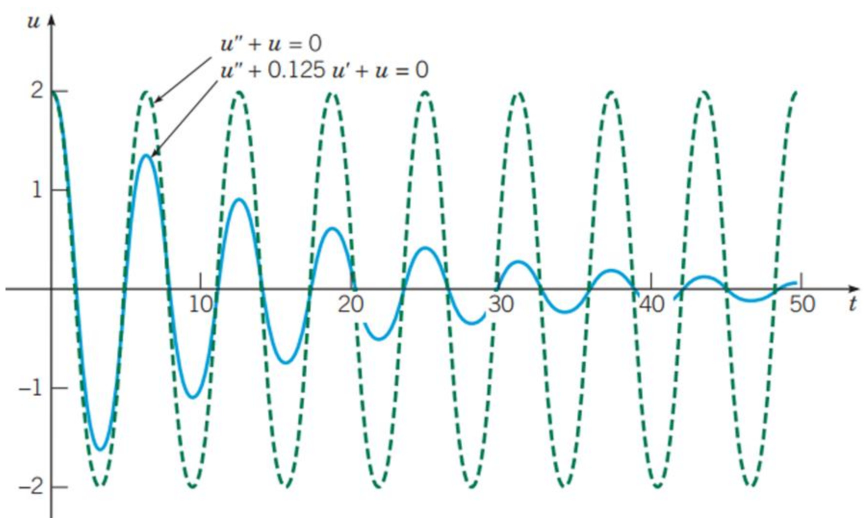
$$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

where

$$\tan\delta = \frac{1}{\sqrt{255}} \Rightarrow \delta \cong 0.06254 \quad (\text{recall } A = R\cos\delta,\; B = R\sin\delta)$$

Example 3.7.3: Quasi Frequency & Period

- The quasi-frequency is $u = \sqrt{255}/16$, and the quasi-period is $T_d = 2\pi/\mu$.
- For the undamped case: $\omega_0 = 1$, $T = 2\pi$
- The graph of this solution, along with solution to the corresponding undamped problem, is given below.

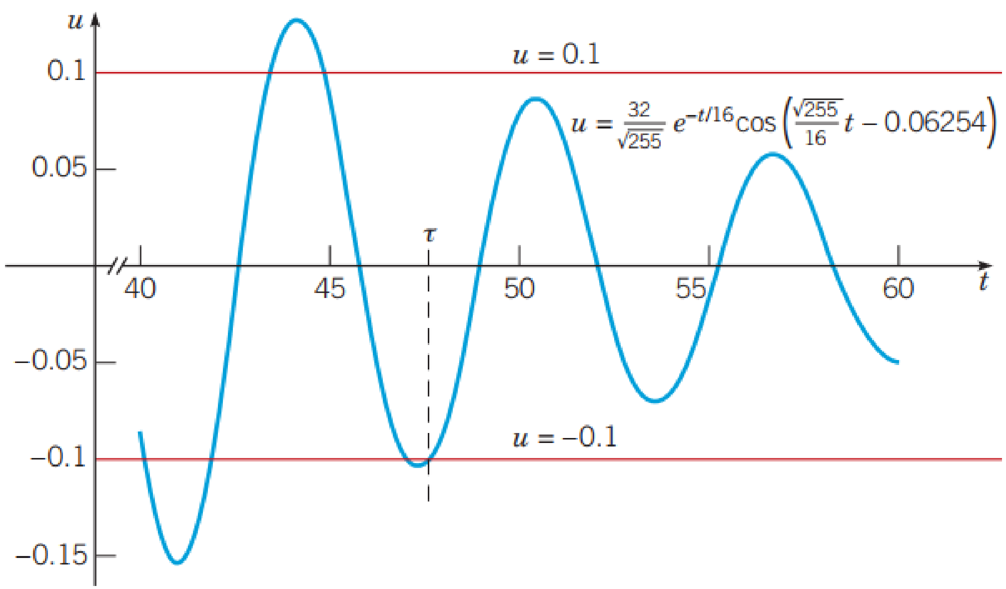


Blue curve = small damping

Dotted green curve = no damping

Example 3.7.3: Damping Coefficient

- The damping coefficient is $\gamma = 0.125 = 1/8$, and this is 1/16 of the critical value $2\sqrt{km} = 2$
- Thus damping is small relative to mass and spring stiffness, nevertheless the oscillation amplitude diminishes quickly.
- Using a solver, we find that $|u(t)| < 0.1$ for $t > \tau \cong 47.5149$ s



Solution for $40 \le t \le 60$, showing time τ after which the absolute value of the mass position is < 0.1

Example 3.7.3: Time for Mass to Pass Through Equilibrium Position

- To find the time at which the mass first passes through the equilibrium position, we must solve

$$u(t) = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta \right) = 0$$

- Or more simply, solve

$$\frac{\sqrt{255}}{16} t - \delta = \frac{\pi}{2}$$

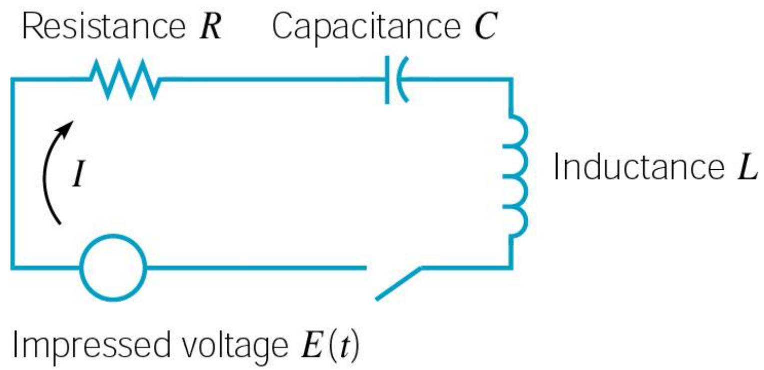
$$\Rightarrow t = \frac{16}{\sqrt{255}} \left(\frac{\pi}{2} + \delta \right) \cong 1.637 \text{ sec}$$

Electric Circuits

- The flow of current in certain basic electrical circuits is modeled by second order linear ODEs with constant coefficients:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

- The flow of current in this circuit is mathematically equivalent to motion of spring-mass system.
- For more details, see text.



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