Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

Boyce

Chapter 2

First-Order Differential Equations

Section 2.1
Linear Differential Equations; Method of Integrating Factors

Definition of First-Order Linear ODE

- A linear first-order ODE has the general form

$$\frac{dy}{dt} = f(t, y)$$

where f is linear in y. Examples include equations with constant coefficients:

$$\frac{dy}{dt} = -ay + b$$

or equations with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Example 2.1.1: Solution by Direct Integration

Solve the linear differential equation: $\left(4+t^2\right)\frac{dy}{dt}+2ty=4t$

Notice that the left-hand side of the equation is the derivative of a product:

$$\left(4+t^2\right)\frac{dy}{dt}+2ty=\frac{d}{dt}\left(\left(4+t^2\right)y\right)$$

Using this relationship allows us to re-write the original ODE as:

$$\frac{d}{dt}\left(\left(4+t^2\right)y\right)=4t$$

Which can then be integrated and re-arranged to give the general solution:

$$\left(4+t^2\right)y=2t^2+c \quad \text{È} \quad y=\frac{2t^2}{4+t^2}+\frac{c}{4+t^2}$$

Example 2.1.2: Method of Integrating Factors (part one)

If a linear ODE cannot be solved by direct integration, the use of a integrating factor can allow a general solution to be found.

Find the general solution of: $\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3}$

<u>Step One</u>: Multiply the original ODE by a function $\mu(t)$ which is yet undetermined:

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}$$

The function $\mu(t)$ is the integrating factor, which is specific to the original ODE and will be determined in Step Two (next slide)

Example 2.1.2: Method of Integrating Factors (part two)

<u>Step Two</u>: Find a $\mu(t)$ which permits us to express $\mu(t)\dfrac{dy}{dt}+\dfrac{1}{2}\mu(t)y$ as the derivative of a product.

$$\frac{d\mu(t)}{dt}=\frac{1}{2}\mu(t) \Rightarrow \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}=\frac{1}{2} \Rightarrow \frac{d}{dt}\ln|\mu(t)|=\frac{1}{2} \Rightarrow \mu(t)=ce^{t/2}$$

<u>Step Three:</u> Apply the integration factor to the original ODE

$$e^{t/2}\frac{dy}{dt}+\frac{1}{2}e^{t/2}y=\frac{1}{2}e^{5t/6} \Rightarrow \frac{d}{dt}\left(e^{t/2}y\right)=\frac{1}{2}e^{5t/6} \Rightarrow e^{t/2}y=\frac{3}{5}e^{5t/6}+c$$

General Solution: $y=\dfrac{3}{5}e^{t/3}+ce^{-t/2}$

Example 2.1.2: Method of Integrating Factors (part three)

Step Four: Use initial conditions to find the particular solution.

Initial Condition: Solution should pass through (0,1) so $t = 0$ when $y = 0$.

$$1 = \frac{3}{5} + c \quad \text{È} \quad c = \frac{2}{5} \quad \text{È} \qquad y = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}$$

Step Five: Examine families of particular solutions graphically

Figure 2.1.1 Direction field and integral curves for $y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}$;

the green curve passes through initial condition point (0,1).

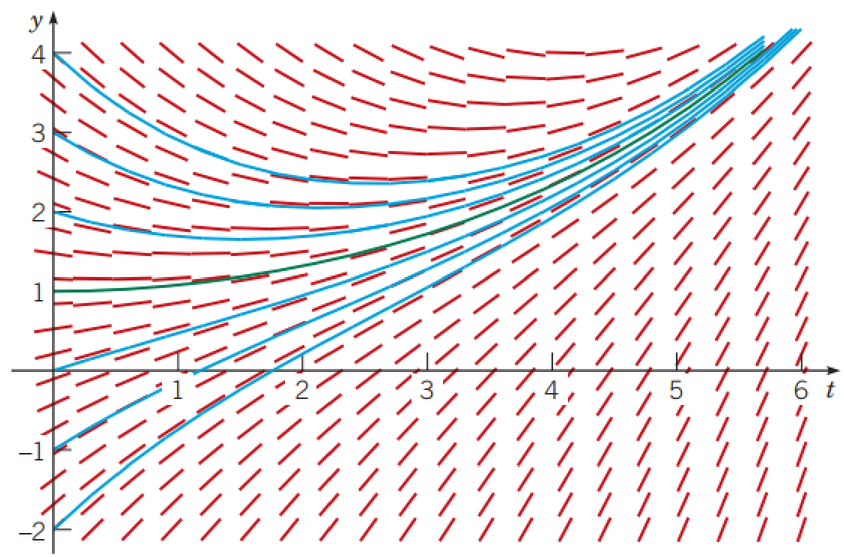


Figure 2.1.1

More General Form For Integrating Factors

We can extended the concept of integrating factors to linear ODE's of the form: $\frac{dy}{dt} + ay = g(t)$

Where a is a constant and $g(t)$ is a given function.

Start with integration factor $\frac{d\mu}{dt} = a\mu$ and apply it to the ODE above:

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t) \Rightarrow \frac{d}{dt}\left(e^{at}y\right) = e^{at}g(t) \Rightarrow e^{at}y = \int e^{at}g(t)\,dt + c$$

General Form Solution: $y = e^{-at}\int_{t_0}^{t} e^{as}g(s)\,ds + ce^{-at}$

Choice of t_0 will determine c and particular solution

Example 2.1.3: Application of General Integrating Factor (part one)

Find the general solution to $\frac{dy}{dt} - 2y = 4 - t$, plot graphs for several particular solutions, and discuss solution behavior as $t \to \infty$.

Step One: Recognize that $a = -2$, so the integrating factor is $u(t) = e^{-2t}$

Step Two: Apply the integration factor and obtain a general solution.

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = 4e^{-2t} - te^{-2t} \quad \Rightarrow \quad \frac{d}{dt}\left(e^{-2t}y\right) = 4e^{-2t} - te^{-2t}$$

$$e^{-2t}y = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c \quad \Rightarrow \quad y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}$$

General Solution

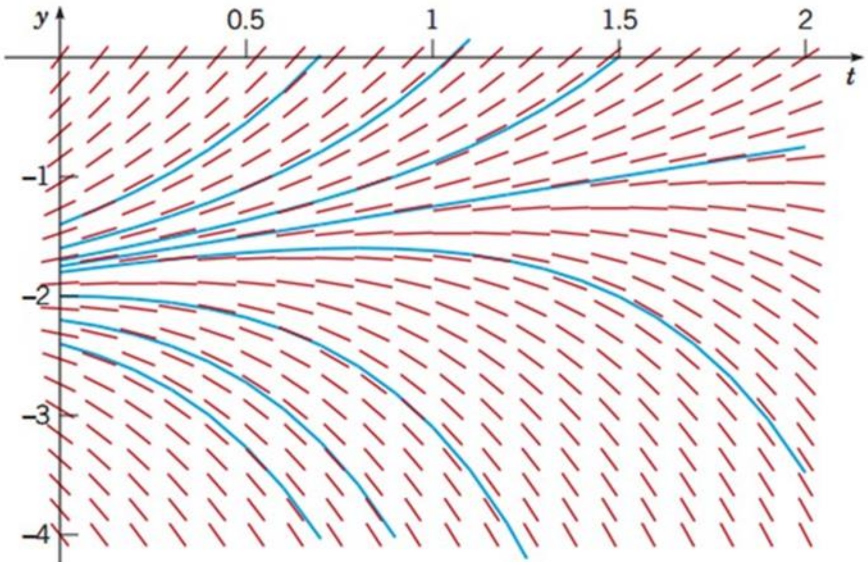
Example 2.1.3: Application of General Integrating Factor (part two)

Step Three: Examine general solution $y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}$ for trends as $t \to \infty$

- If $c \neq 0$, then the ce^{2t} term dominates as t becomes large
- Solutions diverge as t becomes large

Step Four: Examine initial value condition $t = 0$, $c = 0$

- Particular (initial value) solution: $y = -\frac{7}{4} + \frac{1}{2}t$
- Only solution which grows linearly vs. exponentially
- Value $y = -\frac{7}{4}$ separates solutions which grow positively vs. negatively



General From For Integrating Factor Solutions to Linear ODE's (part one)

We can further extend the concept of integrating factors to linear ODE's of the form: $\frac{dy}{dt} + p(t)y = g(t)$ Where both $p(t)$ and $g(t)$ are given functions.

Apply the integration factor $\mu(t)$ to the ODE, then solve for $\mu(t)$ in terms of $p(t)$:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t) \Rightarrow \frac{d\mu(t)}{dt} = p(t)\mu(t) \Rightarrow \frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t)$$

$$\Downarrow$$

Where $\mu(t)$ is positive for all t

$$\mu(t) = \exp\int p(t)\,dt \quad \Leftarrow \quad \ln|\mu(t)| = \int p(t)\,dt + k$$

General From For Integrating Factor Solutions to Linear ODE's (part two)

Now that we have expression $\mu(t) = exp \int p(t)dt$ as an integration factor, we apply it to the original ODE, permitting us to say:

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t) \quad \Rightarrow \quad \mu(t)y = \int \mu(t)g(t)\,dt + c$$

Where c is an arbitrary constant

The general solution can be expressed as: $y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right)$

Note that two integrations are need: one to find $\mu(t)$, and the other to find y

Where t_0 is a convenient lower limit of integration.

Example 2.1.4: Application of General Integrating Factor Method (part one)

Solve the initial value problem $ty' + 2y = 4t^2$, where $y(1) = 2$

Step One: Rewrite the given ODE into the standard form to apply an integrating factor:

$$y' + \frac{2}{t}y = 4t, \text{ where } p(t) = 2/t \text{ and } g(t) = 4t$$

Step Two: Compute integration factor $\mu(t)$:

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = e^{2\ln|t|} = t^2$$

Example 2.1.4: Application of General Integrating Factor Method (part two)

Step Three: Apply the computed integration factor $\mu(t) = t^2$:

$$t^2 y' + 2ty = \left(t^2 y\right)' = 4t^3 \;\Rightarrow\; t^2 y = \int 4t^3 dt = t^4 + c$$

Where c is an arbitrary constant

For $t > 0$, we have the general solution:

$$y = t^2 + \frac{c}{t^2}$$

The value of c will depend on initial conditions.

Example 2.1.4: Application of General Integrating Factor Method (part three)

<u>Step Four</u>: Apply initial conditions $t = 1$, $y = 2$ to find a particular solution and examine solution behavior for other particular solutions.

If $y = 2$ when $t = 0$, then $2 = 1 + c$, so $c = 1 \Rightarrow\ y = t^2 + \dfrac{1}{t^2}$

Integral curves of $ty' + 2y = 4t^2$; where the green curve is the particular solution with $y(1) = 2$; the red curve is the particular solution with $y(1) = 1$.

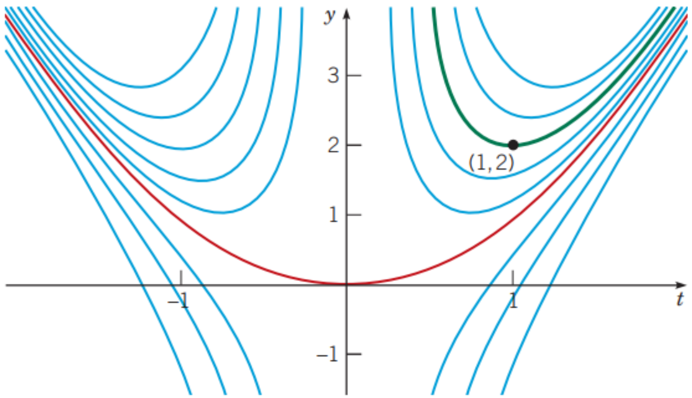


Figure 2.1.3

Example 2.1.5: A Solution in Integral Form (part one)

Solve the initial value problem $2y' + ty = 2,$ where $y(0) = 1$.

Step One: Rewrite the given ODE into the standard form to and compute the integrating factor:

$$y' + \frac{t}{2}y = 1, \text{ where } p(t) = \frac{t}{2} \Rightarrow \quad \mu(t) = e^{t^2/4}$$

Step Two: Apply the integration factor and find a general solution:

$$e^{t^2/4}y' + \frac{t}{2}e^{t^2/4}y = e^{t^2/4}\,y \Rightarrow \quad e^{t^2/4} = \int e^{t^2/4}dt + c$$

Use initial condition $t = 0$ for lower integration limit

$$\Downarrow$$

General Solution $\quad y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4} \quad \Leftarrow \quad e^{t^2/4}y = \int_0^t e^{s^2/4}\,ds + c$

Example 2.1.5: A Solution in Integral Form (part two)

<u>Step Three</u>: Use initial conditions to find a particular solution

$$1 = e^0 \int_0^0 e^{-s^2/4} ds + ce^0$$
$$= 0 + c, \quad \text{so } c = 1$$

Integral must be solved via numerical methods

Integral curves for $2y' + ty = 2$; the green curve is the particular solution satisfying the initial condition $y(0) = 1$.

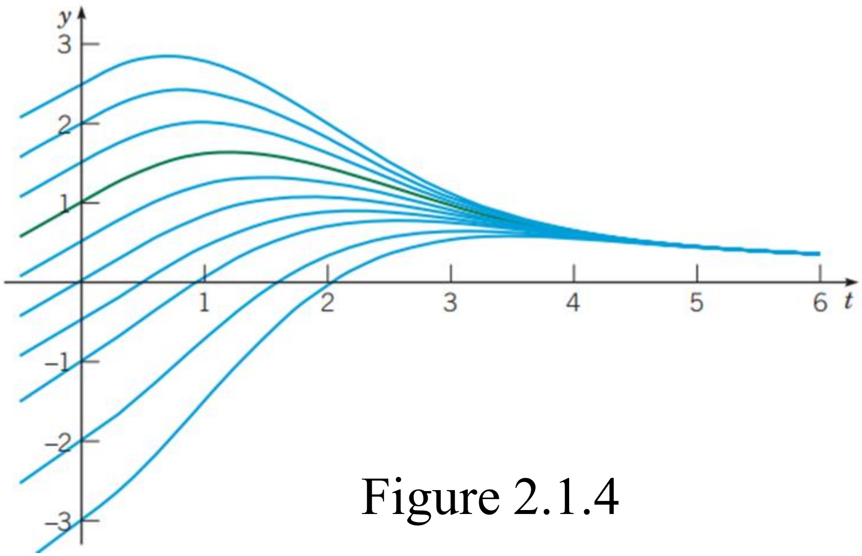


Figure 2.1.4

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