Elementary Differential Equations and Boundary Value Problems

Twelfth Edition

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Chapter 10

Partial Differential Equations and Fourier Series

Section 10.2 Fourier Series

Fourier Series

- We will see that many important problems involving partial differential equations can be solved, provided a given function can be expressed as an infinite sum of sines and/or cosines.
- In this and the following two sections, we explain in detail how this can be done.
- These trigonometric series are called **Fourier series**, and are somewhat analogous to Taylor series in that both types of series provide a means of expressing complicated functions in terms of certain familiar elementary functions.

Fourier Series Representation of Functions

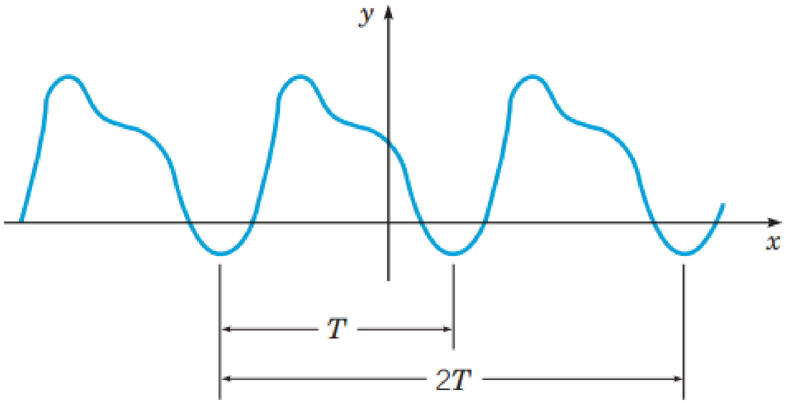
- We begin with a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

- On the set of points where this series converges, it defines a function f whose value at each point x is the sum of the series for that value of x.
- In this case the series is said to be the **Fourier series** of f.
- Our immediate goals are to determine what functions can be represented as a sum of Fourier series, and to find some means of computing the coefficients in the series corresponding to a given function.

Periodic Functions

- We first develop properties of $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer.
- The first property is their periodic character.
- A function is **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if $f(x + T) = f(x)$ for all x.
- See graph below.



Periodicity of the Sine and Cosine Functions

- For a periodic function of period T, $f(x + T) = f(x)$ for all x.
- Note that $2T$ is also a period, and so is any integer multiple of T.
- The smallest value of T for which f is periodic is called the **fundamental period** of f.
- If f and g are two periodic functions with common period T, then any linear combination $c_1 f + c_2 g$ is also periodic with period T.
- In particular, $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1,2,3,\ldots$, are periodic with period $T = 2L/m$.

Orthogonality

- The standard inner product (u, v) of two real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)dx$$

- The functions u and v are **orthogonal** on $\alpha \le x \le \beta$ if their inner product (u, v) is zero:

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)dx = 0$$

- A set of functions is **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

Orthogonality of Sine and Cosine

- The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \ldots$, form a mutually orthogonal set of functions on $-L \le x \le L$, with

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \ne n, \\ L, & m = n; \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0, \text{ all } m, n;$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \ne n, \\ L, & m = n. \end{cases}$$

- These results can be obtained by direct integration; see text.

Finding Coefficients in Fourier Expansion

- Suppose the series converges, and call its sum $f(x)$:

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\left(\frac{m\pi x}{L}\right)+b_m\sin\left(\frac{m\pi x}{L}\right)\right)$$

- The coefficients a_m and b_m, $m = 1, 2, \ldots$, can be found as follows.

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{a_0}{2}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx \quad +\sum_{m=1}^{\infty}a_m\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

Where n is fixed and m ranges over positive integers

$$+\sum_{m=1}^{\infty}b_m\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx.$$

- By orthogonality,

$$\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx=a_n\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx=a_nL,\quad n=1,2,\ldots\ .$$

Coefficient Formulas

- From the previous slide we have

$$a_n \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = a_n L, \quad n = 1, 2, \ldots .$$

- To find the coefficient a_0, we have

$$\int_{-L}^{L} f(x) dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) dx = La_0$$

- Thus the coefficients a_n are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \ldots .$$

- Similarly, the coefficients b_n are given by

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots .$$

- The above equations for a_n and b_n are known as the **Euler-Fourier formulas**

The Euler-Fourier Formulas

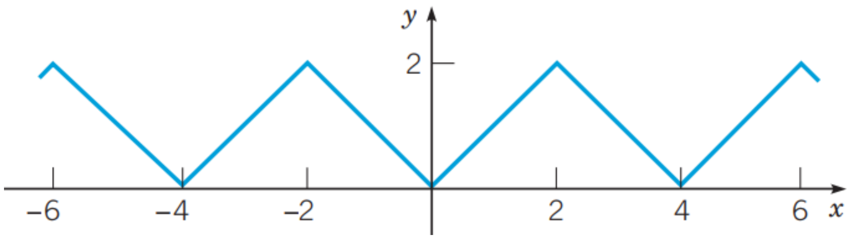
- Note that the Euler-Fourier formulas depend only on the values of $f(x)$ in the interval $-L \le x \le L$.
- Since each term of the Fourier series is periodic with period $2L$, the series converges for all x whenever it converges in $-L \le x \le L$, and its sum is also a periodic function with period $2L$.
- Hence $f(x)$ is determined for all x by its values in the interval $-L \le x \le L$.

Example 10.2.1: A Triangular Wave

Assume that there is a Fourier series converging to the function f defined below. Determine the coefficients in this Fourier series.

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}, \qquad f(x+4) = f(x)$$

- This function represents a triangular wave and is periodic with period $T = 4$. See graph of f below. In this case, $L = 2$.
- Assuming that f has a Fourier series representation, find the coefficients a_m and b_m.



Example 10.2.1: Coefficients

- First, we find a_0:

$$a_0 = \frac{1}{2}\int_{-2}^{0}(-x)\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = 1 + 1 = 2$$

- Then for a_m, $m = 1, 2, \ldots$, we have

$$a_m = \frac{1}{2}\int_{-2}^{0}(-x)\cos\left(\frac{m\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{m\pi x}{2}\right)dx = \begin{cases} -\dfrac{8}{(m\pi)^2}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases}$$

where we have used integration by parts. See text for details.

- Similarly, it can be shown that $b_m = 0$, $m = 1, 2, \ldots$

Example 10.2.1: The Fourier Expansion

- Using the expressions for a_0, a_m, and b_m, we obtain the Fourier series for f:

$$f(x) = 1 - \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \ldots\right)$$

$$= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^2}\cos\left(\frac{m\pi x}{2}\right).$$

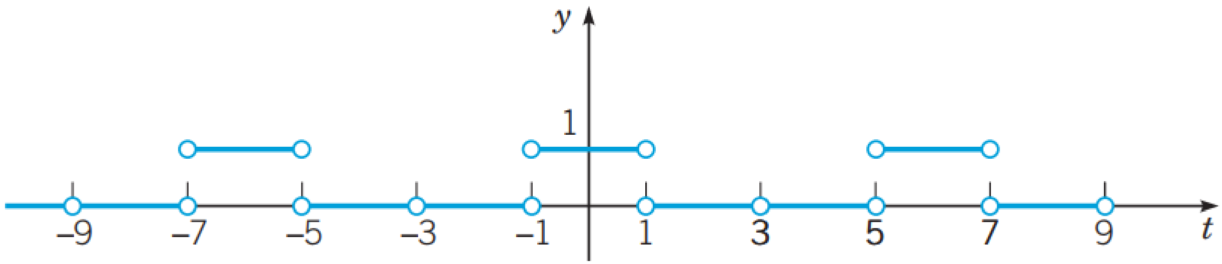
$$= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{2}\right).$$

Example 10.2.2: A Function

Consider the function below and suppose that $f(x+6) = f(x)$. Graph three periods of $y = f(x)$. Find the coefficients in the Fourier series for f.

$$f(x) = \begin{cases} 0, & -3 < x < -1, \\ 1, & -1 < x < 1, \\ 0, & 1 < x < 3 \end{cases}$$

- This function is periodic with period $T = 6$. In this case, $L = 3$. See graph below.
- Assuming that f has a Fourier series representation, find the coefficients a_n and b_n.



Example 10.2.2: Points of Discontinuity

- Note that $f(x)$ is not assigned values at the points of discontinuity, such as $x = -1$ and $x = 1$.
- This has no effect on the values of the Fourier coefficients, because they result from the evaluation of integrals, and the value of an integral is not affected by the value of the integrand at a single point, or at a finite number of points.
- Thus, the coefficients are the same regardless of what value, if any, $f(x)$ is assigned at a point of discontinuity.

Example 10.2.2: Coefficients

- First, we find a_0:

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_{-1}^{1} dx = \frac{2}{3}$$

- Using the Euler-Fourier formulas, we obtain

$$a_n = \frac{1}{3}\int_{-1}^{1} \cos\left(\frac{n\pi x}{3}\right)dx = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Bigg|_{-1}^{1} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right), \quad n = 1, 2, \ldots,$$

$$b_n = \frac{1}{3}\int_{-1}^{1} \sin\left(\frac{n\pi x}{3}\right)dx = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\Bigg|_{-1}^{1} = 0, \quad n = 1, 2, \ldots.$$

Example 10.2.2: Fourier Expansion

- Thus the Fourier series for f is

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{3}\right) \cos\left(\frac{n\pi x}{3}\right)$$

$$= \frac{1}{3} + \frac{\sqrt{3}}{\pi}\left(\cos\left(\frac{\pi x}{3}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{3}\right) - \frac{1}{4}\cos\left(\frac{4\pi x}{3}\right) - \frac{1}{5}\cos\left(\frac{5\pi x}{3}\right) + \mathrm{K}\right)$$

Example 10.2.3: A Triangular Wave

- Consider again the function from Example 10.2.1 (below) and its Fourier series. Investigate the speed with which the series converges. In particular, determine how many terms are needed so that the error is no greater than 0.01 for all x.

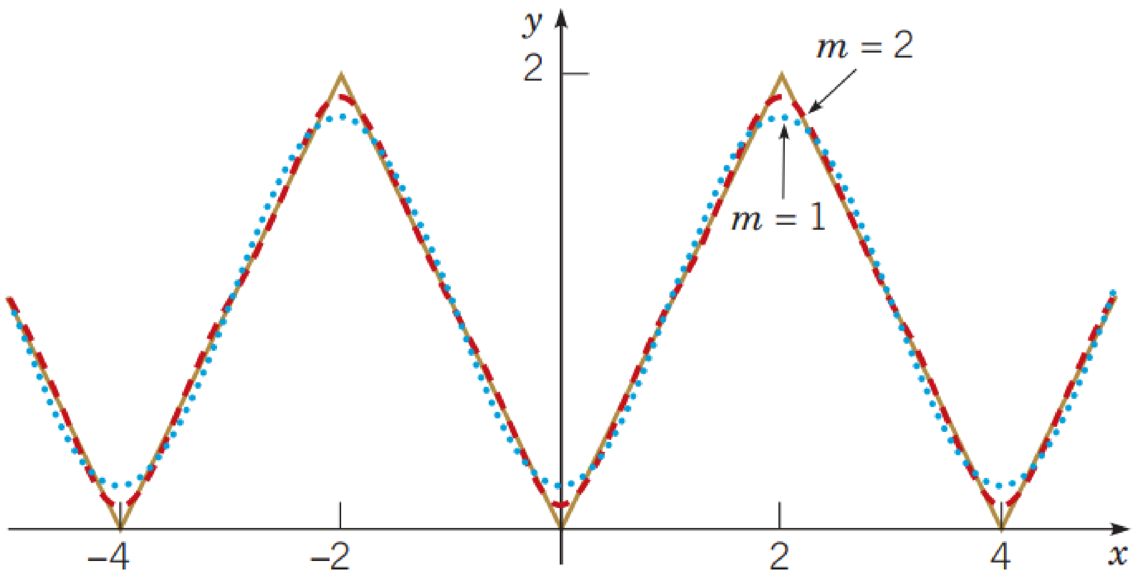
$$f(x)=\begin{cases}-x, & -2\le x<0\\ x, & 0\le x<2\end{cases}, \quad f(x+4)=f(x),$$

- The m^{th} partial sum in this series can be used to approximate the function f.

$$s_m(x)=1-\frac{8}{\pi^2}\sum_{n=1}^{m}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{2}\right)$$

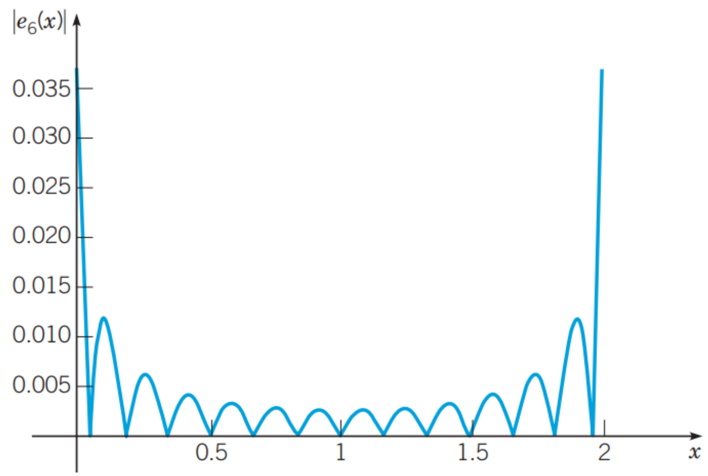
Example 10.2.3: Partial Sums

- The coefficients diminish as $(2n-1)^2$, so the series converges fairly rapidly. This is seen below in the graph below where the partial sums for $m = 1$ (dotted blue) and $m = 2$ (dashed red) are plotted.



Example 10.2.3: Errors

- To investigate the convergence in more detail, we consider the error function $e_m(x) = f(x) - s_m(x)$.
- Given below is a graph of $|e_6(x)|$ on $0 \leq x \leq 2$.
- Note that the error is greatest at $x = 0$ and $x = 2$, where the graph of $f(x)$ has corners.
- Similar graphs are obtained for other values of m.



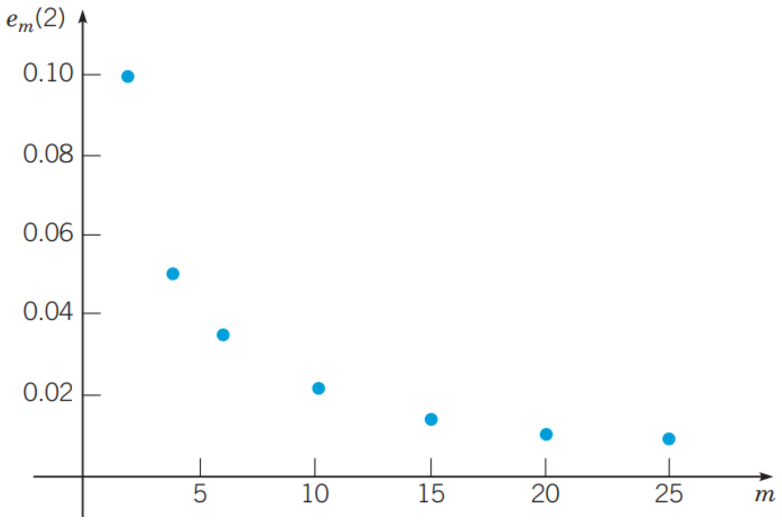
Example 10.2.3: A Uniform Error Bound

- Since the maximum error occurs at $x = 0$ or $x = 2$, we obtain a uniform error bound for each m by evaluating $|e_m(x)|$ at one of these points.
- For example, $e_6(2) = 0.03370$, and hence $|e_6(x)| < 0.034$ on $0 \leq x \leq 2$, and consequently for all x.

Example 10.2.3: The Speed of Convergence

- The table below shows values of $|e_m(2)|$ for other values of m, and these data points are plotted below also.
- From this information, we can begin to estimate the number of terms that are needed to achieve a given level of accuracy.
- To guarantee that $|e_m(2)| \leq 0.01$, we need to choose $m = 21$.

m	$e_m(2)$
2	0.09937
4	0.05040
6	0.03370
10	0.02025
15	0.01350
20	0.01013
25	0.00810



Broad Use of Fourier Series

- We will be using Fourier series as a means of solving certain problems in partial differential equations.
- However, Fourier series have much wider application in science and engineering and, in general, are valuable tools in the investigations of periodic phenomena.
- For example, a basic problem in spectral analysis is to resolve an incoming signal into its harmonic components, which amounts to constructing its Fourier series representation.
- In some frequency ranges the separate terms correspond to different colors or to different audible tones.
- The magnitude of the applicable coefficient determines the amplitude of each component.

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