State space is

$$\dot{X}_i(t) = f_i(t, X_i, u_i) \ \ (0 \le t \le T)$$
$$\dot{X}_i(t) = A * X_i(t) + B * u_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_i(t) \\ v_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t)$$

where, x_1, x_2 is the position for each vehicle, v_1, v_2 is the velocity for each vehicle.

Value function is

$$L_i(X_1, X_2, u_i) = cu_i^2 + \frac{\alpha_i}{\beta[(x_1 - [\frac{L_1}{2} + \frac{R_1}{2}])^2 + (x_2 - [\frac{L_2}{2} + \frac{R_2}{2}])^2] + \gamma}$$
$$V_i(X_1, X_2, t) = F_i(T) + \int_t^T L_i(X_1, X_2, u_i) dt$$
$$F_i(T) = \frac{1}{2} \frac{1}{||x_i(T)||^2}$$

c is the weight for control input. L_1, L_2 is the length of the vehicle. R_1, R_2 is the length of the road.

α_i is the parameter to decide which behavior the i_{th} vehicle will choose: aggressive or non-aggressive. If α_i is bigger, it means that i_{th} vehicle is non-aggressive because there is a higher collision loss, so i_{th} vehicle will avoid the collision and be non-aggressive. If α_i is smaller, the i_{th} vehicle will be aggressive.

We want to minimize the V_i, which is equal to maximize the $-V_i$, so the Value function is changed into:

$$V_i(X_1, X_2, t) = F_i(T) - \int_t^T L_i(X_1, X_2, u_i) dt$$
$$F_i(T) = -\frac{1}{2} \frac{1}{||x_i(T)||^2}$$

where, T is the fixed parameter

$$T = \max(R_1, R_2) / \min(v_1, v_2)$$

v_1, v_2 is the initial velocity for each vehicle. Or we could give the fixed T setting.

HJI derivation process

$$\begin{aligned}
V_i(X_1,X_2,t) \quad &= F_i(T) - \int_t^T L_i(X_1,X_2,u_i)dt \\
&= F_i(T) - \int_t^{t+\triangle t} L_i(X_1,X_2,u_i)dt - \int_{t+\triangle t}^T L_i(X_1,X_2,u_i)dt \\
&= V_i(X_1(t+\triangle t), X_2(t+\triangle t), t+\triangle t) - \int_t^{t+\triangle t} L_i(X_1,X_2,u_i)dt \\
&= V_i(X_1,X_2,t) + [\frac{\partial V_i}{\partial x_1}]^T(X_1(t+\triangle t)-X_1(t)) + [\frac{\partial V_i}{\partial x_2}]^T(X_2(t+\triangle t)-X_2(t)) \\
&+ \frac{\partial V_i}{\partial t}\triangle t + O(\triangle t^2) - L_i(X_1,X_2,u_i)\triangle t - O(\triangle t^2) \\
0 \quad &= \frac{\partial V_i}{\partial t}\triangle t + [\frac{\partial V_i}{\partial X_1}]^T \dot{X}_1 \triangle t + [\frac{\partial V_i}{\partial X_2}]^T \dot{X}_2 \triangle t - L_i(X_1,X_2,u_i)\triangle t \\
0 \quad &= \frac{\partial V_i}{\partial t} + [\frac{\partial V_i}{\partial x_1}]^T f_1(t,X_1,u_1) + [\frac{\partial V_i}{\partial x_2}]^T f_2(t,X_2,u_2) - L_i(X_1,X_2,u_i)
\end{aligned}$$

Hence, Hamiltonian for each vehicle is

$$\begin{aligned}
&H_1(t,X_1,X_2,u_1,u_2,\lambda_{11},\lambda_{12}) \\
&= [V_{1_{X_1}}(X_1,X_2,t)]^T f_1(t,X_1,u_1) + [V_{1_{X_2}}(X_1,X_2,t)]^T f_2(t,X_2,u_2) - L_1(X_1,X_2,u_1) \\
&= \lambda_{11}^T f_1(t,X_1,u_1) + \lambda_{12}^T f_2(t,X_2,u_2) - L_1(X_1,X_2,u_1)
\end{aligned}$$

$$\begin{aligned}
\lambda_{11}(t) &= V_{1_{X_1}}(X_1,X_2,t) \\
\lambda_{12}(t) &= V_{1_{X_2}}(X_1,X_2,t)
\end{aligned}$$

$$\begin{aligned}
&H_2(t,X_1,X_2,u_1,u_2,\lambda_{21},\lambda_{22}) \\
&= [V_{2_{X_1}}(X_1,X_2,t)]^T f_1(t,X_1,u_1) + [V_{2_{X_2}}(X_1,X_2,t)]^T f_2(t,X_2,u_2) - L_2(X_1,X_2,u_2) \\
&= \lambda_{21}^T f_1(t,X_1,u_1) + \lambda_{22}^T f_2(t,X_2,u_2) - L_2(X_1,X_2,u_2)
\end{aligned}$$

$$\begin{aligned}
\lambda_{21}(t) &= V_{2_{X_1}}(X_1,X_2,t) \\
\lambda_{22}(t) &= V_{2_{X_2}}(X_1,X_2,t)
\end{aligned}$$

The optimal control satisfies

$$\begin{aligned}
u_1^* &= \arg\max_{u_1\in U_1} H_1(t,X_1,X_2,u_1,u_2^*,\lambda_{11},\lambda_{12}) = H_{1_{u_1}}(u_2^*) \\
u_2^* &= \arg\max_{u_2\in U_2} H_2(t,X_1,X_2,u_1^*,u_2,\lambda_{21},\lambda_{22}) = H_{2_{u_2}}(u_1^*)
\end{aligned}$$

HJI equation is

$$
\begin{aligned}
V_{1_t} + \lambda_{11}^T f_1(t, X_1, u_1^*) + \lambda_{12}^T f_2(t, X_2, u_2^*) - L_1(X_1, X_2, u_1^*) = 0 \\
V_{2_t} + \lambda_{21}^T f_1(t, X_1, u_1^*) + \lambda_{22}^T f_2(t, X_2, u_2^*) - L_2(X_1, X_2, u_2^*) = 0
\end{aligned}
$$

PMP equation is

$$
\begin{aligned}
&\dot{X}_1(t) = H_{\lambda_{11}} = f_1(t, X_1, u_1^*) \\
&\dot{X}_2(t) = H_{\lambda_{22}} = f_2(t, X_1, u_2^*) \\
&\dot{\lambda}_{11}(t) = -H_{1_{X_1}} = -\nabla_{X_1} f_1(t, X_1, u_1^*) \cdot \lambda_{11} + \nabla_{X_1} L_1(X_1, X_2, u_2^*) = -A^T \lambda_{11} + \nabla_{X_1} L_1(X_1, X_2, u_2^*) \\
&\dot{\lambda}_{12}(t) = -H_{1_{X_2}} = -\nabla_{X_2} f_2(t, X_2, u_2^*) \cdot \lambda_{12} + \nabla_{X_2} L_1(X_1, X_2, u_1^*) = -A^T \lambda_{12} + \nabla_{X_2} L_1(X_1, X_2, u_1^*) \\
&\dot{\lambda}_{21}(t) = -H_{2_{X_1}} = -\nabla_{X_1} f_1(t, X_1, u_1^*) \cdot \lambda_{21} + \nabla_{X_1} L_2(X_1, X_2, u_2^*) = -A^T \lambda_{21} + \nabla_{X_1} L_2(X_1, X_2, u_2^*) \\
&\dot{\lambda}_{22}(t) = -H_{2_{X_2}} = -\nabla_{X_2} f_2(t, X_2, u_2^*) \cdot \lambda_{22} + \nabla_{X_2} L_2(X_1, X_2, u_2^*) = -A^T \lambda_{22} + \nabla_{X_2} L_2(X_1, X_2, u_2^*)
\end{aligned}
$$

Besides, we also consider the differential of value function with respect to time t

$$
\dot{V}_i = L_i(X_1, X_2, u_i^*)
$$

Therefore, we could use below five equations to solve the BVP problem

$$
\begin{aligned}
&\dot{X}_1(t) = H_{\lambda_{11}} = f_1(t, X_1, u_1^*), \quad X_1(0) = X_{1_o} \\
&\dot{X}_2(t) = H_{\lambda_{22}} = f_2(t, X_1, u_2^*), \quad X_2(0) = X_{2_o} \\
&\dot{\lambda}_{11}(t) = -H_{1_{X_1}} = -\nabla_{X_1} f_1(t, X_1, u_1^*) \cdot \lambda_{11} + \nabla_{X_1} L_1(X_1, X_2, u_2^*), \quad \lambda_{11}(T) = \frac{dF_1}{dX_1}(X(T)) \\
&\dot{\lambda}_{12}(t) = -H_{1_{X_2}} = -\nabla_{X_2} f_2(t, X_2, u_2^*) \cdot \lambda_{12} + \nabla_{X_2} L_1(X_1, X_2, u_1^*), \quad \lambda_{12}(T) = \frac{dF_1}{dX_2}(X(T)) \\
&\dot{\lambda}_{21}(t) = -H_{2_{X_1}} = -\nabla_{X_1} f_1(t, X_1, u_1^*) \cdot \lambda_{21} + \nabla_{X_1} L_2(X_1, X_2, u_2^*), \quad \lambda_{21}(T) = \frac{dF_2}{dX_1}(X(T)) \\
&\dot{\lambda}_{22}(t) = -H_{2_{X_2}} = -\nabla_{X_2} f_2(t, X_2, u_2^*) \cdot \lambda_{22} + \nabla_{X_2} L_2(X_1, X_2, u_2^*), \quad \lambda_{22}(T) = \frac{dF_2}{dX_2}(X(T)) \\
&\dot{V}_1(t) = L_1(X_1, X_2, u_1^*), \quad V_1(T) = F_1(X(T)) \\
&\dot{V}_2(t) = L_2(X_1, X_2, u_2^*), \quad V_2(T) = F_2(X(T))
\end{aligned}
$$

For non-cooperative differential game, policy $(u_1,\ u_2)$ should meet below condition, then the policy $(u_1^*,\ u_2^*)$ will be the Nash equilibrium of the game:

$$
\Phi^A(u_1, u_2^*) \leq \Phi^A(u_1^*, u_2^*) \qquad \Phi^B(u_1^*, u_2) \leq \Phi^B(u_1^*, u_2^*)
$$

where Φ^A and Φ^B are the value function of agent A and B respectively.

From above description, we could have policy of Nash equilibrium after solving the HJI equation. That is,

$$
\begin{aligned}
u_1^* = \arg\max_{u_1 \in U_1} H(t, x_1, x_2, u_1, u_2^*, \lambda_{11}, \lambda_{12}) = H_{u_1}(u_2^*) \\
u_2^* = \arg\max_{u_2 \in U_2} H(t, x_1, x_2, u_1^*, u_2, \lambda_{21}, \lambda_{22}) = H_{u_2}(u_1^*)
\end{aligned}
$$

which meets $V_1(u_1, u_2^*) \leq V_1(u_1^*, u_2^*)$ and $V_2(u_1^*, u_2) \leq V_2(u_1^*, u_2^*)$, V_1, V_2 is our defined value function.

$$\dot{\lambda}(t) = \begin{bmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} + \begin{bmatrix} dLdx \\ dLdv \end{bmatrix} = \begin{bmatrix} dLdx \\ \lambda_1(t) \end{bmatrix}$$

$$dLdx_1(X_1, X_2, u_i) = -\frac{2\beta(x_1 - [\frac{L_1}{2} + \frac{R_1}{2}])}{(\beta[(x_1 - [\frac{L_1}{2} + \frac{R_1}{2}])^2 + (x_2 - [\frac{L_2}{2} + \frac{R_2}{2}])^2] + \gamma)^2}$$

$$\begin{aligned} & u_1^* = \arg\max_{u_1 \in U_1} H_1(t, X_1, X_2, u_1, u_2^*, \lambda_{11}, \lambda_{12}) \\ & ==> \frac{H_1}{u_1} = 0 \\ & ==> 2cu_1^* - B^T\lambda_{11} = 0 \\ & ==> u_1^* = \frac{1}{2c}B^T\lambda_{11} \end{aligned}$$

Value function is

$$\begin{aligned} & L_i(X_1, X_2, u_i) = u_i^2 + \frac{1}{\beta[(x_1 - [\frac{L_1}{2} + \frac{R_1}{2}])^2 + (x_2 - [\frac{L_2}{2} + \frac{R_2}{2}])^2] + \gamma} \\ & V_i(X_1, X_2, t) = F_i(T) - \int_t^T L_i(X_1, X_2, u_i)dt \\ & F_i(T) = x_i(T) \end{aligned}$$

Hence, Hamiltonian for each vehicle is

$$\begin{aligned} & H_i(t, X_1, X_2, u_1, u_2, \lambda_{i1}, \lambda_{i2}) \\ & = [V_{i_{X_1}}(X_1, X_2, t)]^T f_1(t, X_1, u_1) + [V_{i_{X_2}}(X_1, X_2, t)]^T f_2(t, X_2, u_2) - L_i(X_1, X_2, u_1) \\ & = \lambda_{i1}^T f_1(t, X_1, u_1) + \lambda_{i2}^T f_2(t, X_2, u_2) - L_i(X_1, X_2, u_1) \end{aligned}$$

$$\begin{aligned} & \lambda_{i1}(t) = V_{i_{X_1}}(X_1, X_2, t) \\ & \lambda_{i2}(t) = V_{i_{X_2}}(X_1, X_2, t) \end{aligned}$$