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## FirstProof # 9

Let  $n \geq 5$ . Let  $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3 \times 4}$  be Zariski-generic. For  $\alpha, \beta, \gamma, \delta \in [n]$ , construct  $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$  so that its  $(i, j, k, \ell)$  entry for  $1 \leq i, j, k, \ell \leq 3$  is given by  $Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$ . Here  $A(i, :)$  denotes the  $i$ th row of a matrix  $A$ , and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors  $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$ .

More precisely, does there exist a polynomial map  $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$  that satisfies the following three properties?

- The map  $\mathbf{F}$  does not depend on  $A^{(1)}, \dots, A^{(n)}$ .
- The degrees of the coordinate functions of  $\mathbf{F}$  do not depend on  $n$ .
- Let  $\lambda \in \mathbb{R}^{n \times n \times n \times n}$  satisfy  $\lambda_{\alpha\beta\gamma\delta} \neq 0$  for precisely  $\alpha, \beta, \gamma, \delta \in [n]$  that are not identical. Then  $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$  holds if and only if there exist  $u, v, w, x \in (\mathbb{R}^*)^n$  such that  $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$  for all  $\alpha, \beta, \gamma, \delta \in [n]$  that are not identical.

## Model Response (Agent B)

### 1. Construction of the Polynomial Map F

Let the input to our polynomial map  $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$  be the formal tensor variables  $X$ , mapping to the scaled quadrifocal tensor:

$$X_{ijkl}^{(\alpha\beta\gamma\delta)} = \lambda_{\alpha\beta\gamma\delta} \det \begin{bmatrix} A^{(\alpha)}(i, :) \\ A^{(\beta)}(j, :) \\ A^{(\gamma)}(k, :) \\ A^{(\delta)}(\ell, :) \end{bmatrix},$$

where  $A^{(\alpha)}(i, :) \in \mathbb{R}^{1 \times 4}$  denotes the  $i$ -th row of the  $3 \times 4$  Zariski-generic matrix  $A^{(\alpha)}$ . There are  $\binom{4}{2} = 6$  symmetric ways to partition the four tensor slots into two evaluated slots and two fixed slots. Consider the bipartition where we evaluate slots 1 and 2, and fix slots 3 and 4. For any pair of fixed matrix indices  $\gamma \neq \delta \in [n]$  and fixed row indices  $k, \ell \in \{1, 2, 3\}$ , we define a  $3n \times 3n$  evaluation matrix  $\mathbf{M}_{k,\ell}^{\gamma,\delta}(X)$ . Its rows are indexed by the multi-index  $(\alpha, i) \in [n] \times \{1, 2, 3\}$  and its columns by  $(\beta, j) \in [n] \times \{1, 2, 3\}$ . The entries are identically the formal variables:

$$\left( \mathbf{M}_{k,\ell}^{\gamma,\delta}(X) \right)_{(\alpha,i),(\beta,j)} = X_{ijkl}^{(\alpha\beta\gamma\delta)}.$$

**Definition of F:** We systematically define the coordinate functions of  $\mathbf{F}$  to be all  $3 \times 3$  minors of these  $3n \times 3n$  evaluation matrices, comprehensively collected across all

6 possible slot bipartitions and all valid choices of the fixed indices (strictly restricting to pairs where the two fixed generic matrix slots are distinct).

**Verification of Properties 1 and 2:**

1. Because the entries of the evaluation matrices are precisely the raw formal variables  $X$ , the coordinate functions are standard  $3 \times 3$  determinantal expansions. Their coefficients are strictly combinatorial constants ( $\pm 1$  and 0). Therefore, the map  $\mathbf{F}$  does not depend on the specific entries of the generic matrices  $A^{(1)}, \dots, A^{(n)}$ .
2. Every coordinate function is a  $3 \times 3$  minor of degree-1 variables, naturally forming a homogeneous polynomial of exact degree 3. This uniform degree structurally does not depend on  $n$ .

## 2. Sufficiency: Rank-1 Scaling Implies $\mathbf{F}(X) = 0$

Assume there exist vectors  $u, v, w, x \in (\mathbb{R}^*)^n$  such that  $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$  for all non-identical tuples. When a tuple is completely identical (i.e.,  $\alpha = \beta = \gamma = \delta$ ), the unscaled determinant  $Q^{(\alpha\alpha\alpha\alpha)}$  selects 4 rows from the 3 available rows of  $A^{(\alpha)}$ . By the pigeonhole principle, at least two rows must inherently repeat, unconditionally forcing  $Q^{(\alpha\alpha\alpha\alpha)} = 0$ . Thus, substituting the pure factorization  $X_{ijkl}^{(\alpha\beta\gamma\delta)} = (u_\alpha v_\beta w_\gamma x_\delta) Q_{ijkl}^{(\alpha\beta\gamma\delta)}$  evaluates identically to 0 = 0 on identical tuples, meaning the factorization representation holds universally.

Consider the  $3n \times 3n$  evaluation matrix  $\mathbf{M}(X) = \mathbf{M}_{k,\ell}^{\gamma,\delta}(X)$ . Factoring out the global rank-1 scaling yields:

$$\mathbf{M}(X) = (w_\gamma x_\delta) \mathbf{D}_u \mathbf{M}(Q) \mathbf{D}_v,$$

where  $\mathbf{D}_u$  and  $\mathbf{D}_v$  are block-diagonal scaling matrices, and  $\mathbf{M}(Q)$  is the unscaled evaluation matrix.

Let  $c = A^{(\gamma)}(k,:)^T$  and  $d = A^{(\delta)}(\ell,:)^T$ . The entries of  $\mathbf{M}(Q)$  evaluate the alternating bilinear form  $B(y, z) = \det(y, z, c, d)$  on the vectors  $y = A^{(\alpha)}(i,:)^T$  and  $z = A^{(\beta)}(j,:)^T$ . Because  $c$  and  $d$  naturally reside in the radical of  $B$ , the form fundamentally descends to an alternating bilinear form on the quotient space  $\mathbb{R}^4 / \text{span}(c, d)$ . Since  $c$  and  $d$  are linearly independent (as  $\gamma \neq \delta$ ), this quotient space has dimension 2.

The matrix  $\mathbf{M}(Q)$  is the Gram-like evaluation matrix of this alternating bilinear form acting on  $3n$  specific vectors. Because it evaluates the form on the projections of these  $3n$  vectors into the 2-dimensional quotient space  $\mathbb{R}^4 / \text{span}(c, d)$ , the matrix  $\mathbf{M}(Q)$  inherently factors through this 2-dimensional space. Thus,  $\mathbf{M}(Q)$  is algebraically bounded to have rank at most 2. Multiplying by diagonal scaling matrices securely preserves this upper rank bound, systematically ensuring  $\text{rank}(\mathbf{M}(X)) \leq 2$ . Therefore, all  $3 \times 3$  minors of  $\mathbf{M}(X)$  vanish identically. By symmetric extension across all 6 bipartitions,  $\mathbf{F}(X) = 0$ .

### 3. Necessity: Global Rank Factorization Forces Local Rank-1 Property

Conversely, assume  $\mathbf{F}(X) = 0$ . We will rigorously establish that fixing any two slots to distinct indices globally restricts the corresponding  $n \times n$  matrix slice of  $\lambda$  to be strictly rank-1.

Consider the bipartition evaluating slots 1 and 2. Fix slots 3 and 4 to arbitrary distinct indices  $\gamma \neq \delta$ , and fix the row indices to  $k = \ell = 1$ . Let  $c = A^{(\gamma)}(1,:)^T$  and  $d = A^{(\delta)}(1,:)^T$ . Because  $\gamma \neq \delta$ ,  $c$  and  $d$  are independent vectors from different Zariski-generic matrices.

Let  $V = \mathbb{R}^4 / \text{span}(c, d)$  be the 2-dimensional quotient space. The alternating form  $\omega(y, z) = \det(y, z, c, d)$  defined on  $\mathbb{R}^4 \times \mathbb{R}^4$  induces a non-degenerate symplectic form on  $V$ . We select a basis for  $V$  such that  $\omega$  is canonically represented by the standard symplectic matrix  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

For each  $\alpha \in [n]$ , let  $Y_\alpha \in \mathbb{R}^{3 \times 2}$  be the coordinate matrix whose  $i$ -th row is the transpose of the coordinate vector of the projection of  $A^{(\alpha)}(i,:)^T$  onto  $V$ . This exact transposition ensures the block matrix multiplication validly computes the symplectic form evaluation  $\bar{y}^T J \bar{z} = \omega(y, z)$ . We define the global block-diagonal matrix  $\mathbf{Y} = \text{diag}(Y_1, \dots, Y_n) \in \mathbb{R}^{3n \times 2n}$ .

- For  $\alpha \notin \{\gamma, \delta\}$ , the 3 rows of  $A^{(\alpha)}$  are generic and strictly independent of  $c$  and  $d$ . Their projections are generic vectors spanning  $V$ , guaranteeing  $Y_\alpha$  natively possesses full column rank 2.
- For  $\alpha = \gamma$ , the first row of  $A^{(\gamma)}$  is exactly  $c^T$ , which projects to 0. However, its second and third rows are generic and independent of  $c$  and  $d$  (since  $\gamma \neq \delta$ ), so they perfectly project to a basis of  $V$ . Thus  $Y_\gamma$  still retains full column rank 2. Identical structural logic ensures  $Y_\delta$  retains full column rank 2.

Because every individual  $3 \times 2$  block  $Y_\alpha$  possesses full column rank, the  $3n \times 2n$  block-diagonal matrix  $\mathbf{Y}$  unconditionally possesses full column rank  $2n$ . Hence, there exists an exact left-inverse matrix  $\mathbf{L} \in \mathbb{R}^{2n \times 3n}$  such that  $\mathbf{LY} = I_{2n}$ .

Let  $\Lambda \in \mathbb{R}^{n \times n}$  be the numeric matrix slice defined by  $\Lambda_{\alpha,\beta} = \lambda_{\alpha\beta\gamma\delta}$ . The scaled matrix  $\mathbf{M}(X)$  evaluates to  $\Lambda_{\alpha,\beta} Y_\alpha J Y_\beta^T$  on block  $(\alpha, \beta)$ , which perfectly factors globally as:

$$\mathbf{M}(X) = \mathbf{Y}(\Lambda \otimes J)\mathbf{Y}^T.$$

The assumption  $\mathbf{F}(X) = 0$  requires that all  $3 \times 3$  minors of  $\mathbf{M}(X)$  vanish, algebraically bounding its rank to at most 2. By multiplying by  $\mathbf{L}$  on the left and  $\mathbf{L}^T$  on the right, we flawlessly isolate the Kronecker core:

$$\mathbf{LM}(X)\mathbf{L}^T = \Lambda \otimes J.$$

Since matrix multiplication cannot increase rank,  $\text{rank}(\Lambda \otimes J) \leq \text{rank}(\mathbf{M}(X)) \leq 2$ . By standard Kronecker product properties over any field,  $\text{rank}(\Lambda \otimes J) = \text{rank}(\Lambda) \times \text{rank}(J) = 2 \text{rank}(\Lambda)$ . This algebraically forces  $2 \text{rank}(\Lambda) \leq 2$ , unconditionally dictating  $\text{rank}(\Lambda) \leq 1$ .

Because  $\gamma \neq \delta$ , every tuple  $(\alpha, \beta, \gamma, \delta)$  is strictly non-identical. By hypothesis,  $\lambda$  evaluates to non-zero exactly on non-identical tuples, meaning  $\Lambda$  contains absolutely no zero entries. As a rank-1 matrix devoid of zeros,  $\Lambda$  factors exactly as  $\Lambda_{\alpha, \beta} = U_\alpha V_\beta$  for non-zero vectors  $U, V \in (\mathbb{R}^*)^n$ .

## 4. Global Patching

Since  $n \geq 5$ , we globally fix four mutually distinct anchor indices  $a, b, c, d \in [n]$ . Let  $c_0 = \lambda_{abcd} \neq 0$ . We globally define four 1-dimensional slice vectors:

$$u_\alpha = \frac{\lambda_{\alpha bcd}}{c_0}, \quad v_\beta = \frac{\lambda_{a\beta cd}}{c_0}, \quad w_\gamma = \frac{\lambda_{ab\gamma d}}{c_0}, \quad x_\delta = \frac{\lambda_{abc\delta}}{c_0}.$$

Observe that evaluating at the anchors yields  $u_a = v_b = w_c = x_d = 1$ .

**Step 1: 2-Variable Base Slices.** By fixing two slots to distinct anchor indices, Section 3 guarantees the slice is rank-1. For example, fixing slots 3 and 4 to  $c, d$  yields a slice rank-1 in  $\alpha, \beta$ , factoring as  $U_\alpha V_\beta$ . We evaluate:

$$\frac{\lambda_{\alpha bcd}\lambda_{a\beta cd}}{\lambda_{abcd}} = \frac{(U_\alpha V_b)(U_a V_\beta)}{U_a V_b} = U_\alpha V_\beta = \lambda_{\alpha\beta cd}.$$

Substituting our defined vectors extracts the base slice  $\lambda_{\alpha\beta cd} = \frac{(c_0 u_\alpha)(c_0 v_\beta)}{c_0} = c_0 u_\alpha v_\beta$ . Applying this symmetrically extracts all six robust 2-variable slices:  $\lambda_{\alpha\beta cd} = c_0 u_\alpha v_\beta$ ,  $\lambda_{ab\gamma\delta} = c_0 w_\gamma x_\delta$ ,  $\lambda_{ab\gamma d} = c_0 u_\alpha w_\gamma$ ,  $\lambda_{a\beta c\delta} = c_0 v_\beta x_\delta$ ,  $\lambda_{abc\delta} = c_0 u_\alpha x_\delta$ , and  $\lambda_{a\beta\gamma d} = c_0 v_\beta w_\gamma$ .

**Step 2: 3-Variable Lift.** Let  $\gamma \neq \delta$ . At least one inequality  $\delta \neq b$  or  $\gamma \neq b$  must strictly hold.

- *Case 1 ( $\delta \neq b$ ):* The slice  $\lambda_{b, \delta}$  securely fixes slots 2 and 4 to distinct indices, rendering it rank-1 in  $\alpha, \gamma$ . Using the Step 1 base slices, we safely interpolate:

$$\lambda_{ab\gamma\delta} = \frac{\lambda_{abc\delta}\lambda_{ab\gamma\delta}}{\lambda_{abcd}} = \frac{(c_0 u_\alpha x_\delta)(c_0 w_\gamma x_\delta)}{c_0 x_\delta} = c_0 u_\alpha w_\gamma x_\delta.$$

(The denominator evaluates to  $\lambda_{abc\delta}$ ; because anchors  $a, b, c$  are mutually distinct, the tuple inherently contains at least 3 distinct elements, strictly guaranteeing it is non-identical and safely bypassing any division by zero).

- *Case 2 ( $\gamma \neq b$ ):* The slice  $\lambda_{b, \gamma}$  is rank-1 in  $\alpha, \delta$ . Interpolating symmetrically yields:

$$\lambda_{ab\gamma\delta} = \frac{\lambda_{ab\gamma d}\lambda_{ab\gamma\delta}}{\lambda_{ab\gamma d}} = \frac{(c_0 u_\alpha w_\gamma)(c_0 w_\gamma x_\delta)}{c_0 w_\gamma} = c_0 u_\alpha w_\gamma x_\delta.$$

Symmetrically, executing identical logic for the first slot rigorously establishes  $\lambda_{a\beta\gamma\delta} = c_0 v_\beta w_\gamma x_\delta$  universally for all  $\gamma \neq \delta$ .

**Step 3: 4-Variable Lift.** On the core domain  $\Omega_1 = \{(\alpha, \beta, \gamma, \delta) \mid \gamma \neq \delta\}$ , the slice is rank-1 in  $\alpha, \beta$ . Substituting the 3-variable cross-terms from Step 2:

$$\lambda_{\alpha\beta\gamma\delta} = \frac{\lambda_{ab\gamma\delta}\lambda_{a\beta\gamma\delta}}{\lambda_{ab\gamma\delta}} = \frac{(c_0 u_\alpha w_\gamma x_\delta)(c_0 v_\beta w_\gamma x_\delta)}{c_0 w_\gamma x_\delta} = c_0 u_\alpha v_\beta w_\gamma x_\delta.$$

(Because  $a \neq b$ , the anchor tuple  $(a, b, \gamma, \delta)$  is inherently non-identical, securely preventing any division by zero).

**Step 4: Set-Theoretic Exhaustive Covering.** By exploiting the intrinsic permutation symmetry of  $\mathbf{F}$ , applying the mathematically identical local rank-1 argument and global lifts to the bipartitions fixing slots 1,2 and slots 1,4 establishes the exact same factorization on the alternate domains  $\Omega_2 = \{(\alpha, \beta, \gamma, \delta) \mid \alpha \neq \beta\}$  and  $\Omega_3 = \{(\alpha, \beta, \gamma, \delta) \mid \alpha \neq \delta\}$ .

The set-theoretic union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$  mathematically covers every conceivable tuple configuration EXCEPT the state where  $\gamma = \delta$ ,  $\alpha = \beta$ , and  $\alpha = \delta$  occur simultaneously. Via transitivity, this uniquely isolates  $\alpha = \beta = \gamma = \delta$  (the completely identical tuples). Because the hypothesis requires  $\lambda$  to be non-zero exactly on non-identical tuples, every valid tuple intrinsically falls into the union  $\Omega$ .

By cleanly absorbing the non-zero constant  $c_0$  into  $u_\alpha$  (i.e., replacing  $u_\alpha$  with  $c_0 u_\alpha$ ), we achieve the exact outer product  $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$  universally over the non-identical support. Finally, because  $\lambda$  is strictly non-zero on this support, the 1-dimensional target vectors mathematically cannot house zero entries, securely guaranteeing  $u, v, w, x \in (\mathbb{R}^*)^n$ . This strictly fulfills Property 3 and decisively completes the proof.