The prompts and responses below have been edited only for formatting compatibility.

FirstProof # 7

Suppose that Γ is a uniform lattice in a real semi-simple group, and that Γ contains some 2-torsion. Is it possible for Γ to be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over the rational numbers $\mathbb{Q}$?

Model Response (Agent A)

1 Introduction and Main Result

Let G be a real semi-simple Lie group with finitely many connected components, and let $K \subset G$ be a maximal compact subgroup. The quotient symmetric space $S = G/K$ is a contractible Hadamard manifold. Let $\Gamma \subset G$ be a uniform lattice containing an element of 2-torsion. In this note, we resolve the question of whether Γ can act as the fundamental group of a compact manifold without boundary M whose universal cover $\widetilde{M}$ is acyclic over the rationals $\mathbb{Q}$.

Theorem 1. *It is impossible for such a uniform lattice Γ to be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over $\mathbb{Q}$.*

We provide two independent proofs of Theorem 1. The first relies on purely topological methods to establish a more general obstruction: no discrete group containing torsion can act as the fundamental group of such a manifold. The second proof engages directly with the geometric structure of the uniform lattice and its proper, cocompact action on the associated symmetric space to construct a rigid geometric contradiction.

2 Topological Obstruction

First Proof of Theorem 1. Suppose for the sake of contradiction that there exists a compact manifold without boundary M of dimension n such that its fundamental group $\Gamma = \pi_1(M)$ contains an element γ of order 2, and its universal cover $X = \widetilde{M}$ is acyclic over $\mathbb{Q}$. Since Γ contains a 2-torsion element, M is not a point, so $n \geq 1$. Because M is a topological manifold, its universal cover X is inherently a simply connected, and hence orientable, n-dimensional topological manifold. By Poincaré duality for non-compact orientable manifolds, the compactly supported rational cohomology of X is isomorphic to its ordinary rational homology:

$$H^k_c(X; \mathbb{Q}) \cong H_{n-k}(X; \mathbb{Q}) \tag{1}$$

for all integers k. Because X is $\mathbb{Q}$-acyclic by hypothesis, its rational homology is concentrated entirely in degree zero, with $H_0(X; \mathbb{Q}) \cong \mathbb{Q}$. Consequently, the compactly supported rational cohomology of X is finite-dimensional and concentrated exclusively in the top degree n:

$$H_c^k(X;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases} \tag{2}$$

The fundamental group Γ acts on X by deck transformations. Because M is a manifold, this properly discontinuous action is strictly free. Therefore, the torsion element γ acts on X as a free, finite-order homeomorphism.
As a homeomorphism of X, γ induces a linear pullback automorphism γ^* on the compactly supported cohomology $H_c^*(X;\mathbb{Q})$. Because $H_c^n(X;\mathbb{Q})$ is a 1-dimensional $\mathbb{Q}$-vector space and $\gamma^2 = \mathrm{id}$, the induced map γ^* must act as multiplication by ± 1. We evaluate the compactly supported Lefschetz number of γ, defined algebraically as the alternating sum of the traces of the induced maps on cohomology. This collapses to a single non-vanishing term at $k = n$, yielding:

$$L_c(\gamma, X) = \sum_{k=0}^{n} (-1)^k \operatorname{Tr}\left(\gamma^* \mid H_c^k(X;\mathbb{Q})\right) = (-1)^n(\pm 1) \neq 0. \tag{3}$$

On the other hand, we can compute $L_c(\gamma, X)$ topologically. Since γ generates a strictly free action of $\mathbb{Z}/2\mathbb{Z}$ on X, the quotient $Y = X/\langle\gamma\rangle$ is a non-compact manifold, and the projection $X \to Y$ is a 2-sheeted regular covering map. The transfer map for compactly supported rational cohomology yields an isomorphism $H_c^*(Y;\mathbb{Q}) \cong H_c^*(X;\mathbb{Q})^\gamma$. Because $H_c^*(X;\mathbb{Q})$ is finite-dimensional, $H_c^*(Y;\mathbb{Q})$ is finite-dimensional, so its compactly supported Euler characteristic $\chi_c(Y)$ is a well-defined integer.
Algebraically, the dimension of this invariant subspace dictates $\dim H_c^k(Y;\mathbb{Q}) = \frac{1}{2}\left(\dim H_c^k(X;\mathbb{Q}) + \operatorname{Tr}(\gamma^* \mid H_c^k(X;\mathbb{Q}))\right)$. Taking the alternating sum over all degrees yields the relation:

$$\chi_c(Y) = \frac{1}{2}\left(\chi_c(X) + L_c(\gamma, X)\right). \tag{4}$$

However, for any finite k-sheeted unbranched covering of locally compact ENRs with finite Betti numbers, the compactly supported Euler characteristic is strictly multiplicative. Thus, $\chi_c(X) = 2\chi_c(Y)$. Substituting this into our relation strictly forces:

$$L_c(\gamma, X) = 0. \tag{5}$$

Equating (3) and (5) yields $0 = \pm 1$, a mathematical contradiction. We conclude that no such manifold M can exist. $\square$

3 Geometric Obstruction via Symmetric Spaces

While the previous section fully resolves the problem, the specific geometric hypothesis that Γ is a uniform lattice in G allows for a rich alternative proof intertwining the algebraic action of the torsion element on $\widetilde{M}$ with its geometric action on S.

Second Proof of Theorem 1. Assume for the sake of contradiction that M exists, and let $X = \widetilde{M}$. Since M is a compact manifold, it possesses the homotopy type of a finite CW complex, thereby equipping X with the structure of a proper, cocompact Γ-CW complex.

Because S is a contractible space, the Borel construction projection of the mixed space $(X \times S)/\Gamma \to X/\Gamma = M$ is a fiber bundle with contractible fiber S. Since the base M has the homotopy type of a finite CW complex, standard obstruction theory guarantees the existence of a continuous global section. Lifting this section to the universal covers yields a continuous Γ-equivariant map $f : X \to S$.

Lemma 1. *The Γ-equivariant map $f : X \to S$ is a proper map.*

Proof. Because Γ acts cocompactly on both X (as M is compact) and S (as Γ is a uniform lattice), we can select compact fundamental domains $D_X \subset X$ and $D_S \subset S$. For any compact subset $K \subset S$, properness of the Γ-action on S ensures that the set of group elements $\Phi = \{g \in \Gamma \mid gf(D_X) \cap K \neq \emptyset\}$ is finite.

If $x \in f^{-1}(K)$, then $x = g \cdot y$ for some $g \in \Gamma$ and $y \in D_X$. By Γ-equivariance, $f(x) = g \cdot f(y) \in K$, enforcing $g \in \Phi$. Thus, the closed set $f^{-1}(K)$ is contained entirely in the finite union of compact sets $\bigcup_{g \in \Phi} g \cdot D_X$. Therefore, $f^{-1}(K)$ is definitively compact, establishing that f is a proper map. $\square$

Since Γ is a finitely generated lattice in a Lie group G with finitely many connected components, its image under the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a finitely generated linear group. By Selberg's Lemma, $\mathrm{Ad}(\Gamma)$ contains a torsion-free normal subgroup of finite index. Because the kernel of the adjoint representation is the finite center $Z(G)$, pulling this subgroup back to Γ and intersecting it with a subgroup avoiding the finite central torsion ensures Γ itself is virtually torsion-free. Thus, Γ contains a torsion-free normal subgroup Γ_0 of finite index. By passing to a further finite-index subgroup if necessary, we may assume Γ_0 acts preserving the chosen orientations of both X and S.

Because S is a Hadamard manifold and Γ_0 is torsion-free, Γ_0 acts strictly freely on S. The respective quotients $M_0 = X/\Gamma_0$ and $S_0 = S/\Gamma_0$ are thus compact orientable manifolds. Let $d = \dim S$. The proper map f descends to a continuous map $f_0 : M_0 \to S_0$.

Because S is contractible, S_0 is an Eilenberg-MacLane space $K(\Gamma_0, 1)$, natively yielding $H^*(S_0; \mathbb{Q}) \cong H^*(\Gamma_0; \mathbb{Q})$. Furthermore, because X is acyclic over $\mathbb{Q}$, the Cartan-Leray spectral sequence for the regular covering $X \to M_0$ collapses at the E_2 page, yielding $H^*(M_0; \mathbb{Q}) \cong H^*(\Gamma_0; \mathbb{Q})$. Since f is Γ-equivariant, f_0 induces the identity homomorphism on $\pi_1 \cong \Gamma_0$. Thus, f_0 induces an isomorphism on rational cohomology: $f_0^* : H^*(S_0; \mathbb{Q}) \xrightarrow{\sim} H^*(M_0; \mathbb{Q})$.

Because S_0 is a compact orientable d-manifold, its top non-vanishing rational cohomology is strictly $H^d(S_0; \mathbb{Q}) \cong \mathbb{Q}$. The isomorphism f_0^* ensures $H^d(M_0; \mathbb{Q}) \cong \mathbb{Q}$. As f_0^* is an isomorphism globally in all degrees, the non-vanishing cohomologies of M_0 and S_0 must perfectly align, strictly forcing the global dimension of M_0 to be exactly d.

Because f_0 is a continuous map between orientable compact d-manifolds that induces an isomorphism in the top degree, it possesses a non-zero mapping degree $D \neq 0$. Consequently, its proper lift $f : X \to S$ similarly carries this non-zero degree. For a proper map between non-compact oriented manifolds of the same dimension, the induced pullback on the top-degree compactly supported rational cohomology is equivalent to multiplication by its mapping degree D. Because $D \neq 0$,

$$f^* : H_c^d(S; \mathbb{Q}) \xrightarrow{\sim} H_c^d(X; \mathbb{Q}) \tag{6}$$

is an isomorphism of 1-dimensional $\mathbb{Q}$-vector spaces.
Let $\gamma \in \Gamma$ be the given element of order 2. Because f is strictly Γ-equivariant, commutativity enforces $\gamma_X^* \circ f^* = f^* \circ \gamma_S^*$. Since f^* is an isomorphism, the traces of γ^* on $H_c^d(X; \mathbb{Q})$ and $H_c^d(S; \mathbb{Q})$ identically match. Because all other compactly supported rational cohomology groups for both X and S identically vanish (by acyclicity and contractibility, respectively), their compactly supported Lefschetz numbers match precisely:

$$L_c(\gamma_X, X) = L_c(\gamma_S, S). \tag{7}$$

As established natively in the first proof via the transfer map and Euler characteristic multiplicativity, the strictly free action of γ_X on X enforces:

$$L_c(\gamma_X, X) = 0. \tag{8}$$

On the symmetric space $S \cong \mathbb{R}^d$, γ_S acts as an isometric involution. By Cartan's fixed-point theorem for complete simply connected manifolds of non-positive curvature, the fixed-point set S^{γ_S} is non-empty. Selecting a fixed point $p \in S^{\gamma_S}$, the exponential map at p provides a γ_S-equivariant diffeomorphism from the tangent space $T_pS \cong \mathbb{R}^d$ to S. Under this identification, γ_S acts as a linear involution. Its $+1$ eigenspace has some dimension k $(0 \leq k \leq d)$ corresponding to the tangent space of S^{γ_S}, and its -1 eigenspace has dimension $d - k$. The induced action of γ_S on the top compactly supported cohomology $H_c^d(S; \mathbb{Q}) \cong \mathbb{Q}$ is equivalent to multiplication by the sign of the determinant of this linear involution, which is precisely $(-1)^{d-k}$. Thus, its compactly supported Lefschetz number evaluates algebraically to:

$$L_c(\gamma_S, S) = (-1)^d \operatorname{Tr}(\gamma_S^* \mid H_c^d(S; \mathbb{Q})) = (-1)^d (-1)^{d-k} = (-1)^k \neq 0. \tag{9}$$

Equating (8) and (9) by way of (7) yields $0 = (-1)^k$, an inescapable contradiction structurally confirming that Γ cannot be the fundamental group of M. $\square$