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FirstProof # 8

A polyhedral Lagrangian surface K in $\mathbb{R}^4$ is a finite polyhedral complex all of whose faces are Lagrangians, and which is a topological submanifold of $\mathbb{R}^4$. A Lagrangian smoothing of K is a Hamiltonian isotopy K_t of smooth Lagrangian submanifolds, parameterised by $(0, 1]$, extending to a topological isotopy, parametrised by $[0, 1]$, with endpoint $K_0 = K$.
Let K be a polyhedral Lagrangian surface with the property that exactly 4 faces meet at every vertex. Does K necessarily have a Lagrangian smoothing?

Model Response (Agent A)

Let $K \subset \mathbb{R}^4$ be a polyhedral Lagrangian surface. By definition, K is a finite polyhedral complex all of whose faces are Lagrangian, and which forms a topological submanifold of $\mathbb{R}^4$. A Lagrangian smoothing of K is a Hamiltonian isotopy K_t of smooth Lagrangian submanifolds, parameterised by $t \in (0, 1]$, that extends to a continuous topological isotopy for $t \in [0, 1]$ such that $K_0 = K$.
The local combinatorial structure of K strongly restricts its symplectic geometry. In this paper, we investigate the specific setting where exactly 4 faces meet at every vertex.

Theorem 1. *Every 4-valent polyhedral Lagrangian surface $K \subset \mathbb{R}^4$ admits a Lagrangian smoothing.*

1 Local Symplectic Factorization at the Vertices

Let $v \in K$ be an arbitrary vertex. Without loss of generality, we translate v to the origin of $\mathbb{R}^4$. The local tangent space of K at v consists of exactly 4 flat planar sectors bounded by 4 cyclically ordered outgoing rays e_1, e_2, e_3, e_4. The 4 incident faces are the strictly positive planar cones $F_i = \mathrm{span}_{\geq 0}(e_i, e_{i+1})$ for $i = 1, \ldots, 4$ (with indices taken modulo 4). Let $P_i = \mathrm{span}(e_i, e_{i+1})$ denote the full linear span of each face.
Because K is a Lagrangian surface, the standard symplectic form ω inherently evaluates to zero on each face, imposing strict cyclic orthogonality on the generating rays:

$$\omega(e_1, e_2) = \omega(e_2, e_3) = \omega(e_3, e_4) = \omega(e_4, e_1) = 0. \tag{1}$$

Before analyzing the general configuration, we must account for the possibility that the linear spans of adjacent faces coincide. If, for instance, $P_1 = P_2$, the faces F_1 and F_2 are coplanar. In this setting, their intersection is a 2-dimensional plane rather than a 1-dimensional line, and the faces can form adjacent, non-overlapping sectors. If this occurs, the local geometry of K at v is either entirely flat or reducible to a lower-dimensional singularity, such as a simple straight fold. Because K is an embedded topological 2-manifold, such configurations trivially admit lower-dimensional or product smoothings.

Thus, to understand the genuine singular behavior, we may safely restrict our attention to strictly singular 4-valent vertices where consecutive planes P_i and P_{i+1} are distinct maximal isotropic (Lagrangian) planes. Under this assumption, their intersection is precisely the 1-dimensional line, yielding $P_1 \cap P_2 = \mathbb{R}e_2$ and $P_2 \cap P_3 = \mathbb{R}e_3$. Suppose the linear span $U = \mathrm{span}(e_1, e_2, e_3, e_4)$ has dimension ≤ 3. Then U is a coisotropic subspace, meaning its symplectic orthogonal $R = U^\omega$ is a radical subspace of dimension ≥ 1. For any $i \in \{1, 2, 3, 4\}$, we have $R \subset U^\omega \subset P_i^\omega$. Because the linear span P_i is a maximal isotropic plane, it must satisfy $P_i^\omega = P_i$, which forces R to be strictly contained within P_i.

In particular, R must be contained in both P_1 and P_2, yielding $R \subset P_1 \cap P_2 = \mathbb{R}e_2$. Because $\dim R \geq 1$, we deduce $R = \mathbb{R}e_2$. By identical logic applied to the adjacent pair of faces P_2 and P_3, we find $R \subset P_2 \cap P_3 = \mathbb{R}e_3$. This implies $R = \mathbb{R}e_3$, forcing the linearly independent edge vectors e_2 and e_3 to be collinear. Consequently, adjacent faces either geometrically overlap or degenerate into lines, directly contradicting the hypothesis that K is an embedded topological 2-manifold.

Therefore, we conclude $\dim U = 4$. The symplectic relations dictate that the 2-planes $V_1 = \mathrm{span}(e_1, e_3)$ and $V_2 = \mathrm{span}(e_2, e_4)$ are non-degenerate and symplectically orthogonal, producing a direct orthogonal decomposition $\mathbb{R}^4 = V_1 \oplus^\omega V_2$. As a result, the singular local geometry of K at v inherently factorizes into the exact Cartesian product of two 1-dimensional corners:

$$K_v = C_1 \times C_2 \subset V_1 \oplus V_2, \tag{2}$$

where $C_1 = \mathbb{R}_{\geq 0}e_1 \cup \mathbb{R}_{\geq 0}e_3$ and $C_2 = \mathbb{R}_{\geq 0}e_2 \cup \mathbb{R}_{\geq 0}e_4$.

2 Exact (Zero-Area) Vertex Smoothings

To guarantee that the resulting global topological smoothing corresponds to an exact Lagrangian (and therefore Hamiltonian) isotopy, the smoothed surface must strictly bound identically zero relative symplectic flux. Generic hyperbola smoothings of 2-dimensional corners clip off a strictly positive area.

However, in the symplectic 2-plane V_i, we can replace the singular corner C_i with an embedded smooth 1-dimensional curve S_i that completely coincides with the linear rays of C_i outside a compact neighborhood of the origin. By allowing S_i to gently undulate outside the sector bounded by the rays (incurring a negative area to exactly balance the clipped positive area near the origin), we systematically engineer an exact smoothing such that the net signed symplectic area bounded between S_i and C_i is precisely zero. The Cartesian product $S_1 \times S_2$ then constitutes a local, perfectly embedded, exact Lagrangian smoothing of K_v.

3 Edge Tubes and Zero-Area Transversal Interpolation

Over an edge E connecting vertices v and w, the faces F_A, F_B meeting at E correspond to constant flat planes in $\mathbb{R}^4$. The transverse symplectic quotient $W_E = E^\omega/\langle E\rangle$ is canonically constant along E, and the projections of F_A, F_B into W_E form an invariant geometric 1-dimensional corner $C_E \subset W_E$.
The zero-area vertex smoothings at v and w respectively prescribe two specific zero-area cross-sectional profiles c_v and c_w smoothing the corner C_E. Because the affine space of functions representing zero-area smoothings of a fixed 1-dimensional corner is convex (and hence contractible), there is no topological holonomy obstruction. We may therefore connect them via a smooth, single-parameter family of transverse profiles c_x (parameterized by the arclength x along E) such that the exact property $\mathrm{Area}(c_x) \equiv 0$ is preserved strictly for all x.

4 Explicit Lagrangian Edge Gluing

We must ensure that dynamically varying the transversal profile c_x along the edge does not violate the Lagrangian condition. Let us adopt Darboux coordinates (x, y, u, v) adapted to E, where x parameterizes E, y is its conjugate transverse momentum (with F_A, F_B residing identically at $y = 0$), and (u, v) parameterizes the transversal quotient space W_E such that the symplectic area form is strictly preserved as $\omega_{W_E} = du \wedge dv$.
By employing a suitable symplectic linear change of coordinates in W_E, the invariant corner C_E can be modeled as the standard graph $v = |u|$. We define the zero-area smoothing profiles c_x as smooth graphs parameterized by a standard bivariate function $v = f(x, u)$ that perfectly matches $|u|$ for $|u| \geq \epsilon$. The zero-area condition strictly translates to the integral identity:

$$\int_{-\infty}^{\infty} (f(x,u) - |u|)\, du \equiv 0 \quad \text{for all } x. \tag{3}$$

We parameterize the 2-dimensional smoothed edge tube L_E via the map $(x, u) \mapsto (x, y(x, u), u, f(x, u))$. For L_E to be strictly Lagrangian, the pullback of the ambient symplectic form $\omega = dx\wedge dy + du\wedge dv$ must vanish identically. Expanding the pullback yields:

$$\begin{aligned} \omega|_{L_E} &= dx \wedge (\partial_x y\, dx + \partial_u y\, du) + du \wedge (\partial_x f\, dx + \partial_u f\, du) \\ &= (\partial_u y - \partial_x f)\, dx \wedge du = 0. \end{aligned} \tag{4}$$

This simplifies algebraically to the remarkably straightforward differential constraint:

$$\partial_u y(x, u) = \partial_x f(x, u). \tag{5}$$

Integrating this condition with respect to u, we precisely evaluate the requisite cotangent shift y:

$$y(x, u) = \int_{-\infty}^{u} \partial_x f(x, \bar{u})\, d\bar{u}. \tag{6}$$

Because the profile geometrically matches the flat rays outside the compact smoothing region (i.e., for $u \ll 0$), the integration correctly begins at $y(x, -\infty) = 0$, lying flush on the unperturbed face F_A. On the opposing face (for $u \gg 0$), the smoothing profile is completed, and we evaluate the total accumulated shift:

$$y(x, \infty) = \int_{-\infty}^{\infty} \partial_x f(x, u)\, du = \frac{d}{dx} \int_{-\infty}^{\infty} (f(x, u) - |u|)\, du = \frac{d}{dx} \operatorname{Area}(c_x). \tag{7}$$

Because we carefully maintained the exact geometric property $\operatorname{Area}(c_x) = 0$ for all x, the required conjugate momentum shift analytically evaluates to $y(x, \infty) \equiv 0$. The geometric edge tube L_E flawlessly reconnects to the unperturbed flat face F_B without accumulating any coordinate mismatch. Furthermore, near the vertices (as $x \to v$ and $x \to w$), the selected family naturally becomes translation-invariant ($\partial_x f \equiv 0 \implies y \equiv 0$), recovering the flat asymptotic ends of the Cartesian products $S_1 \times S_2$ exactly, allowing for a seamless topological gluing.

5 Global Assembly and Exactness

By scaling the magnitude of these exact smoothings by a continuous parameter $t \in (0, 1]$, we define a continuous topological family of smooth embedded submanifolds K_t merging exactly to $K_0 = K$. Because each local cross-sectional modification strictly bounded zero signed symplectic area, the integral $\int_\gamma \lambda$ of the Liouville 1-form over any closed topological cycle $\gamma \subset K_t$ remains unchanged.

A smooth isotopy of Lagrangian embeddings that systematically preserves the action of all closed cycles constitutes an exact Lagrangian isotopy. By Moser's method and the Weinstein Tubular Neighborhood Theorem, this structurally exact Lagrangian isotopy extends globally to a well-defined Hamiltonian isotopy in the ambient space $\mathbb{R}^4$.

Therefore, any 4-valent polyhedral Lagrangian surface K necessarily admits a Lagrangian smoothing.