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## FirstProof # 2

Let  $F$  be a non-archimedean local field with ring of integers  $\mathfrak{o}$ . Let  $N_r$  denote the subgroup of  $\mathrm{GL}_r(F)$  consisting of upper-triangular unipotent elements. Let  $\psi : F \rightarrow \mathbb{C}^\times$  be a nontrivial additive character of conductor  $\mathfrak{o}$ , identified in the standard way with a generic character of  $N_r$ . Let  $\Pi$  be a generic irreducible admissible representation of  $\mathrm{GL}_{n+1}(F)$ , realized in its  $\psi^{-1}$ -Whittaker model  $\mathcal{W}(\Pi, \psi^{-1})$ . Must there exist  $W \in \mathcal{W}(\Pi, \psi^{-1})$  with the following property?

Let  $\pi$  be a generic irreducible admissible representation of  $\mathrm{GL}_n(F)$ , realized in its  $\psi$ -Whittaker model  $\mathcal{W}(\pi, \psi)$ . Let  $\mathfrak{q}$  denote the conductor ideal of  $\pi$ , let  $Q \in F^\times$  be a generator of  $\mathfrak{q}^{-1}$ , and set

$$u_Q := I_{n+1} + Q E_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$

where  $E_{i,j}$  is the matrix with a 1 in the  $(i,j)$ -entry and 0 elsewhere. For some  $V \in \mathcal{W}(\pi, \psi)$ , the local Rankin–Selberg integral

$$\int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1) u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

is finite and nonzero for all  $s \in \mathbb{C}$ .

## Model Response (Agent A)

### Introduction

Let  $F$  be a non-archimedean local field with ring of integers  $\mathfrak{o}$  and maximal ideal  $\mathfrak{p}$ . We fix a uniformizer  $\varpi \in \mathfrak{p}$  and let  $q = |\mathfrak{o}/\mathfrak{p}|$ . Let  $N_r$  denote the subgroup of  $\mathrm{GL}_r(F)$  consisting of upper-triangular unipotent elements, and let  $K_r = \mathrm{GL}_r(\mathfrak{o})$  denote the standard maximal compact subgroup. Let  $\psi : F \rightarrow \mathbb{C}^\times$  be a fixed nontrivial additive character of conductor  $\mathfrak{o}$ .

**Theorem 1.** *Let  $\Pi$  be a generic irreducible admissible representation of  $\mathrm{GL}_{n+1}(F)$ . There exists  $W \in \mathcal{W}(\Pi, \psi^{-1})$  such that for any generic irreducible admissible representation  $\pi$  of  $\mathrm{GL}_n(F)$  with conductor ideal  $\mathfrak{q}$ , and any generator  $Q \in F^\times$  of  $\mathfrak{q}^{-1}$ , there exists  $V \in \mathcal{W}(\pi, \psi)$  such that the local Rankin–Selberg integral*

$$Z(s, W, V) = \int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1) u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

*is finite and nonzero for all  $s \in \mathbb{C}$ , where  $u_Q = I_{n+1} + Q E_{n,n+1}$ .*

*Proof. Step 1: Gelfand–Kazhdan restriction and  $s$ -independence.*

We evaluate the right translation of  $W$  by  $u_Q$ . Block matrix multiplication yields

$$\text{diag}(g, 1)u_Q = U_Q \text{diag}(g, 1), \quad \text{where } U_Q = I_{n+1} + Q \sum_{i=1}^n g_{i,n} E_{i,n+1}.$$

Because  $U_Q \in N_{n+1}$  and its only superdiagonal entry (i.e., immediately above the main diagonal) is located at  $(n, n+1)$  with value  $Qg_{n,n}$ , the left  $N_{n+1}$ -equivariance of the Whittaker model  $\mathcal{W}(\Pi, \psi^{-1})$  entails

$$W(\text{diag}(g, 1)u_Q) = \psi^{-1}(Qg_{n,n})W(\text{diag}(g, 1)) = \psi(-Qg_{n,n})W(\text{diag}(g, 1)).$$

By the Gelfand–Kazhdan restriction theory for the Kirillov model (H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, *Rankin–Selberg Convolutions*, 1983), the restriction map  $W \mapsto W|_{\text{GL}_n(F)}$  surjects onto a space of functions containing  $\mathcal{C}_c^\infty(N_n \backslash \text{GL}_n(F), \psi^{-1})$ .

We define a smooth cut-off function  $\Phi \in \mathcal{C}_c^\infty(N_n \backslash \text{GL}_n(F), \psi^{-1})$  supported precisely on the open and closed double coset  $N_n K_n$  by setting  $\Phi(nk) = \psi^{-1}(n)$  for  $n \in N_n$  and  $k \in K_n$ , and extending it by zero elsewhere. This is well-defined because  $\psi$  is trivial on the intersection  $N_n \cap K_n = N_n \cap \text{GL}_n(\mathfrak{o})$ . We fix a choice of  $W \in \mathcal{W}(\Pi, \psi^{-1})$  satisfying  $W(\text{diag}(g, 1)) = \Phi(g)$ .

Substituting this test vector  $W$  restricts the domain of integration strictly to the compact quotient  $N_n \backslash N_n K_n \simeq (N_n \cap K_n) \backslash K_n$ . For  $k \in K_n$ , we have  $|\det k| = 1$ , which completely eliminates the complex parameter  $s$ . Normalizing the quotient measure appropriately, the integral converges absolutely to a finite,  $s$ -independent functional:

$$L_Q(V) = \int_{K_n} \psi(-Qk_{n,n})V(k) dk.$$

We are reduced to showing that there exists  $V \in \mathcal{W}(\pi, \psi)$  such that  $L_Q(V) \neq 0$ .

### Step 2: The unramified case ( $c = 0$ ).

If  $\pi$  is unramified, its conductor ideal is  $\mathfrak{q} = \mathfrak{o}$ , meaning  $Q \in \mathfrak{o}^\times$ . We evaluate the functional on the normalized spherical vector  $V = V_0$ , which satisfies  $V_0(k) = 1$  for all  $k \in K_n$ . Since  $k_{n,n} \in \mathfrak{o}$  and  $Q \in \mathfrak{o}^\times$ , we have  $-Qk_{n,n} \in \mathfrak{o}$ . Because the additive character  $\psi$  has conductor  $\mathfrak{o}$ , it follows that  $\psi(-Qk_{n,n}) = 1$ . The functional thus yields  $L_Q(V_0) = \text{vol}(K_n) > 0$ .

### Step 3: Finite Fourier analysis setup ( $c \geq 1$ ).

Assume  $\pi$  has conductor  $\mathfrak{q} = \mathfrak{p}^c$  with  $c \geq 1$ . Here,  $Q = \alpha\varpi^{-c}$  for some unit  $\alpha \in \mathfrak{o}^\times$ . Let  $V_0 \in \mathcal{W}(\pi, \psi)$  be the essential newform, properly normalized so that  $V_0(I_n) = 1$ . Suppose, for the sake of contradiction, that  $L_Q(\pi(h^{-1})V_0) = 0$  for all  $h \in K_n$ . Evaluating the functional and making the change of variables  $k \mapsto kh$  yields:

$$\int_{K_n} \psi(-Q(kh)_{n,n})V_0(k) dk = 0 \quad \text{for all } h \in K_n.$$

Let  $e_n = (0, \dots, 0, 1)$  be viewed as a row vector in  $\mathfrak{o}^n$ . Then  $(kh)_{n,n} = e_n k h e_n^T = (e_n k) y$ , where  $y = h e_n^T$  is a column vector. As  $h$  traverses  $K_n$ , the vector  $y$  traverses all unimodular column vectors in  $\mathfrak{o}^n$ .

Because  $Q = \alpha\varpi^{-c}$ , the value  $\psi(-Q(e_n k)y)$  depends on the row vector  $e_n k$  exclusively modulo  $\mathfrak{p}^c$ . We descend to the finite quotient module  $G = (\mathfrak{o}/\mathfrak{p}^c)^n$  by defining a function  $H : G \rightarrow \mathbb{C}$  as follows:

$$H(\eta) = \int_{\{k \in K_n : e_n k \equiv \eta \pmod{\mathfrak{p}^c}\}} V_0(k) dk.$$

If  $\eta$  does not lift to a unimodular vector in  $\mathfrak{o}^n$ , the domain of integration is empty, forcing  $H(\eta) = 0$ . The vanishing assumption dictates that the finite Fourier transform of  $H$  is identically zero on all unimodular vectors  $y \in G$ :

$$\widehat{H}(y) = \sum_{\eta \in G} H(\eta) \psi(-Q\eta y) = 0.$$

**Step 4: Fourier inversion and translation invariance.**

Since  $\widehat{H}(y) = 0$  for all unimodular  $y$ , the support of  $\widehat{H}$  is restricted to non-unimodular vectors. Over the finite module  $G$ , a vector is non-unimodular if and only if all its entries belong to  $\mathfrak{p}/\mathfrak{p}^c$ , meaning  $\widehat{H}$  is supported entirely on  $\mathfrak{p}G$ .

Applying the Fourier inversion formula over  $G$ , we obtain:

$$H(\eta) = \frac{1}{|G|} \sum_{y \in \mathfrak{p}G} \widehat{H}(y) \psi(Q\eta y).$$

Let  $y \in \mathfrak{p}G$ , guaranteeing  $y = \varpi z$  for some column vector  $z \in (\mathfrak{o}/\mathfrak{p}^{c-1})^n$ . For an arbitrary shift  $\delta \in \mathfrak{p}^{c-1}G$ , we may write  $\delta = \varpi^{c-1}x$  with a row vector  $x \in G$ . The inner product computes as:

$$Q\delta y = (\alpha\varpi^{-c})(\varpi^{c-1}x)(\varpi z) = \alpha x z \in \mathfrak{o}.$$

Because the additive character  $\psi$  has conductor  $\mathfrak{o}$ , we deduce  $\psi(Q\delta y) = \psi(\alpha x z) = 1$ . The additive character thus neutralizes the shift, giving  $\psi(Q(\eta + \delta)y) = \psi(Q\eta y)$ . Consequently,  $H(\eta)$  exhibits translation invariance:

$$H(\eta + \delta) = H(\eta) \quad \text{for all } \delta \in \mathfrak{p}^{c-1}G.$$

**Step 5: Level lowering and contradiction.**

We evaluate  $H(e_n)$ . The domain of integration enforces the condition  $e_n k \equiv e_n \pmod{\mathfrak{p}^c}$ , which strictly characterizes the mirabolic congruence subgroup  $K_1(\mathfrak{p}^c)$  consisting of matrices in  $K_n$  whose last row is congruent to  $e_n$  modulo  $\mathfrak{p}^c$ . By foundational theory (H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, *Conducteur des représentations du groupe linéaire*, 1981), the essential newform  $V_0$  is invariant under right translation by  $K_1(\mathfrak{p}^c)$ . Since  $V_0(I_n) = 1$ , we obtain:

$$H(e_n) = \int_{K_1(\mathfrak{p}^c)} V_0(k) dk = \text{vol}(K_1(\mathfrak{p}^c)) > 0.$$

By the translation invariance established above, summing  $H$  over the  $q^n$  mutually distinct shifts  $\delta \in \mathfrak{p}^{c-1}G$  yields:

$$\sum_{\delta \in \mathfrak{p}^{c-1}G} H(e_n + \delta) = q^n H(e_n) > 0.$$

Conversely, returning to the definition of  $H$ , the disjoint union of the integration domains  $\{k \in K_n : e_n k \equiv e_n + \delta \pmod{\mathfrak{p}^c}\}$  systematically aggregates to the coarser condition  $e_n k \equiv e_n \pmod{\mathfrak{p}^{c-1}}$ . This aggregated domain precisely identifies the broader mirabolic congruence subgroup  $K_1(\mathfrak{p}^{c-1})$ . Therefore:

$$\sum_{\delta \in \mathfrak{p}^{c-1} G} H(e_n + \delta) = \int_{K_1(\mathfrak{p}^{c-1})} V_0(k) dk.$$

This integral calculates the evaluation at  $I_n$  of the locally averaged vector

$$v = \int_{K_1(\mathfrak{p}^{c-1})} \pi(k) V_0 dk \in \mathcal{W}(\pi, \psi).$$

Because the conductor of  $\pi$  is exactly  $\mathfrak{p}^c$ , the representation  $\pi$  contains no nonzero vectors invariant under the larger mirabolic congruence subgroup  $K_1(\mathfrak{p}^{c-1})$ . Thus, the vector  $v$  identically vanishes, mandating  $v(I_n) = 0$ .

We are thereby led to the contradiction  $0 = q^n \text{vol}(K_1(\mathfrak{p}^c)) > 0$ . This contradiction demonstrates that the uniform vanishing hypothesis must be false. We conclude that there exists some translate  $V = \pi(h^{-1})V_0$  for which the local Rankin–Selberg integral is finite and nonzero, completing the proof.  $\square$