

The prompts and responses below have been edited only for formatting compatibility.

FirstProof # 7

Suppose that Γ is a uniform lattice in a real semi-simple group, and that Γ contains some 2-torsion. Is it possible for Γ to be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over the rational numbers \mathbb{Q} ?

Model Response (Agent B)

Theorem. *Let G be a real semi-simple Lie group, and let $\Gamma \subset G$ be a uniform lattice containing an element of order 2. Then Γ cannot be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over the rational numbers \mathbb{Q} .*

Proof. The proof proceeds by contradiction. Assume there exists a compact, boundaryless manifold M with fundamental group $\pi_1(M) \cong \Gamma$ whose universal cover \tilde{M} is rationally acyclic. Let K be a maximal compact subgroup of G , and let $X = G/K$ be the associated symmetric space. As a symmetric space of non-compact type, X is a contractible, proper CAT(0) metric space. Let $d = \dim X$. By hypothesis, Γ contains an element γ of order 2.

Step 1: Topological Setup and Equivariant Classifying Map

By Selberg's Lemma, the finitely generated linear group Γ contains a torsion-free normal subgroup of finite index, say Γ_1 . To ensure orientability and connectedness, we refine this subgroup. Let G^0 be the connected identity component of G . Let $\Gamma_X^+ \subset \Gamma$ be the subgroup acting by orientation-preserving isometries on X , and let $\Gamma_M^+ \subset \Gamma$ be the subgroup acting by orientation-preserving deck transformations on \tilde{M} . Because orientations admit exactly two states, these subgroups have index at most 2 in Γ . We define

$$\Gamma_0 = \Gamma_1 \cap G^0 \cap \Gamma_X^+ \cap \Gamma_M^+.$$

As a finite intersection of finite-index normal subgroups, Γ_0 is a torsion-free normal subgroup of Γ of finite index. Crucially, $\Gamma_0 \subset G^0$, and it acts freely and orientation-preservingly on both \tilde{M} and X .

The quotient spaces $M_0 = \tilde{M}/\Gamma_0$ and $X_0 = X/\Gamma_0$ are closed, orientable manifolds. Because X is contractible and the Γ_0 -action is free and cocompact, X_0 natively serves as a compact Eilenberg–MacLane classifying space $B\Gamma_0$.

To rigorously construct a classifying map that is strictly pointwise equivariant, we apply Bredon equivariant obstruction theory on the universal covers. The full group Γ acts strictly freely on \tilde{M} via deck transformations, making \tilde{M} a free Γ -CW complex. Furthermore, X is a contractible proper Γ -space. The geometric obstructions to constructing a Γ -equivariant map reside in the Bredon equivariant cohomology groups $H_F^k(\tilde{M}; \pi_{k-1}(X))$. Since X is contractible, $\pi_{k-1}(X) = 0$ for all $k \geq 1$, meaning all obstructions identically vanish. Thus, there exists a strictly Γ -equivariant continuous

map $\tilde{f} : \tilde{M} \rightarrow X$. Descending this map to the Γ_0 -quotients yields a canonical classifying map $f_0 : M_0 \rightarrow X_0$, which is strictly Γ/Γ_0 -equivariant by geometric construction. Because both \tilde{M} and X are \mathbb{Q} -acyclic, the Cartan–Leray spectral sequence collapses, guaranteeing that f_0 induces an isomorphism on rational homology. Consequently, $\dim M_0 = \dim X_0 = d$, and its proper topological mapping degree $D = \deg(f_0)$ is a strictly non-zero integer.

Because Γ_0 is a normal subgroup, the order-2 element $\gamma \notin \Gamma_0$ projects to a non-trivial involution $\bar{\gamma} \in \Gamma/\Gamma_0$ acting on both M_0 and X_0 . Let $\epsilon_M, \epsilon_X \in \{\pm 1\}$ denote the orientation parity of $\bar{\gamma}$ on M_0 and X_0 , respectively. Because f_0 is strictly $\mathbb{Z}/2$ -equivariant with respect to $\bar{\gamma}$ (i.e., $f_0 \circ \bar{\gamma} = \bar{\gamma} \circ f_0$), evaluating the induced homology maps on the fundamental class $[M_0] \in H_d(M_0; \mathbb{Z})$ yields:

$$f_{0*}(\bar{\gamma}_*[M_0]) = f_{0*}(\epsilon_M[M_0]) = \epsilon_M D[X_0],$$

$$\bar{\gamma}_*(f_{0*}[M_0]) = \bar{\gamma}_*(D[X_0]) = \epsilon_X D[X_0].$$

This algebraically mandates $\epsilon_M D = \epsilon_X D$. Since $D \neq 0$, we immediately obtain $\epsilon_M = \epsilon_X$. Thus, $\bar{\gamma}$ either preserves the orientation of both manifolds or reverses the orientation of both manifolds.

Step 2: The Mapping Degree Parity Constraint via Atiyah–Borel Localization

We now establish that the proper mapping degree D must be an *even integer*.

Because X is a complete CAT(0) metric space, Cartan’s Fixed-Point Theorem ensures that the finite group $\langle \gamma \rangle$ fixes a point in X . This equivariance trivially descends to a fixed point for the involution $\bar{\gamma}$ on the quotient; thus, the fixed-point set $X_0^{\bar{\gamma}} \neq \emptyset$. Conversely, suppose $\bar{\gamma}$ fixed a point $[y] \in M_0$. The fixed-point relation would natively lift to $\gamma \tilde{y} = g_0 \tilde{y}$ for some $g_0 \in \Gamma_0$, where $\tilde{y} \in \tilde{M}$ represents a valid chosen lift of $[y]$. The freeness of the Γ -action on \tilde{M} mandates $\gamma = g_0 \in \Gamma_0$. Since γ has order 2 and Γ_0 is torsion-free, this is mathematically impossible. Thus, $M_0^{\bar{\gamma}} = \emptyset$.

Assume for contradiction that D is an odd integer. We evaluate $\mathbb{Z}/2$ -equivariant Borel cohomology with \mathbb{F}_2 coefficients. To maintain orientability of the Borel constructions, we select the dimension N of the approximating sphere S^N based on the orientation parity ϵ_X :

- If $\epsilon_X = 1$ (orientation-preserving), we choose N to be an odd integer, ensuring the antipodal map on S^N preserves orientation.
- If $\epsilon_X = -1$ (orientation-reversing), we choose N to be an even integer, ensuring the antipodal map on S^N reverses orientation.

In both cases, the diagonal $\mathbb{Z}/2$ -action on the products $M_0 \times S^N$ and $X_0 \times S^N$ strictly preserves orientation. Thus, the Borel quotients $M_N = M_0 \times_{\mathbb{Z}/2} S^N$ and $X_N = X_0 \times_{\mathbb{Z}/2} S^N$ are strictly closed, orientable manifolds.

The equivariant map f_0 induces a proper fiber bundle map $f_N : M_N \rightarrow X_N$ of degree D . The ordinary cohomological Gysin transfer unconditionally satisfies $(f_N)_! \circ f_N^*(x) = (D \bmod 2) \cdot x$. Because D is odd, $D \equiv 1 \pmod{2}$, meaning the composition

acts as the identity on $H^*(X_N; \mathbb{F}_2)$. This algebraically forces the pullback f_N^* to be a split injection.

Taking the inverse limit as $N \rightarrow \infty$ (over the parity-appropriate sequence of spheres), the true equivariant pullback $f_{\mathbb{Z}/2}^* : H_{\mathbb{Z}/2}^*(X_0; \mathbb{F}_2) \rightarrow H_{\mathbb{Z}/2}^*(M_0; \mathbb{F}_2)$ is forced to be injective. By the exactness of localization, passing to the localized rings by inverting the polynomial generator $u \in H^1(B\mathbb{Z}/2; \mathbb{F}_2)$ preserves this injectivity:

$$S^{-1}f_{\mathbb{Z}/2}^* : S^{-1}H_{\mathbb{Z}/2}^*(X_0; \mathbb{F}_2) \hookrightarrow S^{-1}H_{\mathbb{Z}/2}^*(M_0; \mathbb{F}_2).$$

By the Atiyah–Borel Localization Theorem, the localized equivariant cohomology of a space is isomorphic to the localized equivariant cohomology of its fixed-point set:

$$S^{-1}H_{\mathbb{Z}/2}^*(Y; \mathbb{F}_2) \cong H^*(Y^{\tilde{\gamma}}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[u, u^{-1}].$$

Because $X_0^{\tilde{\gamma}} \neq \emptyset$, the localized domain evaluates to a non-zero module. Because $M_0^{\tilde{\gamma}} = \emptyset$, the localized target is identically zero. An exact injection from a non-zero module into a zero module is impossible. Therefore, D must rigidly be an **even integer**.

Step 3: Higher Signature Rigidity via the Strong Novikov Conjecture

We evaluate the symmetric higher signature of M_0 within the real C^* -algebra K-theory $KO_d(C_r^*(\Gamma_0; \mathbb{R}))$ to establish a fundamentally conflicting constraint.

Because \tilde{M} and X are \mathbb{Q} -acyclic, the Universal Coefficient Theorem guarantees they are \mathbb{R} -acyclic. Thus, their real cellular chain complexes $C_*(\tilde{M}; \mathbb{R})$ and $C_*(X; \mathbb{R})$ constitute finite-length projective free resolutions of the trivial module \mathbb{R} over the real group ring $\mathbb{R}\Gamma_0$. By the Fundamental Lemma of Homological Algebra, the chain map $C_*(f; \mathbb{R})$ is inherently a strict chain homotopy equivalence over $\mathbb{R}\Gamma_0$.

The algebraic symmetric Poincaré duality structure of M_0 (defined by capping with $[M_0]$) pushes forward under this chain equivalence to capping with $f_{0*}[M_0] = D[X_0]$. Thus, the algebraic symmetric Poincaré complex defining the higher signature $\sigma(M_0)$ is strictly chain-equivalent to the complex of X_0 globally scaled by D .

Over any real C^* -algebra, scaling a non-degenerate symmetric Poincaré complex by a non-zero real scalar D yields a complex isomorphic to one scaled merely by its sign. Specifically, we can algebraically conjugate the duality structure with the central, self-adjoint, invertible scalar $c = 1/\sqrt{|D|} \in \mathbb{R}$. This canonical isomorphism scales the symmetric form by exactly $c \cdot c^* \cdot D = c^2 \cdot D = D/|D| = \text{sgn}(D)$. Therefore, the analytic symmetric signatures natively satisfy:

$$\sigma(M_0) = \text{sgn}(D)\sigma(X_0) \in KO_d(C_r^*(\Gamma_0; \mathbb{R})).$$

Because $\Gamma_0 \subset G^0$ is a discrete uniform lattice in a connected Lie group, Kasparov’s foundational theorem (G. Kasparov, “Equivariant KK-theory and the Novikov conjecture”, *Inventiones Mathematicae* 91 (1988), 147–201) establishing the Strong Novikov Conjecture for connected Lie groups guarantees that the rational assembly map

$$\mu : \bigoplus_k H_{d-4k}(B\Gamma_0; \mathbb{Q}) \rightarrow KO_d(C_r^*(\Gamma_0; \mathbb{R})) \otimes \mathbb{Q}$$

is injective.

The analytic symmetric signature fundamentally corresponds to the assembly of the Poincaré dual of the Hirzebruch L -class. Explicitly incorporating the functorial push-forward of the homology class via the classifying map f_0 , we have:

$$\sigma(M_0) = \mu(f_{0*}(L(M_0) \cap [M_0])) \quad \text{and} \quad \sigma(X_0) = \mu(L(X_0) \cap [X_0]).$$

Because the assembly map μ is rationally injective, the strict equality of K-theoretic signatures forces strict equality of the underlying rationally assembled homology classes in $H_*(X_0; \mathbb{Q})$:

$$f_{0*}(L(M_0) \cap [M_0]) = \text{sgn}(D)(L(X_0) \cap [X_0]).$$

We evaluate this rigid homological equality strictly at the top topological degree d . Because the degree-0 component of the Hirzebruch L -class is identically $L_0 = 1 \in H^0(Y; \mathbb{Q})$, the top-dimensional component of the mixed-degree class $L(Y) \cap [Y]$ evaluates exactly to the fundamental class $[Y] \in H_d(Y; \mathbb{Q})$. Restricting the graded equality exclusively to $H_d(X_0; \mathbb{Q})$ yields:

$$f_{0*}[M_0] = \text{sgn}(D)[X_0].$$

However, by the geometric definition of the proper mapping degree, $f_{0*}[M_0] = D[X_0]$. Equating the two algebraic coefficients rigorously enforces:

$$D = \text{sgn}(D).$$

Because D is a strictly non-zero integer, this analytic rigidity relation exclusively bounds the proper mapping degree to exactly $D = 1$ **or** $D = -1$.

Step 4: Final Contradiction

In Step 2, the generalized Atiyah–Borel equivariant parity constraint definitively established that the proper mapping degree D must be an *even integer* to structurally sustain the free $\mathbb{Z}/2$ -action without fixed points.

Simultaneously, in Step 3, analytic higher signature rigidity (via the Strong Novikov Conjecture) established that D must equal precisely ± 1 to preserve the acyclic chain equivalence over the real group C^* -algebra.

An integer cannot be simultaneously strictly even and equal to ± 1 . This terminal geometric and algebraic contradiction mathematically proves that our initial assumption was fundamentally false. Therefore, it is definitively impossible for such a manifold M to exist. \square