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FirstProof # 2

Let F be a non-archimedean local field with ring of integers \mathfrak{o} . Let N_r denote the subgroup of $\mathrm{GL}_r(F)$ consisting of upper-triangular unipotent elements. Let $\psi : F \rightarrow \mathbb{C}^\times$ be a nontrivial additive character of conductor \mathfrak{o} , identified in the standard way with a generic character of N_r . Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$, realized in its ψ^{-1} -Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$. Must there exist $W \in \mathcal{W}(\Pi, \psi^{-1})$ with the following property?

Let π be a generic irreducible admissible representation of $\mathrm{GL}_n(F)$, realized in its ψ -Whittaker model $\mathcal{W}(\pi, \psi)$. Let \mathfrak{q} denote the conductor ideal of π , let $Q \in F^\times$ be a generator of \mathfrak{q}^{-1} , and set

$$u_Q := I_{n+1} + Q E_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$

where $E_{i,j}$ is the matrix with a 1 in the (i,j) -entry and 0 elsewhere. For some $V \in \mathcal{W}(\pi, \psi)$, the local Rankin–Selberg integral

$$\int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

is finite and nonzero for all $s \in \mathbb{C}$.

Model Response (Agent B)

Theorem. *Let F be a non-archimedean local field and let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$. There exists a universally defined test vector $W \in \mathcal{W}(\Pi, \psi^{-1})$ such that for any generic irreducible admissible representation π of $\mathrm{GL}_n(F)$ with conductor ideal \mathfrak{q} and a generator $Q \in \mathfrak{q}^{-1}$, there exists $V \in \mathcal{W}(\pi, \psi)$ for which the local Rankin–Selberg integral*

$$I(s, W, V) = \int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

is finite and nonzero for all $s \in \mathbb{C}$.

Proof. Let \mathfrak{o} be the ring of integers of F , \mathfrak{p} its maximal ideal, and ϖ a fixed uniformizer. Let $q = |\mathfrak{o}/\mathfrak{p}|$. Let $G_m = \mathrm{GL}_m(F)$, let N_m be its standard upper-triangular unipotent radical, and let $K_m = \mathrm{GL}_m(\mathfrak{o})$ be the standard maximal compact subgroup.

Step 1: Construction of the universal test vector W

Because the additive character ψ has exact conductor \mathfrak{o} , its inverse ψ^{-1} evaluates identically to 1 on the compact intersection $N_n(\mathfrak{o}) = N_n \cap K_n$. Thus, we may unambiguously define a smooth, locally constant function $\Phi \in C_c^\infty(N_n \backslash G_n, \psi^{-1})$ supported

precisely on the open and closed double coset $N_n K_n$ by setting:

$$\Phi(nk) = \psi^{-1}(n) \quad \text{for all } n \in N_n \text{ and } k \in K_n,$$

and extending by zero outside $N_n K_n$. This function is compactly supported modulo N_n .

Let $P_{n+1} \subset G_{n+1}$ be the mirabolic subgroup (matrices with last row $(0, \dots, 0, 1)$). By the foundational exact sequence of the Kirillov model (see Theorem 5.20 of I. N. Bernstein and A. V. Zelevinsky, *Representations of the group $\mathrm{GL}(n, F)$, where F is a local non-Archimedean field*, Uspekhi Mat. Nauk **31**:3 (1976), 5–70), the restriction map $W \mapsto W|_{P_{n+1}}$ is an isomorphism from the Whittaker space $\mathcal{W}(\Pi, \psi^{-1})$ onto the Kirillov space $\mathcal{K}(\Pi, \psi^{-1})$. A defining structural property of this exact sequence is that for generic representations, $\mathcal{K}(\Pi, \psi^{-1})$ canonically contains the subspace of compactly supported functions $C_c^\infty(N_{n+1} \backslash P_{n+1}, \psi^{-1})$.

Under the natural embedding $G_n \hookrightarrow P_{n+1}$ via $g \mapsto \mathrm{diag}(g, 1)$, we have a canonical topological isomorphism $N_{n+1} \backslash P_{n+1} \cong N_n \backslash G_n$. Thus, the restriction of $\mathcal{W}(\Pi, \psi^{-1})$ to G_n canonically contains $C_c^\infty(N_n \backslash G_n, \psi^{-1})$. We may therefore universally fix a valid pre-image vector $W \in \mathcal{W}(\Pi, \psi^{-1})$ such that $W(\mathrm{diag}(g, 1)) = \Phi(g)$ for all $g \in G_n$. Crucially, W is fixed exclusively by Π and ψ , and is completely independent of s , π , and Q .

Step 2: Algebraic reduction to a compact functional

Let $u_Q = I_{n+1} + Q E_{n,n+1}$, where $E_{i,j}$ denotes the elementary matrix with a 1 in the (i, j) -entry and 0 elsewhere. For any $g \in G_n$, block matrix multiplication yields the exact algebraic factorization:

$$\mathrm{diag}(g, 1) u_Q = U_g \mathrm{diag}(g, 1), \quad \text{where } U_g = I_{n+1} + Q \sum_{i=1}^n g_{i,n} E_{i,n+1} \in N_{n+1}.$$

Using the left (N_{n+1}, ψ^{-1}) -equivariance of W , we evaluate the Whittaker function. The unique non-zero superdiagonal entry of U_g is located at $(n, n+1)$, evaluating exactly to $Q g_{n,n}$. All other superdiagonal entries are zero. Therefore:

$$W(\mathrm{diag}(g, 1) u_Q) = \psi^{-1}(U_g) W(\mathrm{diag}(g, 1)) = \psi^{-1}(Q g_{n,n}) \Phi(g).$$

Substituting this identity into the local Rankin–Selberg integral against an arbitrary $V \in \mathcal{W}(\pi, \psi)$, the compact support of Φ strictly truncates the integration domain exactly to the compact quotient $N_n \backslash N_n K_n \cong N_n(\mathfrak{o}) \backslash K_n$.

For any representative $k \in K_n$, the modular factor $|\det k|^{s-1/2} = 1$, unconditionally eliminating all s -dependence. Because the resulting integrand is natively left $N_n(\mathfrak{o})$ -invariant, we lift the integral directly to K_n . Normalizing compatibly with the probability Haar measure dk on K_n , the integral rigorously reduces to an absolutely convergent, s -independent linear functional:

$$I(s, W, V) = \int_{K_n} \psi^{-1}(Q k_{n,n}) V(k) dk := \ell_Q(V).$$

Step 3: Resolution for unramified π ($c = 0$)

If π is unramified, its exact conductor ideal is \mathfrak{o} , meaning $Q \in \mathfrak{o}^\times$. We naturally choose $V = V_{\text{sph}}$, the normalized spherical vector, so $V_{\text{sph}}(k) = 1$ for all $k \in K_n$. Since $k_{n,n} \in \mathfrak{o}$ for all $k \in K_n$, the product $Qk_{n,n} \in \mathfrak{o}$. Because ψ^{-1} is identically trivial on \mathfrak{o} , the phase evaluates to 1, yielding $\ell_Q(V_{\text{sph}}) = \text{vol}(K_n) = 1 > 0$.

Step 4: Ramified non-vanishing via Hecke convolution ($c \geq 1$)

Assume π is ramified with exact conductor ideal $\mathfrak{q} = \mathfrak{p}^c$. Thus $v(Q) = -c$. We define a localized test function in the Hecke algebra $\mathcal{H}(G_n)$:

$$f_Q(g) = \mathbf{1}_{K_n}(g)\psi^{-1}(Qg_{n,n}).$$

Note that by definition, for any $V \in \mathcal{W}(\pi, \psi)$, we have $\ell_Q(V) = \int_{G_n} f_Q(k)V(k) dk = (\pi(f_Q)V)(I_n)$. We construct an adjoint element $\tilde{f}_Q(g) = \overline{f_Q(g^{-1})} = \mathbf{1}_{K_n}(g)\psi(Q(g^{-1})_{n,n})$.

Let $W^{\text{ess}} \in \mathcal{W}(\pi, \psi)$ be the essential newform of π (see Théorème 5.1 of H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256**:2 (1981), 199–214), uniquely normalized so $W^{\text{ess}}(I_n) = 1$. We construct our test vector $V \in \mathcal{W}(\pi, \psi)$ as the convolution $V = \pi(\tilde{f}_Q)W^{\text{ess}}$.

Evaluating $\ell_Q(V)$ yields the exact action of the convolution operator $F_Q = f_Q * \tilde{f}_Q$ on the newform:

$$\ell_Q(V) = \left(\pi(f_Q)\pi(\tilde{f}_Q)W^{\text{ess}} \right)(I_n) = (\pi(F_Q)W^{\text{ess}})(I_n) = \int_{K_n} F_Q(k)W^{\text{ess}}(k) dk.$$

We explicitly compute $F_Q(k)$ for $k \in K_n$:

$$F_Q(k) = \int_{G_n} f_Q(kh)\tilde{f}_Q(h^{-1}) dh = \int_{K_n} \psi^{-1}(Q(kh)_{n,n})\psi(Qh_{n,n}) dh.$$

Let x^\top be the last row of k , and y be the last column of h . Standard matrix multiplication gives $(kh)_{n,n} = \langle x, y \rangle$ and $h_{n,n} = \langle e_n, y \rangle$. As h varies over K_n , its last column y uniformly covers the sphere of primitive unimodular vectors $S^{n-1} = \mathfrak{o}^n \setminus \mathfrak{p}^n$. The integral descends exactly to the unique invariant probability measure dy on S^{n-1} :

$$F_Q(k) = \int_{S^{n-1}} \psi(Q\langle e_n - x, y \rangle) dy.$$

Let $z = e_n - x$. Since $v(Q) = -c$, the character depends solely on $y \pmod{\mathfrak{p}^c}$. We partition S^{n-1} into fibers over $G = (\mathfrak{o}/\mathfrak{p}^c)^n$. The reduction of S^{n-1} is precisely $S_c = G \setminus \mathfrak{p}G$, which has cardinality $|S_c| = q^{nc} - q^{n(c-1)}$. The integral cleanly descends to a finite geometric average:

$$F_Q(k) = \frac{1}{|S_c|} \left[\sum_{y \in G} \psi(Q\langle z, y \rangle) - \sum_{y \in \mathfrak{p}G} \psi(Q\langle z, y \rangle) \right].$$

By standard orthogonality of characters of finite abelian groups, the sums evaluate to exact values:

1. The first sum yields q^{nc} if $z \equiv 0 \pmod{\mathfrak{p}^c}$, and 0 otherwise.

2. For the second sum, parameterizing $y = \varpi u$ where $u \in (\mathfrak{o}/\mathfrak{p}^{c-1})^n$, the character becomes $\psi(Q\varpi\langle z, u \rangle)$. Since $v(Q\varpi) = -(c-1)$, this yields $q^{n(c-1)}$ if $z \equiv 0 \pmod{\mathfrak{p}^{c-1}}$, and 0 otherwise.

Crucially, the condition $z \equiv 0 \pmod{\mathfrak{p}^m}$ strictly enforces $x \equiv e_n \pmod{\mathfrak{p}^m}$. Because x^\top is the last row of $k \in K_n$, this is identically the defining condition of the mirabolic congruence subgroup $K_1(\mathfrak{p}^m)$. We have thus synthesized the exact geometric indicator identity:

$$F_Q = \frac{q^{nc}}{|S_c|} \mathbf{1}_{K_1(\mathfrak{p}^c)} - \frac{q^{n(c-1)}}{|S_c|} \mathbf{1}_{K_1(\mathfrak{p}^{c-1})}.$$

Step 5: Strict non-vanishing via the exact conductor

Applying the operator $\pi(F_Q)$ to W^{ess} rigorously projects the newform onto mirabolic congruence spaces:

1. By Théorème 5.1 of Jacquet, Piatetski-Shapiro, and Shalika (1981), c is the exact conductor of π . Therefore, π intrinsically possesses strictly zero non-trivial vectors invariant under $K_1(\mathfrak{p}^{c-1})$. Thus, the projection integral $\int_{K_1(\mathfrak{p}^{c-1})} \pi(k) W^{\text{ess}} dk$ vanishes identically.
2. By definition, the essential newform W^{ess} is strictly invariant under $K_1(\mathfrak{p}^c)$. Its integral over $K_1(\mathfrak{p}^c)$ naturally scales it by $\text{vol}(K_1(\mathfrak{p}^c))$.

Combining these orthogonal evaluations, the convolution cleanly collapses the operator into a strictly positive scalar multiple of the newform itself:

$$\pi(F_Q)W^{\text{ess}} = \frac{q^{nc}}{|S_c|} \text{vol}(K_1(\mathfrak{p}^c))W^{\text{ess}}.$$

Evaluating this vector at the identity I_n unconditionally concludes the proof:

$$\ell_Q(V) = (\pi(F_Q)W^{\text{ess}})(I_n) = \frac{q^{nc}}{|S_c|} \text{vol}(K_1(\mathfrak{p}^c))W^{\text{ess}}(I_n) > 0.$$

We have explicitly constructed a generic Whittaker vector $V \in \mathcal{W}(\pi, \psi)$ such that the shifted local Rankin–Selberg integral $I(s, W, V)$ evaluates to a strictly positive constant. It is finite, nonzero, and functionally independent of $s \in \mathbb{C}$, universally completing the proof. \square