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FirstProof # 5

Fix a finite group G. Let $\mathscr{O}$ denote an incomplete transfer system associated to an N_∞ operad. Define the slice filtration on the G-equivariant stable category adapted to $\mathscr{O}$ and state and prove a characterization of the $\mathscr{O}$-slice connectivity of a connective G-spectrum in terms of the geometric fixed points.

Model Response (Agent A)

1 Setup and Structural Invariants

In equivariant stable homotopy theory, commutative ring spectra can be parameterized by N_∞ operads, which specify the sets of admissible norm maps. By the foundational work of Blumberg and Hill [2, Section 4], and its subsequent combinatorial classification by Rubin [5, Theorem 3.7] via indexing systems, the data of an N_∞ operad on a finite group G can be equivalently captured by a partial order formalism known as an *incomplete transfer system*. This formulation was explicitly introduced and formalized by Balchin, Barnes, and Roitzheim [1, Definition 2.4].

Definition 1 (Balchin, Barnes, and Roitzheim [1, Definition 2.4]). A *transfer system* $\mathcal{O}$ on a finite group G is a partial order $\subseteq_{\mathcal{O}}$ on the set of subgroups of G that refines inclusion and satisfies two axioms:

1. **Conjugation:** $K \subseteq_{\mathcal{O}} H \implies gKg^{-1} \subseteq_{\mathcal{O}} gHg^{-1}$ for all $g \in G$.
2. **Restriction:** If $K \subseteq_{\mathcal{O}} H$ and $J \leq H$, then $K \cap J \subseteq_{\mathcal{O}} J$.

Because the set of subgroups $\{K \leq H \mid K \subseteq_{\mathcal{O}} H\}$ is finite and closed under intersection (via the restriction axiom and the transitivity of the partial order), it possesses a unique minimal element, which we denote by $H_{\mathcal{O}}$. This allows us to define a natural index for subgroups adapted to the operad.

Definition 2. The *$\mathcal{O}$-index* of a subgroup $J \leq G$ is defined as the maximal index of an $\mathcal{O}$-admissible subgroup of J, which evaluates to:

$$\|J\|_{\mathcal{O}} := [J : J_{\mathcal{O}}].$$

Definition 3. A finite H-set T is *$\mathcal{O}$-admissible* if the stabilizer of every point $t \in T$ is an $\mathcal{O}$-admissible subgroup of H (i.e., $\mathrm{Stab}_H(t) \subseteq_{\mathcal{O}} H$). A permutation representation is $\mathcal{O}$-admissible if it is isomorphic to $\mathbb{R}[T]$ for an $\mathcal{O}$-admissible H-set T. Let $RO^+_{\mathcal{O}}(H)$ denote the monoid of isomorphism classes of $\mathcal{O}$-admissible permutation representations of H.

We adapt the regular slice filtration (cf. Hill, Hopkins, and Ravenel [3, Section 4.1]; Hill and Yarnall [4, Section 2]) to the incomplete transfer system $\mathcal{O}$ as follows:

Definition 4. For an integer $n \geq 0$, the *$\mathcal{O}$-slice category* $\Sigma^{\mathcal{O}}_{\geq n}$ is the full subcategory of connective genuine G-spectra generated (under arbitrary wedges, extensions, and homotopy colimits) by the $\mathcal{O}$-slice cells:

$$\left\{ G_+ \wedge_H S^V \;\middle|\; H \leq G,\, V \in RO^+_{\mathcal{O}}(H), \text{ and } \dim_{\mathbb{R}}(V) \geq n \right\}.$$

2 The Main Theorem

We generalize the characterization of slice connectivity from Hill and Yarnall [4, Theorem 2.5], providing a sharp equivalence between the $\mathcal{O}$-slice filtration and the connectivity of geometric fixed points.

Theorem 5. *Let $\mathcal{O}$ be an incomplete transfer system on a finite group G, and let $n \geq 0$. A connective G-spectrum X belongs to the $\mathcal{O}$-slice category $\Sigma^{\mathcal{O}}_{\geq n}$ if and only if for every subgroup $J \leq G$, the geometric fixed point spectrum $\Phi^J(X)$ is $\lceil n/\|J\|_{\mathcal{O}} \rceil$-connective (i.e., it belongs to the non-equivariant category $\mathrm{Sp}_{\geq \lceil n/\|J\|_{\mathcal{O}} \rceil}$).*

3 A Combinatorial Lemma

We first establish a strict lower bound on the fixed-point dimensions of $\mathcal{O}$-admissible representations.

Lemma 1. *For any $H \leq G$, $V \in RO^+_{\mathcal{O}}(H)$, and $J \leq H$, the dimension of the J-fixed points satisfies*

$$\dim(V^J) \geq \frac{\dim(V)}{\|J\|_{\mathcal{O}}}.$$

Proof. By additivity over disjoint unions of H-orbits, it suffices to prove this for transitive admissible representations $V = \mathbb{R}[H/K]$ where $K \subseteq_{\mathcal{O}} H$. The dimension $\dim(V^J)$ is precisely the number of J-orbits on the coset space H/K, which corresponds to the number of double cosets $|J\backslash H/K|$.
Consider a generic orbit corresponding to the double coset JxK. The stabilizer in J of the coset $xK \in H/K$ is $L = J \cap xKx^{-1}$. By the conjugation axiom, $xKx^{-1} \subseteq_{\mathcal{O}} xHx^{-1} = H$. By the restriction axiom applied to $J \leq H$, we obtain $L \subseteq_{\mathcal{O}} J$. Because L is $\mathcal{O}$-admissible in J, its index satisfies $[J : L] \leq [J : J_{\mathcal{O}}] = \|J\|_{\mathcal{O}}$.
The size of this generic J-orbit on H/K is $[J : L]$, which is bounded above by $\|J\|_{\mathcal{O}}$. Partitioning the elements of H/K into these orbits yields:

$$\dim(V) = [H : K] = \sum_{\text{orbits}} [J : L] \leq \sum_{\text{orbits}} \|J\|_{\mathcal{O}} = \dim(V^J) \cdot \|J\|_{\mathcal{O}}.$$

Dividing by $\|J\|_{\mathcal{O}}$ yields the desired inequality. $\square$

4 Proof of Necessity

Assume $X \in \Sigma^{\mathcal{O}}_{\geq n}$. Since the geometric fixed point functor Φ^J is exact and preserves arbitrary wedges and homotopy colimits, it suffices to verify the connectivity condition on the generators $Y = G_+ \wedge_H S^V$ with $\dim_{\mathbb{R}}(V) \geq n$.

By the geometric double coset Mackey formula:

$$\Phi^J(Y) \simeq \Phi^J(G_+ \wedge_H S^V) \simeq \bigvee_{x\in J\backslash G/H,\ J\le xHx^{-1}} S^{(xV)^J}.$$

For each double coset x contributing to the wedge, define $J' = x^{-1}Jx \le H$. The space $(xV)^J$ is naturally isomorphic to $V^{J'}$. By Lemma 1, we have $\dim(V^{J'}) \ge \dim(V)/\|J'\|_{\mathcal{O}}$.

Because the transfer system $\mathcal{O}$ is conjugation-invariant, conjugation by x induces a poset isomorphism between the $\mathcal{O}$-admissible subgroups of J' and those of J, giving $\|J'\|_{\mathcal{O}} = \|J\|_{\mathcal{O}}$. Thus, $\dim(V^{J'}) \ge n/\|J\|_{\mathcal{O}}$. Because dimensions of real representations are integers, we obtain the bound $\dim(V^{J'}) \ge \lceil n/\|J\|_{\mathcal{O}}\rceil$.

Therefore, $\Phi^J(Y)$ is a wedge of non-equivariant spheres of dimension at least $\lceil n/\|J\|_{\mathcal{O}}\rceil$, implying $\Phi^J(Y) \in \mathrm{Sp}_{\ge\lceil n/\|J\|_{\mathcal{O}}\rceil}$. Because the subcategory $\mathrm{Sp}_{\ge\lceil n/\|J\|_{\mathcal{O}}\rceil}$ is closed under wedges, extensions, and homotopy colimits, and Φ^J preserves these operations, this connectivity bound necessarily extends from the generators Y to the generic spectrum X. This completes the proof of necessity.

5 Proof of Sufficiency

We proceed by strong induction on the order of G. The base case $G = \{e\}$ is trivial. Assume the theorem holds for all proper subgroups of G. Let X be a G-spectrum satisfying the fixed-point bounds for all $J \le G$.

Let $\mathcal{P}$ denote the family of proper subgroups of G. Consider the G-isotropy separation cofiber sequence:

$$E\mathcal{P}_+ \wedge X \to X \to \tilde{E}\mathcal{P} \wedge X.$$

Since the subcategory $\Sigma^{\mathcal{O}}_{\ge n}$ is closed under extensions, it suffices to show that both outer terms belong to $\Sigma^{\mathcal{O}}_{\ge n}$.

Step 1: The term $E\mathcal{P}_+ \wedge X$

The classifying space $E\mathcal{P}$ is a G-CW complex, and thus $E\mathcal{P}_+$ is equipped with a skeletal filtration where the cofibers of the successive skeletal inclusions $E\mathcal{P}_+^{(k-1)} \to E\mathcal{P}_+^{(k)}$ are finite wedges of cells of the form $(G/H)_+ \wedge S^k$ for proper subgroups $H \in \mathcal{P}$ and $k \ge 0$. Smashing this filtration with X, we deduce that $E\mathcal{P}_+ \wedge X$ is built via wedges, extensions, and homotopy colimits from spectra of the form $(G/H)_+ \wedge S^k \wedge X \simeq \Sigma^k(G_+ \wedge_H i_H^* X)$. Because $\Sigma^{\mathcal{O}}_{\ge n}$ is closed under homotopy colimits, it is closed under suspensions (as ΣZ is the homotopy colimit of $* \leftarrow Z \to *$). Establishing that the unsuspended spectrum $G_+ \wedge_H i_H^* X \in \Sigma^{\mathcal{O}}_{\ge n}$ is strictly sufficient to guarantee that $E\mathcal{P}_+ \wedge X \in \Sigma^{\mathcal{O}}_{\ge n}$.

For any proper subgroup $H < G$, let $\mathcal{O}_H$ denote the transfer system restricted to subgroups of H. For any $J \le H$, we note $\Phi^J(i_H^* X) \simeq \Phi^J(X)$. By hypothesis, this spectrum is $\lceil n/\|J\|_{\mathcal{O}}\rceil$-connective. Because the minimal admissible subgroups in $\mathcal{O}_H$ structurally coincide with those in $\mathcal{O}$, we have $\|J\|_{\mathcal{O}_H} = \|J\|_{\mathcal{O}}$. Thus, $i_H^* X$

perfectly satisfies the connectivity hypothesis for $\mathcal{O}_H$. By the inductive hypothesis, $i_H^* X \in \Sigma^{\mathcal{O}_H}_{\geq n}$.
The exact induction functor $G_+ \wedge_H (-)$ sends $\mathcal{O}_H$-admissible cells identically to $\mathcal{O}$-admissible cells. Specifically, for any $K \leq H$ and any $\mathcal{O}_H$-admissible representation W of K, we have $G_+ \wedge_H (H_+ \wedge_K S^W) \simeq G_+ \wedge_K S^W$. Because W is $\mathcal{O}_H$-admissible, its stabilizers are $\mathcal{O}_H$-admissible in K, which natively implies they are $\mathcal{O}$-admissible. Therefore, $G_+ \wedge_H i_H^* X \in \Sigma^{\mathcal{O}}_{\geq n}$, which guarantees $E\mathcal{P}_+ \wedge X \in \Sigma^{\mathcal{O}}_{\geq n}$.

Step 2: The term $\tilde{E}\mathcal{P} \wedge X$

By standard geometric localization (see, e.g., [4, Lemma 2.2] or [3, Section 2.5]), $\tilde{E}\mathcal{P} \wedge X$ is a geometric G-spectrum naturally determined by its geometric G-fixed points, yielding the equivalence $\tilde{E}\mathcal{P}\wedge X \simeq \tilde{E}\mathcal{P}\wedge \mathrm{infl}(\Phi^G X)$, where infl denotes inflation from non-equivariant spectra.
By hypothesis, $\Phi^G X \in \mathrm{Sp}_{\geq m}$, where $m = \lceil n/\|G\|_{\mathcal{O}} \rceil$. Consequently, $\Phi^G X$ is generated (under wedges, extensions, and homotopy colimits) by non-equivariant spheres S^k for $k \geq m$. Because these operations commute with inflation and smashing with $\tilde{E}\mathcal{P}$, it suffices to prove that $\tilde{E}\mathcal{P} \wedge \mathrm{infl}(S^k) \simeq \tilde{E}\mathcal{P} \wedge S^k \in \Sigma^{\mathcal{O}}_{\geq n}$ for all integers $k \geq m$.
Consider the regular admissible permutation representation $V = k\mathbb{R}[G/G_{\mathcal{O}}]$. Its dimension is $k[G : G_{\mathcal{O}}] = k\|G\|_{\mathcal{O}} \geq m\|G\|_{\mathcal{O}} \geq n$. Consequently, the representation sphere $S^V \in \Sigma^{\mathcal{O}}_{\geq n}$.
We apply the isotropy separation sequence to S^V:

$$E\mathcal{P}_+ \wedge S^V \to S^V \to \tilde{E}\mathcal{P} \wedge S^V.$$

Observe that for any $J \leq G$, $\Phi^J(S^V) \simeq S^{V^J}$. By Lemma 1, for any proper subgroup $J < G$, we have $\dim(V^J) \geq \dim(V)/\|J\|_{\mathcal{O}} \geq n/\|J\|_{\mathcal{O}}$. Thus, S^V perfectly satisfies the geometric fixed point bounds for all proper subgroups. Applying the exact same induction logic established in Step 1 to the spectrum $Y = S^V$, we deduce that $i_H^* S^V \in \Sigma^{\mathcal{O}_H}_{\geq n}$ via the inductive hypothesis, ensuring $E\mathcal{P}_+ \wedge S^V \in \Sigma^{\mathcal{O}}_{\geq n}$. As the subcategory $\Sigma^{\mathcal{O}}_{\geq n}$ is closed under homotopy colimits (and thus cofibers), we deduce from the sequence that $\tilde{E}\mathcal{P} \wedge S^V \in \Sigma^{\mathcal{O}}_{\geq n}$.
Finally, we identify the G-fixed points V^G. Because G acts transitively on the single orbit $G/G_{\mathcal{O}}$, the G-fixed point subspace (the trivial subrepresentation) of $\mathbb{R}[G/G_{\mathcal{O}}]$ is 1-dimensional. Thus, $\dim(V^G) = k \cdot 1 = k$.
Passing to the $\tilde{E}\mathcal{P}$-localization evaluates exactly to these fixed points:

$$\tilde{E}\mathcal{P} \wedge S^V \simeq \tilde{E}\mathcal{P} \wedge \mathrm{infl}(\Phi^G S^V) \simeq \tilde{E}\mathcal{P} \wedge S^{V^G} \simeq \tilde{E}\mathcal{P} \wedge S^k.$$

This formally establishes $\tilde{E}\mathcal{P} \wedge S^k \in \Sigma^{\mathcal{O}}_{\geq n}$. By generation, $\tilde{E}\mathcal{P} \wedge X \in \Sigma^{\mathcal{O}}_{\geq n}$, closing the induction loop.

References

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