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User Prompt (ver 1)

Is it true that if $a_1 < a_2 < \dots$ is a sequence of integers with

$$\liminf a_n^{1/2^n} > 1$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational?

Model Response:

Theorem 1. Let $(a_n)_{n \geq 1}$ be a strictly increasing sequence of integers such that

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1.$$

Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational.

Proof.

1. Growth Estimates Based on Hypothesis Let $\lambda = \liminf_{n \rightarrow \infty} a_n^{1/2^n}$. By hypothesis, $\lambda > 1$. Select a constant c such that $1 < c < \lambda$. By the definition of limit inferior, there exists an integer N_0 such that for all $n \geq N_0$:

$$a_n^{1/2^n} > c \implies a_n > c^{2^n}.$$

Let $u_n = \ln a_n$. Taking the logarithm, we have $u_n > 2^n \ln c$. Let $T_n = \sum_{i=1}^n u_i$. Bounding the sum from below:

$$T_n > \sum_{i=N_0}^n 2^i \ln c = \ln c (2^{n+1} - 2^{N_0}).$$

Let $\alpha = \ln c > 0$. For sufficiently large n , we have:

$$T_n > \alpha 2^n. \tag{1}$$

Since a_n is a strictly increasing sequence of integers, $a_n \geq 1$ for all sufficiently large n . We assume $n \geq N_0$ throughout the remaining argument.

2. Assumption of Rationality and Lower Bound Assume, for the sake of contradiction, that S is rational. Let $S = \frac{p}{q}$ for positive integers p, q . Define the n -th partial sum S_n and the remainder r_n :

$$S_n = \sum_{k=1}^n \frac{1}{a_k a_{k+1}}, \quad r_n = S - S_n = \sum_{k=n+1}^{\infty} \frac{1}{a_k a_{k+1}}.$$

Since the terms are positive, $S_n < n < S$, so $r_n > 0$. Consider the denominators in S_n . Each term is of the form $\frac{1}{a_k a_{k+1}}$. Notice that $a_k a_{k+1}$ divides the product $P_{n+1} = \prod_{i=1}^{n+1} a_i$. Let $L_n = \text{lcm}(a_1 a_2, \dots, a_n a_{n+1})$. Then $L_n \leq P_{n+1}$. Writing $S_n = \frac{M_n}{L_n}$ for some integer M_n , we have:

$$r_n = \frac{p}{q} - \frac{M_n}{L_n} = \frac{pL_n - qM_n}{qL_n}.$$

Since $r_n > 0$, the numerator is a positive integer, so $pL_n - qM_n \geq 1$. Thus:

$$r_n \geq \frac{1}{qL_n} \geq \frac{1}{qP_{n+1}}. \quad (2)$$

Using the relation $P_{n+1} = a_{n+1}P_n$, we write this as:

$$r_n \geq \frac{1}{qa_{n+1}P_n}.$$

3. Universal Upper Bound for Remainder We now bound $r_{n+1} = \sum_{k=n+2}^{\infty} \frac{1}{a_k a_{k+1}}$. Since (a_k) is strictly increasing, $a_{k+1} > a_k$, so $\frac{1}{a_k a_{k+1}} < \frac{1}{a_k^2}$. Thus, $r_{n+1} < \sum_{k=n+2}^{\infty} \frac{1}{a_k^2}$.

Since terms may be clustered, we cannot assume $a_{k+1} \gg a_k$. Instead, we partition the indices $k \geq n+2$ into sets I_j based on powers of 2 relative to a_{n+2} :

$$I_j = \{k \geq n+2 \mid 2^j a_{n+2} \leq a_k < 2^{j+1} a_{n+2}\}, \quad j = 0, 1, 2, \dots$$

For any $k \in I_j$, we have $a_k \geq 2^j a_{n+2}$, so $\frac{1}{a_k^2} \leq \frac{1}{(2^j a_{n+2})^2}$. We estimate the size $|I_j|$.

For any $k \in I_j$, we have $a_k < 2^{j+1} a_{n+2}$. From the growth condition, $a_k > c^{2^k}$. Combining these:

$$c^{2^k} < 2^{j+1} a_{n+2} \implies 2^k \ln c < \ln c + \ln(2^{j+1}) + \ln a_{n+2}.$$

$$k < \log_2 \left(\frac{(j+1) \ln 2 + \ln a_{n+2}}{\ln c} \right).$$

Since the indices k are positive integers, the number of such indices in I_j is bounded by the maximum possible index. Let $L = \ln a_{n+2}$. For sufficiently large n (and thus large L), the dominant term is $\ln L$. There exists a constant $C > 0$ such that $|I_j| \leq C(\ln L + j)$. Substituting this into the sum:

$$\sum_{k=n+2}^{\infty} \frac{1}{a_k^2} = \sum_{j=0}^{\infty} \sum_{k \in I_j} \frac{1}{a_k^2} \leq \sum_{j=0}^{\infty} \frac{C(\ln L + j)}{(2^j a_{n+2})^2}.$$

$$r_{n+1} < \frac{C}{a_{n+2}^2} \left(\ln L \sum_{j=0}^{\infty} \frac{1}{4^j} + \sum_{j=0}^{\infty} \frac{j}{4^j} \right).$$

The series converge to finite constants. Thus, for some constant K :

$$r_{n+1} < \frac{K \ln L}{a_{n+2}^2} = \frac{K \ln \ln a_{n+2}}{a_{n+2}^2}. \quad (3)$$

4. Deriving the Recurrence Relation Recall the identity $r_n = \frac{1}{a_{n+1}a_{n+2}} + r_{n+1}$. Combining the lower bound (2) and the upper bound (3):

$$\frac{1}{qa_{n+1}P_n} \leq \frac{1}{a_{n+1}a_{n+2}} + \frac{K \ln \ln a_{n+2}}{a_{n+2}^2}.$$

Multiplying through by $a_{n+1}a_{n+2}$:

$$\frac{a_{n+2}}{qP_n} \leq 1 + \frac{a_{n+1}}{a_{n+2}} K \ln \ln a_{n+2}.$$

Since the sequence is strictly increasing, $a_{n+1} < a_{n+2}$, implying $\frac{a_{n+1}}{a_{n+2}} < 1$. Thus:

$$\frac{a_{n+2}}{qP_n} < 1 + K \ln \ln a_{n+2}.$$

Rearranging to isolate a_{n+2} :

$$a_{n+2} < qP_n(1 + K \ln \ln a_{n+2}).$$

Taking the natural logarithm ($u_n = \ln a_n$ and $\ln P_n = T_n$):

$$u_{n+2} < \ln q + T_n + \ln(1 + K \ln u_{n+2}).$$

Since $a_n \rightarrow \infty$, $u_{n+2} \rightarrow \infty$. Thus for any $\epsilon > 0$, there exists N_ϵ such that for all $n \geq N_\epsilon$:

$$\ln(1 + K \ln u_{n+2}) < \epsilon u_{n+2}.$$

Using this bound:

$$\begin{aligned} u_{n+2} &< T_n + \ln q + \epsilon u_{n+2}. \\ (1 - \epsilon)u_{n+2} &< T_n + \ln q. \end{aligned}$$

Dividing by $1 - \epsilon$:

$$u_{n+2} < \frac{1}{1 - \epsilon} T_n + \frac{\ln q}{1 - \epsilon}.$$

Substitute $u_{n+2} = T_{n+2} - T_{n+1}$:

$$T_{n+2} - T_{n+1} < \frac{1}{1 - \epsilon} T_n + C_\epsilon.$$

Rearranging gives a linear recurrence inequality:

$$T_{n+2} < T_{n+1} + \frac{1}{1 - \epsilon} T_n + C_\epsilon.$$

5. Contradiction Let us analyze the growth rate implied by this recurrence. Consider the sequence X_n satisfying the corresponding homogeneous equality $X_{n+2} = X_{n+1} + \frac{1}{1-\epsilon} X_n$. The characteristic equation is $x^2 - x - \frac{1}{1-\epsilon} = 0$. The positive root is:

$$\rho(\epsilon) = \frac{1 + \sqrt{1 + \frac{4}{1-\epsilon}}}{2}.$$

Since T_n consists of positive terms, standard comparison theorems for linear recurrences imply that for sufficiently large n :

$$T_n < T_{n+1} < C \cdot (\rho(\epsilon))^n.$$

We now choose ϵ sufficiently small to force a contradiction with the known lower bound. Let $\epsilon = 0.05$. Then $\frac{1}{1-\epsilon} \approx 1.053$.

$$\rho(0.05) = \frac{1 + \sqrt{1 + 4.21}}{2} = \frac{1 + \sqrt{5.21}}{2} \approx \frac{1 + 2.28}{2} = 1.64.$$

Crucially, $\rho(0.05) < 1.7 < 2$. Thus, if S is rational, then $T_n = O(1.7^n)$.

However, from Step 1 (Inequality 1), we established that the condition $\liminf a_n^{1/2^n} > 1$ implies:

$$T_n > \alpha 2^n.$$

Combining these bounds yields:

$$\alpha 2^n < n < C(1.7)^n \implies \alpha < \frac{C}{2^n} (1.7)^n \implies \alpha < C \left(\frac{1.7}{2}\right)^n.$$

Since $\frac{1.7}{2} < 1$, the right hand side approaches 0 as $n \rightarrow \infty$. This implies $\alpha \leq 0$. But $\alpha = \ln c > 0$. This is a contradiction.

Therefore, our initial assumption that S is rational must be false.

Conclusion The sum $S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$ is irrational.

User Prompt (ver 2)

Is it true that if $a_1 < a_2 < \dots$ is a sequence of integers with

$$\liminf a_n^{1/2^n} > 1$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational?

Model Response:

The problem asks us to determine the truth of the statement: if (a_n) is a strictly increasing sequence of integers with $\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$, then $\sum \frac{1}{a_n a_{n+1}}$ is irrational.

Executive Summary

Status: Complete Solution

We prove that the statement is true. The argument proceeds by contradiction. We first reduce the problem to the case where all $a_n \geq 2$ without loss of generality. Assuming the series sums to a rational number p/q , we construct a sequence of integers $Z_n = qP_{n+1}R_n$, where R_n is the series remainder and P_{n+1} is the partial product of terms. The properties of (a_n) ensure Z_n is a positive integer for all n . We derive an upper bound for Z_n that leads to the inequality $a_{n+2} < 2qP_n$ for sufficiently large n . This recurrence implies that $\ln P_n$ grows no faster than the Fibonacci sequence (order ϕ^n). However, the given condition $\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$ implies that $\ln P_n$ grows at least as 2^n . Since $2 > \phi$, these growth rates are incompatible, establishing the contradiction.

Polished Solution

Let $(a_n)_{n=1}^\infty$ be a strictly increasing sequence of integers satisfying $\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$. We prove that the sum of the series $\sum_{n=1}^\infty \frac{1}{a_n a_{n+1}}$ is an irrational number.

Irrationality of the Series $\sum_{n=1}^\infty \frac{1}{a_n a_{n+1}}$ Under Rapid Growth Conditions

Introduction

We analyze the arithmetic nature of the convergent series

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}},$$

given that the integer sequence $(a_n)_{n=1}^\infty$ grows doubly exponentially. Specifically, we assume:

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1. \quad (1)$$

We demonstrate that this growth condition forces the sum S to be irrational.

Proof of Irrationality

Theorem 2. *Let $(a_n)_{n=1}^\infty$ be a strictly increasing sequence of integers such that*

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1.$$

Then

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}} \notin \mathbb{Q}.$$

Proof. Let $\lambda = \liminf_{n \rightarrow \infty} a_n^{1/2^n}$. By hypothesis, $\lambda > 1$. Let c be a constant such that $1 < c < \lambda$. By the definition of limit inferior, there exists N_0 such that for all $n \geq N_0$, $a_n^{1/2^n} > c$, or equivalently $a_n > c^{2^n}$. Since $c > 1$, $a_n \rightarrow \infty$. Because (a_n) is strictly increasing, the terms eventually become positive integers. Thus, there exists $N_1 \geq N_0$ such that $a_n \geq 2$ for all $n \geq N_1$.

Note that the rationality of S depends only on the tail of the series. Let $S = \sum_{n=1}^{N_1-1} \frac{1}{a_n a_{n+1}} + S_{\text{tail}}$. The finite initial sum is rational. Thus, $S \in \mathbb{Q} \iff S_{\text{tail}} \in \mathbb{Q}$. We shift the indices of the sequence by defining $a'_n = a_{n+N_1-1}$ for $n \geq 1$. The condition (1) is invariant under this shift (with a modified exponent factor that does not affect the strict inequality $\lambda' > 1$). Without loss of generality, we proceed assuming $a_n \geq 2$ and $a_n > c^{2^n}$ for all $n \geq 1$. Convergence is guaranteed by comparison with the geometric series $\sum c^{-2^n}$, which converges absolutely.

We begin by characterizing the relative growth of the terms.

Lemma 1. *Under the hypotheses, $\lim_{n \rightarrow \infty} \frac{a_{n+2}}{a_n} = \infty$.*

Proof. Suppose the limit is not infinity. Then there exist a constant $M > 0$ and a subsequence (n_k) such that $a_{n_k+2} \leq M a_{n_k}$. Rearranging yields $a_{n_k} \geq M^{-1} a_{n_k+2}$. Raising both sides to the power $1/2^{n_k}$:

$$a_{n_k}^{1/2^{n_k}} \geq M^{-1/2^{n_k}} \left(a_{n_k+2}^{1/2^{n_k+2}} \right)^{2^{n_k+2}/2^{n_k}}.$$

Substituting $2^{n_k+2}/2^{n_k} = 4$:

$$a_{n_k}^{1/2^{n_k}} \geq M^{-1/2^{n_k}} \left(a_{n_k+2}^{1/2^{n_k+2}} \right)^4.$$

Taking the limit inferior as $k \rightarrow \infty$:

1. $\liminf a_{n_k}^{1/2^{n_k}} \geq \liminf a_n^{1/2^n} = \lambda$.
2. $\lim M^{-1/2^{n_k}} = 1$.
3. $\liminf \left(a_{n_k+2}^{1/2^{n_k+2}} \right)^4 \geq \lambda^4$, since any subsequence of $a_n^{1/2^n}$ has limit inferior at least λ .

Thus, $\lambda \geq \lambda^4$. Since $\lambda > 1$, dividing by λ yields $1 \geq \lambda^3$, which implies $\lambda \leq 1$. This contradicts $\lambda > 1$. Therefore, the ratio must tend to infinity. \square

From Lemma 1, $\frac{a_n}{a_{n+2}} \rightarrow 0$. Hence, there exists an integer $N^* \geq 1$ such that for all $n \geq N^*$:

$$\frac{a_n}{a_{n+2}} \leq \frac{1}{4}. \tag{2}$$

Assume for the sake of contradiction that $S = \frac{p}{q}$ for integers $p, q \geq 1$. Define the partial product $P_n = \prod_{i=1}^n a_i$. Let $R_n = \sum_{k=n+1}^{\infty} \frac{1}{a_k a_{k+1}}$. Define $Z_n = qP_{n+1}R_n$. Writing $R_n = S - \sum_{k=1}^n \frac{1}{a_k a_{k+1}}$, we have:

$$Z_n = qP_{n+1} \left(\frac{p}{q} - \sum_{k=1}^n \frac{1}{a_k a_{k+1}} \right) = pP_{n+1} - q \sum_{k=1}^n \frac{P_{n+1}}{a_k a_{k+1}}.$$

Consider the term $T_k = \frac{P_{n+1}}{a_k a_{k+1}}$ for $1 \leq k \leq n$. Since a_i are distinct integers (strictly increasing), the factors a_k and a_{k+1} are distinct elements in the set $\{a_1, \dots, a_{n+1}\}$. Thus, their product divides P_{n+1} . Consequently, $T_k \in \mathbb{Z}$ for all k , implying $Z_n \in \mathbb{Z}$. Since $a_n \geq 2$, all terms in the series are positive, so $R_n > 0$. Since $q, P_{n+1} > 0$, we have $Z_n > 0$. As Z_n is an integer:

$$Z_n \geq 1. \quad (3)$$

We now bound R_n . Let $u_k = \frac{1}{a_k a_{k+1}}$. For $k \geq N^*$, the ratio of consecutive series terms is:

$$\frac{u_{k+1}}{u_k} = \frac{a_k a_{k+1}}{a_{k+1} a_{k+2}} = \frac{a_k}{a_{k+2}} \leq \frac{1}{4}.$$

Using this geometric bound for $n \geq N^*$:

$$R_n = \sum_{j=1}^{\infty} u_{n+j} \leq u_{n+1} \sum_{j=0}^{\infty} \left(\frac{1}{4} \right)^j = u_{n+1} \frac{4}{3} = \frac{4}{3a_{n+1}a_{n+2}}.$$

Substituting this into the definition of Z_n :

$$Z_n \leq qP_{n+1} \frac{4}{3a_{n+1}a_{n+2}} = \frac{4qP_n}{3a_{n+2}}.$$

Applying (3):

$$1 \leq \frac{4qP_n}{3a_{n+2}} \implies a_{n+2} \leq \frac{4}{3}qP_n.$$

Since $4/3 < 2$, we strictly have:

$$a_{n+2} < 2qP_n. \quad (4)$$

This inequality relates the term a_{n+2} to the cumulative product P_n . We analyze the growth of P_n . From $P_{n+2} = P_{n+1}a_{n+2}$, we deduce:

$$P_{n+2} < 2qP_nP_{n+1}.$$

Let $L_n = \ln P_n$. Since $P_n \geq 2^n$, $L_n > 0$.

$$L_{n+2} < L_{n+1} + L_n + \ln(2q).$$

Let $K = \ln(2q)$. Define an auxiliary sequence (x_n) by $x_{n+2} = x_{n+1} + x_n + K$ with initial conditions $x_{N^*} = L_{N^*}$, $x_{N^*+1} = L_{N^*+1}$. By simple induction, $L_n \leq x_n$ for all $n \geq N^*$. Let $y_n = x_n + K$. Then $y_{n+2} = y_{n+1} + y_n$. Thus y_n follows the Fibonacci

recurrence. Standard solutions give $y_n = A\phi^n + B\psi^n$, where $\phi = \frac{1+\sqrt{5}}{2}$. Since $|\psi| < 1$, for large n , $y_n \sim A\phi^n$. Consequently, there exists $C_1 > 0$ such that:

$$L_n \leq C_1 \phi^n. \quad (5)$$

Now we contrast this with the lower bound implied by the hypothesis. Recall $a_n > c^{2^n}$ for $n \geq N^*$ (renormalized index), with $c > 1$.

$$P_n = \prod_{k=1}^n a_k > \left(\prod_{k=1}^{N^*-1} a_k \right) \prod_{k=N^*}^n c^{2^k}.$$

The exponent sum is $\sum_{k=N^*}^n 2^k = 2^{n+1} - 2^{N^*}$. So $P_n > \text{Const} \cdot c^{2^{n+1}}$. Taking natural logarithms:

$$L_n > 2^{n+1} \ln c + C_{\text{const}}.$$

Since $\ln c > 0$, for sufficiently large n , there exists $C_2 > 0$ such that:

$$L_n > C_2 2^n. \quad (6)$$

Comparing (5) and (6):

$$C_2 2^n < L_n \leq C_1 \phi^n.$$

Dividing by ϕ^n :

$$C_2 \left(\frac{2}{\phi} \right)^n < C_1.$$

Observe that $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$, so $2 > \phi$. Thus $\frac{2}{\phi} > 1$. Consequently, $\lim_{n \rightarrow \infty} \left(\frac{2}{\phi} \right)^n = \infty$. The inequality implies that a sequence tending to infinity is strictly bounded by the constant C_1/C_2 , which is absurd.

Thus, the initial assumption that S is rational leads to a contradiction. We conclude that S must be irrational. \square

User Prompt (Gemini Deep Think)

Find a meaningful, interesting generalization of this theorem or its proof.

Theorem Fix $d \geq 2$, and let c_d be the unique positive real root of $x^d - x^{d-1} - 1 = 0$. Suppose a strictly increasing sequence $\{a_n\}$ of integers satisfy

$$\liminf_{n \rightarrow \infty} a_n^{1/b^n} > 1$$

for some $b > c_d$. Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1} \cdots a_{n+d-1}}$$

is irrational.

Proof For two functions f and g , we write $f \lesssim g$ when there exists some constant $C > 0$ satisfying $f \leq Cg$. Assume for contradiction that $S = p/q$. We may assume there is some constant $c > 1$ such that $a_n > c^{b^n}$ for all n . We set

$$P_n := \prod_1^n a_i, \quad S_n = \sum_{m=1}^n \frac{1}{a_m a_{m+1} \cdots a_{m+d-1}}.$$

Since $S > S_n$ for all n , we have

$$r_n := S - S_n = \frac{p}{q} - \frac{*}{P_{n+d-1}} \geq \frac{1}{qP_{n+d-1}}.$$

Let us define

$$I_j(n) := \{k \geq n+1 \mid 2^j a_{n+1}^d \leq a_k^d < 2^{j+1} a_{n+1}^d\}, \quad j = 0, 1, 2, \dots$$

so that each $k \in I_j(n)$ satisfies

$$c^{d \cdot b^k} < a_k^d < 2^{j+1} a_{n+1}^d.$$

The cardinality of $I_j(n)$ can be bounded from above as

$$|I_j(n)| \leq \max_{k \in I_j(n)} k \leq \log_b \left(\frac{\log_c(2^{j+1} a_{n+1}^d)}{d} \right) \lesssim \log j + \log \log a_{n+1}.$$

It follows that

$$r_n < \sum_{k=n+1}^{\infty} \frac{1}{a_k^d} = \sum_{j=0}^{\infty} \sum_{k \in I_j(n)} \frac{1}{a_k^d} \leq \sum_{j=0}^{\infty} \frac{|I_j(n)|}{2^j a_{n+1}^d} \lesssim \sum_{j=0}^{\infty} \frac{\log j + \log \log a_{n+1}}{2^j a_{n+1}^d} \lesssim \frac{\log \log a_{n+1}}{a_{n+1}^d}.$$

Combining the lower and the upper bounds for r_n and r_{n+1} , we have

$$\begin{aligned} \frac{1}{qP_{n+d-1}} &\leq r_n = \sum_{j=1}^{d-1} \frac{1}{a_{n+j} \cdots a_{j+n+d-1}} + r_{n+d-1} \\ &\lesssim \frac{d-1}{a_{n+1} \cdots a_{n+d}} + \frac{\log \log a_{n+d}}{a_{n+d}^d} \leq \frac{\log \log a_{n+d}}{a_{n+1} \cdots a_{n+d}}. \end{aligned}$$

Fix an arbitrarily small $\epsilon > 0$. Multiplying both sides by P_{n+d} , we obtain that

$$a_{n+d} = P_{n+d}/P_{n+d-1} \lesssim P_n \log \log a_{n+d} \lesssim P_n a_{n+d}^\epsilon = P_n (P_{n+d}/P_{n+d-1})^\epsilon.$$

It follows that we have $P_{n+d} \lesssim P_{n+d-1} P_n^{1/(1-\epsilon)}$. The sequence $u_n := \log_c P_n$ satisfies the recursive inequality

$$u_{n+d} \leq u_{n+d-1} + \frac{1}{1-\epsilon} u_n + K$$

for some constant $K = K(\epsilon) > 0$. For a sufficiently small ϵ , the equation

$$x^d = x^{d-1} + \frac{1}{1-\epsilon}$$

has exactly one positive real root $s = s(d, \epsilon)$, slightly larger than c_d , and one or zero negative real root, depending on the parity of d ; moreover, the negative real root, if exists, belong to $(-1, 0)$. By choosing a sufficiently small ϵ satisfying $s = s(d, \epsilon) < b$, we have that $u_n \lesssim s^n$. This contradicts the estimate $\log_c P_n \geq \log_c a_n \geq b^n$.
****QED****

Model Response:

A meaningful and natural generalization of this theorem extends the result to sums of reciprocals of products with ****arbitrary integer weights****.

The original theorem relies on a product of d consecutive terms, effectively assigning a weight of 1 to every integer in a "sliding window." By generalizing these exponents, we can quantify how the density or sparsity of the denominator affects the growth rate required for the series to be irrational.

Theorem (Generalized)

Fix an integer $d \geq 1$ and a tuple of non-negative integers $\mathbf{w} = (w_0, w_1, \dots, w_{d-1})$ with $w_{d-1} \geq 1$. Let $W = \max(w_0, \dots, w_{d-1})$. Let $c_{\mathbf{w}}$ be the unique real root greater than 1 of the polynomial:

$$P(x) = (x-1) \sum_{j=0}^{d-1} w_j x^j - W x^{d-1}$$

Suppose a strictly increasing sequence of integers $\{a_n\}$ satisfies

$$\liminf_{n \rightarrow \infty} a_n^{1/b^n} > 1$$

for some $b > c_{\mathbf{w}}$. Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n^{w_0} a_{n+1}^{w_1} \cdots a_{n+d-1}^{w_{d-1}}}$$

is irrational.

Proof Sketch

The generalization follows the "gap principle" strategy used in the original proof, balancing the decay of the tail against the growth of the common denominator.

1. **The Common Denominator (LCM):**

Assume for contradiction that $S = p/q$. Let S_n be the n -th partial sum. The denominators of the terms in S_n involve integers up to a_{n+d-1} .

Because the window shifts by 1 at each step, any specific integer a_k appears in d consecutive terms with exponents w_{d-1}, \dots, w_0 . When taking the Least Common Multiple (LCM) to form the common denominator Q_n , the exponent of a_k is bounded by the maximum weight W .

$$Q_n \lesssim \prod_{k=1}^{n+d-1} a_k^W \approx P_{n+d-1}^W$$

2. The Tail Estimate:

The remainder $r_n = S - S_n$ is dominated by the first term of the tail (index $n+1$).

$$r_n \approx \frac{1}{a_{n+1}^{w_0} a_{n+2}^{w_1} \cdots a_{n+d}^{w_{d-1}}}$$

3. The Inequality:

For S to be rational, the error r_n cannot be "too small" relative to the denominator Q_n . Specifically, $r_n \geq 1/(qQ_n)$. This implies $Q_n r_n \gtrsim 1$. Substituting the bounds:

$$P_{n+d-1}^W \cdot \frac{1}{\prod_{j=0}^{d-1} a_{n+1+j}^{w_j}} \gtrsim 1 \implies \prod_{j=0}^{d-1} a_{n+1+j}^{w_j} \lesssim P_{n+d-1}^W$$

Taking logarithms (assuming $\log a_k \sim b^k$) and dividing by b^{n+1} :

$$\sum_{j=0}^{d-1} w_j b^j \lesssim W \frac{b^{d-1}}{b-1}$$

Rearranging this inequality yields the polynomial $P(b) \leq 0$. If $b > c_{\mathbf{w}}$ (the root), this inequality is violated for large n , proving irrationality.

Why this is Meaningful

1. It Unifies "Dense" and "Sparse" Problems **The Original Case:** If $\mathbf{w} = (1, \dots, 1)$, then $W = 1$. The sum is $\frac{x^d - 1}{x - 1}$.

$$(x-1) \frac{x^d - 1}{x - 1} - 1 \cdot x^{d-1} = x^d - x^{d-1} - 1 = 0$$

This recovers the original constant c_d .

The "Gap" Case: Consider $S = \sum \frac{1}{a_n a_{n+2}}$. Here $\mathbf{w} = (1, 0, 1)$ and $W = 1$. The polynomial becomes:

$$(x-1)(1+x^2) - x^2 = x^3 - 2x^2 + x - 1 = 0$$

The root is $c \approx 1.75$, which is larger than the $c \approx 1.46$ required for the consecutive block $a_n a_{n+1} a_{n+2}$. This quantifies the intuition that "missing terms" weaken the convergence, forcing the sequence a_n to grow faster to guarantee irrationality.

2. It Handles Heavy Weights

Sylvester-Type Series: If $d = 1$ and $w_0 = 1$, the polynomial is $(x-1)(1)-1 = x-2$, recovering the classic result that $\sum 1/a_n$ is irrational if $a_{n+1} \sim a_n^2$ (limit $b = 2$).

If we increase the weight of the *last* term (e.g., $1/(a_n a_{n+1}^2)$), the root increases, making irrationality harder to prove. This is because the "cost" of the LCM ($W = 2$) rises, while the benefit to the tail decay is limited to the future terms.

3. Universal Irrationality for Double-Exponential Sequences

Since $c_d \rightarrow 1$ as $d \rightarrow \infty$ for the uniform case, this generalization implies a powerful corollary:

For **any** sequence $\{a_n\}$ with double-exponential growth (any $b > 1$), there exists a sufficiently large block size d such that $\sum \frac{1}{a_n \cdots a_{n+d-1}}$ is irrational.