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FirstProof # 5

Fix a finite group G . Let \mathcal{O} denote an incomplete transfer system associated to an N_∞ operad. Define the slice filtration on the G -equivariant stable category adapted to \mathcal{O} and state and prove a characterization of the \mathcal{O} -slice connectivity of a connective G -spectrum in terms of the geometric fixed points.

Model Response (Agent B)

The slice filtration provides a powerful structural filtration on the category of genuine G -spectra. The regular slice filtration has proven particularly advantageous because its slice connectivity can be characterized strictly in terms of geometric fixed points. When working with N_∞ operads, the permissible norm maps are governed by combinatorial structures defined on the poset of subgroups of G . In this paper, we construct a generalized slice filtration adapted to an arbitrary incomplete transfer system \mathcal{O} . We then rigorously characterize its connective covers utilizing only the dimensions of these geometric fixed point spectra.

1 Transfer Systems and \mathcal{O} -Regular Representations

Combinatorial models for N_∞ operads inherently involve partial orders on the subgroups of G , formally known as *transfer systems*. To align with the working context of our problem statement, we refer to them here as *incomplete transfer systems*.

Definition 1. An incomplete transfer system \mathcal{O} on a finite group G is a partial order \rightarrow on the set of subgroups of G satisfying the following axioms:

1. (*Refinement*) If $K \rightarrow H$, then $K \leq H$.
2. (*Conjugation*) If $K \rightarrow H$, then $gKg^{-1} \rightarrow gHg^{-1}$ for all $g \in G$.
3. (*Restriction*) If $K \rightarrow H$ and $L \leq H$, then $K \cap L \rightarrow L$.

If $K \rightarrow H$, we say that K is an \mathcal{O} -subgroup of H .

Transfer systems possess unique minimal elements, which are central to calculating our geometric connectivity bounds.

Lemma 1. For any subgroup $H \leq G$, the set of \mathcal{O} -subgroups of H is closed under intersection. Thus, there exists a unique minimal \mathcal{O} -subgroup of H , which we denote by K_H . Furthermore, K_H is a normal subgroup of H .

Proof. Let $A \rightarrow H$ and $B \rightarrow H$. Applying the restriction axiom to $A \rightarrow H$ and $B \leq H$, we obtain $A \cap B \rightarrow B$. Since $B \rightarrow H$, the transitivity of the partial order

yields $A \cap B \rightarrow H$. Because H is finite and the partial order is reflexive (meaning the set of \mathcal{O} -subgroups of H is non-empty since $H \rightarrow H$), the intersection of all \mathcal{O} -subgroups of H is a well-defined \mathcal{O} -subgroup, representing the unique minimum K_H . For any $h \in H$, the conjugation axiom implies $hK_Hh^{-1} \rightarrow hHh^{-1} = H$. By minimality and uniqueness, $hK_Hh^{-1} \geq K_H$, which forces $hK_Hh^{-1} = K_H$ by cardinality considerations. Thus $K_H \trianglelefteq H$. \square

Definition 2. For each $H \leq G$, define the \mathcal{O} -regular representation of H to be the permutation representation $\rho_H^{\mathcal{O}} = [H/K_H]$. We define the \mathcal{O} -index of H as $|H|_{\mathcal{O}} = [H : K_H] = \dim(\rho_H^{\mathcal{O}})$.

Following the approach used for the regular slice filtration, we define the \mathcal{O} -slice filtration using these \mathcal{O} -regular representations.

Definition 3. For $n \geq 0$, an \mathcal{O} -slice cell of dimension $\geq n$ is a G -spectrum of the form $G_+ \wedge_H S^{m\rho_H^{\mathcal{O}} - \epsilon}$, where $H \leq G$, $m \in \mathbb{Z}_{\geq 0}$, $\epsilon \in \{0, 1\}$, and $m|H|_{\mathcal{O}} - \epsilon \geq n$. The category of \mathcal{O} -slice $\geq n$ spectra, denoted $\tau_{\geq n}^{\mathcal{O}}$, is the smallest full subcategory of the G -equivariant stable category containing all \mathcal{O} -slice cells of dimension $\geq n$ that is closed under arbitrary wedges, extensions, positive suspensions, and retracts.

2 Geometric Fixed Point Characterization

Recall that a non-equivariant spectrum E is c -connective if $\pi_k(E) = 0$ for all $k < c$. A known result characterizes the regular slice filtration by showing $X \in \tau_{\geq n}$ if and only if $\Phi^L X$ is $(n/|L| - 1)$ -connected. Noting that an x -connected spectrum is equivalently $\lfloor x \rfloor$ -connected, which is in turn $(\lfloor x \rfloor + 1)$ -connective, the condition naturally translates to being $\lfloor n/|L| \rfloor$ -connective. We establish the precise analogue for an arbitrary incomplete transfer system \mathcal{O} .

Theorem 4. Let X be a connective G -spectrum and $n \geq 0$. Then $X \in \tau_{\geq n}^{\mathcal{O}}$ if and only if for all subgroups $L \leq G$, the geometric fixed point spectrum $\Phi^L X$ is $\lfloor n/|L|_{\mathcal{O}} \rfloor$ -connective.

The proof requires a sharp combinatorial lemma governing the L -fixed points of \mathcal{O} -regular representations.

Lemma 2. Let $H, L \leq G$ and $g \in G$ be such that $L \leq gHg^{-1}$. The number of L -orbits on the set gHg^{-1}/gK_Hg^{-1} under left multiplication is bounded below by $|H|_{\mathcal{O}}/|L|_{\mathcal{O}}$.

Proof. Let $H' = gHg^{-1}$ and $K' = gK_Hg^{-1}$. By the conjugation axiom, $K' \rightarrow H'$. Since $K_H \trianglelefteq H$ (Lemma 1), we have $K' \trianglelefteq H'$. Thus H'/K' is a well-defined group quotient, and $L \leq H'$ acts on it by left multiplication. The L -orbits correspond to the right cosets of the image of L inside H'/K' .

Because K' is normal in H' , the stabilizer in L of any coset $h'K'$ is precisely $L \cap h'K'(h')^{-1} = L \cap K'$. Therefore, every single L -orbit has the exact same uniform size: $w_L = [L : L \cap K']$. Applying the restriction axiom to $K' \rightarrow H'$ and $L \leq H'$, we obtain $L \cap K' \rightarrow L$. Because K_L is the unique minimal \mathcal{O} -subgroup of L , it structurally

follows that $K_L \leq L \cap K'$. Thus, the uniform orbit size satisfies $w_L = [L : L \cap K'] \leq [L : K_L] = |L|_{\mathcal{O}}$.

The total number of L -orbits is exactly the cardinality of the underlying set divided by this uniform orbit size: $[H' : K']/w_L = |H|_{\mathcal{O}}/w_L \geq |H|_{\mathcal{O}}/|L|_{\mathcal{O}}$. \square

Proof of Theorem 4. Necessity (\Rightarrow): Because the geometric fixed point functor Φ^L preserves exact triangles, positive suspensions, arbitrary wedges, and retracts, the full subcategory of connective G -spectra satisfying the given connectivity bound is closed under these operations. It thus suffices to verify the claim on the generating \mathcal{O} -slice cells. Let $C = G_+ \wedge_H S^{m\rho_H^{\mathcal{O}} - \epsilon} \in \tau_{\geq n}^{\mathcal{O}}$, so $m|H|_{\mathcal{O}} - \epsilon \geq n$. By the Mackey double coset formula,

$$\Phi^L C \simeq \bigvee_{g \in L \backslash G/H, L \leq gHg^{-1}} \Phi^L \left(S^{m\rho_H^{\mathcal{O}} - \epsilon} \right).$$

For a valid double coset g , the L -fixed points of $g\rho_H^{\mathcal{O}}$ have dimension d_g given precisely by the number of L -orbits on gHg^{-1}/gK_Hg^{-1} . By Lemma 2, $d_g = |H|_{\mathcal{O}}/w_L$ where $w_L = [L : L \cap gK_Hg^{-1}] \leq |L|_{\mathcal{O}}$. The wedge summand corresponding to g is a sphere of dimension $D = md_g - \epsilon$. We compute:

$$D = m \frac{|H|_{\mathcal{O}}}{w_L} - \epsilon \geq \frac{n + \epsilon}{w_L} - \epsilon = \frac{n}{w_L} + \epsilon \left(\frac{1}{w_L} - 1 \right).$$

If $\epsilon = 0$, $D \geq n/w_L$. Since D is an integer (the dimension of a real fixed-point subspace), it is bounded below by the ceiling of n/w_L : $D \geq \lceil n/w_L \rceil \geq \lfloor n/|L|_{\mathcal{O}} \rfloor$.

If $\epsilon = 1$, $D \geq \frac{n+1}{w_L} - 1$. Since D is an integer, $D \geq \lceil \frac{n+1}{w_L} \rceil - 1$. Write $n = qw_L + r$ via the division algorithm with remainder $0 \leq r \leq w_L - 1$. We have $\lfloor n/w_L \rfloor = q$. Then $\lceil \frac{n+1}{w_L} \rceil - 1 = \lceil q + \frac{r+1}{w_L} \rceil - 1 = q + \lceil \frac{r+1}{w_L} \rceil - 1$. Because $1 \leq r+1 \leq w_L$, we have $\lceil \frac{r+1}{w_L} \rceil = 1$. Thus, the expression exactly equals $q = \lfloor n/w_L \rfloor$, which is again $\geq \lfloor n/|L|_{\mathcal{O}} \rfloor$. In both exhaustive cases, $\Phi^L C$ is a wedge of spheres of dimension $\geq \lfloor n/|L|_{\mathcal{O}} \rfloor$, making it $\lfloor n/|L|_{\mathcal{O}} \rfloor$ -connective.

Sufficiency (\Leftarrow): We proceed by contradiction utilizing the exact triangle associated to the \mathcal{O} -slice tower:

$$F \rightarrow X \rightarrow W \rightarrow \Sigma F$$

where $F = \tau_{\geq n}^{\mathcal{O}} X \in \tau_{\geq n}^{\mathcal{O}}$ and $W = P_{< n}^{\mathcal{O}} X \in (\tau_{\geq n}^{\mathcal{O}})^{\perp}$. Because X is connective and F is generated by \mathcal{O} -slice cells C whose geometric fixed points are universally connective (as established in the necessity direction, $\Phi^K C$ is $\lfloor n/|K|_{\mathcal{O}} \rfloor$ -connective for all $K \leq G$, and $\lfloor n/|K|_{\mathcal{O}} \rfloor \geq 0$ since $n \geq 0$), the generators of F are equivariantly connective. Consequently, both X and F are connective spectra. Thus, W sits in a cofiber sequence of connective spectra, meaning W is itself bounded below. By the necessity direction, $\Phi^L F$ is $\lfloor n/|L|_{\mathcal{O}} \rfloor$ -connective. The long exact sequence of homotopy groups combined with the connectivity assumption on X implies $\Phi^L W$ is also $\lfloor n/|L|_{\mathcal{O}} \rfloor$ -connective.

Assume for contradiction that $W \not\simeq *$. The geometric fixed points of a non-trivial bounded-below G -spectrum cannot be universally contractible. Thus, we may choose

a subgroup $L \leq G$ of minimal order such that $\Phi^L W \not\simeq *$. Let c be the lowest integer such that $\pi_c(\Phi^L W) \neq 0$. By the connectivity bound, $c \geq q$, where $q = \lfloor n/|L|_{\mathcal{O}} \rfloor \geq 0$. Let \mathcal{P} denote the family of proper subgroups of L . Because L was chosen to be minimal, $\Phi^K W \simeq *$ for all $K \in \mathcal{P}$. The geometric fixed points of the bounded-below L -spectrum $EP_+ \wedge W$ satisfy $\Phi^K(EP_+ \wedge W) \simeq \Phi^K(EP_+) \wedge \Phi^K W$. For $K \in \mathcal{P}$, this is contractible because $\Phi^K W \simeq *$. For $K = L$, it is contractible because $\Phi^L(EP_+) \simeq (EP^L)_+ \simeq \emptyset_+ \simeq *$. Since its geometric fixed points vanish for all subgroups, $EP_+ \wedge W \simeq *$ by the equivariant Whitehead theorem.

The isotropy separation sequence $EP_+ \wedge W \rightarrow W \rightarrow \tilde{E}\mathcal{P} \wedge W$ then implies that the natural map $W \rightarrow \tilde{E}\mathcal{P} \wedge W$ is an equivalence of L -spectra. Consequently, the true fixed points satisfy $W^L \simeq (\tilde{E}\mathcal{P} \wedge W)^L \simeq \Phi^L W$. We may thus choose a non-nullhomotopic map $S^c \rightarrow \Phi^L W \simeq W^L$.

We claim that for any real representation U of L and $\delta \in \{0, 1\}$, there is a natural isomorphism $[S^{U-\delta}, W]_L \cong [S^{\dim(U^L)-\delta}, W^L]$. Decompose $U = U^L \oplus U'$, where $(U')^L = 0$. The unit sphere $S(U')$ is an L -CW complex with isotropy contained in \mathcal{P} . Consequently, the projection map $S(U') \times EP \rightarrow S(U')$ is an L -equivariant equivalence. This implies $S(U')_+ \wedge EP_+ \simeq S(U')_+$. Smashing $S(U')_+$ with the standard isotropy separation cofiber sequence $EP_+ \rightarrow S^0 \rightarrow \tilde{E}\mathcal{P}$ immediately yields the exact triangle $S(U')_+ \wedge EP_+ \rightarrow S(U')_+ \rightarrow S(U')_+ \wedge \tilde{E}\mathcal{P}$. Because the first map is an equivalence, the desired contractibility $S(U')_+ \wedge \tilde{E}\mathcal{P} \simeq *$ immediately follows. Smashing the standard cofiber sequence $S(U')_+ \rightarrow S^0 \rightarrow S^{U'}$ with $W \simeq W \wedge \tilde{E}\mathcal{P}$ yields an exact triangle $S(U')_+ \wedge W \rightarrow W \rightarrow S^{U'} \wedge W$, where the first term is contractible. Thus, the natural map $W \rightarrow S^{U'} \wedge W \simeq \Sigma^{U'} W$ is an equivalence. Using this, we deduce:

$$\begin{aligned} [S^{U-\delta}, W]_L &\cong [S^{U^L-\delta} \wedge S^{U'}, \Sigma^{U'} W]_L \\ &\cong [S^{U^L-\delta}, W]_L \\ &\cong [S^{\dim(U^L)-\delta}, W^L]. \end{aligned}$$

We define the representation U and parameter δ in two exhaustive cases to construct an equivariant lift:

1. **Case $c > q$:** Let $U = c\rho_L^{\mathcal{O}}$ and $\delta = 0$. The L -fixed points of the transitive permutation representation $\rho_L^{\mathcal{O}} = [L/K_L]$ consist solely of the constant functions, so $\dim(U^L) = c(1) = c$. Under the isomorphism above, the non-nullhomotopic map $S^c \rightarrow W^L$ lifts to a non-trivial L -equivariant map $S^{c\rho_L^{\mathcal{O}}} \rightarrow W$. By the induction-restriction adjunction, this canonically corresponds to a non-trivial G -equivariant map $f : C \rightarrow W$, where $C = G_+ \wedge_L S^{c\rho_L^{\mathcal{O}}}$. Its slice dimension is $c|L|_{\mathcal{O}}$. Since $c > q = \lfloor n/|L|_{\mathcal{O}} \rfloor$, we have $c \geq \lfloor n/|L|_{\mathcal{O}} \rfloor + 1 > n/|L|_{\mathcal{O}}$, ensuring $c|L|_{\mathcal{O}} > n$. Because $c > q \geq 0$, we have $c \geq 1 > 0$, ensuring C is a valid \mathcal{O} -slice cell of dimension $\geq n$. Thus $C \in \tau_{\geq n}^{\mathcal{O}}$.
2. **Case $c = q$:** Let $U = (q+1)\rho_L^{\mathcal{O}}$ and $\delta = 1$. Then $\dim(U^L) - \delta = (q+1)(1) - 1 = q = c$. The non-nullhomotopic map $S^c \rightarrow W^L$ lifts to a non-trivial L -equivariant map $S^{(q+1)\rho_L^{\mathcal{O}}-1} \rightarrow W$. By the induction-restriction adjunction,

this uniquely yields a non-trivial G -equivariant map $f : C \rightarrow W$, where $C = G_+ \wedge_L S^{(q+1)\rho_L^\mathcal{O}-1}$. Its slice dimension is $(q+1)|L|_\mathcal{O} - 1$. Write $n = q|L|_\mathcal{O} + r$ with remainder $0 \leq r \leq |L|_\mathcal{O} - 1$. We have $(q+1)|L|_\mathcal{O} - 1 = q|L|_\mathcal{O} + |L|_\mathcal{O} - 1 \geq q|L|_\mathcal{O} + r = n$. Since $n \geq 0$ and $q \geq 0$, we have $q+1 \geq 1 > 0$. Hence, C is a valid \mathcal{O} -slice cell and $C \in \tau_{\geq n}^\mathcal{O}$.

In both configurations, we found an \mathcal{O} -slice cell $C \in \tau_{\geq n}^\mathcal{O}$ admitting a non-nullhomotopic map $f : C \rightarrow W$. However, $W \in (\tau_{\geq n}^\mathcal{O})^\perp$ fundamentally resides in the right-orthogonal complement of $\tau_{\geq n}^\mathcal{O}$, demanding that any map from C to W be universally nullhomotopic. This constitutes a strict contradiction. Therefore, we must have $W \simeq *$, implying $X \simeq F \in \tau_{\geq n}^\mathcal{O}$. \square