

The prompts and responses below have been edited only for formatting compatibility.

User Prompt

Consider the graded ring $R = \mathbb{Q}[x_1, \dots, x_n]$ where each x_i has degree 2. This has an obvious action of the symmetric group $W = S_n$, by permutations on the variables, so that

$$R^W = \mathbb{Q}[e_1, \dots, e_n]$$

where the e_i are the elementary symmetric polynomials in the x_i . Let

$$\mathfrak{R}_\lambda = \prod_{i=3}^n (x_1 - x_i)(x_2 - x_i).$$

Let $W_\lambda = S_2 \times S_{n-2}$, the subgroup of W permuting $\{x_1, x_2\}$ and $\{x_3, \dots, x_n\}$. For any $f \in R^{W_\lambda}$, we define

$$\int f = \sum_{w \in W/W_\lambda} w(f/\mathfrak{R}_\lambda) \in R^W.$$

Let $N = 2(n-2)$ and $\eta = (x_1 + x_2)^{N+1}$. Consider the map $\nabla_\lambda^\eta: R^W \rightarrow R^W$ defined as

$$\nabla_\lambda^\eta(f) = \int (\eta \partial_\lambda f)$$

where $\partial_\lambda f = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}$.

Consider ∇_λ^η acting on the first augmentation quotient

$$\mathbb{V} = \text{Gr}_{\text{aug}}^1(R^W) / \text{Gr}_{\text{aug}}^2(R^W)$$

for the augmentation filtration on R^W , for which a basis is the elementary symmetric polynomials e_1, \dots, e_n . This \mathbb{V} is graded by degree, and the e_1, \dots, e_n are eigenvectors for the action of ∇_λ^η on the associated graded of \mathbb{V} .

Calculate the associated eigenvalues $\epsilon_1, \dots, \epsilon_n$. In other words, it will be the case that

$$\nabla_\lambda^\eta e_i = \epsilon_i e_i + (\text{polynomial in } e_1, \dots, e_{i-1})$$

calculate the ϵ_i .

Model Response (*Aletheia*)

The problem asks for the eigenvalues ϵ_i of the operator ∇_λ^η acting on the first augmentation quotient $\mathbb{V} = I/I^2$ of the ring of symmetric polynomials $R^W = \mathbb{Q}[e_1, \dots, e_n]$, where $I = (e_1, \dots, e_n)$. The space \mathbb{V} has a basis $\{e_1, \dots, e_n\} \pmod{I^2}$.

The operator is defined as $\nabla_\lambda^\eta(f) = \int (\eta \partial_\lambda f)$. We are given $\deg(x_i) = 2$, $N = 2n - 4$, and $\eta = (x_1 + x_2)^{N+1} = (x_1 + x_2)^{2n-3}$. Let $M = 2n - 3$, and $\partial_\lambda = \partial_{x_1} + \partial_{x_2}$. The

Gysin map \int is defined using $\mathfrak{R}_\lambda = \prod_{i=3}^n (x_1 - x_i)(x_2 - x_i)$.

1 Degree Preservation and Diagonal Action

We check the degree preservation of ∇_λ^η . Given $\deg(\eta) = 2M = 4n - 6$ and $\deg(\mathfrak{R}_\lambda) = 4(n - 2) = 4n - 8$, the Gysin map \int lowers the degree by $\deg(\mathfrak{R}_\lambda)$. Since ∂_λ lowers the degree by 2:

$$\deg(\nabla_\lambda^\eta(f)) = \deg(f) - 2 + \deg(\eta) - \deg(\mathfrak{R}_\lambda) = \deg(f) - 2 + (4n - 6) - (4n - 8) = \deg(f).$$

Since ∇_λ^η preserves the degree, and the basis elements e_i have distinct degrees $2i$, the action on \mathbb{V} is diagonal:

$$\nabla_\lambda^\eta(e_i) \equiv \epsilon_i e_i \pmod{I^2}.$$

2 Calculation using Power Sums

It is convenient to use the power sum basis $p_i = \sum x_j^i$. Modulo I^2 , p_i are related to e_i and the complete homogeneous symmetric polynomials h_i by $p_i \equiv i h_i \equiv (-1)^{i-1} i e_i \pmod{I^2}$. The eigenvalues on p_i are the same as on e_i .

$$\begin{aligned} \partial_\lambda p_i &= i(x_1^{i-1} + x_2^{i-1}) \\ \nabla_\lambda^\eta(p_i) &= i \int (x_1 + x_2)^M (x_1^{i-1} + x_2^{i-1}) \end{aligned}$$

3 Localization Formula and Divided Differences

We use the localization formula for the Gysin map \int . Let $A(z) = \prod_{k=1}^n (z - x_k)$.

$$\int f = - \sum_{a < b} w_{\{a,b\}}(f) \frac{(x_a - x_b)^2}{A'(x_a)A'(x_b)}.$$

Let $Q(u, v) = (u + v)^M (u - v)^2$. $Q(u, v)$ is symmetric.

$$\nabla_\lambda^\eta(p_i) = -i \sum_{a < b} \frac{(x_a^{i-1} + x_b^{i-1})Q(x_a, x_b)}{A'(x_a)A'(x_b)}.$$

Rewriting as a sum over ordered pairs $a \neq b$:

$$\nabla_\lambda^\eta(p_i) = -i \sum_{a \neq b} \frac{x_a^{i-1} Q(x_a, x_b)}{A'(x_a)A'(x_b)} = -iT.$$

Expressing T using divided differences:

$$T = \sum_a \frac{x_a^{i-1}}{A'(x_a)} \sum_{b \neq a} \frac{Q(x_a, x_b)}{A'(x_b)}.$$

Let $P_a(z) = Q(x_a, z) = (x_a + z)^M(x_a - z)^2$. Since $M = 2n - 3 \geq 1$ (for $n \geq 2$), $P_a(x_a) = 0$. The inner sum I_a is the divided difference $[x_1, \dots, x_n]P_a(z)$. Expanding $P_a(z)$, let $P(t) = (1+t)^M(1-t)^2 = \sum_{k=0}^{2n-1} K_k t^k$. The degree is $D = 2n-1$. Using the homogenization $P_a(z) = z^D P(x_a/z) = \sum_k K_k x_a^k z^{2n-1-k}$ and the property $[x_1, \dots, x_n]z^j = h_{j-n+1}$:

$$I_a = \sum_k K_k x_a^k [x_1, \dots, x_n] z^{2n-1-k} = \sum_k K_k x_a^k h_{n-k}.$$

Substituting back into T :

$$T = \sum_k K_k h_{n-k} \sum_a \frac{x_a^{k+i-1}}{A'(x_a)} = \sum_k K_k h_{n-k} h_{k+i-n}.$$

4 Reduction modulo I^2

Analyzing T modulo I^2 , the product $h_{n-k} h_{k+i-n} \in I^2$ unless $k = n$ or $k = n - i$.

$$T \equiv K_n h_0 h_i + K_{n-i} h_i h_0 = (K_n + K_{n-i}) h_i \pmod{I^2}.$$

Using $\nabla_\lambda^\eta(p_i) = -iT$ and $p_i \equiv ih_i \pmod{I^2}$:

$$\nabla_\lambda^\eta(p_i) \equiv -i(K_n + K_{n-i}) h_i = -(K_n + K_{n-i}) p_i \pmod{I^2}.$$

The eigenvalues are $\epsilon_i = -(K_n + K_{n-i})$.

5 Calculation of the Coefficients K_k

K_k is the coefficient of t^k in $P(t) = (1+t)^M(1-2t+t^2)$ where $M = 2n - 3$.

$$K_k = \binom{M}{k} - 2 \binom{M}{k-1} + \binom{M}{k-2}.$$

For K_n :

$$K_n = \binom{M}{n} - 2 \binom{M}{n-1} + \binom{M}{n-2}.$$

Using symmetry $\binom{M}{n} = \binom{M}{n-3}$ and $\binom{M}{n-1} = \binom{M}{n-2}$:

$$K_n = \binom{M}{n-3} - \binom{M}{n-2} = - \left(\binom{2n-3}{n-2} - \binom{2n-3}{n-3} \right) = -C_{n-1}.$$

The eigenvalues are:

$$\epsilon_i = -(-C_{n-1} + K_{n-i}) = C_{n-1} - K_{n-i}.$$

The explicit formula for the eigenvalues $\epsilon_1, \dots, \epsilon_n$ is:

$$\epsilon_i = \frac{1}{n} \binom{2n-2}{n-1} - \left(\binom{2n-3}{n-i} - 2 \binom{2n-3}{n-i-1} + \binom{2n-3}{n-i-2} \right),$$

where $\binom{N}{k} = 0$ if $k < 0$ or $k > N$.