

The prompts and responses below have been edited only for formatting compatibility.

## FirstProof # 2

Let  $F$  be a non-archimedean local field with ring of integers  $\mathfrak{o}$ . Let  $N_r$  denote the subgroup of  $\mathrm{GL}_r(F)$  consisting of upper-triangular unipotent elements. Let  $\psi : F \rightarrow \mathbb{C}^\times$  be a nontrivial additive character of conductor  $\mathfrak{o}$ , identified in the standard way with a generic character of  $N_r$ . Let  $\Pi$  be a generic irreducible admissible representation of  $\mathrm{GL}_{n+1}(F)$ , realized in its  $\psi^{-1}$ -Whittaker model  $\mathcal{W}(\Pi, \psi^{-1})$ . Must there exist  $W \in \mathcal{W}(\Pi, \psi^{-1})$  with the following property?

Let  $\pi$  be a generic irreducible admissible representation of  $\mathrm{GL}_n(F)$ , realized in its  $\psi$ -Whittaker model  $\mathcal{W}(\pi, \psi)$ . Let  $\mathfrak{q}$  denote the conductor ideal of  $\pi$ , let  $Q \in F^\times$  be a generator of  $\mathfrak{q}^{-1}$ , and set

$$u_Q := I_{n+1} + Q E_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$

where  $E_{i,j}$  is the matrix with a 1 in the  $(i,j)$ -entry and 0 elsewhere. For some  $V \in \mathcal{W}(\pi, \psi)$ , the local Rankin–Selberg integral

$$\int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

is finite and nonzero for all  $s \in \mathbb{C}$ .

## Model Response (Agent Af)

### 1. Introduction and the Universal Test Vector

Let  $F$  be a non-archimedean local field with ring of integers  $\mathfrak{o}$ , maximal ideal  $\mathfrak{p}$ , and a fixed uniformizer  $\varpi$ . Let  $G_n = \mathrm{GL}_n(F)$  and let  $K_n = \mathrm{GL}_n(\mathfrak{o})$  denote the standard maximal compact subgroup. Let  $N_k(F)$  signify the subgroup of upper-triangular unipotent matrices in  $\mathrm{GL}_k(F)$ . We are given a non-trivial additive character  $\psi : F \rightarrow \mathbb{C}^\times$  structurally mapped with a conductor exactly  $\mathfrak{o}$ .

Because  $\psi$  has conductor  $\mathfrak{o}$ , its inverse character  $\psi^{-1}$  evaluates identically to 1 on the compact open unipotent intersection  $N_n(\mathfrak{o}) = N_n(F) \cap K_n$ . By the Iwasawa decomposition  $G_n = N_n(F)A_n(F)K_n$ , the double coset  $N_n(F)K_n$  forms an open and closed topological subspace of  $G_n$ . We may therefore define a smooth, locally constant test function  $\Phi : G_n \rightarrow \mathbb{C}$  identically supported on  $N_n(F)K_n$  by setting:

$$\Phi(nk) = \psi^{-1}(n) \quad \text{for } n \in N_n(F), \ k \in K_n.$$

This function is unequivocally well-defined: if  $n_1 k_1 = n_2 k_2$ , then  $n_2^{-1} n_1 = k_2 k_1^{-1} \in N_n(\mathfrak{o})$ , upon which  $\psi^{-1}$  is structurally trivial. Modulo left translation by  $N_n(F)$ , the support of  $\Phi$  natively corresponds to the compact quotient  $N_n(\mathfrak{o}) \backslash K_n$ . Consequently,  $\Phi$  functions as a naturally defined element of the Schwartz space  $C_c^\infty(N_n(F) \backslash G_n, \psi^{-1})$ .

By the exactness of the mirabolic restriction functor in the foundational theory of the Kirillov model (I. N. Bernstein and A. V. Zelevinsky, *Representations of the group*  $\mathrm{GL}(n, F)$ , where  $F$  is a local non-Archimedean field, 1976), the restriction map  $W \mapsto W|_{G_n}$  from the Whittaker model  $\mathcal{W}(\Pi, \psi^{-1})$  strictly contains the compactly supported space  $C_c^\infty(N_n(F) \backslash G_n, \psi^{-1})$ . Thus, we may fix a universally valid test vector  $W \in \mathcal{W}(\Pi, \psi^{-1})$  definitively satisfying:

$$W(\mathrm{diag}(g, 1)) = \Phi(g) \quad \text{for all } g \in G_n.$$

## 2. Annihilation of Analytic Dependence on $s$

Let  $\pi$  be a generic irreducible admissible representation of  $G_n$ , and let  $\mathfrak{q}$  denote its conductor ideal. For any  $g \in G_n$ , we evaluate  $W$  on the shifted unipotent element  $\mathrm{diag}(g, 1)u_Q$ , where  $u_Q = I_{n+1} + QE_{n, n+1}$  and  $Q \in F^\times$  generates  $\mathfrak{q}^{-1}$ . Matrix block multiplication factorizes this sequentially as:

$$\mathrm{diag}(g, 1)u_Q = v_Q(g) \mathrm{diag}(g, 1), \quad \text{where } v_Q(g) = I_{n+1} + Q \sum_{i=1}^n g_{i, n} E_{i, n+1}.$$

The extracted matrix  $v_Q(g)$  natively resides in  $N_{n+1}(F)$ . Its superdiagonal entries strictly vanish everywhere except precisely at the index  $(n, n+1)$ , which maps to  $Qg_{n, n}$ . Because  $W$  transforms equivariantly under left translation by  $N_{n+1}(F)$  via  $\psi^{-1}$ , isolating the additive phase algebraically yields:

$$W(\mathrm{diag}(g, 1)u_Q) = \psi^{-1}(v_Q(g))W(\mathrm{diag}(g, 1)) = \psi^{-1}(Qg_{n, n})\Phi(g).$$

Substituting this universal test vector  $W$  into the local Rankin–Selberg integral  $I(s)$  against an arbitrary generic vector  $V \in \mathcal{W}(\pi, \psi)$  gives:

$$I(s) = \int_{N_n(F) \backslash G_n} \Phi(g) \psi^{-1}(Qg_{n, n}) V(g) |\det g|^{s-1/2} dg.$$

Because  $\Phi(g)$  actively forces the integration domain to rigidly collapse onto the orbit  $N_n(F)K_n$ , any representative factoring as  $g = nk \in N_n(F)K_n$  automatically satisfies  $|\det g| = |\det n| |\det k| = 1$ . The integral completely sheds all analytic dependence on the parameter  $s$ .

Furthermore, utilizing the transformation properties  $\Phi(nk) = \psi^{-1}(n)\Phi(k)$  and  $V(nk) = \psi(n)V(k)$ , the translation phases cleanly cancel. This guarantees that the integrand  $F(k) = \psi^{-1}(Qk_{n, n})V(k)$  is definitively left  $N_n(\mathfrak{o})$ -invariant. Applying the standard quotient integral formula relating Haar measures for an integrand defined natively on  $K_n$  to the quotient, we smoothly transition the domain:

$$\int_{K_n} F(k) dk = \int_{N_n(\mathfrak{o}) \backslash K_n} \left( \int_{N_n(\mathfrak{o})} F(nk) dn \right) d\dot{k} = \mathrm{vol}(N_n(\mathfrak{o})) \int_{N_n(\mathfrak{o}) \backslash K_n} F(\dot{k}) d\dot{k}.$$

Defining the non-zero volume constant  $C = \mathrm{vol}(N_n(\mathfrak{o}))^{-1} > 0$  and designating  $f(k) = \psi^{-1}(Qk_{n, n})$ , the shifted local Rankin–Selberg integral flawlessly descends to a purely

$s$ -independent compact functional smoothly mapped over  $K_n$ :

$$I(s) = C \int_{K_n} f(k) V(k) dk.$$

### 3. Exact Non-Vanishing

Let  $c \geq 0$  denote the exact conductor exponent of  $\pi$ , such that  $\mathfrak{q} = \mathfrak{p}^c$ . We partition the evaluation cleanly based on  $c$ .

#### 3.1. The Unramified Case ( $c = 0$ )

If  $c = 0$ , the conductor natively resolves to  $\mathfrak{o}$ , meaning  $Q \in \mathfrak{o}^\times$ . Since  $k_{n,n} \in \mathfrak{o}$  for all  $k \in K_n$  and  $\psi^{-1}$  vanishes natively on  $\mathfrak{o}$ , the phase trivializes perfectly to  $f(k) \equiv 1$ . Selecting  $V$  as the unramified spherical vector  $V_{\text{sph}} \in \mathcal{W}(\pi, \psi)$  normalized to  $V_{\text{sph}}(I_n) = 1$ , we systematically obtain  $V_{\text{sph}}(k) = 1$  uniformly over  $K_n$ . The functional strictly extracts  $I(s) = C \text{vol}(K_n) \neq 0$ .

#### 3.2. Finite Fourier Analysis of Functional Subspaces ( $c \geq 1$ )

Assume  $c \geq 1$ . We uniquely factorize the generator as  $Q = u_0 \varpi^{-c}$  for some valid unit  $u_0 \in \mathfrak{o}^\times$ . Define the standard principal compact congruence subgroup corresponding to the terminal row:

$$H_c = K_1(\mathfrak{p}^c) = \{x \in K_n \mid e_n^T x \equiv e_n^T \pmod{\mathfrak{p}^c}\}.$$

For any  $h \in H_c$  and  $k \in K_n$ , matrix multiplication enforces  $(hk)_{n,n} \equiv k_{n,n} \pmod{\mathfrak{p}^c}$ . This ensures  $Q(hk)_{n,n} - Qk_{n,n} \in \mathfrak{o}$ , strictly forcing  $f(hk) = f(k)$ . As a strongly left  $H_c$ -invariant function,  $f$  seamlessly bounds itself into  $L^2(X_c)$ , where  $X_c = H_c \backslash K_n$ . The natural projection map  $k \mapsto e_n^T k \pmod{\mathfrak{p}^c}$  canonically bijects  $X_c$  to the finite discrete set of primitive row vectors  $M_{\text{prim}}$  residing inside the module  $M = (\mathfrak{o}/\mathfrak{p}^c)^n$ . Mapped under this exact isomorphism,  $f(v) = \psi^{-1}(u_0 \varpi^{-c} v_n)$ . Let  $V_f \subset L^2(X_c)$  map the cyclic right  $K_n$ -module natively generated by  $f$ . Taking right translations evaluates  $f(vk) = \psi^{-1}(\varpi^{-c} v \cdot w)$ , where  $w = u_0 k e_n \pmod{\mathfrak{p}^c}$ . Because  $k$  transverses  $K_n$ ,  $w$  bijectively bounds all primitive column vectors in  $M$ . Hence,  $V_f$  rigorously spans the restricted characters  $\chi_w(v) = \psi^{-1}(\varpi^{-c} v \cdot w)$  along all completely primitive vectors  $w \in M_{\text{prim}}$ .

Let  $L^2(X_{c-1})$  detail the corresponding explicit subspace of functions strictly pulled back from left-invariance under  $H_{c-1}$ . Translated onto  $M_{\text{prim}}$ , the subspace  $L^2(X_{c-1})$  elegantly mirrors the span of restricted characters  $\chi_{w'}|_{M_{\text{prim}}}$  defined exactly on non-primitive vectors  $w' \in \mathfrak{p}M$ .

For  $c \geq 2$ , we mathematically assert exact orthogonality  $V_f \perp L^2(X_{c-1})$ . Computing the robust  $L^2$  inner product over  $M_{\text{prim}} = M \setminus \mathfrak{p}M$  between a primitive character  $\chi_w$  and a non-primitive character  $\chi_{w'}$  yields:

$$\langle \chi_w, \chi_{w'} \rangle = \sum_{v \in M} \psi^{-1}(\varpi^{-c} v \cdot (w - w')) - \sum_{y \in (\mathfrak{o}/\mathfrak{p}^{c-1})^n} \psi^{-1}(\varpi^{-(c-1)} y \cdot (w - w')).$$

Because  $w$  is explicitly primitive and  $w' \in \mathfrak{p}M$ , the vector disparity  $w - w'$  remains absolutely primitive. Thus,  $v \mapsto \psi^{-1}(\varpi^{-c}v \cdot (w - w'))$  acts as an intrinsically non-trivial character across the abelian group  $M$ , unconditionally evaluating the first sum to 0. Upon restricting into the non-primitive subset  $\mathfrak{p}M$ , the corresponding character on  $(\mathfrak{o}/\mathfrak{p}^{c-1})^n$  continues to assert non-trivial behavior because  $c \geq 2$  and  $w - w' \not\equiv 0 \pmod{\mathfrak{p}}$ . This unconditionally zeroes the second sum, verifying the exact orthogonal decomposition  $L^2(X_c) = V_f \oplus L^2(X_{c-1})$ .

(For the explicit boundary case  $c = 1$ , summing over primitive  $\chi_w$  identically yields  $-1$  for any given element. This intrinsically verifies that the constant function  $\mathbf{1} \in V_f$ , securely confirming  $V_f = L^2(X_1)$  inherently covers the space).

### 3.3. Provable Non-Vanishing via Frobenius Reciprocity

By the core theory of essential local newforms (H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann., 1981),  $\pi$  uniquely houses an essential Whittaker newform  $V_0$  fundamentally right-invariant under  $H_c$ , successfully normalized such that  $V_0(I_n) = 1$ . Most definitively, because the true algorithmic conductor ideal aligns perfectly with exactly  $\mathfrak{p}^c$ ,  $\pi$  analytically houses strictly zero vectors invariant under  $H_{c-1}$ .

Deploy the bounded  $K_n$ -equivariant left-averaging projection mapping  $P_c : \pi|_{K_n} \rightarrow L^2(X_c)$  characterized by:

$$(P_c V)(k) = \frac{1}{\text{vol}(H_c)} \int_{H_c} V(hk) dh.$$

Let  $M_V = P_c(\pi|_{K_n})$  cleanly map its extracted right  $K_n$ -submodule image. Because  $V_0$  behaves completely right  $H_c$ -invariant,  $(P_c V_0)(I_n) = V_0(I_n) = 1$ , practically verifying  $M_V \neq \{0\}$ .

Assume for mathematical contradiction that  $M_V \perp f$ . Since  $M_V$  acts naturally as a fully generic right  $K_n$ -module, this safely dictates unconditionally  $M_V \perp V_f$ . For  $c \geq 2$ , this locks the orthogonal restriction  $M_V \subset L^2(X_{c-1}) \cong \text{Ind}_{H_{c-1}}^{K_n}(\mathbf{1})$ . (For  $c = 1$ , it explicitly forces  $M_V = \{0\}$ , contradicting  $P_c V_0(I_n) = 1$ ). Because smooth complex representations of the totally disconnected profinite group  $K_n$  are unequivocally completely semisimple, the quotient module  $M_V$  injects backwards as a strictly stable right  $K_n$ -submodule over  $\pi|_{K_n}$ . Through standard Frobenius reciprocity:

$$\text{Hom}_{K_n}(M_V, \text{Ind}_{H_{c-1}}^{K_n}(\mathbf{1})) \cong \text{Hom}_{H_{c-1}}(M_V, \mathbf{1}) \neq 0.$$

This mathematically establishes that  $M_V$  (and inherently  $\pi$ ) unequivocally possesses a perfectly non-zero vector strictly fixed by  $H_{c-1}$ , thoroughly contradicting the rigorous primary axiom regarding the exact conductor exponent equaling  $c$ .

Therefore,  $M_V$  strictly cannot be orthogonal to  $f$ . Necessarily, there inherently exists a generic vector  $V \in \mathcal{W}(\pi, \psi)$  dynamically triggering  $\int_{K_n} (P_c V)(k) \overline{f(k)} dk \neq 0$ .

Recovering that  $\overline{f(k)} = \psi(Qk_{n,n})$  strongly retains heavy left-invariance under  $H_c$ , deploying Fubini's Theorem functionally processes the internal left-averaging param-

eter:

$$\int_{K_n} \left( \frac{1}{\text{vol}(H_c)} \int_{H_c} V(hk) dh \right) \overline{f(k)} dk = \int_{K_n} V(k) \overline{f(k)} dk \neq 0.$$

To securely specify the final explicit non-conjugated test vector, we mandate the explicitly right-translated element  $V'(g) = V(gD) \in \mathcal{W}(\pi, \psi)$  governed by the involution  $D = \text{diag}(1, \dots, 1, -1) \in K_n$ . Executing a reliable measure-preserving shift via substitution  $k \mapsto kD^{-1}$  faultlessly returns:

$$\int_{K_n} \psi^{-1}(Qk_{n,n}) V'(k) dk = \int_{K_n} \psi^{-1}(Q(kD^{-1})_{n,n}) V(k) dk.$$

Noticing that  $D^{-1} = D$ , applying the internal matrix evaluation distinctly yields  $(kD^{-1})_{n,n} = -k_{n,n}$ . Thus, the parameters beautifully intertwine to convert  $\psi^{-1}(-Qk_{n,n}) = \psi(Qk_{n,n}) = \overline{f(k)}$ . It fundamentally resolves as:

$$\int_{K_n} f(k) V'(k) dk = \int_{K_n} \overline{f(k)} V(k) dk \neq 0.$$

Consequently, calculated universally against this exactly derived sequence substitution  $V' \in \mathcal{W}(\pi, \psi)$ , the shifted local Rankin–Selberg integral unequivocally bounds into an analytically  $s$ -independent, formally explicitly finite, and uniformly non-zero strict evaluation, affirmatively resolving the inquiry.