

The prompts and responses below have been edited only for formatting compatibility.

### User Prompt (ver 1)

Is it true that if  $a_1 < a_2 < \dots$  is a sequence of integers with

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational?

### Model Response:

**Theorem 1.** Let  $(a_n)_{n \geq 1}$  be a strictly increasing sequence of integers such that

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1.$$

Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational.

#### Proof.

**1. Growth Estimates Based on Hypothesis** Let  $\lambda = \liminf_{n \rightarrow \infty} a_n^{1/2^n}$ . By hypothesis,  $\lambda > 1$ . Select a constant  $c$  such that  $1 < c < 1 < c < \lambda$ . By the definition of limit inferior, there exists an integer  $N_0$  such that for all  $n \geq N_0$ :

$$a_n^{1/2^n} > c \implies a_n > c^{2^n}.$$

Let  $u_n = \ln a_n$ . Taking the logarithm, we have  $u_n > 2^n \ln c$ . Let  $T_n = \sum_{i=1}^n u_i$ . Bounding the sum from below:

$$T_n > \sum_{i=N_0}^n 2^i \ln c = \ln c (2^{n+1} - 2^{N_0}).$$

Let  $\alpha = \ln c > 0$ . For sufficiently large  $n$ , we have:

$$T_n > \alpha 2^n. \tag{1}$$

Since  $a_n$  is a strictly increasing sequence of integers,  $a_n \geq 1$  for all sufficiently large  $n$ . We assume  $n \geq N_0$  throughout the remaining argument.

**2. Assumption of Rationality and Lower Bound** Assume, for the sake of contradiction, that  $S$  is rational. Let  $S = \frac{p}{q}$  for positive integers  $p, q$ . Define the  $n$ -th partial sum  $S_n$  and the remainder  $r_n$ :

$$S_n = \sum_{k=1}^n \frac{1}{a_k a_{k+1}}, \quad r_n = S - S_n = \sum_{k=n+1}^{\infty} \frac{1}{a_k a_{k+1}}.$$

Since the terms are positive,  $S_n < n < S$ , so  $r_n > 0$ . Consider the denominators in  $S_n$ . Each term is of the form  $\frac{1}{a_k a_{k+1}}$ . Notice that  $a_k a_{k+1}$  divides the product  $P_{n+1} = \prod_{i=1}^{n+1} a_i$ . Let  $L_n = \text{lcm}(a_1 a_2, \dots, a_n a_{n+1})$ . Then  $L_n \leq P_{n+1}$ . Writing  $S_n = \frac{M_n}{L_n}$  for some integer  $M_n$ , we have:

$$r_n = \frac{p}{q} - \frac{M_n}{L_n} = \frac{pL_n - qM_n}{qL_n}.$$

Since  $r_n > 0$ , the numerator is a positive integer, so  $pL_n - qM_n \geq 1$ . Thus:

$$r_n \geq \frac{1}{qL_n} \geq \frac{1}{qP_{n+1}}. \tag{2}$$

Using the relation  $P_{n+1} = a_{n+1} P_n$ , we write this as:

$$r_n \geq \frac{1}{qa_{n+1} P_n}.$$

**3. Universal Upper Bound for Remainder** We now bound  $r_{n+1} = \sum_{k=n+2}^{\infty} \frac{1}{a_k a_{k+1}}$ . Since  $(a_k)$  is strictly increasing,  $a_{k+1} > a_k$ , so  $\frac{1}{a_k a_{k+1}} < \frac{1}{a_k^2}$ . Thus,  $r_{n+1} < \sum_{k=n+2}^{\infty} \frac{1}{a_k^2}$ .

Since terms may be clustered, we cannot assume  $a_{k+1} \gg a_k$ . Instead, we partition the indices  $k \geq n+2$  into sets  $I_j$  based on powers of 2 relative to  $a_{n+2}$ :

$$I_j = \{k \geq n+2 \mid 2^j a_{n+2} \leq a_k < 2^{j+1} a_{n+2}\}, \quad j = 0, 1, 2, \dots$$

For any  $k \in I_j$ , we have  $a_k \geq 2^j a_{n+2}$ , so  $\frac{1}{a_k^2} \leq \frac{1}{(2^j a_{n+2})^2}$ . We estimate the size  $|I_j|$ .

For any  $k \in I_j$ , we have  $a_k < 2^{j+1} a_{n+2}$ . From the growth condition,  $a_k > c^{2^k}$ . Combining these:

$$c^{2^k} < 2^{j+1} a_{n+2} \implies 2^k \ln c < \ln c < \ln(2^{j+1}) + \ln a_{n+2}.$$

$$k < \log_2 \left( \frac{(j+1) \ln 2 + \ln a_{n+2}}{\ln c} \right).$$

Since the indices  $k$  are positive integers, the number of such indices in  $I_j$  is bounded by the maximum possible index. Let  $L = \ln a_{n+2}$ . For sufficiently large  $n$  (and thus large  $L$ ), the dominant term is  $\ln L$ . There exists a constant  $C > 0$  such that  $|I_j| \leq C(\ln L + j)$ . Substituting this into the sum:

$$\sum_{k=n+2}^{\infty} \frac{1}{a_k^2} = \sum_{j=0}^{\infty} \sum_{k \in I_j} \frac{1}{a_k^2} \leq \sum_{j=0}^{\infty} \frac{C(\ln L + j)}{(2^j a_{n+2})^2}.$$

$$r_{n+1} < \frac{C}{a_{n+2}^2} \left( \ln L \sum_{j=0}^{\infty} \frac{1}{4^j} + \sum_{j=0}^{\infty} \frac{j}{4^j} \right).$$

The series converge to finite constants. Thus, for some constant  $K$ :

$$r_{n+1} < \frac{K \ln L}{a_{n+2}^2} = \frac{K \ln \ln a_{n+2}}{a_{n+2}^2}. \quad (3)$$

**4. Deriving the Recurrence Relation** Recall the identity  $r_n = \frac{1}{a_{n+1}a_{n+2}} + r_{n+1}$ . Combining the lower bound (2) and the upper bound (3):

$$\frac{1}{qa_{n+1}P_n} \leq \frac{1}{a_{n+1}a_{n+2}} + \frac{K \ln \ln a_{n+2}}{a_{n+2}^2}.$$

Multiplying through by  $a_{n+1}a_{n+2}$ :

$$\frac{a_{n+2}}{qP_n} \leq 1 + \frac{a_{n+1}}{a_{n+2}} K \ln \ln a_{n+2}.$$

Since the sequence is strictly increasing,  $a_{n+1} < a_{n+2}$ , implying  $\frac{a_{n+1}}{a_{n+2}} < 1$ . Thus:

$$\frac{a_{n+2}}{qP_n} < 1 + K \ln \ln a_{n+2}.$$

Rearranging to isolate  $a_{n+2}$ :

$$a_{n+2} < qP_n(1 + K \ln \ln a_{n+2}).$$

Taking the natural logarithm ( $u_n = \ln a_n$  and  $\ln P_n = T_n$ ):

$$u_{n+2} < \ln q + T_n + \ln(1 + K \ln u_{n+2}).$$

Since  $a_n \rightarrow \infty$ ,  $u_{n+2} \rightarrow \infty$ . Thus for any  $\epsilon > 0$ , there exists  $N_\epsilon$  such that for all  $n \geq N_\epsilon$ :

$$\ln(1 + K \ln u_{n+2}) < \epsilon u_{n+2}.$$

Using this bound:

$$u_{n+2} < T_n + \ln q + \epsilon u_{n+2}.$$

$$(1 - \epsilon)u_{n+2} < T_n + \ln q.$$

Dividing by  $1 - \epsilon$ :

$$u_{n+2} < \frac{1}{1 - \epsilon} T_n + \frac{\ln q}{1 - \epsilon}.$$

Substitute  $u_{n+2} = T_{n+2} - T_{n+1}$ :

$$T_{n+2} - T_{n+1} < \frac{1}{1 - \epsilon} T_n + C_\epsilon.$$

Rearranging gives a linear recurrence inequality:

$$T_{n+2} < T_{n+1} + \frac{1}{1 - \epsilon} T_n + C_\epsilon.$$

**5. Contradiction** Let us analyze the growth rate implied by this recurrence. Consider the sequence  $X_n$  satisfying the corresponding homogeneous equality  $X_{n+2} = X_{n+1} + \frac{1}{1-\epsilon}X_n$ . The characteristic equation is  $x^2 - x - \frac{1}{1-\epsilon} = 0$ . The positive root is:

$$\rho(\epsilon) = \frac{1 + \sqrt{1 + \frac{4}{1-\epsilon}}}{2}.$$

Since  $T_n$  consists of positive terms, standard comparison theorems for linear recurrences imply that for sufficiently large  $n$ :

$$T_n < T_n < C \cdot (\rho(\epsilon))^n.$$

We now choose  $\epsilon$  sufficiently small to force a contradiction with the known lower bound. Let  $\epsilon = 0.05$ . Then  $\frac{1}{1-\epsilon} \approx 1.053$ .

$$\rho(0.05) = \frac{1 + \sqrt{1 + 4.21}}{2} = \frac{1 + \sqrt{5.21}}{2} \approx \frac{1 + 2.28}{2} = 1.64.$$

Crucially,  $\rho(0.05) < 1.7 < 2$ . Thus, if  $S$  is rational, then  $T_n = O(1.7^n)$ .

However, from Step 1 (Inequality 1), we established that the condition  $\liminf a_n^{1/2^n} > 1$  implies:

$$T_n > \alpha 2^n.$$

Combining these bounds yields:

$$\alpha 2^n < n < C(1.7)^n \implies \alpha < \text{implies } \alpha < C \left( \frac{1.7}{2} \right)^n.$$

Since  $\frac{1.7}{2} < 1$ , the right hand side approaches 0 as  $n \rightarrow \infty$ . This implies  $\alpha \leq 0$ . But  $\alpha = \ln c > 0$ . This is a contradiction.

Therefore, our initial assumption that  $S$  is rational must be false.

**Conclusion** The sum  $S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$  is irrational.

### User Prompt (ver 2)

Is it true that if  $a_1 < a_2 < \dots$  is a sequence of integers with

$$\liminf a_n^{1/2^n} > 1$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$$

is irrational?

## Model Response:

The problem asks us to determine the truth of the statement: if  $(a_n)$  is a strictly increasing sequence of integers with  $\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$ , then  $\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$  is irrational.

## Executive Summary

Status: Complete Solution

We prove that the statement is true. The argument proceeds by contradiction. We first reduce the problem to the case where all  $a_n \geq 2$  without loss of generality. Assuming the series sums to a rational number  $p/q$ , we construct a sequence of integers  $Z_n = qP_{n+1}R_n$ , where  $R_n$  is the series remainder and  $P_{n+1}$  is the partial product of terms. The properties of  $(a_n)$  ensure  $Z_n$  is a positive integer for all  $n$ . We derive an upper bound for  $Z_n$  that leads to the inequality  $a_{n+2} < 2qP_n$  for sufficiently large  $n$ . This recurrence implies that  $\ln P_n$  grows no faster than the Fibonacci sequence (order  $\phi^n$ ). However, the given condition  $\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$  implies that  $\ln P_n$  grows at least as  $2^n$ . Since  $2 > \phi$ , these growth rates are incompatible, establishing the contradiction.

## Polished Solution

Let  $(a_n)_{n=1}^{\infty}$  be a strictly increasing sequence of integers satisfying  $\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1$ . We prove that the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$  is an irrational number.

### Irrationality of the Series $\sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}$ Under Rapid Growth Conditions

#### Introduction

We analyze the arithmetic nature of the convergent series

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}},$$

given that the integer sequence  $(a_n)_{n=1}^{\infty}$  grows doubly exponentially. Specifically, we assume:

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1. \quad (1)$$

We demonstrate that this growth condition forces the sum  $S$  to be irrational.

#### Proof of Irrationality

**Theorem 2.** Let  $(a_n)_{n=1}^{\infty}$  be a strictly increasing sequence of integers such that

$$\liminf_{n \rightarrow \infty} a_n^{1/2^n} > 1.$$

Then

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}} \notin \mathbb{Q}.$$

*Proof.* Let  $\lambda = \liminf_{n \rightarrow \infty} a_n^{1/2^n}$ . By hypothesis,  $\lambda > 1$ . Let  $c$  be a constant such that  $1 < c < \lambda$ . By the definition of limit inferior, there exists  $N_0$  such that for all  $n \geq N_0$ ,  $a_n^{1/2^n} > c$ , or equivalently  $a_n > c^{2^n}$ . Since  $c > 1$ ,  $a_n \rightarrow \infty$ . Because  $(a_n)$  is strictly increasing, the terms eventually become positive integers. Thus, there exists  $N_1 \geq N_0$  such that  $a_n \geq 2$  for all  $n \geq N_1$ .

Note that the rationality of  $S$  depends only on the tail of the series. Let  $S = \sum_{n=1}^{N_1-1} \frac{1}{a_n a_{n+1}} + S_{\text{tail}}$ . The finite initial sum is rational. Thus,  $S \in \mathbb{Q} \iff S_{\text{tail}} \in \mathbb{Q}$ . We shift the indices of the sequence by defining  $a'_n = a_{n+N_1-1}$  for  $n \geq 1$ . The condition (1) is invariant under this shift (with a modified exponent factor that does not affect the strict inequality  $\lambda' > 1$ ). Without loss of generality, we proceed assuming  $a_n \geq 2$  and  $a_n > c^{2^n}$  for all  $n \geq 1$ . Convergence is guaranteed by comparison with the geometric series  $\sum c^{-2^n}$ , which converges absolutely.

We begin by characterizing the relative growth of the terms.

**Lemma 1.** *Under the hypotheses,  $\lim_{n \rightarrow \infty} \frac{a_{n+2}}{a_n} = \infty$ .*

*Proof.* Suppose the limit is not infinity. Then there exist a constant  $M > 0$  and a subsequence  $(n_k)$  such that  $a_{n_k+2} \leq M a_{n_k}$ . Rearranging yields  $a_{n_k} \geq M^{-1} a_{n_k+2}$ . Raising both sides to the power  $1/2^{n_k}$ :

$$a_{n_k}^{1/2^{n_k}} \geq M^{-1/2^{n_k}} \left( a_{n_k+2}^{1/2^{n_k+2}} \right)^{2^{n_k+2}/2^{n_k}}.$$

Substituting  $2^{n_k+2}/2^{n_k} = 4$ :

$$a_{n_k}^{1/2^{n_k}} \geq M^{-1/2^{n_k}} \left( a_{n_k+2}^{1/2^{n_k+2}} \right)^4.$$

Taking the limit inferior as  $k \rightarrow \infty$ :

1.  $\liminf a_{n_k}^{1/2^{n_k}} \geq \liminf a_n^{1/2^n} = \lambda$ .
2.  $\lim M^{-1/2^{n_k}} = 1$ .
3.  $\liminf \left( a_{n_k+2}^{1/2^{n_k+2}} \right)^4 \geq \lambda^4$ , since any subsequence of  $a_n^{1/2^n}$  has limit inferior at least  $\lambda$ .

Thus,  $\lambda \geq \lambda^4$ . Since  $\lambda > 1$ , dividing by  $\lambda$  yields  $1 \geq \lambda^3$ , which implies  $\lambda \leq 1$ . This contradicts  $\lambda > 1$ . Therefore, the ratio must tend to infinity.  $\square$

From Lemma 1,  $\frac{a_n}{a_{n+2}} \rightarrow 0$ . Hence, there exists an integer  $N^* \geq 1$  such that for all  $n \geq N^*$ :

$$\frac{a_n}{a_{n+2}} \leq \frac{1}{4}. \tag{2}$$

Assume for the sake of contradiction that  $S = \frac{p}{q}$  for integers  $p, q \geq 1$ . Define the partial product  $P_n = \prod_{i=1}^n a_i$ . Let  $R_n = \sum_{k=n+1}^{\infty} \frac{1}{a_k a_{k+1}}$ . Define  $Z_n = qP_{n+1}R_n$ . Writing  $R_n = S - \sum_{k=1}^n \frac{1}{a_k a_{k+1}}$ , we have:

$$Z_n = qP_{n+1} \left( \frac{p}{q} - \sum_{k=1}^n \frac{1}{a_k a_{k+1}} \right) = pP_{n+1} - q \sum_{k=1}^n \frac{P_{n+1}}{a_k a_{k+1}}.$$

Consider the term  $T_k = \frac{P_{n+1}}{a_k a_{k+1}}$  for  $1 \leq k \leq n$ . Since  $a_i$  are distinct integers (strictly increasing), the factors  $a_k$  and  $a_{k+1}$  are distinct elements in the set  $\{a_1, \dots, a_{n+1}\}$ . Thus, their product divides  $P_{n+1}$ . Consequently,  $T_k \in \mathbb{Z}$  for all  $k$ , implying  $Z_n \in \mathbb{Z}$ . Since  $a_n \geq 2$ , all terms in the series are positive, so  $R_n > 0$ . Since  $q, P_{n+1} > 0$ , we have  $Z_n > 0$ . As  $Z_n$  is an integer:

$$Z_n \geq 1. \quad (3)$$

We now bound  $R_n$ . Let  $u_k = \frac{1}{a_k a_{k+1}}$ . For  $k \geq N^*$ , the ratio of consecutive series terms is:

$$\frac{u_{k+1}}{u_k} = \frac{a_k a_{k+1}}{a_{k+1} a_{k+2}} = \frac{a_k}{a_{k+2}} \leq \frac{1}{4}.$$

Using this geometric bound for  $n \geq N^*$ :

$$R_n = \sum_{j=1}^{\infty} u_{n+j} \leq u_{n+1} \sum_{j=0}^{\infty} \left(\frac{1}{4}\right)^j = u_{n+1} \frac{4}{3} = \frac{4}{3a_{n+1}a_{n+2}}.$$

Substituting this into the definition of  $Z_n$ :

$$Z_n \leq qP_{n+1} \frac{4}{3a_{n+1}a_{n+2}} = \frac{4qP_n}{3a_{n+2}}.$$

Applying (3):

$$1 \leq \frac{4qP_n}{3a_{n+2}} \implies a_{n+2} \leq \frac{4}{3}qP_n.$$

Since  $4/3 < 2$ , we strictly have:

$$a_{n+2} < 2qP_n. \quad (4)$$

This inequality relates the term  $a_{n+2}$  to the cumulative product  $P_n$ . We analyze the growth of  $P_n$ . From  $P_{n+2} = P_{n+1}a_{n+2}$ , we deduce:

$$P_{n+2} < 2qP_n P_{n+1}.$$

Let  $L_n = \ln P_n$ . Since  $P_n \geq 2^n$ ,  $L_n > 0$ .

$$L_{n+2} < L_{n+1} + L_n + \ln(2q).$$

Let  $K = \ln(2q)$ . Define an auxiliary sequence  $(x_n)$  by  $x_{n+2} = x_{n+1} + x_n + K$  with initial conditions  $x_{N^*} = L_{N^*}, x_{N^*+1} = L_{N^*+1}$ . By simple induction,  $L_n \leq x_n$  for all  $n \geq N^*$ . Let  $y_n = x_n + K$ . Then  $y_{n+2} = y_{n+1} + y_n$ . Thus  $y_n$  follows the Fibonacci

recurrence. Standard solutions give  $y_n = A\phi^n + B\psi^n$ , where  $\phi = \frac{1+\sqrt{5}}{2}$ . Since  $|\psi| < 1$ , for large  $n$ ,  $y_n \sim A\phi^n$ . Consequently, there exists  $C_1 > 0$  such that:

$$L_n \leq C_1\phi^n. \quad (5)$$

Now we contrast this with the lower bound implied by the hypothesis. Recall  $a_n > c^{2^n}$  for  $n \geq N^*$  (renormalized index), with  $c > 1$ .

$$P_n = \prod_{k=1}^n a_k > \left( \prod_{k=1}^{N^*-1} a_k \right) \prod_{k=N^*}^n c^{2^k}.$$

The exponent sum is  $\sum_{k=N^*}^n 2^k = 2^{n+1} - 2^{N^*}$ . So  $P_n > \text{Const} \cdot c^{2^{n+1}}$ . Taking natural logarithms:

$$L_n > 2^{n+1} \ln c + C_{\text{const}}.$$

Since  $\ln c > 0$ , for sufficiently large  $n$ , there exists  $C_2 > 0$  such that:

$$L_n > C_2 2^n. \quad (6)$$

Comparing (5) and (6):

$$C_2 2^n < L_n \leq C_1 \phi^n.$$

Dividing by  $\phi^n$ :

$$C_2 \left( \frac{2}{\phi} \right)^n < C_1.$$

Observe that  $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ , so  $2 > \phi$ . Thus  $\frac{2}{\phi} > 1$ . Consequently,  $\lim_{n \rightarrow \infty} \left( \frac{2}{\phi} \right)^n = \infty$ . The inequality implies that a sequence tending to infinity is strictly bounded by the constant  $C_1/C_2$ , which is absurd.

Thus, the initial assumption that  $S$  is rational leads to a contradiction. We conclude that  $S$  must be irrational.  $\square$

### User Prompt (Gemini Deep Think)

Find a meaningful, interesting generalization of this theorem or its proof.

**Theorem** Fix  $d \geq 2$ , and let  $c_d$  be the unique positive real root of  $x^d - x^{d-1} - 1 = 0$ . Suppose a strictly increasing sequence  $\{a_n\}$  of integers satisfy

$$\liminf_{n \rightarrow \infty} a_n^{1/b^n} > 1$$

for some  $b > c_d$ . Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1} \cdots a_{n+d-1}}$$

is irrational.

**Proof** For two functions  $f$  and  $g$ , we write  $f \lesssim g$  when there exists some constant  $C > 0$  satisfying  $f \leq Cg$ . Assume for contradiction that  $S = p/q$ . We may assume there is some constant  $c > 1$  such that  $a_n > c^{b^n}$  for all  $n$ . We set

$$P_n := \prod_1^n a_i, \quad S_n = \sum_{m=1}^n \frac{1}{a_m a_{m+1} \cdots a_{m+d-1}}.$$

Since  $S > S_n$  for all  $n$ , we have

$$r_n := S - S_n = \frac{p}{q} - \frac{*}{P_{n+d-1}} \geq \frac{1}{q P_{n+d-1}}.$$

Let us define

$$I_j(n) := \{k \geq n+1 \mid 2^j a_{n+1}^d \leq a_k^d < 2^{j+1} a_{n+1}^d\}, \quad j = 0, 1, 2, \dots$$

so that each  $k \in I_j(n)$  satisfies

$$c^{d \cdot b^k} < a_k^d < 2^{j+1} a_{n+1}^d.$$

The cardinality of  $I_j(n)$  can be bounded from above as

$$|I_j(n)| \leq \max_{k \in I_j(n)} k \leq \log_b \left( \frac{\log_c(2^{j+1} a_{n+1}^d)}{d} \right) \lesssim \log j + \log \log a_{n+1}.$$

It follows that

$$r_n < \sum_{k=n+1}^{\infty} \frac{1}{a_k^d} = \sum_{j=0}^{\infty} \sum_{k \in I_j(n)} \frac{1}{a_k^d} \leq \sum_{j=0}^{\infty} \frac{|I_j(n)|}{2^j a_{n+1}^d} \lesssim \sum_{j=0}^{\infty} \frac{\log j + \log \log a_{n+1}}{2^j a_{n+1}^d} \lesssim \frac{\log \log a_{n+1}}{a_{n+1}^d}.$$

Combining the lower and the upper bounds for  $r_n$  and  $r_{n+1}$ , we have

$$\begin{aligned} \frac{1}{q P_{n+d-1}} &\leq r_n = \sum_{j=1}^{d-1} \frac{1}{a_{n+j} \cdots a_{n+d-1}} + r_{n+d-1} \\ &\lesssim \frac{d-1}{a_{n+1} \cdots a_{n+d}} + \frac{\log \log a_{n+d}}{a_{n+d}^d} \leq \frac{\log \log a_{n+d}}{a_{n+1} \cdots a_{n+d}}. \end{aligned}$$

Fix an arbitrarily small  $\epsilon > 0$ . Multiplying both sides by  $P_{n+d}$ , we obtain that

$$a_{n+d} = P_{n+d}/P_{n+d-1} \lesssim P_n \log \log a_{n+d} \lesssim P_n a_{n+d}^\epsilon = P_n (P_{n+d}/P_{n+d-1})^\epsilon.$$

It follows that we have  $P_{n+d} \lesssim P_{n+d-1} P_n^{1/(1-\epsilon)}$ . The sequence  $u_n := \log_c P_n$  satisfies the recursive inequality

$$u_{n+d} \leq u_{n+d-1} + \frac{1}{1-\epsilon} u_n + K$$

for some constant  $K = K(\epsilon) > 0$ . For a sufficiently small  $\epsilon$ , the equation

$$x^d = x^{d-1} + \frac{1}{1-\epsilon}$$

has exactly one positive real root  $s = s(d, \epsilon)$ , slightly larger than  $c_d$ , and one or zero negative real root, depending on the parity of  $d$ ; moreover, the negative real root, if exists, belong to  $(-1, 0)$ . By choosing a sufficiently small  $\epsilon$  satisfying  $s = s(d, \epsilon) < b$ , we have that  $u_n \lesssim s^n$ . This contradicts the estimate  $\log_c P_n \geq \log_c a_n \geq b^n$ .  
\*\*QED\*\*

### Model Response:

A meaningful and natural generalization of this theorem extends the result to sums of reciprocals of products with \*\*arbitrary integer weights\*\*.

The original theorem relies on a product of  $d$  consecutive terms, effectively assigning a weight of 1 to every integer in a "sliding window." By generalizing these exponents, we can quantify how the density or sparsity of the denominator affects the growth rate required for the series to be irrational.

### Theorem (Generalized)

Fix an integer  $d \geq 1$  and a tuple of non-negative integers  $\mathbf{w} = (w_0, w_1, \dots, w_{d-1})$  with  $w_{d-1} \geq 1$ . Let  $W = \max(w_0, \dots, w_{d-1})$ .

Let  $c_{\mathbf{w}}$  be the unique real root greater than 1 of the polynomial:

$$P(x) = (x-1) \sum_{j=0}^{d-1} w_j x^j - W x^{d-1}$$

Suppose a strictly increasing sequence of integers  $\{a_n\}$  satisfies

$$\liminf_{n \rightarrow \infty} a_n^{1/b^n} > 1$$

for some  $b > c_{\mathbf{w}}$ . Then the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{a_n^{w_0} a_{n+1}^{w_1} \cdots a_{n+d-1}^{w_{d-1}}}$$

is irrational.

### Proof Sketch

The generalization follows the "gap principle" strategy used in the original proof, balancing the decay of the tail against the growth of the common denominator.

#### 1. The Common Denominator (LCM):

Assume for contradiction that  $S = p/q$ . Let  $S_n$  be the  $n$ -th partial sum. The denominators of the terms in  $S_n$  involve integers up to  $a_{n+d-1}$ .

Because the window shifts by 1 at each step, any specific integer  $a_k$  appears in  $d$  consecutive terms with exponents  $w_{d-1}, \dots, w_0$ . When taking the Least Common Multiple (LCM) to form the common denominator  $Q_n$ , the exponent of  $a_k$  is bounded by the maximum weight  $W$ .

$$Q_n \lesssim \prod_{k=1}^{n+d-1} a_k^W \approx P_{n+d-1}^W$$

## 2. The Tail Estimate:

The remainder  $r_n = S - S_n$  is dominated by the first term of the tail (index  $n+1$ ).

$$r_n \approx \frac{1}{a_{n+1}^{w_0} a_{n+2}^{w_1} \cdots a_{n+d}^{w_{d-1}}}$$

## 3. The Inequality:

For  $S$  to be rational, the error  $r_n$  cannot be "too small" relative to the denominator  $Q_n$ . Specifically,  $r_n \geq 1/(qQ_n)$ . This implies  $Q_n r_n \gtrsim 1$ . Substituting the bounds:

$$P_{n+d-1}^W \cdot \frac{1}{\prod_{j=0}^{d-1} a_{n+1+j}^{w_j}} \gtrsim 1 \implies \prod_{j=0}^{d-1} a_{n+1+j}^{w_j} \lesssim P_{n+d-1}^W$$

Taking logarithms (assuming  $\log a_k \sim b^k$ ) and dividing by  $b^{n+1}$ :

$$\sum_{j=0}^{d-1} w_j b^j \lesssim W \frac{b^{d-1}}{b-1}$$

Rearranging this inequality yields the polynomial  $P(b) \leq 0$ . If  $b > c_{\mathbf{w}}$  (the root), this inequality is violated for large  $n$ , proving irrationality.

## Why this is Meaningful

**1. It Unifies "Dense" and "Sparse" Problems** **The Original Case:** If  $\mathbf{w} = (1, \dots, 1)$ , then  $W = 1$ . The sum is  $\frac{x^d - 1}{x - 1}$ .

$$(x - 1) \frac{x^d - 1}{x - 1} - 1 \cdot x^{d-1} = x^d - x^{d-1} - 1 = 0$$

This recovers the original constant  $c_d$ .

**The "Gap" Case:** Consider  $S = \sum \frac{1}{a_n a_{n+2}}$ . Here  $\mathbf{w} = (1, 0, 1)$  and  $W = 1$ . The polynomial becomes:

$$(x - 1)(1 + x^2) - x^2 = x^3 - 2x^2 + x - 1 = 0$$

The root is  $c \approx 1.75$ , which is larger than the  $c \approx 1.46$  required for the consecutive block  $a_n a_{n+1} a_{n+2}$ . This quantifies the intuition that "missing terms" weaken the convergence\*\*, forcing the sequence  $a_n$  to grow faster to guarantee irrationality.

## 2. It Handles Heavy Weights

**Sylvester-Type Series:** If  $d = 1$  and  $w_0 = 1$ , the polynomial is  $(x - 1)(1) - 1 = x - 2$ , recovering the classic result that  $\sum 1/a_n$  is irrational if  $a_{n+1} \sim a_n^2$  (limit  $b = 2$ ). If we increase the weight of the *last* term (e.g.,  $1/(a_n a_{n+1}^2)$ ), the root increases, making irrationality harder to prove. This is because the "cost" of the LCM ( $W = 2$ ) rises, while the benefit to the tail decay is limited to the future terms.

### 3. Universal Irrationality for Double-Exponential Sequences

Since  $c_d \rightarrow 1$  as  $d \rightarrow \infty$  for the uniform case, this generalization implies a powerful corollary:

For **any** sequence  $\{a_n\}$  with double-exponential growth (any  $b > 1$ ), there exists a sufficiently large block size  $d$  such that  $\sum \frac{1}{a_n \cdots a_{n+d-1}}$  is irrational.