

The prompts and responses below have been edited only for formatting compatibility.

### FirstProof # 9

Let  $n \geq 5$ . Let  $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3 \times 4}$  be Zariski-generic. For  $\alpha, \beta, \gamma, \delta \in [n]$ , construct  $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$  so that its  $(i, j, k, \ell)$  entry for  $1 \leq i, j, k, \ell \leq 3$  is given by  $Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$ . Here  $A(i, :)$  denotes the  $i$ th row of a matrix  $A$ , and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors  $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$ .

More precisely, does there exist a polynomial map  $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$  that satisfies the following three properties?

- The map  $\mathbf{F}$  does not depend on  $A^{(1)}, \dots, A^{(n)}$ .
- The degrees of the coordinate functions of  $\mathbf{F}$  do not depend on  $n$ .
- Let  $\lambda \in \mathbb{R}^{n \times n \times n \times n}$  satisfy  $\lambda_{\alpha\beta\gamma\delta} \neq 0$  for precisely  $\alpha, \beta, \gamma, \delta \in [n]$  that are not identical. Then  $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$  holds if and only if there exist  $u, v, w, x \in (\mathbb{R}^*)^n$  such that  $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$  for all  $\alpha, \beta, \gamma, \delta \in [n]$  that are not identical.

### Model Response (Agent A)

#### 1. Construction of the Polynomial Map $\mathbf{F}$

We construct a polynomial map  $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$  whose coordinate functions are homogeneous polynomials of degree 3.

Let  $T$  be a formal tensor of variables in  $\mathbb{R}^{n \times n \times n \times n \times 3 \times 3 \times 3 \times 3}$ . The  $81n^4$  entries of  $T$ , denoted  $T_{ijkl}^{(\alpha\beta\gamma\delta)}$  for positional indices  $\alpha, \beta, \gamma, \delta \in [n]$  and row choices  $i, j, k, \ell \in \{1, 2, 3\}$ , serve as the algebraic inputs to  $\mathbf{F}$ . We will subsequently evaluate  $\mathbf{F}$  under the substitution  $T_{ijkl}^{(\alpha\beta\gamma\delta)} = \lambda_{\alpha\beta\gamma\delta} Q_{ijkl}^{(\alpha\beta\gamma\delta)}$ .

To build the coordinate functions of  $\mathbf{F}$ , we proceed as follows:

1. Fix an ordered pair of distinct tensor modes (e.g., the third and fourth modes). Assign them strictly distinct positional indices  $\gamma, \delta \in [n]$  with  $\gamma \neq \delta$ , alongside arbitrary row indices  $k, \ell \in \{1, 2, 3\}$ .
2. For the remaining two varying modes (e.g., the first and second modes), choose three strictly distinct indices for the first mode:  $\alpha_1 < \alpha_2 < \alpha_3$  in  $[n]$ , and three strictly distinct indices for the second mode:  $\beta_1 < \beta_2 < \beta_3$  in  $[n]$ .
3. Assign arbitrary row configurations  $i_1, i_2, i_3 \in \{1, 2, 3\}$  and  $j_1, j_2, j_3 \in \{1, 2, 3\}$ .

Construct a  $3 \times 3$  matrix  $B$  populated by the corresponding formal variables from  $T$ :

$$B_{r,s} = T_{i_r j_s k \ell}^{(\alpha_r \beta_s \gamma \delta)} \quad \text{for } 1 \leq r, s \leq 3.$$

We define one coordinate function of  $\mathbf{F}$  to be the polynomial  $\det(B)$ . The complete map  $\mathbf{F}$  is formed by collecting all such determinantal polynomials over all  $\binom{4}{2} = 6$  valid pairs of varying modes, and over all valid combinatorial choices of positional and row indices.

**Constraint Verification:** By definition, each coordinate function evaluates a standard  $3 \times 3$  minor of the formal variables. Its coefficients are exclusively  $\pm 1$  or  $0$ . Thus,  $\mathbf{F}$  depends entirely on combinatorial structures and is strictly independent of the entries of the generic matrices  $A^{(m)}$ . Furthermore, every coordinate function is a homogeneous polynomial of degree exactly 3, satisfying the requirement that the degrees of the coordinate functions do not depend on  $n$ .

## 2. Proof of Sufficiency ( $\Leftarrow$ )

Assume there exist parameters  $u, v, w, x \in (\mathbb{R}^*)^n$  such that  $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$  precisely over the domain of tuples  $(\alpha, \beta, \gamma, \delta)$  that are not uniformly identical. Evaluating  $\mathbf{F}$  at  $T = \lambda \odot Q$  (where  $\odot$  denotes the pointwise Hadamard product), the matrix  $B$  constructed above via the first and second varying modes takes the form:

$$B_{r,s} = \lambda_{\alpha_r\beta_s\gamma\delta} Q_{i_r j_s k \ell}^{(\alpha_r\beta_s\gamma\delta)}.$$

Because the construction strictly enforced  $\gamma \neq \delta$ , the evaluated tuple  $(\alpha_r, \beta_s, \gamma, \delta)$  fundamentally avoids identical uniformity. Hence, substituting the rank-1 factorization yields:

$$B_{r,s} = u_{\alpha_r} v_{\beta_s} w_\gamma x_\delta \det [A^{(\alpha_r)}(i_r, :); A^{(\beta_s)}(j_s, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)].$$

This factors the matrix as  $B = c \cdot D_u \tilde{Q} D_v$ , where  $c = w_\gamma x_\delta \neq 0$ , the diagonal scaling matrices  $D_u = \text{diag}(u_{\alpha_1}, u_{\alpha_2}, u_{\alpha_3})$  and  $D_v = \text{diag}(v_{\beta_1}, v_{\beta_2}, v_{\beta_3})$  are non-singular, and  $\tilde{Q}$  is the core multilinear  $4 \times 4$  determinant matrix defined by

$$\tilde{Q}_{r,s} = \det [A^{(\alpha_r)}(i_r, :); A^{(\beta_s)}(j_s, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)].$$

Because the matrices  $A^{(m)}$  are Zariski-generic and  $\gamma \neq \delta$ , the selected fixed rows  $A^{(\gamma)}(k, :)$  and  $A^{(\delta)}(\ell, :)$  are linearly independent and naturally span a 2-dimensional subspace  $V \subset \mathbb{R}^4$ . The  $4 \times 4$  determinant evaluates the standard volume form on  $\mathbb{R}^4$ , which mathematically descends to an alternating bilinear form on the 2-dimensional quotient space  $\mathbb{R}^4/V \cong \mathbb{R}^2$ . Letting  $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^4/V$  be the canonical projection, there exists a non-zero scalar  $C$  (dependent strictly on  $V$ ) such that:

$$\tilde{Q}_{r,s} = C \det \left( \pi(A^{(\alpha_r)}(i_r, :)), \pi(A^{(\beta_s)}(j_s, :)) \right),$$

where the outer determinant represents the canonical  $2 \times 2$  volume form on the quotient space. Algebraically, this structural equivalence dictates that the  $3 \times 3$  matrix  $\tilde{Q}$  factors precisely through an intermediate 2-dimensional vector space. Consequently,  $\text{rank}(\tilde{Q}) \leq 2$ . Thus, all evaluated  $3 \times 3$  minors identically vanish, forcing  $\det(B) = 0$  universally across all valid mode and index selections. This establishes sufficiency.

### 3. Proof of Necessity ( $\implies$ ): Local Rank-1 Separability

Assume  $\mathbf{F}(\lambda \odot Q) \equiv 0$ . Expanding the constraint  $\det(B) = 0$  via the Leibniz formula yields:

$$\sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma) \prod_{r=1}^3 \left( \lambda_{\alpha_r \beta_{\sigma(r)} \gamma_\delta} \tilde{Q}_{r, \sigma(r)} \right) = 0.$$

Let  $v_r = \pi(A^{(\alpha_r)}(i_r, :))$  and  $u_s = \pi(A^{(\beta_s)}(j_s, :))$ . Factoring out the volume scalar  $C^3 \neq 0$  and denoting the coefficient component by  $W_\sigma = \prod_{r=1}^3 \lambda_{\alpha_r \beta_{\sigma(r)} \gamma_\delta}$ , the sum defines a multilinear map  $L$  evaluating to zero:

$$L(v_1, v_2, v_3, u_1, u_2, u_3) = \sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma) W_\sigma \prod_{r=1}^3 \det(v_r, u_{\sigma(r)}) = 0.$$

For each fixed index  $r \in \{1, 2, 3\}$ , the set of possible values for the vector  $v_r$  is generated by independently varying the assigned row choice  $i_r \in \{1, 2, 3\}$ . Because each single matrix  $A^{(\alpha_r)}$  is Zariski-generic, its three available rows span a 3-dimensional subspace of  $\mathbb{R}^4$ . The intersection of this 3-dimensional subspace with the 2-dimensional kernel  $V$  of the projection  $\pi$  is generically 1-dimensional. Consequently, as  $i_r$  varies, the set of three available projections  $\{\pi(A^{(\alpha_r)}(1, :)), \pi(A^{(\alpha_r)}(2, :)), \pi(A^{(\alpha_r)}(3, :))\}$  dynamically spans the entirety of the 2-dimensional target space  $\mathbb{R}^4/V \cong \mathbb{R}^2$ .

Since this independent spanning property holds universally for the domain of every argument slot  $v_r$  as  $i_r$  varies (and symmetrically for  $u_s$  as  $j_s$  varies), the multilinear map  $L$  evaluates directly to zero on the full Cartesian product of these structurally full independent spanning sets. Standard multilinear algebra therefore rigorously dictates that  $L \equiv 0$  identically everywhere on  $(\mathbb{R}^2)^6$ .

By the First and Second Fundamental Theorems of Invariant Theory for  $\text{SL}(2, \mathbb{R})$  (see H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939), the vector space of multilinear  $\text{SL}(2, \mathbb{R})$ -invariants on six generic vectors in  $\mathbb{R}^2$  has dimension exactly equal to the Catalan number  $C_3 = 5$ . There are precisely  $3! = 6$  uniquely generated multi-determinant bipartite invariants  $P_\sigma = \prod_{r=1}^3 \det(v_r, u_{\sigma(r)})$ . Because these bipartite invariants span the entire 5-dimensional invariant space (linearly generating all non-crossing perfect matchings via successive Grassmann-Plücker syzygies), the space of linear relations strictly spanning them is exactly 1-dimensional ( $6 - 5 = 1$ ). This unique relation is universally generated by the primary Grassmann-Plücker identity:

$$\sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma) P_\sigma = 0.$$

Because the syzygy space is strictly 1-dimensional, the identically vanishing linear combination  $\sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma) W_\sigma P_\sigma \equiv 0$  functionally mandates that the coefficients  $\text{sgn}(\sigma) W_\sigma$  are strictly proportional to the syzygy coefficients  $\text{sgn}(\sigma)$ . This systematically yields  $W_\sigma = K$ , where  $K$  is a constant completely independent of  $\sigma$ . Equating

$W_{\text{id}} = W_{(12)}$  systematically yields:

$$\lambda_{\alpha_1\beta_1\gamma\delta}\lambda_{\alpha_2\beta_2\gamma\delta}\lambda_{\alpha_3\beta_3\gamma\delta} = \lambda_{\alpha_1\beta_2\gamma\delta}\lambda_{\alpha_2\beta_1\gamma\delta}\lambda_{\alpha_3\beta_3\gamma\delta}.$$

Because  $\gamma \neq \delta$ , the tuple  $(\alpha_3, \beta_3, \gamma, \delta)$  categorically avoids identical uniformity. By hypothesis,  $\lambda$  strictly vanishes exclusively on uniformly identical tuples, reliably guaranteeing  $\lambda_{\alpha_3\beta_3\gamma\delta} \neq 0$ . Factoring it out isolates the local  $2 \times 2$  rank-1 relation over the varying domains:

$$\lambda_{\alpha_1\beta_1\gamma\delta}\lambda_{\alpha_2\beta_2\gamma\delta} = \lambda_{\alpha_1\beta_2\gamma\delta}\lambda_{\alpha_2\beta_1\gamma\delta}.$$

## 4. Global Rank-1 Gluing (Necessity)

We now uncouple the variables and bridge domains to synthesize the local rank-1 conditions into a global factorization. To streamline notation, we re-index the tensor positional modes as  $(a, b, c, d) \in [n]^4$ . Let  $\Omega \subset [n]^4$  be the domain of non-identical tuples. By hypothesis,  $\lambda_{abcd} \neq 0$  for all  $(a, b, c, d) \in \Omega$ .

Fix an arbitrary index  $a_0 \in [n]$ . Consider the normalized ratio:

$$R(a \mid b, c, d) = \frac{\lambda_{abcd}}{\lambda_{a_0bcd}}.$$

This ratio is universally well-defined provided  $(a_0, b, c, d) \in \Omega$ , which holds unconditionally whenever  $b, c, d$  are not all simultaneously equal to  $a_0$ .

By symmetrically evaluating the local rank-1 relation across different pairs of varying modes, we deduce the following uncoupled constraints:

1. When  $c \neq d$ , the local rank-1 factorization (derived by varying the first and second modes) implies the submatrix  $M_{a,b}^{(cd)} = \lambda_{abcd}$  natively has rank 1. Thus,  $\lambda_{abcd} = \tilde{A}_{c,d}(a)\tilde{B}_{c,d}(b)$ . Therefore, the ratio  $R(a \mid b, c, d) = \tilde{A}_{c,d}(a)/\tilde{A}_{c,d}(a_0)$  unconditionally filters out the variable  $b$ , allowing us to safely denote it as a function  $P(a, c, d)$ .
2. When  $b \neq d$ , by symmetric application (varying the first and third modes), the ratio strictly uncouples independently of the variable  $c$ , yielding a function  $Q(a, b, d)$ .
3. When  $b \neq c$ , by symmetry (varying the first and fourth modes), the ratio strictly uncouples independently of the variable  $d$ , yielding a function  $S(a, b, c)$ .

For any tuple satisfying both  $c \neq d$  and  $b \neq d$ , we have  $P(a, c, d) = R(a \mid b, c, d) = Q(a, b, d)$ . Because  $P$  is independent of  $b$  and  $Q$  is independent of  $c$ , their common structural value must be functionally independent of both  $b$  and  $c$ , strictly reducing to a function  $U(a, d)$ . For any fixed pair  $c \neq d$ , since  $n \geq 5$ , we can fundamentally select an index  $b \notin \{c, d, a_0\}$ . For this choice of  $b$ , the conditions  $(a_0, b, c, d) \in \Omega$ ,  $c \neq d$ , and  $b \neq d$  hold simultaneously. This mathematically forces  $P(a, c, d) = U(a, d)$  identically for all  $c \neq d$ .

Similarly, on the intersection spanning  $c \neq d$  and  $b \neq c$ , overlapping yields  $U(a, d) = S(a, b, c)$ . By an analogous valid index selection for  $c$ , this structurally

isolates the variables completely independent of  $d$ . Therefore,  $U$  inherently restricts to a univariate global function dependent solely on  $a$ :  $U(a, d) \equiv u_1(a)$ . Consequently,  $P(a, c, d) = u_1(a)$  for all  $c \neq d$ , meaning  $R(a \mid b, c, d) = u_1(a)$  universally over the subspace where  $c \neq d$ .

This establishes domain relations seamlessly across three operative partitions of  $\Omega$ :

- $D_1 = \{(a, b, c, d) \in \Omega \mid c \neq d\}$ :  $R = u_1(a) \implies \lambda_{abcd} = u_1(a)\lambda_{a_0bcd} = u_1(a)F_1(b, c, d)$ . Since  $M_{a,b}^{(cd)}$  has rank 1,  $F_1$  must factor out the dependence on  $b$ . By defining an analogous ratio for the second mode and applying identical symmetric reasoning, we globally extract a univariate function  $u_2(b)$ . Thus, the relation uncouples strictly into  $\lambda_{abcd} = u_1(a)u_2(b)G_1(c, d)$ .
- $D_2 = \{(a, b, c, d) \in \Omega \mid a \neq b\}$ : By complete corresponding symmetry, this structurally uncouples into  $\lambda_{abcd} = u_3(c)u_4(d)G_2(a, b)$ , extracting global univariate functions  $u_3(c)$  and  $u_4(d)$ .
- $D_3 = \{(a, b, c, d) \in \Omega \mid b \neq d\}$ : Symmetrically, the variables uniquely uncouple to  $\lambda_{abcd} = u'_1(a)u'_3(c)G_3(b, d)$ . Overlapping exactly with  $D_1$  and  $D_2$  logically guarantees  $u'_1$  and  $u'_3$  correspond directly (up to constant scaling) to  $u_1$  and  $u_3$ , yielding  $\lambda_{abcd} = u_1(a)u_3(c)G_3(b, d)$ .

On the intersection bridging  $D_1 \cap D_2 = \{c \neq d \text{ and } a \neq b\}$ , evaluation strictly crosses conditions:

$$u_1(a)u_2(b)G_1(c, d) = u_3(c)u_4(d)G_2(a, b).$$

Because  $\lambda$  categorically avoids 0 on  $\Omega$ , the functional components are identically non-zero. This cross-condition fundamentally separates the localized functions, forcing the fraction  $\frac{G_1(c, d)}{u_3(c)u_4(d)} = \frac{G_2(a, b)}{u_1(a)u_2(b)} = K$  for some strictly non-zero constant  $K$ . Thus  $G_1(c, d) = Ku_3(c)u_4(d)$ . For any given pair  $(c, d) \in D_1$ , we can select  $a, b$  such that  $a \neq b$ , implying  $(a, b, c, d) \in D_1 \cap D_2$ . Thus  $G_1(c, d) = Ku_3(c)u_4(d)$  holds identically everywhere on  $D_1$ . Absorbing  $K$  structurally into  $u_4(d)$  seamlessly unifies the mapping precisely on  $D_1 \cup D_2$ :

$$\lambda_{abcd} = u_1(a)u_2(b)u_3(c)u_4(d).$$

Validating continuity onto the third uncoupled plane naturally extends into  $D_3$ . On  $(D_1 \cup D_2) \cap D_3$ :

$$u_1(a)u_3(c)G_3(b, d) = u_1(a)u_2(b)u_3(c)u_4(d) \implies G_3(b, d) = u_2(b)u_4(d).$$

Because for any  $(b, d) \in D_3$ , we can reliably select an index  $c$  such that  $c \neq d$ , every element mapping in  $D_3$  securely lies in  $D_1 \cap D_3$ . This definitively anchors the exact global rank-1 factorization over the united domain  $D_1 \cup D_2 \cup D_3$ .

**Completeness of Domain Coverage:** If any valid tuple  $(a, b, c, d) \in \Omega$  were to fall outside the spanned domain  $D_1 \cup D_2 \cup D_3$ , its exact mathematical omission logically structures the rigorous constraints:  $c = d$  (violating  $D_1$ ),  $a = b$  (violating  $D_2$ ), and  $b = d$  (violating  $D_3$ ). These three constraints collectively restrict the index boundaries exclusively to  $a = b = c = d$ , fully characterizing identical uniformity.

However, uniformly identical tuples are unconditionally excluded from  $\Omega$  strictly by hypothesis.

Thus, the global union  $D_1 \cup D_2 \cup D_3$  perfectly equals  $\Omega$ , entirely covering all valid non-identical subsets and firmly securing the exact uncoupled relation  $\lambda_{\alpha\beta\gamma\delta} = u_1(\alpha)u_2(\beta)u_3(\gamma)u_4(\delta)$ . Concurrently, since structural evaluation over  $\Omega$  categorically rejects evaluation to zero, it unequivocally enforces the required continuous functional constraints  $u_1, u_2, u_3, u_4 \in (\mathbb{R}^*)^n$  (formally corresponding to the vectors  $u, v, w, x$ ). This comprehensively establishes necessity and completes the full proof.