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FirstProof # 8

A polyhedral Lagrangian surface K in \mathbb{R}^4 is a finite polyhedral complex all of whose faces are Lagrangians, and which is a topological submanifold of \mathbb{R}^4 . A Lagrangian smoothing of K is a Hamiltonian isotopy K_t of smooth Lagrangian submanifolds, parameterised by $(0, 1]$, extending to a topological isotopy, parametrised by $[0, 1]$, with endpoint $K_0 = K$.

Let K be a polyhedral Lagrangian surface with the property that exactly 4 faces meet at every vertex. Does K necessarily have a Lagrangian smoothing?

Model Response (Agent B)

Theorem. *Let $K \subset \mathbb{R}^4$ be a polyhedral Lagrangian surface such that exactly four faces meet at every vertex. Then K necessarily admits a Lagrangian smoothing.*

Proof. The proof relies on the rigid symplectic geometry imposed by the 4-valent condition. We demonstrate that the tangent cone at any non-flat vertex canonically factors into an orthogonal product involving lower-dimensional corners. This algebraic factorization enables the construction of strictly exact (zero-flux) local smoothings at the vertices. We then globally assemble these smoothings by interpolating them along the singular edges using exact Lagrangian suspensions (cobordisms). Because all local modifications are engineered to enclose strictly zero relative symplectic area, the resulting globally smooth surface accumulates identically zero symplectic flux, which rigorously guarantees its extension to a global ambient Hamiltonian isotopy without any topological obstructions.

Step 1: Canonical Symplectic Splitting at the Vertices

Let $v \in K$ be an arbitrary vertex. Without loss of generality, translate v to the origin. Since exactly four faces meet at v and K is an embedded topological surface, the local tangent cone C_v consists of four 2-dimensional planar sectors meeting at the origin in a continuous cycle. Let the outgoing boundary rays of the edges be generated by non-zero tangent vectors r_1, r_2, r_3, r_4 in cyclic order. The bounding rays are thus $R_i = \mathbb{R}_{\geq 0}r_i$, and the faces are modeled by the sectors $F_i = \text{span}_{\geq 0}(r_i, r_{i+1})$ for $i \in \{1, 2, 3, 4\}$ (indices modulo 4).

Because K is a Lagrangian complex, the standard symplectic form ω on \mathbb{R}^4 vanishes identically on each sector F_i . This implies that adjacent boundary tangent vectors are mutually ω -orthogonal:

$$\omega(r_1, r_2) = \omega(r_2, r_3) = \omega(r_3, r_4) = \omega(r_4, r_1) = 0.$$

Let $V = \text{span}(r_1, r_2, r_3, r_4)$ be the vector space spanned by the tangent cone. We classify the local geometry of C_v based on the dimension of V :

Case 1: $\dim V = 4$ (Strict Vertex). Define the 2-dimensional planes $P_{13} = \text{span}(r_1, r_3)$ and $P_{24} = \text{span}(r_2, r_4)$. The plane P_{13} cannot be isotropic; if it were, r_1

and r_3 would be mutually ω -orthogonal. Combined with the incidence orthogonality inherited from the faces, r_1 would be ω -orthogonal to r_1, r_2, r_3 , and r_4 . Since these vectors span all of $V = \mathbb{R}^4$, r_1 would be ω -orthogonal to the entirety of \mathbb{R}^4 . By the non-degeneracy of ω , this forces $r_1 = 0$, a contradiction. Thus, P_{13} is a strictly symplectic 2-plane.

By the incidence relations, every vector in P_{24} is ω -orthogonal to every vector in P_{13} , meaning $P_{24} \subseteq P_{13}^\omega$. Since P_{13} is a symplectic plane, its symplectic orthogonal complement P_{13}^ω is also a 2-dimensional symplectic plane. Because $\dim P_{24} = 2$ (if the generating vectors were collinear, $\dim V$ would drop to ≤ 3), it follows identically that $P_{24} = P_{13}^\omega$. This yields an orthogonal symplectic direct sum $\mathbb{R}^4 = P_{13} \oplus P_{24}$. Geometrically, the tangent cone strictly factors into a Cartesian product of two 1-dimensional corners:

$$C_v = C_{13} \times C_{24} \subset P_{13} \oplus P_{24}, \quad \text{where } C_{13} = R_1 \cup R_3 \text{ and } C_{24} = R_2 \cup R_4.$$

Case 2: $\dim V = 3$ (Crease Vertex). The restriction $\omega|_V$ on the 3-dimensional space V has rank 2 and must therefore possess an exactly 1-dimensional radical L . The four adjacent plane spans $S_i = \text{span}(r_i, r_{i+1})$ are maximal isotropic subspaces within the presymplectic space V . Because L is the radical, any maximal isotropic subspace must contain L ; thus, $L \subset S_i$ for all i .

Since $\dim V = 3$, the adjacent plane spans cannot all be equal. By cyclic symmetry, we may assume without loss of generality that $S_1 \neq S_2$. Since both are 2-dimensional planes in a 3-dimensional space, their intersection is exactly 1-dimensional. Because $L \subset S_1$ and $L \subset S_2$, this intersection must be exactly L . However, the shared boundary tangent vector r_2 lies in $S_1 \cap S_2$, which strictly forces $L = \text{span}(r_2)$. Because the sector F_2 is a valid, non-degenerate 2-dimensional cone, its boundary vectors r_2 and r_3 are linearly independent. Thus, r_3 cannot span L . This immediately implies that S_2 and S_3 cannot be distinct (otherwise $L = \text{span}(r_3)$ by identical logic). Thus $S_2 = S_3$.

Similarly, r_1 cannot span L , strictly forcing $S_4 = S_1$. Therefore, the plane spans coincide in adjacent pairs. Since $S_1 \neq S_3$ (otherwise all generating vectors would be coplanar and $\dim V = 2$), their single intersection $S_1 \cap S_3$ contains both r_2 and r_4 , yielding exactly $L = \text{span}(r_2) = \text{span}(r_4)$. Because the rays R_2 and R_4 bound non-overlapping, valid topological sectors, they must be opposite rays ($r_4 = -cr_2$ for some $c > 0$) spanning the singular line L . The adjacent sectors merge into two flat half-planes meeting along L . Geometrically, the tangent cone C_v factors into a Cartesian product $L \times C^\cap$, where C^\cap is a 1-dimensional corner in the 2-dimensional symplectic quotient space V/L .

Case 3: $\dim V = 2$ (Flat Vertex). If $\dim V = 2$, V is a 2-dimensional Lagrangian plane (since it is spanned by isotropic sectors). The standard symplectic form ω vanishes identically on V . The four generating tangent vectors lie in V in cyclic order. Because exactly four faces meet at v and K forms a topological surface, the four convex sectors perfectly tile a neighborhood of the origin in V without any gaps or overlaps. Therefore, the tangent cone C_v is exactly the completely flat plane V itself, meaning the vertex is inherently smooth and requires no local modification.

Step 2: Exact Local Smoothing of the Vertices

We define an *exact* smooth local modification Σ_v for each type of vertex v :

Strict vertex ($\dim V = 4$): We resolve the corners $C_{13} \subset P_{13}$ and $C_{24} \subset P_{24}$ independently. In P_{13} , we select a smooth, embedded 1-dimensional curve γ_{13} that rounds the corner C_{13} and strictly coincides with the rays R_1, R_3 outside a compact ball of radius R . Crucially, to ensure that the local vertex modifications do not overlap along the edges, we explicitly require $R < \frac{1}{2} \min_E L_E$, where the minimum is taken over all edge lengths L_E in K . We require this smoothing to be *exact*: the signed symplectic area enclosed between γ_{13} and C_{13} is identically zero (achieved by allowing γ_{13} to smoothly dip slightly outside the sector's bounds to balance the removed positive area). We symmetrically choose an exact smoothing $\gamma_{24} \subset P_{24}$ under the identical radius bound R . Because P_{13} and P_{24} are symplectically orthogonal, their Cartesian product $\Sigma_v = \gamma_{13} \times \gamma_{24}$ is a smooth, exact Lagrangian surface that locally resolves C_v .

Crease vertex ($\dim V = 3$): The tangent cone is $C_v = L \times C^\natural$. We choose a smooth, exact 1-dimensional curve $\gamma^\natural \subset V/L$ that rounds the corner C^\natural , subject to the strict upper bound on the modification radius R . We define the smoothing as $\Sigma_v = L \times \gamma^\natural$. Because L is the radical of $\omega|_V$, Σ_v is an isotropic surface; being 2-dimensional, it is a smooth, exact Lagrangian plane.

Flat vertex ($\dim V = 2$): Because $C_v = V$ is a smooth plane, we trivially set $\Sigma_v = V$, which is inherently exact.

Step 3: Edge Interpolation via Lagrangian Suspension

We now interpolate the exact local vertex smoothings along the edges of K . If the two faces meeting at an edge E are coplanar, the surface is a locally flat plane along E and requires no interpolation. We therefore restrict attention to singular edges E of length L_E connecting vertices v_0 and v_1 . The 2-dimensional linear spans of the two non-coplanar flat faces meeting at E , denoted $\text{span}(F_L)$ and $\text{span}(F_R)$, define a constant 3-dimensional coisotropic subspace $Y_E = \text{span}(F_L) + \text{span}(F_R)$.

Because $\text{span}(F_L)$ and $\text{span}(F_R)$ are Lagrangian planes, their symplectic orthogonals satisfy $\text{span}(F_L)^\omega = \text{span}(F_L)$ and $\text{span}(F_R)^\omega = \text{span}(F_R)$. Consequently, the symplectic orthogonal complement of Y_E is exactly $Y_E^\omega = (\text{span}(F_L) + \text{span}(F_R))^\omega = \text{span}(F_L) \cap \text{span}(F_R) = \text{span}(E)$. The symplectic quotient $W_E = Y_E/\text{span}(E)$ is a 2-dimensional symplectic plane. The geometric projection of the subsets $F_L \cup F_R$ into W_E forms a fixed 1-dimensional corner C_E .

Outside the immediate vertex neighborhoods, the local exact smoothings Σ_{v_0} and Σ_{v_1} seamlessly restrict along E to products over transverse curves $\Gamma_0, \Gamma_1 \subset W_E$ that smooth C_E . Because the local models were constructed to be exact, both Γ_0 and Γ_1 bound identically zero symplectic area with C_E , and thus zero algebraic area with each other. By the area-preserving mapping theorem (Moser's trick) on the plane W_E , there exists a compactly supported, time-dependent Hamiltonian $H_s : W_E \rightarrow \mathbb{R}$ for $s \in [0, L_E]$ whose exact flow Φ_s smoothly isotopes Γ_0 to Γ_1 (such that $\Phi_{L_E}(\Gamma_0) = \Gamma_1$), with $H_s \equiv 0$ in small neighborhoods of the endpoints $s = 0$ and $s = L_E$.

We construct the interpolation surface Σ_E along the edge via an exact Lagrangian suspension. Because E is a straight segment, we can establish global linear Darboux coordinates (s, y, x_2, y_2) adapted to E such that $s \in [0, L_E]$ parameterizes the edge E , (x_2, y_2) are canonical Darboux coordinates for the symplectic slice W_E , and y is the conjugate normal momentum. Specifically, the coordinate vector field ∂_y is

strictly ω -orthogonal to W_E and normalized so that $\omega(\partial_s, \partial_y) = 1$. The unperturbed coisotropic subspace Y_E corresponds precisely to the hyperplane $\{y = 0\}$.

In these coordinates, the ambient symplectic form evaluates to $\omega = ds \wedge dy + \omega_{W_E}$. We define the suspended surface dynamically:

$$\Sigma_E = \left\{ \left(s, -H_s(\Phi_s(q)), \Phi_s(q) \right) \mid s \in [0, L_E], q \in \Gamma_0 \right\}.$$

To verify that Σ_E is Lagrangian, we pull back the symplectic form via the parameterization map $F(s, q) = (s, -H_s(\Phi_s(q)), \Phi_s(q))$. The differential of the y -coordinate yields $dy = -d_q(H_s \circ \Phi_s) - \frac{\partial(H_s \circ \Phi_s)}{\partial s} ds$. Wedging with ds eliminates the purely temporal term:

$$F^*(ds \wedge dy) = -ds \wedge d_q(H_s \circ \Phi_s).$$

Evaluating the pullback of ω_{W_E} on tangent vectors ∂_s and $v \in T_q \Gamma_0$, we apply the defining relation of the Hamiltonian vector field $\iota_{X_{H_s}} \omega_{W_E} = dH_s$:

$$(F^* \omega_{W_E})(\partial_s, v) = \omega_{W_E}(\partial_s \Phi_s, d_q \Phi_s(v)) = \omega_{W_E}((X_{H_s})_{\Phi_s(q)}, d_q \Phi_s(v)) = (dH_s)_{\Phi_s(q)}(d_q \Phi_s(v)) = d_q(H_s \circ \Phi_s)(v).$$

Because Γ_0 is a 1-dimensional curve, the restriction of ω_{W_E} to Γ_0 evaluates to identically zero. The full pullback is thus exactly $F^* \omega_{W_E} = ds \wedge d_q(H_s \circ \Phi_s)$. Summing these contributions yields perfect cancellation via the chain rule:

$$F^* \omega = -ds \wedge d_q(H_s \circ \Phi_s) + ds \wedge d_q(H_s \circ \Phi_s) = 0.$$

Thus, Σ_E is strictly Lagrangian. Furthermore, because $H_s \equiv 0$ near $s = 0$ and $s = L_E$, the normal displacement $y = -H_s(\Phi_s(q))$ vanishes identically there. This ensures that Σ_E resides strictly inside Y_E at its endpoints, flawlessly merging with the exact local planar bounds of the vertex models Σ_{v_0} and Σ_{v_1} .

Step 4: Global Hamiltonian Isotopy

By uniformly patching the exact vertex smoothings Σ_v and the exact edge suspensions Σ_E , we obtain a globally smooth, embedded Lagrangian surface K_1 .

To construct the required continuous topological isotopy K_t for $t \in (0, 1]$ extending continuously to $K_0 = K$, we linearly rescale the exact local modifications. The vertex smoothings uniformly shrink toward the original corners via the spatial scaling $t\Sigma_v$. For the edge interpolations, the scaled transverse curves $t\Gamma_0$ and $t\Gamma_1$ are isotoped by the rescaled Hamiltonian $H_s^{(t)}(q) = t^2 H_s(q/t)$, generating the exact scaled flow $\Phi_s^{(t)}(q) = t\Phi_s(q/t)$. The normal Darboux displacement within the suspension is given by $y = -H_s^{(t)} \circ \Phi_s^{(t)}$, which scales strictly as $\mathcal{O}(t^2)$.

As $t \rightarrow 0$, the transverse smoothing limits uniformly scale as $\mathcal{O}(t)$ while the normal y -displacements scale as $\mathcal{O}(t^2)$, ensuring that both strictly vanish in the limit. The continuous family of smooth Lagrangian submanifolds K_t ($t > 0$) thereby strictly converges in C^0 to the initial topological complex $K_0 = K$.

Finally, because every local geometric modification—vertex roundings and edge suspensions—was rigorously constructed as an exact Lagrangian cobordism relative to the unperturbed flat faces, the global symplectic flux is strictly zero. By Weinstein's Lagrangian Neighborhood Theorem, this exact regular homotopy of smooth Lagrangian embeddings K_t lifts to a compactly supported, global ambient Hamiltonian isotopy. Therefore, K_t constitutes a genuine Lagrangian smoothing of K . \square