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FirstProof # 9

Let $n \geq 5$. Let $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3 \times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ so that its (i, j, k, ℓ) entry for $1 \leq i, j, k, \ell \leq 3$ is given by $Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$. Here $A(i, :)$ denotes the i th row of a matrix A , and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$.

More precisely, does there exist a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$ that satisfies the following three properties?

- The map \mathbf{F} does not depend on $A^{(1)}, \dots, A^{(n)}$.
- The degrees of the coordinate functions of \mathbf{F} do not depend on n .
- Let $\lambda \in \mathbb{R}^{n \times n \times n \times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

Model Response (Agent Af)

1 Construction of the Polynomial Map F

Let $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3 \times 4}$ be Zariski-generic matrices. We identify the formal input space \mathbb{R}^{81n^4} with the vector space of 4-way tensors $\mathbf{X} \in \mathbb{R}^{3n \times 3n \times 3n \times 3n}$. We index the components of this tensor via composite multi-indices $I = (\alpha, i)$, $J = (\beta, j)$, $K = (\gamma, k)$, and $L = (\delta, \ell)$, where $\alpha, \beta, \gamma, \delta \in [n]$ specify the choice of matrices and $i, j, k, \ell \in \{1, 2, 3\}$ specify their localized row indices. Evaluated over the parameters of the problem, the tensor coordinates map to:

$$\mathbf{X}_{IJKL} = \lambda_{\alpha\beta\gamma\delta} Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \lambda_{\alpha\beta\gamma\delta} \det \begin{bmatrix} A^{(\alpha)}(i, :) \\ A^{(\beta)}(j, :) \\ A^{(\gamma)}(k, :) \\ A^{(\delta)}(\ell, :) \end{bmatrix}.$$

Remark 1.1. We say a parameter tuple $(\alpha, \beta, \gamma, \delta)$ is *monochromatic* if $\alpha = \beta = \gamma = \delta$, and *non-monochromatic* otherwise. For any $\alpha \in [n]$, the array $Q^{(\alpha\alpha\alpha\alpha)}$ evaluates the determinant of a 4×4 matrix whose four rows are drawn entirely from the same 3×4 matrix $A^{(\alpha)}$. By the Pigeonhole Principle, at least two of these row choices must evaluate to the identical row of $A^{(\alpha)}$, making the determinant unconditionally vanish. Thus, $Q^{(\alpha\alpha\alpha\alpha)} = 0$ universally.

Consequently, $\mathbf{X}_{IJKL} = 0$ unconditionally for all monochromatic tuples. The problem dictates that $\lambda_{\alpha\beta\gamma\delta} \neq 0$ solely across non-monochromatic configurations; nevertheless,

extending the scalar parameters $\lambda_{\alpha\alpha\alpha\alpha}$ arbitrarily over the monochromatic bounds leaves the evaluated tensor \mathbf{X} completely unaltered.

We define four principal multilinear matrix flattenings of \mathbf{X} , each mapping naturally to a structured matrix of dimensions $3n \times 27n^3$:

- $M^{(1)}$: Rows indexed by I , columns by $C_1 = (J, K, L)$.
- $M^{(2)}$: Rows indexed by J , columns by $C_2 = (I, K, L)$.
- $M^{(3)}$: Rows indexed by K , columns by $C_3 = (I, J, L)$.
- $M^{(4)}$: Rows indexed by L , columns by $C_4 = (I, J, K)$.

Definition 1. We define the polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$, where $N = 4\binom{3n}{5}\binom{27n^3}{5}$, such that its coordinate functions evaluate all 5×5 minors across the four flattenings $M^{(1)}, M^{(2)}, M^{(3)}$, and $M^{(4)}$.

This multilinear representation immediately secures the problem's first two requisite properties:

- **Property 1:** The coordinate functions of \mathbf{F} are standard determinantal minor expansions evaluated strictly over the formal tensor variables \mathbf{X}_{IJKL} . Their coefficients consist exclusively of the constants ± 1 and 0. Thus, the polynomial map \mathbf{F} operates completely independently of the underlying generic matrices $A^{(1)}, \dots, A^{(n)}$.
- **Property 2:** Each coordinate function extracts a 5×5 minor, rigorously defining it as a homogeneous polynomial of exact degree 5 over the tensor inputs. This uniform degree is invariant and strictly independent of n .

2 Proof of Property 3: Sufficiency

Assume there exist scalar vectors $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ holds across all valid non-monochromatic configurations. By Remark 1.1, since $Q^{(\alpha\alpha\alpha\alpha)} = 0$, applying the identically factored substitution $\lambda_{\alpha\alpha\alpha\alpha} = u_\alpha v_\alpha w_\alpha x_\alpha$ over the excluded monochromatic bounds leaves \mathbf{X} perfectly unaltered. Absorbing these parameters via the multilinearity of the determinant globally yields:

$$\mathbf{X}_{IJKL} = \det \begin{bmatrix} u_\alpha A^{(\alpha)}(i,:) \\ v_\beta A^{(\beta)}(j,:) \\ w_\gamma A^{(\gamma)}(k,:) \\ x_\delta A^{(\delta)}(\ell,:) \end{bmatrix}.$$

For the first flattening $M^{(1)}$, let the localized row vector $U_I = u_\alpha A^{(\alpha)}(i,:) \in \mathbb{R}^4$. Expanding the determinant via Laplace expansion along this leading row extracts:

$$M_{I,C_1}^{(1)} = \sum_{m=1}^4 (U_I)_m \cdot \text{cofactor}_{1,m} \begin{bmatrix} U_I \\ v_\beta A^{(\beta)}(j,:) \\ w_\gamma A^{(\gamma)}(k,:) \\ x_\delta A^{(\delta)}(\ell,:) \end{bmatrix}.$$

The four scalar cofactor terms intrinsically evaluate using exclusively the column configuration C_1 and remain completely decoupled from the localized row index I . Hence, $M^{(1)}$ structurally factors into the matrix product of a $3n \times 4$ matrix and a $4 \times 27n^3$ matrix. This mathematically guarantees $\text{rank}(M^{(1)}) \leq 4$, geometrically forcing all of its 5×5 minors to evaluate to zero. Symmetric parity across the exterior maps subsequently ensures $\text{rank}(M^{(m)}) \leq 4$ for all flattenings $m \in \{1, 2, 3, 4\}$, unconditionally verifying $\mathbf{F}(\mathbf{X}) = \mathbf{0}$.

3 Proof of Property 3: Necessity

Assume $\mathbf{F}(\mathbf{X}) = \mathbf{0}$. The universal vanishing of all 5×5 minors strictly bounds the rank identically as $\text{rank}(M^{(m)}) \leq 4$ across all four principal flattenings.

3.1 Subspace Intersections and the Evaluation Map

Let $S \subset \mathbb{R}^{27n^3}$ be the row space of $M^{(1)}$, which inherently satisfies $\dim S \leq 4$. Let $U_\alpha = \text{rowspan}(A^{(\alpha)}) \subset \mathbb{R}^4$ denote the generic 3-dimensional row space of matrix $A^{(\alpha)}$. We define a linear evaluation map $T_\alpha : U_\alpha \rightarrow S$ that maps a generic spatial vector $y = \sum_{i=1}^3 c_i A^{(\alpha)}(i,:) \in U_\alpha$ into the equivalent linear combination of the corresponding rows within S . Evaluated locally on a subset of columns forming a fixed block $B = (\beta, \gamma, \delta) \in [n]^3$, this equivalently leverages multilinearity to output:

$$T_\alpha(y)_B = \lambda_{\alpha B} \Psi_B(y), \quad \text{where} \quad \Psi_B(y)_{jkl} = \det \begin{bmatrix} y \\ A^{(\beta)}(j,:) \\ A^{(\gamma)}(k,:) \\ A^{(\delta)}(\ell,:) \end{bmatrix},$$

and $\lambda_{\alpha B}$ abbreviates $\lambda_{\alpha\beta\gamma\delta}$. Evaluating $\Psi_B(y) = 0$ is algebraically equivalent to stating that $y \wedge w_1 \wedge w_2 \wedge w_3 = 0$ within the exterior algebra $\Lambda^4 \mathbb{R}^4$ for all valid combinations $w_1 \in U_\beta, w_2 \in U_\gamma, w_3 \in U_\delta$.

Lemma 1. *Let $V = \mathbb{R}^4$, and let $A^{(1)}, \dots, A^{(n)}$ be generic 3×4 matrices with row spaces $U_i = \text{rowspan}(A^{(i)})$.*

- (i) *If $B = (\beta, \gamma, \delta)$ is non-monochromatic, then $\ker \Psi_B = \{0\}$.*
- (ii) *If $B = (\beta, \beta, \beta)$ is monochromatic, then $\ker \Psi_B = U_\beta$.*

Proof. The constraint $\Psi_B(y) = 0$ requires $y \wedge w_1 \wedge w_2 \wedge w_3 = 0$ for all $w_1 \in U_\beta, w_2 \in U_\gamma, w_3 \in U_\delta$.

(i) Assume B is non-monochromatic. Since the wedge product is commutative up to sign, we may assume without loss of generality that $\beta \neq \delta$. We consider the structural span of the 2-forms $w_1 \wedge w_2$. If $\beta \neq \gamma$, U_β and U_γ are distinct generic 3-dimensional subspaces intersecting in a 2-dimensional subspace within V . Constructing a basis adapted to this intersection yields 6 linearly independent 2-forms, proving the span of $w_1 \wedge w_2$ covers the entirety of $\Lambda^2 V$. If $\beta = \gamma$, the span of $w_1 \wedge w_2$ for $w_1, w_2 \in U_\beta$ evaluates exactly to $\Lambda^2 U_\beta$, a 3-dimensional subspace natively housed within $\Lambda^2 V$.

In both cases, the span contains $\Lambda^2 U_\beta$. Consequently, the overarching span of $w_1 \wedge w_2 \wedge w_3$ contains $\Lambda^2 U_\beta \wedge U_\delta$. Since $\beta \neq \delta$, the generic 3-dimensional subspaces U_β and U_δ reliably intersect in a 2-dimensional subspace. By decomposing this space as $U_\delta = (U_\beta \cap U_\delta) \oplus \text{span}(v)$ for a specific $v \in U_\delta \setminus U_\beta$, we deduce:

$$\Lambda^2 U_\beta \wedge U_\delta = (\Lambda^2 U_\beta \wedge (U_\beta \cap U_\delta)) \oplus (\Lambda^2 U_\beta \wedge v) = \Lambda^3 U_\beta \oplus (\Lambda^2 U_\beta \wedge v).$$

It is immediate that $\Lambda^3 U_\beta$ is exactly 1-dimensional. Furthermore, since $v \notin U_\beta$, wedging with v injectively maps $\Lambda^2 U_\beta$ into $\Lambda^3 V$, meaning $\Lambda^2 U_\beta \wedge v$ is strictly 3-dimensional. To verify the trivial intersection parity, suppose an element $0 \neq \eta \in \Lambda^3 U_\beta$ satisfies $\eta = \omega \wedge v$ for some $\omega \in \Lambda^2 U_\beta$. Given any $x \in U_\beta$, evaluating $\eta \wedge x = 0$ strictly forces $\omega \wedge x \wedge v = 0$. Since $V = U_\beta \oplus \text{span}(v)$, we must assert $\omega \wedge x = 0$ in $\Lambda^3 U_\beta$ uniformly over all $x \in U_\beta$. The non-degenerate pairing dictates this is only possible if $\omega = 0$, yielding $\eta = 0$, forming a contradiction.

Therefore, the algebraic sum directly establishes itself over $1 + 3 = 4$ dimensions. Because $\dim \Lambda^3 V = 4$, the established span encompasses exactly $\Lambda^3 V$. Enforcing $y \wedge \Omega = 0$ for all valid $\Omega \in \Lambda^3 V$ unconditionally forces $y = 0$.

(ii) If $B = (\beta, \beta, \beta)$, the span corresponding to $w_1 \wedge w_2 \wedge w_3$ converges exclusively to $\Lambda^3 U_\beta$, representing the 1-dimensional volume form bounding U_β . Resolving $y \wedge \Lambda^3 U_\beta = 0$ structurally enforces $y \in U_\beta$. \square

Given $n \geq 5$, for any isolated generic index $\alpha \in [n]$, we explicitly choose a non-monochromatic block $B = (\sigma, \sigma, \tau)$ mapping elements strictly disjoint from α (requiring exactly $3 \leq n$ distinct indices). Because the evaluated tuple $(\alpha, \sigma, \sigma, \tau)$ is strictly non-monochromatic, the premise guarantees $\lambda_{\alpha B} \neq 0$. Bounded against Lemma 1(i), evaluating $T_\alpha(y)_B = 0 \implies y = 0$, validating that T_α is universally injective. Its equivalently mapped image $W_\alpha = T_\alpha(U_\alpha) \subset S$ firmly maintains dimension 3. Anchored dynamically against $\dim S \leq 4$, Grassmann's formula for the dimension of subspace intersections necessitates:

$$\dim(W_\alpha \cap W_\mu) = \dim W_\alpha + \dim W_\mu - \dim(W_\alpha + W_\mu) \geq 3 + 3 - 4 = 2 \quad \text{for any } \alpha \neq \mu.$$

3.2 Universal Local Factoring

Let $E_{\alpha, \mu} = T_\alpha^{-1}(W_\alpha \cap W_\mu) \subset U_\alpha$. Grounded strictly by injectivity, $\dim E_{\alpha, \mu} \geq 2$. For any vector $x \in E_{\alpha, \mu}$, there universally exists a unique vector $y \in U_\mu$ firmly satisfying $T_\alpha(x) = T_\mu(y)$. Pivoting on $n \geq 5$, we securely configure a non-monochromatic block $B_0 = (\sigma, \sigma, \tau)$ mutually disjoint from both bounds α and μ (leveraging exactly $2 + 2 = 4 \leq n$ indices). Extracting locally outputs $T_\alpha(x)_{B_0} = T_\mu(y)_{B_0}$, mapping identically onto $\lambda_{\alpha B_0} \Psi_{B_0}(x) = \lambda_{\mu B_0} \Psi_{B_0}(y)$.

Applying the multilinearity of Ψ_{B_0} enforces $\Psi_{B_0}(\lambda_{\alpha B_0} x - \lambda_{\mu B_0} y) = 0$. Validating against $\ker \Psi_{B_0} = \{0\}$ and knowing the corresponding scalars unconditionally correspond to non-monochromatic configurations (thus are non-zero), we extract $\lambda_{\alpha B_0} x = \lambda_{\mu B_0} y$. Structuring $c_{\alpha, \mu} = \lambda_{\alpha B_0} / \lambda_{\mu B_0} \neq 0$, we unconditionally isolate $y = c_{\alpha, \mu} x$. Since $y \in U_\mu$ and $c_{\alpha, \mu} \neq 0$, it implies $x \in U_\mu$. Thus, $E_{\alpha, \mu} \subseteq U_\alpha \cap U_\mu$. Bounding the intersection of two generic 3-dimensional spaces in \mathbb{R}^4 caps the dimension at exactly 2, ensuring $E_{\alpha, \mu} = U_\alpha \cap U_\mu$.

Using the explicit relation $T_\alpha(x) = T_\mu(c_{\alpha,\mu}x)$, we logically evaluate the mappings globally over a generalized tracking block B :

$$T_\alpha(x)_B = T_\mu(c_{\alpha,\mu}x)_B \implies (\lambda_{\alpha B} - c_{\alpha,\mu}\lambda_{\mu B})\Psi_B(x) = 0 \quad \text{for all } x \in U_\alpha \cap U_\mu.$$

Because $\dim(U_\alpha \cap U_\mu) = 2$, we mathematically isolate the coefficients $\lambda_{\alpha B} = c_{\alpha,\mu}\lambda_{\mu B}$ by filtering against $\ker \Psi_B$:

- If B is non-monochromatic, $\ker \Psi_B = \{0\}$. Consequently, for any valid non-zero $x \in U_\alpha \cap U_\mu$, resolving $\Psi_B(x) \neq 0$ securely enforces $\lambda_{\alpha B} = c_{\alpha,\mu}\lambda_{\mu B}$.
- If $B = (\beta, \beta, \beta)$ with $\beta \notin \{\alpha, \mu\}$, Lemma 1 forces $\ker \Psi_B = U_\beta$. The generic intersection $(U_\alpha \cap U_\mu) \cap U_\beta$ yields exactly dimension $2 + 3 - 4 = 1$. Since $\dim(U_\alpha \cap U_\mu) = 2$, there exists an element $x \in (U_\alpha \cap U_\mu) \setminus U_\beta$, universally validating $\Psi_B(x) \neq 0$. This rigorously forces $\lambda_{\alpha B} = c_{\alpha,\mu}\lambda_{\mu B}$.

Therefore, the mapped equivalence holds cleanly for all valid evaluations $B \notin \{(\alpha, \alpha, \alpha), (\mu, \mu, \mu)\}$.

To decipher the transitive cocycle condition $c_{\alpha,\nu} = c_{\alpha,\mu}c_{\mu,\nu}$ for three distinct variable indices $\alpha, \mu, \nu \in [n]$, we purposefully select a 2-element non-monochromatic block $B_2 = (\rho, \rho, \kappa)$ mutually disjoint from α, μ , and ν . This geometric verification guarantees applicability because $3+2=5 \leq n$. Resolving outside monochromatic boundaries yields $\lambda_{\alpha B_2} = c_{\alpha,\mu}\lambda_{\mu B_2}$, $\lambda_{\mu B_2} = c_{\mu,\nu}\lambda_{\nu B_2}$, and $\lambda_{\nu B_2} = c_{\nu,\alpha}\lambda_{\alpha B_2}$. Directly dividing these inherently non-zero quantities verifies the cocycle property $c_{\alpha,\nu} = c_{\alpha,\mu}c_{\mu,\nu}$.

We define $u_1 = 1$ and $u_\alpha = c_{\alpha,1}$ for $\alpha \geq 2$, meaning $c_{\alpha,\mu} = u_\alpha/u_\mu$. We securely decouple $Y_B = \lambda_{1B}$ evaluating $B \neq (1, 1, 1)$, alongside bounds $Y_{111} = \lambda_{2,1,1,1}/u_2$. This strictly limits coordinates globally as $\lambda_{\alpha B} = u_\alpha Y_B$ over all non-monochromatic tuples (α, B) :

- Bounding $B \notin \{(1, 1, 1), (\alpha, \alpha, \alpha)\}$, we obtain $\lambda_{\alpha B} = c_{\alpha,1}\lambda_{1B} = u_\alpha Y_B$.
- Bounding over $B = (1, 1, 1)$, the evaluated tuple $(\alpha, 1, 1, 1)$ strictly mandates non-monochromatic parity, inherently forcing $\alpha \neq 1$. Fixing $\mu = 2$ (valid using $n \geq 5$), resolving $\alpha \neq 2$ outputs $\lambda_{\alpha,1,1,1} = c_{\alpha,2}\lambda_{2,1,1,1} = (u_\alpha/u_2)\lambda_{2,1,1,1} = u_\alpha Y_{111}$. For $\alpha = 2$, identity holds trivially.

Mirroring sequential deductions identically over equivalent matrix flattenings $M^{(2)}$, $M^{(3)}$, and $M^{(4)}$ guarantees the existence of complementary vectors $v, w, x \in (\mathbb{R}^*)^n$ mapped over spatial tracking tensors Z, P, Q , uniformly restricting parameters universally across valid subsets:

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha Y_{\beta\gamma\delta} = v_\beta Z_{\alpha\gamma\delta} = w_\gamma P_{\alpha\beta\delta} = x_\delta Q_{\alpha\beta\gamma}.$$

3.3 Global Connectedness of the Valid Configuration Graph

Let $\mathcal{T} \subset [n]^4$ denote the discrete subset of exclusively non-monochromatic valid parameter multi-tuples. We formulate the universally normalized relational map $H : \mathcal{T} \rightarrow \mathbb{R}$ explicitly by:

$$H(T) = \frac{\lambda_{\alpha\beta\gamma\delta}}{u_\alpha v_\beta w_\gamma x_\delta},$$

evaluated strictly over $T = (\alpha, \beta, \gamma, \delta) \in \mathcal{T}$. Leveraging our preceding factorizations cleanly parses $H(T) = \frac{Y_{\beta\gamma\delta}}{v_\beta w_\gamma x_\delta}$, which is manifestly independent of the leading coordinate α . Consequently, $H(T)$ functionally persists invariantly under dynamic shifting of the first localized coordinate element natively assuming the newly formed tuple remains bounded strictly within \mathcal{T} . Extrapolating symmetric multilinear independence logically dictates that $H(T)$ is invariant across alterations to any single isolated coordinate, provided the intermediate tuples strictly evaluate inside \mathcal{T} .

We conceptualize \mathcal{T} topologically as a configuration graph network connecting multi-tuples differing exactly by a single localized coordinate. The map $H(T)$ evaluates trivially to a constant value across any connected component of this graph. We now strictly establish that \mathcal{T} is entirely globally connected. Let $T \in \mathcal{T}$. Because T is non-monochromatic, it contains at most 3 identical coordinate values.

1. If T contains exactly 3 identical coordinates (e.g., matching (a, a, a, b) with $a \neq b$), we can shift one of the identical coordinates to a uniquely evaluated constant $c \notin \{a, b\}$. Since $n \geq 5 \geq 3$, such a generic c is universally valid. The resulting adjacent tuple (e.g., (c, a, a, b)) inherently remains within \mathcal{T} and correctly houses at most 2 identical coordinates. Thus, every bounded tuple with 3 identical constraints trivially bounds adjacent to a tuple harboring at most 2 identical coordinates.
2. Over localized constraints containing at most 2 identical elements, we sequentially alter coordinates one-by-one, scanning left-to-right, to strictly match a non-intersecting distinctly evaluated target sequence, specifically $(1, 2, 3, 4)$. Because $n \geq 5$, such a discrete target sequence is naturally guaranteed to exist. At any traversal step $k \in \{1, 2, 3, 4\}$, the intermediate tuple identically comprises a successfully mapped prefix of length k containing strictly unique target elements, and an unchanged trailing suffix of length $4 - k$ natively drawn from the original values of T . Because the prefix sequence is constrained to strictly distinct entries, any given element can appear at most *once* across the prefix. Since the foundational original tuple T maintained at most 2 identical evaluations, any mapped element can occur at most *twice* within the unchanged tracking suffix. Consequently, the maximum element occurrence frequency across the entire intermediate tuple rigorously bounds to 1 (from the distinct prefix) + 2 (from the unchanged suffix) = 3. This combinatorially ensures that no intermediate topological step can theoretically collapse to a frequency of 4 (identically monochromatic parity), guaranteeing that every sequence transition evaluates safely and dynamically within the strict confines of the valid graph \mathcal{T} .

Since every multi-tuple topologically connects directly to the valid boundary path $(1, 2, 3, 4)$, the entire configuration graph \mathcal{T} is globally connected. Therefore, the function $H(T)$ equates universally to a non-zero symmetric mapping scalar K exclusively over all strictly valid configurations $T \in \mathcal{T}$. Scaling out the identity via $u_\alpha \leftarrow Ku_\alpha$ rigorously establishes that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ identically bridges all generic, non-monochromatic parameterizations. This fully satisfies Property 3 and conclusively completes the proof.