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FirstProof # 2

Let F be a non-archimedean local field with ring of integers \mathfrak{o} . Let N_r denote the subgroup of $\mathrm{GL}_r(F)$ consisting of upper-triangular unipotent elements. Let $\psi : F \rightarrow \mathbb{C}^\times$ be a nontrivial additive character of conductor \mathfrak{o} , identified in the standard way with a generic character of N_r . Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$, realized in its ψ^{-1} -Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$. Must there exist $W \in \mathcal{W}(\Pi, \psi^{-1})$ with the following property?

Let π be a generic irreducible admissible representation of $\mathrm{GL}_n(F)$, realized in its ψ -Whittaker model $\mathcal{W}(\pi, \psi)$. Let \mathfrak{q} denote the conductor ideal of π , let $Q \in F^\times$ be a generator of \mathfrak{q}^{-1} , and set

$$u_Q := I_{n+1} + Q E_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$

where $E_{i,j}$ is the matrix with a 1 in the (i,j) -entry and 0 elsewhere. For some $V \in \mathcal{W}(\pi, \psi)$, the local Rankin–Selberg integral

$$\int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

is finite and nonzero for all $s \in \mathbb{C}$.

Model Response (Agent A)

Introduction

Let F be a non-archimedean local field with ring of integers \mathfrak{o} and maximal ideal \mathfrak{p} . We fix a uniformizer $\varpi \in \mathfrak{p}$ and let $q = |\mathfrak{o}/\mathfrak{p}|$. Let N_r denote the subgroup of $\mathrm{GL}_r(F)$ consisting of upper-triangular unipotent elements, and let $K_r = \mathrm{GL}_r(\mathfrak{o})$ denote the standard maximal compact subgroup. Let $\psi : F \rightarrow \mathbb{C}^\times$ be a fixed nontrivial additive character of conductor \mathfrak{o} .

Theorem 1. *Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$. There exists $W \in \mathcal{W}(\Pi, \psi^{-1})$ such that for any generic irreducible admissible representation π of $\mathrm{GL}_n(F)$ with conductor ideal \mathfrak{q} , and any generator $Q \in F^\times$ of \mathfrak{q}^{-1} , there exists $V \in \mathcal{W}(\pi, \psi)$ such that the local Rankin–Selberg integral*

$$Z(s, W, V) = \int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)u_Q) V(g) |\det g|^{s-\frac{1}{2}} dg$$

is finite and nonzero for all $s \in \mathbb{C}$, where $u_Q = I_{n+1} + Q E_{n,n+1}$.

Proof. Step 1: Gelfand–Kazhdan restriction and s -independence.

We evaluate the right translation of W by u_Q . Block matrix multiplication yields

$$\text{diag}(g, 1)u_Q = U_Q \text{diag}(g, 1), \quad \text{where} \quad U_Q = I_{n+1} + Q \sum_{i=1}^n g_{i,n} E_{i,n+1}.$$

Because $U_Q \in N_{n+1}$ and its only superdiagonal entry (i.e., immediately above the main diagonal) is located at $(n, n+1)$ with value $Qg_{n,n}$, the left N_{n+1} -equivariance of the Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$ entails

$$W(\text{diag}(g, 1)u_Q) = \psi^{-1}(Qg_{n,n})W(\text{diag}(g, 1)) = \psi(-Qg_{n,n})W(\text{diag}(g, 1)).$$

By the Gelfand–Kazhdan restriction theory for the Kirillov model (H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, *Rankin–Selberg Convolutions*, 1983), the restriction map $W \mapsto W|_{\text{GL}_n(F)}$ surjects onto a space of functions containing $\mathcal{C}_c^\infty(N_n \backslash \text{GL}_n(F), \psi^{-1})$.

We define a smooth cut-off function $\Phi \in \mathcal{C}_c^\infty(N_n \backslash \text{GL}_n(F), \psi^{-1})$ supported precisely on the open and closed double coset $N_n K_n$ by setting $\Phi(nk) = \psi^{-1}(n)$ for $n \in N_n$ and $k \in K_n$, and extending it by zero elsewhere. This is well-defined because ψ is trivial on the intersection $N_n \cap K_n = N_n \cap \text{GL}_n(\mathfrak{o})$. We fix a choice of $W \in \mathcal{W}(\Pi, \psi^{-1})$ satisfying $W(\text{diag}(g, 1)) = \Phi(g)$.

Substituting this test vector W restricts the domain of integration strictly to the compact quotient $N_n \backslash N_n K_n \simeq (N_n \cap K_n) \backslash K_n$. For $k \in K_n$, we have $|\det k| = 1$, which completely eliminates the complex parameter s . Normalizing the quotient measure appropriately, the integral converges absolutely to a finite, s -independent functional:

$$L_Q(V) = \int_{K_n} \psi(-Qk_{n,n})V(k) dk.$$

We are reduced to showing that there exists $V \in \mathcal{W}(\pi, \psi)$ such that $L_Q(V) \neq 0$.

Step 2: The unramified case ($c = 0$).

If π is unramified, its conductor ideal is $\mathfrak{q} = \mathfrak{o}$, meaning $Q \in \mathfrak{o}^\times$. We evaluate the functional on the normalized spherical vector $V = V_0$, which satisfies $V_0(k) = 1$ for all $k \in K_n$. Since $k_{n,n} \in \mathfrak{o}$ and $Q \in \mathfrak{o}^\times$, we have $-Qk_{n,n} \in \mathfrak{o}$. Because the additive character ψ has conductor \mathfrak{o} , it follows that $\psi(-Qk_{n,n}) = 1$. The functional thus yields $L_Q(V_0) = \text{vol}(K_n) > 0$.

Step 3: Finite Fourier analysis setup ($c \geq 1$).

Assume π has conductor $\mathfrak{q} = \mathfrak{p}^c$ with $c \geq 1$. Here, $Q = \alpha\varpi^{-c}$ for some unit $\alpha \in \mathfrak{o}^\times$. Let $V_0 \in \mathcal{W}(\pi, \psi)$ be the essential newform, properly normalized so that $V_0(I_n) = 1$. Suppose, for the sake of contradiction, that $L_Q(\pi(h^{-1})V_0) = 0$ for all $h \in K_n$. Evaluating the functional and making the change of variables $k \mapsto kh$ yields:

$$\int_{K_n} \psi(-Q(kh)_{n,n})V_0(k) dk = 0 \quad \text{for all } h \in K_n.$$

Let $e_n = (0, \dots, 0, 1)$ be viewed as a row vector in \mathfrak{o}^n . Then $(kh)_{n,n} = e_n k h e_n^T = (e_n k) y$, where $y = h e_n^T$ is a column vector. As h traverses K_n , the vector y traverses all unimodular column vectors in \mathfrak{o}^n .

Because $Q = \alpha\varpi^{-c}$, the value $\psi(-Q(e_n k)y)$ depends on the row vector $e_n k$ exclusively modulo \mathfrak{p}^c . We descend to the finite quotient module $G = (\mathfrak{o}/\mathfrak{p}^c)^n$ by defining a function $H : G \rightarrow \mathbb{C}$ as follows:

$$H(\eta) = \int_{\{k \in K_n : e_n k \equiv \eta \pmod{\mathfrak{p}^c}\}} V_0(k) dk.$$

If η does not lift to a unimodular vector in \mathfrak{o}^n , the domain of integration is empty, forcing $H(\eta) = 0$. The vanishing assumption dictates that the finite Fourier transform of H is identically zero on all unimodular vectors $y \in G$:

$$\widehat{H}(y) = \sum_{\eta \in G} H(\eta) \psi(-Q\eta y) = 0.$$

Step 4: Fourier inversion and translation invariance.

Since $\widehat{H}(y) = 0$ for all unimodular y , the support of \widehat{H} is restricted to non-unimodular vectors. Over the finite module G , a vector is non-unimodular if and only if all its entries belong to $\mathfrak{p}/\mathfrak{p}^c$, meaning \widehat{H} is supported entirely on $\mathfrak{p}G$.

Applying the Fourier inversion formula over G , we obtain:

$$H(\eta) = \frac{1}{|G|} \sum_{y \in \mathfrak{p}G} \widehat{H}(y) \psi(Q\eta y).$$

Let $y \in \mathfrak{p}G$, guaranteeing $y = \varpi z$ for some column vector $z \in (\mathfrak{o}/\mathfrak{p}^{c-1})^n$. For an arbitrary shift $\delta \in \mathfrak{p}^{c-1}G$, we may write $\delta = \varpi^{c-1}x$ with a row vector $x \in G$. The inner product computes as:

$$Q\delta y = (\alpha\varpi^{-c})(\varpi^{c-1}x)(\varpi z) = \alpha xz \in \mathfrak{o}.$$

Because the additive character ψ has conductor \mathfrak{o} , we deduce $\psi(Q\delta y) = \psi(\alpha xz) = 1$. The additive character thus neutralizes the shift, giving $\psi(Q(\eta + \delta)y) = \psi(Q\eta y)$. Consequently, $H(\eta)$ exhibits translation invariance:

$$H(\eta + \delta) = H(\eta) \quad \text{for all } \delta \in \mathfrak{p}^{c-1}G.$$

Step 5: Level lowering and contradiction.

We evaluate $H(e_n)$. The domain of integration enforces the condition $e_n k \equiv e_n \pmod{\mathfrak{p}^c}$, which strictly characterizes the mirabolic congruence subgroup $K_1(\mathfrak{p}^c)$ consisting of matrices in K_n whose last row is congruent to e_n modulo \mathfrak{p}^c . By foundational theory (H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika, *Conducteur des représentations du groupe linéaire*, 1981), the essential newform V_0 is invariant under right translation by $K_1(\mathfrak{p}^c)$. Since $V_0(I_n) = 1$, we obtain:

$$H(e_n) = \int_{K_1(\mathfrak{p}^c)} V_0(k) dk = \text{vol}(K_1(\mathfrak{p}^c)) > 0.$$

By the translation invariance established above, summing H over the q^n mutually distinct shifts $\delta \in \mathfrak{p}^{c-1}G$ yields:

$$\sum_{\delta \in \mathfrak{p}^{c-1}G} H(e_n + \delta) = q^n H(e_n) > 0.$$

Conversely, returning to the definition of H , the disjoint union of the integration domains $\{k \in K_n : e_n k \equiv e_n + \delta \pmod{\mathfrak{p}^c}\}$ systematically aggregates to the coarser condition $e_n k \equiv e_n \pmod{\mathfrak{p}^{c-1}}$. This aggregated domain precisely identifies the broader mirabolic congruence subgroup $K_1(\mathfrak{p}^{c-1})$. Therefore:

$$\sum_{\delta \in \mathfrak{p}^{c-1}G} H(e_n + \delta) = \int_{K_1(\mathfrak{p}^{c-1})} V_0(k) dk.$$

This integral calculates the evaluation at I_n of the locally averaged vector

$$v = \int_{K_1(\mathfrak{p}^{c-1})} \pi(k) V_0 dk \in \mathcal{W}(\pi, \psi).$$

Because the conductor of π is exactly \mathfrak{p}^c , the representation π contains no nonzero vectors invariant under the larger mirabolic congruence subgroup $K_1(\mathfrak{p}^{c-1})$. Thus, the vector v identically vanishes, mandating $v(I_n) = 0$.

We are thereby led to the contradiction $0 = q^n \text{vol}(K_1(\mathfrak{p}^c)) > 0$. This contradiction demonstrates that the uniform vanishing hypothesis must be false. We conclude that there exists some translate $V = \pi(h^{-1})V_0$ for which the local Rankin–Selberg integral is finite and nonzero, completing the proof. \square