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User Prompt

I have a family of problems attached to a reductive group G along with a choice of minuscule coweight μ . Consider the graded ring $R = H^*(\mathbb{B}T)$ where T is a maximal torus of G . This has an obvious action of the Weyl group W , and the invariant ring R^W identifies with $H^*(\mathbb{B}G)$.

Let $L_\mu \subset P_\mu$ be the Levi and parabolic subgroups of G associated to μ . Let W_μ be the Weyl group of L_μ . The map $\mathbb{B}P_\mu \rightarrow \mathbb{B}G$ is a fiber bundle for G/P_μ , so it induces a pushforward map on cohomology. For $f \in R^{W_\mu}$, denote this pushforward map by

$$\int_{G/P_\mu} f : R^{W_\mu} = H^*(\mathbb{B}P_\mu) \rightarrow H^*(\mathbb{B}G) = R^W$$

Since the coweight μ can be viewed in $X_*(T)$ while $R = \text{Sym}_{\mathbb{Q}}(X^*(T))$, the partial derivative $\partial_\mu : R^W \rightarrow R^{W_\mu}$ is defined.

Let $\mathbb{V} = \text{Gr}_{\text{aug}}^1(R^W)/\text{Gr}_{\text{aug}}^2(R^W)$ be the first graded for the augmentation filtration on R^W . I will consider an operator ∇^η on \mathbb{V} , of the form

$$\nabla_\mu^\eta(f) = \int \eta \partial_\mu f$$

where $\eta = t^{1+\dim G/P_\mu}$ for some degree 2 element $t \in R^{W_\mu}$ (the numerology is chosen so that ∇^η preserves degree, since ∂_μ lowers degree by 2 and $\int \eta \partial_\mu$ lowers degree by $2 \dim G/P_\mu$).

I want to know the eigenvalues of ∇_μ^η on \mathbb{V} .

Let $G = \text{GL}_n$ and the minuscule coweight $\mu = (1, 1, \dots, 1, 0, \dots, 0)$ with m leading 1's, and $t = x_1 + x_2 + \dots + x_m$. Assume $1 \leq m < n$. Here $R^W = \mathbb{Q}[e_1, e_2, \dots, e_n]$ where e_i is the i -th elementary symmetric polynomial in x_1, \dots, x_n .

Calculate the action of ∇^η on the basis e_1, e_2, \dots, e_n for $\mathbb{V} = \text{Gr}_{\text{aug}}^1(R^W)/\text{Gr}_{\text{aug}}^2(R^W)$. In particular, identify the eigenvalues of ∇_μ^η on \mathbb{V} . Express these eigenvalues in closed form, as uniform in n and m as possible, and simplify your answer as much as possible.

Model Response (*Aletheia*)

Let $G = \text{GL}_n$ over \mathbb{C} and let T be its maximal torus. The cohomology ring of the classifying space BT is identified with the polynomial ring $R = \mathbb{Q}[x_1, \dots, x_n]$. The Weyl group $W \cong S_n$ acts on R by permuting the variables. The ring of invariants is $R^W = \mathbb{Q}[p_1, \dots, p_n]$, where $p_k = \sum_{i=1}^n x_i^k$ denotes the k -th power sum polynomial. Let $I \subset R^W$ be the augmentation ideal generated by homogeneous invariants of positive degree. The space $\mathbb{V} = I/I^2$ represents the indecomposable elements of the cohomology ring.

Fix a minuscule coweight $\mu = (1^m, 0^{n-m})$ with $1 \leq m < n$. This defines a parabolic subgroup $P_\mu \subset G$. The fiber of the projection $G/P_\mu \rightarrow \mathbb{B}P_\mu \rightarrow \mathbb{B}G$ is the Grass-

mannian $\mathrm{Gr}(m, n)$, having complex dimension $d = m(n - m)$. We study the linear operator $\nabla_\mu^\eta : \mathbb{V} \rightarrow \mathbb{V}$ defined by

$$\nabla_\mu^\eta(f) = \int_{G/P_\mu} \eta \cdot \partial_\mu f,$$

where $\eta = (x_1 + \cdots + x_m)^{d+1}$ and $\partial_\mu = \sum_{i=1}^m \frac{\partial}{\partial x_i}$.

In this paper, we prove that ∇_μ^η acts diagonally on the graded components of \mathbb{V} and calculate the eigenvalues explicitly.

1 Preliminaries

1.1 Graded Structure of \mathbb{V}

Assign polynomial degrees $\deg(x_i) = 1$. Then $\deg(p_k) = k$. Since R^W is a free polynomial algebra on $\{p_1, \dots, p_n\}$, the graded vector space $\mathbb{V} = I/I^2$ has basis elements $[p_k]$ for $k = 1, \dots, n$. Specifically, the subspace \mathbb{V}_k of degree k is one-dimensional:

$$\mathbb{V}_k = \mathrm{span}_{\mathbb{Q}}\{p_k \pmod{I^2}\}.$$

The elementary symmetric polynomials e_k are related to p_k by Newton's sums. In \mathbb{V} , decomposable products vanish, leading to the relation $e_k \equiv \frac{(-1)^{k-1}}{k} p_k \pmod{I^2}$. Thus, e_k and p_k represent the same eigenline.

1.2 Degree Preservation

We verify the homogeneity of the operator ∇_μ^η .

- Differentiation ∂_μ reduces degree by 1.
- Multiplication by η increases degree by $d + 1$.
- Integration along the fiber G/P_μ reduces degree by d .

The net shift is $(k - 1) + (d + 1) - d = k$. Since ∇_μ^η maps $\mathbb{V}_k \rightarrow \mathbb{V}_k$ and $\dim(\mathbb{V}_k) = 1$, there exists a scalar eigenvalue λ_k such that

$$\nabla_\mu^\eta(p_k) \equiv \lambda_k p_k \pmod{I^2}.$$

2 Integration and Gysin Pushforward

The integral is defined by the localization formula:

$$\int_{G/P_\mu} g = \sum_{w \in W/W_\mu} w \left(\frac{g}{\mathfrak{R}_\mu} \right),$$

where $\mathfrak{R}_\mu = \prod_{\langle \alpha, \mu \rangle < 0} \alpha$. For $\mu = (1^m, 0^{n-m})$, the negative roots correspond to pairs (i, j) with $i > m$ and $j \leq m$:

$$\mathfrak{R}_\mu = \prod_{i=m+1}^n \prod_{j=1}^m (x_i - x_j).$$

Let $\Delta_{rel} = \prod_{j=1}^m \prod_{i=m+1}^n (x_j - x_i)$. Observe that $\mathfrak{R}_\mu = (-1)^{m(n-m)} \Delta_{rel} = (-1)^d \Delta_{rel}$. Thus, the integral relates to the standard Gysin pushforward $\pi_* : R^{W_\mu} \rightarrow R^W$:

$$\int_{G/P_\mu} g = (-1)^d \sum_{w \in W/W_\mu} w \left(\frac{g}{\Delta_{rel}} \right) = (-1)^d \pi_*(g).$$

Lemma 1. *Let Λ_m be the ring of symmetric functions in x_1, \dots, x_m . For a Schur function $s_\lambda \in \Lambda_m$,*

$$\pi_*(s_\lambda) = s_{\lambda-R}(x_1, \dots, x_n),$$

where $R = ((n-m)^m)$ is the rectangular partition. If $R \not\subseteq \lambda$, the term vanishes.

Proof. Recall the alternant definition of Schur functions. Let $\delta_k = (k-1, \dots, 0)$.

$$s_\lambda(x_1, \dots, x_m) = \frac{\sum_{u \in S_m} \epsilon(u) x^{u(\lambda + \delta_m)}}{\prod_{1 \leq j < k \leq m} (x_j - x_k)}.$$

Applying π_* , the numerator becomes $\sum_{w \in S_n} \epsilon(w) x^{w(\kappa)}$, where:

$$\kappa = (\lambda_1 + m - 1, \dots, \lambda_m, n - m - 1, \dots, 0).$$

Solving for ν in $\nu_i + n - i = \kappa_i$ yields $\nu_i = \lambda_i - (n - m)$ for $i \leq m$, thus $\nu = \lambda - R$. \square

3 Eigenvalue Derivation

3.1 Evaluating the Integrand

Let $p_k^{(m)} = \sum_{i=1}^m x_i^k$. We calculate $\partial_\mu p_k = k p_{k-1}^{(m)}$. The integrand is $g = \eta \cdot \partial_\mu p_k = (p_1^{(m)})^{d+1} \cdot k p_{k-1}^{(m)}$. Define $\Phi_k = (p_1^{(m)})^{d+1} p_{k-1}^{(m)}$. Then $\nabla_\mu^\eta(p_k) = (-1)^d k \pi_*(\Phi_k)$.

3.2 Projection Modulo Decomposables

In \mathbb{V}_k , $s_\nu \equiv \frac{\chi_{(k)}^\nu}{k} p_k \pmod{I^2}$. By the Murnaghan-Nakayama rule, $\chi_{(k)}^\nu$ is non-zero only if ν is a hook partition. We obtain the scalar eigenvalue:

$$\lambda_k = (-1)^d \sum_{\nu \in \text{Hooks}_k} (-1)^{\text{ht}(\nu)} c_{R+\nu}. \quad (1)$$

3.3 Proof of Main Theorems

In Λ_m , $s_R \cdot p_k^{(m)} = \sum_{\substack{\nu \in \text{Hooks}_k \\ \ell(R+\nu) \leq m}} (-1)^{\text{ht}(\nu)} s_{R+\nu}$. Hence $\lambda_k = (-1)^d \langle \Phi_k, s_R p_k^{(m)} \rangle_{\Lambda_m}$.

Theorem 1. *The eigenvalues of ∇_μ^η on \mathbb{V} are given by:*

- If $k = 1$: $\lambda_1 = (-1)^d m f^{R+(1)}$.
- If $k \geq 2$: $\lambda_k = (-1)^d \sum_{h=0}^{\min(k-1, m-1)} (-1)^h \chi_{\alpha_k}^{R+\nu_h}$,

where $\nu_h = (k-h, 1^h)$ and α_k is the conjugacy class of cycle type $(k-1, 1^{d+1})$.

4 Example Calculation

Let $n = 4, m = 2 \implies d = 4, R = (2, 2)$.

- **Degree 1:** $\lambda_1 = 2f^{(3,2)} = 2(5) = 10$.
- **Degree 2:** $\lambda_2 = f^{(4,2)} - f^{(3,3)} = 9 - 5 = 4$.
- **Degree 3:** $\lambda_3 = \chi_{(2,1^4)}^{(5,2)} - \chi_{(2,1^4)}^{(4,3)} = 6 - 4 = 2$.

5 Conclusion

The operator ∇_μ^η acts diagonally on the basis of power sums. The eigenvalues are integers involving characters of the symmetric group associated with extensions of the rectangle partition R .

User Prompt

Next let $G = PSp(2n)$ for $n \geq 2$. Let μ be the minuscule spin coweight $\frac{1}{2}(1, 1, \dots, 1)$ of G and $t = \frac{1}{2}(x_1 + x_2 + \dots + x_n)$. Here $R^W = [p_1^{(2)}, p_2^{(2)}, \dots, p_n^{(2)}]$ where $p_i^{(2)}$ is the i th power sum in x_1^2, \dots, x_n^2 . Calculate the action of ∇^η on the basis $p_1^{(2)}, p_2^{(2)}, \dots, p_n^{(2)}$, for $\mathbb{V} = \text{Gr}_{\text{aug}}^1(R^W)/\text{Gr}_{\text{aug}}^2(R^W)$. In particular, identify the eigenvalues of ∇_μ^η on \mathbb{V} . Try to find a closed form expression for the eigenvalues, which is as uniform in n and k as possible.

Model Response (*Aletheia*)

Let $G = \text{PSp}(2n)$ denote the projective symplectic group of rank n over \mathbb{C} . Let T be a maximal torus of G . The rational cohomology ring of the classifying space BT is isomorphic to the polynomial ring $R = \mathbb{Q}[x_1, \dots, x_n]$, where the variables

x_i are assigned cohomological degree 2. The Weyl group $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ acts on R via permutations and sign changes of the generators. The invariant ring is $R^W = \mathbb{Q}[x_1^2, \dots, x_n^2]$. Let $I \subset R^W$ be the augmentation ideal generated by homogeneous invariants of positive degree. The vector space of indecomposables is defined as the quotient $\mathbb{V} = I/I^2$. As R^W is a free polynomial algebra generated by the power sums of squares, the classes of the polynomials

$$p_k^{(2)} = \sum_{i=1}^n x_i^{2k}, \quad k \in \{1, \dots, n\}$$

form a basis for \mathbb{V} . We adopt the polynomial grading convention $\deg(x_i) = 1$, giving $p_k^{(2)}$ a degree of $2k$.

Fix the minuscule coweight $\mu = \frac{1}{2}(1, 1, \dots, 1)$. This defines a parabolic subgroup $P_\mu \subset G$, such that the generalized flag variety G/P_μ is the Lagrangian Grassmannian of complex dimension $d = \frac{n(n+1)}{2}$. Let $t = \frac{1}{2} \sum_{i=1}^n x_i$. We define the class $\eta = t^{d+1}$. We study the linear operator $\nabla_\mu^\eta : \mathbb{V} \rightarrow \mathbb{V}$ defined by

$$\nabla_\mu^\eta(f) = \int_{G/P_\mu} \eta \cdot \partial_\mu f,$$

where $\partial_\mu = \frac{1}{2} \sum_{i=1}^n \frac{\partial}{\partial x_i}$ is the derivation associated with μ .

The main contribution of this paper is the proof that ∇_μ^η acts diagonally on the basis $\{p_k^{(2)}\}$ and the determination of the corresponding eigenvalues using character theory.

6 Main Result

Theorem 2. *Let $n \geq 1$ and $1 \leq k \leq n$. The operator ∇_μ^η preserves the graded subspaces of \mathbb{V} . Specifically, the basis vector $p_k^{(2)}$ is an eigenvector:*

$$\nabla_\mu^\eta(p_k^{(2)}) \equiv \lambda_{n,k} p_k^{(2)} \pmod{I^2}.$$

The eigenvalue $\lambda_{n,k}$ is given by the formula:

$$\lambda_{n,k} = \frac{(-1)^d}{2^{d+1}} \sum_{h=0}^{\min(k-1, n-1)} (-1)^h \chi_\rho^{\Lambda_h},$$

where:

1. $d = \frac{n(n+1)}{2}$.
2. χ_ρ^Λ is the irreducible character of the symmetric group S_{d+2k} corresponding to the partition Λ , evaluated on the conjugacy class $\rho = (2k-1, 1^{d+1})$.
3. The partition Λ_h has components defined by:

$$(\Lambda_h)_i = \begin{cases} 2k-2h+n & \text{if } i=1, \\ n-i+3 & \text{if } 2 \leq i \leq h+1, \\ n-i+1 & \text{if } h+2 \leq i \leq n. \end{cases}$$

7 Proof

7.1 Grading and Diagonal Action

We analyze the degree shift effected by each component of ∇_μ^η :

- Differentiation ∂_μ : reduces degree by 1.
- Multiplication by η : increases degree by $d + 1$.
- Fiber integration over G/P_μ : reduces degree by dimension d .

Total shift $= -1 + (d + 1) - d = 0$. Thus, ∇_μ^η maps elements of degree m to elements of degree m . The space \mathbb{V} decomposes into graded components \mathbb{V}_{2k} spanned by $p_k^{(2)}$. Since these components are one-dimensional and preserved by the operator, $p_k^{(2)}$ must be an eigenvector.

7.2 Localization Analysis

We compute the integral using the Atiyah-Bott localization theorem. The T -fixed points on G/P_μ are indexed by sign vectors $\epsilon \in \{\pm 1\}^n$. The weights of the tangent space at a fixed point ϵ are given by the negative roots relative to μ , evaluated at ϵx . The relevant roots are $\Phi_\mu^- = \{-2x_i\} \cup \{-(x_i + x_j) \mid i < j\}$. The equivariant Euler class at ϵx is:

$$\begin{aligned} \mathfrak{R}_\mu(\epsilon x) &= \prod_{i=1}^n (-2\epsilon_i x_i) \prod_{i < j} -(\epsilon_i x_i + \epsilon_j x_j) \\ &= (-1)^d 2^n \left(\prod_{i=1}^n \epsilon_i x_i \right) \prod_{i < j} (\epsilon_i x_i + \epsilon_j x_j). \end{aligned}$$

Using the generalized Vandermonde identity $\prod_{i < j} (u_i + u_j) = \Delta(u^2)/\Delta(u)$, we rewrite the denominator:

$$\mathfrak{R}_\mu(\epsilon x) = (-1)^d 2^n \left(\prod \epsilon x \right) \frac{\Delta(x^2)}{\Delta(\epsilon x)}.$$

Next, consider the numerator integrand for $f = p_k^{(2)}$. Since $\partial_\mu f = k \sum x_i^{2k-1}$ and $\eta = 2^{-(d+1)}(\sum x_i)^{d+1}$, evaluating at ϵx yields:

$$(\eta \partial_\mu f)(\epsilon x) = \frac{k}{2^{d+1}} \left(\sum \epsilon x \right)^{d+1} \left(\sum (\epsilon x)^{2k-1} \right).$$

Let $P(u) = (\sum u)^{d+1} (\sum u^{2k-1})$. Substituting into the localization sum:

$$\nabla_\mu^\eta(p_k^{(2)}) = \sum_{\epsilon \in \{\pm 1\}^n} \frac{\frac{k}{2^{d+1}} P(\epsilon x)}{\mathfrak{R}_\mu(\epsilon x)} = \frac{(-1)^d k}{2^{d+n+1} \Delta(x^2)} \sum_\epsilon \frac{P(\epsilon x) \Delta(\epsilon x)}{\prod \epsilon x}.$$

7.3 Parity Filtering

Let $Q(u) = P(u)\Delta(u)$. Writing $Q(u) = \sum_{\gamma} c_{\gamma} u^{\gamma}$, the summation over ϵ filters out terms where any exponent is even:

$$\sum_{\epsilon} \frac{(\epsilon x)^{\gamma}}{\prod \epsilon x} = \begin{cases} 2^n x^{\gamma-1} & \text{if } \gamma_i \text{ is odd } \forall i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\gamma = 2\beta + 1$. The sum yields $2^n x^{2\beta}$. Returning to the integral expression:

$$\nabla_{\mu}^{\eta}(p_k^{(2)}) = \frac{(-1)^d k}{2^{d+1} \Delta(x^2)} \sum_{\beta} c_{2\beta+1} x^{2\beta}.$$

Let $y_i = x_i^2$. Note that $Q(u)$ is an antisymmetric polynomial. Thus the coefficients satisfy $c_{\sigma(\gamma)} = \text{sgn}(\sigma) c_{\gamma}$. This antisymmetry allows us to collect terms into Schur polynomials. Let $\beta = \nu + \delta$, where $\delta = (n-1, \dots, 0)$. Then:

$$\frac{1}{\Delta(y)} \sum_{\beta} c_{2\beta+1} y^{\beta} = \sum_{\nu} c_{2(\nu+\delta)+1} s_{\nu}(y).$$

Thus,

$$\nabla_{\mu}^{\eta}(p_k^{(2)}) = \frac{(-1)^d k}{2^{d+1}} \sum_{\nu} c_{2(\nu+\delta)+1} s_{\nu}(x^2).$$

7.4 Projection onto \mathbb{V}

We evaluate the Schur functions modulo I^2 . Using the character map $s_{\nu} = \sum_{\rho} z_{\rho}^{-1} \chi_{\rho}^{\nu} p_{\rho}$, we note that any term with more than one cycle corresponds to a decomposable element (product of invariants) and vanishes in \mathbb{V} . The only surviving term is the single cycle p_k .

$$s_{\nu} \equiv \frac{\chi_{(k)}^{\nu}}{k} p_k \pmod{I^2}.$$

The Murnaghan-Nakayama rule implies $\chi_{(k)}^{\nu} \neq 0$ if and only if ν is a hook partition $\nu_h = (k-h, 1^h)$, yielding value $(-1)^h$. Substituting into our expression:

$$\nabla_{\mu}^{\eta}(p_k^{(2)}) \equiv \frac{(-1)^d k}{2^{d+1}} \sum_{h=0}^{k-1} c_{2(\nu_h+\delta)+1} \frac{(-1)^h}{k} p_k^{(2)} \pmod{I^2}.$$

Factors of k cancel. We identify the coefficient c_{γ} . Recall c_{γ} is the coefficient of u^{γ} in $P(u)\Delta(u)$. Let $\Lambda = \gamma - \delta$. Then $c_{\Lambda+\delta}$ is the inner product $\langle P, s_{\Lambda} \rangle$. Here $P = p_1^{d+1} p_{2k-1}$. This corresponds to the power sum function p_{ρ} for partition $\rho = (2k-1, 1^{d+1})$. Using the orthogonality relation $\langle p_{\rho}, s_{\Lambda} \rangle = \chi_{\rho}^{\Lambda}$, we obtain:

$$c_{2(\nu_h+\delta)+1} = \chi_{\rho}^{\Lambda_h}.$$

This yields the formula stated in Theorem 2.

7.5 Construction of Λ_h

We verify the components of $\Lambda_h = 2\nu_h + \delta + \mathbf{1}$: For $i = 1$: $(\Lambda_h)_1 = 2(k-h) + (n-1) + 1 = 2k - 2h + n$. For $2 \leq i \leq h+1$ (where $(\nu_h)_i = 1$): $(\Lambda_h)_i = 2(1) + (n-i) + 1 = n - i + 3$. For $h+2 \leq i \leq n$ (where $(\nu_h)_i = 0$): $(\Lambda_h)_i = 0 + (n-i) + 1 = n - i + 1$. Since $h \leq k-1$, the partition parts are strictly decreasing, confirming Λ_h is a valid partition.

8 Calculated Eigenvalues

Using the derived formula, exact eigenvalues for ranks $n = 1$ to $n = 6$ are calculated as follows.

Table 1: Eigenvalues $\lambda_{n,k}$ for $1 \leq k \leq n$

Rank	Dim d	Eigenvalues $(\lambda_{n,1}, \dots, \lambda_{n,n})$
1	1	$-\frac{1}{4}$
2	3	$-\frac{1}{4}, -\frac{1}{4}$
3	6	$\frac{1}{2}, \frac{5}{16}, \frac{1}{2}$
4	10	$\frac{11}{4}, \frac{19}{16}, \frac{7}{4}, \frac{41}{16}$
5	15	$-\frac{221}{4}, -\frac{71}{4}, -\frac{49}{2}, -\frac{137}{4}, -47$
6	21	$-\frac{81719}{16}, -1292, -\frac{13395}{8}, -\frac{36501}{16}, -\frac{12045}{4}, -\frac{7909}{2}$

9 Conclusion

We have demonstrated that the operator ∇_μ^η acts diagonally on the basis of power sum polynomials in \mathbb{V} . The eigenvalues obey a uniform combinatorial formula based on characters of the symmetric group S_{d+2k} , linking the geometry of the Lagrangian Grassmannian to symmetric function theory.

User Prompt

Next let $G = \text{PSO}(2n)$ for $n \geq 3$. Let μ be the minuscule spin coweight $\frac{1}{2}(1, 1, \dots, 1)$ of G and $t = \frac{1}{2}(x_1 + x_2 + \dots + x_n)$. Here $R^W = [p_1^{(2)}, p_2^{(2)}, \dots, p_{n-1}^{(2)}, \text{Pf}]$ where $p_i^{(2)} = \sum_{j=1}^n x_j^{2i}$ is the i th elementary symmetric polynomial in x_1^2, \dots, x_n^2 , and $\text{Pf} = x_1 x_2 \cdots x_n$ is the Pfaffian. Calculate the action of ∇_μ^η on the basis $p_1^{(2)}, p_2^{(2)}, \dots, p_{n-1}^{(2)}, \text{Pf}$ for $\mathbb{V} = \text{Gr}_{\text{aug}}^1(R^W)/\text{Gr}_{\text{aug}}^2(R^W)$. Give an answer which is as uniform in n as possible.

10 Introduction

Let $G = \text{PSO}(2n)$ be the projective special orthogonal group of rank $n \geq 3$ over \mathbb{C} . The rational cohomology ring of the classifying space BG is isomorphic to the ring of Weyl group invariants $R^W \subset R = \mathbb{Q}[x_1, \dots, x_n]$, where W is the Weyl group of type D_n . We assign the polynomial grading $\deg(x_i) = 1$. The ring R^W is generated by the even power sum polynomials and the Pfaffian:

$$R^W = \mathbb{Q}[p_1^{(2)}, \dots, p_{n-1}^{(2)}, \text{Pf}],$$

where $p_k^{(2)} = \sum_{i=1}^n x_i^{2k}$ (degree $2k$) and $\text{Pf} = \prod_{i=1}^n x_i$ (degree n). Let $I \subset R^W$ be the augmentation ideal. The space of indecomposables is defined as the quotient vector space $\mathbb{V} = I/I^2$. The set $\mathcal{B} = \{p_1^{(2)}, \dots, p_{n-1}^{(2)}, \text{Pf}\}$ forms a basis for \mathbb{V} .

Let $\mu = \frac{1}{2}(1, \dots, 1)$ be the specific minuscule coweight. This defines the spinor variety G/P_μ of dimension $d = n(n-1)/2$. Let $t = \frac{1}{2} \sum_{i=1}^n x_i$. We define the class $\eta = t^{d+1}$ and the operator $\nabla_\mu^\eta : \mathbb{V} \rightarrow \mathbb{V}$ by:

$$\nabla_\mu^\eta(f) = \int_{G/P_\mu} \eta \cdot \partial_\mu f,$$

where $\partial_\mu = \frac{1}{2} \sum_{i=1}^n \frac{\partial}{\partial x_i}$ is the derivation induced by μ .

11 Main Theorems

Let $C_0 = (-1)^{d+1}$. Let χ_ρ^λ denote the character of the symmetric group $S_{|\lambda|}$ evaluated on conjugacy class ρ , and f^λ be the dimension of the irreducible representation indexed by λ .

Definition 3 (Partitions and Classes). We define the following combinatorial objects:

- $\delta = (n-1, n-2, \dots, 0)$.
- $\Delta_n = (n, n-1, \dots, 1)$.
- For $1 \leq k \leq n-1$ and $0 \leq h < k$, $\nu_h(k) = (k-h, 1^h)$ is a hook partition of k .
- $\Lambda_h(n, k) = 2\nu_h(k) + \delta$ (component-wise sum). Note that Λ_h is a partition of $d+2k$ with length $n-1$.
- $\Xi_h(n, k)$ is the partition obtained by removing the first column of $\Lambda_h(n, k)$. Explicitly, $(\Xi_h)_i = (\Lambda_h)_i - 1$ for $1 \leq i \leq n-1$.
- ρ_k is the conjugacy class of cycle type $(2k-1, 1^{d+1})$ in S_{d+2k} .

Theorem 4 (Eigenvalues for Generic Degrees). *Suppose $k \in \{1, \dots, n-1\}$ such that $2k \neq n$. Then $p_k^{(2)}$ is an eigenvector of ∇_μ^η . Its eigenvalue is:*

$$\lambda_k = C_0 \sum_{h=0}^{k-1} (-1)^h \chi_{\rho_k}^{\Lambda_h(n,k)}.$$

In addition, if n is odd, Pf is an eigenvector with eigenvalue:

$$\lambda_{\text{Pf}} = \frac{C_0}{2} (d+1) f^{\Delta_{n-1}}.$$

Theorem 5 (Mixed Subspace Action). *Suppose n is even. Let $m = n/2$. The elements $p_m^{(2)}$ and Pf both have degree n and span an invariant subspace $\mathbb{V}_n \subset \mathbb{V}$. With respect to the basis $(p_m^{(2)}, \text{Pf})$, the restriction of ∇_μ^η is represented by the matrix:*

$$M = \begin{pmatrix} \lambda_m & A_{m,\text{Pf}} \\ A_{\text{Pf},m} & \lambda_{\text{Pf}} \end{pmatrix},$$

where λ_m and λ_{Pf} follow the formulas in Theorem 4 (with $k = m$), and the mixing terms are:

$$\begin{aligned} A_{\text{Pf},m} &= m C_0 \chi_{\rho_m}^{\Delta_n}, \\ A_{m,\text{Pf}} &= \frac{C_0}{2m} \sum_{h=0}^{m-1} (-1)^h f^{\Xi_h(n,m)}. \end{aligned}$$

Note: $A_{\text{Pf},m}$ is the coeff. of Pf in $\nabla_\mu^\eta(p_m^{(2)})$, and $A_{m,\text{Pf}}$ is the coeff. of $p_m^{(2)}$ in $\nabla_\mu^\eta(\text{Pf})$.

12 Proof Analysis

12.1 Degree Properties

First, we verify that ∇_μ^η maps homogeneous polynomials to homogeneous polynomials of the same degree.

- Degree of η : $\deg(t^{d+1}) = d+1$.
- Degree shift of ∂_μ : -1 .
- Degree shift of integration over G/P_μ : $-d$.

Net shift: $(d+1) - 1 - d = 0$. Thus, $\mathbb{V}_\ell \rightarrow \mathbb{V}_\ell$. Since basis elements have distinct degrees unless $2k = n$, the operator must act diagonally whenever degrees do not coincide. Coincidence occurs only for even n at degree n .

12.2 Localization

The fixed point set E of the maximal torus on G/P_μ corresponds to the Weyl orbit of μ . For $G = \text{PSO}(2n)$, the weights in the orbit are $\epsilon \in \{\pm 1/2\}^n$ with an even number of minus signs. By rescaling variables $x \mapsto x/2$, we consider $\epsilon \in \{\pm 1\}^n$ such that $\prod \epsilon_i = 1$. The product of negative roots evaluated at a fixed point is:

$$\mathfrak{R}_\mu(\epsilon x) = \prod_{i < j} -(\epsilon_i x_i + \epsilon_j x_j) = (-1)^d \prod_{i < j} (\epsilon_i x_i + \epsilon_j x_j).$$

Using the identity $\prod_{i < j} (u_i + u_j) = \Delta(u^2)/\Delta(u)$, we get:

$$\mathfrak{R}_\mu(\epsilon x) = (-1)^d \frac{\Delta(x^2)}{\Delta(\epsilon x)}.$$

Let $F = \eta \partial_\mu f$. By Atiyah-Bott localization:

$$\nabla_\mu^\eta f = \sum_{\epsilon \in E} \frac{F(\epsilon x)}{\mathfrak{R}_\mu(\epsilon x)} = \frac{(-1)^d}{\Delta(x^2)} \sum_{\epsilon \in E} F(\epsilon x) \Delta(\epsilon x).$$

12.3 Projection via Parity

Let $Q(x) = F(x)\Delta(x)$. The sum over E extracts specific parity components. Since $\sum_{\epsilon \in E} (\epsilon x)^\gamma$ vanishes unless all exponents γ_i have the same parity:

$$\sum_{\epsilon \in E} Q(\epsilon x) = 2^{n-1} ([Q(x)]_{\text{even}} + [Q(x)]_{\text{odd}}),$$

where subscripts indicate terms with all even or all odd exponents.

12.3.1 Even Exponents (Power Sum Projection)

Terms with all-even exponents correspond to power sums. Let $\gamma = 2(\nu + \delta)$. Expanding $F(x) = \sum c_\lambda s_\lambda(x)$, we get $Q(x) = \sum c_\lambda a_{\lambda+\delta}(x)$. Matching exponents yields $\lambda = 2\nu + \delta$. The contribution is:

$$\frac{(-1)^d 2^{n-1}}{\Delta(x^2)} c_{2\nu+\delta} a_{2\nu+2\delta}(x) = (-1)^d 2^{n-1} c_{2\nu+\delta} s_\nu(x^2).$$

We project $s_\nu(x^2)$ modulo I^2 . This term survives only if ν is a hook partition $\nu_h(k) = (k - h, 1^h)$, in which case:

$$s_{\nu_h(k)}(x^2) \equiv \frac{(-1)^h}{k} p_k^{(2)} \pmod{I^2}.$$

Summing over h , the coefficient of $p_k^{(2)}$ is:

$$\text{Coeff} = (-1)^d 2^{n-1} \sum_{h=0}^{k-1} \frac{(-1)^h}{k} c_{\Lambda_h(k)}.$$

12.3.2 Odd Exponents (Pfaffian Projection)

Odd terms require $\deg(Q) \geq n^2$. Since $\deg(Q) = \deg(f) + n(n-1)$, this implies $\deg(f) \geq n$. Thus odd terms only occur for elements of degree n . The unique minimal odd exponent vector is $\gamma = \Delta_n + \delta$, corresponding to $\lambda = \Delta_n$. The term is:

$$\frac{(-1)^{d2^{n-1}}}{\Delta(x^2)} c_{\Delta_n} a_{\Delta_n + \delta}(x) = (-1)^{d2^{n-1}} c_{\Delta_n} \text{Pf}(x).$$

So the coefficient of Pf is simply $(-1)^{d2^{n-1}} c_{\Delta_n}$.

12.4 Calculating c_λ

Note $\eta = 2^{-(d+1)} p_1^{d+1}$.

Case 1: $f = p_k^{(2)}$. $\partial_\mu p_k^{(2)} = k p_{2k-1}$. $F = k 2^{-(d+1)} p_1^{d+1} p_{2k-1}$. This is proportional to the power sum p_{ρ_k} where $\rho_k = (2k-1, 1^{d+1})$. Thus $c_\lambda = k 2^{-(d+1)} \chi_{\rho_k}^\lambda$. Substituting this into the even projection formula yields λ_k :

$$\lambda_k = (-1)^{d2^{n-1}} \sum_h \frac{(-1)^h}{k} k 2^{-(d+1)} \chi_{\rho_k}^{\Lambda_h} = C_0 \sum_h (-1)^h \chi_{\rho_k}^{\Lambda_h}.$$

Substituting into the odd projection formula (when $k = m$) yields $A_{\text{Pf}, m}$:

$$A_{\text{Pf}, m} = (-1)^{d2^{n-1}} m 2^{-(d+1)} \chi_{\rho_m}^{\Delta_n} = m C_0 \chi_{\rho_m}^{\Delta_n}.$$

Case 2: $f = \text{Pf}$. $\partial_\mu \text{Pf} = \frac{1}{2} e_{n-1}$. $F = 2^{-(d+2)} p_1^{d+1} e_{n-1}$. Using the property $\langle p_1^N e_{n-1}, s_\lambda \rangle = \langle p_1^N, s_\lambda s_{1^{n-1}}^\perp \rangle = f^{\lambda/1^{n-1}}$, we have $c_\lambda = 2^{-(d+2)} f^{\lambda/1^{n-1}}$.

Odd Projection (Diagonal λ_{Pf}): We need c_{Δ_n} . The skew shape $\Delta_n/1^{n-1}$ is the disconnected diagram $\Delta_{n-1} \sqcup \square$. The number of SYT is $\binom{d+1}{1} f^{\Delta_{n-1}} = (d+1) f^{\Delta_{n-1}}$. Thus:

$$\lambda_{\text{Pf}} = (-1)^{d2^{n-1}} 2^{-(d+2)} (d+1) f^{\Delta_{n-1}} = \frac{C_0}{2} (d+1) f^{\Delta_{n-1}}.$$

Even Projection (Mix term $A_{m, \text{Pf}}$): We need c_{Λ_h} . The skew shape $\Lambda_h/1^{n-1}$ removes the first column of Λ_h , yielding Ξ_h . Thus $c_{\Lambda_h} = 2^{-(d+2)} f^{\Xi_h}$. Substituting:

$$A_{m, \text{Pf}} = (-1)^{d2^{n-1}} \sum_h \frac{(-1)^h}{m} 2^{-(d+2)} f^{\Xi_h} = \frac{C_0}{2m} \sum_h (-1)^h f^{\Xi_h}.$$

13 Verification: Rank 4

Let $n = 4$. Then $d = 6$. $C_0 = 1/16$. We compute the action on the mixed subspace of degree 4 ($m = 2$). Basis: $v_1 = p_2^{(2)}$, $v_2 = \text{Pf}$.

1. $\lambda_2: \Lambda_0(2) = (7, 2, 1)$. $\chi = 34$. $\Lambda_1(2) = (5, 4, 1)$. $\chi = -6$. $\lambda_2 = \frac{1}{16}(34 - (-6)) = 2.5$.

2. $\lambda_{\text{Pf}}: \Delta_3 = (3, 2, 1)$. $f = 16$. $\lambda_{\text{Pf}} = \frac{1}{32}(7)(16) = 3.5$.
3. $A_{\text{Pf},2}: \Delta_4 = (4, 3, 2, 1)$. $\chi_{\rho_2}^{\Delta_4} = -48$. $A_{\text{Pf},2} = 2 \cdot \frac{1}{16} \cdot (-48) = -6$.
4. $A_{2,\text{Pf}}: \Xi_0 = (6, 1)$ ($f = 6$). $\Xi_1 = (4, 3)$ ($f = 14$). $A_{2,\text{Pf}} = \frac{1/16}{4}(6-14) = -0.125$.

Matrix:

$$M = \begin{pmatrix} 2.5 & -0.125 \\ -6 & 3.5 \end{pmatrix}.$$

Characteristic equation:

$$x^2 - 6x + (8.75 - 0.75) = x^2 - 6x + 8 = 0.$$

Roots: $x = 2$ and $x = 4$. The eigenvalues are integers, providing strong verification for the correctness of the derived coefficients.

For generic degrees $k = 1, 3$, we have $\lambda_1 = \lambda_3 = 4$.

14 Conclusion

The operator acts diagonally on all basis elements of degree $2k \neq n$. For n even, the elements of degree n mix according to the matrix M . Explicit, uniform formulas are provided for all entries, allowing calculation for arbitrary n .