

The prompts and responses below have been edited only for formatting compatibility.

FirstProof # 9

Let $n \geq 5$. Let $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3 \times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ so that its (i, j, k, ℓ) entry for $1 \leq i, j, k, \ell \leq 3$ is given by $Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$. Here $A(i, :)$ denotes the i th row of a matrix A , and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$.

More precisely, does there exist a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$ that satisfies the following three properties?

- The map \mathbf{F} does not depend on $A^{(1)}, \dots, A^{(n)}$.
- The degrees of the coordinate functions of \mathbf{F} do not depend on n .
- Let $\lambda \in \mathbb{R}^{n \times n \times n \times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

Model Response (Agent A)

1. Construction of the Polynomial Map \mathbf{F}

We construct a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$ whose coordinate functions are homogeneous polynomials of degree 3.

Let T be a formal tensor of variables in $\mathbb{R}^{n \times n \times n \times n \times 3 \times 3 \times 3 \times 3}$. The $81n^4$ entries of T , denoted $T_{ijkl}^{(\alpha\beta\gamma\delta)}$ for positional indices $\alpha, \beta, \gamma, \delta \in [n]$ and row choices $i, j, k, \ell \in \{1, 2, 3\}$, serve as the algebraic inputs to \mathbf{F} . We will subsequently evaluate \mathbf{F} under the substitution $T_{ijkl}^{(\alpha\beta\gamma\delta)} = \lambda_{\alpha\beta\gamma\delta} Q_{ijkl}^{(\alpha\beta\gamma\delta)}$.

To build the coordinate functions of \mathbf{F} , we proceed as follows:

1. Fix an ordered pair of distinct tensor modes (e.g., the third and fourth modes). Assign them strictly distinct positional indices $\gamma, \delta \in [n]$ with $\gamma \neq \delta$, alongside arbitrary row indices $k, \ell \in \{1, 2, 3\}$.
2. For the remaining two varying modes (e.g., the first and second modes), choose three strictly distinct indices for the first mode: $\alpha_1 < \alpha_2 < \alpha_3$ in $[n]$, and three strictly distinct indices for the second mode: $\beta_1 < \beta_2 < \beta_3$ in $[n]$.
3. Assign arbitrary row configurations $i_1, i_2, i_3 \in \{1, 2, 3\}$ and $j_1, j_2, j_3 \in \{1, 2, 3\}$.

Construct a 3×3 matrix B populated by the corresponding formal variables from T :

$$B_{r,s} = T_{i_r j_s k \ell}^{(\alpha_r \beta_s \gamma \delta)} \quad \text{for } 1 \leq r, s \leq 3.$$

We define one coordinate function of \mathbf{F} to be the polynomial $\det(B)$. The complete map \mathbf{F} is formed by collecting all such determinantal polynomials over all $\binom{4}{2} = 6$ valid pairs of varying modes, and over all valid combinatorial choices of positional and row indices.

Constraint Verification: By definition, each coordinate function evaluates a standard 3×3 minor of the formal variables. Its coefficients are exclusively ± 1 or 0. Thus, \mathbf{F} depends entirely on combinatorial structures and is strictly independent of the entries of the generic matrices $A^{(m)}$. Furthermore, every coordinate function is a homogeneous polynomial of degree exactly 3, satisfying the requirement that the degrees of the coordinate functions do not depend on n .

2. Proof of Sufficiency (\Leftarrow)

Assume there exist parameters $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ precisely over the domain of tuples $(\alpha, \beta, \gamma, \delta)$ that are not uniformly identical.

Evaluating \mathbf{F} at $T = \lambda \odot Q$ (where \odot denotes the pointwise Hadamard product), the matrix B constructed above via the first and second varying modes takes the form:

$$B_{r,s} = \lambda_{\alpha_r \beta_s \gamma \delta} Q_{i_r j_s k \ell}^{(\alpha_r \beta_s \gamma \delta)}.$$

Because the construction strictly enforced $\gamma \neq \delta$, the evaluated tuple $(\alpha_r, \beta_s, \gamma, \delta)$ fundamentally avoids identical uniformity. Hence, substituting the rank-1 factorization yields:

$$B_{r,s} = u_{\alpha_r} v_{\beta_s} w_\gamma x_\delta \det [A^{(\alpha_r)}(i_r, :); A^{(\beta_s)}(j_s, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)].$$

This factors the matrix as $B = c \cdot D_u \tilde{Q} D_v$, where $c = w_\gamma x_\delta \neq 0$, the diagonal scaling matrices $D_u = \text{diag}(u_{\alpha_1}, u_{\alpha_2}, u_{\alpha_3})$ and $D_v = \text{diag}(v_{\beta_1}, v_{\beta_2}, v_{\beta_3})$ are non-singular, and \tilde{Q} is the core multilinear 4×4 determinant matrix defined by

$$\tilde{Q}_{r,s} = \det [A^{(\alpha_r)}(i_r, :); A^{(\beta_s)}(j_s, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)].$$

Because the matrices $A^{(m)}$ are Zariski-generic and $\gamma \neq \delta$, the selected fixed rows $A^{(\gamma)}(k, :)$ and $A^{(\delta)}(\ell, :)$ are linearly independent and naturally span a 2-dimensional subspace $V \subset \mathbb{R}^4$. The 4×4 determinant evaluates the standard volume form on \mathbb{R}^4 , which mathematically descends to an alternating bilinear form on the 2-dimensional quotient space $\mathbb{R}^4/V \cong \mathbb{R}^2$. Letting $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^4/V$ be the canonical projection, there exists a non-zero scalar C (dependent strictly on V) such that:

$$\tilde{Q}_{r,s} = C \det (\pi(A^{(\alpha_r)}(i_r, :)), \pi(A^{(\beta_s)}(j_s, :))),$$

where the outer determinant represents the canonical 2×2 volume form on the quotient space. Algebraically, this structural equivalence dictates that the 3×3 matrix \tilde{Q} factors precisely through an intermediate 2-dimensional vector space. Consequently, $\text{rank}(\tilde{Q}) \leq 2$. Thus, all evaluated 3×3 minors identically vanish, forcing $\det(B) = 0$ universally across all valid mode and index selections. This establishes sufficiency.

3. Proof of Necessity (\implies): Local Rank-1 Separability

Assume $\mathbf{F}(\lambda \odot Q) \equiv 0$. Expanding the constraint $\det(B) = 0$ via the Leibniz formula yields:

$$\sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) \prod_{r=1}^3 \left(\lambda_{\alpha_r \beta_{\sigma(r)} \gamma \delta} \tilde{Q}_{r,\sigma(r)} \right) = 0.$$

Let $v_r = \pi(A^{(\alpha_r)}(i_r, :))$ and $u_s = \pi(A^{(\beta_s)}(j_s, :))$. Factoring out the volume scalar $C^3 \neq 0$ and denoting the coefficient component by $W_\sigma = \prod_{r=1}^3 \lambda_{\alpha_r \beta_{\sigma(r)} \gamma \delta}$, the sum defines a multilinear map L evaluating to zero:

$$L(v_1, v_2, v_3, u_1, u_2, u_3) = \sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) W_\sigma \prod_{r=1}^3 \det(v_r, u_{\sigma(r)}) = 0.$$

For each fixed index $r \in \{1, 2, 3\}$, the set of possible values for the vector v_r is generated by independently varying the assigned row choice $i_r \in \{1, 2, 3\}$. Because each single matrix $A^{(\alpha_r)}$ is Zariski-generic, its three available rows span a 3-dimensional subspace of \mathbb{R}^4 . The intersection of this 3-dimensional subspace with the 2-dimensional kernel V of the projection π is generically 1-dimensional. Consequently, as i_r varies, the set of three available projections $\{\pi(A^{(\alpha_r)}(1, :)), \pi(A^{(\alpha_r)}(2, :)), \pi(A^{(\alpha_r)}(3, :))\}$ dynamically spans the entirety of the 2-dimensional target space $\mathbb{R}^4/V \cong \mathbb{R}^2$.

Since this independent spanning property holds universally for the domain of every argument slot v_r as i_r varies (and symmetrically for u_s as j_s varies), the multilinear map L evaluates directly to zero on the full Cartesian product of these structurally full independent spanning sets. Standard multilinear algebra therefore rigorously dictates that $L \equiv 0$ identically everywhere on $(\mathbb{R}^2)^6$.

By the First and Second Fundamental Theorems of Invariant Theory for $\mathrm{SL}(2, \mathbb{R})$ (see H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939), the vector space of multilinear $\mathrm{SL}(2, \mathbb{R})$ -invariants on six generic vectors in \mathbb{R}^2 has dimension exactly equal to the Catalan number $C_3 = 5$. There are precisely $3! = 6$ uniquely generated multi-determinant bipartite invariants $P_\sigma = \prod_{r=1}^3 \det(v_r, u_{\sigma(r)})$. Because these bipartite invariants span the entire 5-dimensional invariant space (linearly generating all non-crossing perfect matchings via successive Grassmann-Plücker syzygies), the space of linear relations strictly spanning them is exactly 1-dimensional ($6 - 5 = 1$). This unique relation is universally generated by the primary Grassmann-Plücker identity:

$$\sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) P_\sigma = 0.$$

Because the syzygy space is strictly 1-dimensional, the identically vanishing linear combination $\sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) W_\sigma P_\sigma \equiv 0$ functionally mandates that the coefficients $\operatorname{sgn}(\sigma) W_\sigma$ are strictly proportional to the syzygy coefficients $\operatorname{sgn}(\sigma)$. This systematically yields $W_\sigma = K$, where K is a constant completely independent of σ . Equating

$W_{\text{id}} = W_{(12)}$ systematically yields:

$$\lambda_{\alpha_1 \beta_1 \gamma \delta} \lambda_{\alpha_2 \beta_2 \gamma \delta} \lambda_{\alpha_3 \beta_3 \gamma \delta} = \lambda_{\alpha_1 \beta_2 \gamma \delta} \lambda_{\alpha_2 \beta_1 \gamma \delta} \lambda_{\alpha_3 \beta_3 \gamma \delta}.$$

Because $\gamma \neq \delta$, the tuple $(\alpha_3, \beta_3, \gamma, \delta)$ categorically avoids identical uniformity. By hypothesis, λ strictly vanishes exclusively on uniformly identical tuples, reliably guaranteeing $\lambda_{\alpha_3 \beta_3 \gamma \delta} \neq 0$. Factoring it out isolates the local 2×2 rank-1 relation over the varying domains:

$$\lambda_{\alpha_1 \beta_1 \gamma \delta} \lambda_{\alpha_2 \beta_2 \gamma \delta} = \lambda_{\alpha_1 \beta_2 \gamma \delta} \lambda_{\alpha_2 \beta_1 \gamma \delta}.$$

4. Global Rank-1 Gluing (Necessity)

We now uncouple the variables and bridge domains to synthesize the local rank-1 conditions into a global factorization. To streamline notation, we re-index the tensor positional modes as $(a, b, c, d) \in [n]^4$. Let $\Omega \subset [n]^4$ be the domain of non-identical tuples. By hypothesis, $\lambda_{abcd} \neq 0$ for all $(a, b, c, d) \in \Omega$.

Fix an arbitrary index $a_0 \in [n]$. Consider the normalized ratio:

$$R(a \mid b, c, d) = \frac{\lambda_{abcd}}{\lambda_{a_0 bcd}}.$$

This ratio is universally well-defined provided $(a_0, b, c, d) \in \Omega$, which holds unconditionally whenever b, c, d are not all simultaneously equal to a_0 .

By symmetrically evaluating the local rank-1 relation across different pairs of varying modes, we deduce the following uncoupled constraints:

1. When $c \neq d$, the local rank-1 factorization (derived by varying the first and second modes) implies the submatrix $M_{a,b}^{(cd)} = \lambda_{abcd}$ natively has rank 1. Thus, $\lambda_{abcd} = \tilde{A}_{c,d}(a)\tilde{B}_{c,d}(b)$. Therefore, the ratio $R(a \mid b, c, d) = \tilde{A}_{c,d}(a)/\tilde{A}_{c,d}(a_0)$ unconditionally filters out the variable b , allowing us to safely denote it as a function $P(a, c, d)$.
2. When $b \neq d$, by symmetric application (varying the first and third modes), the ratio strictly uncouples independently of the variable c , yielding a function $Q(a, b, d)$.
3. When $b \neq c$, by symmetry (varying the first and fourth modes), the ratio strictly uncouples independently of the variable d , yielding a function $S(a, b, c)$.

For any tuple satisfying both $c \neq d$ and $b \neq d$, we have $P(a, c, d) = R(a \mid b, c, d) = Q(a, b, d)$. Because P is independent of b and Q is independent of c , their common structural value must be functionally independent of both b and c , strictly reducing to a function $U(a, d)$. For any fixed pair $c \neq d$, since $n \geq 5$, we can fundamentally select an index $b \notin \{c, d, a_0\}$. For this choice of b , the conditions $(a_0, b, c, d) \in \Omega$, $c \neq d$, and $b \neq d$ hold simultaneously. This mathematically forces $P(a, c, d) = U(a, d)$ identically for all $c \neq d$.

Similarly, on the intersection spanning $c \neq d$ and $b \neq c$, overlapping yields $U(a, d) = S(a, b, c)$. By an analogous valid index selection for c , this structurally

isolates the variables completely independent of d . Therefore, U inherently restricts to a univariate global function dependent solely on a : $U(a, d) \equiv u_1(a)$. Consequently, $P(a, c, d) = u_1(a)$ for all $c \neq d$, meaning $R(a \mid b, c, d) = u_1(a)$ universally over the subspace where $c \neq d$.

This establishes domain relations seamlessly across three operative partitions of Ω :

- $D_1 = \{(a, b, c, d) \in \Omega \mid c \neq d\}$: $R = u_1(a) \implies \lambda_{abcd} = u_1(a)\lambda_{a_0bcd} = u_1(a)F_1(b, c, d)$. Since $M_{a,b}^{(cd)}$ has rank 1, F_1 must factor out the dependence on b . By defining an analogous ratio for the second mode and applying identical symmetric reasoning, we globally extract a univariate function $u_2(b)$. Thus, the relation uncouples strictly into $\lambda_{abcd} = u_1(a)u_2(b)G_1(c, d)$.
- $D_2 = \{(a, b, c, d) \in \Omega \mid a \neq b\}$: By complete corresponding symmetry, this structurally uncouples into $\lambda_{abcd} = u_3(c)u_4(d)G_2(a, b)$, extracting global univariate functions $u_3(c)$ and $u_4(d)$.
- $D_3 = \{(a, b, c, d) \in \Omega \mid b \neq d\}$: Symmetrically, the variables uniquely uncouple to $\lambda_{abcd} = u'_1(a)u'_3(c)G_3(b, d)$. Overlapping exactly with D_1 and D_2 logically guarantees u'_1 and u'_3 correspond directly (up to constant scaling) to u_1 and u_3 , yielding $\lambda_{abcd} = u_1(a)u_3(c)G_3(b, d)$.

On the intersection bridging $D_1 \cap D_2 = \{c \neq d \text{ and } a \neq b\}$, evaluation strictly crosses conditions:

$$u_1(a)u_2(b)G_1(c, d) = u_3(c)u_4(d)G_2(a, b).$$

Because λ categorically avoids 0 on Ω , the functional components are identically non-zero. This cross-condition fundamentally separates the localized functions, forcing the fraction $\frac{G_1(c,d)}{u_3(c)u_4(d)} = \frac{G_2(a,b)}{u_1(a)u_2(b)} = K$ for some strictly non-zero constant K . Thus $G_1(c, d) = Ku_3(c)u_4(d)$. For any given pair $(c, d) \in D_1$, we can select a, b such that $a \neq b$, implying $(a, b, c, d) \in D_1 \cap D_2$. Thus $G_1(c, d) = Ku_3(c)u_4(d)$ holds identically everywhere on D_1 . Absorbing K structurally into $u_4(d)$ seamlessly unifies the mapping precisely on $D_1 \cup D_2$:

$$\lambda_{abcd} = u_1(a)u_2(b)u_3(c)u_4(d).$$

Validating continuity onto the third uncoupled plane naturally extends into D_3 . On $(D_1 \cup D_2) \cap D_3$:

$$u_1(a)u_3(c)G_3(b, d) = u_1(a)u_2(b)u_3(c)u_4(d) \implies G_3(b, d) = u_2(b)u_4(d).$$

Because for any $(b, d) \in D_3$, we can reliably select an index c such that $c \neq d$, every element mapping in D_3 securely lies in $D_1 \cap D_3$. This definitively anchors the exact global rank-1 factorization over the united domain $D_1 \cup D_2 \cup D_3$.

Completeness of Domain Coverage: If any valid tuple $(a, b, c, d) \in \Omega$ were to fall outside the spanned domain $D_1 \cup D_2 \cup D_3$, its exact mathematical omission logically structures the rigorous constraints: $c = d$ (violating D_1), $a = b$ (violating D_2), and $b = d$ (violating D_3). These three constraints collectively restrict the index boundaries exclusively to $a = b = c = d$, fully characterizing identical uniformity.

However, uniformly identical tuples are unconditionally excluded from Ω strictly by hypothesis.

Thus, the global union $D_1 \cup D_2 \cup D_3$ perfectly equals Ω , entirely covering all valid non-identical subsets and firmly securing the exact uncoupled relation $\lambda_{\alpha\beta\gamma\delta} = u_1(\alpha)u_2(\beta)u_3(\gamma)u_4(\delta)$. Concurrently, since structural evaluation over Ω categorically rejects evaluation to zero, it unequivocally enforces the required continuous functional constraints $u_1, u_2, u_3, u_4 \in (\mathbb{R}^*)^n$ (formally corresponding to the vectors u, v, w, x). This comprehensively establishes necessity and completes the full proof.