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FirstProof # 9

Let $n \geq 5$. Let $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3 \times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ so that its (i, j, k, ℓ) entry for $1 \leq i, j, k, \ell \leq 3$ is given by $Q_{ijkl}^{(\alpha\beta\gamma\delta)} = \det[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)]$. Here $A(i, :)$ denotes the i th row of a matrix A , and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$.

More precisely, does there exist a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$ that satisfies the following three properties?

- The map \mathbf{F} does not depend on $A^{(1)}, \dots, A^{(n)}$.
- The degrees of the coordinate functions of \mathbf{F} do not depend on n .
- Let $\lambda \in \mathbb{R}^{n \times n \times n \times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

Model Response (Agent B)

1. Construction of the Polynomial Map \mathbf{F}

Let the input to our polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \rightarrow \mathbb{R}^N$ be the formal tensor variables X , mapping to the scaled quadrifocal tensor:

$$X_{ijkl}^{(\alpha\beta\gamma\delta)} = \lambda_{\alpha\beta\gamma\delta} \det \begin{bmatrix} A^{(\alpha)}(i, :) \\ A^{(\beta)}(j, :) \\ A^{(\gamma)}(k, :) \\ A^{(\delta)}(\ell, :) \end{bmatrix},$$

where $A^{(\alpha)}(i, :) \in \mathbb{R}^{1 \times 4}$ denotes the i -th row of the 3×4 Zariski-generic matrix $A^{(\alpha)}$. There are $\binom{4}{2} = 6$ symmetric ways to partition the four tensor slots into two evaluated slots and two fixed slots. Consider the bipartition where we evaluate slots 1 and 2, and fix slots 3 and 4. For any pair of fixed matrix indices $\gamma \neq \delta \in [n]$ and fixed row indices $k, \ell \in \{1, 2, 3\}$, we define a $3n \times 3n$ evaluation matrix $\mathbf{M}_{k, \ell}^{\gamma, \delta}(X)$. Its rows are indexed by the multi-index $(\alpha, i) \in [n] \times \{1, 2, 3\}$ and its columns by $(\beta, j) \in [n] \times \{1, 2, 3\}$. The entries are identically the formal variables:

$$\left(\mathbf{M}_{k, \ell}^{\gamma, \delta}(X) \right)_{(\alpha, i), (\beta, j)} = X_{ijkl}^{(\alpha\beta\gamma\delta)}.$$

Definition of \mathbf{F} : We systematically define the coordinate functions of \mathbf{F} to be all 3×3 minors of these $3n \times 3n$ evaluation matrices, comprehensively collected across all

6 possible slot bipartitions and all valid choices of the fixed indices (strictly restricting to pairs where the two fixed generic matrix slots are distinct).

Verification of Properties 1 and 2:

1. Because the entries of the evaluation matrices are precisely the raw formal variables X , the coordinate functions are standard 3×3 determinantal expansions. Their coefficients are strictly combinatorial constants (± 1 and 0). Therefore, the map \mathbf{F} does not depend on the specific entries of the generic matrices $A^{(1)}, \dots, A^{(n)}$.
2. Every coordinate function is a 3×3 minor of degree-1 variables, naturally forming a homogeneous polynomial of exact degree 3. This uniform degree structurally does not depend on n .

2. Sufficiency: Rank-1 Scaling Implies $\mathbf{F}(X) = 0$

Assume there exist vectors $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all non-identical tuples. When a tuple is completely identical (i.e., $\alpha = \beta = \gamma = \delta$), the unscaled determinant $Q^{(\alpha\alpha\alpha\alpha)}$ selects 4 rows from the 3 available rows of $A^{(\alpha)}$. By the pigeonhole principle, at least two rows must inherently repeat, unconditionally forcing $Q^{(\alpha\alpha\alpha\alpha)} = 0$. Thus, substituting the pure factorization $X_{ijk\ell}^{(\alpha\beta\gamma\delta)} = (u_\alpha v_\beta w_\gamma x_\delta) Q_{ijk\ell}^{(\alpha\beta\gamma\delta)}$ evaluates identically to $0 = 0$ on identical tuples, meaning the factorization representation holds universally.

Consider the $3n \times 3n$ evaluation matrix $\mathbf{M}(X) = \mathbf{M}_{k,\ell}^{\gamma,\delta}(X)$. Factoring out the global rank-1 scaling yields:

$$\mathbf{M}(X) = (w_\gamma x_\delta) \mathbf{D}_u \mathbf{M}(Q) \mathbf{D}_v,$$

where \mathbf{D}_u and \mathbf{D}_v are block-diagonal scaling matrices, and $\mathbf{M}(Q)$ is the unscaled evaluation matrix.

Let $c = A^{(\gamma)}(k, :)^T$ and $d = A^{(\delta)}(\ell, :)^T$. The entries of $\mathbf{M}(Q)$ evaluate the alternating bilinear form $B(y, z) = \det(y, z, c, d)$ on the vectors $y = A^{(\alpha)}(i, :)^T$ and $z = A^{(\beta)}(j, :)^T$. Because c and d naturally reside in the radical of B , the form fundamentally descends to an alternating bilinear form on the quotient space $\mathbb{R}^4 / \text{span}(c, d)$. Since c and d are linearly independent (as $\gamma \neq \delta$), this quotient space has dimension 2.

The matrix $\mathbf{M}(Q)$ is the Gram-like evaluation matrix of this alternating bilinear form acting on $3n$ specific vectors. Because it evaluates the form on the projections of these $3n$ vectors into the 2-dimensional quotient space $\mathbb{R}^4 / \text{span}(c, d)$, the matrix $\mathbf{M}(Q)$ inherently factors through this 2-dimensional space. Thus, $\mathbf{M}(Q)$ is algebraically bounded to have rank at most 2. Multiplying by diagonal scaling matrices securely preserves this upper rank bound, systematically ensuring $\text{rank}(\mathbf{M}(X)) \leq 2$. Therefore, all 3×3 minors of $\mathbf{M}(X)$ vanish identically. By symmetric extension across all 6 bipartitions, $\mathbf{F}(X) = 0$.

3. Necessity: Global Rank Factorization Forces Local Rank-1 Property

Conversely, assume $\mathbf{F}(X) = 0$. We will rigorously establish that fixing any two slots to distinct indices globally restricts the corresponding $n \times n$ matrix slice of λ to be strictly rank-1.

Consider the bipartition evaluating slots 1 and 2. Fix slots 3 and 4 to arbitrary distinct indices $\gamma \neq \delta$, and fix the row indices to $k = \ell = 1$. Let $c = A^{(\gamma)}(1, :)^T$ and $d = A^{(\delta)}(1, :)^T$. Because $\gamma \neq \delta$, c and d are independent vectors from different Zariski-generic matrices.

Let $V = \mathbb{R}^4 / \text{span}(c, d)$ be the 2-dimensional quotient space. The alternating form $\omega(y, z) = \det(y, z, c, d)$ defined on $\mathbb{R}^4 \times \mathbb{R}^4$ induces a non-degenerate symplectic form on V . We select a basis for V such that ω is canonically represented by the standard symplectic matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For each $\alpha \in [n]$, let $Y_\alpha \in \mathbb{R}^{3 \times 2}$ be the coordinate matrix whose i -th row is the transpose of the coordinate vector of the projection of $A^{(\alpha)}(i, :)^T$ onto V . This exact transposition ensures the block matrix multiplication validly computes the symplectic form evaluation $\bar{y}^T J \bar{z} = \omega(y, z)$. We define the global block-diagonal matrix $\mathbf{Y} = \text{diag}(Y_1, \dots, Y_n) \in \mathbb{R}^{3n \times 2n}$.

- For $\alpha \notin \{\gamma, \delta\}$, the 3 rows of $A^{(\alpha)}$ are generic and strictly independent of c and d . Their projections are generic vectors spanning V , guaranteeing Y_α natively possesses full column rank 2.
- For $\alpha = \gamma$, the first row of $A^{(\gamma)}$ is exactly c^T , which projects to 0. However, its second and third rows are generic and independent of c and d (since $\gamma \neq \delta$), so they perfectly project to a basis of V . Thus Y_γ still retains full column rank 2. Identical structural logic ensures Y_δ retains full column rank 2.

Because every individual 3×2 block Y_α possesses full column rank, the $3n \times 2n$ block-diagonal matrix \mathbf{Y} unconditionally possesses full column rank $2n$. Hence, there exists an exact left-inverse matrix $\mathbf{L} \in \mathbb{R}^{2n \times 3n}$ such that $\mathbf{L}\mathbf{Y} = I_{2n}$.

Let $\Lambda \in \mathbb{R}^{n \times n}$ be the numeric matrix slice defined by $\Lambda_{\alpha, \beta} = \lambda_{\alpha\beta\gamma\delta}$. The scaled matrix $\mathbf{M}(X)$ evaluates to $\Lambda_{\alpha, \beta} Y_\alpha J Y_\beta^T$ on block (α, β) , which perfectly factors globally as:

$$\mathbf{M}(X) = \mathbf{Y}(\Lambda \otimes J)\mathbf{Y}^T.$$

The assumption $\mathbf{F}(X) = 0$ requires that all 3×3 minors of $\mathbf{M}(X)$ vanish, algebraically bounding its rank to at most 2. By multiplying by \mathbf{L} on the left and \mathbf{L}^T on the right, we flawlessly isolate the Kronecker core:

$$\mathbf{L}\mathbf{M}(X)\mathbf{L}^T = \Lambda \otimes J.$$

Since matrix multiplication cannot increase rank, $\text{rank}(\Lambda \otimes J) \leq \text{rank}(\mathbf{M}(X)) \leq 2$. By standard Kronecker product properties over any field, $\text{rank}(\Lambda \otimes J) = \text{rank}(\Lambda) \times \text{rank}(J) = 2 \text{rank}(\Lambda)$. This algebraically forces $2 \text{rank}(\Lambda) \leq 2$, unconditionally dictating $\text{rank}(\Lambda) \leq 1$.

Because $\gamma \neq \delta$, every tuple $(\alpha, \beta, \gamma, \delta)$ is strictly non-identical. By hypothesis, λ evaluates to non-zero exactly on non-identical tuples, meaning Λ contains absolutely no zero entries. As a rank-1 matrix devoid of zeros, Λ factors exactly as $\Lambda_{\alpha, \beta} = U_{\alpha} V_{\beta}$ for non-zero vectors $U, V \in (\mathbb{R}^*)^n$.

4. Global Patching

Since $n \geq 5$, we globally fix four mutually distinct anchor indices $a, b, c, d \in [n]$. Let $c_0 = \lambda_{abcd} \neq 0$. We globally define four 1-dimensional slice vectors:

$$u_{\alpha} = \frac{\lambda_{\alpha bcd}}{c_0}, \quad v_{\beta} = \frac{\lambda_{a\beta cd}}{c_0}, \quad w_{\gamma} = \frac{\lambda_{ab\gamma d}}{c_0}, \quad x_{\delta} = \frac{\lambda_{abc\delta}}{c_0}.$$

Observe that evaluating at the anchors yields $u_a = v_b = w_c = x_d = 1$.

Step 1: 2-Variable Base Slices. By fixing two slots to distinct anchor indices, Section 3 guarantees the slice is rank-1. For example, fixing slots 3 and 4 to c, d yields a slice rank-1 in α, β , factoring as $U_{\alpha} V_{\beta}$. We evaluate:

$$\frac{\lambda_{\alpha bcd} \lambda_{a\beta cd}}{\lambda_{abcd}} = \frac{(U_{\alpha} V_b)(U_a V_{\beta})}{U_a V_b} = U_{\alpha} V_{\beta} = \lambda_{\alpha\beta cd}.$$

Substituting our defined vectors extracts the base slice $\lambda_{\alpha\beta cd} = \frac{(c_0 u_{\alpha})(c_0 v_{\beta})}{c_0} = c_0 u_{\alpha} v_{\beta}$. Applying this symmetrically extracts all six robust 2-variable slices: $\lambda_{\alpha\beta cd} = c_0 u_{\alpha} v_{\beta}$, $\lambda_{ab\gamma\delta} = c_0 w_{\gamma} x_{\delta}$, $\lambda_{\alpha b\gamma d} = c_0 u_{\alpha} w_{\gamma}$, $\lambda_{a\beta c\delta} = c_0 v_{\beta} x_{\delta}$, $\lambda_{\alpha bc\delta} = c_0 u_{\alpha} x_{\delta}$, and $\lambda_{a\beta\gamma d} = c_0 v_{\beta} w_{\gamma}$.

Step 2: 3-Variable Lift. Let $\gamma \neq \delta$. At least one inequality $\delta \neq b$ or $\gamma \neq b$ must strictly hold.

- *Case 1 ($\delta \neq b$):* The slice $\lambda_{b\cdot\delta}$ securely fixes slots 2 and 4 to distinct indices, rendering it rank-1 in α, γ . Using the Step 1 base slices, we safely interpolate:

$$\lambda_{\alpha b\gamma\delta} = \frac{\lambda_{\alpha bc\delta} \lambda_{ab\gamma\delta}}{\lambda_{abc\delta}} = \frac{(c_0 u_{\alpha} x_{\delta})(c_0 w_{\gamma} x_{\delta})}{c_0 x_{\delta}} = c_0 u_{\alpha} w_{\gamma} x_{\delta}.$$

(The denominator evaluates to $\lambda_{abc\delta}$; because anchors a, b, c are mutually distinct, the tuple inherently contains at least 3 distinct elements, strictly guaranteeing it is non-identical and safely bypassing any division by zero).

- *Case 2 ($\gamma \neq b$):* The slice $\lambda_{b\cdot\gamma}$ is rank-1 in α, δ . Interpolating symmetrically yields:

$$\lambda_{\alpha b\gamma\delta} = \frac{\lambda_{\alpha b\gamma d} \lambda_{ab\gamma\delta}}{\lambda_{ab\gamma d}} = \frac{(c_0 u_{\alpha} w_{\gamma})(c_0 w_{\gamma} x_{\delta})}{c_0 w_{\gamma}} = c_0 u_{\alpha} w_{\gamma} x_{\delta}.$$

Symmetrically, executing identical logic for the first slot rigorously establishes $\lambda_{a\beta\gamma\delta} = c_0 v_{\beta} w_{\gamma} x_{\delta}$ universally for all $\gamma \neq \delta$.

Step 3: 4-Variable Lift. On the core domain $\Omega_1 = \{(\alpha, \beta, \gamma, \delta) \mid \gamma \neq \delta\}$, the slice is rank-1 in α, β . Substituting the 3-variable cross-terms from Step 2:

$$\lambda_{\alpha\beta\gamma\delta} = \frac{\lambda_{\alpha b\gamma\delta} \lambda_{a\beta\gamma\delta}}{\lambda_{ab\gamma\delta}} = \frac{(c_0 u_{\alpha} w_{\gamma} x_{\delta})(c_0 v_{\beta} w_{\gamma} x_{\delta})}{c_0 w_{\gamma} x_{\delta}} = c_0 u_{\alpha} v_{\beta} w_{\gamma} x_{\delta}.$$

(Because $a \neq b$, the anchor tuple (a, b, γ, δ) is inherently non-identical, securely preventing any division by zero).

Step 4: Set-Theoretic Exhaustive Covering. By exploiting the intrinsic permutation symmetry of \mathbf{F} , applying the mathematically identical local rank-1 argument and global lifts to the bipartitions fixing slots 1,2 and slots 1,4 establishes the exact same factorization on the alternate domains $\Omega_2 = \{(\alpha, \beta, \gamma, \delta) \mid \alpha \neq \beta\}$ and $\Omega_3 = \{(\alpha, \beta, \gamma, \delta) \mid \alpha \neq \delta\}$.

The set-theoretic union $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ mathematically covers every conceivable tuple configuration EXCEPT the state where $\gamma = \delta$, $\alpha = \beta$, and $\alpha = \delta$ occur simultaneously. Via transitivity, this uniquely isolates $\alpha = \beta = \gamma = \delta$ (the completely identical tuples). Because the hypothesis requires λ to be non-zero exactly on non-identical tuples, every valid tuple intrinsically falls into the union Ω .

By cleanly absorbing the non-zero constant c_0 into u_α (i.e., replacing u_α with $c_0 u_\alpha$), we achieve the exact outer product $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ universally over the non-identical support. Finally, because λ is strictly non-zero on this support, the 1-dimensional target vectors mathematically cannot house zero entries, securely guaranteeing $u, v, w, x \in (\mathbb{R}^*)^n$. This strictly fulfills Property 3 and decisively completes the proof.