Near-Field to Far-Field Transformations for Metasurfaces

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We wish to determine the scattering amplitude [1] of an infinite metasurface illuminated with an infinite plane wave. The scattering amplitude from such a structure would itself be infinite and so we define the normalised scattering amplitude as

$$E_{sca} = \lim_{N\to\infty} \frac{E_0^{sca}}{NE_{inc}}, \tag{1}$$

where N is the number of resonators and E_{inc} is the electric field incident on the unit cell.

We take as our starting point the Stratton-Chu formula, [3, 4] which gives the electric field at some observation point ρ (not neccessarily in the far-field) which lies within some closed surface S, which is in the near-field above the metasurface. If the vector field over the surface S is known then we can calculate the field at ρ by integrating over S:

$$\vec{\mathbf{E}}(\vec{\rho}) \;=\; \frac{-1}{4\pi} \oiint_S \left[i\omega\mu(\hat{n} \times \vec{\mathbf{H}})G + (\hat{n} \times \vec{\mathbf{E}}) \times \vec{\boldsymbol{\nabla}}G + (\hat{n} \cdot \vec{\mathbf{E}})\vec{\boldsymbol{\nabla}}G \right] \mathrm{d}S, \tag{2}$$

where

$$G = \frac{\exp(ik\rho)}{\rho}. \tag{3}$$

$\vec{\rho}$ is the vector from the observation point to the calcuclation point on S. $\hat{n}$ is the unit normal vector on S pointing *outwards*.

Stratton and Chu note in their paper that the closed surface S can be closed at infinity and we take this to be the case by making S a sphere whose center moves to infinity while its radius increases correspondingly. That is, S becomes a flat, infinite plane above the metasurface. In this case $\hat{n}$ points towards the metasurface and the observation point ρ is on the opposite side of S to the metasurface. We then need to calculate the integral in (2) over an infinite surface.

In the case of a metasurface this can be acheived by noting that, because of the Floquet boundaries of the unit cell, the integral, if S is in the near-field, can be broken up into sub-integrals - one for each cell of the metasurface. By the indistinguishability of the cells we can see that each of these sub-integrals will have the same value. In the light of (1) we can then conclude that the normalised field at ρ is

$$\vec{\mathbf{E}}_{norm}(\vec{\rho}) = \frac{-1}{4\pi E_0} \iint_{S'} \Big[i\omega\mu(\hat{n} \times \vec{\mathbf{H}})G + (\hat{n} \times \vec{\mathbf{E}}) \times \vec{\boldsymbol{\nabla}} G + (\hat{n} \cdot \vec{\mathbf{E}})\vec{\boldsymbol{\nabla}} G \Big] \mathrm{d}A, \quad (4)$$

where S' is the finite segment of S within the unit cell.

To take the observation point to the far-field we first substitute (3) into (4) and note that The $1/\rho^2$ terms vanish in the far-field and also that the far-field conditions [2] require that the field at ρ be perpendicular to the radial vector. Equation (4) thus becomes

$$\vec{\mathbf{E}}_{norm}(\vec{\rho}) = \frac{-ik}{4\pi E_0} \iint_{S'} \Big[- \eta\hat{\rho} \times \hat{\rho} \times (\hat{n} \times \vec{\mathbf{H}}) - \hat{\rho} \times (\hat{n} \times \vec{\mathbf{E}}) \Big] \frac{\exp(ik\rho)}{\rho} \mathrm{d}A,$$

where we have taken the projection of the $\vec{\mathbf{H}}$ field onto the plane perpendicular to $\hat{\rho}$.

Now, the far-field conditions also require that $\vec{\rho}$ is invariant across the resonator, except with respect to phase. This can be taken into account with the substitution

$$\frac{\exp(ik\rho)}{\rho} \rightarrow \frac{\exp(ik\rho)}{\rho} \exp(ik\hat{\rho} \cdot \vec{\rho}_s),$$

where $\vec{\rho}_s$ is the vector from the origin to the calculation point on S. Because we are in the far field we have reversed the vector $\hat{\rho}$ for convenience and taken it to be the unit vector from the origin pointing in the direction of the observation point.

By making this substitution and reversing the direction of $\hat{n}$ we find that the normalised scattering amplitude is

$$\vec{\mathbf{E}}_{sca}(\vec{\rho}) = \frac{ik}{4\pi E_0} \hat{\rho} \times \iint_{S'} \Big\{ (\hat{n} \times \vec{\mathbf{E}}) - \eta\hat{\rho} \times (\hat{n} \times \vec{\mathbf{H}}) \Big\} \exp(ik\hat{\rho} \cdot \vec{\rho}_s) \mathrm{d}A, \quad (5)$$

where $\hat{n}$ now points *away* from the metasurface.

References

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[2] Michael I. Mishchenko. Far-field approximation in electromagnetic scattering. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 100(1-3):268–276, 2006.

[3] J. A. Stratton and L. J. Chu. Diffraction theory of electromagnetic waves. *Physical Review*, 56(1):99–107, 1939.

[4] Peter Török, Peter R. T. Munro, and Emmanouil E. Kriezis. Rigorous near- to far-field transformation for vectorial diffraction calculations and its numerical implementation. *JOSA A*, 2006.