Applied math Project2

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1 Code

All implementations for the simulation of the two-patch environment discussed in this report are available on GitHub:

`https://github.com/dayforday2468/Applied_math_Project2`

This repository contains the following script:

- `two_patches_environment.py`: Implements the two-patch competition model and performs simulations across a range of environmental capacities and diffusion rates. It generates and saves time series plots for various initial conditions to analyze species behavior under different parameter settings.
- `multi_patches_environment.py`: Extends the two-patch model to a multi-patch spatial environment. By increasing the number of patches, it approximates spatial diffusion effects and simulates how species compete and spread over space. This script focuses on a fixed scenario and saves an animation showing the temporal evolution of both species.

The script is located in the `code` folder, and its output figures are saved in the `result` folder.

2 Problem

Consider the case when two species are competing in two patches environment. Let u_1, u_2 be the slow species in the first and second patch. By the same way, let v_1, v_2 be the fast species in the first and second patch. We will see whether the fast movement is better to survive.

2.1 Competition model for two patches environment

The competition model for two species u, v in two patches environment is represented as follow:

$$\dot{u_1} = u_1(1 - \frac{u_1 + v_1}{K_1}) - d_u(u_1 - u_2)$$
$$\dot{u_2} = u_2(1 - \frac{u_2 + v_2}{K_2}) - d_u(u_2 - u_1)$$
$$\dot{v_1} = v_1(1 - \frac{u_1 + v_1}{K_1}) - d_v(v_1 - v_2)$$
$$\dot{v_2} = v_2(1 - \frac{u_2 + v_2}{K_2}) - d_v(v_2 - v_1)$$

Here K_1, K_2 are environmental capacities for each patch and d_u, d_v are diffusivity of each species. The second terms in each equation represent mobility between two patches.

3 Approach

3.1 Search Range

3.1.1 Environmental Capacities

We fix the environmental capacity of the first patch as

$$K_1 = 500$$

and vary the second patch capacity to explore the effect of environmental heterogeneity:

$$K_2 \in \{250,\ 500,\ 1000\}$$

3.1.2 Diffusivity

The diffusivity of the slow species is fixed at

$$d_u = 0.01$$

to represent limited movement. The diffusivity of the fast species is varied to study the impact of mobility:

$$d_v \in \{0.01,\ 0.03, \dots,\ 0.19\}$$

3.1.3 Initial Population

Both species start with the same initial population of 3. We consider three types of initial spatial configurations:

- **same_patch**: $u_1 = v_1 = 3, \quad u_2 = v_2 = 0$
- **separate_patch_A**: $u_1 = v_2 = 3, \quad u_2 = v_1 = 0$
- **separate_patch_B**: $u_2 = v_1 = 3, \quad u_1 = v_2 = 0$

3.2 ODE Solve

Once model parameters and initial values are fixed, we numerically solve the differential equations:

$$\begin{aligned}
\dot{u_1} &= u_1(1 - \frac{u_1 + v_1}{K_1}) - d_u(u_1 - u_2) \\
\dot{u_2} &= u_2(1 - \frac{u_2 + v_2}{K_2}) - d_u(u_2 - u_1) \\
\dot{v_1} &= v_1(1 - \frac{u_1 + v_1}{K_1}) - d_v(v_1 - v_2) \\
\dot{v_2} &= v_2(1 - \frac{u_2 + v_2}{K_2}) - d_v(v_2 - v_1)
\end{aligned}$$

We integrate the equation over the time period $[0, 150]$ using the ‘solve ivp‘ function in Python. The time period is chosen to observe the convergence of the model.

4 Result

4.1 same patch, $K_2 = 250$

When $d_v = 0.01$, which is equal to d_u, we observe that the fast and slow species exhibit identical behavior. This is expected because they have the same initial population and the same diffusivity. In the first patch with environmental capacity $K_1 = 500$, both species converge to 250. In the second patch, due to the lower capacity $K_2 = 250$, they converge to 125.

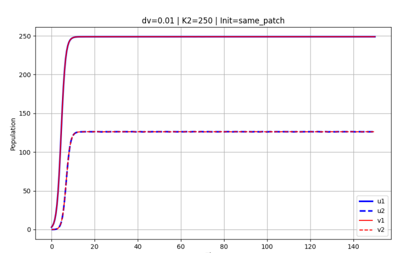


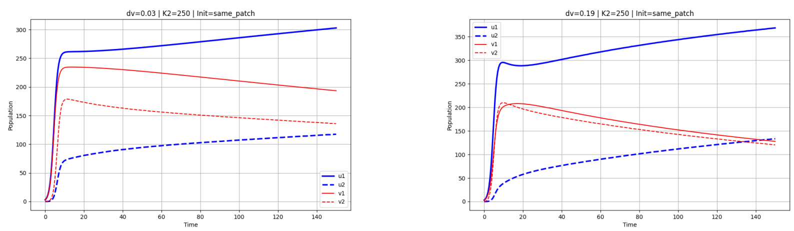
Figure 1: same patch, $K_2 = 250$, $d_v = 0.01$

Except for the special case above, the system tends to exhibit the following behaviors:

- **The slow species dominates the starting patch, which has better environment.**
- **The fast species spreads to the second patch.**

- **Growth of the fast species is limited due to poor environment.**
- **The slow species diffuses into the second patch, gradually increasing its presence.**
- **The fast species declines, while the slow species dominates both patches.**

As the diffusivity of the fast species increases, these trends become more pronounced.



(a) same patch, $K_2 = 250$, $d_v = 0.03$ (b) same patch, $K_2 = 250$, $d_v = 0.19$

Figure 2: General behavior of same patch with $K_2 = 250$

4.2 same patch, $K_2 = 500$

When $d_v = 0.01$, the diffusivity of the fast species matches that of the slow species. Combined with identical initial populations, both species behave symmetrically across the two patches. Since the environmental capacities of both patches are equal to 500, each species converges to 250 in both patches, resulting in a fully symmetric steady state.

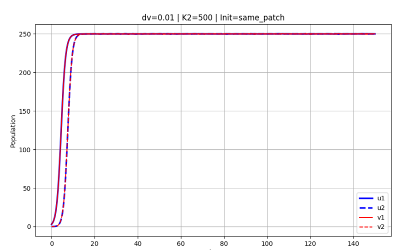


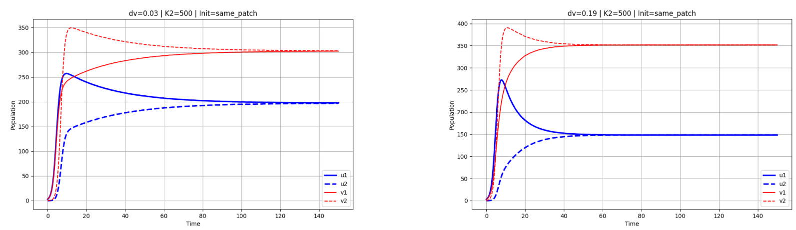
Figure 3: same patch, $K_2 = 500$, $d_v = 0.01$

Except for the special case above, the system tends to exhibit the following behaviors:

- **The slow species grows in the starting patch, but faces strong competition.**

- **The fast species escapes to the second patch, where competition is low.**
- **Faster growth in the second patch gives the fast species an early advantage**.
- **The fast species diffuses back to the first patch, reinforcing its dominance.**

As the diffusivity of the fast species increases, these trends become more pronounced.



(a) same patch, $K_2 = 500$, $d_v = 0.03$

(b) same patch, $K_2 = 500$, $d_v = 0.19$

Figure 4: General behavior of same patch with $K_2 = 500$

4.3 same patch, $K_2 = 1000$

When $d_v = 0.01$, the fast and slow species start with equal initial conditions and identical diffusivity. Due to this symmetry and equal intrinsic dynamics, both species behave identically over time.

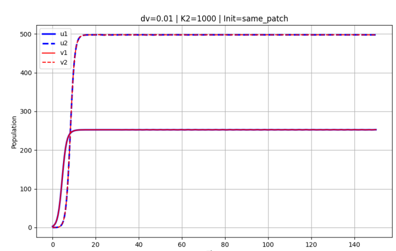


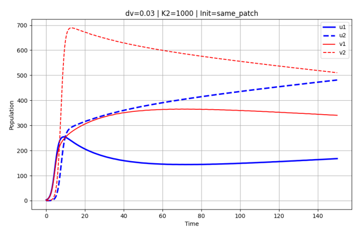
Figure 5: same patch, $K_2 = 1000$, $d_v = 0.01$

Except for the special case above, the system tends to exhibit the following behaviors:

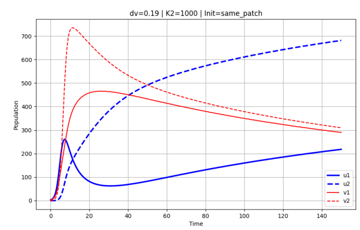
- **Rapid early growth of the fast species in the second patch**
- **Stable retention of the slow species in the first patch**

- **Late-stage decline of the fast species in the second patch due to excessive diffusion**
- **Gradual takeover of the second patch by the slow species, enabled by its stable presence**
- **Subsequent reinforcement of the slow species in the first patch via diffusion from the second patch**

As the diffusivity of the fast species increases, these trends become more pronounced.



(a) same patch, $K_2 = 1000$, $d_v = 0.03$



(b) same patch, $K_2 = 1000$, $d_v = 0.19$

Figure 6: General behavior of same patch with $K_2 = 1000$

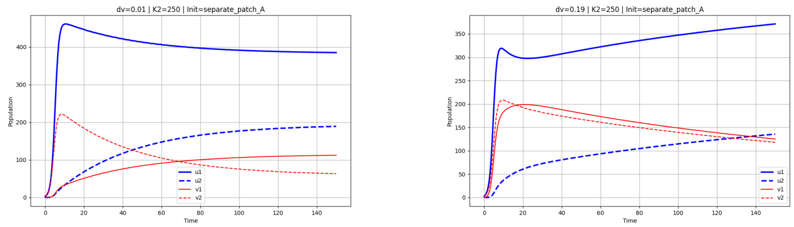
4.4 separate patch A, $K_2 = 250$

In this setting, the fast species is initially placed in the second patch, while the slow species starts in the first patch. Due to their separate starting locations and differing environment capacities, the species no longer exhibit symmetric behavior.

The system tends to exhibit the following patterns:

- **Rapid initial dominance of each species in their respective starting patches**
- **Long-term advantage of the slow species due to occupation of the better patch**
- **Gradual growth of the slow species in the second patch despite its initial scarcity, fueled by diffusion from the first patch**
- **Decline of the fast species due to inability to retain dominance in the better patch**

As the diffusivity of the fast species increases, these trends become more pronounced.



(a) separate patch A, $K_2 = 250$, $d_v = 0.01$

(b) separate patch A, $K_2 = 250$, $d_v = 0.19$

Figure 7: General behavior of separate patch A with $K_2 = 250$

4.5 separate patch A, $K_2 = 500$

When $d_v = 0.01$, the diffusivity of the fast species matches that of the slow species. Additionally, the environmental capacities of both patches are the same.

In this setting, although the species begin in different patches, their movement and interaction are symmetric. As a result, both species exhibit identical population dynamics. Each species grows rapidly in its starting patch, and due to equal diffusivity, they gradually spread into the other patch.

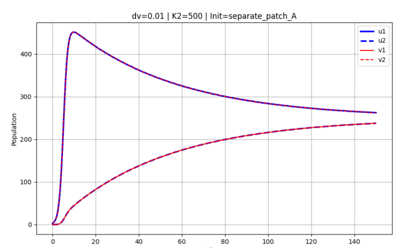
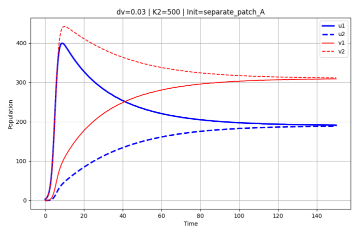


Figure 8: separate patch A, $K_2 = 500$, $d_v = 0.01$

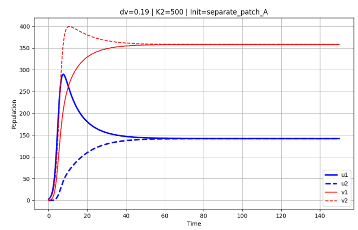
Except for the special case where $d_v = d_u$, the system generally displays the following behaviors:

- **Each species initially dominates its starting patch**
- **The fast species rapidly spreads to the other patch, quickly introducing competition in the region dominated by the slow species.**
- **The slow species spreads more slowly, and thus has limited influence in disrupting the fast species' dominance in its own patch.**
- **As a result, the fast species maintains a higher population at equilibrium**

As the diffusivity of the fast species increases, these trends become more pronounced.



(a) separate patch A, $K_2 = 500$, $d_v = 0.03$



(b) separate patch A, $K_2 = 500$, $d_v = 0.19$

Figure 9: General behavior of separate patch A with $K_2 = 500$

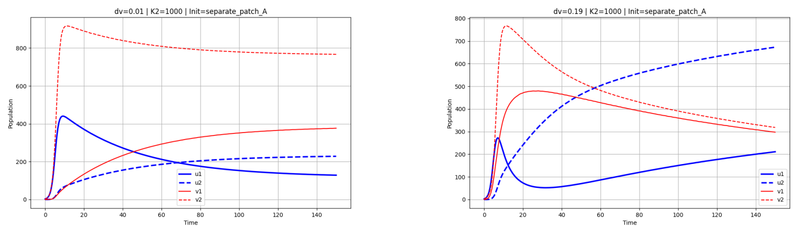
4.6 separate patch A, $K_2 = 1000$

Unlike the case with $K_2 = 500$, even when $d_v = d_u = 0.01$, the two species do not exhibit symmetric behavior. This asymmetry arises because the second patch has a significantly higher environmental capacity.

The system tends to exhibit the following patterns:

- **Initial dominance of the slow species in the poor patch, and the fast species in the rich patch**
- **Rapid growth of the fast species leads to sharp population gradient and outward diffusion**
- **Diffusion weakens the fast species in the rich patch, allowing slow species to gradually dominate**
- **Population of the slow species in the rich patch eventually surpasses that of the fast species**
- **Inflow from the rich patch boosts the slow species in the poor patch, resulting in overall dominance**
- **Fast species declines in both patches due to unsustainable diffusion-driven pressure**

As the diffusivity of the fast species increases, these trends become more pronounced.



(a) separate patch A, $K_2 = 1000$, $d_v = 0.01$

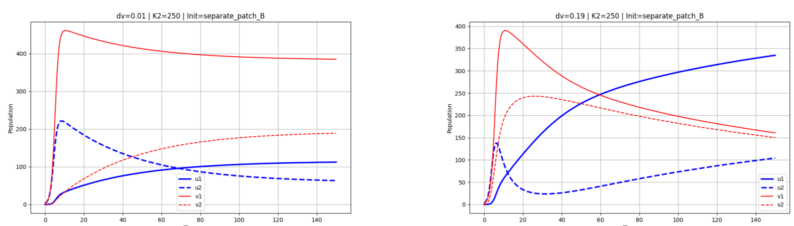
(b) separate patch A, $K_2 = 1000$, $d_v = 0.19$

Figure 10: General behavior of separate patch A with $K_2 = 1000$

4.7 separate patch B, $K_2 = 250$

There is no symmetric behavior observed in this case.

In general cases, the models exhibit the same tendencies shown at *separate patch A* $K_2 = 1000$. Only size of population and location are differ.



(a) separate patch B, $K_2 = 250$, $d_v = 0.01$

(b) separate patch B, $K_2 = 250$, $d_v = 0.19$

Figure 11: Behavior of separate patch B with $K_2 = 250$

4.8 separate patch B, $K_2 = 500$

When $d_v = 0.01$, the fast species behaves identically to the slow species, as in the case of *separate patch A*, $K_2 = 500$. Despite starting in different patches, both species show the same dynamics due to equal diffusivity and symmetric environmental capacities.

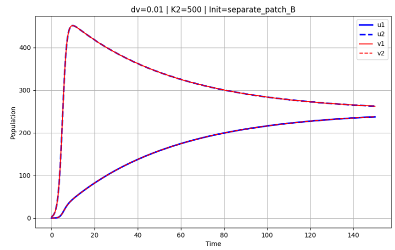
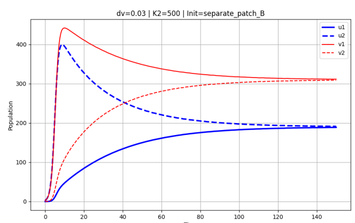
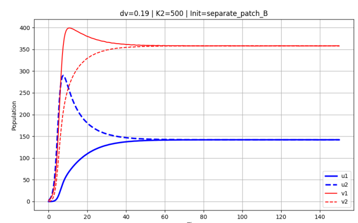


Figure 12: separate patch B, $K_2 = 500$, $d_v = 0.01$

In general cases, the dynamics are nearly identical to those observed in the *separate patch A,* $K_2 = 500$ configuration. The only difference lies in the initial locations of the two species.



(a) separate patch B, $K_2 = 500$, $d_v = 0.03$



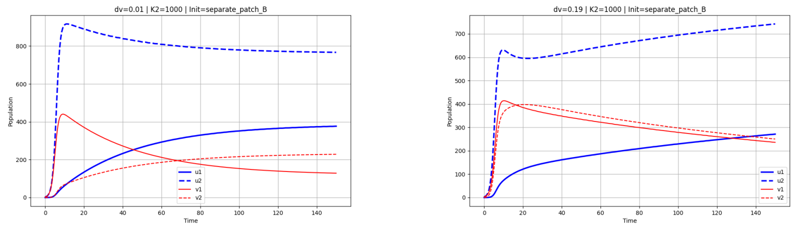
(b) separate patch B, $K_2 = 500$, $d_v = 0.19$

Figure 13: General behavior of separate patch B with $K_2 = 500$

4.9 separate patch B, $K_2 = 1000$

Unlike some previous cases, no symmetric behavior is observed for any value of d_v.

In general cases, the behaviors are almost identical to those observed in the *separate patch A* $K_2 = 250$ configuration. Only size of population and location are differ.



(a) separate patch B, $K_2 = 1000$, $d_v = 0.01$

(b) separate patch B, $K_2 = 1000$, $d_v = 0.19$

Figure 14: General behavior of separate patch B with $K_2 = 1000$

5 Conclusion

5.1 Who is the winner?

Across different initializations, we observe consistent long-term behavior despite variations in the early dynamics.

When $K_2 = 250$ or $K_2 = 1000$, the fast species gradually fades from both patches, while the slow species eventually dominates. This suggests that the slow species has a long-term advantage when there is environmental inequality between the patches.

However, when $K_2 = 500$, which is equal to K_1, the outcome changes significantly. In this balanced environment, neither species goes extinct. Instead, they appear to coexist stably. Most notably, the fast species ultimately dominates both patches which is in contrast to the other cases.

5.2 Model's Limitations

While the model provides useful insights into population dynamics, it has several limitations that should be acknowledged:

- **Overly symmetric behavior due to deterministic modeling:** In several scenarios, the model shows perfectly symmetric behavior between the species when initialized under identical conditions. This outcome is likely a result of modeling populations as continuous quantities. In reality, individual-based stochastic models would introduce randomness, making perfect symmetry unlikely even under the same conditions.
- **Simplified diffusion mechanism:** The model assumes that individuals diffuse solely based on population gradients. However, in biological systems, movement is often driven by factors like competition intensity or resource availability. Organisms are more likely to move away from highly competitive areas, not merely from crowded ones. Our model does not capture this nuance.

- **Assumption of static environmental capacities:** The model assumes that the environmental capacities K_1 and K_2 remain constant over time. In real-world ecosystems, however, factors like seasonal changes, resource depletion, or environmental disturbances may cause these capacities to fluctuate, which could significantly alter population dynamics.

6 Appendix

6.1 Coexistence Equilibrium

Assume $K_1 = K_2 = K$. From simulation results, we observe symmetric coexistence at:

$$u_1 = u_2 = u, \quad v_1 = v_2 = v$$

At this state, diffusion vanishes, and the system reduces to:

$$\dot{u} = u\left(1 - \frac{u+v}{K}\right), \quad \dot{v} = v\left(1 - \frac{u+v}{K}\right)$$

By setting the derivatives to zero, we get the following equilibrium states:

$$u = v = 0 \quad \text{or} \quad u + v = K$$

To analyze stability, we compute the Jacobian of the reduced system:

$$J = \begin{pmatrix} \partial_u \dot{u} & \partial_v \dot{u} \\ \partial_u \dot{v} & \partial_v \dot{v} \end{pmatrix} = \begin{pmatrix} 1 - \frac{2u+v}{K} & -\frac{u}{K} \\ -\frac{v}{K} & 1 - \frac{u+2v}{K} \end{pmatrix}$$

At equilibrium $u + v = K$, let $u = a$, $v = K - a$, then:

$$J = \begin{pmatrix} -\frac{a}{K} & -\frac{a}{K} \\ -\frac{K-a}{K} & -\frac{K-a}{K} \end{pmatrix}$$

The eigenvalues of J are 0 and -1, which shows the equilibrium is **semi-stable**: the system is attracted to the line $u + v = K$, but the final point depends on initial conditions.

6.2 Extinction Equilibrium

When $K_1 \neq K_2$, simulations converge to a state where fast species goes extinct. That is:

$$v_1 = v_2 = 0, \quad u_1, u_2 > 0$$

Assume $K_1 > K_2$. Then, at equilibrium, we typically have $u_1 < K_1$ and $u_2 > K_2$. To assess the stability of this extinction state, we consider a small perturbation in the fast species population: let $v_1 = \epsilon$ and $v_2 = 0$ with $\epsilon > 0$

small. Then, $\dot{v}_1 = \epsilon\left(1 - \frac{u_1+\epsilon}{K_1} - d_v\right)$. The sign of this derivative depends on u_1, ϵ, and d_v. If u_1 is not sufficiently close to K_1 and d_v is small, $\dot{v}_1$ can be positive for small ϵ, meaning the fast species population may initially grow. By these cases, we can say that the extinction equilibrium is **unstable** and the system may instead approach a state of very low-level coexistence.

6.3 Multi-patch competition model and its connection to continuous space

We consider the competition dynamics of two species u and v in a spatially structured environment consisting of N patches. The population in each patch evolves according to the following system:

$$
\begin{aligned}
\dot{u}_1 &= u_1\left(1 - \frac{u_1+v_1}{K_1}\right) - d_u(u_1 - u_2)\\
\dot{u}_i &= u_i\left(1 - \frac{u_i+v_i}{K_i}\right) + d_u(u_{i-1} - 2u_i + u_{i+1}) \quad (i = 2, \dots, N-1)\\
\dot{u}_N &= u_N\left(1 - \frac{u_N+v_N}{K_N}\right) - d_u(u_N - u_{N-1})\\
\dot{v}_1 &= v_1\left(1 - \frac{u_1+v_1}{K_1}\right) - d_v(v_1 - v_2)\\
\dot{v}_i &= v_i\left(1 - \frac{u_i+v_i}{K_i}\right) + d_v(v_{i-1} - 2v_i + v_{i+1}) \quad (i = 2, \dots, N-1)\\
\dot{v}_N &= v_N\left(1 - \frac{u_N+v_N}{K_N}\right) - d_v(v_N - v_{N-1})
\end{aligned}
$$

Here, K_i denotes the environmental capacity of the i-th patch, and d_u, d_v represent the diffusivity of each species. The diffusion terms describe movement between adjacent patches, and Neumann (zero-flux) boundary conditions are assumed at both ends.

To relate this model to a continuous spatial system, we interpret the patch index i as corresponding to a spatial location $x_i = i \cdot \Delta x$, where $\Delta x = L/(N-1)$ is the patch spacing over domain length L. We assume $u_i(t) \approx u(x_i, t)$, and apply the central difference approximation for the second spatial derivative:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t) \approx \frac{u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)}{(\Delta x)^2}.$$

Substituting into the discrete model, the equation becomes:

$$\dot{u}_i = u_i\left(1 - \frac{u_i+v_i}{K_i}\right) + d_u \cdot \frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2}.$$

Taking the limit as $\Delta x \to 0$, the model converges to the continuous diffusion competition system:

$$\dot{u}_i = d_u \frac{\partial^2 u}{\partial x^2}(x_i, t) + u_i(1 - \frac{u_i + v_i}{K_i})$$

Thus, the multi-patch competition model can be interpreted as a finite-difference approximation of a continuous-space diffusion competition model, where the number of patches and spatial resolution determine the accuracy of the approximation.