Applied math Project2

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1 Code

All implementations for the simulation of the two-patch environment discussed in this report are available on GitHub:

`https://github.com/dayforday2468/Applied_math_Project2`

This repository contains the following script:

- `two_patches_environment.py`: Implements the two-patch competition model and performs simulations across a range of environmental capacities and diffusion rates. It generates and saves time series plots for various initial conditions to analyze species behavior under different parameter settings.
- `multi_patches_environment.py`: Extends the two-patch model to a multi-patch spatial environment. By increasing the number of patches, it approximates spatial diffusion effects and simulates how species compete and spread over space. This script focuses on a fixed scenario and saves an animation showing the temporal evolution of both species.

The script is located in the `code` folder, and its output figures are saved in the `result` folder.

2 Problem

Consider the case when two species are competing in two patches environment. Let u_1, u_2 be the slow species in the first and second patch. By the same way, let v_1, v_2 be the fast species in the first and second patch. We will see whether the fast movement is better to survive.

2.1 Competition model for two patches environment

The competition model for two species u, v in two patches environment is represented as follow:

$$\dot{u_1} = u_1(1 - \frac{u_1 + v_1}{K_1}) - d_u(u_1 - u_2)$$
$$\dot{u_2} = u_2(1 - \frac{u_2 + v_2}{K_2}) - d_u(u_2 - u_1)$$
$$\dot{v_1} = v_1(1 - \frac{u_1 + v_1}{K_1}) - d_v(v_1 - v_2)$$
$$\dot{v_2} = v_2(1 - \frac{u_2 + v_2}{K_2}) - d_v(v_2 - v_1)$$

Here K_1, K_2 are environmental capacities for each patch and d_u, d_v are diffusivity of each species. The second terms in each equation represent mobility between two patches.

3 Approach

3.1 Search Range

3.1.1 Environmental Capacities

We fix the environmental capacity of the first patch as

$$K_1 = 500$$

and vary the second patch capacity to explore the effect of environmental heterogeneity:

$$K_2 \in \{250,\ 500,\ 1000\}$$

3.1.2 Diffusivity

The diffusivity of the slow species is fixed at

$$d_u = 0.01$$

to represent limited movement. The diffusivity of the fast species is varied to study the impact of mobility:

$$d_v \in \{0.01,\ 0.03, \ldots,\ 0.19\}$$

3.1.3 Initial Population

Both species start with the same initial population of 3. We consider three types of initial spatial configurations:

- **same_patch**: $u_1 = v_1 = 3, \quad u_2 = v_2 = 0$
- **separate_patch_A**: $u_1 = v_2 = 3, \quad u_2 = v_1 = 0$
- **separate_patch_B**: $u_2 = v_1 = 3, \quad u_1 = v_2 = 0$

3.2 ODE Solve

Once model parameters and initial values are fixed, we numerically solve the differential equations:

$$
\begin{aligned}
\dot{u_1} &= u_1(1 - \frac{u_1 + v_1}{K_1}) - d_u(u_1 - u_2) \\
\dot{u_2} &= u_2(1 - \frac{u_2 + v_2}{K_2}) - d_u(u_2 - u_1) \\
\dot{v_1} &= v_1(1 - \frac{u_1 + v_1}{K_1}) - d_v(v_1 - v_2) \\
\dot{v_2} &= v_2(1 - \frac{u_2 + v_2}{K_2}) - d_v(v_2 - v_1)
\end{aligned}
$$

We integrate the equation over the time period $[0, 150]$ using the ‘solve ivp‘ function in Python. The time period is chosen to observe the convergence of the model.

4 Result

4.1 Symmetric Behavior

In certain simulations, we observed symmetric dynamics between the two species. Symmetry was consistently observed under the following conditions:

- Diffusion rates were equal: $d_u = d_v = 0.01$
- Initial distributions were either the same for both species or mirrored

Table 1 summarizes the simulation conditions where such symmetry emerged.

Initial Condition	K_2	d_v
same patch	250	0.01
same patch	500	0.01
same patch	1000	0.01
separate patch A	500	0.01
separate patch B	500	0.01

Table 1: Simulation conditions under which symmetric behavior was observed.

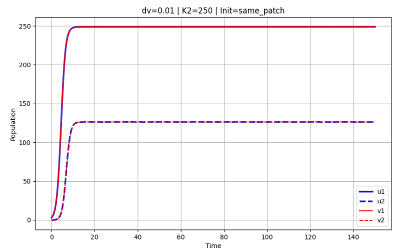
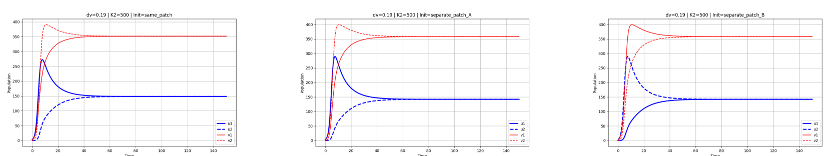


Figure 1: Symmetric dynamics when $K_2 = 250$ and $d_v = 0.01$ with same patch condition.

4.2 When $K_1 = K_2$

When the environmental capacities are homogeneous, the competition outcome is determined by the difference in diffusion rates. In all such cases, the fast species becomes dominant across the spatial domain.

Notably, the slow species does not go extinct, and the two species coexist with the fast species in a clearly dominant position.



(a) Same patch (b) Separate patch A (c) Separate patch B

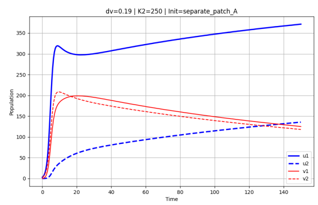
Figure 2: Simulation results when $K_1 = K_2 = 500$ and $d_v = 0.19$, under three different initial conditions.

Figure 2 shows the simulation results when equal environmental capacities. To see their tendencies clearly, we pick $d_v = 0.19$ which is the highest diffusivity.

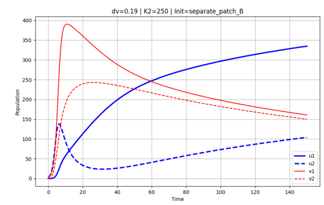
4.3 When $K_1 \neq K_2$

When the environmental capacities are heterogeneous, the slow species tends to dominate regardless of the initial condition. The fast species, due to its high mobility, fails to maintain its population in the favorable region and eventually declines.

In most cases, the slow species fully occupies the entire space, while the fast species goes extinct or survives only at a minimal level. This pattern consistently appeared across all initial conditions.



(a) Consistent dominance



(b) Reversed dominance

Figure 3: Simulation results when $K_1 \neq K_2$ and $d_v = 0.19$.

Figure 3 shows the simulation results under heterogeneous environments with $d_v = 0.19$, the highest diffusion rate tested. The left figure represents a case where the fast species is disadvantaged from the beginning, while the right figure illustrates a case where the fast species initially dominates but is eventually overtaken by the slow species.

5 Conclusion

5.1 Who is the winner?

Across different initializations, we observe consistent long-term behavior despite variations in early dynamics.

When $K_2 = 250$ or $K_2 = 1000$, introducing environmental heterogeneity, the fast species gradually fades from both patches while the slow species eventually dominates. This suggests that the slow species gains a long-term advantage under spatially uneven conditions. In contrast, when $K_2 = 500$, which matches K_1, both species coexist, but the fast species ultimately dominates.

This contrast reveals two distinct mechanisms of competitive advantage:

- **When $K_1 = K_2$:** The environment is uniform, so the key to success is **rapidly occupying space and exploiting resources early**. The fast species spreads quickly across the domain and suppresses the slower competitor through early access.

- **When $K_1 \neq K_2$:** The environment is heterogeneous, making **residency in high-quality regions more important than speed**. The slow species, with its lower diffusion rate, stays longer in the more favorable patch and outcompetes the fast species locally. Once dominant there, it **logistically grows and later expands into less favorable areas**, maintaining a competitive edge despite slower movement.

This analysis highlights the trade-off between mobility and local competitiveness: **high diffusion is beneficial in uniform environments**, while **low diffusion is advantageous in heterogeneous environments**.

5.2 Model's Limitations

While the model provides useful insights into population dynamics, it has several limitations that should be acknowledged:

- **Overly symmetric behavior due to deterministic modeling:** In several scenarios, the model shows perfectly symmetric behavior between the species when initialized under identical conditions. This outcome is likely a result of modeling populations as continuous quantities. In reality, individual-based stochastic models would introduce randomness, making perfect symmetry unlikely even under the same conditions.
- **Simplified diffusion mechanism:** The model assumes that individuals diffuse solely based on population gradients. However, in biological systems, movement is often driven by factors like competition intensity or resource availability. Organisms are more likely to move away from highly competitive areas, not merely from crowded ones. Our model does not capture this nuance.
- **Assumption of static environmental capacities:** The model assumes that the environmental capacities K_1 and K_2 remain constant over time. In real-world ecosystems, however, factors like seasonal changes, resource depletion, or environmental disturbances may cause these capacities to fluctuate, which could significantly alter population dynamics.

6 Appendix

6.1 Coexistence Equilibrium

Assume $K_1 = K_2 = K$. From simulation results, we observe symmetric coexistence at:

$$u_1 = u_2 = u, \quad v_1 = v_2 = v$$

At this state, diffusion vanishes, and the system reduces to:

$$\dot{u} = u\left(1 - \frac{u+v}{K}\right), \quad \dot{v} = v\left(1 - \frac{u+v}{K}\right)$$

By setting the derivatives to zero, we get the following equilibrium states:

$$u = v = 0 \quad \text{or} \quad u + v = K$$

To analyze stability, we compute the Jacobian of the reduced system:

$$J = \begin{pmatrix} \partial_u \dot{u} & \partial_v \dot{u} \\ \partial_u \dot{v} & \partial_v \dot{v} \end{pmatrix} = \begin{pmatrix} 1 - \frac{2u+v}{K} & -\frac{u}{K} \\ -\frac{v}{K} & 1 - \frac{u+2v}{K} \end{pmatrix}$$

At equilibrium $u + v = K$, let $u = a$, $v = K - a$, then:

$$J = \begin{pmatrix} -\frac{a}{K} & -\frac{a}{K} \\ -\frac{K-a}{K} & -\frac{K-a}{K} \end{pmatrix}$$

The eigenvalues of J are 0 and -1, which shows the equilibrium is **semi-stable**: the system is attracted to the line $u + v = K$, but the final point depends on initial conditions.

The below plot shows the gradient of u, v for each u, v values.

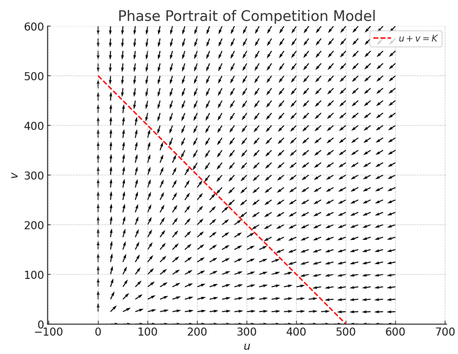


Figure 4: Phase plot of the reduced system

We observe that the vector field consistently points toward the line $u+v = K$, indicating that the system is attracted to this manifold. This confirms that small perturbations in u or v will eventually decay and the system will settle back onto the equilibrium line.

6.2 Extinction Equilibrium

We consider the extinction equilibrium where the fast species vanishes:

$$v_1 = v_2 = 0, \quad u_1, u_2 > 0$$

According to the theorem in Dockery et al. [1], when two species differ only in their diffusion rates, the equilibrium in which only the slow-diffusing species survives is **globally asymptotically stable**. This result holds under the assumptions that the environment is spatially heterogeneous and that there is no mutation between species.

Therefore, rather than analyzing the Jacobian directly, we cite this result to conclude that the extinction equilibrium is **stable**.

> *"For two phenotypes with different diffusion rates, the equilibrium with only the slow diffuser is globally asymptotically stable. All other equilibria are unstable."* — Dockery et al. [1]

This theoretical guarantee fully supports the extinction behaviors observed in our simulations.

6.3 Multi-patch competition model and its connection to continuous space

We consider the competition dynamics of two species u and v in a spatially structured environment consisting of N patches. The population in each patch evolves according to the following system:

$$
\begin{aligned}
\dot{u}_1 &= u_1\left(1-\frac{u_1+v_1}{K_1}\right) - d_u(u_1-u_2) \\
\dot{u}_i &= u_i\left(1-\frac{u_i+v_i}{K_i}\right) + d_u(u_{i-1}-2u_i+u_{i+1}) \quad (i=2,\dots,N-1) \\
\dot{u}_N &= u_N\left(1-\frac{u_N+v_N}{K_N}\right) - d_u(u_N-u_{N-1}) \\
\dot{v}_1 &= v_1\left(1-\frac{u_1+v_1}{K_1}\right) - d_v(v_1-v_2) \\
\dot{v}_i &= v_i\left(1-\frac{u_i+v_i}{K_i}\right) + d_v(v_{i-1}-2v_i+v_{i+1}) \quad (i=2,\dots,N-1) \\
\dot{v}_N &= v_N\left(1-\frac{u_N+v_N}{K_N}\right) - d_v(v_N-v_{N-1})
\end{aligned}
$$

Here, K_i denotes the environmental capacity of the i-th patch, and d_u, d_v represent the diffusivity of each species. The diffusion terms describe movement between adjacent patches, and Neumann (zero-flux) boundary conditions are assumed at both ends.

To relate this model to a continuous spatial system, we interpret the patch index i as corresponding to a spatial location $x_i = i\cdot\Delta x$, where $\Delta x = L/(N-1)$ is the patch spacing over domain length L. We assume $u_i(t) \approx u(x_i,t)$, and apply the central difference approximation for the second spatial derivative:

$$
\frac{\partial^2 u}{\partial x^2}(x_i,t) \approx \frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{(\Delta x)^2}.
$$

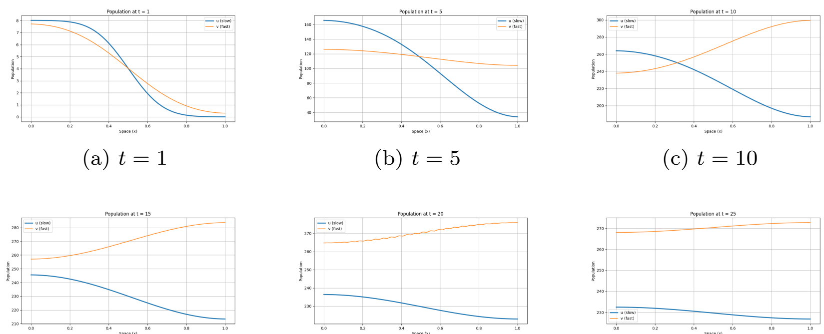
Substituting into the discrete model, the equation becomes:

$$
\dot{u}_i = u_i\left(1-\frac{u_i+v_i}{K_i}\right) + d_u\cdot\frac{u_{i-1}-2u_i+u_{i+1}}{(\Delta x)^2}.
$$

Taking the limit as $\Delta x \to 0$, the model converges to the continuous diffusion competition system:

$$
\dot{u}_i = d_u\frac{\partial^2 u}{\partial x^2}(x_i,t) + u_i\left(1-\frac{u_i+v_i}{K_i}\right)
$$

Thus, the multi-patch competition model can be interpreted as a finite-difference approximation of a continuous-space diffusion competition model, where the number of patches and spatial resolution determine the accuracy of the approximation.



(a) $t = 1$ (b) $t = 5$ (c) $t = 10$

(d) $t = 15$ (e) $t = 20$ (f) $t = 25$

Figure 5: Population distributions $u(x, t)$ and $v(x, t)$ at different time points

Figure 5 shows the spatiotemporal population dynamics of the two species under the multi-patch model with $K_1 = K_2 = 500$, $d_u = 0.01$, and $d_v = 0.03$ across 50 spatial locations. Despite being evaluated on discrete patches, the resulting profiles appear nearly continuous, indicating the validity of using this discrete model as an approximation of a continuous spatial system.

Initially, both species expand from their initial positions. Over time, we observe that the fast species v gradually spreads across the domain more quickly than the slow species u. As time progresses, the fast species occupies more of the space and attains higher density than the slow species.

This behavior aligns with observations from the two-patch model, where co-existence occurs under symmetric environmental conditions, and the fast species tends to dominate in the long run due to its mobility. The multi-patch simulation confirms this tendency in a more spatially resolved context.

References

[1] John Dockery et al. “Evolution of slow dispersal rates: a reaction diffusion model”. In: *Journal of Mathematical Biology* 37.1 (1998), pp. 61–83.