

1 Probability Potpourri

1. We want to prove the covariance matrix Σ is always positive semi-definite, i.e. $a^T \Sigma a \geq 0$ for all $n \times 1$ vectors a . Given $\Sigma = E[(Z - \mu)(Z - \mu)^T]$, set the following equation:

$$\begin{aligned} a^T E[(Z - \mu)(Z - \mu)^T] a &\geq 0 \\ E[a^T (Z - \mu)(Z - \mu)^T a] &\geq 0 \end{aligned}$$

by linearity of expectation

$$\begin{aligned} E[((Z - \mu)a)^T ((Z - \mu)a)] &\geq 0 \\ E[|(Z - \mu)a|^2] &\geq 0 \end{aligned}$$

2. Let W represent the event that there is a gust of wind and let H represent the event that the archer hits her target. We know that $P(H|W) = 0.4$, $P(H|W^C) = 0.7$, and $P(W) = 0.3$.

1. $P(\text{there is gust of wind and she hits target}) = P(W \cap H)$
 $= P(W) * P(H|W)$
 $= 0.3 * 0.4$
 $= 0.12$
2. $P(\text{hits the target on the first shot})$
 $= P(H)$
 $= P(H \cap W) + P(H \cap W^C)$
 $= 0.12 + P(W) * P(H|W^C)$
 $= 0.12 + 0.7 * 0.7$
 $= 0.62$
3. $P(\text{hits the target exactly once in two shots})$
 $= P(H_1 \cap H_2^C) + P(H_1^C \cap H_2)$
 $= (0.62)(1 - 0.62) + (1 - 0.62)(0.62)$
 $= 0.4712$
4. $P(\text{there was no gust of wind on an occasion when she missed})$
 $= P(W^C | H^C)$
 $= \frac{P(W^C \cap H^C)}{P(H^C)}$
 $= \frac{P(W^C)P(H^C|W^C)}{P(H^C)}$
 $= \frac{0.7 * 0.3}{1 - 0.62}$
 $= 0.55$

3. Let Y represent the score of a single strike, or $Y = g(X = x)$ such that

$$g(X = x) = \begin{cases} 4, & \text{if } x \leq \frac{1}{\sqrt{3}} \\ 3, & \text{if } \frac{1}{\sqrt{3}} < x \leq 1 \\ 2, & \text{if } 1 < x \leq \sqrt{3} \\ 0, & \text{otherwise.} \end{cases}$$

Find expectation $E[Y] = \int yf(y)dy = \int g(x)f(x)dx = \int_0^\infty g(x)\frac{2}{\pi(1+x^2)}dx$

$$\begin{aligned}
 &= 4 * \frac{2}{\pi} \arctan(x) \Big|_0^{\frac{1}{\sqrt{3}}} + 3 * \frac{2}{\pi} \arctan(x) \Big|_{\frac{1}{\sqrt{3}}}^1 + 2 * \frac{2}{\pi} \arctan(x) \Big|_1^{\sqrt{3}} \\
 &= 4 * \frac{2}{\pi} \left(\frac{\pi}{6} - 0\right) + 3 * \frac{2}{\pi} \left(\frac{\pi}{4} - \frac{\pi}{6}\right) + 2 * \frac{2}{\pi} \left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
 &= 2.167
 \end{aligned}$$

2 Properties of Gaussians

1. We want to prove that $E[e^{\lambda X}] = e^{\sigma^2 \lambda^2 / 2}$