1 Multivariate Gaussians: A Review

We want to determine whether $Z_1 = X_1$ and $Z_2 = X_2$ are jointly Gaussian and calculate their JG covariance matrix.
Because X_1 and X_2 are each a standard normal RV, we already know that Z_1 and Z_2 are each marginally Gaussian. Because X_1 and X_2 are independent, $Z_1|Z_2 = Z_1$, which is Gaussian/standard normal. By symmetry, $Z_2|Z_1$ is also Gaussian/standard normal. Thus, we have satisfied both conditions that allow for $Z \in \mathbb{R}^2$ to be jointly Gaussian. Let us now calculate the covariance matrix.

$$\begin{aligned}\Sigma_Z &= \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\text{since } X_1 \text{ and } X_2 \text{ are independent and standard normal.}\end{aligned}$$

We want to determine whether $Z_1 = X_1$ and $Z_2 = X_1 + X_2$ are jointly Gaussian and calculate their JG covariance matrix.
We can use the second characterization - a JG RV $Z \in \mathbb{R}^2$ can be written as $Z = AX$ where $A \in \mathbb{R}^{2\times 2}$ is a transition matrix and $X \in \mathbb{R}^2$ is a collection of i.i.d. standard normal RVs. We can decompose $Z = (X_1\ X_1 + X_2)^T$ into

$$Z = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Let us now calculate the covariance matrix.

$$\begin{aligned}\Sigma_Z &= \begin{bmatrix} \Sigma_{X_1 X_1} & \Sigma_{X_1\ X_1+X_2} \\ \Sigma_{X_1+X_2\ X_1} & \Sigma_{X_1+X_2\ X_1+X_2} \end{bmatrix} \\ \Sigma_{X_1\ X_1+X_2} &= E[(X_1 - E(X_1))(X_1 + X_2 - E(X_1 + X_2))] \\ &= E[(X_1)(X_1 + X_2)] \\ &= E[X_1^2] + E[X_1 X_2] \\ &= \sigma_{X_1}^2 + E[X_1]^2 + E[X_1]E[X_2] \text{ because } X_1 \text{ and } X_2 \text{ are independent.} \\ &= 1 + 0^2 + 1 \cdot 1 \\ \Sigma_{X_1\ X_1+X_2} &= 2 \\ \Sigma_Z &= \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\end{aligned}$$

We want to determine whether $Z_1 = X_1$ and $Z_2 = -X_1$ are jointly Gaussian and calculate their JG covariance matrix. Again, we can use the second characterization - a JG RV $Z \in \mathbb{R}^2$ can be written as $Z = AX$ where $A \in \mathbb{R}^{2\times 2}$ is a transition matrix and $X \in \mathbb{R}^2$ is a collection of i.i.d. standard normal RVs. We can decompose $Z = (X_1\ X_1 + X_2)^T$ into

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_1 \end{pmatrix}$$