EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 4

This homework is due September 24, 2021, at 23:59.
Self-grades are due September 27, 2021, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit each file to its respective assignment on Gradescope.

1. Reading Assignment

For this homework, please review Note 5 and read Notes 6, 7. The notes 5 and 6 provide an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps, and matrix inversion. Note 7 provides an introduction to vector spaces. You are always welcome and encouraged to read beyond this as well. Note 8 discusses column spaces and nullspaces, so it might be useful to read that for this homework as well.

You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

Solution: Each column in the state transition matrix must sum to one.

2. Feedback on your study groups

Please help us understand how your study groups are going! Fill out the following survey to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

https://forms.gle/xhPwbFZzMWN1dBrn7

To get full credit for this question you must (1) fill out the survey (it will record your email) and (2) indicate in your homework submission that you filled out the survey.

3. Easing into Proofs

(Contributors: Urmita Sikder, Gireeja Ranade)

Learning Objectives: *This is an opportunity to practice your proof development skills.*

(a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.** Let us use the structure delineated in **Note 4** to approach this proof. This problem has 4 sub-parts and the following is a chart showing the sequential steps we are going to take to approach this proof.

In a text book you might see the steps in a proof written out in the order in the middle column of the table. But when you are building a proof you usually want to go in another order — this is the order of the subparts in this problem.

Proof steps		Corresponding problem sub-parts
1	Write what is known	Sub-part (i)
2	Manipulate what is known	Sub-part (iii)
3	Connecting it up	Sub-part (iv)
4	What is to be shown	Sub-part (ii)

(i) **First Step: write what you know**
Think about the *information we already know* from the problem statement. We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions. Infinitely many solutions are hard to work with, but perhaps we can simplify to something that we can work with. If the system has infinite number of solutions, it must have at least ___ distinct solutions (Fill in the blank).
So let us assume that $\vec{u}$ and $\vec{v}$ are two different vectors, both of which are solutions to $\mathbf{A}\vec{x} = \vec{b}$.
Express the sentence above in a mathematical form (Just writing the equations will suffice; no need to take do further mathematical manipulation).
Solution: If the system has infinite number of solutions, it must have at least two distinct solutions.
(Self-grading comment: Do not reduce points if you forgot to write the answer to the blank in your solutions)
$\vec{u}$ and $\vec{v}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{b}, \quad \mathbf{A}\vec{v} = \vec{b}. \tag{1}$$

$$\vec{u} \neq \vec{v}. \tag{2}$$

(ii) **What we want to show:**
Now consider *what we need to show*. We have to show that the columns of $\mathbf{A}$ are linearly dependent. Let us assume that $\mathbf{A}$ has columns $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.
Solution: According to the definition of linear dependence:

$$\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n = \vec{0}. \tag{3}$$

where at least one α_i is not equal to zero.

(iii) **Manipulating what we know:**
Now let us try to start from the **First step: equations from (i)**, make mathematically logical steps and reach the **What we want to show: equations from (ii)**. Since your answer to (ii) is expressed in terms of the column vectors of $\mathbf{A}$, let us try to express the mathematical equations from (i), in terms of the the column vectors too. For example, we can write

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{b} \\ \implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} &= \vec{b} \\ \implies x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n &= \vec{b} \end{aligned}$$

Notice that $x_1,\ldots x_n$ etc are scalars. Now use your answer to part (i) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$. Note that this is proceeding slightly differently from how we did this proof in lecture. This is fine — there are often many correct ways to do a proof.

Solution:

$$\begin{aligned}
\mathbf{A}\vec{u} &= \vec{b} \\
\implies \begin{bmatrix}\vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_n\end{bmatrix}\begin{bmatrix}u_1 \\ u_2 \\ \ldots \\ u_n\end{bmatrix} &= \vec{b} \\
\implies u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n &= \vec{b}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}\vec{v} &= \vec{b} \\
\implies \begin{bmatrix}\vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_n\end{bmatrix}\begin{bmatrix}v_1 \\ v_2 \\ \ldots \\ v_n\end{bmatrix} &= \vec{b} \\
\implies v_1\vec{c}_1 + v_2\vec{c}_2 + \ldots + v_n\vec{c}_n &= \vec{b}
\end{aligned}$$

(iv) **Connecting it up:**

Now think about how you can mathematically manipulate your answer from part (iii) (**Manipulating what we know**) to **match the pattern** of your answer from part (ii) (**What we want to show**).

Solution: Subtracting the second equation from the first equation in part (iii), we have

$$u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n - v_1\vec{c}_1 - v_2\vec{c}_2 - \ldots - v_n\vec{c}_n = \vec{b} - \vec{b} \tag{4}$$

$$\implies (u_1 - v_1)\vec{c}_1 + (u_2 - v_2)\vec{c}_2 + \ldots + (u_n - v_n)\vec{c}_n = \vec{0} \tag{5}$$

Let $\alpha_1 = u_1 - v_1$, ..., and $\alpha_n = u_n - v_n$, i.e. $\vec{\alpha} = \vec{u} - \vec{v}$. Here, at least one α_i is not equal to zero since $\vec{u} \neq \vec{v}$. Hence the mathematical expression from part (ii) (the **Final Step**) is satisfied, i.e. the proof is complete!

(b) **[PRACTICE]** Now try this proof on your own. Similar proofs will also be covered in your discussion section 3A. Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \operatorname{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In order to show this, you have to proof the two following statements:

- If a vector $\vec{q}$ belongs in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\operatorname{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belong is $\operatorname{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to proof the problem statement from both directions. Now use the method developed in part (a) to proof these statements.

Solution:

We start with $\vec{q} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Choosing some scalars a_i, we can write this statement out in mathematical form:

$$\begin{aligned}
\vec{q} &= a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \\
&= a_1\vec{v}_1 + a_1\vec{v}_2 + -a_1\vec{v}_2 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \\
&= a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n
\end{aligned}$$

Since a_1, $(-a_1 + a_2)$ etc are all scalars in the above equation, we can decide that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have finished proving the first statement. So $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

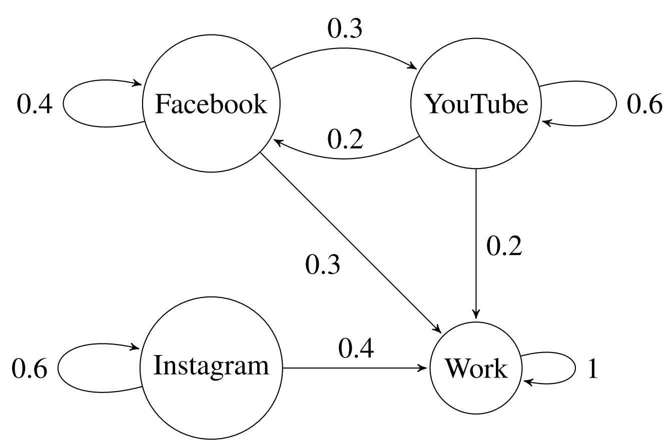
$$\begin{aligned}\vec{r} &= b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n \\ &= b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.\end{aligned}$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

4. Social Media

Learning Objective: *Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.*

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.



(a) Let us define $x_F[n]$ as the number of students on Facebook at timestep n, $x_Y[n]$ as the number of students on YouTube at timestep n, $x_I[n]$ as the number of students on Instagram at timestep n, and $x_W[n]$ as the number of students working at timestep n. Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_F[n] \\ x_Y[n] \\ x_I[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix.

Solution:

Let us explicitly write the equations that we can then use to determine the state transition matrix.

$x_F[n+1] = 0.4x_F[n] + 0.2x_Y[n]$
$x_Y[n+1] = 0.3x_F[n] + 0.6x_Y[n]$

$x_I[n+1] = 0.6x_I[n]$
$x_W[n+1] = 0.3x_F[n] + 0.2x_Y[n] + 0.4x_I[n] + x_W[n]$
Let $\vec{x}[n] = \begin{bmatrix} x_F[n] \\ x_Y[n] \\ x_I[n] \\ x_W[n] \end{bmatrix}$.
We can now solve for the state transition matrix A such that:
$\vec{x}[n+1] = A\vec{x}[n]$.
A is therefore equal to:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix}$$

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Facebook, 450 on YouTube, 200 on Instagram, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

Solution:

In order to calculate the state vector at the next timestep, we can use the equation $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$. Substituting the values for A and $\vec{x}[n]$, we get the following:

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 700 \\ 450 \\ 200 \\ 150 \end{bmatrix} = \begin{bmatrix} 370 \\ 480 \\ 120 \\ 530 \end{bmatrix}$$

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

Solution:

Since each column's sum is equal to 1, the system is conservative. This means that we aren't losing students after each time step.

5. Mechanical Inverses

Learning Objectives: *Matrices represent linear transformations, and their inverses represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following values of matrix A:

i Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii **For parts (a)-(d)**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ transforms an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

For example, if $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x \\ y \end{bmatrix}$ across the x axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii **For parts (a)-(d)**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x \\ y \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

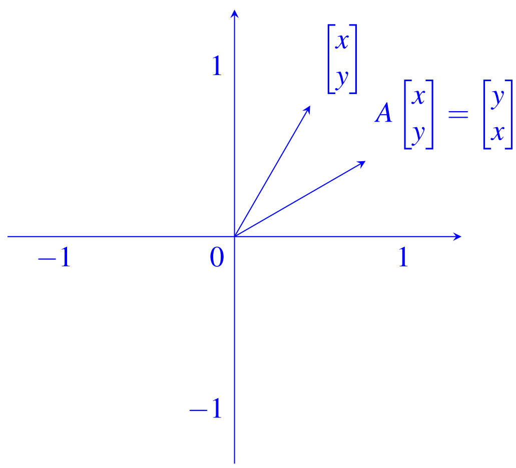
(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \text{swap } \mathrm{R}_1, \mathrm{R}_2$$

The inverse does exist.

$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



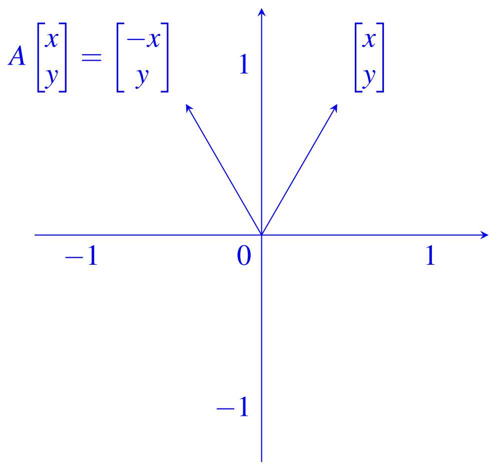
The original matrix $\mathbf{A}$ flips the x and y components of the vector. Any correct equivalent sequence of operations (such as reflecting the vector across the $x = y$ line) warrants full credit. Notice how the inverse does the exact same thing–that is, it switches the x and y components of the vector it's applied to. This makes sense–switching x and y twice on a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ gives us the same vector $\begin{bmatrix} x \\ y \end{bmatrix}$. So the transformation done by $\mathbf{A}$ is reversible.

(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] R_1 \leftarrow -R_1$$

The inverse does exist.
$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



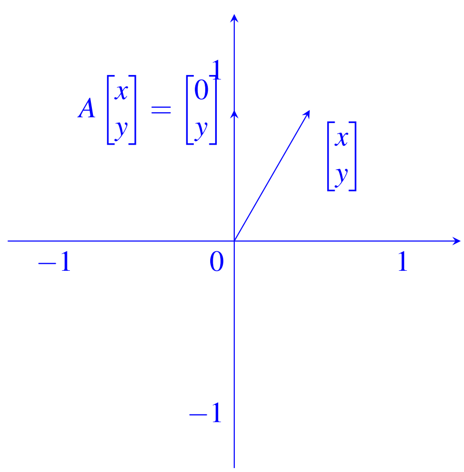
The original matrix $\mathbf{A}$ reflects the vector across the y-axis, i.e. it multiplies the vector's x-component by a factor of -1. Reflecting the vector across the y-axis again with $\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ will give you the original vector, i.e. the transformation done by $\mathbf{A}$ is reversible.

(c) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{swap } \mathrm{R}_1, \mathrm{R}_2$$

We see here that the inverse does not exist because the second row represents an inconsistent equation. Another way to see that the inverse does not exist is by realizing that the first column (and first row) of the original matrix are the zero vector, so the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.



The original matrix $\mathbf{A}$ removes the x-component of the vector it's applied to and keeps the same y-component. Graphically speaking, this matrix can be thought of as taking the "shadow" of the vector on the y-axis if you were to shine a light perpendicular to the y-axis.

Since the x-component of the vector is completely lost after the transformation, the process is not reversible.

(d) $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Assume $\cos\theta \neq 0$. *Hint:* $\cos^2\theta + \sin^2\theta = 1$.

Solution:

The inverse does exist.

$$\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1/\cos\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_1 \times \sin\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \frac{1}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \qquad \cos^2\theta + \sin^2\theta = 1$$

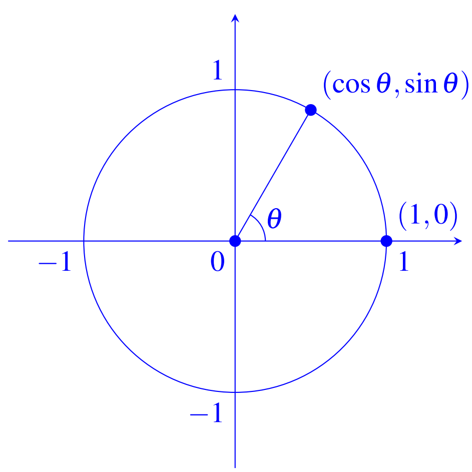
$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \qquad R_2 \leftarrow R_2 \times \cos\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \qquad R_1 \leftarrow R_1 + R_2 \times \sin\theta/\cos\theta$$

$\mathbf{A}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

The original matrix $\mathbf{A}$ is the two-dimensional rotation matrix as seen in discussion 3B. The rotation matrix rotates a vector in the counter-clockwise direction, and its inverse rotates a vector in the clockwise direction. Take the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for example:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



The inverse matrix can also be found from the rotation matrix that rotates a vector by an angle $-\theta$. The inverse matrix can also be found as follows:

$$\mathbf{A}^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

So the transformation done by $\mathbf{A}$ is a reversible process.

(e) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - 2R_1$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \qquad R_2 \leftarrow -R_2/2$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \qquad R_1 \leftarrow R_1 - R_2$$

Inverse exists: $\mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$.

(f) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 4 & -1 & 0 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - 2R_2$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the second and third columns are equal, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(g) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$
\left[\begin{array}{ccc|ccc} -1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 \times -1
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2/2
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \quad R_3 \leftarrow 2R_3/3; R_2 \rightarrow R_2/2
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \quad R_2 \leftarrow R_2 + R_3/2; R_1 \leftarrow R_1 - R_3/2
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \quad R_1 \leftarrow R_1 + R_2
$$

Inverse exists: $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution:

$$
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] R_3 \leftarrow R_3 + R_2
$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the third column is the negative of the second column, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(i) **(OPTIONAL)**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix? Hint 2: We're reasonable people!

Solution:

Inverse does not exist because $\text{column}_1 + \text{column}_2 + \text{column}_3 = \text{column}_4$, which means that the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

6. Finding Null Spaces and Column Spaces

Learning Objectives: *Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $\mathrm{N}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{N}(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of $\mathbf{A}$'s columns. The column space can be notated as $\mathrm{C}(\mathbf{A})$ or $\mathrm{range}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{C}(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors - i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $\mathrm{C}(\mathbf{A})$?

Solution: If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with $*$ are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

Here all 5 columns are $\in \mathbb{R}^3$. The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span $\mathbb{R}^3$, therefore any choice of fourth and fifth columns, also in $\mathbb{R}^3$, can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, if $m < n$, then the columns of $\mathbf{A} \in \mathbb{R}^{m\times n}$ will always be linearly dependent, since you cannot have more than m linearly independent columns in $\mathbb{R}^m$.

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{A})$ (i.e. a basis for $\mathrm{C}(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $\mathrm{C}(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

Solution: $\mathrm{C}(\mathbf{A})$ is the space spanned by its columns, so the set of all columns is a valid span for $\mathrm{C}(\mathbf{A})$. However, we are asking you to choose a subset of the columns and still span $\mathrm{C}(\mathbf{A})$, as we showed in part (a). To find the minimum number of columns needed and determine the dimension of $\mathrm{C}(\mathbf{A})$, we can remove vectors from the set of columns until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the dimension of $\mathbf{A}$ is 2.

One set spanning $\mathrm{C}(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span $\mathrm{C}(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of $\mathbf{A}$ appear. Despite this, the span of this set will still be equal to $\mathrm{C}(\mathbf{A})$, which for this matrix is the set of all vectors in $\mathbb{R}^3$ with zero third entry. Geometrically, both of these solutions span the same plane, i.e. the *xy*-plane in the 3D space.

Give yourself full credit if you recognized that the dimension was 2, and if you had a *minimum* set of vectors that spans $\mathrm{C}(\mathbf{A})$.

(c) Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{A})$ (i.e. a basis for $\mathrm{N}(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $\mathrm{N}(\mathbf{A})$?

Solution:

Finding $\mathrm{N}(\mathbf{A})$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2, x_4, and x_5 are free variables, since they correspond to the columns with no pivots.

Thus, we let $x_2 = a$, $x_4 = b$, and $x_5 = c$. Now we rewrite the equations as:

$$\begin{aligned}
x_1 &= -a + 2b - 3c \\
x_2 &= a \\
x_3 &= b - c \\
x_4 &= b \\
x_5 &= c
\end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\mathrm{N}(\mathbf{A})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of $\mathrm{N}(\mathbf{A})$ is 3, as it is the minimum number of vectors we need to span it.

(d) Find the sum of the dimensions of $\mathrm{N}(\mathbf{A})$ and $\mathrm{C}(\mathbf{A})$. What do you notice about this sum in relation to the dimensions of $\mathbf{A}$?

Solution: The dimensions of $\mathrm{C}(\mathbf{A})$ and $\mathrm{N}(\mathbf{A})$ add up to the number of columns in $\mathbf{A}$. This is true of all matrices. This relates to what is known as the rank-nullity theorem; however we will not be covering this in 16A. You'll get to explore this in 16B.

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{B})$ (i.e. a basis for $\mathrm{C}(\mathbf{B})$). What is the minimum number of vectors required to span the $\mathrm{C}(\mathbf{B})$?

Solution:

We see that the first two column vectors of $\mathbf{B}$ are linearly independent and sufficient to span $\mathrm{C}(\mathbf{B})$, since the third column is trivial (all zeros) and does not contribute anything to the span. Therefore, $\mathrm{C}(\mathbf{B})$ has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(f) You are given the following matrix $\mathbf{G}$. Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{G})$, i.e. a basis for $\mathrm{N}(\mathbf{G})$.

$$\mathbf{G} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution: To find $\mathrm{N}(\mathbf{G})$, we wish to solve for all $\vec{x}$ such that $\mathbf{G}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 2 & -4 & 4 & 8 & 0 \\ 1 & -2 & 3 & 6 & 0 \\ 2 & -4 & 5 & 10 & 0 \\ 3 & -6 & 7 & 14 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 1 & -2 & 3 & 6 & 0 \\ 2 & -4 & 5 & 10 & 0 \\ 3 & -6 & 7 & 14 & 0 \end{array}\right] \qquad R_1 \leftarrow \frac{1}{2}R_1$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 - 3R_1 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_1 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2 \end{array}$$

Vectors in $\mathrm{N}(\mathbf{G})$ satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \implies \begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

We then assign free variables $x_2 = a$ and $x_4 = b$ and substitute in:

$$\begin{aligned} x_1 &= 2a \\ x_2 &= a \\ x_3 &= -2b \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, $\mathrm{N}(\mathbf{G})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(g) (**OPTIONAL**) For the following matrix $\mathbf{D}$, find $\mathrm{C}(\mathbf{D})$ and its dimension, and $\mathrm{N}(\mathbf{D})$ and its dimension.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find $\mathrm{C}(\mathbf{D})$, we identify the linearly independent columns of $\mathbf{D}$ by inspection. The second column is a scaled version of the first column. The third column is linearly independent from the first and second columns, since it is not a scaled version of the first column. Finally, the fourth column is simply the first column minus the third column and thus is linearly dependent with respect to prior columns.

So we conclude that the linearly independent columns of $\mathbf{D}$ are the first and third columns so that a basis for $\mathrm{C}(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$$

and thus the dimension of $\mathrm{C}(\mathbf{D})$ is 2.

To find $\mathrm{N}(\mathbf{D})$, we can row reduce the matrix to find solutions to $\mathbf{D}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & -1 & -3 & 4 & 0 \\ 3 & -3 & -5 & 8 & 0 \\ 1 & -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[\text{rref}]{} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since we only have pivots in the first and third columns, we can assign the free variables $x_2 = s$ and $x_4 = t$. We can write all solutions to $\mathbf{D}\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

A basis for $\mathrm{N}(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and thus the dimension of $\mathrm{N}(\mathbf{D})$ is 2.

7. Cubic Polynomials

Learning Goal: *This problem shows us that we can treat fixed-degree polynomials as a vector space. Furthermore, many operations on polynomials are linear operations in this vector space and can be represented by matrices.*

(a) Show that the set of all cubic polynomials

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3,$$

where $t \in [a,b]$ and the coefficients p_k are real scalars, forms a vector space. Call this vector space V.

Solution:

No escape (scaling) property:

$$\alpha p(t) = \alpha p_0 + \alpha p_1 t + \alpha p_2 t^2 + \alpha p_3 t^3$$

The function above is, itself, a cubic polynomial, so the no escape property holds.

No escape (addition) property:
Let us define two cubic polynomials:

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$
$$q(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$$

$$p(t) + q(t) = (p_0 + q_0) + (p_1 + q_1)t + (p_2 + q_2)t^2 + (p_3 + q_3)t^3$$

The function above is also a cubic polynomial, so this no escape property also holds.
It can be shown from the properties of scalar multiplication and addition that the remaining properties of vector spaces hold for the set of all p(t):
Commutativity: $p(t) + q(t) = q(t) + p(t)$
Associativity of vector addition: $(p(t) + q(t)) + r(t) = p(t) + (q(t) + r(t))$
Additive identity: There exists 0 in the set of all p(t) such that for all $p(t)$, $0 + p(t) = p(t) + 0 = p(t)$
Existence of inverse: For every p(t), there is element $-p(t)$ such that $p(t) + -p(t)$=0
Associativity of scalar multiplication: $c(d(p(t))) = (cd)p(t)$
Distributivity of scalar sums: $(c+d)p(t) = cp(t) + dp(t)$
Distributivity of vector sums: $c(p(t) + q(t)) = cp(t) + cq(t)$
Scalar multiplication identity: $q(t) = 1 \implies q(t)p(t) = 1p(t) = p(t)$
Since all properties of a vector space hold, the set of cubic polynomials is a vector space.

(b) Consider the set of real-valued monomials given below:

$$\varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \varphi_2(t) = t^2, \quad \varphi_3(t) = t^3,$$

where $t \in \mathbb{R}$.
Show that every real-valued cubic polynomial

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

defined over the interval $[a,b]$ can be written as a linear combination of the monomials $\varphi_0(t)$, $\varphi_1(t)$, $\varphi_2(t)$, and $\varphi_3(t)$. In particular, show that

$$p(t) = \vec{c}^T \vec{\varphi}(t),$$

where

$$\vec{c}^T = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}$$

is a vector of appropriately chosen coefficients and

$$\vec{\varphi}(t) = \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}.$$

Solution:
$\vec{c}$ is chosen exactly as $c_i = p_i$, so

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 = p_0 + p_1 t + p_2 t^2 + p_3 t^3 = p(t).$$

(c) The monomials $\varphi_k(t) = t^k$, for $k = 0, 1, 2, 3$, constitute a basis for the vector space of real-valued cubic polynomials defined over the interval $[a, b]$. Justify why this is true. What is the dimension of V?
Solution:
First, none of them can be written as a linear combination of the other monomials, i.e. you can't write $\varphi_0(t)$ using scalar multiples of $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$. Hence they are linearly independent.
They also span the space of possible polynomials, as shown in the solution to part (b).
There are four basis monomials, so the dimension of V must be four.

(d) Express the derivatives of the basis polynomials $\varphi_i(t)$ for $i = 0, 1, 2, 3$ in terms of the $\varphi_i(t)$ for $i = 0, 1, 2, 3$. **Solution:** Manually computing their derivatives, we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^0 = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^1 = 1 \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^2 = 2t \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi_3(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^3 = 3t^2. \end{aligned}$$

Now, re-expressing the right-hand-sides of the above in terms of monomials, we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(t) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t) &= \varphi_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t) &= 2\varphi_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi_3(t) &= 3\varphi_2(t). \end{aligned}$$

(e) Let $\mathbf{D}$ be a 4x4 matrix. Use the previous part to help you find the entries of $\mathbf{D}$, such that for any polynomial

$$p(t) = \vec{c}^T \vec{\varphi}(t),$$

its derivative can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = (D\vec{c})^T \vec{\varphi}(t).$$

Hint: What are the dimensions of $(\mathbf{D}\vec{c})^T$? (Reminder: The dimensions of a matrix or vector is a different concept than the dimensions of a vector space).
Solution: Letting

$$\vec{c} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}^T,$$

we see by its definition that

$$p(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t).$$

Since we showed that differentiation was a linear operator, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = c_0\frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(t) + c_1\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t) + c_2\frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t) + c_3\frac{\mathrm{d}}{\mathrm{d}t}\varphi_3(t).$$

Now, substituting in the results from the previous part, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = c_0 \cdot 0 + c_1\varphi_0(t) + 2c_2\varphi_1(t) + 3c_3\varphi_2(t).$$

Pulling the coefficients out into their own vector, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = \begin{bmatrix} c_1 & 2c_2 & 3c_3 & 0 \end{bmatrix} \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} = \begin{bmatrix} c_1 & 2c_2 & 3c_3 & 0 \end{bmatrix} \vec{\varphi}(t)$$

Now, observe that

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}.$$

Thus, we can rearrange our previous equation to become

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \right)^T \vec{\varphi}(t),$$

so we find that our desired matrix

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.