1 Probability Potpourri

1. We want to prove the covariance matrix Σ is always positive semi-definite, i.e. $a^T\Sigma a >= 0$ for all $n \times 1$ vectors a. Given $\Sigma = E[(Z-\mu)(Z-\mu)^T]$, set the following equation:

$$a^T E[(Z-\mu)(Z-\mu)^T]a >= 0$$
$$E[a^T(Z-\mu)(Z-\mu)^T a] >= 0$$

   by linearity of expectation

$$E[((Z-\mu)a)^T((Z-\mu)a)] >= 0$$
$$E[||(Z-\mu)a||^2] >= 0$$

2. Let W represent the event that there is a gust of wind and let H represent the event that the archer hits her target. We know that $P(H|W) = 0.4$, $P(H|W^C) = 0.7$, and $P(W) = 0.3$.

   1. P(there is gust of wind and she hits target) $= P(W \cap H)$
      $= P(W) * P(H|W)$
      $= 0.3 * 0.4$
      $= 0.12$
   2. P(hits the target on the first shot)
      $= P(H)$
      $= P(H \cap W) + P(H \cap W^C)$
      $= 0.12 + P(W) * P(H|W^C)$
      $= 0.12 + 0.7 * 0.7$
      $= 0.62$
   3. P(hits the target exactly once in two shots)
      $=P(H_1 \cap H_2^C) + P(H_1^C \cap H_2)$
      $= (0.62)(1-0.62) + (1-0.62)(0.62)$
      $= 0.4712$
   4. P(there was no gust of wind on an occasion when she missed)
      $=P(W^C|H^C)$
      $= \frac{P(W^C \cap H^C)}{P(H^C)}$
      $= \frac{P(W^C)P(H^C|W^C)}{P(H^C)}$
      $= \frac{0.7*0.3}{1-0.62}$
      $= 0.55$

3. Let Y represent the score of a single strike, or $Y = g(X = x)$ such that

$$g(X=x) = \begin{cases} 4, & \text{if } x \leq \frac{1}{\sqrt{3}} \\ 3, & \text{if } \frac{1}{\sqrt{3}} < x \leq 1 \\ 2, & \text{if } 1 < x \leq \sqrt{3} \\ 0, & \text{otherwise.} \end{cases}$$

Find expectation $E[Y] = \int yf(y)dy = \int g(x)f(x)dx = \int_0^\infty g(x)\frac{2}{\pi(1+x^2)}dx$

$$
\begin{aligned}
&= 4 * \frac{2}{\pi} arctan(x) \mid_0^{\frac{1}{\sqrt{3}}} + 3 * \frac{2}{\pi} arctan(x) \mid_{\frac{1}{\sqrt{3}}}^{1} + 2 * \frac{2}{\pi} arctan(x) \mid_1^{\sqrt{3}} \\
&= 4 * \frac{2}{\pi}(\frac{\pi}{6} - 0) + 3 * \frac{2}{\pi}(\frac{\pi}{4} - \frac{\pi}{6}) + 2 * \frac{2}{\pi}(\frac{\pi}{3} - \frac{\pi}{4}) \\
&= 2.167
\end{aligned}
$$

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2 Properties of Gaussians

1. We want to prove that $E[e^{\lambda X}] = e^{\sigma^2 \lambda 2/2}$