# EECS 16A Designing Information Devices and Systems I
# Fall 2021 Homework 5

**This homework is due Friday, October 1, 2021, at 23:59.**
**Self-grades are due Monday, October 4, 2021, at 23:59.**

**Submission Format**

Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

   For this homework, please read Note 8 through 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. You are always welcome and encouraged to read beyond this as well.

2. **Subspaces, Bases and Dimension**

   For each of the sets $\mathbb{U}$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

   (a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

   **Solution:** We test the three properties of a subspace:

   i. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha\vec{v}_1$, where $\alpha$ is a scalar. Here

   $$\vec{u}_1 = \alpha\vec{v}_1 = \alpha \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_1 + \alpha y_1) \\ \alpha x_1 \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} 2(x_u+y_u) \\ x_u \\ y_u \end{bmatrix},$$

   where $x_u = \alpha x_1$ and $y_u = \alpha y_1$. Hence, $\vec{u}_1 = \alpha\vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

   ii. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2(x_2+y_2) \\ x_2 \\ y_2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume

$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 2(x_1+y_1)+2(x_2+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_1+x_2+y_1+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_3+y_3) \\ x_3 \\ y_3 \end{bmatrix},$$

where $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$ Hence, $\vec{v}_3$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} 2(x_0+y_0) \\ x_0 \\ y_0 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v}_0 = \begin{bmatrix} 2(0+0) \\ 0 \\ 0 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

where $x$ and $y$ are free variables. So $\mathbb{U}$ can be expressed as span $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$. Hence the basis is given by the set: $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$. Dimension = 2.

(b) **(PRACTICE, OPTIONAL)** $\mathbb{U} = \left\{\begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x,y,z \in \mathbb{R}\right\}$

**Solution:**

Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha\vec{v}_1$, where $\alpha$ is a scalar. Here

$$\vec{u}_1 = \alpha\vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 + \alpha \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ (\alpha z_1 + \alpha - 1) + 1 \end{bmatrix} = \begin{bmatrix} x_u \\ y_u \\ z_u + 1 \end{bmatrix},$$

where $x_u = \alpha x_1$, $y_u = \alpha y_1$ and $z_u = \alpha z_1 + \alpha - 1$. Hence, $\vec{u}_1 = \alpha\vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2+1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2+2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ (z_1+z_2+1)+1 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ z_3+1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$ and $z_3 = z_1 + z_2 + 1$. Hence, $\vec{v}_3$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0+1 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$, $y_0 = 0$ and $z_0 = -1$. So the vector $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ -1+1 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (z+1)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_{new}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where $x$, $y$ and $z_{new} = z+1$ are free variables. So $\mathbb{U}$ can be expressed as span $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Hence the basis is given by the set: $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$. The dimension is 3, which makes $\mathbb{U}$ the same as $\mathbb{R}^3$.

(c) $\mathbb{U} = \left\{\begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R}\right\}$

**Solution:** Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha\vec{v}_1$, where $\alpha$ is a scalar. Here

$$\vec{u}_1 = \alpha\vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha \end{bmatrix} \neq \begin{bmatrix} x_u \\ y_u \\ x_u+1 \end{bmatrix},$$

where $x_u = \alpha x_1$ and $y_u = \alpha y_1$. Hence, $\vec{u}_1 = \alpha\vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2+1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ x_1+x_2+2 \end{bmatrix} \neq \begin{bmatrix} x_3 \\ y_3 \\ x_3+1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v}_3$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0+1 \end{bmatrix}$ be a member of the set. The first and third elements cannot both be zero regardless of the value chosen for $x_0$. So the zero vector is not contained in this set.

Hence we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$. **Note that for full credit you only have to show that one of the properties is violated, you don't have to show all three.**

(d) **(PRACTICE, OPTIONAL)** $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

**Solution:** Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1^2 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{u}_1 = \alpha \vec{v}_1$, where $\alpha$ is a scalar. Here

$$\vec{u}_1 = \alpha \vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha y_1^2 \end{bmatrix} \neq \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + (\alpha y_1)^2 \end{bmatrix} = \begin{bmatrix} x_u \\ y_u \\ x_u + y_u^2 \end{bmatrix},$$

where $x_u = \alpha x_1$ and $y_u = \alpha y_1$. Hence, $\vec{u}_1 = \alpha \vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1^2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2 + y_2^2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + y_1^2 + y_2^2 \end{bmatrix} \neq \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) + (y_1 + y_2)^2 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ x_3 + y_3^2 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v}_3$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0 + y_0^2 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 + 0^2 \end{bmatrix} = \vec{0}$ is contained in this set.

Since two of the three properties do not hold, we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$.

**Just showing that one of the three properties is violated is enough to prove that a subset is not a subspace. However, in order to prove that a subset is a subspace, you have to show that all three properties hold.**

## 3. Introduction to Eigenvalues and Eigenvectors

***Learning Goal:*** *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

**Solution:**

**Self-grading note:** For this subproblem and the following subproblems which involve computing eigenvectors, give yourself full credit if the eigenvector(s) you calculated is/are a scaled (i.e, multiplied by a real valued $\alpha$) version of the eigenvector(s) given in the solutions.

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det\left(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

where $x$ is a free variable.

Any vector in $\text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 5$.

$\lambda = 2$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

where $y$ is a free variable.

Any vector in $\text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 2$.

(b) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

**Solution:**

Here, it is hard to guess the answers.

$$\det\left(\begin{bmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} (22-\lambda)(13-\lambda) - 36 &= 0 \\ 250 - 35\lambda + \lambda^2 &= 0 \\ (\lambda - 10)(\lambda - 25) &= 0 \\ \implies \lambda &= 10, 25 \end{aligned}$$

$\lambda = 10$:

$$\mathbf{A}\vec{x} = 10\vec{x} \implies (\mathbf{A} - 10\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2x + y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix}$$

where $x$ is a free variable.

Any vector that lies in span$\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda = 10$.

$\lambda = 25$:

$$\mathbf{A}\vec{x} = 25\vec{x} \implies (\mathbf{A} - 25\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix} - \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2y = x \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix}$$

where $y$ is a free variable.

Any vector that lies in span$\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 25$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

**Solution:**

This can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, which belongs in the **nullspace of the matrix**.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can explicitly calculate.

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} (1-\lambda)(4-\lambda) - 4 &= 0 \\ \lambda^2 - 5\lambda &= 0 \implies \lambda(\lambda - 5) = 0 \\ \lambda &= 0, 5 \end{aligned}$$

$\lambda = 0$:

$$\mathbf{A}\vec{x} = 0\vec{x} \implies \mathbf{A}\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = -2y \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix}$$

where $y$ is a free variable.

Any vector that lies in span$\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}-4 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\implies \begin{bmatrix}2 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies y = 2x \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\2x\end{bmatrix}$$

where $x$ is a free variable.

Any vector that lies in span$\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is an eigenvector corresponding to eigenvalue $\lambda = 5$.

(d) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

**Solution:**

Recall the definition of a matrix subspace from Note 8. A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ if it contains the zero vector, is closed under scalar multiplication, and is closed under vector addition.

i. Zero vector: The zero vector is contained in this set since $\mathbf{A}\vec{0} = \vec{0} = \lambda\vec{0}$.

ii. Scalar multiplication: Let $\vec{v}_1$ be a member of the set. Let $\vec{u} = \alpha\vec{v}_1$. Note that $\vec{u} \in \mathbb{R}^n$, thus a possible value of $\vec{x}$. Now, $\mathbf{A}\vec{u} = \mathbf{A}\alpha\vec{v}_1 = \alpha\mathbf{A}\vec{v}_1 = \alpha\lambda\vec{v}_1 = \lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

iii. Vector addition: Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Observe below that the set is closed under vector addition as well.

$$\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

Note that $\vec{v}_1 + \vec{v}_2$ is also a vector in $\mathbb{R}^n$, which corresponds to how $\vec{x}$ is defined in this setup.

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

## 4. The Dynamics of Romeo and Juliet's Love Affair

***Learning Goal:*** *Eigenvalues and eigenvectors of state transition matrices tend to reveal useful information about the dynamical systems they model. This problem serves as an example of extracting useful information through analysis of the eigenvalues of the state transition matrix of a dynamical system.*

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day $n$, and let $J[n]$ denote Juliet's feelings about Romeo on day $n$, where $R[n]$ and $J[n]$ are **scalars**. The **sign** of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The **magnitude** (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger magnitude of $R[n]$ means that Romeo has a stronger emotion towards Juliet (strong love if $R[n] > 0$ or strong hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \ldots$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denotes the state transition matrix for our dynamic system model.

The selection of the parameters $a, b, c, d$ results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. $a, b, c, d$) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a + b = c + d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue $\lambda_1$.
Show that

$$\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue $\lambda_2$.
***Hint:*** *Consider $\mathbf{A}\vec{v}_1$. Is it equal to a scalar multiple of $\vec{v}_1$? Repeat a similar process for $\vec{v}_2$.*
**Solution:**

$$\begin{aligned} \mathbf{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} a+b \\ c+d \end{bmatrix} \\ &= (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (c+d)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Let $\lambda_1 = a + b = c + d$. So, we find that,

$$\left(\lambda_1 = a+b = c+d,\ \mathit{Eigenspace}(\lambda_1) = \mathit{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right)$$

To determine the other eigenvalues and corresponding eigenvectors $(\lambda_2, \vec{v}_2)$, we test the assumption that $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$. Note that by modifying the constraint $a+b=c+d$, we can also get $a-c=d-b$, which helps simplify the following:

$$\begin{aligned} \mathbf{A}\begin{bmatrix} b \\ -c \end{bmatrix} &= \begin{bmatrix} ab-bc \\ cb-dc \end{bmatrix} \\ &= \begin{bmatrix} b(a-c) \\ -c(d-b) \end{bmatrix} \\ &= (a-c)\begin{bmatrix} b \\ -c \end{bmatrix} \\ &= (d-b)\begin{bmatrix} b \\ -c \end{bmatrix} \end{aligned}$$

Therefore, we have our second eigenvalue and corresponding eigensapce:

$$\left(\lambda_2 = a-c = d-b,\ Eigenspace(\lambda_2) = span\left(\begin{bmatrix} b \\ -c \end{bmatrix}\right)\right).$$

For parts (b) - (d), consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(b) Determine the eigenvalues and corresponding eigenvectors (i.e. $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

**Solution:**

From the results of part (a), we know that the eigenvalues and eigenvectors of this matrix are

$$\left(\lambda_1 = a+b = 0.75+0.25 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a-c = 0.75-0.25 = 0.5, \vec{v}_2 = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}\right).$$

**Note:** If your choice of eigenvector $\vec{v}_1$ and $\vec{v}_2$ is a scaled version of the ones given in this solution, that is fine.

(c) Determine all of the non-zero *steady states* of the system. That is, find all possible state vectors $\vec{s}_*$ such that if Romeo and Juliet start at, or enter, any of those state vectors, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

**Solution:** Any $\vec{s}_* \in \text{span}\{\vec{v}_1\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is an eigenvector which corresponds to the steady state, because $\vec{v}_1$ corresponds to the eigenvalue $\lambda_1 = 1$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

**Solution:**

We note that $\vec{s}[0] \in \text{span}\{\vec{v}_2\}$. Therefore,

$$\begin{aligned} \vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_2, \vec{v}_2 \end{aligned}$$

where $\vec{s}[0] = \alpha\vec{v}_2$.

If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_2^n\vec{v}_2 \end{aligned}$$

In this case $\lambda_2 = 0.5$. This means that as $n \to \infty$, $\lambda_2^n \to 0$.

Therefore,

$$\begin{aligned} \vec{s}[n] &= \alpha\lambda_2^n\vec{v}_2 \\ &= \alpha \cdot 0 \cdot \vec{v}_2 \\ &= \vec{0} \end{aligned}$$

which means that

$$\lim_{n\to\infty}(R[n], J[n]) = (0, 0)$$

So, ultimately, Romeo and Juliet will become neutral to each other.

(e) Now suppose we have a different state-transition matrix $\mathbf{A}$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$, with a corresponding eigenvalue $\lambda_1 = 2$. Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\vec{s}[0] \neq \vec{0}$. What happens to their relationship over time in this setup? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

**Solution:** We note that $\vec{s}[0] \in \text{span}\{\vec{v}_1\}$. Therefore,

$$\begin{aligned} \vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_1\vec{v}_1 \end{aligned}$$

where $\vec{s}[0] = \alpha\vec{v}_1$.

If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

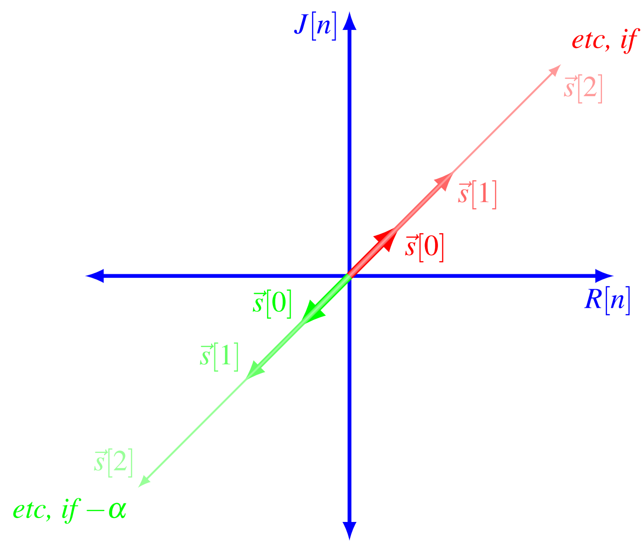
$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_1^n\vec{v}_1 \end{aligned}$$

In this problem, $\lambda_1 = 2$. Therefore,

$$\vec{s}[n] = \alpha 2^n \vec{v}_1$$

This means that as $n \to \infty$, $\lambda_1^n \to \infty$. Essentially, the elements of the state vector continue to double at each time step and grow without bound to either $+\infty$ or $-\infty$.

Therefore, what happens to Romeo and Juliet depends on $\vec{s}[0]$. If $\vec{s}[0]$ is in the first quadrant, Romeo and Juliet will become "infinitely" in love with each other. On the other hand, if $\vec{s}[0]$ is in the third quadrant, then Romeo and Juliet will have "infinite" hatred for each other. Graphically, the dynamics of Romeo and Juliet's love affair for this example are illustrated below. The red vectors are the first three state vectors corresponding to the case where $\alpha$ is a *positive* value and therefore $\vec{s}[0]$ is in the first quadrant. Similarly, the green vectors are the first three state vectors corresponding to the case where $\alpha$ is a *negative* value and therefore $\vec{s}[0]$ is in the third quadrant. In the end of the story of Romeo and Juliet, both of them died after being "infinitely" in loved with each other.




## 5. Noisy Images

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***Learning Goal:*** *The imaging lab uses the eigenvalues of the masking matrix to understand which masks are better than others for image reconstruction in the presence of additive noise. This problem explores the underlying mathematics.*

In lab, we used a single pixel camera to capture many measurements of an image $\vec{i}$. A single scalar measurement $s_i$ is captured using a mask $\vec{h}_i$ such that $s_i = \vec{h}_i^T \vec{i}$. Many measurements can be expressed as a matrix-vector multiplication of the masks with the image, where the masks lie along the rows of the matrix.

$$\begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} = \begin{bmatrix} \vec{h}_1^T \\ \vdots \\ \vec{h}_N^T \end{bmatrix} \vec{i} \tag{1}$$

$$\vec{s} = \mathbf{H}\vec{i} \tag{2}$$

In the real world, noise, $\vec{w}$, creeps into our measurements, so instead we have,

$$\vec{s} = \mathbf{H}\vec{i} + \vec{w}. \tag{3}$$

(a) Express $\vec{i}$ in terms of $\mathbf{H}$ (or its inverse), $\vec{s}$, and $\vec{w}$. Assume $\mathbf{H}$ is invertible. (*Hint:* Think about what you did in the imaging lab.)

**Solution:**

$$\vec{s} = \mathbf{H}\vec{i} + \vec{w} \tag{4}$$
$$\mathbf{H}\vec{i} = \vec{s} - \vec{w} \tag{5}$$
$$\mathbf{H}^{-1}\mathbf{H}\vec{i} = \mathbf{H}^{-1}(\vec{s} - \vec{w}) \tag{6}$$
$$\vec{i} = \mathbf{H}^{-1}(\vec{s} - \vec{w}) \tag{7}$$
$$\vec{i} = \mathbf{H}^{-1}\vec{s} - \mathbf{H}^{-1}\vec{w} \tag{8}$$

(b) It turns out that the eigenvalues of $\mathbf{H}$ and $\mathbf{H}^{-1}$ impact how well we can reconstruct the image from the measurements $\vec{s}$. We will see this in subsequent parts of the problem. First, let us compute the eigenvalues of $\mathbf{H}^{-1}$. The eigenvalues of $\mathbf{H}^{-1}$ are actually related to the eigenvalues of $\mathbf{H}$! Show that if $\lambda$ is an eigenvalue of a matrix $\mathbf{H}$, then $\frac{1}{\lambda}$ is an eigenvalue of the matrix $\mathbf{H}^{-1}$.
*Hint:* Start with an eigenvalue $\lambda$ and one corresponding eigenvector $\vec{v}$, such that they satisfy $\mathbf{H}\vec{v} = \lambda\vec{v}$.

**Solution:**

Since we're showing that $\frac{1}{\lambda}$ is an eigenvalue, we need to first show that $\lambda \neq 0$. We know that $\mathbf{H}$ is invertible,
$\Rightarrow \mathbf{H}\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$.
$\Rightarrow \mathbf{H}\vec{x} = \vec{0}$ has a unique solution.
$\Rightarrow \vec{x} = \vec{0}$ is the only solution to $\mathbf{H}\vec{x} = \vec{0}$.
$\Rightarrow \mathbf{H}\vec{x} = \vec{0}$ has no non-zero vectors $\vec{x}$ that satisfy it.
$\Rightarrow \mathbf{H}\vec{x} = 0\vec{x}$ has no non-zero vectors $\vec{x}$ that satisfy it.
Therefore, 0 is not an eigenvalue. Let $\vec{v}$ be the eigenvector of $\mathbf{A}$ corresponding to $\lambda$.

$$\mathbf{H}\vec{v} = \lambda\vec{v}$$

You may give yourself full-credit during self-grades even if you did not explicitly prove that $\lambda \neq 0$, but make sure you understand why it is necessary to show this.
Since we know that $\mathbf{H}$ is invertible, we can left-multiply both sides by $\mathbf{H}^{-1}$.

$$\mathbf{H}^{-1}\mathbf{H}\vec{v} = \lambda\mathbf{H}^{-1}\vec{v}$$

$$\vec{v} = \lambda\mathbf{H}^{-1}\vec{v}$$

$$\mathbf{H}^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

(c) We are going to try different $\mathbf{H}$ matrices in this problem and compare how they deal with noise. Run all of the cells in the attached IPython notebook. Observe the **plots and the printed results.** Which matrix performs best in reconstructing the original image and why? What do you observe regarding the eigenvalues of matrices $\mathbf{H_1}, \mathbf{H_2}$ and $\mathbf{H_3}$? What special matrix is $\mathbf{H}_1$? (*Notice that each plot in the* *iPython notebook returns the result of trying to image a noisy image as well as the minimum absolute value of the eigenvalue of each matrix.*) Comment on the effect of small eigenvalues on the noise in the image.

**Solution:**

Notice that we are printing the eigenvalue with the smallest absolute value. Comparing results from $\mathbf{H_1}$, $\mathbf{H_2}$ and $\mathbf{H_3}$, we see that $\mathbf{H_1}$ has the largest value of minimum absolute eigenvalue, while $\mathbf{H_3}$ has the smallest. As the absolute value of the smallest eigenvalue of $\mathbf{H}$ decreases, the absolute value of the largest eigenvalue of $\mathbf{H^{-1}}$ increases (see why this happens in part d). Hence, the noise in the recovered image increases from $\mathbf{H_1}$ to $\mathbf{H_3}$.

The matrix $\mathbf{H}_1$ is the identity matrix.

(d) Now, because there is noise in our measurements, there will be noise in our recovered image. However, the noise is scaled. From the results of part (a), you know that: $\vec{i} = \mathbf{H}^{-1}\vec{s} - \mathbf{H}^{-1}\vec{w}$, so the impact of the noise on the image $\vec{i}$ is given by $\mathbf{H}^{-1}\vec{w}$.

Let us call this quantity $\hat{\vec{w}}$, often called “w-hat”.

$$\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} \tag{9}$$

To analyze how this transformation alters $\vec{w}$, we represent $\vec{w}$ as a linear combination of the eigenvectors of $\mathbf{H}^{-1}$,

$$\vec{w} = \alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N, \tag{10}$$

where, $\vec{b}_i$ is $\mathbf{H}^{-1}$’s eigenvector corresponding to eigenvalue $\frac{1}{\lambda_i}$.

Show that we can express the noise in the recovered image as the following linear combination of the vectors $\vec{b}_i$:

$$\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} = \alpha_1\frac{1}{\lambda_1}\vec{b}_1 + \ldots + \alpha_N\frac{1}{\lambda_N}\vec{b}_N. \tag{11}$$

Now, if $\lambda_i$ is very large, will the coefficient of $\vec{b}_i$ be large or small in $\hat{\vec{w}}$? If we want $\hat{\vec{w}}$ to be as small as possible, do we prefer large $\lambda_i$’s or small $\lambda_i$’s

**Solution:** To show this, we will write $\vec{w}$ as a linear combination of the eigenvectors of $\mathbf{H^{-1}}$ and then use the distributivity property of matrix-vector multiplication operation:

$$\begin{aligned}
\hat{\vec{w}} = \mathbf{H}^{-1}\vec{w} &= \mathbf{H}^{-1}(\alpha_1\vec{b}_1 + \ldots + \alpha_N\vec{b}_N) \\
&= \alpha_1\mathbf{H}^{-1}\vec{b}_1 + \ldots + \alpha_N\mathbf{H}^{-1}\vec{b}_N \\
&= \alpha_1\frac{1}{\lambda_1}\vec{b}_1 + \ldots + \alpha_N\frac{1}{\lambda_N}\vec{b}_N
\end{aligned}$$

For eigenvectors with large $\mathbf{H^{-1}}$ eigenvalues (i.e. small $\mathbf{H}$ eigenvalues $\lambda_i$), the noise will be amplified. This is bad and could corrupt the recovered image significantly.

For eigenvectors with small $\mathbf{H^{-1}}$ eigenvalues (i.e. large $\mathbf{H}$ eigenvalues $\lambda_i$), the noise will be attenuated. This is better and will not corrupt the recovered image as much.

So we strongly prefer matrices $\mathbf{H}$ that have large eigenvalues as this minimizes the impact of noise.

**6. Homework Process and Study Group**

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

**Solution:**

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.