EECS 16A Designing Information Devices and Systems I
Fall 2021 Homework 6

This homework is due Friday, October 8, 2021, at 23:59.
Self-grades are due Tuesday, October 12, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Note 11, which introduces the basics of circuit analysis and node voltage analysis. Please also read Note 12, which introduces using circuits for modelling. You are always welcome and encouraged to read beyond this as well. **Question to answer: What is the value of having a systematic procedure for solving circuits?**

Solution:
A systematic procedure for solving circuits gives an algorithm to follow when solving by hand, which is useful for more complicated circuits. In addition, this algorithm can be implemented on a computer, enabling fast analysis of circuits that are too large for manual calculations to be practical.

2. Page Rank

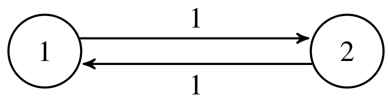
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram, as follows: entries t_{ji} represent the *proportion* of the people who are at website i that click the link for website j.

The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

(a) For graph A shown below, what are the steady-state frequencies (i.e. fraction of visitors in steady-state) for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

Solution:

To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0 \tag{2}$$

$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

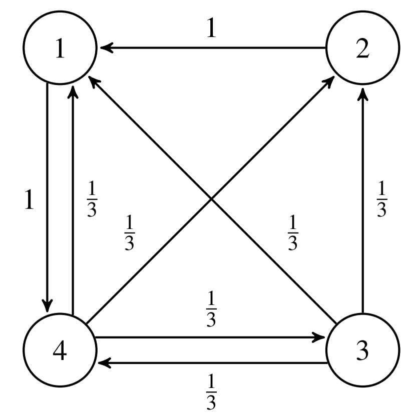
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

The sum of the values of the vector should equal 1 since the number of people is conserved, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. Carefully read the Python documentation for `numpy.linalg.eig` to understand what this function does and what it returns. Graph B is shown below, with weights in place to help you construct the transition matrix.



Graph B

Solution:

To determine the steady-state frequencies, we need to create the transition matrix $\mathbf{T}$ first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

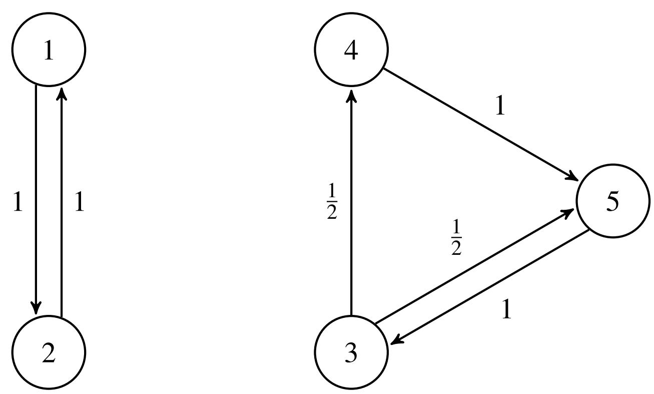
The eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython). Scaling it by

$$\frac{1}{(-0.61 + (-0.31) + (-0.23) + (-0.69))}$$

so the elements sum to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$

These are the steady-state frequencies for the pages.

(c) Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? Again, graph C with weights in place is shown below. You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

Using IPython, we find that the eigenspace associated with $\lambda = 1$ is spanned by the vectors $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$. While any linear combination of these vectors is an eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially “two internets”, or two disjoint sets of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

3. Properties of Pump Systems - II

Learning Objectives: *This problem builds on the pump examples we have been doing, but is meant to help you learn to do proofs in a step by step fashion. Can you generalize intuition from a simple example?*

We consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water moved is written on top of the edge.

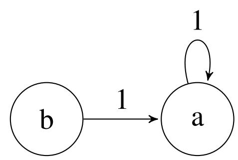


Figure 1: Pump system

We want to prove the following theorem. We will do this step by step.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(a) Rewrite the theorem statement for a graph with only two reservoirs.

Solution: Consider a system consisting of 2 reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then total amount of water at timestep $n+1$ will also be s.

(b) Since the problem does not specify the transition matrix, let us consider the transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and the state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$. Write out what is "known" or what is given to you in the theorem statement in mathematical form.

Hint: In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph.

Solution: Each column of the transition matrix sums to one:

$$a_{11} + a_{21} = 1, \qquad a_{12} + a_{22} = 1$$

The total amount of water in the system is s at timestep n:

$$x_1[n] + x_2[n] = s$$

We know that the state vector at the next timestep is equal to the transition matrix applied to the state vector at the current timestep:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$$

(c) Now write out what is to be proved in mathematical form.

Solution: We want to prove that the total amount of water at timestep $n+1$ will also be s:

$$x_1[n+1]+x_2[n+1]=s$$

(d) Prove the statement for the case of two reservoirs.

Solution: Consider the product $\mathbf{A}\vec{x}[n]=\vec{x}[n+1]$:

$$\mathbf{A}\vec{x}[n]=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1[n]\\ x_2[n]\end{bmatrix}=\begin{bmatrix}a_{11}x_1[n]+a_{12}x_2[n]\\ a_{21}x_1[n]+a_{22}x_2[n]\end{bmatrix}$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\sum_{i=1}^{2}x_i[n+1]=(a_{11}x_1[n]+a_{12}x_2[n])+(a_{21}x_1[n]+a_{22}x_2[n])$$

Regrouping terms:

$$(a_{11}+a_{21})x_1[n]+(a_{12}+a_{22})x_2[n]=x_1[n]+x_2[n]=s$$

(e) Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about the columns.

Solution:

Let $\vec{x}[n]=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_k\end{bmatrix}$ be the amount of water in each reservoir at timestep n. We know:

$$x_1[n]+x_2[n]+\cdots+x_k[n]=s$$

Let $\vec{a}_j$ be the j-th column of the state transition matrix $\mathbf{A}$.

$$\mathbf{A}=\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k\end{bmatrix}$$

We know that every column of $\mathbf{A}$ sums to one, so we know for all j,

$$a_{1j}+a_{2j}+\cdots+a_{kj}=1$$

Now, consider the product $\mathbf{A}\vec{x}[n]$:

$$\mathbf{A}\vec{x}[n]=\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k\end{bmatrix}\begin{bmatrix}x_1[n]\\ x_2[n]\\ \vdots\\ x_k[n]\end{bmatrix}=x_1[n]\vec{a}_1+x_2[n]\vec{a}_2+\cdots+x_k[n]\vec{a}_k=\vec{x}[n+1]$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\begin{aligned} x_1[n+1]+x_2[n+1]+\cdots+x_k[n+1] =& (a_{11}x_1[n]+a_{12}x_2[n]+\cdots+a_{1k}x_k[n]) \\ &+(a_{21}x_1[n]+a_{22}x_2[n]+\cdots+a_{2k}x_k[n]) \\ &+\ldots \\ &+(a_{k1}x_1[n]+a_{k2}x_2[n]+\cdots+a_{kk}x_k[n]) \end{aligned}$$

Factoring out each element of $x[n]$ gives:

$$\begin{aligned} &x_1[n](a_{11}+a_{21}+\cdots+a_{k1})+x_2[n](a_{12}+a_{22}+\cdots+a_{k2})+\cdots+x_k[n](a_{1k}+a_{2k}+\cdots+a_{kk}) \\ &= x_1[n]+x_2[n]+\cdots+x_k[n] = s \end{aligned}$$

4. Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.
Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

Solution:

Recall that we find the eigenvalues of a matrix $\mathbf{A}$ by setting the determinant of $\mathbf{A} - \lambda\mathbf{I}$ to 0.

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\left((\mathbf{A}-\lambda\mathbf{I})^T\right) = \det\left(\mathbf{A}^T-\lambda\mathbf{I}\right) = 0$$

Since the two determinants are equal, the characteristic polynomials of the two matrices must also be equal. Therefore, they must have the same eigenvalues.

(b) Let a square matrix $\mathbf{A}$ have, for each row, entries that sum to one. Show that $\vec{1} = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

Solution:

Recall that if the rows of $\mathbf{A}$ sum to one, then $\mathbf{A}\vec{1} = \vec{1}$. Therefore, the corresponding eigenvalue is $\lambda = 1$.

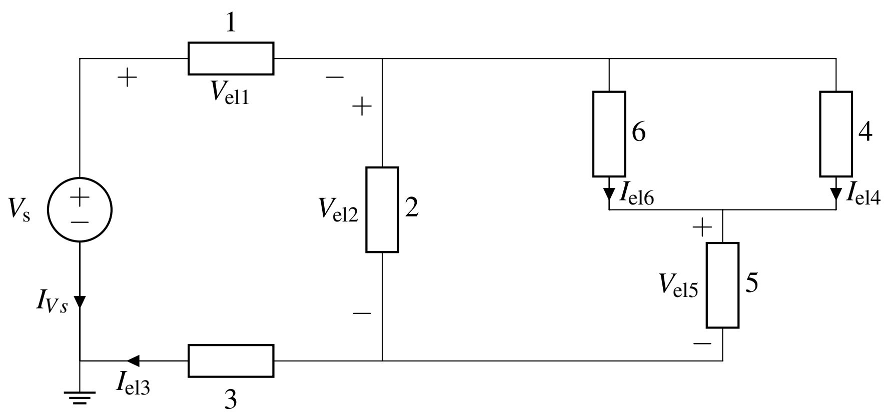
(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices have, for each column, entries that sum to 1.

Solution:

If we transpose a conservative state transition matrix $\mathbf{A}$, then the rows of $\mathbf{A}^T$ (or the columns of $\mathbf{A}$) sum to one by definition of a conservative system. Then, from part (b), we know that $\mathbf{A}^T$ has the eigenvalue $\lambda = 1$. Furthermore, from part (a), we know that the $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues, so $\mathbf{A}$ also has the eigenvalue $\lambda = 1$.

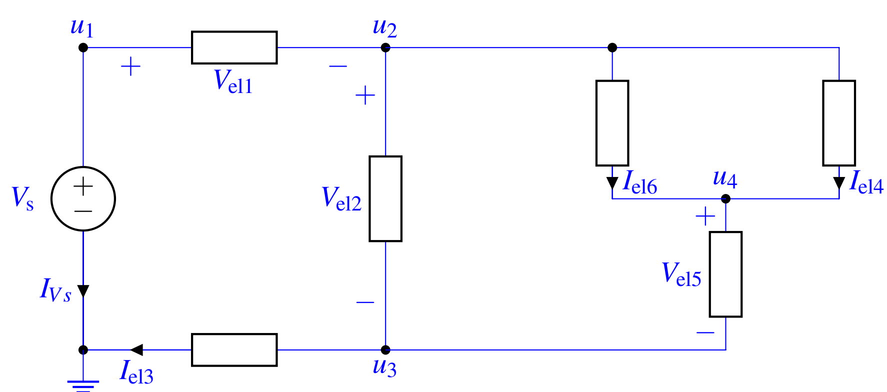
5. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KCL and KVL equations.*



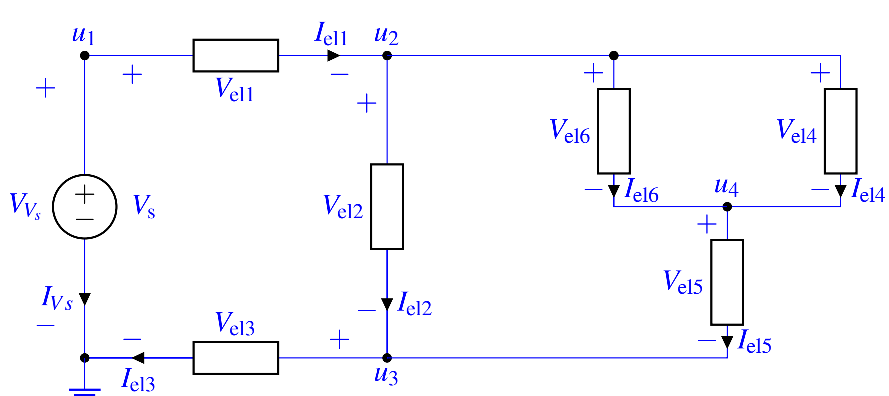
(a) How many nodes does the above circuit have? Label them.
Note: The ground node has been selected for you, so you don't need to label that, but you need to include it in your node count.

Solution: There is a total of 5 nodes in the circuit, including the ground node. They are labeled u_1 - u_4 below:



(b) Notice that elements 1 - 6 and the voltage source V_s have either the *voltage across* or the *current through* them not labeled. Label the missing *voltages across* or *currents through* for elements 1 - 6, and the voltage source V_s, so that they all follow **passive sign convention**.

Solution: The passive sign convention dictates that the current flows from the positive to the negative terminal of the element (or equivalently exiting the negative terminal / entering the positive terminal if you prefer):



(c) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

Solution: For our specific node labeling we can write:

$$V_{V_s} = u_1 - 0 = u_1 (= V_s)$$

$$V_{\text{el1}} = u_1 - u_2$$

$$V_{\text{el2}} = u_2 - u_3$$

$$V_{\text{el3}} = u_3 - 0 = u_3$$

$$V_{\text{el4}} = u_2 - u_4$$

$$V_{\text{el5}} = u_4 - u_3$$

$$V_{\text{el6}} = u_2 - u_4$$

Notice that the element voltage is always of the form: $V_{\text{el}} = u_+ - u_-$.

(d) Write one KCL equation that involves the currents through elements 1 and 2.
*Hint: This will **not** be specific to your node labeling. Your answer may contain currents through other elements too.*

Solution: The only node for which we can write a KCL equation involving both elements 1 and 2 is node u_2, since they only intersect on that node:

$$I_{el1} = I_{el2} + I_{el6} + I_{el4}$$

(e) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

Solution: Notice that there are in fact 3 loops that contain the voltage source V_s, for which we can write the following equations, starting each time from the ground node and ending at the ground node:

$$V_s - V_{\text{el1}} - V_{\text{el2}} - V_{\text{el3}} = 0$$

$$V_s - V_{\text{el1}} - V_{\text{el6}} - V_{\text{el5}} - V_{\text{el3}} = 0$$

$$V_s - V_{\text{el1}} - V_{\text{el4}} - V_{\text{el5}} - V_{\text{el3}} = 0$$

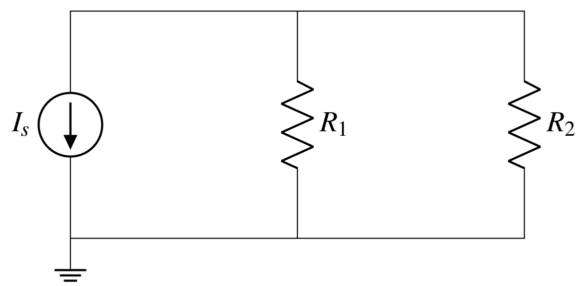
The reason this is not specific to our labeling is that the polarity of all elements is either given or set through the passive sign convention.

6. Circuit Analysis

Learning Goal: *This problem will help you practice circuit analysis using the node voltage analysis (NVA) method.*

Using the steps outlined in lecture or in Note 11, analyze the following circuits to calculate the currents through each element and the voltages at each node. Use the ground node labelled for you. You may use a numerical tool such as IPython to solve the final system of linear equations.

(a) $I_s = 3\,\text{mA}, R_1 = 2\,\text{k}\Omega, R_2 = 4\,\text{k}\Omega$



Solution:

Step 1) Define Reference Node

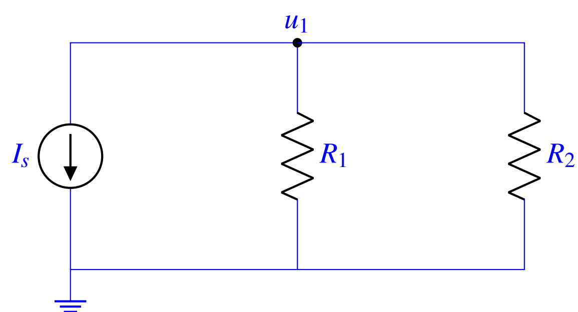
Select a reference (ground) node. Any node can be chosen for this purpose. This has already been done for you in this circuit.

Step 2) Label Nodes with Voltage Set by Sources

We don't have any other voltage sources in this circuit, so we can skill this step.

Step 3) Label Remaining Nodes

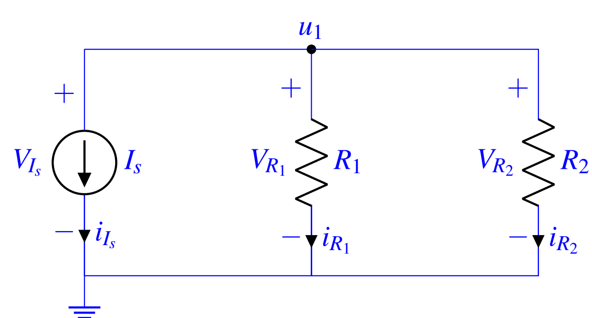
We only have one other node here, and we label the corresponding source u_1 (names are arbitrary, but must be unique).



Step 4) Label Element Voltages and Currents

Next we mark all element voltages and currents.

Start with the current. The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps). Then mark the element voltages following the passive sign convention, i.e. the voltage and current point in the "same" direction.



Step 5) KCL Equations

Write KCL equations for all nodes with unknown voltage, which is only u_1.

$$i_{I_s} + i_1 + i_2 = 0$$

Step 6) Element Currents

Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In this circuit there are three, R_1, R_2, and I_s.

$$\begin{aligned} i_{I_s} &= I_s \\ i_{R_1} &= \frac{V_{R_1}}{R_1} \\ i_{R_2} &= \frac{V_{R_2}}{R_2} \end{aligned}$$

Step 7) Element Voltages

Rewrite the element voltages using the node differences.

$$\begin{aligned} i_{I_s} &= I_s \\ i_{R_1} &= \frac{u_1}{R_1} \\ i_{R_2} &= \frac{u_1}{R_2} \end{aligned}$$

Step 8) Substitute Element Currents in KCL Equations

Now we substitute the expressions derived in Step 7 into the KCL equations from Step 5.

$$I_s + \frac{u_1}{R_1} + \frac{u_1}{R_2} = 0$$

We can isolate the unknown terms (u_1) on the left and the known on the right

$$\frac{u_1}{R_1} + \frac{u_1}{R_2} = -I_s$$

We only have one equation to solve. Setting up the matrix equation would just be the same as solving this equation. Solving for u_1, we get

$$u_1 = -I_s \frac{R_1 R_2}{R_1 + R_2}$$

Plugging in the values we were given, we get

$$\begin{aligned} u_1 &= -3\,\mathrm{mA} \frac{2\,\mathrm{k\Omega} \cdot 4\,\mathrm{k\Omega}}{2\,\mathrm{k\Omega} + 4\,\mathrm{k\Omega}} \\ &= -4\,\mathrm{V} \end{aligned}$$

Node u_1 is $-4\,\text{V}$ relative to the ground node we defined. If we had defined the top node as ground, then the bottom node would have measured as $4\,\text{V}$. Similarly, as drawn, we have $V_{R_1} = V_{R_2} = -4\,\text{V}$; if we flipped the polarities, i.e. swapped + and -, we would have $V_{R_1} = V_{R_2} = 4\,\text{V}$.

The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

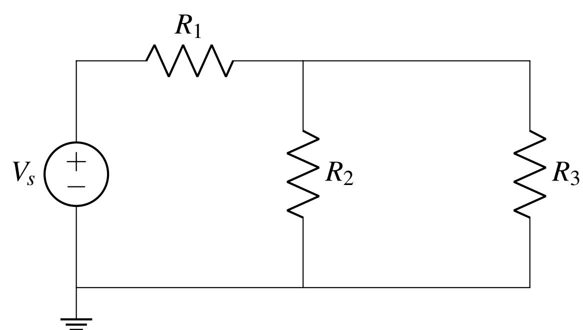
$$i_{R_1} = \frac{u_1}{R_1} = -2\,\text{mA}$$
$$i_{R_2} = \frac{u_1}{R_2} = -1\,\text{mA}$$

Note that

$$i_{R_1} = -I_s \frac{R_2}{R_1+R_2} \qquad i_{R_2} = -I_s \frac{R_1}{R_1+R_2}$$

These are very similar equations to the voltage divider circuit. We call this circuit a current divider.

(b) $V_s = 5\,\text{V}, R_1 = R_2 = 2\,\text{k}\Omega, R_3 = 4\,\text{k}\Omega$

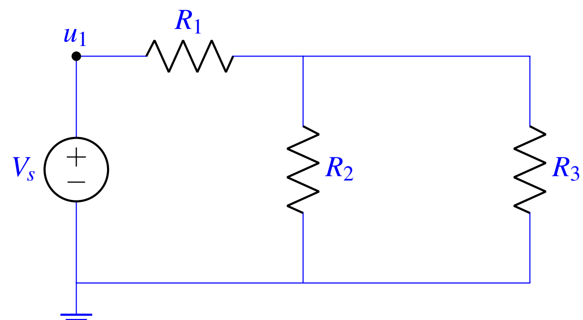


Solution:

Step 1) Define Reference Node

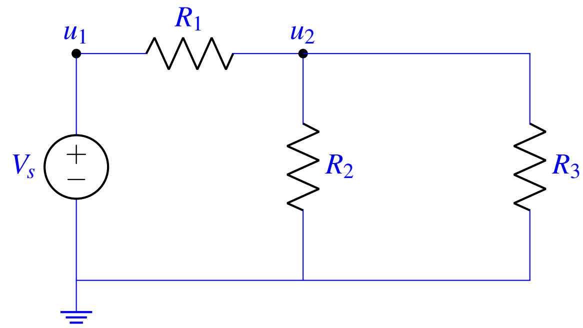
Select a reference (ground) node. Any node can be chosen for this purpose. This has already been done for you in this circuit.

Step 2) Label Nodes with Voltage Set by Sources

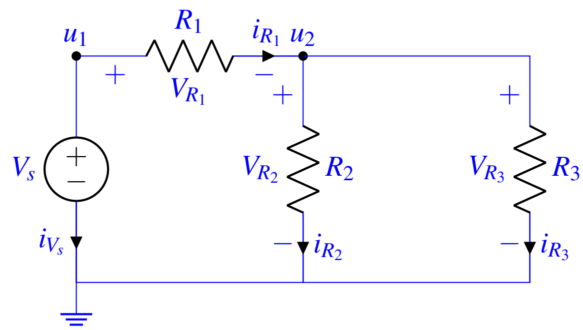


Step 3) Label Remaining Nodes

We only have one other node here, and we label the corresponding source u_2 (names are arbitrary, but must be unique).



Step 4) Label Element Voltages and Currents



Step 5) KCL Equations
Write KCL equations for all nodes with unknown voltage, which is only u_2.

$$i_1 = i_2 + i_3$$

Step 6) Element Currents
Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In this circuit there are three, R_1, R_2, and R_3.

$$i_{R_1} = \frac{V_{R_1}}{R_1}$$
$$i_{R_2} = \frac{V_{R_2}}{R_2}$$
$$i_{R_3} = \frac{V_{R_3}}{R_3}$$

Step 7) Element Voltages

Rewrite the element voltages using the node differences.

$$\begin{aligned} u_1 &= V_s \\ i_{R_1} &= \frac{V_s - u_2}{R_1} \\ i_{R_2} &= \frac{u_2}{R_2} \\ i_{R_3} &= \frac{u_2}{R_3} \end{aligned}$$

Step 8) Substitute Element Currents in KCL Equations
Now we substitute the expressions derived in Step 7 into the KCL equations from Step 5.

$$\frac{V_s - u_2}{R_1} = \frac{u_2}{R_2} + \frac{u_2}{R_3}$$

We can isolate the unknown terms (u_2) on the left and the known on the right

$$\frac{u_2}{R_1} + \frac{u_2}{R_2} + \frac{u_2}{R_3} = \frac{V_s}{R_1}$$

We only have one equation to solve. Setting up the matrix equation would just be the same as solving this equation. Solving for u_2, we get

$$\begin{aligned} u_2 &= \frac{V_s}{R_1} \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \\ &= V_s \frac{1}{1 + R_1(\frac{1}{R_2} + \frac{1}{R_3})} \end{aligned}$$

Plugging in the values we were given, we get

$$\begin{aligned} u_2 &= 5\,\mathrm{V} \frac{1}{1 + 2\,\mathrm{k}\Omega(\frac{1}{2\,\mathrm{k}\Omega} + \frac{1}{4\,\mathrm{k}\Omega})} \\ &= 2\,\mathrm{V} \end{aligned}$$

The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

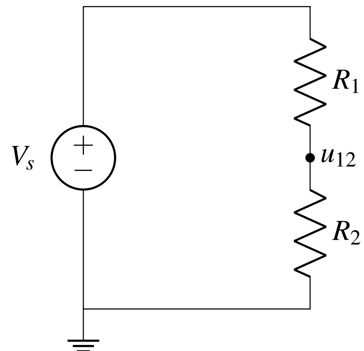
$$\begin{aligned} i_{R_1} &= \frac{u_1 - u_2}{R_1} = \frac{5V - 2V}{2\,\mathrm{k}\Omega} = 1.5\,\mathrm{mA} \\ i_{R_2} &= \frac{u_2}{R_2} = \frac{2V}{2\,\mathrm{k}\Omega} = 1\,\mathrm{mA} \\ i_{R_3} &= \frac{u_2}{R_3} = \frac{2V}{4\,\mathrm{k}\Omega} = 0.5\,\mathrm{mA} \end{aligned}$$

7. Voltage divider

Learning Goal: *This problem will help you practice designing circuits under given conditions using the analysis tools you've learned.*

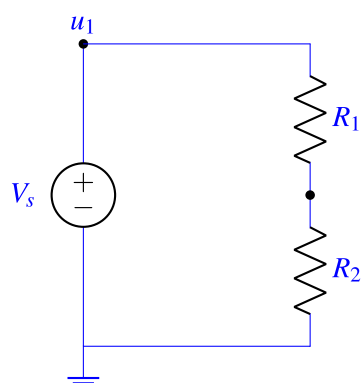
In the following parts, $V_s = 12\,\text{V}$. **Choose resistance values such that the current through each element is $\leq$ 0.8A.**

(a) Select values for R_1 and R_2 in the circuit below such that $u_{12} = 6\,\text{V}$. This is a **design problem**, so there can be more than one set of correct answers to this problem.

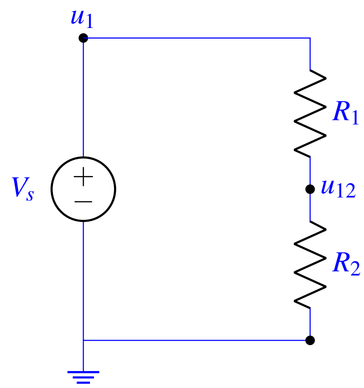


Solution: Step 1: Reference Node
We notice that the ground node has already been selected for us in the question.

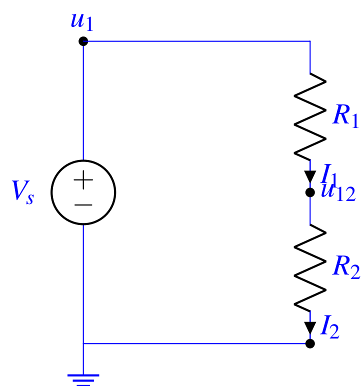
Step 2: Label nodes with voltage set by sources.



Step 3: Label other nodes.



Step 4: Label element voltages and currents.
Let V_{R1}, V_{R2} be the voltage drop across R_1 and R_2 respectively. Let I_1 be the current through R_1. Let I_2 be the current through R_2.



Step 5: KCL Equations
From KCL, at u_{12},

$$I_1 = I_2.$$

Step 6: Find element currents
Using Ohm's law, we have that

$$I_1 = \frac{V_{R1}}{R_1}$$

$$I_2 = \frac{V_{R2}}{R_2}.$$

Writing the element voltages in terms of the node voltages, we have that $V_{R1} = V_s - u_{12}$ and $V_{R2} = u_{12}$.

Step 7: Substitute element currents into KCL Equations.
Substituting back, we obtain $\frac{V_s - u_{12}}{R_1} = \frac{u_{12}}{R_2}$. Solving, we find that $u_{12} = \frac{R_2}{R_1+R_2}V_s$. Plugging in $u_{12} = 6V$ and $V_s = 12V$, we see that $R_1 = R_2$ must be true.

To choose R_1 and R_2 such that the current through each element is ≤ 0.8A, use KVL to write an expression for I_1, I_2 as a function of R_1, R_2:

$V_s - I_1R_1 - I_2R_2 = 0$, with $I_1 = I_2 = I_s$

$V_s = I_s(R_1 + R_2)$
$I_s = \frac{V_s}{(R_1+R_2)}$
We need,
$I_s \leq 0.8\text{A}$
Therefore,
$\frac{12\text{V}}{(R_1+R_2)} \leq 0.8\text{A}$
$R_1 + R_2 \geq \frac{12\text{V}}{0.8\text{A}}$
$R_1 + R_2 \geq 15\,\Omega$
As $R_1 + R_2$ must be at least $15\,\Omega$, and $R_1 = R_2$, we choose $R_1 = R_2 = 7.5\,\Omega$. Any other solution with $R_1 = R_2 = R \geq 7.5\Omega$ is also a valid solution.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.