1 Probability Potpourri

1. We want to prove the covariance matrix Σ is always positive semi-definite, i.e. $a^T \Sigma a >= 0$ for all $n \times 1$ vectors a. Given $\Sigma = E[(Z-\mu)(Z-\mu)^T]$, set the following equation:

$$
\begin{aligned}
a^T E[(Z-\mu)(Z-\mu)^T]a &>= 0 \\
E[a^T(Z-\mu)(Z-\mu)^T a] &>= 0
\end{aligned}
$$

   by linearity of expectation

$$
\begin{aligned}
E[((Z-\mu)a)^T((Z-\mu)a)] &>= 0 \\
E[||(Z-\mu)a||^2] &>= 0
\end{aligned}
$$

2. Let W represent the event that there is a gust of wind and let H represent the event that the archer hits her target. We know that $P(H|W) = 0.4$, $P(H|W^C) = 0.7$, and $P(W) = 0.3$.

   1. P(there is gust of wind and she hits target) $= P(W \cap H)$
      $= P(W) * P(H|W)$
      $= 0.3 * 0.4$
      $= 0.12$
   2. P(hits the target on the first shot)
      $= P(H)$
      $= P(H \cap W) + P(H \cap W^C)$
      $= 0.12 + P(W) * P(H|W^C)$
      $= 0.12 + 0.7 * 0.7$
      $= 0.62$
   3. P(hits the target exactly once in two shots)
      $=P(H_1 \cap H_2^C) + P(H_1^C \cap H_2)$
      $= (0.62)(1-0.62) + (1-0.62)(0.62)$
      $= 0.4712$
   4. P(there was no gust of wind on an occasion when she missed)
      $=P(W^C|H^C)$
      $= \frac{P(W^C \cap H^C)}{P(H^C)}$
      $= \frac{P(W^C)P(H^C|W^C)}{P(H^C)}$
      $= \frac{0.7*0.3}{1-0.62}$
      $= 0.55$

3. Let Y represent the score of a single strike, or $Y = g(X = x)$ such that

$$
g(X = x) = \begin{cases} 4, & \text{if } x \leq \frac{1}{\sqrt{3}} \\ 3, & \text{if } \frac{1}{\sqrt{3}} < x \leq 1 \\ 2, & \text{if } 1 < x \leq \sqrt{3} \\ 0, & \text{otherwise.} \end{cases}
$$

Find expectation $E[Y] = \int yf(y)dy = \int g(x)f(x)dx = \int_0^\infty g(x)\frac{2}{\pi(1+x^2)}dx$

$$= 4 * \frac{2}{\pi} arctan(x) \mid_0^{\frac{1}{\sqrt{3}}} + 3 * \frac{2}{\pi} arctan(x) \mid_{\frac{1}{\sqrt{3}}}^{1} + 2 * \frac{2}{\pi} arctan(x) \mid_1^{\sqrt{3}}$$
$$= 4 * \frac{2}{\pi}(\frac{\pi}{6} - 0) + 3 * \frac{2}{\pi}(\frac{\pi}{4} - \frac{\pi}{6}) + 2 * \frac{2}{\pi}(\frac{\pi}{3} - \frac{\pi}{4})$$
$$= 2.167$$

2 Properties of Gaussians

1. We want to prove that $E[e^{\lambda X}] = e^{\sigma^2 \lambda 2/2}$

3 Linear Algebra Review

1. We will prove equivalence between these three different definitions of PSD using a cycle. First, we will prove that given (b), (b) $\implies$ (a).

$$x^T Ax = x^T \lambda x = \lambda x^T x = \lambda |x|_2 \geq 0$$

We have shown that given A has nonnegative eigenvalues (from (b)), we maintain the inequality given in (a).

Second, we will prove that given (a), (a) $\implies$ (c).

$$x^T UU^T x = (U^T x)^T U^T x = |U^T x|_2 \geq 0$$

We have shown that given the inequality in (a), we can find a matrix U from (c) that upholds (a).

Third, we will prove that given (c), (c) $\implies$ (b).

A is a symmetric matrix

4 Gradients and Norms

1. First, we will prove that $\frac{1}{\sqrt{n}}||x||_2 \leq ||x||_\infty$. We know that $||x||_\infty = max(x)$. Let's use the Cauchy-Schwartz inequality to produce an upper bound for $\frac{1}{\sqrt{n}}||x||_2$. Let $\mathbf{1}$ be the ones $n \times 1$ vector. The Cauchy-Schwarz inequality states that for all vectors u and v of an inner product space, it is true that

$$\begin{aligned}
| < u, v > |^2 \leq &< u, u > \cdot < v, v > \\
| < x, \mathbf{1} > |^2 \leq &< x, x > \cdot < \mathbf{1}, \mathbf{1} > \\
||x||_1^2 \leq &||x||_2^2 \cdot n \\
||x||_1 \leq &\sqrt{n}||x||_2
\end{aligned}$$

Next, we will prove that $||x||_\infty \leq ||x||_1$. We know that $||x||_\infty = \max(x)$ and $||x||_1 = \sum_{i=1}^n |x_i|$, which includes the max and is, by default, at least as large as the L-infinity norm.