Note 4: Inductors and RLC Circuits

1 Inductors

Here, we introduce a new passive component, the inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.
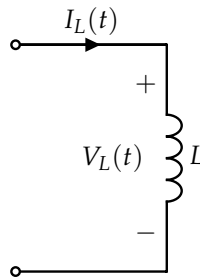
1.1 Basics

Figure 1: Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t}. \tag{1}$$

where L is the *inductance* of the inductor. The SI unit of inductance is the Henry (H). Looking at eq. (1), we observe that an inductor behaves as a capacitor with the roles of current and voltage reversed.

Concept Check: The current across the inductor cannot change instantaneously. Why? *Answer:* If our current changes instantaneously, then $\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \to \infty$, and from eq. (1) the voltage across the inductor $V_L \to \infty$, which is not possible. Hence, our current cannot change instantaneously.

In steady state, when the current flowing through an inductor is constant, there is no voltage drop across the inductor. This may be confusing if we think about resistors, where a constant current corresponds to a *nonzero* constant voltage. But remember, the inductor is a different element! It's essentially a spool of wire wrapped around a conductor. If the current through the inductor is changing, then there will be a voltage drop across the inductor. The energy stored in the inductor turns out to be $E_L = \frac{1}{2}LI^2$, but we won't be using this very much in EECS16B. We are only mentioning it here because it helps us interpret what is happening later.

1.2 Physics behind Inductors
(not in scope for EECS 16B, just for information)

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an electric field ($\vec{E}$), anytime current flows down a conductor, it creates a magnetic field ($\vec{B}$), and this magnetic field can store energy. Inductors' behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries to create useful inductances. A **solenoid** is a good example, where we wind a wire around a conductor like a copper rod:
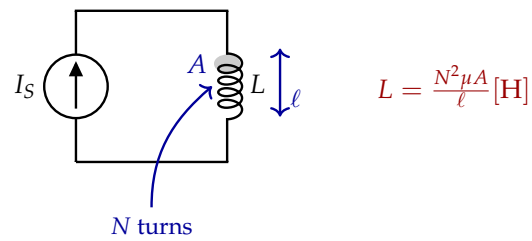
Figure 2: The Inductance of a Solenoid: a wire coiled around something.

Note that the inductance (L) depends on the **geometry** and a material property called **magnetic permeability** (μ) of the solenoid core material. In the case of the solenoid in fig. 2, the inductance depends on the number of turns (N), the length of the solenoid (l) and the area (A) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit (analogous to capacitors).

1.3 Equivalence Relations

Now, let's derive the equivalence relations for series and parallel combinations of inductors. We will find that these are similar to those of resistors, because the law governing an inductor $V_L = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t}$ involves a proportionality constant L that multiplies a current-like quantity to give a voltage. In a resistor, the resistance R multiplies current to give a voltage.

1.3.1 Series Equivalence
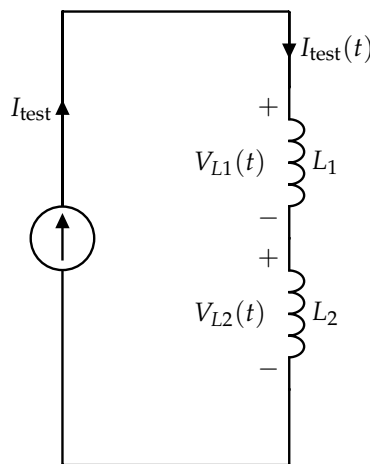
Figure 3: Series Inductor Circuit

Let's apply a $\frac{\mathrm{d}I_{\text{test}}}{\mathrm{d}t}$ through the two inductors, then

$$V_{L1}(t) + V_{L2}(t) = V_L(t)$$

where, $V_L(t)$ is the voltage across the two inductors. From I-V relationship for inductors, we get

$$L_1\frac{\mathrm{d}I_{\text{test}}}{\mathrm{d}t} + L_2\frac{\mathrm{d}I_{\text{test}}}{\mathrm{d}t} = V_L(t) \tag{2}$$

$$(L_1 + L_2)\frac{\mathrm{d}I_{\text{test}}}{\mathrm{d}t} = V_L(t) \tag{3}$$

2

$$L_{\text{eq}} \frac{\mathrm{d}I_{\text{test}}}{\mathrm{d}t} = V_L(t) \tag{4}$$

where, $L_{\text{eq}} = L_1 + L_2$.
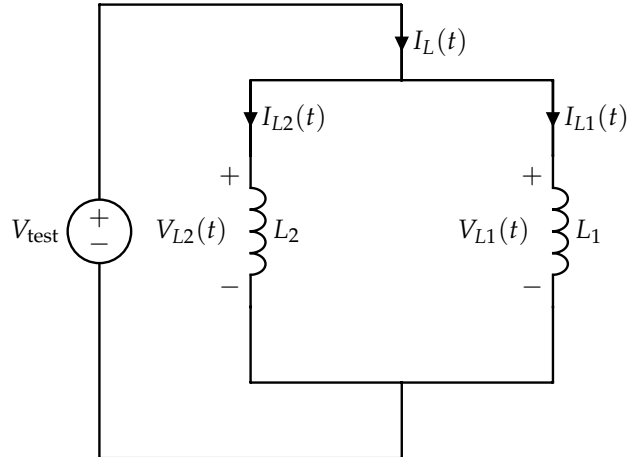
1.3.2 Parallel Equivalence

Figure 4: Parallel Inductor Circuit

We apply at V_{test} across the parallel combination. We have

$$V_{L1}(t) = V_{L2}(t) = V_{\text{test}}(t)$$

$$L_1 \frac{\mathrm{d}I_{L1}}{\mathrm{d}t} = L_2 \frac{\mathrm{d}I_{L2}}{\mathrm{d}t} = L_{\text{eq}} \frac{\mathrm{d}I_L}{\mathrm{d}t}$$

and from KCL, we have

$$I_L(t) = I_{L1}(t) + I_{L2}(t)$$

Differentiating with respect to time, and substituting from the above equality,

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{\mathrm{d}I_{L1}}{\mathrm{d}t} + \frac{\mathrm{d}I_{L2}}{\mathrm{d}t} \tag{5}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{L_{\text{eq}}}{L_1} \frac{\mathrm{d}I_L}{\mathrm{d}t} + \frac{L_{\text{eq}}}{L_2} \frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{6}$$

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} \tag{7}$$

2 LC Tank

In our two resistor, two capacitor circuit example, we found that our eigenvalues were real. But, we could also encounter a system whose eigenvalues are complex. In this section, we will explore a circuit, commonly known as an LC tank, whose matrix will have purely imaginary eigenvalues.

In the circuit in fig. 5, we have an inductor $L = 10\,\text{nH}$ and capacitor $C = 10\,\text{pF}$ in parallel. Let $I_L(0) = 50\,\text{mA}$ and $V_{\text{out}}(0) = 0\,\text{V}$ as the initial conditions: Since the inductor and capacitor are in parallel:
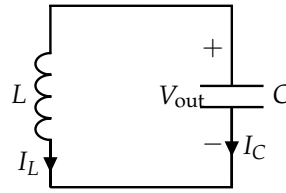
$$V_L = V_C = V_{\text{out}} \tag{8}$$

Figure 5: An LC Tank.

KCL gives:

$$I_L = -I_C = -C\frac{dV_{out}}{dt} \tag{9}$$

$$\implies \frac{dV_{out}}{dt} = -\frac{1}{C}I_L \tag{10}$$

$$V_L = V_{out} = L\frac{dI_L}{dt} \tag{11}$$

$$\implies \frac{dI_L}{dt} = \frac{1}{L}V_{out} \tag{12}$$

Putting it into matrix form (treating the differentiated quantities, V_{out} and I_L as state variables):

$$\begin{bmatrix} \frac{dV_{out}}{dt} \\ \frac{dI_L}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} V_{out} \\ I_L \end{bmatrix} \tag{13}$$

Finding the eigenvalues:

$$\det\left(\begin{bmatrix} -\lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0 \tag{14}$$

$$\implies \lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}} \tag{15}$$

Next, we can find the eigenvectors of the above matrix as $v_1 = \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. We can use these vectors to transform our coordinates to one where the matrix becomes diagonal. More concretely,

$$\begin{bmatrix} V_{out} \\ I_L \end{bmatrix} = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} \widetilde{V}_{out} \\ \widetilde{I}_L \end{bmatrix} \tag{16}$$

As discussed before, once in these new coordinates, our system becomes uncoupled, and we can solve for V_{out} and I_L as follows:

$$\begin{bmatrix} \frac{d\widetilde{V}_{out}}{dt} \\ \frac{d\widetilde{I}_L}{dt} \end{bmatrix} = \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \begin{bmatrix} \widetilde{V}_{out} \\ \widetilde{I}_L \end{bmatrix} \tag{17}$$

$$\implies \frac{d\widetilde{V}_{out}}{dt} = j\frac{1}{\sqrt{LC}}\widetilde{V}_{out}, \qquad \frac{d\widetilde{I}_L}{dt} = -j\frac{1}{\sqrt{LC}}\widetilde{I}_L \tag{18}$$

$$\implies \widetilde{V}_{out} = \widetilde{k}_1 e^{\frac{j}{\sqrt{LC}}t}, \qquad \widetilde{I}_L = \widetilde{k}_2 e^{-\frac{j}{\sqrt{LC}}t} \tag{19}$$

Next, we need to find initial conditions in this new coordinate system. Substituting the given values,

$$\begin{bmatrix} \widetilde{V}_{out}(0) \\ \widetilde{I}_L(0) \end{bmatrix} = \begin{bmatrix} j\sqrt{\frac{L}{C}} & -j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} V_{out}(0) \\ I_L(0) \end{bmatrix} \tag{20}$$

$$= \frac{1}{j20\sqrt{10}} \begin{bmatrix} 1 & j10\sqrt{10} \\ -1 & j10\sqrt{10} \end{bmatrix} \begin{bmatrix} 0 \\ 0.05 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} 2.5 \times 10^{-2} \\ 2.5 \times 10^{-2} \end{bmatrix} \tag{22}$$

Hence, $\widetilde{k}_1 = 2.5 \times 10^{-2}$ and $\widetilde{k}_1 = 2.5 \times 10^{-2}$. Finally, we tranform back to our original coordinate system:

$$\begin{bmatrix} V_{\text{out}} \\ I_L \end{bmatrix} = \begin{bmatrix} j10\sqrt{10} & -j10\sqrt{10} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2.5 \times 10^{-2} e^{j\sqrt{10} \times 10^9 t} \\ 2.5 \times 10^{-2} e^{-j\sqrt{10} \times 10^9 t} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} j0.25\sqrt{10} e^{j\sqrt{10} \times 10^9 t} - j0.25\sqrt{10} e^{-j\sqrt{10} \times 10^9 t} \\ 2.5 \times 10^{-2} e^{j\sqrt{10} \times 10^9 t} + 2.5 \times 10^{-2} e^{-j\sqrt{10} \times 10^9 t} \end{bmatrix} \tag{24}$$

Concept Check: Write the above sum of exponentials as sine and cosine. *Hint: Use the Euler form of sine and cosine we encountered in Note j.*

Based on the intuition we have gained above, let's guess a solution with pure sines and cosines, as follows:

$$V_{\text{out}}(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}} t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}} t\right) \tag{25}$$

Next, plugging in initial conditions to solve for the constants:

$$V_{\text{out}}(0) = 0 = c_1 \tag{26}$$

$$I_C(0) = -I_L(0) = -50 \times 10^{-3} \tag{27}$$

$$\frac{\mathrm{d}V_{\text{out}}(0)}{\mathrm{d}t} = \frac{1}{C} I_c(0) = \frac{-50 \times 10^{-3}}{10^{-11}} = \frac{c_2}{\sqrt{10^{-8} \times 10^{-11}}} \tag{28}$$

$$c_1 = 0 \tag{29}$$

$$\implies c_2 = -\frac{5}{\sqrt{10}} = -0.5\sqrt{10} \tag{30}$$

$$\implies V_{\text{out}}(t) = -0.5\sqrt{10} \sin\left(\sqrt{10} \times 10^9 t\right) \tag{31}$$

Notice that the amplitude of V_{out} is constant.[1]

Concept Check: Follow the same steps above to find the current, $I_L(t)$. *Hint: The current will also be of the form in eq. (25), but with different constants.*

Answer: $I_L(t) = 50 \times 10^{-3} \cos\left(\sqrt{10} \times 10^9 t\right)$

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[1]And, in case the algebra is confusing, the $\frac{c_2}{\sqrt{10^{-8} \times 10^{-11}}}$ part comes from evaluating the derivative of the output voltage at time $t = 0$. That is, $\frac{d}{dt} V_{\text{out}}(t) = c_2 \cos\left(\frac{1}{\sqrt{LC}} t\right)$, and we know the value for this at $t = 0$. Plugging in L, C, gives this equation.
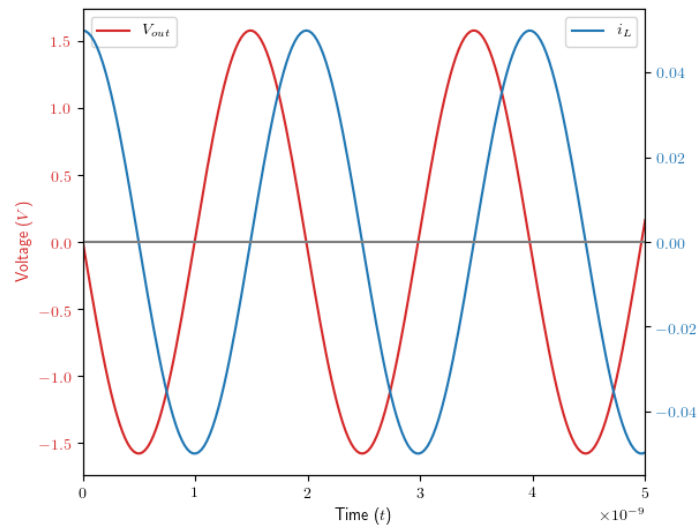
Figure 6: Voltage and Current response of LC Tank

Figure 6 plots the above solutions for the capacitor voltage and inductor current. This system is also called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

From the above plots, we can see that the current and voltage are $90°$ out of phase, i.e. when the current is at its maximum or minimum, the voltage is at 0V, and vice versa. What does this mean for the energy stored in these components? We know the energy in the capacitor is $E_C = \frac{1}{2}CV^2 = 1.25 \times 10^{-11} \sin^2\left(\sqrt{10} \times 10^9 t\right)$ and the energy in the inductor is $E_L = 1.25 \times 10^{-11} \cos^2\left(\sqrt{10} \times 10^9 t\right)$. Figure 7 plots the these energies. The total energy seems to be sloshing back and forth between the inductor and capacitor.
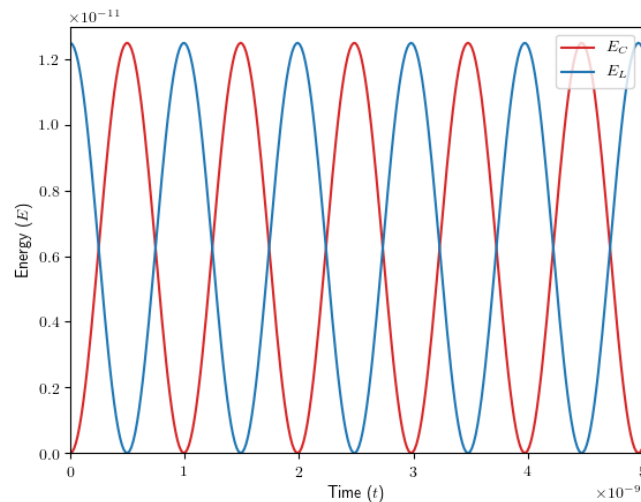
Figure 7: Energy stored in Inductor and Capacitor. Notice the sum is constant.

3   Finding Initial Conditions for Circuits

When analyzing RLC circuits, it can be very useful to use inspection techniques to understand the initial conditions for the circuit. For example, if a problem begins with a statement such as "V_{in} has been $0\,V$ for all time before $t = 0$, at which point it turns on and becomes $2\,V$", it is critical to understand the behavior of the circuit elements before $t = 0$.

Remember the following critical idea to understanding initial conditions (steady-state conditions): *In steady-state, time-derivatives are zero because things are not changing, by definition.* So, if your circuit has quantities like capacitor voltages and inductor currents, know that your steady-state condition must not only satisfy KCL/KVL, but also satisfies the fact that time-differentiated quantities are zero.

The transient analysis that we have seen prior to now is closely related to this concept of steady-state conditions, albeit much more general. For example, transient analysis will reveal that the voltage on the capacitor of an RC circuit decays exponentially when the voltage source turns off. If we want to know the exact waveform of the decay in this way, then transient analysis (setting up and solving a differential equation) is required. However, in many cases, we do not need this full level of detail. In situations where the circuit has been "left alone" for a long time (that is, it has reached steady-state), we can actually solve for certain values in the circuit, such as capacitor voltages or inductor currents, without solving any differential equations! Let's proceed to get some intuition for how this works through common examples.

3.1   RC-RC Circuit: Voltage Source On, Steady-State

We will start with the example of a voltage source connected to the RC-RC circuit in section 3.1 for a long time. More precisely, suppose the voltage source has been on at $V_{in} = 3\,V$ for all time $t < 0$. What are the capacitor voltages? Recall that when we perform transient analysis, solving the differential equation gives us the general shape of the waveform (exponential decay, or asymptotically approaching a specific value), but the initial conditions are necessary to precisely describe the waveform. The values of the capacitor voltages at $t = 0$ will set the initial conditions that inform any later analysis.

Here, we can appeal to our intuition for how capacitors behave. They are "sinks" that can hold a fixed amount of charge, and when they are initially uncharged, the current flowing into them will be high. This rate of current will slowly decrease as the capacitor voltage increases (approaching its steady-state value); this phenomenon is what the governing equation $I_C = C\frac{dV_C(t)}{dt}$ describes. When the capacitor is holding as much charge Q as it can (a limit that is set according to $Q = CV$), no more current can flow into the capacitor (and by the capacitor current equation, $\frac{dV_C(t)}{dt} = 0$.

We can then find that the capacitors have reached their steady-state voltages when the current through each of them is zero ($I_1 = I_2 = 0\,A$). Observing the circuit below, what can we now do to solve for what numerical value this steady-state voltage is? Let's approach it with KVL, first forming a loop clockwise through the voltage source, R_1 and C_1. We find that:

$$V_{in} - I_1 R_1 - V_{C_1} = 0 \tag{32}$$

Since $I_1 = 0$, we rearrange and find $V_{C_1} = V_{in}$. Nice! Repeating the same process for C_2, we find:

$$V_{in} - I_1 R_1 - I_2 R_2 - V_{C_1} = 0 \tag{33}$$

And by the same logic ($I_1 = 0$, $I_2 = 0$), we have $V_{C_2} = V_{in}$. We now have solved for the initial conditions of the given circuit, and if a problem statement said, "What would be the capacitor voltages as a function of time when we turn the voltage source off, and $V_{in} = 0\,V$ for $t \geq 0$?", we would be ready to perform the transient analysis.
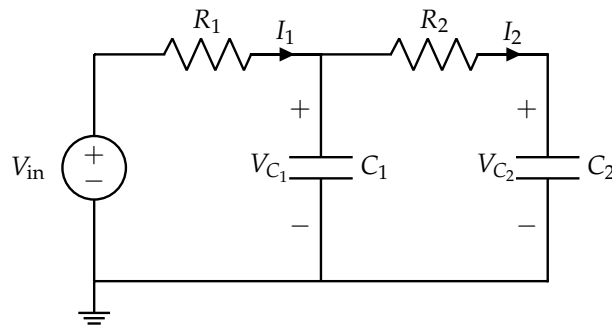
3.2 RC-RC Circuit: Voltage Source Off, Steady-State

There are two ways to approach the question of finding the steady-state capacitor voltages after the voltage source has been *off* for a long time. We could follow the same logic in the previous section and find that $V_{C_2} = V_{C_1} = V_{in} = 0$. Or, we could notice that if the voltage source is off, there's no current in the circuit! The current in the circuit could not be anything other than 0 A, so the capacitors must have a zero steady-state voltage. Even if the capacitors once had some nonzero voltage, the voltage source being turned off means the capacitors would discharge through the resistors (leading to exponential decay of their voltage over time).

Either way, the capacitors' steady-state conditions would be that $V_{C_2} = V_{C_1} = 0 \, \text{V}$ if the voltage source stays off for a long time.

3.3 RL Circuit: Voltage Source On, Steady-State

Suppose we have the RL circuit shown in fig. 8, with a voltage source that has been on at $V_{in} = 5 \, \text{V}$ for a long time. For all $t \leq 0$, the inductor is connected only to the voltage source (the right switch is open, so the resistor is not connected here). What is the steady-state current through the inductor?
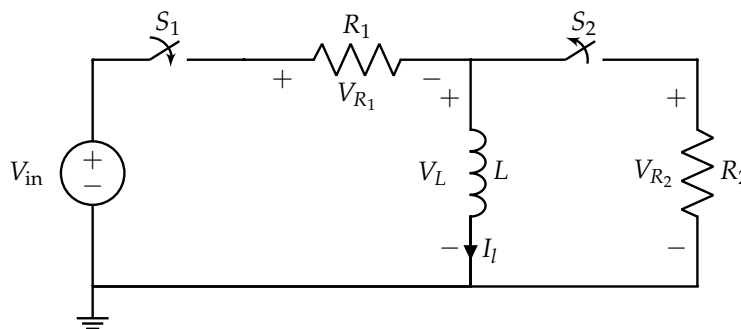

Figure 8: Inductor in series with a resistor and voltage source.

This is a good chance to review some fundamentals of inductor behavior, which are likely less familiar to you than the properties of capacitors. The inductor's governing equation is $V_L(t) = L\frac{dI_L(t)}{dt}$, in which the voltage and current play opposite roles as compared to the capacitor equation. Here, if the voltage source has been constant for a long time, what does that mean for the inductor's steady-state current and voltage?

While we could write out KVL, form a non-homogeneous differential equation, and solve for the current after a long time by plugging in $t = \infty$, we can also use our intuition about inductors. Inductors will have no voltage drop across them once their current reaches steady-state because the rate-of-change of current in steady-state is zero ($\frac{dI_L(t)}{dt} = 0$). So, in steady-state, $V_L = 0 \, \text{V}$. Then, by KVL, the entire voltage drop of the source must be across R_1. So the current in the loop is $\frac{V_S}{R_1}$ by Ohm's Law. Finally, we can write that the

current in the inductor must be the same constant, $\frac{V_S}{R_1}$, in the steady-state.

3.4 RL Circuit: Source Off, Steady-State

Suppose that we now flip a switch after reaching the steady-state solved for above, and $I_L(0) = \frac{V_S}{R_1}$. Flipping the switch disconnects the voltage source and R_1, and connects R_2 to L. A diagram of this is shown in fig. 9. Now, what is the new steady-state current and voltage of the inductor?[2]
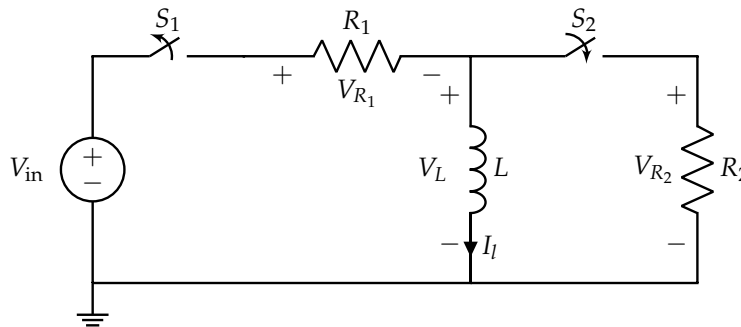

Figure 9: Inductor in series with a resistor.

We can approach this question with our intuition; writing KVL (carefully noting *passive* sign convention for the inductor, and element polarities), we find:

$$-V_L(t) + V_R(t) = 0 \tag{34}$$
$$-V_L(t) + I_R(t)R_2 = 0 \tag{35}$$
$$-L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} - I_L(t)R_2 = 0 \tag{36}$$
$$-L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = I_L(t)R_2 \tag{37}$$
$$\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = -\frac{R_2}{L}I_L(t) \tag{38}$$

Note that $I_L(t) = -I_R(t)$ by KCL at the bottom node, since the resistor's current direction is set by passive sign convention and therefore points down.

This is a homogenous differential equation for $I_L(t)$! We can see that the solution is an exponential, and after considering the initial condition of $I_L(0) = \frac{V_S}{R_1}$ A, we find that $I_L(t) = \frac{V_S}{R_1}e^{-\frac{R_2}{L}t}$. Then, since $V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t}$, we can also see that $V_L(t) = -\frac{V_S}{R_1}L\frac{R_2}{L}e^{-\frac{R_2}{L}t} = -V_S\frac{R_2}{R_1}e^{-\frac{R_2}{L}t}$. Since the inductor is connected only to another passive element (R_2), both its voltage and current will decay exponentially, and after enough time the (steady-state) current and voltage will both equal 0.

One final note: we have seen in previous circuit problems that when a current source is attached to a capacitor, the voltage on that capacitor increases without bound ($I_C = C\frac{\mathrm{d}V_C(t)}{\mathrm{d}t}$ and so if I_C is constant, the rate of increase of V_C is constant). Similarly, we can use the inductor equation $V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t}$ to see that if we connect a *voltage source* to an inductor, that inductor's *current* will also increase without bound. This is why we had to include a resistor R_1 in the circuit above to induce a steady-state current; without it, there would be no steady-state!

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[2]Note that to go from the inductor steady-state conditions to those for the resistor, we write KCL and KVL. KCL dictates that the currents must be the same magnitude; KVL dictates that the voltages must be equal and opposite.

4 RLC Circuits and Higher Order Differential Equations

The LC tank we studied in the previous section was an ideal case where we assumed there was no resistor in the system. But this is rarely true, and we will need to understand how adding this third component will modify our differential equations.

To motivate our discussions, consider the following circuit, with $V_s = 4\text{V}$, $C = 2\text{fF}$, $R = 60\text{k}\Omega$, and $L = 1\mu\text{H}$. Before $t = 0$, switch S_1 is on while S_2 is off. At $t = 0$, both switches flip state (S_1 turns off and S_2 turns on):
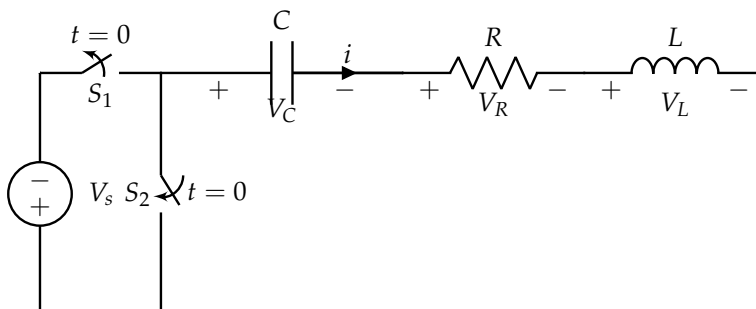
This is something that you will work out in the homework. The key is to follow the steps marking anything with a derivative on it as a state variable, writing out the differential equations, and solving the system.

Contributors:

- Neelesh Ramachandran.
- Kristofer Pister.
- Utkarsh Singhal.
- Aditya Arun.
- Kyle Tanghe.
- Anant Sahai.
- Kareem Ahmad.
- Nikhil Shinde.