Chapter 7. Series of Real num

(7-1 Convergence Tests)

Thm 7.1.1 If $\sum_{k=1}^{\infty} a_k = \alpha$ and $\sum_{k=1}^{\infty} b_k = \beta$, then (a) $\sum_{k=1}^{\infty} c a_k = c\alpha \ \forall c \in \mathbb{R}$

(b) $\sum_{k=1}^{\infty} (a_k + b_k) = \alpha + \beta$

(pf) $\forall n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$

Since the series conv to α and β, respectively, $\lim_{n\to\infty} s_n = \alpha$, $\lim_{n\to\infty} t_n = \beta$.

Therefore $\lim_{n\to\infty}(s_n + t_n) = \alpha + \beta$.

$s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$

$s_n + t_n$: n-th partial sum of series $\sum(a_k + b_k)$

Since $\{s_n + t_n\}$: Conv to $\alpha+\beta$, $\sum_{k=1}^{\infty}(a_k + b_k) = \alpha + \beta$ ∎

(Comparison Test)

Thm 7.1.2 Supp) $\sum a_k$ and $\sum b_k$: non-negative real num, $0 \le a_k \le M b_k$ for some $M > 0$, $\forall k \ge k_0$, fixed $k_0 \in \mathbb{N}$

-(a) If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$

-(b) If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.

(pf) Supp) $\{a_k\}$ $\{b_k\}$ satisfy $a_k \le M b_k \ \forall k \ge k_0$, $M > 0$

then $0 \le \sum_{k=m+1}^{n} a_k \le M \sum_{k=m+1}^{n} b_k \ \forall n > m \ge k_0$

Supp) $\sum b_k$: Conv.

$\forall \varepsilon > 0$, by cauchy criterion, $\exists n_0 \ge k_0$ s.t $\sum_{k=m+1}^{n} b_k < \frac{\varepsilon}{M} \ \forall n > m \ge n_0$

Thus $0 \le \sum_{k=m+1}^{n} a_k < \varepsilon \ \forall n > m \ge n_0$

by cauchy criterion $\sum a_k$: Conv. ∎

Corol 7.1.3 (Limit comparison Test)

Supp/ $\Sigma a_k, \Sigma b_k$: positive real num

-(a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then Σa_k : Conv $\Leftrightarrow$ Σb_k : Conv

-(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_k : Conv $\Rightarrow$ Σa_k : Conv

(pf) -(a) $\exists n_1 \in \mathbb{N}$ s.t $\left| \frac{a_n}{b_n} - L \right| < \frac{1}{2} L$ $\forall n \geq n_1$

$\Rightarrow \frac{1}{2} L < \frac{a_n}{b_n} < \frac{3}{2} L$

$\Rightarrow \frac{1}{2} L b_n < a_n < \frac{3}{2} L b_n$

Then $\Sigma a_n < \infty \Rightarrow \Sigma b_n < \infty$ and $\Sigma b_n < \infty \Rightarrow \Sigma a_n < \infty$

-(b) $\exists n_1 \in \mathbb{N}$ s.t $\left| \frac{a_n}{b_n} - 0 \right| < 1$ $\forall n \geq n_1 \Rightarrow a_n < b_n$

Then. $\Sigma b_n < \infty \Rightarrow \Sigma a_n < \infty$

Ex 7.1.4 -ⓐ $\sum_{k=1}^{\infty} \frac{k}{3^k}$

(sol) Compare with $\Sigma \left(\frac{1}{2}\right)^k$.

$\exists k_0 \in \mathbb{N}$ s.t $\left(\frac{k}{3^k}\right) \leq \left(\frac{1}{2}\right)^k$ $\forall k \geq k_0$. then $\lim_{k\to\infty} k\left(\frac{2}{3}\right)^k = 0$

by taking $\varepsilon = 1$. $\exists k_0$ s.t $k\left(\frac{2}{3}\right)^k \leq 1$ $\forall k \geq k_0$.

$\frac{k}{3^k} \leq \frac{1}{2^k}$ $\forall k \geq k_0$.

Since $\Sigma \left(\frac{1}{2}\right)^k$: Conv. by the Comparison test, $\sum_{k=1}^{\infty} \frac{k}{3^k}$: Conv. ∎

(Integral Test)

Thm 7.1.5) Let $\{a_k\}_{k=1}^{\infty}$: decreasing, non-negative real num

let f : M−D on $[1,\infty)$, $f(k)=a_k$ $\forall k\in\mathbb{N}$

then $\sum_{k=1}^{\infty} a_k < \infty \Leftrightarrow \int_1^{\infty} f(x)dx < \infty$

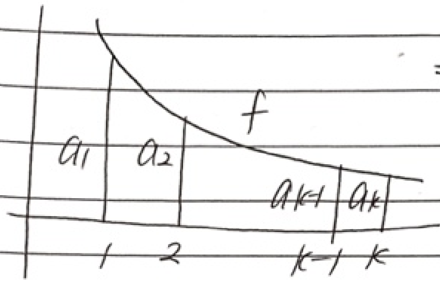
(pf) Since f: Monotone on $[1,\infty)$, by Thm 6.1.8 $f\in R[1,c]$ $\forall c>1$

Let $n\in\mathbb{N}$, $n\geq 2$ $P=\{1,2,\cdots,n\}$: partition of $[1,n]$

Since f is decreasing, $\forall k=2,3,\cdots,n$

$\sup\{f(t) \mid t\in[k-1,k]\} = f(k-1) = a_{k-1}$

$\inf\{f(t) \mid t\in[k-1,k]\} = f(k) = a_k$



$\Rightarrow \sum_{k=2}^{n} a_k = L(P,f) \leq \int_1^n f(x)dx$

$\leq U(P,f) = \sum_{k=1}^{n-1} a_k$ ∎

(Ex 7.1.6) ⓐ $\sum_{k=1}^{\infty} \frac{1}{k^p}$

(sol) when $p=1 \rightarrow$ harmonic series.

If $p\leq 0$, then $\{k^{-p}\}$: not conv to 0, $\Rightarrow \sum \frac{1}{k^p}$: div

Supp/ $p>0$, $p\neq 1$. let $f(x)=x^{-p}$: decreasing $[1,\infty)$

Then $\int_1^c x^{-p}dx = \frac{1}{p-1}\left[1-\frac{1}{c^{p-1}}\right]$

$\int_1^{\infty} x^{-p}dx = \lim_{c\to\infty}\int_1^c x^{-p}dx = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$

By integral test, $\sum \frac{1}{k^p}$: div $p<1$

: conv $p>1$

$\int_1^c \frac{1}{x}dx$ $p=1$

$(=\ln c)$ ∎

Thm 7.1.7) (Ratio and root test)
(Ratio-) Let $\sum a_k$: "+" terms. $R = \overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$, $r = \underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$

(a) If $R<1$, $\sum_{k=1}^{\infty} a_k < \infty$

(b) If $r>1$, $\sum_{k=1}^{\infty} a_k = \infty$

(c) If $r \le 1 \le R$, the test is inconclusive

Thm 7.1.8) (Root test)
Let $\sum a_k$: Non-negative. $\alpha = \overline{\lim_{k\to\infty}} \sqrt[k]{a_k}$

(a) If $\alpha < 1$, $\sum_{k=1}^{\infty} a_k < \infty$

(b) If $\alpha > 1$, $\sum_{k=1}^{\infty} a_k = \infty$

(c) If $\alpha = 1$, then test is inconclusive.

Thm 7.1.10) let $\{a_n\}$: positive num. then
$$\underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} \le \underline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$$

(pf) let $R = \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$

If $R = \infty \Rightarrow$ Nothing to prove
Assume) $R < \infty$, let $\beta > R$ be arbitrary.
$\exists n_0 > 0$ s.t $\frac{a_{n+1}}{a_n} \le \beta \quad \forall n \ge n_0$

$\Rightarrow a_n \le M\beta^n \quad \forall n \ge n_0$, where $M = \frac{a_{n_0}}{\beta^{n_0}}$

$\Rightarrow \sqrt[n]{a_n} \le \beta \sqrt[n]{M} \quad \forall n \ge n_0$
Since $\lim_{n\to\infty} \sqrt[n]{M} = 1$, $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \beta$.

Since $\beta > R$ was arbitrary, $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le R$ ∎

3 - (a) $\sum_{k=0}^{\infty} (\sin p)^k$

(sol) For $p = 2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}$ $\forall n \in \mathbb{N}$. $\sum (\sin p)^k$: div

For $p \in \mathbb{R} \cap \{2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3}{2}\pi\}^c$ $n \in \mathbb{N}$.

$\sum (\sin p)^k = \dfrac{\sin p}{1 - \sin p}$ ■

- (b) $\sum_{k=1}^{\infty} \left(\frac{p}{3}\right)^{2k}$

(sol) $p \geq 3$, $p \leq -3$, $\sum_{k=1}^{\infty} \left(\frac{p}{3}\right)^{2k} = \infty$.

$-3 < p < 3$, $\sum_{k=1}^{\infty} \left(\frac{p}{3}\right)^{2k} = \dfrac{\frac{p^2}{9}}{1 - \frac{p^2}{9}} = \dfrac{p^2}{9 - p^2}$ ■

- (C) $\sum_{k=0}^{\infty} \left(\frac{1+p}{1-p}\right)^k$, $(p \neq 1)$

(sol) * Necessary condition $-1 < \frac{1+p}{1-p} < 1$.

For $1-p > 0$ (i.e. $p < 1$) $-1+p < 1+p < 1-p \Rightarrow p < 0$ (HE)
$1-p < 0$ (i.e $1 < p$) $-1+p < 1+p < 1-p \Rightarrow p > 1$
Thus, for $p > 1$, Conv to
$$\frac{\frac{1+p}{1-p}}{1 - \frac{1+p}{1-p}} = \frac{1+p}{1-p-1-p} = \frac{1+p}{-2p}$$ ■

5 Determine all values of p and q $\sum_{k=2}^{\infty} \frac{1}{k^q (\ln k)^p}$
(Hint : $q > 1$, $q = 1$, $q < 1$)

(pf) Consider $q < 1$, since $\exists k_1 \in \mathbb{N}$ s.t $\ln k \leq k^{\frac{1-q}{p}}$ $\forall k \geq k_1$
Then, $\frac{1}{k^q (\ln k)^p} \geq \frac{1}{k}$ for all $k \geq k_1$, but $\sum_{k=1}^{\infty} \frac{1}{k}$: div

By comparison test, $\sum_{k=2}^{\infty} \frac{1}{k^q (\ln k)^p}$: div $\forall q < 1$, $\forall p$.

Consider $q > 1$. If $p \geq 0$ $\sum_{k=2}^{\infty} \frac{1}{k^q (\ln k)^p}$: Conv.

For $p<0$, $\exists k_1 \in \mathbb{N}$ s.t $k^{\frac{q-1}{2}} > (\ln k)^{-p}$ $\forall k \geq k_1$

Then $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{q+1}{2}}}$: Conv

By Comparison test, $\sum_{k=2}^{\infty} \frac{1}{k^q (\ln k)^p}$: Conv

Consider $q=1$. i.e $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$

Let $f(x) = \frac{1}{x(\ln x)^p}$, then $\int_2^{\infty} f(x)dx = \int_{\ln 2}^{\infty} \frac{1}{u^p} du$.

$u = \ln x$, $du = \frac{1}{x} dx$. it conv $p>1$.

By integral test, $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$: Conv $p>1$. ■

8. Suppl $\{a_n\}$: Seq in $\mathbb{R}$, $a_n > 0$ $\forall n \in \mathbb{N}$ $\forall k \in \mathbb{N}$, $b_k = \frac{1}{k}\sum_{n=1}^{k} a_n$

Proove that $\sum_{k=1}^{\infty} b_k$: div

(pf) $\forall k$, $b_k \geq \frac{1}{k} a_1 > 0$.

$\sum_{k=1}^{\infty} \frac{1}{k} a_1 = \infty$. By Comparison test, $\sum_{k=1}^{\infty} b_k$: div. ■

14 $a_k = \begin{cases} \frac{1}{2^k} & \text{when } k: \text{even} \\ \frac{1}{2^{k+2}} & k: \text{odd} \end{cases}$ Apply root & ratio test

(sol) ① root test

let $\alpha = \overline{\lim_{k\to\infty}} \sqrt[k]{a_k} \to \begin{cases} \overline{\lim_{k\to\infty}} \left(\frac{1}{2^k}\right)^{\frac{1}{k}} & k: \text{even} \\ \overline{\lim_{k\to\infty}} \left(\frac{1}{2^{k+2}}\right)^{\frac{1}{k}} & k: \text{odd} \end{cases} = \frac{1}{2}$: Conv

② ratio test

$R = \overline{\lim_{k\to\infty}} \frac{\frac{1}{2^{k+2}}}{\frac{1}{2^k}} = \overline{\lim_{k\to\infty}} \frac{2^k}{2^{k+2}} = \frac{1}{4} < 1 \Rightarrow$ Conv.

$\underline{\lim_{k\to\infty}} \frac{\frac{1}{2^k}}{\frac{1}{2^{k+2}}} = \frac{1}{4}$

15. Suppl $a_k \ge 0 \ \forall k \in \mathbb{N}$. Proove $\sum a_k$ conv $\Leftrightarrow \{S_{n_k}\}$ of $\{S_n\}$: Conv

(pf) $\forall \varepsilon > 0$. Suppl $\sum_{k=1}^{\infty} a_k = S$. By Cauchy criterion.

($\Rightarrow$) $\exists n_1 \in \mathbb{N}$ s.t $\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon \quad \forall m > n \ge n_1$

$$|S_m - S_n| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon.$$

By cauchy. $\{S_n\}$:conv $\Rightarrow \{S_{n_k}\}$: Conv

($\Leftarrow$) Suppl $S_{n_k} \to S$

Then by cauchy. $\exists n_1 \in \mathbb{N}$ s.t $|S_{n_p} - S_{n_q}| < \varepsilon \quad \forall p > q \ge n_1$

Since $a_k \ge 0 \ \forall k$. $|S_{n_2} - S_{m_2}| < \varepsilon \quad \forall n_2, m_2$ with $n_p \ge n_2 > m_2 \ge n_q$

$$\Rightarrow \forall n_2 > m_2 \ge n_1, \quad \left|\sum_{k=m_2+1}^{n_2} a_k\right| < \varepsilon.$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k : \text{Conv}$$

(7-2 The dirichlet Test)

Thm 7.2.1) (Abel partial sum Fomula) Let $\{a_k\}, \{b_k\}$ be seq of real num.

Set $A_0 = 0$ and $A_n = \sum_{k=1}^{n} a_k$ if $n \ge 1$

Then if $1 \le p \le q$,

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p.$$

(pf) Since $a_k = A_k - A_{k-1}$,

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (A_k - A_{k-1}) b_k$$

$$= \sum_{k=p}^{q} A_k b_k - A_{k-1} b_k = \sum_{k=p}^{q} A_k b_k - \sum_{k=p-1}^{q-1} A_k b_{k+1}$$

$$= \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p \quad \blacksquare$$

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Thm 1.2.2 (Dirichlet Test)

Supp/ $\{a_k\}, \{b_k\}$: seqs of real num, satisfying

- (a) the partial sums $A_n = \sum_{k=1}^{n} a_k$ form a boundd seq
- (b) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and
- (c) $\lim_{k\to\infty} b_k = 0$

Then $\sum_{k=1}^{\infty} a_k b_k$: Conv

(pf) Since $\{A_n\}$: bdd seq. choose $M>0$ s.t $|A_n| \leq M$ $\forall n$.
Also since $b_n \to 0$, $\forall \varepsilon>0$, $\exists n_0>0$ s.t $b_n < \frac{\varepsilon}{2M}$ $\forall n \geq n_0$.

$\ni$ if $n_0 \leq p \leq q$, by the partial sum formula.

$$\left|\sum_{k=1}^{q} a_k b_k\right| = \left|\sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p\right|$$

$$\leq \sum_{k=p}^{q-1} |A_k|(b_k - b_{k+1}) + |A_q| b_q + |A_{p-1}| b_p$$

$$\leq M\left(\sum_{k=p}^{q-1}(b_k - b_{k+1}) + b_q + b_p\right)$$

$$\leq 2M b_p < \varepsilon.$$

By Cauchy criterion, $\sum a_k b_k$: Conv ∎

(Alternating series)

Thm 1.2.3) If $\{b_k\}$: seq of real num satisfying

- (a) $b_1 \geq b_2 \geq \cdots \geq 0$
- (b) $\lim_{k\to\infty} b_k = 0$

then $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$: Conv

(pf) ↓

(pf) Let $a_k = (-1)^{k+1}$.
Then $|A_n| \le 1 \ \forall n$, and the dirichlet test applies. ■

Thm 7.2.4) Consider $\Sigma (-1)^{k+1} b_k$, $\{b_k\}$ satisfies Thm 7.2.3
let $S_n = \sum_{k=1}^{n} (-1)^{k+1} b_k$, $S = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$.

Then $|S - S_n| \le b_{n+1} \ \forall n \in \mathbb{N}$.

(pf) Consider $\{S_{2n}\}$
Since $S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} b_k = (b_1 - b_2) + \cdots + (b_{2n-1} - b_{2n})$
and $(b_{k-1} - b_k) \ge 0 \ \forall k$.
$\Rightarrow \{S_{2n}\}$: M-I.
Similarly $\{S_{2n+1}\}$: M-D.
Since $\{S_n\} \to S$, so do the subseqs $\{S_{2n}\}, \{S_{2n+1}\}$
$\Rightarrow S_{2n} \le S \le S_{2n+1} \ \forall n \in \mathbb{N}$
$\Rightarrow |S - S_k| \le |S_{k+1} - S_k| = b_k \ \forall k \in \mathbb{N}$. ■

Thm 7.2.6) (Trigonometric series)
Supp $\{b_k\}$: $b_1 \ge b_2 \ge \cdots \ge 0$ and $\lim_{k\to\infty} b_k = 0$

-(a) $\sum_{k=1}^{\infty} b_k \sin kt$ conv $\forall t \in \mathbb{R}$

-(b) $\sum_{k=1}^{\infty} b_k \cos kt$ conv $\forall t \in \mathbb{R}$. except $t = 2p\pi$, $p \in \mathbb{Z}$

(pf) For $t \ne 2p\pi$, $p \in \mathbb{Z}$. $\sum_{k=1}^{n} \sin kt = \dfrac{\cos\frac{1}{2}t - \cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}$ -(1)

$\sum_{k=1}^{n} \cos kt = \dfrac{\sin(n+\frac{1}{2})t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}$ -(2)

-(1) : set $A_n = \sum_{k=1}^{n} \sin kt$.

Using $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$$(\sin\tfrac{1}{2}t)A_n = \sum_{k=1}^{n} \sin\tfrac{1}{2}t \sin kt$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\cos(k-\tfrac{1}{2})t - \cos(k+\tfrac{1}{2})t\right)$$

$$= \frac{1}{2}\left(\cos\tfrac{1}{2}t - \cos(n+\tfrac{1}{2})t\right)$$

Thus for $t \neq 2p\pi,\ p \in \mathbb{Z}$. $A_n = \dfrac{\cos\frac{1}{2}t - \cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}$

Therefore $|A_n| \leq \dfrac{|\cos\frac{1}{2}t| + |\cos(n+\frac{1}{2})t|}{2|\sin\frac{1}{2}t|} \leq \dfrac{1}{|\sin\frac{1}{2}t|}$

$t \neq 2p\pi,\ p \in \mathbb{Z}$.

$\Rightarrow$ By dirichlet test. $\sum b_k \sin kt$: Conv $\forall t \neq 2p\pi, p \in \mathbb{Z}$.
If $t = 2p\pi,\ p \in \mathbb{Z}$. $\sin kt = 0\ \ \forall k$.
$\Rightarrow \sum b_k \sin kt$: Conv $\forall t \in \mathbb{R}$.

-(b) When $t = 2p\pi$. $\cos k\pi = 1\ \forall k \in \mathbb{N} \Rightarrow$ may or may not conv. ∎

(7-3 Absolute and Conditional Conv)

Def 7.3.1) $\sum a_k$: absolute conv if $\sum |a_k|$: Conv

$\sum a_k$: Conditionally conv if $\sum a_k$: Conv but not absolute conv.

(Ex 7.3.2) (a) Since $\{\frac{1}{k}\}$: decreases to "0". $\sum \frac{(-1)^{k+1}}{k}$: Conv.
However. $\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty}\frac{1}{k} = \infty$.

Thus $\sum (-1)^{k+1}/k$ is conditionally conv. ∎

(b) $\sum \frac{(-1)^{k+1}}{k^2}$. By the alternating series conv. (Thm 7.2.3)

Furthermore. since $\sum \frac{1}{k^2} < \infty$. the series : Absolutely conv. ∎

Thm 7.3.3) If Σa_k : Conv absolutely, then Σa_k : Conv and

$$\left|\sum_{k=1}^{\infty} a_k\right| \le \sum_{k=1}^{\infty} |a_k|$$

(pf) Suppl Σa_k : conv absolutely i.e) $\Sigma |a_k| < \infty$.

By T-I, for $1 \le p \le q$,

$$\left|\sum_{k=p}^{q} a_k\right| \le \sum_{k=p}^{q} |a_k|$$

Thus By cauchy criterion. Σa_k : conv.

$p=1 \Rightarrow \left|\sum_{k=1}^{\infty} a_k\right| = \lim_{q\to\infty} \left|\sum_{k=1}^{q} a_k\right| \le \lim_{q\to\infty} \sum_{k=1}^{q} |a_k|$

$$\le \sum_{k=1}^{\infty} |a_k| .$$

P301 Rearrangement of sequence

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Def 7.3.6) $\Sigma a_k'$: Re arrangement of Σa_k
(재 배 열)

$\Leftrightarrow \exists j : \mathbb{N} \to \mathbb{N}$: 1-1 & onto s.t $a_k' = a_{j(k)}$ $\forall k \in \mathbb{N}$

Q) If Σa_k : Conv $\Rightarrow$ $\Sigma a_k'$: Conv? No
If so, $\Sigma a_k = \Sigma a_k'$? No

Ex 7.3.7) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$: Conv to $\ln 2$

$\frac{1}{k} = b_k$ 로 두고 교대급수 판정 or Dirichelt Test

* 디리클레 판정법
실수열 (a_n)이 감소수열이며 0으로 수렴한다 하자.
어떤 복소수 수열 (b_n)의 부분합 $\sum_{k=0}^{n} b_k$ 이 유계
즉 $\left|\sum_{k=0}^{n} b_k\right| \leq M$ 인 상수 M이 존재하면 $b_0 a_0 + b_1 a_1 + \cdots$ 은 수렴.

* 교대급수판정법
$b_n = (-1)^n$ 이면 부분합이 0 또는 1이므로, 디리클레 판정 적용 가능//

$\Sigma a_k' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ -(*)

let $S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

짝수까지 홀수까지
$\underset{\text{증가. bdd above}}{\underline{S_{2n}}} < S < \underset{\text{감소. bdd below}}{\underline{S_{2n+1}}}$

$\Rightarrow S < S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
Let S_n' : nth Partial sum of (*)

$S_{3n}' = \sum_{k=1}^{n}\left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k}\right) = \sum_{k=1}^{n} \frac{8k-3}{2k(4k-1)(4k-3)} \to$ Conv as $n \to \infty$

$0 < \frac{8k-3}{2k(4k-1)(4k-3)} < \frac{M}{k^2}$ (비교판정)
↳ 양이라는 보장 있어야 사용 가능

let $S' = \lim_{n\to\infty} S'_{3n}$

$S'_{3n+1} = S'_{3n} + \frac{1}{4n+1} \to S'$ as $n\to\infty$

$S'_{3n+2} = S'_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3} \to S'$ as $n\to\infty$

$\therefore \lim_{n\to\infty} S_n' = S'$

since $\frac{5}{6} = S_3' < S_6' < \cdots < S'$

$\Rightarrow S' = \lim_{n\to\infty} S_n' > \underset{\underset{S_3}{\parallel}}{\frac{5}{6}} > S$

$\therefore S \neq S'$

p302

Thm 1.3.8) Σa_k : Conv Absolutely (절대수렴).
$\forall a_k'$: Conv to the same "sum"
($\Leftrightarrow$ If $\Sigma|a_k| < \infty$, $\forall \Sigma a_k'$, $\Sigma a_k' = \Sigma a_k$)
양항급수에서 급수가 수렴 $\Leftrightarrow$ 유계이다
(동치)
'단조정리'에 의해 - 급수가 증가하고 수렴하면 유계

(pf) $\Sigma a_k'$: rearrangement of Σa_k
since $\Sigma |a_k| < \infty$ $\forall \varepsilon > 0$
$\exists N \in \mathbb{N}$ s.t $\sum_{k=n}^{m} |a_k| < \varepsilon$ $\forall m \geq n \geq N$

suppl $a_k' = a_{j(k)}$
$j : \mathbb{N} \to \mathbb{N}$: 1-1 & onto 이므로 항상 p가 존재.
Choose $p \geq N$ s.t $\{1, 2, \cdots, N\} \subset \{j(1), j(2), \cdots, j(p)\}$
let $S_n = \sum_{k=1}^{n} a_k$, $S_n' = \sum_{k=1}^{n} a_k'$

If $n \geq p$, $S_n - S_n' = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{j(k)}$

$\Rightarrow \{a_1, a_2, \cdots, a_N\} \subseteq \{a_1, \cdots, a_n\}$ & $\{a_1, a_2, \cdots, a_N\} \subseteq \{a_{j(1)}, \cdots, a_{j(p)}, \cdots, a_{j(n)}\}$

$\Rightarrow |S_n - S_n'| = |\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{j(k)}|$

$\le \sum_{k=N+1}^{n} |a_k| + \sum_{\{j(1), \cdots, j(n)\} \setminus \{1, \cdots, N\}} |a_{j(k)}|$

$< \varepsilon + \varepsilon = 2\varepsilon$

$\therefore \lim_{n\to\infty} S_n' = \lim_{n\to\infty} S_n$

$\therefore \forall \Sigma a_k'$: 재배열 $\Rightarrow \Sigma a_k' = \Sigma a_k$

p303 Thm 7.3.9) (결론)

Σa_k : Conditionally conv (조건부 수렴)

($\Leftrightarrow \Sigma a_k$: Conv But $\Sigma |a_k| = +\infty$)

Suppl $\forall \alpha \in \mathbb{R}$, $\exists \Sigma a_k'$: rearrangement of Σa_k

s.t $\Sigma a_k' = \alpha$ $(-\infty \le \alpha \le \infty)$

(EX) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$: Conv conditionally

(i) "+"-Terms (ii) "−"-Terms

$p_k = \frac{1}{2k-1}, k \in \mathbb{N}$ $Q_k = \frac{1}{2k}, k \in \mathbb{N}$

$\Sigma p_k = \Sigma Q_k = +\infty$ (div)

$(\lim_{k\to\infty} p_k = \lim_{k\to\infty} Q_k = 0)$

let $\alpha = \frac{3}{2}$, let $m_1 \in \mathbb{N}$: the smallest s.t

$p_1 + p_2 + \cdots + p_{m_1} > \alpha$ $(1 + \frac{1}{3} + \frac{1}{5} > \frac{3}{2}, m_1 = 3)$

& let $n_1 \in \mathbb{N}$: the smallest s.t

$p_1 + \cdots + p_{m_1} - Q_1 - \cdots - Q_{n_1} < \alpha$

$(1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} < \frac{3}{2} \Rightarrow n_1 = 1)$

$\vdots$

Choose the smallest m_k, n_k at each step.

$\Rightarrow \Sigma a_k' = \alpha$ //

(pf of Thm 7.39) " Σa_k "

(Assume) $a_k \neq 0 \quad \forall k \geq 0$

let $p_k = \frac{1}{2}(|a_k| + a_k)$, $q_k = \frac{1}{2}(|a_k| - a_k)$

$\Rightarrow p_k - q_k = a_k$, $p_k + q_k = |a_k|$

If $a_k > 0 \Rightarrow q_k = 0$, $p_k = a_k$

If $a_k < 0 \Rightarrow p_k = 0$, $q_k = |a_k|$

*) Σp_k, Σq_k : Div to ∞

($\because \Sigma(p_k + q_k) = \Sigma|a_k| = +\infty$

$\Rightarrow \Sigma p_k$ or Σq_k : Div to ∞

& $\Sigma(p_k - q_k) = \Sigma a_k$: Conv

$\Rightarrow$ (Σp_k, Σq_k : Conv

Σp_k, 또 q_k : Div ✓

Let $p_1, p_2, p_3, \cdots$: +- Terms of Σa_k

($p_j \neq 0$, $j \geq 1$)

$Q_1, Q_2, Q_3 \cdots$: Absolute Values of "$-$"-terms of Σa_k

($Q_i \neq 0$, $i \geq 1$)

Σp_k, ΣQ_k : Div to $+\infty$

$$\begin{aligned}\Sigma a_k' = &\overbrace{p_1 + p_2 + \cdots + p_{m_1}}^{>\alpha} \overbrace{- Q_1 - \cdots - Q_{n_1}}^{<\alpha} \\ &\overbrace{+ p_{m_1+1} + \cdots + p_{m_2}}^{>\alpha} \overbrace{- Q_{n_1+1} - \cdots - Q_{n_2}}^{<\alpha} \\ &\overbrace{+ p_{m_2+1} + \cdots + p_{m_3}}^{>\alpha} \overbrace{- Q_{n_2+1} - \cdots - Q_{n_3}}^{<\alpha}\end{aligned}$$

$\lim p_k, Q_k \to 0$ as $k \to \infty$ 이므로 갈수록 정밀해짐

$-\infty \leq \alpha \leq \infty$ α는 어떤 수 잡든 가능 (Σp_k, ΣQ_k : Div 이므로)

Let m_1 : the smallest s.t $X_1 = p_1 + \cdots + p_{m_1} > \alpha$

(m_1 : Exist ($\because \Sigma p_k = \infty$))

n_1 : the smallest s.t $Y_1 = X_1 - Q_1 - Q_2 - \cdots - Q_{n_1} < \alpha$

(n_1 : Exist ($\because \Sigma Q_k = \infty$))

Suppl $\{m_1, \cdots, m_k\}$ $\{n_1, \cdots, n_k\}$: have been chosen.
let m_{k+1}, n_{k+1} : the smallest s.t

$\begin{cases} X_{k+1} = Y_k + P_{m_k+1} + \cdots + P_{m_{k+1}} > \alpha \\ X_{k+1} - P_{m_{k+1}} \le \alpha \quad (i) \end{cases}$

$\begin{cases} Y_{k+1} = X_{k+1} - Q_{n_k+1} - \cdots - Q_{n_{k+1}} < \alpha \\ Y_{k+1} + Q_{n_{k+1}} \ge \alpha \quad (ii) \end{cases}$

$\Rightarrow 0 < X_{k+1} - \alpha \le P_{m_{k+1}}$ By (i)
& $0 < \alpha - Y_{k+1} \le Q_{n_{k+1}}$ By (ii)
Since Σa_k : Conv. $\lim_{k\to\infty} P_k = \lim_{k\to\infty} Q_k = 0$

$\Rightarrow \lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k = \alpha$

Let S_n' : nth partial sum of $\Sigma a_k'$
(i) If $a_n' = P_{\square}$
$\cdots Y_k + P + \cdots + \underset{a_n'}{P_{m_{k+1}}}$

$\Rightarrow \exists k \in \mathbb{N}$ s.t $\underset{\downarrow \alpha}{Y_k} < S_n' \le \underset{\downarrow \alpha}{X_{k+1}}$

(ii) $a_n' = -Q_{\square}$
$Y_{k+1} = X_{k+1} - Q - \cdots - \underset{a_n'}{Q_{\square}}$
$\Rightarrow \exists k \in \mathbb{N}$ s.t $\underset{\downarrow \alpha}{Y_{k+1}} \le S_n < \underset{\downarrow \alpha}{X_{k+1}}$

By (i), (ii) $\lim_{n\to\infty} S_n' = \Sigma a_k' = \alpha$ ∎

Rmk) (암용X) Σa_k : Conditionally Conv
$\forall \alpha, \beta \in \mathbb{R}$, $-\infty \le \alpha \le \beta \le \infty$.
$\exists \Sigma a_k'$: Rearrangement of Σa_k s.t $\underline{\lim}_{n\to\infty} S_n = \alpha$, $\overline{\lim}_{n\to\infty} S_n = \beta$
$(S_n' = \sum_{k=1}^{n} a_k')$

(pf) $\Sigma a_k' = \overbrace{P_1 + \cdots + P_{m_1}}^{>\beta} - \overbrace{Q_1 - \cdots - Q_{n_1}}^{<\alpha}$
$+ \overbrace{P_{m_1+1} + \cdots + P_{m_2}}^{>\beta} - \overbrace{Q_{n_1+1} - \cdots - Q_{n_2}}^{<\alpha}$