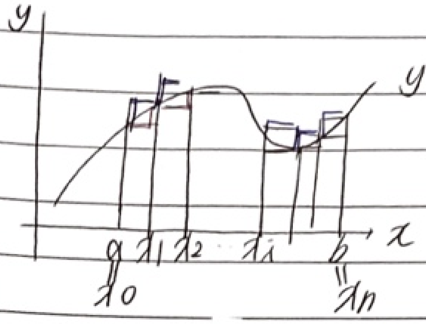
Chapter 6. The Riemann. R-Stieltjes Integral



$L \le S \le U$

< Upper & Lower Sums >
(상합) (하합)

let $[a,b]$ $(a<b)$: closed bdd in $\mathbb{R}$

def) $P = \{x_0, x_1, \cdots, x_n\}$: partition (분할) of $[a,b]$ (with $x_0 = a$, $x_n = b$)

$\Leftrightarrow a = x_0 < x_1 < x_2 < \cdots < x_n = b$

denote) $i = 1, 2, \cdots, n$

$\Delta x_i = x_i - x_{i-1}$

* supp/ $f : [a,b] \to \mathbb{R}$: bdd ft.

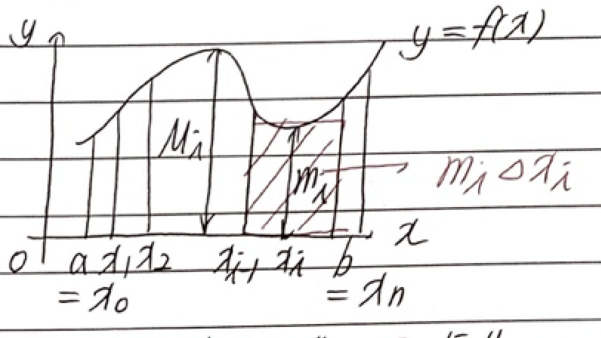
$P = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

$m_i = \inf\{f(t) \mid t \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\}$

: Exist for $i = 1, 2, \cdots, n$

→ f 그래프 구멍 뚫려도 그냥 그래프대로 하고 연속일 땐 max, min 인데 연속일 필요 없으니 → sup, inf.



If $f : [a,b] \to \mathbb{R}$: Conti then $\exists s_i \in [x_{i-1}, x_i]$ s.t. $m_i = f(s_i)$

& $\exists t_i \in [x_{i-1}, x_i]$ s.t. $M_i = f(t_i)$

def) $P = \{x_0, x_1, \cdots, x_n\}$: Partition of $[a,b]$ &

$f : [a,b] \to \mathbb{R}$: Bdd ft

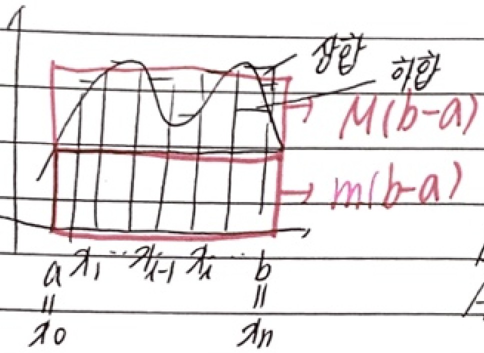
① $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$: "The upper sum" of f for P.
(상합)

(ii) $L(p,f) = \sum_{i=1}^{n} m_i \Delta x_i$: "The Lower sum" of f for p

Since $m_i \le M_i \ \forall i = 1, 2, \cdots, n$
$\Rightarrow L(p,f) \le U(p,f) \ \forall p$

<Upper & Lower Integrals>
(상적분) (하적분)

If $f : [a,b] \to \mathbb{R}$: bdd then $\exists\, m (\le) M \in \mathbb{R}$ s.t
$m \le f(t) \le M \ \forall t \in [a,b]$



$m(b-a) \le L(p,f) \le U(p,f) \le M(b-a)$
$\forall p$: partition of $[a,b]$
let $A = \{U(p,f) \mid p : \text{partition of } [a,b]\}$
$B = \{L(p,f) \mid p : \text{partition of } [a,b]\}$
A, B : Bdd below & above.

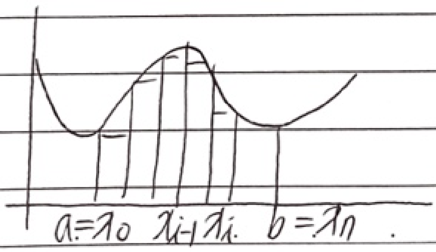
Def 6.1.1) $f : [a,b] \to \mathbb{R}$: bdd ft

(i) $\overline{\int_a^b} f = \inf\{U(p,f) \mid p : \text{partition of } [a,b]\}$
: "The upper integral of f" (상적분)

(ii) $\underline{\int_a^b} f = \sup\{L(p,f) \mid p : \text{partition of } [a,b]\}$
: "The Lower integral of f" (하적분)

Def 6.1.2) p, p^* : patitions of $[a,b]$. p^* : refinement of $p \Leftrightarrow p \subset p^*$

(ex) p_1, p_2 = partitions of $[a,b] \Rightarrow p_1 \cup p_2$: refinement of p_1 & p_2



$p = \{x_0, x_1, \cdots, x_{i-1}, x_i, \cdots, x_n\}$
$p^* = \{x_0, x_1, \cdots, x_{i-1}, x^*, x_i, \cdots, x_n\}$
$L(p,f) \le L(p^*,f) \le U(p^*,f) \le U(p,f)$

lemma 6.1.2 P, P^* : partitions of $[a,b]$ s.t $P \subset P^*$
then $\mathcal{L}(P,f) \le \mathcal{L}(P^*,f) \le \mathcal{U}(P^*,f) \le \mathcal{U}(P,f)$

(pf) $P = \{x_0, x_1, \cdots, x_n\}$, $P^* = P \cup \{x^*\}$, $x^* \ne x_j$ $(j = 0, 1, \cdots, n)$
$\exists k \in \{1, 2, \cdots, n\}$ s.t $x_{k-1} < x^* < x_k$
P^* : refinement of P
let $m_k = \inf\{f(t) \mid t \in [x_{k-1}, x_k]\}$
$m_k^1 = \inf\{f(t) \mid t \in [x_{k-1}, x^*]\}$
$m_k^2 = \inf\{f(t) \mid t \in [x^*, x_k]\}$
$m_k \le m_k^1, m_k^2$
$\Rightarrow m_k^1(x^* - x_{k-1}) + m_k^2(x_k - x^*) \ge m_k \underbrace{(x_k - x_{k-1})}_{\Delta x_k}$

Thus. $\mathcal{L}(P^*,f) = \sum_{i=1}^{k-1} m_i \Delta x_i$
$+ m_k^1(x^* - x_{k-1}) + m_k^2(x_k - x^*)$
$+ \sum_{l=k+1}^{n} m_l \Delta x_l$
$\ge \sum_{l=1}^{k-1} m_l \Delta x_l + m_k \Delta x_k + \sum_{l=k+1}^{n} m_l \Delta x_l$
$= \mathcal{L}(P,f)$
$\therefore P^*$: refinement of P

세분하면 상합은 감소.

$\Rightarrow \mathcal{L}(P,f) \le \mathcal{L}(P^*,f) \le \mathcal{U}(P^*,f) \le \mathcal{U}(P,f)$

Thm 6.1.4 $f : [a,b] \to \mathbb{R}$: Bdd ft $\Rightarrow \underline{\int_a^b} f \le \overline{\int_a^b} f$

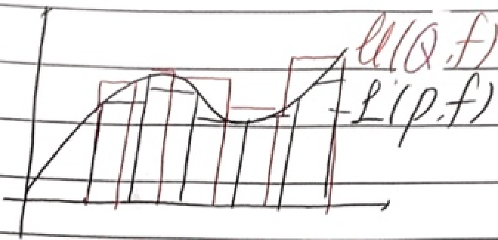
(pf) $\forall P, Q$: partitions of $[a,b]$

상합 세분하면 감소이므로

$\mathcal{L}(P,f) \le \mathcal{L}(P \cup Q, f) \le \mathcal{U}(P \cup Q, f) \le \mathcal{U}(Q,f)$

P.Q 는 서로 독립.

$\Rightarrow \mathcal{L}(P,f) \le \mathcal{U}(Q,f) \quad \forall P,Q$: partition of $[a,b]$



* 세분하면 상합은 감소하다

(pf) Thus $\mathcal{L}(P^*,f) = \sum_{i=1}^{k-1} m_i \Delta x_i + m_k^1(x^* - x_{k-1})$

$$+ m_k^2(x_k - x^*) + \sum_{i=k+1}^{n} m_i \Delta x_i$$

$$\ge \sum_{i=1}^{k-1} m_i \Delta x_i + m_k \Delta x_k + \sum_{i=k+1}^{n} m_i \Delta x_i$$

$$= \mathcal{L}(P,f)$$

$\therefore \mathcal{L}(P^*,f) \ge \mathcal{L}(P,f)$

$\therefore P^*$: refinement of P

$\Rightarrow \mathcal{L}(P,f) \le \mathcal{L}(P^*,f) \le \mathcal{U}(P^*,f) \le \mathcal{U}(P,f)$

$\Rightarrow P^*$: refinement of P. $\boxed{M_k \ge M_k^1, M_k^2}$

let $M_k = \sup\{f(t) \mid t \in [x_{k-1}, x_k]\}$

$M_k^1 = \sup\{f(t) \mid t \in [x_{k-1}, x^*]\}$

$M_k^2 = \sup\{f(t) \mid t \in [x^*, x_k]\}$

$\Rightarrow \mathcal{U}(P^*,f) = \sum_{i=1}^{k-1} M_i \Delta x_i + M_k^1(x^* - x_{k-1})$

$\le M_k(x_{k-1} - x_k)$

$$+ M_k^2(x_k - x^*) + \sum_{i=k+1}^{n} M_i \Delta x_i$$

$$\le \sum_{i=1}^{k-1} M_i \Delta x_i + M_k(x_{k+1}-x_k) + \sum_{i=k+1}^{n} M_i \Delta x_i$$

$$= U(P,f)$$

$\therefore U(P^*,f) \le U(P,f)$ ■

$A = \{ U(P,f) \mid P : \text{partition of } [a,b] \}$
$B = \{ L(P,f) \mid P : \text{partition of } [a,b] \}$

$$\underbrace{\qquad}_{B}\ \underbrace{\qquad}_{A}$$

$m(b-a)$ l l $M(b-a)$

A의 하계 B의 상계

$\sup B \le \inf A$

$\therefore \underline{\int_a^b} f = \sup L(P,f)$

$\le \inf U(Q,f) = \overline{\int_a^b} f$

$\therefore \underline{\int_a^b} f \le \overline{\int_a^b} f$ (=이 된다면 리만적분가능) ■

$f : [a,b] \to \mathbb{R}$: Bdd $\Rightarrow \exists \underline{\int_a^b} f$, $\exists \overline{\int_a^b} f$ s.t $\underline{\int_a^b} f \le \overline{\int_a^b} f$

def 6.1.5) $f : [a,b] \to \mathbb{R}$: bdd ft.
f : (Riemann) Integrable on $[a,b]$ $\Leftrightarrow \underline{\int_a^b} f = \overline{\int_a^b} f$

denote) f : R-Integrable on $[a,b]$

① $\int_a^b f$ or $\int_a^b f(x)dx$

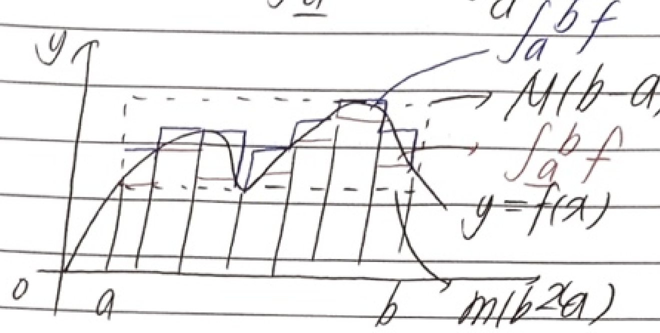
② $R[a,b] = \{ f \mid f : \text{R-Integrable on } [a,b] \}$

④. $f \in R[a,b]$. $(a<b)$ $\int_a^b f = -\int_b^a f$

* $f: [a,b] \to \mathbb{R}$: bdd ft

$\Rightarrow \exists\, m (\le) M \in \mathbb{R}$ s.t $m \le f(t) \le M \quad \forall t \in [a,b]$

$\Rightarrow m(b-a) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le M(b-a)$



If $f \in R[a,b]$, $m(b-a) \le \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \le M(b-a)$

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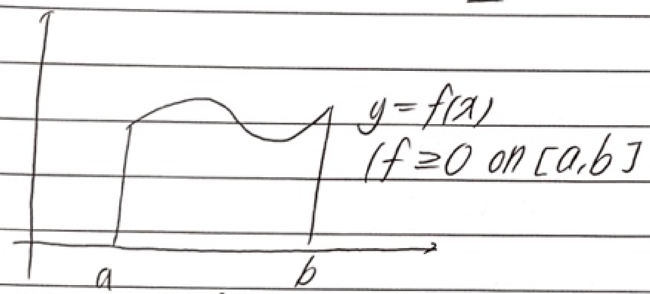


(Ex 6.1.6) ⓐ Dirichlet ft $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Supp/ $a<b$. $P=\{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

$L(P,f)=0$. & $U(P,f)=1$

$b-a>0$ //

p450 측도론 참고

- 리만적분이 cover 불가한 부분을 가능하게

$\Rightarrow \underline{\int_a^b} f = \sup\{L(P,f) \mid P: \text{partition of } [a,b]\}$

$= 0$

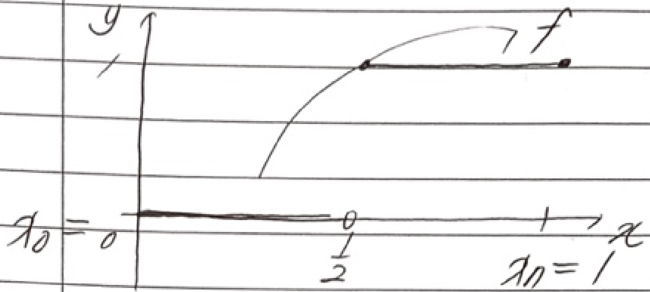
& $\overline{\int_a^b} f = \inf\{U(P,f) \mid P: \text{partition of } [a,b]\}$

$= b-a$

$$\therefore \overline{\int_a^b} f \neq \underline{\int_a^b} f$$

$$\therefore f \notin R[a,b]$$

ⓑ $f:[0,1]\to\mathbb{R}$: defined by $f(x)=\begin{cases}0, & 0\le x<\frac{1}{2}\\ 1, & \frac{1}{2}\le x\le 1\end{cases}$



$P=\{x_0,x_1,\cdots,x_{k-1},x_k,\cdots,x_n\}$: partition of $[a,b]$

$k\in\{1,2,\cdots,n\}$ s.t. $\underbrace{x_{k-1}<\frac{1}{2}\le x_k}$

사이에 $\frac{1}{2}$을 끼울 수 있다는 것

let $m_i=\inf\{f(t)\mid t\in[x_{i-1},x_i]\}$

$M_i=\sup\{f(t)\mid t\in[x_{i-1},x_i]\}$

$m_i=\begin{cases}0, & i=1,\cdots,k\\ 1, & i=k+1,\cdots,n\end{cases}$ $\quad M_i=\begin{cases}0, & i=1,\cdots,k-1\\ 1, & i=k,\cdots,n\end{cases}$

$$\Rightarrow \mathcal{L}(P,f)=\sum_{i=k+1}^{n} 1\cdot\Delta x_i = 1-x_k$$

$$(=(\cancel{x_{k+1}}-x_k)+(\cancel{x_{k+2}}-\cancel{x_{k+1}})+\cdots+(1-\cancel{x_{n-1}}))$$

$$\mathcal{U}(P,f)=\sum_{i=k}^{n} 1\cdot\Delta x_i = 1-x_{k-1}$$

$$(=(\cancel{x_k}-x_{k-1})+(\cancel{x_{k+1}}-\cancel{x_k})+\cdots+(1-\cancel{x_{n-1}}))$$

$$\Rightarrow \mathcal{L}(P,f)=1-x_k\le\frac{1}{2}<1-x_{k-1}=\mathcal{U}(P,f)$$

$$\forall P \Rightarrow \underline{\int_0^1} f\le\frac{1}{2}\le\overline{\int_0^1} f$$

$\forall P, Q$: partitions of $[0,1]$

$L(P,f) \le L(P\cup Q, f) \le \frac{1}{2} < U(P\cup Q, f) \le U(Q,f)$
($P\cup Q$: Refinement of P, Q)
$\Rightarrow$ P, Q는 독립적으로 움직이므로 $\frac{1}{2}$은 L의 상계, Q의 하계
$\Rightarrow$ 상계·하계가 모두 존재하므로 상적·하적으로 볼 수 있다.

$$\Rightarrow \underline{\int_0^1} f = \sup L(P,f) \le \frac{1}{2} \le \inf U(Q,f) = \overline{\int_0^1} f$$

Claim) $f \in R[0,1]$
$U(P,f) = L(P,f) + x_k - x_{k-1}$
$\forall \varepsilon > 0$, P: any partition of $[0,1]$ with $\Delta x_i < \varepsilon$ $i = 1, 2, \cdots, n$

$$\Rightarrow U(P,f) = L(P,f) + \underbrace{x_k - x_{k-1}}_{<\varepsilon}$$

$$< L(P,f) + \varepsilon$$

$$\Rightarrow \overline{\int_0^1} f = \inf_P U(P,f) \le U(P,f)$$

$$< L(P,f) + \varepsilon$$

$$\le \sup_P L(P,f) + \varepsilon$$

$$= \underline{\int_0^1} f + \varepsilon$$

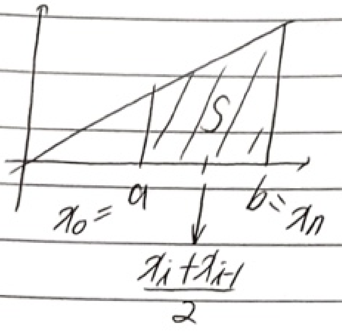
$$\Rightarrow 0 \le \overline{\int_0^1} f - \underline{\int_0^1} f < \varepsilon$$

($\varepsilon > 0$: Arbitrary)

$$\therefore \overline{\int_0^1} f = \underline{\int_0^1} f$$

$$\Leftrightarrow f \in R[0,1] \text{ s.t } \int_0^1 f = \frac{1}{2} \quad \blacksquare$$

ⓒ $f(x) = x \in R[a,b]$ s.t $\int_a^b x = \frac{1}{2}(b^2-a^2)$ $(=S)$



(pf) $P = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

$f(x) = x$: conti on $[a,b]$

$m_i = \inf\{f(t) \mid t \in [x_{i-1}, x_i]\}$

$= f(x_{i-1}) = x_{i-1}$

$M_i = f(x_i) = x_i$

$\Rightarrow \mathcal{L}(P,f) = \sum_{i=1}^n x_{i-1} \Delta x_i$

$\mathcal{U}(P,f) = \sum_{i=1}^n x_i \Delta x_i$

since $x_{i-1} \le \frac{x_i + x_{i-1}}{2} \le x_i$

$\forall P$: partition of $[a,b]$,

$\mathcal{L}(P,f) \le \sum_{i=1}^n \frac{x_i + x_{i-1}}{2} \Delta x_i$

$= \sum_{i=1}^n \frac{x_i^2 - x_{i-1}^2}{2} = \frac{x_n^2 - x_0^2}{2} = \frac{b^2-a^2}{2}$

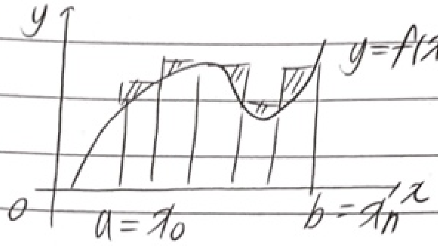
$\le \mathcal{U}(P,f)$

$\Rightarrow \underline{\int_a^b} f \le \frac{1}{2}(b^2-a^2) \le \overline{\int_a^b} f$

claim) $f(x) = x \in R[a,b]$

$0 \le \overline{\int_a^b} x - \underline{\int_a^b} x \le \mathcal{U}(P,f) - \mathcal{L}(P,f) = \sum_{i=1}^n (x_i - x_{i-1}) \Delta x_i$ ■

⟨Riemann's Criterion for Integrability⟩



→ partition 분점을 늘려서 오차 < ε 으로 만든것

$$^{*}\ L(P,f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P,f)\ ^{*}$$

⋮

$$\underline{\int_a^b} f = \overline{\int_a^b} f \iff f \in R[a,b]$$

Thm 6.1.7) $f : [a,b] \to \mathbb{R}$: bdd ft.

$f \in R[a,b] \iff \forall \varepsilon > 0,\ \exists P$: partition of $[a,b]$

s.t $U(P,f) - L(P,f) < \varepsilon$ — $(*)$

$(*) \Rightarrow P'$: refinement of P, $U(P',f) - L(P',f) < \varepsilon$

(pf) ($\Leftarrow$) If $\exists P$: partition of $[a,b]$ s.t $U(P,f) - L(P,f) < \varepsilon$

$\Rightarrow$ since $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P,f) - L(P,f) < \varepsilon$

($\varepsilon > 0$: Arbitrary) — 상적·하적은 bdd 이므로 R-적분 가능이고 아니고를 떠나서 무조건 존재하여 < ε 만족함

$\therefore \overline{\int_a^b} f = \underline{\int_a^b} f$

$\therefore f \in R[a,b]$

($\Rightarrow$) let $f \in R[a,b]$

$\forall \varepsilon > 0,\ \exists P_1, P_2$: partition of $[a,b]$ s.t $U(P_1,f) - \int_a^b f < \frac{\varepsilon}{2}$

since $f \in R[a,b]$, $\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$ 이고, $\int_a^b f - L(P_2,f) < \frac{\varepsilon}{2}$

$\int_a^b f = \overline{\int_a^b} f = \inf\{U(P,f) \mid P: \text{partition of } [a,b]\}$

$\left(\because \int_a^b f = \underline{\int_a^b} f = \sup\{\mathcal{L}(P,f) \mid P: \text{partition of } [a,b]\}\right)$

(P_1, P_2가 다르므로 세분 $P_1 \cup P_2$를 활용한다)

let $P = P_1 \cup P_2 \Rightarrow P$: refinement of P_1, P_2

$$\begin{cases} \mathcal{U}(P,f) \le \mathcal{U}(P_1,f) & \text{감소} \\ \mathcal{L}(P_2,f) \le \mathcal{L}(P,f) & \text{증가} \end{cases}$$

i.e) $$\begin{cases} \mathcal{U}(P,f) - \int_a^b f < \frac{\varepsilon}{2}, \\ \int_a^b f - \mathcal{L}(P,f) < \frac{\varepsilon}{2} \end{cases}$$

$\therefore \mathcal{U}(P,f) - \mathcal{L}(P,f) < \varepsilon$ ■

<Integrability of Conti & Monotone fts.>

Thm 6.1.8) $f: [a,b] \to \mathbb{R}$

- (a) f: Conti on $[a,b] \Rightarrow f \in \mathcal{R}[a,b]$
- (b) f: Monotone on $[a,b] \Rightarrow f \in [a,b]$

(pf-a) f: Conti on $[a,b] \Rightarrow f$: Uniformly conti on $[a,b]$

$\forall \varepsilon > 0,\ \exists \eta > 0$ s.t $(b-a)\eta < \varepsilon$

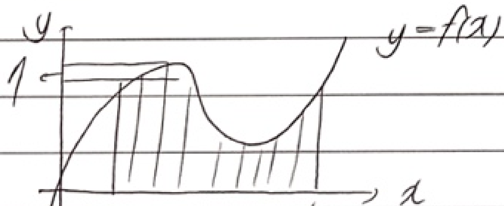
$M_i - m_i$가 됨

$\exists \delta > 0,\ \delta = \delta(\varepsilon)$ s.t $|f(x) - f(t)| < \eta$,

$\forall x, t \in [a,b]$, with $|x - t| < \delta$.

Choose a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$

s.t $\Delta x_i < \delta,\ 1 \le i \le n$



균등연속 ⇒ 어떤 partition의 분점에서도 함숫값의 차가 η보다 더 작다.

$\Rightarrow M_i - m_i \leq \eta,\ 1 \leq i \leq n$

$(\because M_i - m_i = \sup\{|f(t) - f(x)| \mid t, x \in [x_{i-1}, x_i]\}$

$\mapsto |x_n - x_{i-1}| < \delta$ 이므로 전부 η 보다 작고,

"equal"은 될 수 있으나 부등호는 안바뀜)

Thus. $\mathcal{U}(P,f) - \mathcal{L}(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$

$\leq \eta \sum_{i=1}^{n} \Delta x_i = \eta(b-a) < \varepsilon$

$\mathcal{U}(P,f) - \mathcal{L}(P,f) < \varepsilon$

$\therefore f \in \mathcal{R}[a,b]$ ■

(pf-b) Suppl f: M-I on $[a,b]$.

$n \in \mathbb{N}$. s.t $h = \frac{b-a}{n}$ (구분구적 같이 등구간으로 쪼개자)

$x_i = a + ih \ (0 \leq i \leq n)$

$\Rightarrow P = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

f: M-I on $[a,b]$

$\Rightarrow \begin{cases} M_i = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\} \\ \quad = f(x_i) \\ \text{(단조증가이므로 } x_i \text{ 에서의 함수값이 sup)} \\ m_i = \inf\{f(t) \mid t \in [x_{i-1}, x_i]\} \\ \quad = f(x_{i-1}) \end{cases}$

$\mathcal{U}(P,f) - \mathcal{L}(P,f) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))h$

constant이므로

$= \frac{b-a}{n} \cdot \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$ cancle되고 양끝만 남음.

$= \frac{b-a}{n}(f(b) - f(a))$ $(< \varepsilon)$ 인 P가 크하므로

즉. n만 크게 해주면 ε보다 작게 할 수 있음.

$\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t $\frac{(b-a)(f(b)-f(a))}{n} < \varepsilon$

For this n, corresponding P, $\mathcal{U}(P,f) - \mathcal{L}(P,f) < \varepsilon$.

$\therefore f \in \mathcal{R}[a,b]$ ■

<The composition Theorem>

Thm 6.1.9) $f \in [a,b] \to \mathbb{R}$: Bdd & $f \in \mathcal{R}[a,b]$
with $\text{Range}(f) \subset [c,d]$.
If φ : conti on $[c,d]$, $\Rightarrow \varphi \circ f \in \mathcal{R}[a,b]$
[파이]

(pf) φ : conti on $[c,d] \Rightarrow \varphi$: Bdd on $[c,d]$

let $K = \sup\{|\varphi(t)| \mid t \in [c,d]\}$
$\forall \varepsilon > 0$, set $\varepsilon' = \frac{\varepsilon}{b-a+2K} > 0$

Since φ : Unifly conti on $[c,d]$,
$\exists \delta > 0$ with $0 < \delta < \varepsilon'$, $|\varphi(s) - \varphi(t)| < \varepsilon'$
$\forall s, t \in [c,d]$ with $|s-t| < \delta$
& $f \in \mathcal{R}[a,b]$ 리만-판정
$\Rightarrow \exists P = \{x_0, x_1, \cdots x_n\}$: partition of $[a,b]$
s.t $\mathcal{U}(P,f) - \mathcal{L}(P,f) < \delta^2 \; (< (\varepsilon')^2)$
(*)

let $m_k = \inf\{f(t) \mid t \in [x_{k-1}, x_k]\}$
$M_k = \sup\{f(t) \mid t \in [x_{k-1}, x_k]\}$

& $m_k' = \inf\{\varphi(f(t)) \mid t \in [x_{k-1}, x_k]\}$
$M_k' = \sup\{\varphi(f(t)) \mid t \in [x_{k-1}, x_k]\}$

(M_k 와 m_k 의 차가 일반적으로는 δ 보다 클수도 작을수도 있는데
φ가 conti 라는 조건이 있어야 δ 보다 작다)

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┌ A,B는 서로소 집합

$\{1,2,\cdots,n\} = A \cup B \ (A\cap B = \emptyset)$

$A = \{k \mid M_k - m_k < \delta\}$ → φ의 연속성 활용을 위해 모았고 처리 OK

$B = \{k \mid M_k - m_k \geq \delta\}$

↳ φ의 연속성 활용 불가. (※) 활용

Since $|f(t)-f(s)| \leq M_k - m_k \quad \forall t,s \in [x_{k-1}, x_k]$

If $k\in A$, ($\Leftrightarrow M_k - m_k < \delta$)

$\Rightarrow |\varphi(f(t)) - \varphi(f(s))| < \varepsilon' \quad \forall t,s\in[x_{k-1}, x_k]$

$\Rightarrow M_k^* - m_k^* = \sup\{|\varphi(f(t)) - \varphi(f(s))| \mid t,s\in[x_{k-1}, x_k]\}$
$\leq \varepsilon'$

If $k\in B$, $M_k^* - m_k^* \leq 2K$

$(K = \sup\{|\varphi(t)| \mid t\in[c,d]\})$

Thus $U(P, \varphi\circ f) - L(P, \varphi\circ f)$

$$= \sum_{k\in A} \underbrace{(M_k^* - m_k^*)}_{\leq \varepsilon'} \Delta x_k + \sum_{k\in B} \underbrace{(M_k^* - m_k^*)}_{\leq 2K} \Delta x_k$$

$$\leq \varepsilon' \cdot \sum_{k\in A} \Delta x_k + 2K \underbrace{\sum_{k\in B} \Delta x_k}_{(※)}$$

(※) $k\in B \Rightarrow M_k - m_k \geq \delta$

$$\sum_{k\in B} \Delta x_k \leq \frac{1}{\delta} \sum_{k\in B} (M_k - m_k)\Delta x_k$$

(k가 전체가 아니라 B에만 속하니까 $M\Delta x_i$와 $m\Delta x_i$는
각각 상합, 하합의 일부분인 셈 ⇒ 전체와 비교, 부등호 발생)

$$\leq \frac{1}{\delta}(U(P,f) - L(P,f))$$

$$< \frac{1}{\delta}\cdot\delta^2 = \delta < \varepsilon'$$

Hence, $U(P, \varphi\circ f) - L(P, \varphi\circ f) \leq \varepsilon'(b-a) + 2K \underbrace{\sum_{k\in B}\Delta x_k}_{\leq \varepsilon'}$

$$< \varepsilon'(b-a) + 2K\varepsilon'$$
$$= \varepsilon$$

$\therefore \varphi\circ f \in R[a,b]$ ■

Corol 6.1.10) If $f \in R[a,b] \Rightarrow |f|, f^2, f^3, \cdots f^n \in R[a,b]$
(i) (ii) (iii)

(pf) (i) $\varphi(x) = |x|$: conti on $\mathbb{R}$
$\varphi \circ f = |f| \in R[a,b]$

(ii) $\varphi(x) = x^2$

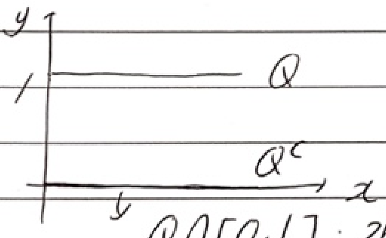
Rmk) $f, g \in R[a,b] \not\Rightarrow f \circ g \in R[a,b]$
(EX 6.1.14 (b))

※ Lebesgue's Theorem

(굉장히 진동하는 함수라 매우 쪼개도 불가능
→ y축으로 쪼개서 역상을 적분 ; 르베그)

"measure : 측도"

* 리만적분은 안되나 르베그 적분은 되는 예



$Q \cap [0,1]$: 가산집합

$m(Q \cap [0,1]) = 0$ (측도가 0)

Def 6.1.11) $E (\subset \mathbb{R})$: "Measure 0"
$\Leftrightarrow \forall \varepsilon > 0, \exists \{I_n\}$ (I_n : open interval) s.t $E \subset \bigcup_n I_n$ &
$\sum_n l(I_n) < \varepsilon$ (I_n 의 길이 length)

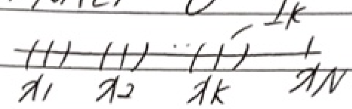
(EX 6.1.12) (a) $E = \{\overbrace{x_1, x_2, \cdots x_N}^{\text{유한개}}\} \subset \mathbb{R}$
let $I_n = \left(x_n - \frac{\varepsilon}{2N}, x_n + \frac{\varepsilon}{2N}\right)$

$\left(l(I_n) = \frac{\varepsilon}{N}\right)$

$\Rightarrow E \subset \bigcup_n I_n$ & $\sum_{n=1}^{N} l(I_n) = \varepsilon$

$\therefore$ E has "Measure 0"

$\mu(E) = 0$



(b) $E = \{x_n\}_{n=1}^{\infty}$ $(\subset \mathbb{R})$: Countable set

$n \in \mathbb{N}$. $I_n = (x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}})$

Since $x_n \in I_n$, $\forall n \geq 1$

$\Rightarrow E \subset \bigcup_{n=1}^{\infty} I_n$ & $\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

$\therefore E = \{x_n\}_{n=1}^{\infty}$

E has "Measure "0" ∎

(EX) $\mathbb{N}, \mathbb{Q}$: Countable sets

$\Rightarrow \mathbb{N}, \mathbb{Q}$: Measure "0"

Thm 6.1.13) (Lebesgue)

$f : [a,b] \to \mathbb{R}$: Bdd & $f \in R[a,b]$

$\Leftrightarrow$ let $D = \{p \in [a,b] \mid f$: Not conti at $p\}$

D : has "Measure 0"

Rmk) ① $f \in C[a,b]$ $(D = \emptyset)$ $\Rightarrow f \in R[a,b]$

"Measure 0"

② $f : [a,b] \to \mathbb{R}$: B.dd & Conti except at "finite pts" in $[a,b]$

($\Rightarrow$ D : finite set, D has "Measure 0")

③ $f : [a,b] \to \mathbb{R}$: Monotone ft

f : M-I $\Rightarrow f(p^-) = \sup\{f(x) \mid x < p\}$

$f(p^+) = \inf\{f(x) \mid x > p\}$

$\Rightarrow$ D : (At most) countable

$\Rightarrow$ D has "Measure 0"

$\therefore f \in R[a,b]$. ∎

(Ex 6.1.14) ⓐ $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \\ \frac{1}{n}, & x = \frac{m}{n} \in [0,1] \quad (n,m)=1 \end{cases}$

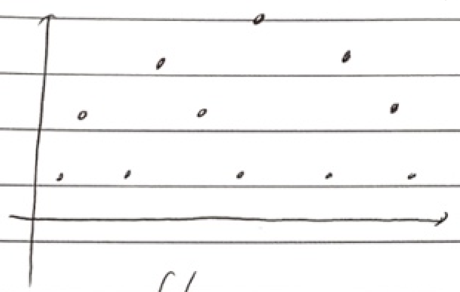
(: Tomae ft)

f : Conti exept at $p \in \mathbb{Q} \cap [0,1]$ where $\mathbb{Q} \cap [0,1]$ "Measure 0"

By Lebesgue Thm. $f \in R[0,1]$

Since $\forall P = \{x_0, x_1, \cdots, x_n\}$: partition of $[0,1]$

$\mathcal{L}(P,f) = 0 \Rightarrow \underline{\int_0^1} f = \sup_P \mathcal{L}(P,f) = 0$



$\therefore \int_0^1 f = 0$

$\mathcal{U}(P,f) < \varepsilon$

$\Rightarrow \mathcal{U}(P,f) - \mathcal{L}(P,f) < \varepsilon$

ⓑ $f \in R[0,1]$ (f : Tomae ft ⓐ)

let $g : [0,1] \to \mathbb{R}$ defined by $g(x) = \begin{cases} 0, & x = 0 \\ 1, & x \in [0,1] \end{cases}$

$\Rightarrow$ g : Conti except a "0"

$\Rightarrow$ $g \in R[0,1]$

But $x \in [0,1]$

$(g \circ f)(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

$\notin R[0,1]$

Chapter 6.2 Property of Riemann Integral

Thm 6.2.1) $f, g \in R[a,b]$ then

(a) $f+g \in R[a,b]$ with $\int_a^b f+g = \int_a^b f + \int_a^b g$

(b) $c \in \mathbb{R}$. $cf \in R[a,b]$ with $\int_a^b cf = c\int_a^b f$

(c) $f, g \in R[a,b]$

$f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

(pf - a) $P = \{x_0, x_1, \cdots, x_n\}$; partition of $[a,b]$

$M_i(f) = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\}$

$M_i(g) = \sup\{g(t) \mid t \in [x_{i-1}, x_i]\}$ $(1 \le i \le n)$ → Δx_i 를 곱해주자

$\Rightarrow f(t) + g(t) \le M_i(f) + M_i(g)$ $\forall t \in [x_{i-1}, x_i]$

$\Rightarrow \sup\{f(t) + g(t) \mid t \in [x_{i-1}, x_i]\} \le M_i(f) + M_i(g)$

임의의 partition 을 잡아 만족이므로 상한을 취해도 바뀌지 않는다.

$U(P, f+g) \le U(P,f) + U(P,g)$

Since $f, g \in R[a,b]$ $\forall \varepsilon > 0$. $\exists P_f, P_g$: Partition of $[a,b]$ s.t

$U(P_f, f) < \underbrace{\int_a^b f} + \frac{\varepsilon}{2}$

$\int_a^b f$ 를 상적분으로 간주하면 $\inf U(P_f, f)$ 인데 $\frac{\varepsilon}{2}$(양수)를 더했으므로 더이상 Not L-B. '완비성 공리'에 의해 이것보다 작은 것이 반드시 존재.

$\left(\because \int_a^b f = \overline{\int_a^b} f = \inf\{U(P,f) \mid P : \text{partition of } [a,b]\} + \underbrace{\frac{\varepsilon}{2}}_{\text{Not L-B}}\right)$

$U(P_g, g) < \int_a^b g + \frac{\varepsilon}{2}$

let $Q = P_f \cup P_g$ ($\Rightarrow Q$: Refinement of P_f, P_g)

$U(Q, f+g) \le U(Q,f) + U(Q,g)$

$\le U(P_f, f) + U(P_g, g)$

$< \int_a^b f + \int_a^b g + \varepsilon$

$\Rightarrow \overline{\int_a^b} (f+g) = \inf_Q \{ U(Q, f+g) \}$
↘ 하한이니까

$\le U(Q, f+g)$

$< \int_a^b f + \int_a^b g + \varepsilon$

$(\varepsilon > 0$: Arbitrary$)$

$\therefore \overline{\int_a^b} (f+g) \le \int_a^b f + \int_a^b g$ — ① ✱

Similiary. $\underline{\int_a^b} (f+g) \ge \int_a^b f + \int_a^b g$.

$L(P, f+g) \ge L(P, f) + L(P, g)$

Since $f, g \in R[a,b]$ $\forall \varepsilon > 0$, $\exists P_f, P_g$: partition of $[a,b]$ s.t

$L(P_f, f) > \int_a^b f - \frac{\varepsilon}{2}$

$(\because \int_a^b f = \underline{\int_a^b} f = \sup \{ L(P, f) \mid P$: partition of $[a,b] \}$

$\underset{\sim}{- \frac{\varepsilon}{2}} \rightarrow$ Not U-B

$L(P_g, g) > \int_a^b g - \frac{\varepsilon}{2}$

let $Q = P_f \cup P_g$ $(\Rightarrow Q$: refinement of $P_f, P_g)$

$L(Q, f+g) \ge L(Q, f) + L(Q, g)$

$\ge L(P_f, f) + L(P_g, g)$

$> \int_a^b f + \int_a^b g - \varepsilon$

$\Rightarrow \underline{\int_a^b} (f+g) = \sup_Q L(Q, f+g)$

$\ge L(Q, f+g)$

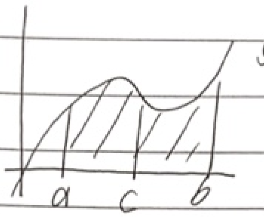
$> \int_a^b f + \int_a^b g - \varepsilon$ $(\forall \varepsilon > 0$: Arbitrary$)$

$\therefore \underline{\int_a^b} (f+g) \ge \int_a^b f + \int_a^b g$ — ② ✱

Thm6.2.3) $f:[a,b]\to\mathbb{R}$: Bdd ft $(a<c<b)$

$\Rightarrow f\in R[a,b] \Leftrightarrow f\in R[a,c]$ & $f\in[c,b]$

with $\int_a^b f = \int_a^c f + \int_c^b f$

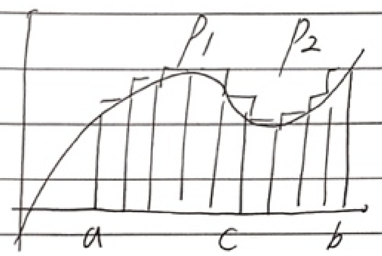


(pf) ① $\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$

② $\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f$

let p_1, p_2 : partition of $[a,c],[c,b]$, respectively.

$\Rightarrow P = P_1 \cup P_2$: partition of $[a,b]$ with $c\in P$



$$U(P,f) = U(P_1,f) + U(P_2,f) \geq \overline{\int_a^c} f + \overline{\int_c^b} f$$

$\forall Q$: Partition of $[a,b]$

let $P = Q\cup\{c\}$: refinement of Q

$$U(Q,f) \geq U(P,f) \geq \overline{\int_a^c} f + \overline{\int_c^b} f$$

$$\Rightarrow \overline{\int_a^b} f = \inf_Q U(Q,f) \geq \overline{\int_a^c} f + \overline{\int_c^b} f \quad -①$$

$\forall \varepsilon>0,\ \exists p_1, p_2$: partition of $[a,c],[c,b]$

$$U(p,f) < \underbrace{\overline{\int_a^c} f + \frac{\varepsilon}{2}}_{\text{↳ Not L-B}}$$

$$U(p_2,f) < \overline{\int_c^b} f + \frac{\varepsilon}{2}$$

$$\Rightarrow U(p_1,f) + U(p_2,f) < \overline{\int_a^c} f + \overline{\int_c^b} f + \varepsilon$$

let $P = P_1 \cup P_2$

$$\overline{\int_a^b} f \le U(P,f) = U(P_1,f) + U(P_2,f)$$

$$< \overline{\int_a^c} f + \overline{\int_c^b} f + \varepsilon$$

$$\Rightarrow \overline{\int_a^b} f \le \overline{\int_a^c} f + \overline{\int_c^b} f + \varepsilon$$

$(\varepsilon > 0 :$ Arbitrary$)$

$$\therefore \overline{\int_a^b} f \le \overline{\int_a^c} f + \overline{\int_c^b} f \quad - \text{(I)}$$

(ii) let P_1, P_2 : partition of $[a,c], [c,b]$ respectively.
& $P : P_1 \cup P_2$

$$L(P,f) = L(P_1,f) + L(P_2,f)$$

$$\le \underline{\int_a^c} f + \underline{\int_c^b} f$$ → 하합들의 상한 = 하적분

For any Q : partition of $[a,b]$ · 임의의 분할
let $P = Q \cup \{c\}$ → 세분 : 하합은 증가 · 상합은 감소

by 완비성공리
상계ㅋ → 상한ㅋ

$$L(Q,f) \le L(P,f) \le \underline{\int_a^c} f + \underline{\int_c^b} f$$

세분

$$\sup_Q L(Q,f) = \underline{\int_a^b} f \le \underline{\int_a^c} f + \underline{\int_c^b} f \quad - \text{(a)}$$

$\forall \varepsilon > 0$, $\exists P_1, P_2$: partitions of $[a,c], [c,b]$

$$L(P_1,f) > \underline{\int_a^c} f - \frac{\varepsilon}{2} \quad (*_1)$$

↳ Not U-B

$\underline{L}(P_2, f) > \underline{\int_a^c} f - \frac{\varepsilon}{2}$ — ($*_2$)

let $P = P_1 \cup P_2 \Rightarrow c \in P$

$$\underline{L}(P, f) = \underline{L}(P_1, f) + \underline{L}(P_2, f)$$

$$> \underline{\int_a^c} f + \underline{\int_c^b} f - \varepsilon$$

$\therefore$ Since $\underline{\int_a^b} f \geq \underline{L}(P, f)$ $\forall P$: partition of $[a, b]$

$\underline{\int_a^b} f > \underline{\int_a^c} f + \underline{\int_c^b} f - \varepsilon$ ($\varepsilon > 0$: Arbitrary)

i.e) $\underline{\int_a^b} f \geq \underline{\int_a^c} f + \underline{\int_c^b} f$ — ⓑ

$\therefore$ By ⓐ, ⓑ, $\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f$ ∎

$\therefore f \in R[a, b]$.

$$\Leftrightarrow \underline{\int_a^b} f = \overline{\int_a^b} f$$

$$\Leftrightarrow \underline{\int_a^c} f + \underline{\int_c^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$$

$\Rightarrow f \in R[a, c], [c, b]$

$$\Leftrightarrow \underbrace{\left(\overline{\int_a^c} f - \underline{\int_a^c} f\right)}_{\geq 0} + \underbrace{\left(\overline{\int_c^b} f - \underline{\int_c^b} f\right)}_{\geq 0} = 0$$

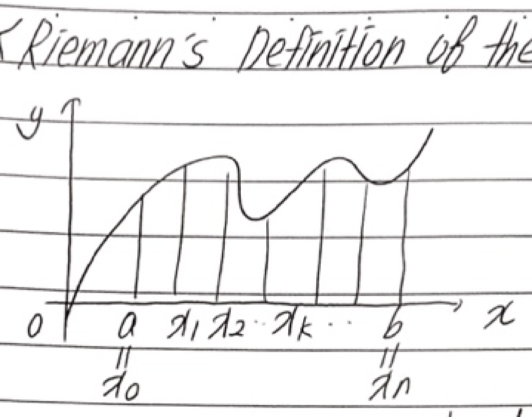
≥ 0, ≥ 0 를 더해 0 되려면 $=0$, $=0$.

$\Leftrightarrow f \in R[a, c]$ & $f \in R[c, b]$

with $\int_a^b f = \int_a^c f + \int_c^b f$. ∎

⟨Riemann's Definition of the Integral⟩

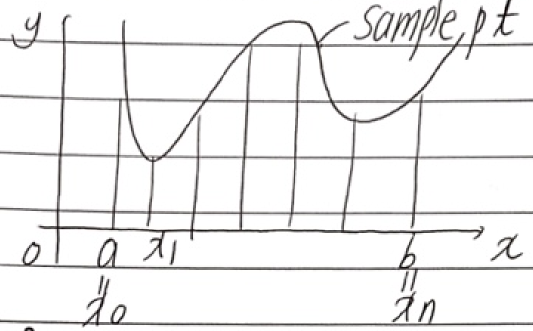
'Idea'



$U \rightarrow \overline{\int_a^b} f$

$L \rightarrow \underline{\int_a^b} f$

분점을 잘게 쪼갤수록 오차↓



$\mathcal{P}(P,f)$ → 파티션에 대한 f의 리만합

※ t_i을 랜덤으로 어떻게나 잡아도 같은 극한으로 갈수밖에 없다

Thm 6.2.4) $f : [a,b] \rightarrow \mathbb{R}$: Bdd ft

let $P = \{x_0, x_1, \cdots, x_{i-1}, x_i, \cdots, x_n\}$: partition of $[a,b]$.

$t_i \in [x_{i-1}, x_i]$ $(i : 1 \le i \le n)$

$$\mathcal{P}(P,f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

[파이]

: Riemann Sum of f (f의 리만합)

w.r.to P (→ P에 대한)

Claim) 리만합의 극한이 적분으로 감

def) let $P = \{x_i \mid x_1, x_2, \cdots, x_n\}$: partition of $[a,b]$

The "Norm" or "Mesh" of the P, denoted $\|P\|$

$\Leftrightarrow \|P\| = \max\{\Delta x_i \mid 1 \le i \le n\}$

→ 등구간일 필요는 없으므로 Δx_i 중 max가 norm.

→ Norm 이 0 으로 가면 $\forall x_i \rightarrow 0$ (가장 큰게 0으로 가므로)

→ 리만합이 수렴한다 : partition을 잘게 쪼개면 t_i와 무관하게 적분값으로 수렴한다.

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Def 6.2.5) $f: [a,b] \to \mathbb{R}$: Bdd ft

$\lim_{\|P\| \to 0} \varphi(P, f) = I$ → t_i의 선택에 무관

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - I \right| < \varepsilon$

$\forall P$: partition of $[a,b]$ with $\|P\| < \delta$
& for any choice of $t_i \in [x_{i-1}, x_i]$

Thm 6.2.6) $f: [a,b] \to \mathbb{R}$: Bdd ft

If $\lim_{\|P\| \to 0} \varphi(P, f) = I$ then $f \in R[a,b]$ with $\int_a^b f = I$

O

difficult to prove

& If $f \in R[a,b]$, then $\exists \lim_{\|P\| \to 0} \varphi(P, f)$ s.t

$\lim_{\|P\| \to 0} \varphi(P, f) = \int_a^b f$.

(pf) ($\Rightarrow$) supp) $\lim_{\|P\| \to 0} \varphi(P, f) = I$

then $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - I \right| < \varepsilon$

$\forall P$: partition of $[a,b]$ with $\|P\| < \delta$
& any choices $t_i \in [x_{i-1}, x_i]$ $1 \le i \le n$

let $M_i = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\}$

$\Rightarrow \exists \xi_i \in [x_{i-1}, x_i]$ s.t $f(\xi_i) > M_i - \varepsilon$

↳ Not U-B

f가 bdd 이므로
완비성공리에 의해 상한 존재.

($P^* = \{x_0, \cdots, x_n\}$: partition of $[a,b]$ with $\|P\| < \delta$)

$$\mathcal{U}(P^*, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$< \sum_{i=1}^{n} f(\xi_i)\Delta x_i + \sum_{i=1}^{n} \varepsilon \Delta x_i$$

$$< I + \varepsilon + \varepsilon(b-a)$$

$$= I + \varepsilon(1+b-a)$$

(since $\overline{\int_a^b} f \leq \mathcal{U}(P^*, f)$.)

$\Rightarrow \overline{\int_a^b} f < I + \varepsilon(1+b-a)$

($\varepsilon > 0$: Arbitrary)

$\Rightarrow \overline{\int_a^b} f \leq I$ — ⓐ 〈상합〉

ⓑ 하합 $\forall P^{**}$: partition of $[a,b]$ with $\|P^{**}\| < \delta$
& any choices of $t_i \in [x_{i-1}, x_i]$, $1 \leq i \leq n$

let $m_i = \inf \{ f(t) \mid t \in [x_{i-1}, x_i] \}$

inf + ε 이므로

$\Rightarrow \exists\, \xi_i \in [x_{i-1}, x_i]$ s.t $f(\xi_i) < m_i + \varepsilon$

↳ Not L-B

($P^{**} = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$ with $\|P\| < \varepsilon$)

$$\mathcal{L}(P^{**}, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$> \sum_{i=1}^{n} f(\xi_i)\Delta x_i - \sum_{i=1}^{n} \varepsilon \Delta x_i$$

$$> I - \varepsilon - \varepsilon(b-a)$$

$$= I - \varepsilon(1+b-a)$$

(since $\underline{\int_a^b} f \geq \mathcal{L}(P^{**}, f)$)

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No.

$\Rightarrow \underline{\int_a^b} f \geq I - \varepsilon(1+b-a)$

($\forall \varepsilon > 0$: Arbitrary)

$\Rightarrow \underline{\int_a^b} f \geq I$ — ⓑ

$\therefore \underline{\int_a^b} f = \overline{\int_a^b} f = I$

$\therefore$ By ⓐⓑ $f \in R[a,b]$ with $\int_a^b f = I$

let $P = P^* \cup P^{**}$

$\Rightarrow \mathcal{U}(P,f) - \mathcal{L}(P,f) < I + \varepsilon(1+b-a) - (I - \varepsilon(1+b-a))$
$= \underbrace{2\varepsilon(1+b-a)}_{\text{put this } '\varepsilon'}$

If $f \in R[a,b]$, let $M > 0$ s.t
$|f(x)| \leq M \quad \forall x \in [a,b]$. $\forall \varepsilon > 0$.
$\exists Q = \{\underset{a}{x_0}, x_1, \cdots, \underset{b}{x_N}\}$: partition of $[a,b]$

s.t $\int_a^b f - \varepsilon < \mathcal{L}(Q,f) \leq \mathcal{U}(Q,f) < \int_a^b f + \varepsilon$

let $\delta = \dfrac{\varepsilon}{NM}$ → 유계일때 M 앞에 있으므로
Q 분점의 개수

& $P = \{\underset{a}{y_0}, y_1, \cdots, \underset{b}{y_n}\}$: partition of $[a,b]$ with $\|P\| < \delta$.

claim) $\lim_{\|P\| \to 0} \mathcal{U}(P,f) = \int_a^b f$

let $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ $(1 \leq i \leq n)$
For each $[y_{k-1}, y_k]$ $(y_{k-1}, y_k \in P)$
: may or may not contain Q - points.
Q : contains N+1 - points.
$\Rightarrow$ At most N−1 - intervals $[y_{k-1}, y_k]$ contain Q - points.

$$\underset{\underset{a}{\parallel}}{y_0} \quad y_1 \ \cdots \ y_{k-1} \quad y_k \ \cdots \ \underset{\underset{b}{\parallel}}{y_n}$$

Suppose $\{\underbrace{x_j, x_{j+1}, \cdots, x_{j+m}}_{\in Q}\} \subset [y_{k-1}, y_k]$ Q의 분점이 이 구간에 들어간다 가정.

Set $M_k' = \sup\{f(x) \mid x \in [y_{k-1}, x_j]\}$ $(y_{k-1} \neq x_j)$

$M_k^2 = \sup\{f(t) \mid t \in [x_{j+m}, y_k]\}$ $(x_{j+m} \neq y_k)$

$t_k \in [y_{k-1}, y_k]$: Arbitrary.

Since $|f(t) - f(s)| \le |f(t)| + |f(s)|$

$\le M + M = 2M \quad \forall t, s \in [y_{k-1}, y_k]$

$$\Rightarrow \begin{cases} f(t_k) \le 2M + M_{j+s}, & s = 1, 2, \cdots, m \\ f(t_k) \le 2M + M_k^i & i = 1, 2 \end{cases}$$

$(\because\ f(t_k) = f(t_k) - f(t) + f(t) \quad t \in [x_{j+s-1}, x_{j+s}]$

$\downarrow$ T-I

$\le |f(t_k) - f(t)| + f(t)$ 이 중 $\sup = f(t)$

$\le 2M + M_{j+s}$

$f(t_k) = f(t_k) - f(x) + f(x), \quad x \in [y_{k-1}, x_j]$

$\downarrow$ T-I

$\le |f(t_k) - f(x)| + f(x)$ 이중에 속할때 $\sup = f(x)$

$\le 2M + M_k'\ )$

$\Delta y_k = y_k - y_{k-1}$

$\Rightarrow f(t_k) \cdot \Delta y_k = f(t_k)(x_j - y_{k-1}) + \sum_{s=1}^{m} f(t_k) \Delta x_{j+s}$

$+ f(t_k)(y_k - x_{j+m})$

[a,b]의 부분partition의 상합

$\le 2M\Delta y_k + M_k'(x_j - y_{k-1}) + \sum_{s=1}^{m} M_{j+s} \Delta x_{j+s}$
$+ M_k^2 (y_k - x_{j+m})$

$< 2M\delta + U(P_k, f)$

$(\uparrow \|P\| < \delta)$

$\cdot\ P_k = \{y_{k-1}, x_j, x_{j+1}, \cdots, x_{j+m}, y_k\}$
$\quad :$ partition of $[y_{k-1}, y_k]$

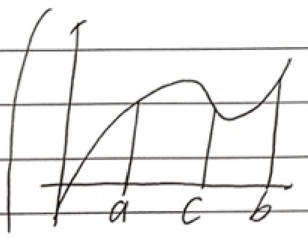
책에 없다 $*$ If $[y_{k-1}, y_k] \cap Q = \emptyset \Rightarrow P_k = \{y_{k-1}, y_k\}$
let $P' = \bigcup_{i=1}^{n} P_k \Rightarrow P'$: refinement of Q
$\qquad P' = P \cup Q$

At most "$N-1$" - interval of P
$\underline{Q}$의 점들을 포함할 수 있는 개수.

$\varphi(P, f) = \sum_{k=1}^{n} f(t_k)\Delta y_k$

$\delta = \frac{\varepsilon}{N \cdot M}$ $\quad < 2M(N-1)\delta + \sum \mathcal{U}(P_k, f)$ → 상합은 그대로

$< 2\varepsilon + \mathcal{U}(P, f)$ ← 각각의 상합의 합
합집합



$\int_a^c f + \int_c^b f = \int_a^b f$

$\le 2\varepsilon + \mathcal{U}(Q, f)$ (세분하면 상합은 감소)

$< 3\varepsilon + \int_a^b f$

상합 사용 $\Rightarrow \varphi(P, f) - \int_a^b f < 3\varepsilon \quad - (\mathrm{I})$

(반대 방향) claim) $-3\varepsilon < \varphi(P, f) - \int_a^b f \quad - (\mathrm{II})$
└ 하합 사용

$f \in R[a,b] \Rightarrow \exists Q = \{x_0, \cdots, x_N\}$: partition of $[a,b]$
$\int_a^b f - \varepsilon < \mathcal{L}(Q, f) \le \mathcal{U}(Q, f) < \int_a^b f + \varepsilon$

let $\delta = \frac{\delta}{M \cdot N}$ ($|f(x)| < M$ $\forall x \in [a,b]$)

let $P = \{y_0, y_1, \cdots, y_n\}$: partition of $[a,b]$ with $\|P\| < \delta$

$[\ y_{k-1} \quad | \ x_j \quad | \ x_{j+1} \quad | \ x_{j+N} \quad]\ y_k$

set $m_k^1 = \inf\{f(x) \mid x \in [y_{k-1}, x_j]\}$ $(y_{k-1} \neq x_j)$

$m_k^2 = \inf\{f(t) \mid t \in [x_{j+m}, y_k]\}$ $(x_{j+m} \neq y_k)$

$t_k \ (\in [y_{k-1}, y_k])$ 는 임의
$f(t_k) \geq -2M + m_{j+s}$, $s = 1, 2, \cdots, m$
$f(t_k) \geq -2M + m_k^i$, $i = 1, 2$

($\because$ $f(t_k) = f(t_k) - f(t) + f(t)$
$\geq -|f(t_k) - f(t)| + f(t)$ $\quad t \in [y_{k-1}, x_j]$
$\geq -2M + m_k^1$ $\quad t \in [y_{k-1}, x_j]$)

$$\Rightarrow f(t_k)\Delta y_k = f(t_k)(x_j - y_{k-1}) + \sum_{s=1}^{m} f(t_k)\Delta x_{j+s} + f(t_k)(y_k - x_{j+m})$$

$$\geq -2M\Delta y_k + m_k^1(x_j - y_{k-1}) + \sum_{s=1}^{m} m_{j+s} + m_k^2(y_k - x_{j+m})$$

$$> -2M\delta + \mathcal{L}(P_k, f)$$

let $P' = \bigcup_{i=1}^{n} P_i \Rightarrow P'$: refinement of Q

$\Rightarrow \mathcal{R}(P, f) = \sum_{k=1}^{n} f(t_k)\Delta y_k$ $\quad \delta = \frac{\varepsilon}{NM}$

$> -2M(N-1)\delta + \mathcal{L}(P', f)$

$> -2\varepsilon + \mathcal{L}(P', f)$ $\quad (P' \supset Q)$

$> -2\varepsilon + L(Q,f)$

$> -2\varepsilon + \int_a^b f - \varepsilon$

$> -3\varepsilon + \int_a^b f$

$\therefore\ \varphi(p,f) - \int_a^b f > -3\varepsilon$ — ②

By ①, ②. $-3\varepsilon < \varphi(p,f) - \int_a^b f < 3\varepsilon$

$\Leftrightarrow\ \left|\varphi(p,f) - \int_a^b f\right| < 3\varepsilon \quad \forall p \text{ with } \|p\| < \delta$.

$\therefore\ \lim_{\|p\|\to 0} \varphi(p,f) = \int_a^b f$ ∎

(Ex 6.2.7) $f(x) = x \in R[a,b]$

$\int_a^b x\,dx = \lim_{\|p\|\to 0} \varphi(p,f)$

$p = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

let $t_k = \frac{x_k + x_{k+1}}{2} \in [x_k, x_{k+1}]$ $(1 \le k \le n)$ ← sample pt

$\varphi(p,f) = \sum_{k=1}^{n} \frac{x_k + x_{k+1}}{2} \cdot \underbrace{x_k - x_{k+1}}_{\Delta x_k}$

$= \frac{1}{2} \sum_{k=1}^{n} (x_k^2 - x_{k+1}^2)$

$= \frac{1}{2}(b^2 - a^2)$ → 계산하기 쉬운 sample pt t_k를 choice 한 것.

* 리만적분 가능 ⇒ 리만합으로 표현 가능.

(EX) 6.2 2, 3, 5, 6, 7, 8, 9, 10

상적 = 하적

$\neq$

No.

Chapter 6-3 fundamental Thm of Calculus (f-c-t)

Def 6.3.1) $f: I \to \mathbb{R}$ (I: interval)
F: anti-derivative of f on I
$\Leftrightarrow F'(x) = f(x) \ \forall x \in I$

Rmk) F, G: Anti-derivative of f on I.
$\Rightarrow G(x) = F(x) + C \ (\forall x \in I)$

Thm 6.3.2 (F-T-C) (부정적분)
If $f \in R[a,b]$ & F: Anti-deriv of f on $[a,b]$
then $\int_a^b f(x)dx = F(b) - F(a)$
$= F(x)\Big|_{x=a}^{x=b}$

(pf) let $P = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$
& $F'(x) = f(x) \quad x \in [a,b]$
By the M-V-T. $\exists t_i \in [x_{i-1}, x_i]$ s.t
$F(x_i) - F(x_{i-1}) = F'(t_i) \cdot \Delta x_i$
$= f(t_i) \cdot \Delta x_i$
$\Rightarrow \underbrace{\sum_{i=1}^{n} f(t_i)\Delta x_i}_{S(P,f)} = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$
$= F(b) - F(a)$

Since $L(P,f) \le \sum_{i=1}^{n} f(t_i)\Delta x_i \le U(P,f)$
$\Rightarrow \underline{\int_a^b} f \le F(b) - F(a) \le \overline{\int_a^b} f$
Since $f \in R[a,b]$
$\therefore \int_a^b f = F(b) - F(a)$ ∎

Rmk) f의 연속성: 불필요

(EX6.3.3)

ⓐ $f(x) = x^n \quad n \in \mathbb{N}$

$\Rightarrow F(x) = \dfrac{1}{n+1} x^{n+1} + C$

: Anti-Deriv of f.

$\Rightarrow \displaystyle\int_a^b x^n \, dx = \frac{1}{n+1} (b^{n+1} - a^{n+1})$

ⓑ $F : [0,1] \to \mathbb{R}$, defined by $F(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

: Anti deriv of f.

$\Rightarrow f(x) = F'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

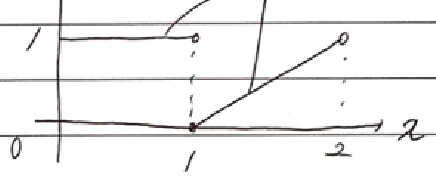
$F'(0) = \displaystyle\lim_{h \to 0} \frac{F(h) - F(0)}{h}$

$= \displaystyle\lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$

$\displaystyle\int_0^1 f(x) \, dx = F(1) - F(0)$

$= \sin 1 - 0 = \sin 1$

ⓒ $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -1 + x, & 1 \le x < 2 \end{cases}$



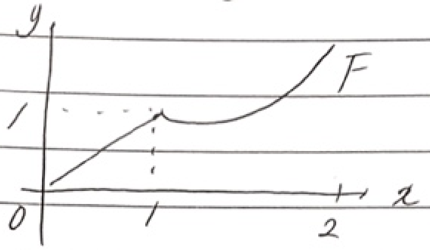
let $F(x) = \displaystyle\int_0^x f(t) \, dt$

ⓘ $0 \le x < 1 \quad F(x) = \displaystyle\int_0^x 1 \, dx = x$

ⓘⓘ $1 \le x < 2 \quad F(x) = \displaystyle\int_0^1 1 \, dt + \int_1^x (t-1) \, dt$

$= \frac{1}{2}x^2 - x + \frac{3}{2}$.

$\Rightarrow F(x) = \begin{cases} x, & 0 \le x \le 1 \\ \frac{1}{2}x^2 - x + \frac{3}{2}, & 1 \le x \le 2 \end{cases}$



f is not conti at $x=1$ &
F is not diff-able at $x=1$.

Thm6.3.4 $f \in R[a,b]$ & $F(x) = \int_a^x f(t)\,dt \Rightarrow F$: $\overset{\text{Unifly conti!}}{\text{conti}}$ on $[a,b]$

then. F: diff at c s.t $F'(c) = f(c)$.

(pf) Since $f \in R[a,b]$, $\exists M > 0$ s.t
$|f(x)| \le M \quad \forall x \in [a,b]$
$\forall \varepsilon > 0$. Take $\delta = \frac{\varepsilon}{M} > 0$ s.t

$|F(t) - F(s)| = \left|\int_s^t f(\eta)\,d\eta\right|$

$\le \int_s^t |f(\eta)|\,d\eta$

$\le M \int_s^t d\eta = M(t-s)$ ↙ 립시츠 조건 → 무조건 균등연속

$< \varepsilon$ ε에 의존하는 δ를 무조건 take가능

$\forall t (>) s \in [a,b]$ with $|t-s| < \delta$
$\therefore$ F: Unifly conti on $[a,b]$
& suppl f: Conti at $c \in [a,b]$.

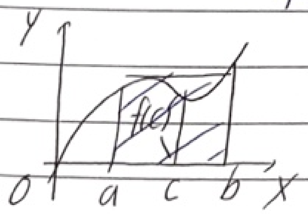
claim) $F'_+(c) = f(c)$
let $h > 0$. $(c, c+h) \in [a,b]$
Then $F(c+h) - F(c) = \int_a^{c+h} f(t)\,dt - \int_a^c f(t)\,dt$

<Consequences of F-T-C>

Thm6.3.6) $f \in C[a,b] \Rightarrow \exists c \in [a,b]$

s.t $\int_a^b f = f(c)(b-a)$

→ 적분값과 같은 직사각형이 존재



(pf) let $F(x) = \int_a^x f(t)\,dt$

$f \in C[a,b] \Rightarrow F'(x) = f(x),\ x \in [a,b]$

Thus by the M-V-T, $\exists c \in [a,b]$ s.t

$$\int_a^b f \underbrace{=}_{\text{F-T-C}} F(b) - F(a)$$

$$\left.\begin{aligned} &= F'(c)(b-a) \end{aligned}\right\} \text{M-V-T}$$

$$= f(c)(b-a) \blacksquare$$

II) I-V-T

$f \in C[a,b] \Rightarrow m \le f(x) \le M \quad \forall x \in [a,b]$

$$\Rightarrow m(b-a) = \int_a^b m \le \int_a^b f \le \int_a^b M$$

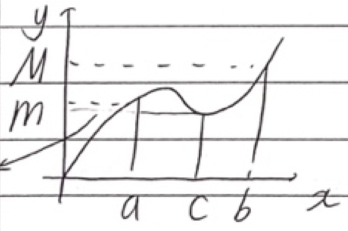
$$= M(b-a)$$

$$\Rightarrow m \le \frac{1}{b-a}\int_a^b f \le M$$

$\therefore$ By M-V-T of conti-ft,

$\exists c \in [a,b]$ s.t $\frac{1}{b-a}\int_a^b f = f(c) \blacksquare$

$\frac{1}{b-a}\int_a^b f$



Thm 6.3.8) (Change of variable)

φ : diff on $[a,b]$ with $\varphi' \in R[a,b]$

if f : conti on $I = \varphi([a,b])$ then

$$\int_a^b f(\varphi(t)) \cdot \varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

(pf) $\varphi \in C[a,b] \Rightarrow I = \varphi([a,b])$: Interval

$\varphi' \in R[a,b]$, $f \circ \varphi \in C[a,b] \subset R[a,b]$

$\Rightarrow (f \circ \varphi) \cdot \varphi' \in R[a,b]$

Ⓘ If $\varphi = C$ (: constant) on $[a,b]$

$\Rightarrow \varphi' = 0$ on $[a,b]$

$$\Rightarrow 0 = \int_a^b f(\varphi(t)) \cdot \underbrace{\varphi'(t)}_{=0}\,dt$$

$$= \int_{\varphi(a)=c}^{\varphi(b)=c} f(x)\,dx = 0.$$

Ⓘⓘ φ : Not constant on $[a,b]$

let $F(x) = \int_{\varphi(a)}^{x} f(t)\,dt$

$\Rightarrow F'(x) = f(x)$ $\forall x \in I = \varphi([a,b])$

→ f가 연속이기 때문.

(By chain-Rule),

$$\frac{d}{dt} F(\varphi(t)) = F'(\varphi(t)) \cdot \varphi'(t)$$

$$= f(\varphi(t)) \cdot \varphi'(t)$$

Thus, $\int_a^b \underbrace{f(\varphi(t))\varphi'(t)}_{\frac{d}{dt}F(\varphi(t))}\,dt \underset{\text{By F-T-C}}{=} F(\varphi(b)) - \underbrace{F(\varphi(a))}_{0} \left(= \int_{\varphi(a)}^{\varphi(a)} f(t)\,dt\right)$

$$= \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx \blacksquare$$

(Ex 6.3.9) $\int_0^2 \frac{t}{1+t^2}\,dt = \frac{1}{2}\int_1^5 \frac{1}{x}\,dx$

$= \frac{1}{2}\ln|x|\Big|_{x=1}^{x=5} = \frac{1}{2}\ln 5$

(Ex 6.3) #16 (Cauchy-Schwarz-Inequality)

$\int_a^b (f+tg)^2\,dt \ge 0$

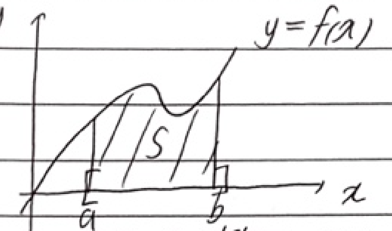
$\Leftrightarrow \int_a^b f^2 + 2t\int_a^b fg + t^2\int_a^b g^2 \ge 0 \quad \forall t$

→ t에 대한 이차 방정식이라 생각

$\frac{D}{4} \le 0 \Leftrightarrow$ C-S-Ineq

Chapter 6.4 Improper Riemann Integrals

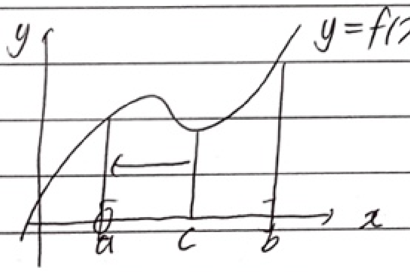
(Intro)



If $(0,+\infty)$라면, partition을 쪼개는 것: meaningless 해진다.

→ partition으로 쪼개야 하므로 폐구간이 기본 조건이며

" + 발산하지 않고 bdd 해야 한다.

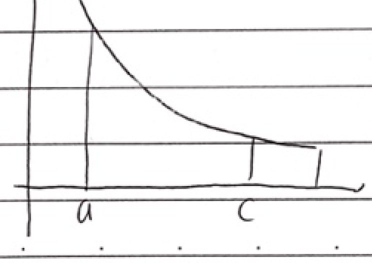


$c \to a^+$ 닫혀있는 c에서 a로 보내자.

⇒ 수렴한다면 적분값

발산하면 적분 불가.

<유계인데 한쪽이 개구간인 경우>



∞ 구간일 때, $c \to \infty$ 로 보내자.

$\lim_{c\to\infty}\int_a^c f(x)\,dx$

<∞로 가는 경우>

$*$ $f \in R[a,b]$ with $|f(x)| \le M.\ \forall x \in [a,b]$

$\Rightarrow f \in R[c,b]\ \forall c \in (a,b)$

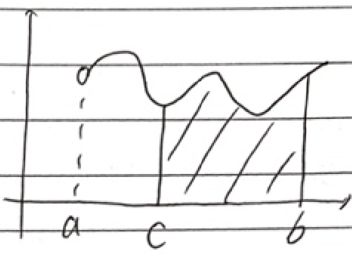
$$\& \left| \int_a^b f - \int_c^b f \right| = \left| \int_a^c f \right|$$

$$\le \int_a^c |f| \le M(c-a)$$

$(c \to a^+)$

$$\therefore \lim_{c \to a^+} \int_c^b f = \int_a^b f \quad //$$

Suppl $f : [a,b] \to \mathbb{R}$: bdd with $f \in R[c,b].\ \forall c \in (a,b)$



$y = f(x)$ define) "$f(a) = 0$"

$\Rightarrow f$: bdd on $[a,b]$ & $f \in R[c,b]$ $\forall c \in (a,b)$

s.t $\displaystyle\int_a^b f = \lim_{c \to a^+} \int_c^b f \quad //$

$$(\because \overline{\int_a^b} f = \overline{\int_a^c} f + \int_c^b f$$

$$\& \underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f \quad |f(x)| \le M \ \forall x \in [a,b]$$

상적은 상적끼리 같고, 하적은 하적끼리 같다. $f \in R[c,b]$이므로
상적 = 하적이므로. 구분하지 않음.

$$\Rightarrow 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f = \overline{\int_a^c} f - \underline{\int_a^c} f$$

$$\le 2M(c-a).\ c \to a^+$$

$c \to a^+$. $\displaystyle\overline{\int_a^b} f = \underline{\int_a^b} f \Leftrightarrow f \in R[a,b]$ &

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f\ )$$

Def 6.4.1) $f : [a,b] \to \mathbb{R}$ (May not bdd)
s.t. $f \in R[c,b]$ $\forall c \in (a,b)$

"the improper integral of f on $(a,b]$
denoted $\int_a^b f$"

$\Leftrightarrow \int_a^b f = \lim_{c \to a+} \int_c^b f$ $\begin{cases} \text{conv} \\ \text{div} \end{cases}$ or

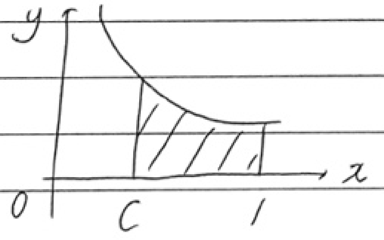
Rmks) ⓘ improper integral of f on $[a,b)$

$\Leftrightarrow \int_a^b f = \lim_{c \to b-} \int_a^c f$ $\begin{cases} \text{conv} \\ \text{div} \end{cases}$ or

ⓘⓘ Improper integral of f on $[a,p) \cup (p,b]$

$\int_a^b f = \lim_{c \to p-} \int_a^c f + \lim_{d \to p+} \int_d^b f$
: Conv or div

[EX 6.4.2) (a) $f(x) = \frac{1}{x} : (0,1] \to \mathbb{R}$: Unbdd at 0



& $f \in R[c,1]$ $\forall c \in (0,1)$

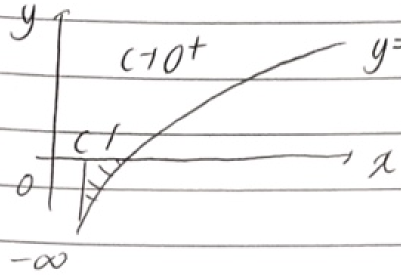
$\int_0^1 f = \lim_{c \to 0+} \int_c^1 f$

$\int_c^1 \frac{1}{x} dx = -\ln c$ $\quad c \in (0,1)$

$\Rightarrow \lim_{c \to 0+} \int_c^1 \frac{1}{x} dx = \lim_{c \to 0+} (-\ln c) = \infty$ (div)

$\therefore$ Improper integral of $\frac{1}{x}$ on $(0,1]$: div (존재 X)

(b) $L(x)=\ln x : (0,1] \to \mathbb{R}$: Unbdd at 0



& $L(x) \in R[c,1]$ $\forall c \in (0,1)$

($\because L$: Conti on $[c,1]$)

s.t $\displaystyle\int_c^1 \ln x\,dx = [x\ln x - x]_{x=c}^{x=1}$

$= -c\ln c + c - 1$

$\Rightarrow \displaystyle\int_0^1 \ln x\,dx = \lim_{c\to 0+} \int_c^1 \ln x\,dx$

$= \displaystyle\lim_{c\to 0+} (-c\ln c + c - 1)$

$= -1$ conv

$\left(\displaystyle\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \left(\because \frac{\infty}{\infty}\right) = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0 \right)$

(c) $f : [-1,1] \to \mathbb{R}$ $f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ \frac{1}{x} & 0 < x \le 1 \end{cases}$

$\displaystyle\int_{-1}^1 f\,dx = \int_{-1}^0 0 + \underbrace{\lim_{c\to 0+} \int_c^1 \frac{1}{x}\,dx}_{\to \text{div}}$

$\Rightarrow \displaystyle\int_{-1}^1 f\,dx$: div

(d) i) $f \in R[a,b] \Rightarrow f^2 \in R[a,b]$

But Improper integral : Not always true.

(ex ①) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{\sqrt{x}}\,dx$

$= \displaystyle\lim_{c\to 0+} [2\sqrt{x}]_{x=c}^{x=1}$

$= 2$: Conv

But $\int_0^1 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_0^1 \frac{1}{x} dx$: Div ∎

ii) $f \in R[a,b] \Rightarrow |f| \in R[a,b]$

But improper integral : Not always True (p.244 참고)

< Infinite intervals >

Def 6.4.3) $f : [a,\infty) \to \mathbb{R}$ & $f \in R[a,c]$ $\forall c \in (a,\infty)$

$\int_0^\infty f$: improper integral of f on $[a,\infty)$:

given by $\int_a^\infty f = \lim_{c\to\infty} \int_a^c f(x)dx$: Conv or div

Similarly $\int_{-\infty}^b f = \lim_{c\to-\infty} \int_c^b f$

$$\int_{-\infty}^\infty f = \lim_{c\to-\infty}\int_c^p f + \lim_{d\to\infty}\int_p^d f \quad (\text{for some } p \in \mathbb{R})$$

; Conv (if those limits Exist)

otherwise $\Rightarrow$ div

Rmk) (Generally,) $\int_{-\infty}^\infty f \neq \lim_{c\to\infty}\int_{-c}^c f$

($*$ P.V $\int_{-\infty}^\infty f = \lim_{c\to\infty}\int_{-c}^c f$)

; Cauchy principle value

(EX) $f(x) = x$

$\lim_{c\to\infty}\int_{-c}^c x\,dx = \lim_{c\to\infty}\frac{1}{2}(c^2-c^2) = 0$

But $\int_{\underset{p}{\underbrace{0}}}^\infty x = \lim_{R\to\infty}\int_0^R x = \lim_{R\to\infty}\frac{R^2}{2} = \infty$ (Div)

$\therefore \int_{-\infty}^{\infty} x$: Div

(ex) $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

$= \int_{p}^{\infty} \frac{1}{1+x^2} dx + \int_{-\infty}^{p} \frac{1}{1+x^2} dx$

$= \lim_{R\to\infty} \tan^{-1}x \Big|_{p}^{R} + \lim_{L\to-\infty} \tan^{-1}x \Big|_{L}^{p}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

Rmk) $f(x) \geq 0 \quad \forall x \in [a, \infty)$ with $f \in R[a, c] \quad \forall c > a$

$F(c) = \int_{a}^{c} f$: M-I f.t of "c"

$\Rightarrow \lim_{c\to\infty} \int_{a}^{c} f$: Conv $\Leftrightarrow \int_{a}^{\infty} f < \infty$

or $\lim_{c\to\infty} \int_{a}^{c} f$: div $\Leftrightarrow \int_{a}^{\infty} f = +\infty$

(EX 6.4.4) -(b) $f(x) = \frac{\sin x}{x}$, $x \in [\pi, \infty)$

1) $\int_{0}^{\infty} \frac{\sin x}{x} dx = \boxed{\int_{0}^{\pi} \frac{\sin x}{x} dx}^{\text{(i)}} + \int_{\pi}^{\infty} \frac{\sin x}{x} dx$

$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ - (*) (ii)

(i) $0 \leq \frac{\sin x}{x} \leq 1 \quad \forall x \in (0, \pi]$

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No.

$\Rightarrow \begin{cases} \frac{\sin x}{x} : \text{Bdd on } (0, \pi] \\ \& \ \frac{\sin x}{x} : \text{conti on } [c, \pi] \ \forall c \in (0, \pi) \\ \Rightarrow \frac{\sin x}{x} \in R[c, \pi] \end{cases}$

$\therefore \int_0^{\pi} \frac{\sin x}{x} dx = \lim_{c \to 0+} \int_c^{\pi} \frac{\sin x}{x} dx$: 존재

$(*)$ $I(\alpha) = \int_0^{\infty} e^{-\alpha x} \cdot \frac{\sin x}{x} dx \quad (\alpha > 0)$

$I'(\alpha) = -\int_0^{\infty} e^{-\alpha x} \sin x \, dx$

$= -\frac{1}{\alpha^2+1} \Rightarrow I(\alpha) = -\tan^{-1}(\alpha) + C$

$I(\infty) = -\tan^{-1}(\infty) + C$

$= -\frac{\pi}{2} + C = 0$

$\therefore C = \frac{\pi}{2}$

$\therefore I(0) = -\overbrace{\tan^{-1}(0)}^{0} + \frac{\pi}{2}$

$= \int_0^{\infty} \frac{\sin x}{x} dx$

Thus. $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$: Conv

But $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx$: Div

$n \in \mathbb{N}$. $\int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx = \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx$

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \geq \int_{(k+\frac{1}{4})\pi}^{(k+\frac{3}{4})\pi} \frac{|\sin x|}{x}\,dx$$

since $|\sin x| > \frac{1}{\sqrt{2}}$, $x \in [(k+\frac{1}{4})\pi, (k+\frac{3}{4})\pi]$

$$\int_{(k+\frac{1}{4})\pi}^{(k+\frac{3}{4})\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{\sqrt{2}} \frac{1}{(k+\frac{3}{4})\pi} \cdot \frac{\pi}{2}$$

$$\geq \frac{\sqrt{2}}{4} \frac{1}{(k+1)}$$

$$\Rightarrow \int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx \geq \frac{\sqrt{2}}{4} \sum_{k=1}^{n} \frac{1}{k+1}$$

(무한대로 간다면 → 조화급수)

since $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$: Div

$$\therefore \int_{\pi}^{\infty} \frac{|\sin x|}{x}\,dx = \lim_{n\to\infty} \int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx = \infty$$

: Div

def) $f: [a,\infty) \to \mathbb{R}$: Absolutely Integrable on $[a,\infty)$

$\Leftrightarrow \int_a^\infty |f| < \infty$

Corol) $f: [a,\infty) \to \mathbb{R}$

$\int_a^\infty |f| < \infty \;\underset{\not\Leftarrow}{\Rightarrow}\; \int_a^\infty f$: Conv

(ex) $\frac{\sin x}{x}$

(pf) $\forall x \in [a,\infty)$, $0 \leq |f(x)| + f(x) \leq 2|f(x)|$

If $\int_a^\infty 2|f| = 2\int_a^\infty |f| < \infty$

$\Rightarrow$ (비교판정) : $\int_a^\infty (|f|+f)$: Conv

$$\int_a^\infty (|f|+f) < \infty.$$

$$\int_0^\infty f = \lim_{c\to\infty} \int_0^c (|f|+f-|f|)$$

$$= \underbrace{\lim_{c\to\infty} \int_0^c (|f|+f)}_{\text{Conv}} - \underbrace{\lim_{c\to\infty} \int_0^c |f|}_{\text{Conv}} \quad : \text{Conv}$$

Thm 6.4.5) (Comparison Test)

let $g : [a,\infty) \to \mathbb{R}$, $g(x) \geq 0 \ \forall x \in [a,\infty)$.

$g \in R[a,c] \ \forall c > a$ & $\int_a^\infty g < \infty$ (Conv)

If $f : [a,\infty) \to \mathbb{R}$: satisfies this ↓

(a) $f \in R[a,c] \ \forall c > a$

(b) $|f(x)| \leq g(x) \ \forall x \in [a,+\infty)$

$\Rightarrow \int_a^\infty f$: Conv s.t

$$\left(\left| \int_a^\infty f \right| \leq \int_a^\infty g \ _{(<\infty)} \right)$$

(pf) $\int_a^\infty |f(x)| = \lim_{c\to\infty} \int_a^c |f| \leq \lim_{c\to\infty} \int_a^c g < \infty$

Hence, $\int_a^\infty |f| < \infty$ (f : Absolutely integrable on $[a,\infty)$)

$\Rightarrow \int_a^\infty f$: Conv

(*) $\int_a^c f = \int_a^c |f|+f - \int_a^c |f|$

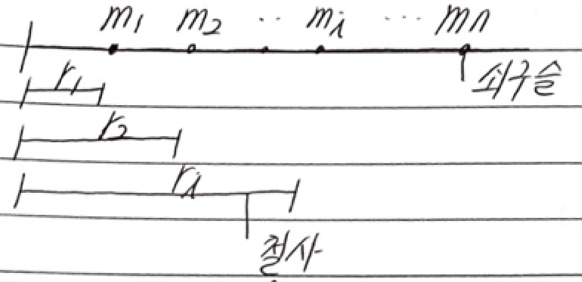
$\to \lim_{c\to\infty} \int_a^c f = \lim_{c\to\infty} \int_a^c |f|+f - \lim_{c\to\infty} \int_a^c |f|$: Conv

$\forall c > a.\ \left|\int_a^c f\right| \le \int_a^c |f|$

$$\left|\int_a^\infty f\right| = \lim_{c\to\infty}\left|\int_a^c f\right| \le \lim_{c\to\infty}\int_a^c |f|$$

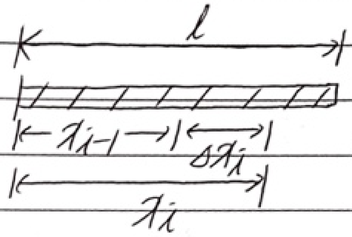
$$\le \lim_{c\to\infty}\int_a^c g = \int_a^\infty g$$

(EX 6.5.1) (관성모멘트)
(Intro)



I : moment of inertia (관성)

$\Rightarrow I = \sum_{i=1}^{n} r_i^2 m_i$ (이산형 - Discrete)



$\Delta x_i = x_i - x_{i-1}$

$$\Rightarrow I = \int_0^l x^2 \underbrace{\rho(x)\,dx}_{\parallel \atop dm(x)} \rightarrow \text{연속형 - conti}$$

($\rho(x)$: cross-sectional density at x)

$\rho(x)\,dx = dm(x) = m'(x)\,dx$ (m이 미가일때만)

$:\ \Delta m_i = m(x_i) - m(x_{i-1})$

x_i의 질량변화율

"질량"은 재질의 밀도에 의존하게 되어있음 (솜, 나무, 쇠볼이)
미가가 안되는 경우의 예? 재질이 솜이면 솜 나무면 나무가 아닌
철이었다 솜이었다 합성소재일 때 · 무게 일정 X

<Definition of the Riemann-Stieltjes integral>

let α : M-I on $[a,b]$

$f : [a,b] \to \mathbb{R}$: Bdd ft

For each $P = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

Set $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

α : M-I on $[a,b]$

$\Rightarrow \Delta\alpha_i \geq 0 \quad \forall i = 1, 2, \cdots, n$

let $m_i = \inf\{f(t) \mid t \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\}$

def ①) $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

: Upper-R-S-Sum

②) $L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

: Lower-R-S-Sum

Since $m_i \leq M_i$, $\Delta\alpha_i \geq 0$, $1 \leq i \leq n$

$\Rightarrow L(P,f,\alpha) \leq U(P,f,\alpha) \quad \forall P$

If $m \leq f(x) \leq M$, $\forall x \in [a,b]$

$$m[\alpha(b) - \alpha(a)] = \sum_{i=1}^{n} m \Delta\alpha_i$$

$$\leq L(P,f,\alpha) \leq U(P,f,\alpha)$$

$$\leq \sum_{i=1}^{n} M \cdot \Delta\alpha_i = M[\alpha(b) - \alpha(a)]$$

$\{L(P,f,\alpha), U(P,f,\alpha) \mid P$: partition of $[a,b]\}$

: B.dd below & above

$\Rightarrow$ define) ① $\overline{\int_a^b} f\,d\alpha = \inf\{\mathcal{U}(P,f,\alpha) \mid P : \text{partition of } [a,b]\}$

: Upper R-S-integral of "f" w.r.to "α" on $[a,b]$

② $\underline{\int_a^b} f\,d\alpha = \sup\{\mathcal{L}(P,f,\alpha) \mid P : \text{partition of } [a,b]\}$

: Lower R-S-integral of "f" w.r.to "α" over $[a,b]$.

Thm 6.5.2) $f : [a,b] \to \mathbb{R}$: B.dd
α : M-I on $[a,b]$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$

(pf) P, P^* : partition of $[a,b]$ s.t
$P \subset P^*$ (P^* : refinement of P)

$$\Rightarrow \mathcal{L}(P,f,\alpha) \le \mathcal{L}(P^*,f,\alpha)$$

$$\le \mathcal{U}(P^*,f,\alpha) \le \mathcal{U}(P,f,\alpha)$$

If P, Q : any partitions of $[a,b]$

$$\Rightarrow \mathcal{L}(P,f,\alpha) \le \mathcal{L}(P\cup Q,f,\alpha)$$
$$\le \mathcal{U}(P\cup Q,f,\alpha) \le \mathcal{U}(Q,f,\alpha)$$

$$\Rightarrow \mathcal{L}(P,f,\alpha) \le \mathcal{U}(P,f,\alpha)$$

Therefore, $\underline{\int_a^b} f\,d\alpha = \sup\limits_P \mathcal{L}(P,f,\alpha)$

$$\le \inf_Q \mathcal{U}(Q,f,\alpha) = \overline{\int_a^b} f\,d\alpha$$

$$\therefore \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$$

Def 6.5.4) $f : [a,b] \to \mathbb{R}$: bdd

α : M-I on $[a,b]$

f : R-S-I w.r.to "α" on $[a,b]$

$$\Leftrightarrow \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \,.$$

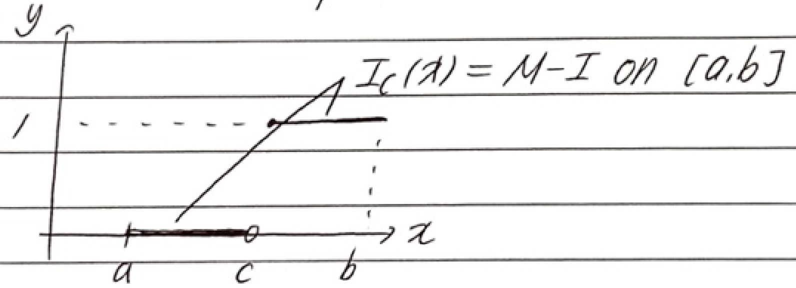
denote) $\int_a^b f\,d\alpha$ or $\int_a^b f(x)\,d\alpha(x)$

※ $\alpha(x) = x \Rightarrow$ Riemann Integral

Ex 6.5.4) ⓐ $a < c \le b$ let $I_c(x) = I(x-c)$

$$I_c(x) = \begin{cases} 0 & , x < c \\ 1 & , x \ge c \end{cases}$$

: Unit Jump f.t (한 단계 도약함수)



Set $\alpha(x) = I_c(x)$: M-I on $[a,c]$

If $f : [a,b] \to \mathbb{R}$: Conti at $c \in [a,b]$

$$\Rightarrow \int_a^b f\,d\alpha = \int_a^b f\,dI_c = f(c) \,.$$

(∵ let $P = \{x_0, x_1, \cdots, x_{k-1}, x_k, \cdots, x_n\}$

: partition of $[a,b]$ with $x_{k-1} < c \le x_k$

$$\Rightarrow \Delta\alpha_i = \begin{cases} \alpha(x_k) - \alpha(x_{k-1}) = 1 & , i = k \\ 0 & , i \ne k \end{cases})$$

$\Rightarrow \mathcal{U}(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \to \begin{cases} 1, & i=k \\ 0, & i\neq k \end{cases}$

$= M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$

' $x_k - x_{k-1} < \delta$ 라면 가정'

$\mathcal{L}(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

$= m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}$

f: Conti at c

$\forall \varepsilon > 0, \exists \delta > 0$ s.t $f(c) - \varepsilon < f(t) < f(c) + \varepsilon$

$\forall \varepsilon \in [a,b]$ with $|t - c| < \delta$

$\Rightarrow t \in (c - \delta, c + \delta)$

P: any partition of $[a,b]$

$\|P\| < \delta$, $\mathcal{L}(P, f, \alpha)$ $\mathcal{U}(P, f, \alpha)$

$\Rightarrow f(c) - \varepsilon \le m_k \le M_k \le f(c) + \varepsilon$

$\Rightarrow f(c) - \varepsilon \le \mathcal{L}(P, f, \alpha) \le \mathcal{U}(P, f, \alpha) \le f(c) + \varepsilon$

$\Rightarrow f(c) - \varepsilon \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le f(c) + \varepsilon$

(하적분) (상적분 : 하합보다 크고 상합보다 작다)

$(\varepsilon > 0$: Arbitrary$)$

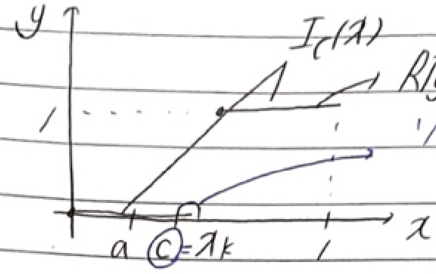
$\Rightarrow \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

$\Leftrightarrow f$: R-S-I w.r.to α & $\int_a^b f\,d\alpha = f(c)$

Rmk) 좌연속만 되도 된다. 우연속일 필요는 없다.

f: left conti at $c \in [a,b]$

$\Rightarrow \int_a^b f\,d\alpha = \int_a^b f\,dI_c = f(c)$



Right-conti at c

'$M_k = 1$, $m_k = 0$' 이므로 항상 상≠하합 ⇒ 리만스틸체 적분 불가

$I_c(x)$가 우연속이니까 좌연속만 된다는 것이지

좌연속이라면 우연속이 되어야 함.

Partition 안에 꼭 c를 첨가

$\Delta\alpha_i = 0 \quad i \neq k$

$|f(t) - f(c)| < \varepsilon, \ t \in (c-\delta, c]$

(*) 1°. P: any partition of $[a,b]$

let $P^* = P \cup \{c\} \Rightarrow (P \subset P^*)$

→ 임의의 파티션에 c를 첨가

$L(P,f,\alpha) \le L(P^*,f,\alpha)$

$\le U(P^*,f,\alpha) \le U(P,f,\alpha)$

Since $c \in P^*$, $\exists k$, $1 \le k \le n$ s.t $x_k = c$

$$\Delta\alpha_i = \begin{cases} \alpha(\underbrace{c}_{= x_k}) - \alpha(x_{k-1}) = 1, & i = k \\ 0 & i \neq k \end{cases}$$

$f : [a,b] \to \mathbb{R}$: left conti at $c \in [a,b]$

$\Leftrightarrow \forall \varepsilon > 0, \ \exists \delta > 0$ s.t

$f(c) - \varepsilon < f(t) < f(c) + \varepsilon \quad \forall t \in [a,b]$

with $c - \delta < t \le c \quad (t \in (c-\delta, c])$

* c를 포함하는 partition P 에서, $\|P\| < \delta$ 이면, $x_k - x_{k-1} < \delta$

$x_k - x_{k-1}$ 사이에 속하는 t는 이를 모두 만족.

하한 취해도 부등호 그대로, 상한 취해도 부등호 그대로

For any $Q > P^*$ $(c \in P^*)$ with $\|Q\| < \delta$ $(\Rightarrow x_k - x_{k-1} < \delta)$

$\Rightarrow f(c) - \varepsilon \le m_k \le M_k \le f(c) + \varepsilon$

$= f(c) - \varepsilon \le L(Q,f,\alpha) \le U(Q,f,\alpha) \le f(c) + \varepsilon$

($\varepsilon > 0$ Arbitrary)

$\therefore f$: R-S-I w.r.to α & $\int_a^b f\,d\alpha = \int_a^b f\,dI_c = f(c)$

(대칭 → 만약 f가 좌연속이면 우연속이어야.)

(EX 6.5.4) ⓑ $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

α: Not const M-I on $[0,1]$

$\Rightarrow f$: Not R-S-I w.r.to α

* α: 상수 $\Rightarrow \Delta\alpha_i = 0,\ 1 \le i \le n$

$\Rightarrow L(P,f,\alpha) = U(P,f,\alpha) = 0$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = 0$$

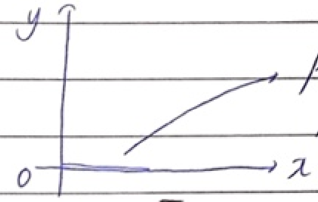
상수 아니면 리만적분불가

$\therefore P = \{x_0, x_1, \cdots, x_n\}$: any partition of $[0,1]$

$$\begin{cases} U(P,f,\alpha) = \sum_{i=1}^{n} 1 \cdot \Delta\alpha_i = \alpha(b) - \alpha(a) > 0 \\ L(P,f,\alpha) = \sum_{i=1}^{n} 0 \cdot \Delta\alpha_i = 0 \end{cases}$$

모든점에서 상합 ≠ 하합

$\Rightarrow$ Not R-S-I



partition을 쪼개더라도

$M_k = 1,\ m_k = 0$

$$\Rightarrow \overline{\int_0^1} f\,d\alpha = 1 \ne 0 = \underline{\int_0^1} f\,d\alpha$$

$\therefore f$: Not R-S-I w.r.to α. ■

Thm 6.5.5) $f : [a,b] \to \mathbb{R}$: b.dd
α : M-I on $[a,b]$
f : R-S-I w.r.to "α"
$\Leftrightarrow \forall \varepsilon > 0, \exists P$: partition of $[a,b]$
s.t $\mathcal{U}(P,f,\alpha) - \mathcal{L}(P,f,\alpha) < \varepsilon$

* $\mathcal{U}(P,f,\alpha) - \mathcal{L}(P,f,\alpha) < \varepsilon$
$\Rightarrow$ f : R-S-I w.r.to α
$\Rightarrow$ ε 보다 작으므로 가능함.

($\Rightarrow$) $\forall \varepsilon > 0, \exists P_1$ s.t

$$\int_a^b f\,d\alpha + \frac{\varepsilon}{2} > \mathcal{U}(P_1,f,\alpha) \geq \mathcal{U}(P,f,\alpha) \quad -①$$

refinement of P_1

$\exists P_2$: partition of $[a,b]$ s.t

$$\int_a^b f\,d\alpha - \frac{\varepsilon}{2} < \mathcal{L}(P,f,\alpha) \leq \mathcal{L}(P_2,f,\alpha) \quad -②$$

let $P_1 \cup P_2 = P$

① - ② $\Rightarrow$ By ①,② $\mathcal{U}(P,f,\alpha) - \mathcal{L}(P,f,\alpha) < \frac{\varepsilon}{2} - (-\frac{\varepsilon}{2}) = \varepsilon$ ∎

Thm 6.5.6) $f : [a,b] \to \mathbb{R}$
α : M-I on $[a,b]$
(a) f : conti on $[a,b]$ $\Rightarrow$ f : R-S-I w.r.to α
(b) f : monotone ft on $[a,b]$
& α : conti on $[a,b]$
$\Rightarrow$ f : R-S-I w.r.to α $\to$ 증명은 리만판정으로
* f가 단조이기만 하면 보장 X
α가 M-I 이고 conti 이어야 가능.

(pf)

(pf) ⓐ Assume) α : Not const M-I on $[a,b]$
$(\alpha(b)-\alpha(a)>0)$

f : conti on $[a,b]$

$\Rightarrow \forall \varepsilon>0,\ \exists \delta=\delta(\varepsilon)>0$ s.t

$|f(t)-f(s)|<\eta \quad \forall s,t\in[a,b]$ with $|t-s|<\delta$

$\eta : 0<[\alpha(b)-\alpha(a)]\eta<\varepsilon$

(예) $\eta = \dfrac{\varepsilon}{2(\alpha(b)-\alpha(a))}$

$\forall P=\{x_0,x_1,\cdots,x_n\}$: partition with $\|P\|<\delta$

$\Rightarrow M_k-m_k = \sup\{\underbrace{|f(t)-f(s)|}_{<\eta} \mid t,s\in[x_{k-1},x_k]\}$

$\le \eta \quad 1\le k\le n$

상한 취하면 등호는 붙을 수 있어도 방향은 그대로.

$\Rightarrow U(P,f,\alpha)-L(P,f,\alpha)$

$= \sum_{k=1}^{n}\underbrace{(M_k-m_k)}_{\le\eta}\Delta\alpha_k$

$\le \eta\sum_{k=1}^{n}\Delta\alpha_k = \eta[\alpha(b)-\alpha(a)]<\varepsilon$

$\therefore f$: R-S-I w.r.to α

ⓑ Assume) f : M-I on $[a,b]$

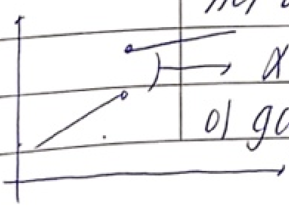
$n\in\mathbb{N}$. Choose $P=\{x_0,x_1,\cdots,x_n\}$: partition of $[a,b]$

with $\Delta\alpha_i = \alpha(x_i)-\alpha(x_{i-1})$

$= \frac{1}{n}(\alpha(b)-\alpha(a))$ ($\because \alpha$: conti)

지금 이 partition P 잡을 수 있는 이유: α : conti 이므로 n이 아무리 커도 원하는 구간 잡기 가능.

α가 비스듬히 감-등을 고려하기 위한 용도



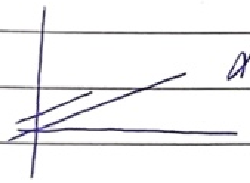
α가 disconti 라면 이 gap을 메울 수 없음.

Since $M_i = f(x_i) \quad 1 \le i \le n$
$m_i = f(x_{i-1}) \quad 1 \le i \le n$

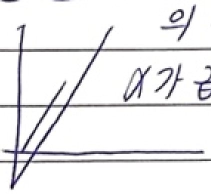
$U(P,f,\alpha) - L(P,f,\alpha)$

$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \frac{1}{n}\underbrace{(\alpha(b)-\alpha(a))}$

의 의미를 주기 위해 존재.



α가 비스듬히 갈때



α가 급격할 때

$= \frac{1}{n}[\alpha(b)-\alpha(a)] \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$

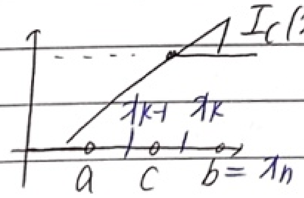
$= \frac{1}{n}[\underbrace{\alpha(b)-\alpha(a)}][f(b)-f(a)] < \epsilon \quad \forall n \ge n_0$

상수이므로 n만 크게 해주면 ϵ보다 작게 가능.

$\therefore$ f : R-S-I w.r.to α (α : M-I & conti) ■

Rmk) f : Monotone on $[a,b]$
α : M-I & Not conti on $[a,b]$
$\not\Rightarrow f \in R(\alpha)$

(EX) $a < c \le b$
$f(x) = I_c(x)$, $\alpha(x) = I_c(x)$ → 리만스틸체 상적, 하적 존재
$I_c(x)$ → c에서 도약 But 상적 ≠ 하적



$\Delta\alpha_i = \begin{cases} 1, & i = k \\ 0, & i \ne k \end{cases}$

$\forall P = \{x_0, x_1, \dots, x_n\}$: partition of $[a,b]$
$\cdots x_{k-1}, x_k \cdots$

$\exists k, 1 \le k \le n$ s.t. $x_{k-1} < c \le x_k$

$M_k = \sup\{f(t) = I_c(t) \mid t \in [x_{k-1}, x_k]\} = 1$

$m_k = \inf\{f(t) = I_c(t) \mid t \in [x_{k-1}, x_k]\} = 0$

$$\Rightarrow \begin{cases} \mathcal{U}(P, f, \alpha) = M_k = 1 \\ \qquad\qquad\qquad\quad \neq \\ \mathcal{L}(P, f, \alpha) = m_k = 0 \end{cases}$$

$\therefore \nexists P$: satisfying $\mathcal{U}(P, f, \alpha) - \mathcal{L}(P, f, \alpha) < \varepsilon \quad (0 < \varepsilon < \frac{1}{2})$

or $\overline{\int_a^b} f\,d\alpha = 1 \neq 0 = \underline{\int_a^b} f\,d\alpha$

$\Rightarrow f \notin R(\alpha)$

* α는 M-I 일 뿐만 아니라 연속이 되어야 한다.

<Property of the R-S-I>

Def 6.5.1) $R(\alpha) = \{f \mid f : \text{R-S-I w.r.to } \alpha \text{ on } [a, b]\}$

Thm 6.5.8 (a) $f, g \in R(\alpha)$, then $f+g$, $cf\,(c \in \mathbb{R}) \in R(\alpha)$

& $\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$

& $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$

(b) $f \in R(\alpha_i)$, $i = 1, 2$

$\Rightarrow f \in R(\alpha_1 + \alpha_2)$

& $\int_a^b f\,d(\alpha_1 + \alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$

ⓒ $f\in R(\alpha),\ a<c\le b. \Rightarrow f$: R-S-I w.r.to α
on $[a,c]$ and $[c,b]$

with $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

ⓓ $f,g\in R(\alpha)$ with $f(x)\le g(x)\ \forall x\in[a,b]$

$$\Rightarrow \int_a^b f\,d\alpha \le \int_a^b g\,d\alpha$$

$\forall P,Q$: partition of $[a,b]$, $L(P,f,\alpha)\le U(Q,f,\alpha)$

ⓔ If $|f(x)|\le M,\ \forall x\in[a,b]$
& $f\in R(\alpha) \Rightarrow |f|\in R(\alpha)$

& $$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$$

$$\le M[\alpha(b)-\alpha(a)]$$

(pf) ⓑ $f\in R(\alpha_i),\ i=1,2$
$\forall \varepsilon>0,\ \exists P_i,\ i=1,2$: partition of $[a,b]$
s.t. $U(P_i,f,\alpha_i)-L(P_i,f,\alpha_i)<\frac{\varepsilon}{2}$

let $P=P_1\cup P_2 \Rightarrow P_1,P_2\subset P$ (refinement)

$U(P,f,\alpha_i)-L(P,f,\alpha_i)<\frac{\varepsilon}{2}$ $i=1,2$

Since $\Delta(\alpha_1+\alpha_2)_i = \Delta(\alpha_1)_i + \Delta(\alpha_2)_i$

$\|$

$(\alpha_1+\alpha_2)x_i-(\alpha_1+\alpha_2)x_{i-1}$
$=(\alpha_1 x_i-\alpha_1 x_{i-1})+(\alpha_2 x_i-\alpha_2 x_{i-1})$
$=\Delta(\alpha_1)_i-\Delta(\alpha_2)_i$

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No.

$\mathcal{U}(P,f,\alpha_1+\alpha_2) - \mathcal{L}(P,f,\alpha_1+\alpha_2)$

$= \mathcal{U}(P,f,\alpha_1) - \mathcal{L}(P,f,\alpha_1) + \mathcal{U}(P,f,\alpha_2) - \mathcal{L}(P,f,\alpha_2)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow f \in R(\alpha_1+\alpha_2)$

$\forall P, Q$: partitions of $[a,b]$.

$\mathcal{L}(P,f,\alpha_1+\alpha_2) = \mathcal{L}(P,f,\alpha_1) + \mathcal{L}(P,f,\alpha_2)$

$\leq \int_{\underline{a}}^{b} f d\alpha_1 + \int_{a}^{b} f d\alpha_2$

하적분으로 간주하면 부등호 발생.

$\leq \mathcal{U}(Q,f,\alpha_1) + \mathcal{U}(Q,f,\alpha_2)$

$\Rightarrow \int_{\underline{a}}^{b} f d(\alpha_1+\alpha_2) = \sup_{P} \mathcal{L}(P,f,\alpha_1+\alpha_2)$

$\leq \int_{a}^{b} f d\alpha_1 + \int_{a}^{b} f d\alpha_2$

$\leq \inf_{Q} \mathcal{U}(Q,f,\alpha_1+\alpha_2)$

$= \int_{a}^{\bar{b}} f d(\alpha_1+\alpha_2)$

Since $f \in R(\alpha_1+\alpha_2)$, $\Rightarrow \int_{a}^{b} f d(\alpha_1+\alpha_2) = \int_{a}^{b} f d\alpha_1 + \int_{a}^{b} f d\alpha_2$ ∎

(pb) - (e) $f \in R(\alpha)$, $P = \{x_0, x_1, \cdots, x_n\}$: partition of $[a,b]$

let $\begin{cases} M_i = \sup\{f(t) \mid t \in [x_{i-1}, x_i]\} \\ m_i = \inf\{f(t) \mid t \in [x_{i-1}, x_i]\} \end{cases}$

$\begin{cases} M_i^* = \sup\{|f(t)| \mid t \in [x_{i-1}, x_i]\} \\ m_i^* = \inf\{|f(t)| \mid t \in [x_{i-1}, x_i]\} \end{cases}$

$\Rightarrow \forall t, x \in [x_{i-1}, x_i]$

$$\big|\, |f(t)| - |f(x)| \,\big| \overset{(T\text{-}I)}{\leq} |f(x) - f(t)|$$
$$\leq M_i - m_i$$
$\Rightarrow M_i^* - m_i^* = \sup \big|\, |f(t)| - |f(x)| \,\big| \quad x, t \in [x_{i-1}, x_i]$
$$\leq M_i - m_i \quad (1 \leq i \leq n)$$

$\Rightarrow \mathcal{U}(P, |f|, \alpha) - \mathcal{L}(P, |f|, \alpha)$

$\leq \mathcal{U}(P, f, \alpha) - \mathcal{L}(P, f, \alpha) < \varepsilon, \ \exists P$

Thus, $|f| \in R(\alpha)$ i.e) $c = \pm 1$ s.t $c \int_a^b f d\alpha \geq 0$

$\Rightarrow \left| \int_a^b f d\alpha \right| = c \int_a^b f d\alpha = \int_a^b c f d\alpha \quad (c f \leq |f|)$ R-S-I 성질 - 부등호 유지

$$\leq \int_a^b |f| d\alpha$$

$$\leq \int_a^b M d\alpha \quad (M : \text{Const})$$

$$= M \int_a^b 1 \cdot d\alpha = M(\alpha(b) - \alpha(a))$$

Thm 6.5.9) (M-V-T)
$f \in C[a,b]$ & α : M-I on $[a,b]$
$\Rightarrow \exists c \in [a,b]$ s.t $\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha(a)]$

(pf) $M = \max\{f(x) \mid x \in [a,b]\}$
$m = \min\{f(x) \mid x \in [a,b]\}$

$\Rightarrow m \le f(x) \le M \quad \forall x \in [a,b]$

$\Rightarrow m[\alpha(b) - \alpha(a)] = \int_a^b m\,d\alpha \le \int_a^b f\,d\alpha \le \int_a^b M\,d\alpha$

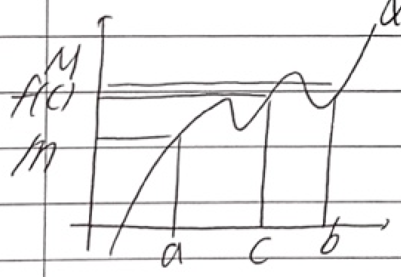
'극단적 case' 상수함수인 경우 trivial $\qquad = M[\alpha(b) - \alpha(a)]$

① $\alpha(b) - \alpha(a) = 0$. then

$$0 = \int_a^b f\,d\alpha = f(c)[\alpha(b) - \alpha(a)] \quad \forall c \in [a,b]$$

② If $\alpha(b) - \alpha(a) > 0$

$$m \le \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f\,d\alpha \le M$$



By the I-V-T of cont ft.
$\exists c \in [a,b]$ s.t

$$f(c) = \frac{1}{[\alpha(a) - \alpha(b)]} \int_a^b f\,d\alpha$$

$$\therefore \int_a^b f\,d\alpha = f(c)[\alpha(a) - \alpha(b)]$$

Thm 6.5.10) let α, β : M-I on $[a,b] \Rightarrow \alpha \in R(\beta), \beta \in R(\alpha)$
If this is the case,

$$\int_a^b \alpha\,d\beta = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_a^b \beta\,d\alpha$$

$$\left(\text{v.s } \int_a^b fg' = [fg]_a^b - \int_a^b f'g\right)$$

(pf) $\forall p$: partition of $[a,b]$.

$$\text{④} \begin{cases} \mathcal{U}(p, \alpha, \beta) = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \mathcal{L}(p, \beta, \alpha) \\ \mathcal{L}(p, \alpha, \beta) = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \mathcal{U}(p, \beta, \alpha) \end{cases}$$

$(\because$ (*) α, β: M-I on $[a,b]$

$$M_i^{\alpha} = \sup\{\alpha(t) \mid t \in [x_{i-1}, x_i]\} = \alpha(x_i)$$
$$M_i^{\beta} = \sup\{\beta(t) \mid t \in [x_{i-1}, x_i]\} = \beta(x_i)$$

$$m_i^{\alpha} = \inf\{\alpha(t) \mid t \in [x_{i-1}, x_i]\} = \alpha(x_{i-1})$$
$$m_i^{\beta} = \inf\{\beta(t) \mid t \in [x_{i-1}, x_i]\} = \beta(x_{i-1})$$

$$\begin{aligned}
U(P, \alpha, \beta) &= \sum_{i=1}^{n} M_i^{\alpha} \Delta\beta_i \\
&= \sum_{i=1}^{n} \alpha(x_i)\left(\beta(x_i) - \beta(x_{i-1})\right) \\
&= \sum_{i=1}^{n} \alpha(x_i)\beta(x_i) - \sum_{i=1}^{n} \beta(x_{i-1})\left(\alpha(x_i) - \alpha(x_{i-1})\right) \\
&\quad - \sum_{i=1}^{n} \beta(x_i)\alpha(x_{i-1}) \\
&= \alpha(b)\beta(b) - \alpha(a)\beta(a) - L(P, \beta, \alpha))
\end{aligned}$$