p301 Rearrangement of Seq

Def 7.3.6) $\sum a_k'$: Re arrangement of $\sum a_k$

$\Leftrightarrow$ $\exists j : \mathbb{N} \to \mathbb{N}$: 1-1 and onto s.t $a_k' = a_{j(k)}$ $\forall k \in \mathbb{N}$

Q) If $\sum a_k$: Conv $\Rightarrow$ $\sum a_k'$: Conv? NO

If so, $\sum a_k = \sum a_k'$? : NO

(absolute : O, Conditional : X)

(EX 7.3.7) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$: Conv.

($\because$ $\frac{1}{k} = b_k$: 교대급수 판정 of 디리클레 판정.)

$\sum a_k' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ — (*)

let $S = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

$\underbrace{S_{2n}}_{\text{bdd above}} < S < \underbrace{S_{2n+1}}_{\text{bdd below}}$ $\forall n \geq 1$.

$\Rightarrow$ $S < S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

Let S_n' : nth partial sum of (*).

$S_{3n}' = \displaystyle\sum_{k=1}^{n} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) = \underbrace{\sum_{k=1}^{n} \frac{8k-3}{2k(4k-1)(4k-3)}}_{\text{Conv as } n \to \infty}$

$0 < \dfrac{8k-3}{2k(4k-1)(4k-3)} < \dfrac{M}{k^2}$ (비교판정)

(since $\displaystyle\sum_{k=1}^{\infty} \frac{M}{k^2} < \infty$.)

let $S' = \lim_{n\to\infty} S_{3n}'$.

$$S_{3n+1}' = S_{3n}' + \frac{1}{4n+1} \to S' \text{ as } n\to\infty$$

$$S_{3n+2}' = S_{3n}' + \frac{1}{4n+1} + \frac{1}{4n+3} \to S' \text{ as } n\to\infty$$

$\therefore \lim_{n\to\infty} S_n' = S'$

since $\frac{5}{6} = S_3' < S_6' < \cdots < S' \Rightarrow S' = \lim_{n\to\infty} S_n' > \frac{5}{6} > S$

$\therefore S \neq S'$

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Thm 7.3.8) Σa_k : Conv Absolutely. $\forall a_k'$: Conv to the same "sum"

($\Leftrightarrow$ If $\Sigma |a_k| < \infty$, $\forall \Sigma a_k'$, $\Sigma a_k' = \Sigma a_k$.)

(pf) $\Sigma a_k'$: rearrangement of Σa_k

since $\Sigma |a_k| < \infty$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$$\sum_{k=n}^{m} |a_k| < \varepsilon \quad \forall m \geq n \geq N.$$

suppl $a_k' = a_{j(k)}$. $j : \mathbb{N} \to \mathbb{N}$: 1-1 & onto.

Choose $p \geq N$ s.t $\{1, 2, \cdots N\} \subset \{j(1), j(2), \cdots j(p)\}$

let $S_n = \sum_{k=1}^{n} a_k$, $S_n' = \sum_{k=1}^{n} a_k'$

If $n \geq p$, $S_n - S_n' = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{j(k)}$

$p \leq n$ 이니까, $a_{j(p)}$

$\Rightarrow \{a_1, a_2, \cdots a_N\} \subseteq \{a_1, \cdots, a_n\}$ & $\{a_1, a_2, \cdots a_N\} \subseteq \{a_{j(1)}, \cdots, a_{j(p)}, \cdots, a_{j(n)}\}$

$$\Rightarrow |S_n - S_n'| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{j(k)} \right| \quad \text{T-I}$$

$$\leq \sum_{k=N+1}^{n} |a_k| + \sum_{\{j(1) \cdots j(n)\} \setminus \{1, \cdots N\}} |a_{j(k)}|$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

$\therefore \lim_{n\to\infty} S_n' = \lim_{n\to\infty} S.$ $\quad \therefore {}^{\forall}\Sigma a_k'$: 재배열 $\Rightarrow \Sigma a_k = \Sigma a_k'$

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Thm 7.3.9) Σa_k : Conditionally conv ($\Leftrightarrow \Sigma a_k$: conv but $\Sigma |a_k| = +\infty$)

Suppl $\forall \alpha \in \mathbb{R}$, $\exists \Sigma a_k'$: rearrangement of Σa_k

s.t $\Sigma a_k' = \alpha$. $(-\infty \leq \alpha \leq \infty)$

(EX) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$; conv conditionally.

① "+"-Terms : $\qquad$ ⑪ "−"-Terms

$P_k = \frac{1}{2k-1}$. $k \in \mathbb{N}$ $\qquad$ $Q_k = \frac{1}{2k}$. $k \in \mathbb{N}$

$\Sigma P_k = \Sigma Q_k = +\infty$ (div)

$(\lim_{k\to\infty} P_k = \lim_{k\to\infty} Q_k = 0)$

let $\alpha = \frac{3}{2}$, let $m_1 \in \mathbb{N}$: the smallest s.t

$P_1 + \cdots + P_{m_1} \cancel{- Q_1 - \cdots - Q_{m_1}} > \alpha$ $\quad (1 + \frac{1}{3} + \frac{1}{5} > \frac{3}{2}. m_1 = 3)$

& let $n_1 \in \mathbb{N}$: the smallest s.t

$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{n_1} < \alpha$ $\quad (1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} < \frac{3}{2} \Rightarrow n_1 = 1)$

$\vdots$

Choose the smallest m_k, n_k at each step

$\Rightarrow \sum a_k' = \alpha.$

(pf of Thm 1.3.9) "$\sum a_k$".

(Assume) $a_k \neq 0,\ \forall k \geq 0.$ let $p_k = \frac{1}{2}(|a_k| + a_k),\ q_k = \frac{1}{2}(|a_k| - a_k)$

$\Rightarrow p_k - q_k = a_k,\quad p_k + q_k = |a_k|$

If $a_k > 0 \Rightarrow q_k = 0,\ p_k = a_k$

If $a_k < 0 \Rightarrow p_k = 0,\ q_k = |a_k|$

*) $\sum p_k, \sum q_k$: Div to ∞

($\because \sum (p_k + q_k) = \sum |a_k| = +\infty.$

$\Rightarrow \sum p_k$ or $\sum q_k$: Div to ∞.

& $\sum (p_k - q_k) = \sum a_k$: Conv

$\Rightarrow \sum p_k, \sum q_k$: Div.)

let $P_1, P_2, P_3 \cdots$: "+" Terms of $\sum a_k$

($P_j \neq 0,\ j \geq 1$)

$Q_1, Q_2, Q_3, \cdots$; Absolute values of "−" terms of $\sum a_k$.

($Q_j \neq 0,\ j \geq 1$).

$\sum P_k, \sum Q_k$: Div to "∞"

$$\Sigma a_k' = \underbrace{p_1 + p_2 + \cdots + p_{m_1}}_{>\alpha} - \underbrace{Q_1 - \cdots - Q_{n_1}}_{<\alpha}$$

$$+ \underbrace{p_{m_1+1} + p_{m_1+2} + \cdots + p_{m_2}}_{>\alpha} - \underbrace{Q_{n_1+1} - \cdots - Q_{n_2}}_{<\alpha}$$

$$+ \underbrace{p_{m_2+1} + \cdots + p_{m_3}}_{>\alpha} - \underbrace{Q_{n_2+1} - \cdots - Q_{n_3}}_{<\alpha}$$

갈수록 정밀.

$\lim p_k, Q_k \to 0$ as $k \to \infty$.

$\Sigma p_k, \Sigma Q_k$: div 이므로 α는 어떤 수나 가능.

Let m_1 : the smallest s.t $X_1 = p_1 + \cdots + p_{m_1} > \alpha$

(m_1 : Exist ($\because \Sigma p_k = \infty$))

n_1 : the smallest s.t $Y_1 = X_1 - Q_1 - \cdots - Q_{n_1} < \alpha$

(n_1 : Exists ($\because \Sigma Q_k = \infty$))

suppl $\{m_1, \cdots, m_k\}, \{n_1, \cdots, n_k\}$: have been chosen.

let m_{k+1}, n_{k+1} : the smallest s.t

$$\begin{cases} X_{k+1} = Y_k + p_{m_k+1} + \cdots + p_{m_{k+1}} > \alpha \\ X_{k+1} - p_{m_{k+1}} \le \alpha \quad \text{①} \end{cases}$$

$$\begin{cases} Y_{k+1} = X_{k+1} - Q_{n_k+1} - \cdots - Q_{n_{k+1}} < \alpha \\ Y_{k+1} + Q_{n_{k+1}} \ge \alpha \quad \text{Ⅱ} \end{cases}$$

$\Rightarrow \ 0 < X_{k+1} - \alpha \leq P_{m_{k+1}}$ By ①

& $0 < \alpha - Y_{k+1} \leq Q_{n_{k+1}}$ By ⑪

Since Σa_k : Conv. $\lim_{k\to\infty} P_k = \lim_{k\to\infty} Q_k = 0 \Rightarrow \lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k = \alpha$.

Let S_n' : nth partial sum of $\Sigma a_k'$

① If $a_n' = P_\square$ (If the last term of S_n' : P_n

$X_{k+1} = Y_k + P + \cdots + P_n + \cdots$)

$\Rightarrow \exists k \in \mathbb{N}$ s.t $Y_k < S_n' \leq X_{k+1}$

⑪ $a_n' = -Q_\square$

$\Rightarrow \exists k \in \mathbb{N}$ s.t $\underset{\alpha}{\underset{\downarrow}{Y_{k+1}}} \leq S_n < \underset{\alpha}{\underset{\downarrow}{X_{k+1}}}$

By ①.⑪, $\lim_{n\to\infty} S_n' = \Sigma a_k' = \alpha$ ∎

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Rmk) Σa_k : Conditionally Conv. $\forall \alpha, \beta \in \mathbb{R}$, $-\infty \leq \alpha \leq \beta \leq \infty$

$\exists \Sigma a_k'$: Rearrangement of Σa_k s.t $\underline{\lim}_{n\to\infty} S_n = \alpha$, $\overline{\lim}_{n\to\infty} S_n = \beta$

($S_n = \sum_{k=1}^{n} a_k'$)

(pf) $\Sigma a_k' = \overbrace{P_1 + \cdots + P_{m_1}}^{>\beta} \overbrace{- Q_1 - \cdots - Q_{n_1}}^{<\alpha}$

$+ \underbrace{P_{m_1+1} + \cdots + P_{m_2}}_{>\beta} \underbrace{- Q_{n_1+1} - \cdots - Q_{n_2}}_{<\alpha}$

Chapter 8. Seq and Series of functions

8.1 Pointwise conv and Interchange of Limits.

Def 8.1.1) $\{f_n\}$: seq of fts on E. $f_n : E \to \mathbb{R}$ $(n \in \mathbb{N})$

$\{f_n\}$: Conv Pointwise on E (점별수렴)

$\Leftrightarrow$ $\{f_n(x)\}$: Conv $\forall x \in E$.

f : defined by $f(x) = \lim_{n\to\infty} f_n(x)$. $x \in E$

$\Rightarrow$ f: Pointwise limit of $\{f_n\}$

$\Leftrightarrow$ for each $x \in E$. $\forall \varepsilon > 0$. $\exists n_0 = n_0(x, \varepsilon) \in \mathbb{N}$ s.t

$|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq n_0(x, \varepsilon)$.

If $\{f_n\}$: Seq of fts on E.

(i) $x \in E$. $S_n(x) = (f_1 + f_2 + \cdots + f_n)(x)$

$= \sum_{k=1}^{n} f_k(x)$: nth partial sum.

(ii) $\sum_{k=1}^{\infty} f_k = \lim_{n\to\infty} S_n$: Series of fts

(iii) $\sum_{k=1}^{\infty} f_k$ $(= \sum f_k)$: Conv Pointwisely on E

$\Leftrightarrow$ $x \in E$. $\sum_{k=1}^{\infty} f_k(x) = \lim_{n\to\infty} S_n(x)$: Conv.

Suppl $f_n : [a,b] \to \mathbb{R}$. $f_n(x) \to f(x)$ as $n \to \infty$. $x \in [a,b]$

(f: 점별극한.

(a) If f_n: Conti at $p \in E$. $\forall n \geq 1$.

$\Rightarrow$ f: Conti at $p \in E$? No

that is. ($f(x) = \lim_{n\to\infty} f_n(t)$ 일때 $\lim_{t\to p}\left(\lim_{n\to\infty} f_n(t)\right) \overset{?}{=} f(p) \overset{?}{=} \lim_{n\to\infty} \underbrace{f_n(p)}_{\lim_{t\to p} f_n(t)}$

(b) $n \in \mathbb{N}$, f_n : Diff at $p \in [a,b]$.

$\Rightarrow$ f: Diff at $p \in [a,b]$?

If so. $f'(p) = \left(\lim_{n\to\infty} f_n(x)\right)'\Big|_{x=p} \overset{?}{=} \lim_{n\to\infty} f_n'(p)$

(c) $n \in \mathbb{N}$. $f_n \in R[a,b] \Rightarrow f \in R[a,b]$? No

If so, $\int_a^b f\,dx = \int_a^b \lim_{n\to\infty} f_n\,dx \overset{?}{=} \lim_{n\to\infty} \int_a^b f_n\,dx$

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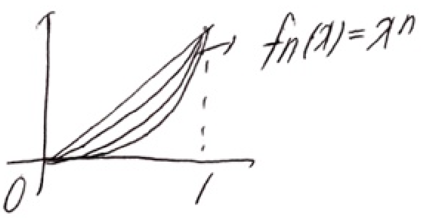
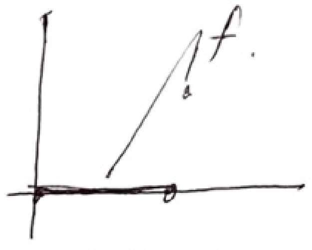
(EX 8.1.2) (a) $E = [0,1]$, $x \in E$.

$f_n(x) = x^n$, $n \in \mathbb{N}$ $\Rightarrow$ $f_n(x)$: Conti on E

However. $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0. & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$



f: Conti on E



f: Not conti on E

$\Rightarrow$ f_n이 연속이라 해도 그에 대한 점별극한이 연속일 필요X.

But. $\underbrace{0 = \int_0^1 f\,dx}_{\displaystyle \| \atop \displaystyle \int_0^1 \lim_{n\to\infty} f_n(x)\,dx} \neq \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n}{n+1}\right) = \frac{1}{2}$.

$*$ put $1-x^2 = t$. $\Rightarrow$ $-2x\,dx = dt$

$x\,dx = -\frac{1}{2}dt$

$\int_0^1 nx(1-x^2)^n$

$= \int_1^0 -\frac{n}{2} t^n dt = \int_0^1 \frac{n}{2} t^n dt = \frac{1}{2}\left(\frac{n}{n+1}\right)$

EX-(e) $f_n(x) = \dfrac{\sin nx}{n}$. $x \in \mathbb{R}\ (=E)$

$|\sin nx| \le 1$ $\forall n. \forall x$. $f(x) = \lim_{n\to\infty} f_n(x) = 0$ $\forall x \in \mathbb{R}$.

f_n, f: Diff in $\mathbb{R}$. But $\underbrace{f'(0)}_{\displaystyle \| \atop \displaystyle \left(\lim_{n\to\infty} f_n(x)\right)'\Big|_{x=0}} = 0 \neq \lim_{n\to\infty} f_n'(0) = \cos 0 = 1$.

Exercise 8.1 #1-(a) $\left\{\dfrac{nx}{1+nx}\right\}$. $x \in [0,\infty)$

$x=0$ 일때 0, $x>0$ 일때 1. 점별극한: $0, 1$

-(b) $\left\{\dfrac{\sin nx}{1+nx}\right\}$

$x=0$ 일때 0. & since $|\sin nx| \le 1$. $\lim_{n\to\infty} \dfrac{\sin nx}{1+nx} = 0$

-(c) $\{(\cos x)^{2n}\}$. $x \in \mathbb{R}$

$\lim_{n\to\infty} (\cos x)^{2n} = \begin{cases} 0, & \text{o.w} \\ 1, & x = k\pi + \frac{1}{2}\pi,\ k \in \mathbb{Z} \end{cases}$

(EX) -(b) $\{f_k\}$. $f_k(x) = \dfrac{x^2}{(1+x^2)^k}$. $x \in \mathbb{R}$. $\rightarrow$ $\dfrac{x^2}{1-\frac{1}{1+x^2}} = \dfrac{x^2(1+x^2)}{\cancel{1+x^2-1}}$

let $f(x) = \lim_{n\to\infty} S_n(x)$ (S_n : conti on $\mathbb{R}$)

$$= \sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} \frac{x^2}{(1+x^2)^k} = \begin{cases} 0, & x=0 \\ 1+x^2, & x \neq 0. \end{cases}$$

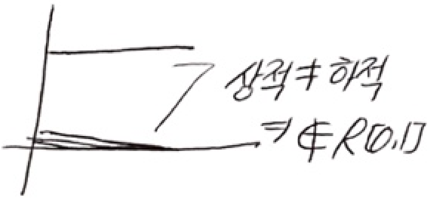
: Not conti on $\mathbb{R}$.

-(c) $E = [0,1]$ $\{x_k\}$: 유리수열 in E

Define) $f_n : [0,1] \to \mathbb{R}$. $f_n(x) = \begin{cases} 0 & \text{If } x = x_k,\ 1 \le k \le n \\ 1, & \text{otherwise.} \end{cases}$

$\in R[0,1]$ (유한개 불연속점 가지고 있으므로).

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0, & x \in [0,1] \cap \mathbb{Q} \\ 1, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

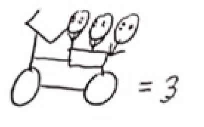


$\notin R[0,1]$

-(d) $x \in [0,1]$, $n \in \mathbb{N}$

$f_n(x) = nx(1-x^2)^n$: conti on $[0,1]$ $\Rightarrow$ $f_n \in R[0,1]$

let $f(x) = \lim_{n\to\infty} f_n(x) = ?$



<점별극한 구하는 법> ⓘ $f_n(0) = f_n(1) = 0$ $\forall n \ge 1$

ⓘⓘ $0 < x < 1 \Rightarrow 0 < 1-x^2 < 1$, $\lim_{n\to\infty} nx(1-x^2)^n = 0$

ⓘ, ⓘⓘ $\Rightarrow$ $f(x) = \lim_{n\to\infty} f_n(x) = 0$. $x \in [0,1]$

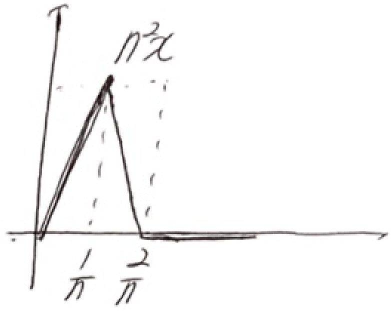
$\Rightarrow$ $f \in R[0,1]$.

1-1(d) $\{nxe^{-nx^2}\}$, $x \in \mathbb{R}$

(sol) By L'Hospital's Law. $\lim\limits_{n\to\infty} \dfrac{nx}{e^{nx^2}} = \lim\limits_{n\to\infty} \dfrac{x}{2nx\, e^{nx^2}}$

$x \neq 0$, $\lim\limits_{n\to\infty} nxe^{-nx^2} = 0$

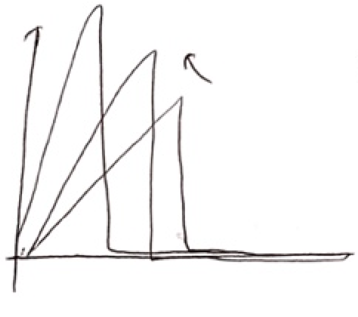
3 $f_n(x) = \begin{cases} n^2x, & 0 \le x \le \frac{1}{n} \\ 2n - n^2x, & \frac{1}{n} < x \le \frac{2}{n} \\ 0, & \frac{2}{n} < x \le 1 \end{cases}$



① $f_n(0) = 0 \quad \forall n \ge 1$

② $f(x) = 0$, $\frac{2}{n} < x \le 1 \Rightarrow \lim\limits_{n\to\infty} f_n(x) = 0$, $x \in (0,1]$

①, ② $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0$



$f_n, f \in R[0,1]$

$\int_0^1 f(x)dx = 0 \neq \lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 1$

8.2 Uniform Convergence

*) $f_n : E \to \mathbb{R}$ $(n \ge 1)$, f_n: Pointwise conv to f (점별수렴)

$\Leftrightarrow$ $x \in E$, $\forall \varepsilon > 0$, $\exists n_0 = n_0(x, \varepsilon)$ s.t $|f(x) - f_n(x)| < \varepsilon$, $\forall n \ge n_0$

If $n_0 = n_0(\varepsilon)$ $\Rightarrow$ 균등.

Def 8.2.1) $f_n : E \to \mathbb{R}$ $(n \geq 1)$ $\{f_n\}$: Conv unifly to f (균등수렴)

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \varepsilon$.

$\forall n \geq n_0$, $\forall x \in E$

* $\sum_{k=1}^{\infty} f_k$: Conv unifly on E

$\Leftrightarrow$ $\{S_n\}$: Conv unifly on E $\left(S_n = \sum_{k=1}^{n} f_k \right)$

(Ex 8.2.2) ⓐ $x \in [0,1]$, $n \in \mathbb{N}$, $f_n(x) = x^n$

$$f_n(x) \to f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1. \end{cases}$$

: 점별수렴 but not unifly.

($\because$ Assume) $f_n \to f$: Conv unifly.

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \varepsilon$. $\forall n \geq n_0$, $\forall x \in [0,1]$

But $x = 1 - \frac{1}{n}$ $(\in [0,1])$

$\Rightarrow \left| f_n(1-\frac{1}{n}) - f(1-\frac{1}{n}) \right| = (1-\frac{1}{n})^n \to \frac{1}{e} (\neq 0)$ $\forall n \geq N > \frac{1}{2e}$)

Rmk) $f_n(x) = x^n$: $[0,a] \to \mathbb{R}$ $(0 < a < 1)$

$f_n(x) \to f(x) = 0$, $n \to \infty$

$|f_n(x) - \underset{0}{\underset{\shortparallel}{f(x)}}| = |f_n(0)| = x^n \leq a^n \to 0$ as $n \to \infty$ $\forall x \in [0,a]$

$\to$ x에 선택에 의존 X.

$\therefore$ $f_n \to f$: Conv unifly on $[0,a]$.

ⓑ $S(x) = \sum_{k=1}^{\infty} [kxe^{-kx^2} - (k-1)xe^{-(k-1)x^2}]$ $(0 \le x < \infty)$

$S_n(x) = \sum_{k=1}^{n} [kxe^{-kx^2} - (k-1)xe^{-(k-1)x^2}] = nxe^{-nx^2}$. $x \in [0, \infty)$

$S_n(x) \to S(x) = 0$ as $n \to \infty$: 점별수렴.

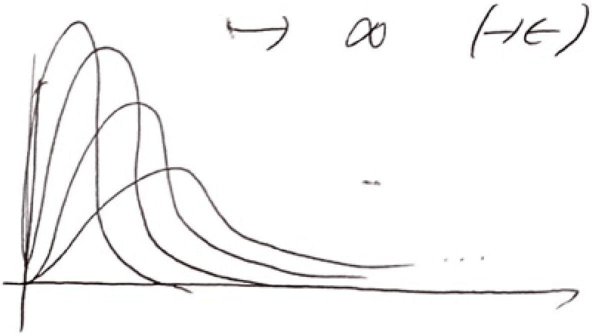
$\{S_n\}$: Not conv unifly to S.

($\because$ Assume) $\{S_n\}$: Conv unifly to S.

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t $|S_n(x) - S(x)| < \varepsilon$.
$(= n_0(\varepsilon))$ $\forall n \ge n_0$ $\forall x \in [0, \infty)$.

But $\max_{x \in [0,\infty)} S_n(x) = \sqrt{\frac{n}{2e}}$. $x = \sqrt{\frac{1}{2n}} \in [0, \infty)$ $\forall n \ge 1$.

$\to \infty$ $(\to\leftarrow)$



< The Cauchy criterion >

Thm 8.2.3) Cauchy criterion.

$\{f_n\}$: Conv unifly on E $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t
$|f_n(x) - f_m(x)| < \varepsilon$. $\forall n, m \ge n_0$. $\forall x \in E$

(pf) If $\{f_n\}$: Conv unifly on E

$\Rightarrow$ $\forall x \in E$. $\{f_n(x)\}$: Conv $\Rightarrow$ $\{f_n(x)\}$: Cauchy seq $\forall x \in E$.

$\Rightarrow |f_n(x)-f_m(x)| < \varepsilon \quad \forall n,m \ge n_0$: O.K.

$\Leftarrow$) If $|f_n(x)-f_m(x)| < \varepsilon$. $\forall n,m \ge n_0$: O.K $\Rightarrow \forall x \in E$, $\{f_n(x)\}$: Cauchy seq.

$\Rightarrow f(x) = \lim_{n\to\infty} f_n(x)$: Exists. $x \in E$.

$\Rightarrow f$: 점별극한.

By $|f_n(x)-f_m(x)| < \varepsilon$. $\forall n,m \ge n_0$,

$\forall \varepsilon > 0$. $\exists n_0 \in \mathbb{N}$ s.t $m \ge n_0$: Fix.

$\Rightarrow |f(x)-f_m(x)| = \lim_{n\to\infty} |f_n(x)-f_m(x)| \le \varepsilon \quad \forall x \in E$.

$\therefore$ $\{f_n\}$: Conv unifly to f on E. ■

Corol 8.2.4) $\sum_{k=1}^{\infty} f_k$: Conv unifly on E.

$\Leftrightarrow \forall \varepsilon > 0$. $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t $\overset{|S_m(x)-S_n(x)|}{\overset{\|}{\left|\sum_{k=n+1}^{m} f_k(x)\right|}} < \varepsilon$. $\forall m > n \ge n_0$, $\forall x \in E$

(pf) Σf_k : Conv unifly on E

$\Leftrightarrow S_n = \sum_{k=1}^{n} f_k$: Conv unifly on E.

$\Leftrightarrow \{S_n\}$: Satisfies the cauchy-criterion.

$\Leftrightarrow |S_m(x)-S_n(x)| = \left|\sum_{k=n+1}^{m} f_k(x)\right| < \varepsilon \quad \forall m > n \ge n_0$. $\forall x \in E$

$$\left.\begin{array}{l} |f_k(x)| \le M_k \\ \Sigma M_k < \infty \end{array}\right) \text{ M-Test.}$$

Thm 8.2.5) $f_n : E \to \mathbb{R}$. $f_n \to f$: 점별수렴

set $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ ☆

$\{f_n\}$: Conv unifly to f on E $\Leftrightarrow$ $M_n \to 0$ as $n \to \infty$.

(pf) If $f_n \to f$: Conv unifly on E.

$\Rightarrow \forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t $|f_n(x) - f(x)| < \varepsilon$ $\forall n \ge n_0,\ \forall x \in E$.

$$\Rightarrow M_n = \sup_{x \in E} |f_n(x) - f(x)| \le \varepsilon \quad \forall n \ge n_0.$$

$\therefore M_n \to 0$ as $n \to \infty$.

If $M_n \to 0$ as $n \to \infty$, $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t

$|f_n(x) - f(x)| \le M_n < \varepsilon$ $\forall n \ge n_0$, $\underline{\forall x \in E}$. ☆

$\therefore \{f_n\}$: Conv unifly to f on E. ■

Ex 8.2.6) (a) $S_n(x) = nxe^{-nx^2}$, $n = 1, 2, \cdots$ $x \in [0, \infty)$

$S_n(x) \to S(x) = 0$ as $n \to \infty$

But $M_n = \sup_{x \in [0,\infty)} S_n(x) = \sqrt{\frac{n}{2e}} \to \infty$ as $n \to \infty$.

$\therefore \{S_n\}$: 균등수렴 X.

$\boxed{\begin{array}{l} \text{※ ii) } f_n(x) = x^n \text{ on } [0,a]. \ (0<a<1) \\ \lim_{n\to\infty} f_n(x) = f(x) = 0 \\ M_n = \sup_{x\in[0,a]} |f_n(x) - f(x)| = 0 \text{ as } n \to \infty \end{array}}$

(b) $f_n(x) = x^n$ on $[0,1]$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases} \Rightarrow M_n = \sup |f_n(x) - f(x)| = 1 \not\to 0 \text{ as } n \to \infty.$$

$\therefore \{f_n\}$: 균등수렴 X.

p327 <The Weierstrass M-Test>

Thm 8.2.1) $f_k : E \to \mathbb{R}$. $\{M_k\} \subset [0,\infty)$ s.t $|f_k(x)| \le M_k$. $\forall x \in E$

If $\sum_{k=1}^{\infty} M_k < \infty \Rightarrow \sum_{k=1}^{\infty} f_k(x)$: Conv unifly on E.

(pf) since $\sum_{k=1}^{\infty} M_k < \infty$

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t $\sum_{k=n+1}^{m} M_k < \varepsilon \quad \forall m > n \ge n_0$

$$\Rightarrow \left| \sum_{k=n+1}^{m} f_k(x) \right| \le \sum_{k=n+1}^{m} |f_k(x)|$$

$$\le \sum_{k=n+1}^{m} M_k < \varepsilon \quad \forall m > n \ge n_0, \ \forall x \in E$$

$\Rightarrow$ By Cauchy criterion, $\sum f_k(x)$: Conv unifly on E. ∎

(EX 8.3.8) ⓐ $p > 1$. $\forall x \in \mathbb{R}$ $\left| \frac{\cos kx}{k^p} \right| \le \frac{1}{k^p}$ & $\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty$

$$\left| \frac{\sin kx}{k^p} \right| \le \frac{1}{k^p}$$

By the W-M-Test. $\sum_{k=1}^{\infty} \frac{\cos kx}{k^p}$, $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$: Conv unifly on $\mathbb{R}$

ⓑ ⓘ $\sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^k$: Conv unifly on $[-a, a]$ $(0 < a < 2)$

$\left(\because \left| \left(\frac{x}{2}\right)^k \right| \le \left(\frac{a}{2}\right)^k, \ \forall x \in [-a, a] \right.$

$$\& \sum_{k=1}^{\infty} \left(\frac{a}{2}\right)^k = \frac{\frac{a}{2}}{1 - \frac{a}{2}} = \frac{a}{2-a} < \infty.$$

ⅰ (W-M-Test).

$\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^k$: Conv unifly on $[a,d]$ (뒷장에) →

8-2 Exercise

※4. Let $f_n(x)=nx(1-x^2)^n$, $0\le x\le 1$. Show that $\{f_n\}$: does not conv unifly to 0 on $[0,1]$.

(pf) If $x=\sqrt{\frac{1}{2n+1}}$. $f_n'(x)=0 \Rightarrow \max_{x\in[0,1]} f_n(x)=f_n\left(\sqrt{\frac{1}{2n+1}}\right)$

$\Rightarrow f_n\left(\frac{1}{\sqrt{2n+1}}\right)=\frac{n}{\sqrt{2n+1}}\left(\frac{2n}{2n+1}\right)^n.$

$\Rightarrow \lim_{n\to\infty}\left(\frac{2n}{2n+1}\right)^n<\infty,\ \lim_{n\to\infty}\frac{n}{\sqrt{2n+1}}=\infty.$

Supp/ $\{f_n\}$: Conv unifly to 0.

Then $\exists n_0\in\mathbb{N}$ s.t $\forall n\ge n_0$. $|f_n(x)-0|<1$

However, since $\lim_{n\to\infty}\frac{n}{\sqrt{2n+1}}\left(\frac{2n}{2n+1}\right)^n=\infty$. $\exists n_1\in\mathbb{N}$ s.t

$\frac{n}{\sqrt{2n+1}}\left(\frac{2n}{2n+1}\right)^n\ge 1 \quad \forall n\ge n_1$. $(\rightarrow\leftarrow)$

$\Rightarrow \{f_n\}$: 균등연속 X on $[0,1]$. ■

→ ⓘ $\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^k$: Conv unifly on $[-a,a]$ $(0<a<2)$

$\left|\left(\frac{x}{2}\right)^k\right| \le \left(\frac{a}{2}\right)^k$ & $\sum_{k=1}^{\infty}\left(\frac{a}{2}\right)^k < \infty$ (W-M-Test)

ⓘⓘ $\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^k$: Not conv unifly on $(-2,2)$

For $x\in(-2,2)$, $\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^k = \dfrac{\frac{x}{2}}{1-\frac{x}{2}} = \dfrac{x}{2-x}$.

$f_n(x) = \sum_{k=1}^{n}\left(\frac{x}{2}\right)^k = \dfrac{\frac{x}{2}\left(1-\left(\frac{x}{2}\right)^n\right)}{1-\frac{x}{2}} = \dfrac{x}{2-x}\left(1-\left(\frac{x}{2}\right)^n\right)$

$|f_n(x)-f(x)| = \left|\left(\frac{x}{2}\right)^n\frac{x}{2-x}\right|$.

idea) $x = 2-\frac{1}{n}\in(-2,2)$.

$\left|f_n\left(2-\frac{1}{n}\right)-f\left(2-\frac{1}{n}\right)\right| = \left(1-\frac{1}{2n}\right)^n n\left(2-\frac{1}{n}\right) \to \infty$ as $n\to\infty$.

$\therefore \sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^k$: Not conv unifly on $(-2,2)$.

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(EX 8.2.9) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. $0\le x\le 1$.

1) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$: Conv unifly on $[0,1]$

let $S_n(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\frac{x^n}{n}$.

$S_{2k}(x) = \underbrace{\left(x-\frac{x^2}{2}\right)}_{\ge 0}+\underbrace{\left(\frac{x^3}{3}-\frac{x^4}{4}\right)}_{\ge 0}+\cdots+\underbrace{\left(\frac{x^{2k-1}}{2k-1}-\frac{x^{2k}}{2k}\right)}_{\ge 0}$

$\Rightarrow \{S_{2k}(x)\}$: M-I. $x\in[0,1]$.

$$S_{2k+1}(x) = x - \underbrace{\left(\frac{x^2}{2} - \frac{x^3}{3}\right)}_{\geq 0} - \underbrace{\left(\frac{x^4}{4} - \frac{x^5}{5}\right)}_{\geq 0} - \cdots - \left(\frac{x^{2k}}{2k} - \frac{x^{2k+1}}{2k+1}\right)$$

$\Rightarrow \{S_{2k+1}(x)\}$: M-D (교대급수 판정)

$$S_n(x) \to S(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot x^k \quad \text{(점별극한)}.$$

$S_{2k}(x) \leq S(x) \leq S_{2k+1}(x)$, $x \in [0,1]$

$$\Rightarrow \forall n \geq 1, \ |S_n(x) - S(x)| \leq |S_{n+1}(x) - S_n(x)|$$

$$= \left| \frac{(-1)^n \cdot x^{n+1}}{n+1} \right| \leq \frac{1}{n+1} \to 0 \text{ as } n \to \infty$$

$\forall x \in [0,1]^{*}$

$\therefore \{S_n(x)\}$: Conv unifly on $[0,1]$

$\Leftrightarrow \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{x^k}{k}$: Conv Unifly on $[0,1]$.

※ 1. M-Test는 충분조건이지 필요조건이 아니다.

2. 균등수렴한다고 절대수렴하는 것은 아니다.

(Exercise ~~8~~ 8.2)

※ 2-(b) $\{f_n\}$: Conv unifly on E, $\{g_n\}$: Conv unifly on E,

$|f_n(x)| \leq M$. $|g_n(x)| \leq N$ $\{f_n g_n\}$: 균등수렴.

(pf))

(pf) $\forall \varepsilon > 0$. $\exists n_0 \in \mathbb{N}$ s.t $|f_m(x) - f_n(x)| < \varepsilon$.

& $|g_m(x) - g_n(x)| < \varepsilon$ $\forall m, n \geq n_0$.

$\therefore$

$|f_n(x)g_n(x) - f_m(x)g_m(x)|$

$= |f_n(x)g_n(x) - f_n(x) \cdot g_m(x) + f_n(x)g_m(x) - f_m(x)g_m(x)|$

$\leq |f_n(x)||g_n(x) - g_m(x)| + |g_m(x)||f_n(x) - f_m(x)|$

$\leq (M+N)(|g_n(x) - g_m(x)| + |f_n(x) - f_m(x)|)$

$< (M+N) \cdot 2\varepsilon' = \varepsilon$ ■

2-(c) f, g : conv unifly but fg : Not conv unifly.

$f_n(x) = x^2 + \frac{1}{n}$. $E = \mathbb{R}$.

$g_n(x) = \frac{1}{n} + \sin x$.

$\Rightarrow f_n(x)g_n(x) = \frac{1}{n^2} + \frac{1}{n}(x^2 + \sin x) + x^2 \sin x$

if $x = n \to \infty$ ■

8-(a) $\sum_{k=1}^{\infty} \frac{1}{k^2 + x^2}$, $0 \leq x < \infty$

(sol) For all x. $\frac{1}{k^2 + x^2} \leq \frac{1}{k^2}$ & since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$.

(~~X~~-W-M-Test)

$\sum_{k=1}^{\infty} \frac{1}{k^2 + x^2}$: conv unifly.

-(b) $\sum_{k=1}^{\infty} e^{-kx} x^k$. $0 \leq x < \infty$.

8-(b) $\sum_{k=1}^{\infty} e^{-k\lambda}\lambda^k \quad 0 \le \lambda < \infty$

(sol) since $e^\lambda > \lambda$. $0 < \frac{\lambda}{e^\lambda} < 1 \Rightarrow \sum_{k=1}^{n} e^{-k\lambda}\lambda^k = \frac{1-\left(\frac{\lambda}{e^\lambda}\right)^n}{1-\frac{\lambda}{e^\lambda}}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} e^{-k\lambda}\lambda^k = \frac{1}{1-\frac{\lambda}{e^\lambda}}.$$

$$\left| \cancel{\lim_{n\to\infty}} \sum_{k=1}^{n} e^{-k\lambda}\lambda^k - \frac{1}{1-\frac{\lambda}{e^\lambda}} \right| = \left| \frac{\left(\frac{\lambda}{e^\lambda}\right)^n}{1-\frac{\lambda}{e^\lambda}} \right| \le \left| \left(\frac{\lambda}{e^\lambda}\right)^n \right|$$

$$(\lambda e^{-\lambda})' = e^{-\lambda} + \lambda(-1)e^{-\lambda} = e^{-\lambda} - \lambda e^{-\lambda}$$
$$= e^{-\lambda}(1-\lambda) \quad \lambda > 1.$$
$$< 0$$

For $\lambda = 1$. $\frac{\lambda}{e^\lambda} \Rightarrow \frac{1}{e} < 1$ (max)

let $\varepsilon > 0$. (By A-P). $\exists n_0 \in \mathbb{N}$ s.t $\log \varepsilon < n_0$.

$$\left| \sum_{k=1}^{n} e^{-k\lambda}\lambda^k - \frac{1}{1-\frac{\lambda}{e^k}} \right| \le \left| \left(\frac{\lambda}{e^\lambda}\right)^n \right| \le \left| \left(\frac{1}{e}\right)^n \right| < \varepsilon \quad \begin{array}{l} \forall n \ge n_0, \\ \forall \lambda \in [0, \infty) \end{array}$$

$\Rightarrow \sum_{k=1}^{n} e^{-k\lambda}\lambda^k$: Conv unifly to $\frac{1}{1-\frac{\lambda}{e^\lambda}}$. ∎

8-1(c) $\sum_{k=1}^{\infty} k^2 e^{-kx}$. $1 \le x < \infty$.

(sol) since $1 \le x < \infty$, $k^2 e^{-kx} \le k^2 e^{-k}$.

& $\sqrt[k]{k^2 e^{-k}} = (k^{\frac{1}{k}})^2 \frac{1}{e} \Rightarrow \overline{\lim_{k \to \infty}} \sqrt[k]{k^2 e^{-k}} = \frac{1}{e} < 1$.

By root test, $\sum_{k=1}^{\infty} k^2 e^{-k} < \infty$.

(W-M-Test).

$\sum_{k=1}^{\infty} k^2 e^{-kx}$: conv unifly. ∎

8-1(d) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}$ $0 \le x < \infty$.

(sol) 교대급수에 의해, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}$: conv $\forall x \in [0, \infty)$

let $S(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}$. if $S_n(x)$: n-th partial sum.

$|S(x) - S_n(x)| \le \frac{1}{n+x} \le \frac{1}{n}$ $\forall x \in [0, \infty)$

let $\epsilon > 0$. (By A-P), $\exists n_0 \in \mathbb{N}$ s.t $n_0 \epsilon > 1$.

$\Rightarrow |S(x) - S_n(x)| \le \frac{1}{n} < \epsilon$ $\forall n \ge n_0$. $\forall x \in [0, \infty)$

$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}$: conv unifly. ∎

9-(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin 2kx}{(2k+1)^{\frac{3}{2}}}$. $x \in \mathbb{R}$

(sol) $\displaystyle\left|\frac{\sin 2kx}{(2k+1)^{3/2}}\right| \le \frac{1}{(2k+1)^{3/2}} \le c \cdot \frac{1}{k^{\frac{3}{2}}}$ for some $c>0$.

Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$. (W-M-T).

$\displaystyle\sum_{k=1}^{\infty} \frac{\sin 2kx}{(2k+1)^{3/2}}$: Conv unifly. ∎

9-(b) $\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{k(\ln k)^2}$. $|x| \le 1$

(sol) $\displaystyle\left|\frac{x^k}{k(\ln k)^2}\right| \le \frac{1}{k(\ln k)^2}$, and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} < \infty$.

(By W-M-T). $\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{k(\ln k)^2}$: Conv unifly $\forall |x| \le 1$ ∎

9-(c) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2}\right)$. $0 \le x \le 1$.

(pf) $\displaystyle\sum_{k=0}^{n} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2}\right) = \frac{1}{2} - \frac{1}{(n+1)x+2}$

if $x=0$. $\hookrightarrow 0$

if $0<x\le 1$. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2}\right) = \frac{1}{2}$. $\lim \sqrt[k]{k} = 1$.

let $f(x) = \begin{cases} 0, & x=0 \\ \frac{1}{2}, & 0<x\le 1 \end{cases}$

Supp/ $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2}\right)$: conv unifly to $f(x)$. $\forall n \ge n_0, \forall x \in [0,1]$

Then $\exists n_0 \in \mathbb{N}$ s.t $\displaystyle |f_n(x) - f(x)| = \left|\frac{1}{2} - \frac{1}{(n+1)x+2} - f(x)\right| < \frac{1}{3}$

9-(d) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot x^{2k+1}}{2k+1}$, $|x| \le 1$.

(sol) let $\varepsilon > 0$. (교대급수 판정) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot x^{2k+1}}{2k+1}$. conv $\forall x \in [-1, 1]$

let $S(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot x^{2k+1}}{2k+1}$. $S_n(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1} \cdot x^{2k+1}}{2k+1}$.

By thm 7.2.4. $|S(x) - S_n(x)| \le \frac{|x|^{2n+3}}{2n+3} \le \frac{1}{2n+3} < \frac{1}{2n}$.

(A-P). $\exists n_0 \in \mathbb{N}$ s.t $\frac{1}{2} < n_0 \varepsilon$.

$\Rightarrow |S(x) - S_n(x)| < \frac{1}{2n} < \varepsilon$.

$\Rightarrow \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot x^{2k+1}}{2k+1}$: conv unifly on $[-1, 1]$. ■

9-(e) $\sum_{k=1}^{\infty} \sin\left(\frac{x}{k^p}\right)$. $p > 1$. $|x| \le 2$

(sol) $\left| \sin\left(\frac{x}{k^p}\right) \right| \le \left| \frac{x}{k^p} \right| \le \frac{2}{|k^p|}$

since $p > 1$, $\sum_{k=1}^{\infty} \frac{2}{|k^p|} < \infty$.

(W-M-Test).

$\sum_{k=1}^{\infty} \sin\left(\frac{x}{k^p}\right)$: conv unifly on $[-2, 2]$. ■

※10-(a) $\sum_{k=0}^{\infty} \frac{1}{1+k^2x}$. $x \in (a, \infty)$

(sol) For $x \in [a, \infty)$, $\left|\frac{1}{1+k^2x}\right| \le \frac{1}{k^2x} \le \frac{1}{k^2a}$.

since $\sum_{k=1}^{\infty} \frac{1}{k^2} \left(=\frac{\pi^2}{6}\right) < \infty$. (W-M-T).

$$\sum_{k=0}^{\infty} \frac{1}{1+k^2x} \text{ : Conv unifly on } [a,\infty).$$

Consider $S_n(x) = \sum_{k=0}^{n} \frac{1}{1+k^2x} \Rightarrow S_n(x) - S_{n-1}(x) = \frac{1}{1+n^2x}$.

If S_n : Conv $\forall \varepsilon$: ~~Arbitrary~~, >0 $\exists n_0 \in \mathbb{N}$

s.t $\frac{1}{1+n^2x} < \varepsilon$ $\forall n \ge n_0$.

But. (By A-P). $\exists n_1 \in \mathbb{N}$ s.t $n_1 > \frac{\varepsilon n^2}{1-\varepsilon}$. $\varepsilon < 1$.

Take $0 < x < \frac{1}{n_1}$. $S_n - S_{n-1} \not\to 0$ as $n \to \infty$.

∴ 균등연속 X on $(0, \infty)$ ∎

※10-(b) $\sum_{k=1}^{\infty} \frac{1}{k^{1+x}}$.

(sol) $\sum_{k=n+1}^{m(=2n)} \frac{1}{k^{1+x}}$, $x = \frac{1}{n} \in (0, \infty)$.

$$> \underbrace{\frac{1}{(2n)^{1+\frac{1}{n}}} + \cdots + \frac{1}{(2n)^{1+\frac{1}{n}}}}_{n\text{개}} = \frac{\cancel{2n}}{2\cancel{n}} \frac{1}{(2n)^{\frac{1}{n}}} \approx 1 > \frac{1}{4} \quad n \ge N.$$

Cauchy criterion 만족X

⇒ 균등수렴 X on $(0, \infty)$. ∎

15. $S_n(x) = \sum_{k=1}^{n} g_k(x)$ $x \in E$; $|S_n(x)| \le M$ $\forall x \in E$.

ⓑ $f_k(x) \ge f_{k+1}(x) \ge \cdots \ge 0$

ⓒ $f_n(x) \to 0$: Unifly on E. $\Rightarrow \sum_{k=1}^{\infty} f_k(x) g_k(x)$: Conv unifly.

(pf) $\left| \sum_{k=n+1}^{m} f_k(x) g_k(x) \right| = \left| \sum_{k=n+1}^{m} (S_k(x) - S_{k-1}(x)) f_k(x) \right|$

$$= \left| \sum_{k=n+1}^{m} S_k(x) f_k(x) - \sum_{k=n}^{m-1} S_k(x) f_{k+1}(x) \right|$$

$$= \left| \sum_{k=n+1}^{m-1} S_k(x)(f_k(x) - f_{k+1}(x)) - S_n(x) f_{n+1}(x) + S_m(x) f_m(x) \right|$$

$$\le M \left[\sum_{k=n+1}^{m-1} (f_k(x) - f_{k+1}(x)) + f_{n+1}(x) + f_m(x) \right]$$

$= 2M f_{n+1}(x) \longrightarrow 0$ as $n \to \infty$ $\forall x \in E$.

(By Cauchy criterion) Conv unifly ∎

16. $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$ $(p > 0)$ ($p > 1$ 일때는 M-Test 되지만 $0 < p < 1$ 이면 불가)

ⓘ $x = 2p\pi$ $(p \in \mathbb{Z}) \Rightarrow$ 균등수렴.

ⓘⓘ $x \ne 2p\pi$ $\left| \sum_{k=1}^{n} \sin kx \right| \le M$ $\forall n \ge 1$ (Bdd)

ⓘⓘⓘ $f_k(x) = \frac{1}{k^p}$: Monotone & $\to 0$ as $k \to \infty$.

∴ 균등수렴 $\forall x \in \mathbb{R}$. ∎