34. Some identities of "log"

Thm) z_1, z_2 : Non zero Complex num

① $\log(z_1 z_2) = \log z_1 + \log z_2$

② $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

③ $\log\left(\frac{1}{z_1}\right) = -\log z_1$

(pf) ① $\log(z_1 z_2) = \log(|z_1 z_2| \cdot e^{i\arg(z_1 z_2)})$ 크기 $\times e^{i\arg(\ )}$

$= \log(|z_1||z_2| e^{i\arg(z_1 z_2)})$

$= \ln(|z_1||z_2|) + i\arg(z_1 z_2)$

두 복소수 곱의 arg는 합.

$(z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2} \Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)})$

$= \ln|z_1| + \ln|z_2| + i\arg(z_1) + i\arg(z_2)$

$= \ln|z_1| + i\arg(z_1) + \ln|z_2| + i\arg(z_2)$

$= \log z_1 + \log z_2$ //

② $\log\left(\frac{z_1}{z_2}\right) = \log\left(\left|\frac{z_1}{z_2}\right| e^{i\arg\left(\frac{z_1}{z_2}\right)}\right)$

$= \ln\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right)$

$= \ln|z_1| - \ln|z_2| + i\arg(z_1) - i\arg(z_2)$

$= \log z_1 - (\ln|z_2| + i\arg(z_2))$

$= \log z_1 - \log z_2$ //

③ $\log\left(\frac{1}{z_1}\right) = \log\left(\left|\frac{1}{z_1}\right| e^{i\arg\left(\frac{1}{z_1}\right)}\right)$

$= \ln\frac{1}{|z_1|} + i\arg\left(\frac{1}{z_1}\right)$ $\quad \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

$= -\ln|z_1| - i\arg(z_1)$

$= -\log z_1$

■

(Ex) $z_1 = z_2 = -1$

① $\log((-1)\cdot(-1)) = \log 1$

$= \ln 1 + i\arg(1)$

$= 0 + i(\overset{\text{주편각 } 0}{2n\pi}) = 2n\pi \quad (n = 0, \pm1, \pm2, \cdots)$

② $\log(-1) = \ln|-1| + i \arg(-1)$

$= 0 + i(\pi + 2n_1\pi)$

$= \pi i + 2n_1\pi i \quad (n_1 = 0, \pm 1, \pm 2, \cdots)$

$\Rightarrow \quad \log(-1) + \log(-1)$

$= \pi i + 2n_1\pi i + \pi i + 2n_2\pi i$

$= 2\pi i + 2(n_1+n_2)\pi i$

$= 2(\overbrace{1+n_1+n_2}^{n})\pi i \quad (n_1, n_2 \in \mathbb{Z})$

$= 2k\pi i \quad (k = 0, \pm 1, \pm 2, \cdots)$

$\therefore \quad \log((-1)\cdot(-1)) = \log(-1) + \log(-1)$ ∎

(Ex) z_1, z_2 : Nonzero Complex num.

$\mathrm{Log}(z_1 z_2) = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 \quad$ iff $\; -\pi < \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \le \pi$.

$(\because \quad \mathrm{Log}(z_1 z_2) = \mathrm{Log}(|z_1 z_2| e^{i\,\mathrm{Arg}(z_1 z_2)})$

$= \ln|z_1 z_2| + i\,\mathrm{Arg}(z_1 z_2)$

$= \ln|z_1| + \ln|z_2| + i\,\mathrm{Arg}(z_1) + i\,\mathrm{Arg}(z_2) \quad$ (iff $\; -\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \le \pi$)

$= \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$

(Ex) $z_1 = z_2 = -1$;

① $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\, 1 = \ln 1 + i\,\mathrm{Arg}(1)$

$= 0 + i\,0 = 0$

② $\mathrm{Log}(z_1) = \mathrm{Log}(-1)$

$= \ln|-1| + i\,\mathrm{Arg}(-1)$

$= 0 + \pi i = \pi i$

$\Rightarrow \quad \mathrm{Log}(z_1 z_2) \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$

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Thm) $z \neq 0$

① $z^n = e^{n \log z}$ $(n = 0, \pm 1, \pm 2, \cdots)$

② $z^{1/n} = e^{\frac{1}{n}\log z}$ $(n = 1, 2, \cdots)$

(pf) ① $z = re^{i\theta}$ $(r \neq 0)$

$z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$= e^{n\ln r} \cdot e^{in\theta}$

$= e^{n\underbrace{(\ln r + i\theta)}_{\log z}} = e^{n\log z} = z^n$ □

② $z^{1/n} = (re^{i\theta})^{1/n}$

$= r^{1/n} e^{i\frac{1}{n}\theta} = e^{\frac{1}{n}\ln r} \cdot e^{\frac{1}{n}i\theta}$

$= e^{\frac{1}{n}(\ln r + i\theta)} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + i\arg z)}$

$= e^{\frac{1}{n}(\ln|z| + i(\Theta + 2k\pi))}$ $(k = 0, \pm 1, \pm 2, \cdots)$

$= \{ |z|^{\frac{1}{n}} e^{\frac{i(\Theta + 2k\pi)}{n}} \mid k = 0, 1, \cdots, n-1 \}$

(Exercise 99) #4, #5

EXERCISES

1. Show that for any two nonzero complex numbers z_1 and z_2,
$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$
where N has one of the values $0, \pm 1$. (Compare with Example 2 in Sec. 34.)

2. Verify expression (4), Sec. 34, for $\log(z_1/z_2)$ by
(a) using the fact that $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ (Sec. 9);
(b) showing that $\log(1/z) = -\log z$ $(z \neq 0)$, in the sense that $\log(1/z)$ and $-\log z$ have the same set of values, and then referring to expression (1), Sec. 34, for $\log(z_1 z_2)$.

3. By choosing specific nonzero values of z_1 and z_2, show that expression (4), Sec. 34, for $\log(z_1/z_2)$ is not always valid when *log* is replaced by *Log*.

4. Show that property (6), Sec. 34, also holds when n is a negative integer. Do this by writing $z^{1/n} = (z^{1/m})^{-1}$ $(m = -n)$, where n has any one of the negative values $n = -1, -2, \ldots$ (see Exercise 9, Sec. 11), and using the fact that the property is already known to be valid for positive integers.

5. Let z denote any nonzero complex number, written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$, and let n denote any fixed positive integer $(n = 1, 2, \ldots)$. Show that all of the values of $\log(z^{1/n})$ are given by the equation
$$\log(z^{1/n}) = \frac{1}{n}\ln r + i\frac{\Theta + 2(pn + k)\pi}{n},$$

where $p = 0, \pm 1, \pm 2, \ldots$ and $k = 0, 1, 2, \ldots, n-1$. Then, after writing
$$\frac{1}{n}\log z = \frac{1}{n}\ln r + i\frac{\Theta + 2q\pi}{n},$$
where $q = 0, \pm 1, \pm 2, \ldots$, show that the set of values of $\log(z^{1/n})$ is the same as the set of values of $(1/n)\log z$. Thus show that $\log(z^{1/n}) = (1/n)\log z$ where, corresponding to a value of $\log(z^{1/n})$ taken on the left, the appropriate value of $\log z$ is to be selected on the right, and conversely. [The result in Exercise 5, Sec. 33, is a special case of this one.]

Suggestion: Use the fact that the remainder upon dividing an integer by a positive integer n is always an integer between 0 and $n-1$, inclusive; that is, when a positive integer n is specified, any integer q can be written $q = pn + k$, where p is an integer and k has one of the values $k = 0, 1, 2, \ldots, n-1$.

35. THE POWER FUNCTION

35. Complex Exponent

Def) $z = 0$, $c \in \mathbb{C}$

$z^c = e^{c\log z} = \exp(c(\ln|z| + i\arg(z)))$

$= \exp(c(\ln|z| + i(\Theta + 2n\pi)))$ $(n \in \mathbb{Z})$

: Multi - Valued.

(Ex) $i^{-2i} = ?$

(sol) $i^{-2i} = e^{-2i\log i}$

$= e^{-2i(\underbrace{\ln|i|}_{0} + i\arg(i))}$

$= e^{-2i(0 + i(\frac{\pi}{2} + 2n\pi))}$ $(n = 0, \pm 1, \pm 2, \cdots)$

$= e^{(\pi + 4n\pi)}$ $(n = 0, \pm 1, \pm 2, \cdots)$

(Quiz) $(-5i)^{\frac{i}{\pi}} = ?$

(sol) $(-5i)^{\frac{i}{\pi}} = e^{\frac{i}{\pi}\cdot\log(-5i)}$

$= e^{\frac{i}{\pi}(\ln|-5i| + i\arg(-5i))}$

$= e^{\frac{i}{\pi}(\ln 5 + i(\frac{3}{2}\pi + 2n\pi))}$ $(n = 0, \pm1, \pm2, \cdots)$

$= e^{\frac{i}{\pi}(\ln 5 + \frac{3}{2}i\pi + 2n\pi i)}$ $(n = 0, \pm1, \pm2, \cdots)$

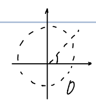
$= e^{\frac{\ln 5}{\pi}i - \frac{3}{2} - 2n}$ $(n = 0, \pm1, \pm2, \cdots)$

※ $\frac{1}{e^z} = e^{-z}$

$\frac{1}{z^c} = \frac{1}{e^{c\log z}} = e^{-c\log z} = z^{-c}$

※ $\frac{d}{dz}(z^c) = ?$

Note) $\log z = \ln|z| + i\theta$ $(|z|>0,\ \alpha<\theta<2\pi+\alpha)$



⇒ $z^c = e^{c\log z}$; Single valued & Analytic in D

$D = \{re^{i\theta} \mid r>0,\ \alpha<\theta<2\pi+\alpha\}$ s.t. $\frac{d}{dz}(z^c) = \frac{d}{dz}[\exp(c\log z)]$

$= \exp[c\log z]\ \frac{d}{dz}[c\log z]$

$= \frac{c}{z}\exp[c\log z] = \frac{c}{\exp(\log z)}\exp[c\log z]$

$= \frac{c}{z}z^c$ $\qquad$ $c\exp[(c-1)\log z]$

$= c\cdot z^{c-1}$ $\qquad$ $c\cdot z^{c-1}$

⇒ 복소수로 미분하면 같다. (단, branch cut 범위 내에서. $\alpha<\theta<\alpha+2\pi$)

def) Principle value of z^c : denoted P.V. z^c

⇔ P.V. $z^c = e^{c\,\mathrm{Log}\,z}$

$= e^{c(\ln|z| + i\Theta)}$ $(-\pi<\Theta\le\pi)$

(Ex) P.V. $(-i)^i$

(sol) P.V$(-i)^i = \exp[i\log(-i)]$

$= \exp[i(\underbrace{\ln|-i|}_{0} + i\underbrace{\mathrm{Arg}(-i)}_{-\frac{\pi}{2}})]$

$= e^{\frac{\pi}{2}}$

(Ex) P.V. $z^{\frac{2}{3}}$ $(z\neq0)$

(sol) P.V. $z^{2/3} = \exp[\frac{2}{3}\mathrm{Log}\,z]$

$= \exp[\frac{2}{3}(\ln|z| + i\Theta)]$

$= r^{\frac{2}{3}}e^{\frac{2}{3}\Theta i} = r^{\frac{2}{3}}\cos(\frac{2}{3}\Theta) + i\,r^{\frac{2}{3}}\sin(\frac{2}{3}\Theta)$

(Ex) $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$

1) $P.V.(z_1 z_2)^i = P.V(z_1)^i \cdot P.V.(z_2)^i$

2) $P.V.(z_2 z_3)^i \neq P.V(z_2)^i \, P.V(z_3)^i$

1) - ① $P.V(z_1 z_2)^i = P.V.(2)^i$

$= \exp[\,i \, Log\, 2\,]$

$= \exp[\,i(\ln 2 + i\,\overset{0}{\overset{\parallel}{Arg(2)}})\,]$

$= e^{i \ln 2}$

② - ① $P.V.(z_1)^i = P.V(1+i)^i$

$= \exp[\,i \, Log(1+i)\,]$

$= \exp[\,i(\ln\sqrt{2} + \frac{\pi}{4} i)\,] = e^{i\ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}}$

② - ② $P.V.(z_2)^i = P.V.(1-i)^i$

$= \exp[\,i \, Log(1-i)\,]$

$= \exp[\,i(\ln\sqrt{2} - \frac{\pi}{4} i)\,]$

$= e^{i\ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}}$

① × ② ⇒ $e^{i\ln\sqrt{2}} = \overset{P.V(z_1)^i \cdot P.V(z_2)^i}{\overset{\parallel}{P.V(z_1 z_2)^i}}$ ■

(Ex 4) $z_2 = 1-i$, $z_3 = -1-i$

2) $P.V(z_2 z_3)^i \neq P.V.(z_2)^i \, P.V.(z_3)^i$

① $P.V.(z_2 z_3)^i = P.V(-2)^i$

$= \exp[\,i\, Log(-2)\,]$

$= \exp[\,i(\ln 2 + \pi i)\,]$

$= e^{i\ln 2} e^{-\pi}$

② $P.V.(z_2)^i = P.V.(1-i)^i$

$= \exp[\,i\, Log(1-i)\,]$

$= \exp[\,i(\ln\sqrt{2} - \frac{\pi}{4} i)\,]$

$= e^{\frac{i}{2}\ln 2} \cdot e^{\frac{\pi}{4}}$

& $P.V(z_3)^i = P.V(-1-i)^i$

$= \exp[\,i\, Log(\underset{-\frac{3}{4}\pi}{-1-i})\,]$

$= \exp[\,i(\ln\sqrt{2} - \frac{3}{4}\pi i)\,]$

$= e^{\frac{i}{2}\ln 2} \cdot e^{\frac{3}{4}\pi}$

$\therefore P.V(z_2 z_3)^i = e^{i\ln 2} \cdot e^{-\pi}$

$\neq e^{i\ln 2} e^{\pi} = P.V(z_2)^i \, P.V(z_3)^i$

def) $c^z = e^{z \log c}$ $(c \in \mathbb{C})$

: Exponential ft with base c.

ㅊㅊ

* the branch of $\log c$: 결정

(분지)

⇒ $c^z = e^{z \log c}$: Entire ft s.t. $\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z \log c} \cdot \log c$

$\therefore \frac{d}{dz} c^z = c^z \log c$

* $\frac{d}{dx}(a^x) = a^x \ln a$

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(Exercise) ㅊ1, ㅊ2, ㅊ3, ㅊ4, ㅊ8, ㅊ9

Hence

$$(z_2z_3)^i = \left[e^{\pi/4}e^{i(\ln 2)/2}\right]\left[e^{3\pi/4}e^{i(\ln 2)/2}\right]e^{-2\pi},$$

or

$$(z_2z_3)^i = z_2^i z_3^i e^{-2\pi}. \tag{2}$$

EXERCISES

1. Show that

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4}+2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)$ $(n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\frac{1}{i^{2i}} = \exp[(4n+1)\pi]$ $(n = 0, \pm 1, \pm 2, \ldots)$.

2. Find the principal value of

(a) $(-i)^i$; (b) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; (c) $(1-i)^{4i}$.

Ans. (a) $\exp(\pi/2)$; (b) $-\exp(2\pi^2)$; (c) $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.

3. Use definition (1), Sec. 35, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing

(a) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;

(b) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 10 is the same as the principal value of $z_0^{1/n}$ defined by equation (3), Sec. 35.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a \ln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Ans. c is real.

8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then

(a) $z^{c_1}z^{c_2} = z^{c_1+c_2}$; (b) $\frac{z^{c_1}}{z^{c_2}} = z^{c_1-c_2}$;

(c) $(z^c)^n = z^{cn}$ $(n = 1, 2, \ldots)$.

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

37. THE TRIGONOMETRIC FUNCTIONS sin z AND cos z

Euler's formula (Sec. 7) tells us that

$$e^{ix} = \cos x + i\sin x \quad \text{and} \quad e^{-ix} = \cos x - i\sin x$$

37. Trigonometric ft in $\mathbb{C}$

Recall) Euler's Fomula

$$\begin{cases} e^{ix} = \cos x + i \sin x \\ e^{-ix} = \cos x - i \sin x \end{cases} \Rightarrow \begin{cases} \cos x = \dfrac{e^{ix}+e^{-ix}}{2} \\ \sin x = \dfrac{e^{ix}-e^{-ix}}{2i} \end{cases}$$

def) $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$ $(z \in \mathbb{C})$ & $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

* $\dfrac{d}{dz}\sin z = \dfrac{d}{dz}\left(\dfrac{e^{iz}-e^{-iz}}{2i}\right) = \dfrac{i(e^{iz}+e^{-iz})}{2i} = \dfrac{e^{iz}+e^{-iz}}{2} = \cos z$

* $\dfrac{d}{dz}\cos z = \dfrac{d}{dz}\left(\dfrac{e^{iz}+e^{-iz}}{2}\right) = \dfrac{i(e^{iz}-e^{-iz})}{2} = -\dfrac{(e^{iz}-e^{-iz})}{2i} = -\sin z$ //

* $\sin(-z) = \dfrac{e^{i(-z)}-e^{-i(-z)}}{2i} = \dfrac{e^{-iz}-e^{iz}}{2i}$ (sin은 기함수)

$= -\dfrac{e^{iz}-e^{-iz}}{2i} = -\sin z$

* $\cos(-z) = \dfrac{e^{i(-z)}+e^{-i(-z)}}{2}$

$= \dfrac{e^{-iz}+e^{iz}}{2} = \cos z$ (우함수)

* $\sin^2 z + \cos^2 z = 1$

$= \left(\dfrac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\dfrac{e^{iz}+e^{-iz}}{2}\right)^2 = -\dfrac{\cancel{e^{2iz}}+\cancel{e^{-2iz}}-2}{4} + \dfrac{\cancel{e^{2iz}}+\cancel{e^{-2iz}}+2}{4} = \dfrac{2}{4}+\dfrac{2}{4} = 1$ //

* $\sin(z+\frac{\pi}{2})$

$= \dfrac{e^{i(z+\frac{\pi}{2})}-e^{-i(z+\frac{\pi}{2})}}{2i} = \dfrac{\cancel{i}e^{iz}+\cancel{i}e^{-iz}}{2\cancel{i}} = \dfrac{e^{iz}+e^{-iz}}{2} = \underline{\cos z}$

* $\cos(z+\frac{\pi}{2}) = -\sin z$

* $\begin{cases} \sin(z+\pi) = -\sin z \\ \cos(z+\pi) = -\cos z \end{cases}$

* $\sin(z_1 \pm z_2)$

$= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

* $\cos(z_1 \pm z_2)$

$= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

(ex) $\cos(z_1+z_2) = \dfrac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2}$

$= \dfrac{(e^{iz_1}+e^{-iz_1})(e^{iz_2}+e^{-iz_2})}{4} + \dfrac{(e^{iz_1}+e^{-iz_1})(e^{iz_2}-e^{-iz_2})}{4}$, (Recall) $\sinh y = \dfrac{e^{y} - e^{-y}}{2}$

$\cosh y = \dfrac{e^{y} + e^{-y}}{2}$

$*$ $\sin(iy) = \dfrac{e^{i(iy)} - e^{-i(iy)}}{2i}$

$= \dfrac{e^{-y} - e^{y}}{2i} = i\left(\dfrac{e^{y} - e^{-y}}{2}\right) = i\sinh y$

$*$ $\cos(iy) = \dfrac{e^{i(iy)} + e^{-i(iy)}}{2} = \dfrac{e^{-y} + e^{y}}{2}$

$= \cosh y$ //

$*$ $\sin z = \sin(x + iy)$

$= \sin x \cos(iy) + \cos x \sin(iy)$

$= \underbrace{\sin x \cosh y}_{u(x,y)} + i\,\underbrace{\cos x \sinh y}_{v(x,y)}$

$\begin{cases} u_x = v_y = \cos x \cosh y \\ u_y = -v_x = \sin x \sinh y \end{cases}$; C-R-Eqs : O.K &

$u_x \sim v_y$: Exist & Conti

$\Rightarrow$ $\sin z$: Entire ft.

$*$ $\cos z = \cos(x + iy)$

$= \cos x \cos(iy) - \sin x \sin(iy)$

$= \underbrace{\cos x \cosh y}_{u(x,y)} - i\,\underbrace{\sin x \sinh y}_{v(x,y)}$

$*$ $|\sin z|^2 = u^2 + v^2$

$= \sin^2 x \underbrace{\cosh^2 y}_{1+\sinh^2 y} + \cos^2 x \sinh^2 y$

$= \sin^2 x\,(1 + \sinh^2 y) + \cos^2 x \sinh^2 y$

$= \sin^2 x + \sin^2 x \sinh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y$

$= \sin^2 x + \sinh^2 y$

$\Rightarrow$ $|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$: Un bdd.

$*$ $|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x\,(1 + \sinh^2 y) + \sin^2 x \sinh^2 y$

$= \cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y$

$= \cos^2 x + \sinh^2 y$

$\Rightarrow$ $|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$: Un bdd.

Quiz) $\sin(\bar{z})$: Analytic ft or Not?

(sol) $\sin(\bar{z}) = \sin(\overline{x+iy})$

$= \sin(x-iy) = \sin x \cos(iy) - \cos x \sin(iy)$

$= \underbrace{\sin x \cosh y}_{u(x,y)} - i \cos x \underbrace{\sinh y}_{v(x,y)}$

$|\sin \bar{z}|^2 = u^2 + v^2$

$= \sin^2 x \underbrace{\cosh^2 y}_{1+\sinh^2 y} + \cos^2 x \sinh^2 y$

$= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y$

$= \sin^2 x + \sin^2 x \sinh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x + \sinh^2 y (\underbrace{\sin^2 x + \cos^2 x}_{1}) = \sin^2 x + \sinh^2 y$

$\Rightarrow |\sin \bar{z}| = \sqrt{\sin^2 x + \sin^2 y}$: Unbdd

$\Rightarrow$ Not Analytic ft ∎

$*$ $|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$: Un.bdd

$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$: Un.bdd

$\Rightarrow \sin z = 0 \Leftrightarrow \sin x = \sinh y = 0$

$\Leftrightarrow \begin{cases} x = n\pi & (n \in \mathbb{Z}) \\ y = 0 \end{cases}$

$\therefore z = n\pi \ (n \in \mathbb{Z})$

$\cos z = 0 \Leftrightarrow \cos x = \sinh y = 0$

$\Leftrightarrow \begin{cases} x = (n+\frac{1}{2})\pi & (n \in \mathbb{Z}) \\ y = 0 \end{cases}$

$\therefore z = (n+\frac{1}{2})\pi \ (n \in \mathbb{Z})$

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EXERCISES

1. Give details in the derivation of expressions (2), Sec. 37, for the derivatives of $\sin z$ and $\cos z$.

2. (*a*) With the aid of expression (4), Sec. 37, show that

$$e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 37, to show how it follows that

$$e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(*b*) Use the results in part (*a*) and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 37.

3. According to the final result in Exercise 2(*b*),

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 37.

4. Verify identity (9) in Sec. 37 using
   (*a*) identity (6) and relations (3) in that section;
   (*b*) the lemma in Sec. 28 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

5. Use identity (9) in Sec. 37 to show that
   (*a*) $1 + \tan^2 z = \sec^2 z$; (*b*) $1 + \cot^2 z = \csc^2 z$.

6. Establish differentiation formulas (3) and (4) in Sec. 38.

7. In Sec. 37, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.
   Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

8. Point out how it follows from expressions (15) and (16) in Sec. 37 for $|\sin z|^2$ and $|\cos z|^2$ that
   (*a*) $|\sin z| \geq |\sin x|$; (*b*) $|\cos z| \geq |\cos x|$.

9. With the aid of expressions (15) and (16) in Sec. 37 for $|\sin z|^2$ and $|\cos z|^2$, show that
   (*a*) $|\sinh y| \leq |\sin z| \leq \cosh y$; (*b*) $|\sinh y| \leq |\cos z| \leq \cosh y$.

10. (*a*) Use definitions (1), Sec. 37, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

   (*b*) With the aid of the identity obtained in part (*a*), show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

12. Use the reflection principle (Sec. 29) to show that for all z,
   (*a*) $\overline{\sin z} = \sin \bar{z}$; (*b*) $\overline{\cos z} = \cos \bar{z}$.

13. With the aid of expressions (13) and (14) in Sec. 37, give direct verifications of the relations obtained in Exercise 12.

14. Show that
   (*a*) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;
   (*b*) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

15. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and then the imaginary parts of $\sin z$ and $\cosh 4$.
   Ans. $\left(\dfrac{\pi}{2} + 2n\pi\right) \pm 4i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

16. With the aid of expression (14), Sec. 37, show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2 + \sqrt{3}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

39. HYPERBOLIC FUNCTIONS

The ***hyperbolic sine and cosine functions*** of a complex variable z are defined as they are with a real variable:

$$(1) \qquad \sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore,

$$(2) \qquad \frac{d}{dz}\sinh z = \cosh z, \qquad \frac{d}{dz}\cosh z = \sinh z.$$

Because of the way in which the exponential function appears in definitions (1) and in the definitions (Sec. 37)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

of $\sin z$ and $\cos z$, the hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$$(3) \qquad -i\sinh(iz) = \sin z, \qquad \cosh(iz) = \cos z,$$

$$(4) \qquad -i\sin(iz) = \sinh z, \qquad \cos(iz) = \cosh z.$$

Note how it follows readily from relations (4) and the periodicity of $\sin z$ and $\cos z$ that $\sinh z$ and $\cosh z$ are *periodic with period* $2\pi i$.

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$(5) \qquad \sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z,$$

$$(6) \qquad \cosh^2 z - \sinh^2 z = 1,$$

$$(7) \qquad \sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$

$$(8) \qquad \cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

def) $\begin{cases} \tan z = \dfrac{\sin z}{\cos z}, & \cot z = \dfrac{\cos z}{\sin z} \\ \sec z = \dfrac{1}{\cos z}, & \csc z = \dfrac{1}{\sin z} \end{cases}$

p109 **39. HYPERBOLIC Fts.**

def) $\begin{cases} \sinh z = \dfrac{e^z - e^{-z}}{2} \\ \cosh z = \dfrac{e^z + e^{-z}}{2} \end{cases}$

* $\dfrac{d}{dz} \sinh z = \dfrac{e^z + e^{-z}}{2} = \cosh z$

$\dfrac{d}{dz} \cosh z = \dfrac{e^z - e^{-z}}{2} = \sinh z$

* $-i \sinh iz = -i \dfrac{e^{iz} - e^{-iz}}{2}$ $\quad \dfrac{-i}{2} = \dfrac{-i^2}{2i} = \dfrac{1}{2i}$

$= \dfrac{e^{iz} - e^{-iz}}{2i} = \sin z$

$\cosh(iz) = \dfrac{e^{iz} + e^{-iz}}{2} = \cos z$

* $\sinh z = \underbrace{\sinh x \cos y}_{u(x,y)} + i \underbrace{\cosh x \sin y}_{v(x,y)}$

$(\because \sinh z = \dfrac{e^z - e^{-z}}{2}$

$= \dfrac{e^{x+iy} - e^{-x-iy}}{2}$

$= \dfrac{1}{2}(e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y))$

$= \dfrac{e^x - e^{-x}}{2} \cos y + i \left(\dfrac{e^x + e^{-x}}{2} \right) \sin y)$

* $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$(\because \cosh z = \dfrac{e^z + e^{-z}}{2}$

$= \dfrac{e^{x+iy} + e^{-x-iy}}{2}$

$= \dfrac{1}{2} \left(e^x(\cos y + i \sin y) + e^{-x}(\cos y - i \sin y) \right)$

$= \left(\dfrac{e^x + e^{-x}}{2} \right) \cos y + i \left(\dfrac{e^x - e^{-x}}{2} \right) \sin y$

$= \underbrace{\cosh x \cos y}_{u(x,y)} + i \underbrace{\sinh x \sin y}_{v(x,y)}$

$\begin{cases} u_x = v_y = \sinh x \cos y \\ v_x = -u_y = \cosh x \sin y \end{cases}$ $\therefore$ C-R-EQS O.K → 전함수

* $\cosh z = \cosh x \cos y + i \sinh x \sin y$

$|\cosh z|^2 = \underbrace{\cosh^2 x}_{1+\sinh^2 x} \cos^2 y + \sinh^2 x \sin^2 y$

↳ 실수부 제곱 + 허수부 제곱

$= \cos^2 y + \cos^2 y \sinh^2 x + \sinh^2 x \sin^2 y$

$= \cos^2 y + \sinh^2 x \, (\underbrace{\cos^2 y + \sin^2 y}_{1})$

$= \sinh^2 x + \cos^2 y$

$\therefore \; |\cosh z| = \sqrt{\sinh^2 x + \cos^2 y}$ $\qquad \therefore \; \cosh z = 0 \iff \sinh x = 0 \;\&\; \cos y = 0$

$\iff x = 0, \; y = (n + \frac{1}{2})\pi, \; n \in \mathbb{Z}$ $\qquad \therefore \; z = i(n + \frac{1}{2})\pi \quad (n \in \mathbb{Z})$

p110 * $\sinh z = \sinh x \cos y + i \cosh x \sin y$

$|\sinh z|^2 = \sinh^2 x \cos^2 y + \underbrace{\cosh^2 x}_{=1+\sinh^2 x} \sin^2 y$

$= \sinh^2 x \cos^2 y + \sin^2 y + \sin^2 y \sinh^2 x$

$= \sinh^2 x(\overbrace{\cos^2 y + \sin^2 y}^{1}) + \sin^2 y = \sinh^2 x + \sin^2 y$

$\therefore |\sinh z| = \sqrt{\sinh^2 x + \sin^2 y}$

& $\sinh z = 0 \iff \sinh x = 0$ & $\sin y = 0$

$\iff x = 0,\ y = n\pi.\ (n \in \mathbb{Z})$

def) $\tanh z = \dfrac{\sinh z}{\cosh z}$, $\coth z = \dfrac{\cosh z}{\sinh z}$

$\operatorname{sech} z = \dfrac{1}{\cosh z}$, $\operatorname{csch} z = \dfrac{1}{\sinh z}$

* $\dfrac{d}{dz} \tanh z = \dfrac{d}{dz}\left(\dfrac{\sinh z}{\cosh z}\right)$

$= \dfrac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z} = \operatorname{sech}^2 z$

p111 Ex) *6, *7, *9, *10, *12, *16, *17

The ***hyperbolic tangent*** of z is defined by means of the equation

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{16}$$

and is analytic in every domain in which $\cosh z \neq 0$. The functions $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are the reciprocals of $\tanh z$, $\cosh z$, and $\sinh z$, respectively. It is straightforward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable:

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z, \tag{17}$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z. \tag{18}$$

EXERCISES

1. Verify that the derivatives of $\sinh z$ and $\cosh z$ are as stated in equations (2), Sec. 39.
2. Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with
   (*a*) definitions (1), Sec. 39, of $\sinh z$ and $\cosh z$;
   (*b*) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 37) and using relations (3) in Sec. 39.
3. Show how identities (6) and (8) in Sec. 39 follow from identities (9) and (6), respectively, in Sec. 37.
4. Write $\sinh z = \sinh(x + iy)$ and $\cosh z = \cosh(x + iy)$, and then show how expressions (9) and (10) in Sec. 39 follow from identities (7) and (8), respectively, in that section.
5. Derive expression (11) in Sec. 39 for $|\sinh z|^2$.
6. Show that $|\sinh x| \leq |\cosh z| \leq \cosh x$ by using
   (*a*) identity (12), Sec. 39;
   (*b*) the inequalities $|\sinh y| \leq |\cos z| \leq \cosh y$, obtained in Exercise 9(*b*), Sec. 38.
7. Show that
   (*a*) $\sinh(z + \pi i) = -\sinh z$; (*b*) $\cosh(z + \pi i) = -\cosh z$;
   (*c*) $\tanh(z + \pi i) = \tanh z$.
8. Give details showing that the zeros of $\sinh z$ and $\cosh z$ are as in the theorem in Sec. 39.
9. Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.
10. Show that $\tanh z = -i \tan(iz)$.
    Suggestion: Use identities (4) in Sec. 39.
11. Derive differentiation formulas (17), Sec. 39.
12. Use the reflection principle (Sec. 29) to show that for all z,
    (*a*) $\overline{\sinh z} = \sinh \bar{z}$; (*b*) $\overline{\cosh z} = \cosh \bar{z}$.
13. Use the results in Exercise 12 to show that $\overline{\tanh z} = \tanh \bar{z}$ at points where $\cosh z \neq 0$.

14. By accepting that the stated identity is valid when z is replaced by the real variable x and using the lemma in Sec. 28, verify that
    (*a*) $\cosh^2 z - \sinh^2 z = 1$; (*b*) $\sinh z + \cosh z = e^z$.
    [Compare with Exercise 4(*b*), Sec. 38.]
15. Why is the function $\sinh(e^z)$ entire? Write its real component as a function of x and y, and state why that function must be harmonic everywhere.
16. By using one of the identities (9) and (10) in Sec. 39 and then proceeding as in Exercise 15, Sec. 38, find all roots of the equation
    (*a*) $\sinh z = i$; (*b*) $\cosh z = \dfrac{1}{2}$.
    Ans. (*a*) $z = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;
    (*b*) $z = \left(2n \pm \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.
17. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 38.)
    Ans. $z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

40. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function $\sin^{-1} z$, we write

$$w = \sin^{-1} z \quad \text{when} \quad z = \sin w.$$

That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is quadratic in e^{iw}, and solve for e^{iw} [see Exercise 8(*a*), Sec. 11], we find that

$$e^{iw} = iz + (1 - z^2)^{1/2} \tag{1}$$

where $(1 - z^2)^{1/2}$ is, of course, a double-valued function of z. Taking logarithms of each side of equation (1) and recalling that $w = \sin^{-1} z$, we arrive at the expression

$$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]. \tag{2}$$

The following example emphasizes the fact that $\sin^{-1} z$ is a multiple-valued function, with infinitely many values at each point z.

SEC 39, Exercise

※6

6. Show that $|\sinh x| \le |\cosh z| \le \cosh x$ by using

(a) identity (12), Sec. 39;

(b) the inequalities $|\sinh y| \le |\cos z| \le \cosh y$, obtained in Exercise 9(b), Sec. 38.

(pf) (a) since we are given the identity $|\cosh z|^2 = \sinh^2 x + \cos^2 y$ for any arbitrary complex number $z = x + iy$, from that we can obtain ;

$$\Leftrightarrow \ |\cosh z|^2 - \sinh^2 x \ge 0$$

$$\Leftrightarrow \ \sinh^2 x \le |\cosh z|^2 \ \Leftrightarrow \ |\sinh x| \le |\cosh z| . \quad (1)$$

We can also write $|\cosh z|^2$ similarity :

$$|\cosh z|^2 = \cosh^2 x - 1 + \cos^2 y$$

$$= \cosh^2 x - (\underbrace{1 - \cos^2 y}_{\sin^2 y}) = \cosh^2 x - \sin^2 y$$

From there we can conclude that $|\cosh z|^2 - \cosh^2 x \le 0$.

hence $|\cosh z|^2 \le \cosh^2 x \ \Leftrightarrow \ |\cosh z| \le \cosh x \quad (2)$

Applying results (1) and (2) we finally get $|\sinh x| \le |\cosh z| \le |\cosh x|$ ∎

(pf) (b) In this case we are given the inequallity $|\sinh y| \le |\cos z| \le |\cosh y|$, so from there we can obtain

the said ineq ' $|\sinh x| \le |\cosh z| \le |\cosh x|$ ' by replacing z with iz . and by using the identity $\cos(iz) = \cosh z$

which is true for any complex number z. ∎

7.

7. Show that

(a) $\sinh(z + \pi i) = -\sinh z$; (b) $\cosh(z + \pi i) = -\cosh z$;

(c) $\tanh(z + \pi i) = \tanh z$.

(pf) (a) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_1 + \cosh z_1 \sinh z_2$ & $\sinh(iz) = i \sin z$, $\cosh(iz) = \cos z$.

Hence. $\sinh(z + i\pi) = \sinh z \cosh(i\pi) + \cosh z \sinh(i\pi)$

$$= \sinh z \cdot \underbrace{\cos \pi}_{(-1)} + \cosh z \, (i \underbrace{\sin \pi}_{0})$$

$$= -\sinh z \quad ∎$$

(b) $\cosh(z + i\pi) = \cosh z \cosh(i\pi) + \sinh z \sinh(i\pi)$

$$= \cosh z \underbrace{\cos \pi}_{(-1)} + \cancel{\sinh z \cdot i \underbrace{\sin \pi}_{0}}$$

$$= -\cosh z \quad ∎$$

(c) $\tanh(z + \pi i) = \dfrac{\sinh(z + \pi i)}{\cosh(z + \pi i)} = \dfrac{-\sinh z}{-\cosh z} = \tanh z$ ∎

9. **9.** Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.

(9) Since $\tanh z = \frac{\sinh z}{\cosh z}$, it follows that the zeros of $\tanh z$ are the same as the zeroes of $\sinh z$

and singularities of $\tan z$ are exactly the zeroes of $\cosh z$.

Now by Exercise 8, the zeroes of $\tanh z$ are $k\pi i$, $k \in \mathbb{Z}$.

and it's singularities are $(\frac{\pi}{2} + k\pi) i$, $k \in \mathbb{Z}$. ∎

10. **10.** Show that $\tanh z = -i \tan(iz)$.
Suggestion: Use identities (4) in Sec. 39.

12. Use the reflection principle (Sec. 29) to show that for all z,
(*a*) $\overline{\sinh z} = \sinh \bar{z}$; (*b*) $\overline{\cosh z} = \cosh \bar{z}$.

40. Inverse Trigonometric Hyperbolic Fts.

def) $w = \sin^{-1} z$ iff $\sin w = z$

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$) 양쪽에 $2i$ 곱하고. e^{iw} 곱하여 e에 대한 이차식으로

$$2iz e^{iw} = e^{2iw} - 1 \Leftrightarrow e^{2iw} - 2iz e^{iw} - 1 = 0$$

근의 공식 use

$$\Rightarrow e^{iw} = \frac{2iz + (4 - 4z^2)^{1/2}}{2} = iz + (1-z^2)^{1/2}$$

$$\therefore w = -i \log (iz + (1-z^2)^{1/2})$$

: Multi-Valued.

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Ex) $\sin^{-1}(-i) = ?$

(sol) $\sin^{-1}(-i) = -i \log (i(-i) + (1-(-i)^2)^{\frac{1}{2}})$

$$= -i \log (-i^2 + (1-i^2)^{\frac{1}{2}}) = -i \log (1 \pm \sqrt{2})$$

$$= -i \log (1+\sqrt{2}) \text{ or } -i \log (1-\sqrt{2})$$

$$= -i (\ln (1+\sqrt{2}) + 2n\pi i) \text{ or } -i (\underbrace{\ln (\overset{\frac{1}{\sqrt{2}+1}}{\sqrt{2}-1})}_{-\ln(\sqrt{2}+1)} + (2n+1)\pi i) \quad (n \in \mathbb{Z})$$

$$* \ \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$$

$$= -i (\ln (1+\sqrt{2}) + 2n\pi i) \text{ or } -i (-\ln (1+\sqrt{2}) + (2n+1)\pi i) \quad (n \in \mathbb{Z})$$

$$= -i ((-1)^n \ln (1+\sqrt{2}) + n\pi i) \quad (n \in \mathbb{Z})$$

$$= n\pi + i (-1)^{n+1} \ln (1+\sqrt{2}) \quad (n = 0, \pm 1, \pm 2, \cdots)$$

Quiz) $\sin^{-1}(2i) = ?$

$$\sin^{-1}(2i) = -i \log (i \cdot 2i + (1 - \underbrace{(2i)^2}_{4i^2 = -4})^{1/2})$$

$$= -i \log (-2 + (\underbrace{1+4}_{5})^{1/2})$$

$$= -i \log (-2 \pm \sqrt{5})$$

$$= -i \log (-2+\sqrt{5}) \text{ or } -i \log (-2-\sqrt{5})$$

$$= -i (\ln (-2+\sqrt{5}) + 2n\pi i) \text{ or } -i (\ln (2+\sqrt{5}) + (2n+1)\pi i) \quad (n \in \mathbb{Z})$$

$$\frac{1}{\sqrt{5}-2} \frac{(\sqrt{5}+2)}{(\sqrt{5}+2)} = \sqrt{5}+2$$

$$= -i (-\ln (2+\sqrt{5}) + 2n\pi i) \text{ or } -i (\ln (2+\sqrt{5}) + (2n+1)\pi i) \quad (n \in \mathbb{Z})$$

$$= -i ((-1)^n \ln (2+\sqrt{5}) + n\pi i) \quad (n \in \mathbb{Z})$$

$$= n\pi + i (-1)^n \ln (2+\sqrt{5}) \quad (n = 0, \pm 1, \pm 2 \cdots)$$

def) $w = \cos^{-1} z$ iff $\cos w = z$

$z = \cos w = \dfrac{e^{iw} + e^{-iw}}{2}$

$\Leftrightarrow e^{2iw} - 2z e^{iw} + 1 = 0$

$\Rightarrow e^{iw} = \dfrac{2z + (4z^2 - 4)^{\frac{1}{2}}}{2}$

$= z^2 + (z^2 - 1)^{1/2}$

$\therefore w = -i \log\left(z + (z^2-1)^{1/2}\right)$

* $w = \tan^{-1} z \Leftrightarrow \tan w = z$

$\Leftrightarrow z = \dfrac{\sin w}{\cos w}$

$= \dfrac{\frac{e^{iw} - e^{-iw}}{2i}}{\frac{e^{iw} + e^{-iw}}{2}} = \dfrac{2(e^{iw} - e^{-iw})}{2i(e^{iw} + e^{-iw})} = \dfrac{e^{2iw} - 1}{i(e^{2iw} + 1)}$

$\Rightarrow iz(e^{2iw} + 1) = e^{2iw} - 1$

$(iz - 1)e^{2iw} = -1 - iz$?

$e^{2iw} = \dfrac{-2i - 1}{2i - 1} = \dfrac{-z + i}{z + i}$

$\Rightarrow w = \dfrac{1}{2i} \log\left(\dfrac{i - z}{i + z}\right) = \dfrac{i}{2} \log\left(\dfrac{i - z}{i + z}\right)$

* $\dfrac{d}{dz} \sin^{-1} z = \dfrac{1}{(1 - z^2)^{1/2}}$

($\because$ $w = \sin^{-1} z \Leftrightarrow \sin w = z$

미분

Diff at $\cos w \cdot \dfrac{dw}{dz} = 1$

$\Rightarrow \dfrac{dw}{dz} = \dfrac{d}{dz}(\sin^{-1} z) = \dfrac{1}{\cos w}$

$(\cos^2 w + \sin^2 w = 1)$

$= \dfrac{1}{(1 - \sin^2 w)^{1/2}} = \dfrac{1}{(1 - z^2)^{1/2}}$)

* $\dfrac{d}{dz} \cos^{-1} z = -\dfrac{1}{(1 - z^2)^{1/2}}$

($\because$ $w = \cos^{-1} z \Leftrightarrow \cos w = z$

Diff at $-\sin w \dfrac{dw}{dz} = 1$

$\dfrac{dw}{dz} = \dfrac{d}{dz}(\cos^{-1} z) = \dfrac{1}{-\sin w}$

$(\cos^2 w + \sin^2 w = 1)$

$= \dfrac{1}{-(1 - \cos^2 w)^{1/2}} = \dfrac{1}{-(1 - z^2)^{1/2}}$)

* $\dfrac{d}{dz} \tan^{-1} z = \dfrac{1}{1 + z^2}$

($w = \tan^{-1} z \Leftrightarrow \tan w = z$

Diff at $\sec^2 w \dfrac{dw}{dz} = 1$

$\Rightarrow \dfrac{dw}{dz} = \dfrac{1}{\sec^2 w} = \dfrac{1}{\tan^2 w + 1} = \dfrac{1}{z^2 + 1}$)

* $\sinh^{-1} z = \log\left(z + (z^2 + 1)^{1/2}\right)$

($\because$ $w = \sinh^{-1} z \Leftrightarrow \sinh w = z$

$\Leftrightarrow \dfrac{e^w - e^{-w}}{2} = z$

$\Leftrightarrow e^{2w} - 2z e^w - 1 = 0$

$\therefore e^w = \dfrac{2z + (4z^2 + 4)^{1/2}}{2}$

$= z + (z^2 + 1)^{1/2}$

$\therefore w = \sin^{-1} z = \log\left(z + (z^2 + 1)^{1/2}\right)$)

* $\cosh^{-1} z = \log\left(z + (z^2 - 1)^{1/2}\right)$

($\because$ $w = \cosh^{-1} z \Leftrightarrow \cosh w = z$

$\Leftrightarrow \dfrac{e^w + e^{-w}}{2} = z$

$\Leftrightarrow e^{2w} - 2z e^w + 1 = 0$

$\Rightarrow e^w = \dfrac{2z + (4z^2 - 4)^{1/2}}{2}$

$= z + (z^2 - 1)^{1/2}$

$\therefore w = \log\left(z + (z^2 - 1)^{1/2}\right)$

$\therefore w = \sin^{-1} z = \log\left(z + (z^2 - 1)^{1/2}\right)$)

18강.

Mon. May 18

(Review) $\sinh^{-1} z = \log(z + (z^2+1)^{1/2})$, $\cosh^{-1} z = \log(z + (z^2-1)^{1/2})$

$*$ $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

($\because$ $w = \tanh^{-1} z \Leftrightarrow \tanh w = z$

$\Leftrightarrow \frac{e^{2w}-1}{e^{2w}+1} = z$

(Exercise) #1, #2, #3 (4-7은 했음)

114 ELEMENTARY FUNCTIONS CHAP. 3

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

(8) $$\sinh^{-1} z = \log[z + (z^2+1)^{1/2}],$$

(9) $$\cosh^{-1} z = \log[z + (z^2-1)^{1/2}],$$

and

(10) $$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}.$$

Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

EXERCISES

1. Find all the values of
(a) $\tan^{-1}(2i)$; (b) $\tan^{-1}(1+i)$; (c) $\cosh^{-1}(-1)$; (d) $\tanh^{-1} 0$.
Ans. (a) $\left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln 3\ (n = 0, \pm 1, \pm 2, \ldots)$;
(d) $n\pi i\ (n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by
(a) equating real parts and then imaginary parts in that equation;
(b) using expression (2), Sec. 40, for $\sin^{-1} z$.
Ans. $z = \left(2n+\frac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3})\ (n = 0, \pm 1, \pm 2, \ldots)$.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Derive expression (5), Sec. 40, for the derivative of $\sin^{-1} z$.

5. Derive expression (4), Sec. 40, for $\tan^{-1} z$.

6. Derive expression (7), Sec. 40, for the derivative of $\tan^{-1} z$.

7. Derive expression (9), Sec. 40, for $\cosh^{-1} z$.

SEC 40, Exercise

1\.

1. Find all the values of
(a) $\tan^{-1}(2i)$; (b) $\tan^{-1}(1+i)$; (c) $\cosh^{-1}(-1)$; (d) $\tanh^{-1} 0$.
Ans. (a) $\left(n+\frac{1}{2}\right)\pi + \frac{i}{2}\ln 3\ (n = 0, \pm 1, \pm 2, \ldots)$;
(d) $n\pi i\ (n = 0, \pm 1, \pm 2, \ldots)$.

(a) $\tan^{-1} z = \frac{i}{2}\log\left(\frac{i-z}{i+z}\right)$

replacing z by $2i$ $\Rightarrow$ $\tan^{-1}(2i) = \frac{i}{2}\log\left(\frac{i-2i}{i+2i}\right)$

$= \frac{i}{2}\log\left(\frac{-i}{3i}\right) = \frac{i}{2}\log\left(-\frac{1}{3}\right)$

$= \frac{i}{2}\left(\ln\left|-\frac{1}{3}\right| + i\underbrace{\arg\left(-\frac{1}{3}\right)}_{+2n\pi}\right) = \frac{i}{2}\ln 3 + \left(n+\frac{1}{2}\right)\pi$?

(b) $\tan^{-1}(1+i) = \frac{i}{2}\log\left(\frac{i-i-1}{i+i+1}\right) = \frac{i}{2}\log\left(\frac{-1}{2i+1}\right)$

$= \frac{i}{2}\log\left(\frac{1}{5}(2i-1)\right)$

(d) $\tanh^{-1} 0 = \frac{1}{2}\log\left(\frac{1}{1}\right) = \frac{1}{2}\underbrace{(\log 1)}_{2n\pi i} = n\pi i$

3. Solve the equation $\cos z = \sqrt{2}$ for z.

(pf) $w = \cos z = \sqrt{2} \Leftrightarrow z = \cos^{-1} w = \cos^{-1}\sqrt{2}$

$\Rightarrow -i \log\left(\sqrt{2} + (2-1)^{1/2}\right)$

$= -i \log(\sqrt{2}+1)$

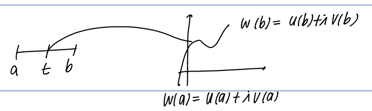
$= -i\left(\ln(\sqrt{2}+1) + i \arg(\sqrt{2}+1)\right)$

Chapter 4. Integrals p115

41. Deriv. of fts $w(t) = u(t) + iv(t)$

* 1- parameter complex ft ;

$w(t) = u(t) + i v(t) \quad (t \in \mathbb{R})$



$w'(t) = \frac{d}{dt} w(t) = u'(t) + i v'(t)$

(ex) $\frac{d}{dt} e^{z_0 t} \quad (z_0 = x_0 + i y_0 \in \mathbb{C})$

$= \frac{d}{dt} e^{x_0 t + i y_0 t}$

$= \frac{d}{dt} (e^{x_0 t})(e^{i y_0 t})$

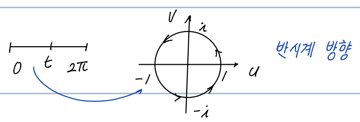
$= x_0 e^{x_0 t} \cdot e^{i y_0 t} + i y_0 e^{x_0 t} e^{i y_0 t}$

$= (x_0 + i y_0) e^{x_0 t} e^{i y_0 t} = z_0 e^{z_0 t}$

< 복소의 미분에서 다른 점 - M-V-T를 항상 만족하지는 않음. >

(Ex) (M-V-T : Not always True in Complex ft : $w(t) = u(t) + iv(t)$)

$w(t) = e^{it} = \underbrace{\cos t}_{u(t)} + i \underbrace{\sin t}_{v(t)} \quad (0 \le t \le 2\pi)$



Since $w'(t) = i e^{it}, \quad t \in [0, 2\pi]$

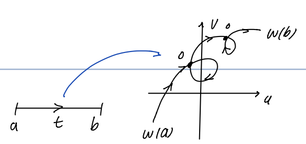
$\Rightarrow |w'(t)| = |i e^{it}| = 1 \quad t \in [0, 2\pi]$

$\therefore \nexists\, t_1$ in $[0, 2\pi]$

$(\because\ w'(t_1) \neq \frac{w(2\pi) - w(0)}{2\pi - 0} = \frac{1-1}{2\pi} = 0$

값다 돌아옴

$\therefore$ M-V-T : Not satisfied



Wed, May 18.

42. Definite Integrals of fts (정적분)

$w(t) = u(t) + i\,v(t)$

def) Definite integral of $w(t) = u(t) + iv(t)$ on $a \le t \le b$.

$$\Leftrightarrow \int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

(Ex) $\int_0^1 \underbrace{(1+it)^2}_{w(t)}\,dt = \int_0^1 (1-t^2+2it)\,dt$

$= \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$

$= \left[t-\frac{t^3}{3}\right]_0^1 + i\left[t^2\right]_0^1 = \frac{2}{3}+i$ //

방법①

방법② put $k = 1+it \Rightarrow dk = i\,dt$

$\int_1^{1+i} k^2(-i\,dk) = -i\int_1^{1+i} k^2\,dk$

$= -i\left[\frac{k^3}{3}\right]_{k=1}^{k=1+i}$ //

(Fundamental Thm of Calculus : F-T-C)

(미적분학 기본정리)

for $w(t) = u(t) + iv(t)$

let U, V : Indefinite integrals of u, v

$\Rightarrow W = U + iV$: Indefinite integral of "w" s.t $W' = w$

& $\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

$= U\Big|_a^b + i\,V\Big|_a^b = W\Big|_a^b$

(ex) $\int_0^{\frac{\pi}{4}} e^{it}\,dt = ?$

(sol) $e^{it} = \underbrace{\cos t}_{u(t)} + i\underbrace{\sin t}_{v(t)}$

Since $\underbrace{\sin t}_{U(t)}, \underbrace{-\cos t}_{V(t)}$: Anti-Deriv of $\cos t, \sin t$,

$U + iV = \sin t - i\cos t$

$= -i(\cos t + i\sin t) = -ie^{it} = W(t)$

$\therefore \int_0^{\frac{\pi}{4}} e^{it}\,dt = \left[-ie^{it}\right]_{t=0}^{t=\frac{\pi}{4}}$

$= -ie^{\frac{\pi}{4}i} + i = -i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + i$ ■

* f : Conti on $[a,b]$

$\Rightarrow f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad c \in [a,b]$

: M-V-T for Integrals.

(Ex) (M-V-T for Integrals : Not always True for $w(t) = u(t) + i v(t)$)

let $w(t) = e^{it}$ $(0 \le t \le 2\pi)$

$\int_0^{2\pi} e^{it}\,dt = -i\left[e^{it}\right]_0^{2\pi}$

$= -i\,[1-1] = 0$

But $|e^{it}| = 1 \quad \forall t \in [0, 2\pi]$

$e^{it} \ne \frac{1}{2\pi - 0}\int_0^{2\pi} e^{it}\,dt = 0 \quad \forall t \in [0, 2\pi]$

$\therefore$ M-V-T for Integral : Not Satisfied.

Quiz) $\int_0^{\infty} e^{-zt}\,dt = \frac{1}{z}$ $(\mathrm{Re}\, z > 0)$ '증명'

(pf) put $z = x + iy$

$\Rightarrow \int_0^{\infty} e^{(-x-iy)t}\,dt$

$= \lim_{c\to\infty}\int_0^{c} e^{-xt}\cdot e^{-iyt}\,dt$

$e^{i(-yt)}$

$= \cos(-yt) + i\sin(-yt)$

⋮

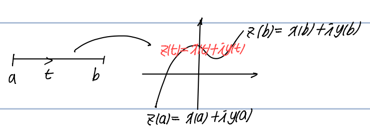
?

18강 (중간 이후 첫 수업) Thu. May 21.

43. Contours p120

def) A set of pts $z = (x, y)$: arc (곡선)

$\Leftrightarrow x = x(t),\ y = y(t) \quad (a \le t \le b)$: Conti

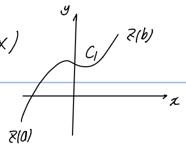


def) The arc $C: z(t) = x(t) + iy(t) \quad (a \le t \le b)$

C: Simple arc or (Jordan arc) 단순곡선 → 즉 꼬이지 X

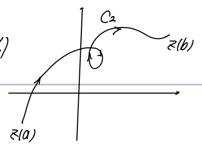
⟺ No Cross Point Itself.

(ex)



C_1 : Simple arc

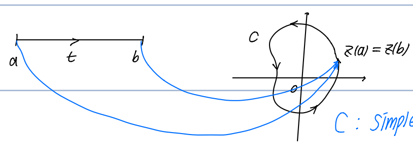
(ex)



C_2 : Not Simple arc.

def) The arc $C: z(t) = x(t) + iy(t) \quad (a \le t \le b)$: Simple Closed curve or simple closed Contour

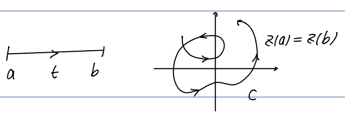
⟺ C: simple curve except for $z(a) = z(b)$.



C : Simple Closed curve

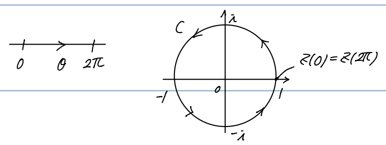
(단순폐곡선)

(ex)



C : Closed but not Simple.

(EX) $C: z = e^{i\theta} \quad (0 \le \theta \le 2\pi)$



C: Simple Closed Curve

def) ① $C: z(t) \quad (a \le t \le b)$: Contour

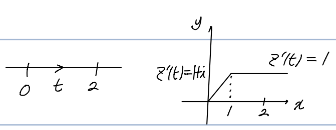
⟺ C: piecewise smooth arc

($C: z(t)$, $z(t)$: Conti , $z'(t)$: Piecewise conti)

② $C: z(t) \quad (a \le t \le b)$: Simple Closed contour

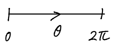
⟺ C: simple arc except for $z(a) = z(b)$

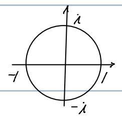
(EX) $z = \begin{cases} x + ix, & 0 \le x \le 1 \\ x + i, & 1 \le x \le 2 \end{cases}$



C: Simple arc ; Contour & Not Closed.

(EX) $C: z = e^{i\theta}\ (0 \le \theta \le 2\pi)$





C: Simple Closed Contour.