Vector Analysis.

Summary: Vectors and vector space.

If $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

1. Vector equality : $x = y \Leftrightarrow x_i = y_i$ $(i = 1, 2, 3)$
2. Vector addition : $x + y = z \Leftrightarrow x_i + y_i = z_i$ (")
3. Scalar multiplication : $ax \Leftrightarrow (ax_1, ax_2, ax_3)$ $a \in \mathbb{R}$
4. Negative of vector : $-x = (-1)x \Leftrightarrow (-x_1, -x_2, -x_3)$
5. Null vector : There exists a null vector $0 = (0, 0, 0)$

Since our vector components are real num (or complex), the followings hold.

1. addition of vector is commutative : $x + y = y + x$
2. " associative : $(x+y)+z = x+(y+z)$
3. Scalar multiplication is distribute : $a(x+y) = ax + ay$
4. Scalar multiplication is associative : $(ab)x = a(bx)$

Def 1.6) Fundamental Unit Vectors.

$e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$ $\cdots$ $e_n = (0, \cdots, 1)$

→ Fundamental unit vector of vector space $\mathbb{R}^n$.

Thm 1.3) Linear Combination

Any vector of $\mathbb{R}^n$ can be expressed by Linear combination of e_i.

$\Rightarrow v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, $v = \sum_{i=1}^{n} v_i e_i$

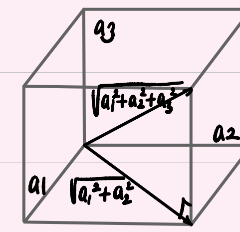
1.2 Inner product (내적)

Thm 1.5) Cosine Rule

세 변 길이 a, b, c $\angle C = \theta$ 인 임의의 삼각형 ΔABC

$c^2 = a^2 + b^2 - 2ab\cos\theta$

Thm 1.6) Norm of a vector



Vector of $\mathbb{R}^3$, $a = (a_1, a_2, a_3)$

length of a $\|a\| := \sqrt{a_1^2 + a_2^2 + a_3^2}$

Thm 1.7) Norm Property (a, b : vector, α : scalar)

(1) $\|a\| = 0 \Leftrightarrow a = 0$

(2) $\|\alpha a\| = |\alpha| \, \|a\|$

(3) $\|a + b\| \le \|a\| + \|b\|$

Def 1.7) Inner product

$\angle(a, b) = \theta$ $(0 \le \theta \le \pi)$,

define $a \cdot b = \|a\| \|b\| \cos\theta$

$= 0 \Leftrightarrow$ a or b is zero vector

Thm 1.8) Cosine Rule

$\forall a, b$: vector, $a \cdot b = \frac{1}{2}(\|a\|^2 + \|b\|^2 - \|b - a\|^2)$

Thm 1.9) Parallel and orthogonal Property

a, b are not zero vector, e_i : unit vector

(1) $e_i \cdot e_j = \delta_{ij}$, $\delta_{ij} = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$

(2) $a \parallel b \Leftrightarrow a \cdot b = \pm \|a\| \|b\|$

(3) $a \perp b \Leftrightarrow a \cdot b = 0$

Thm 1.10) Inner product

$a = (a_1, a_2, a_3)$ $b = (b_1, b_2, b_3)$

$\Rightarrow a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

Thm 1.11) Inner Product Property

(1) $a \cdot b = b \cdot a$ (2) $a \cdot (b + c) = a \cdot b + a \cdot c$

(3) $(\alpha a) \cdot b = a \cdot (\alpha b) = \alpha (a \cdot b)$ (4) $\|a\| = \sqrt{a \cdot a}$

Thm 1.12) Schwartz Ineq.

$a=(a_1,a_2,a_3)$ $b=(b_1,b_2,b_3)$.

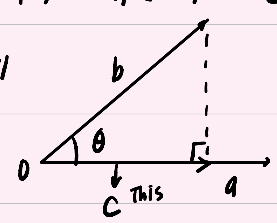
$|a\cdot b| \le \|a\|\|b\|$ (pf by def of inner product)

(Projection - 정사영)

Def 1.8) Orthogonal Projection

define) '벡터 b를 벡터 a가 놓여있는 직선에 수직으로 내린' 벡터 c를

a 위로 b의 정사영, $P_a b$ 로 표기



Thm 1.13) Orthogonal Projection

a,b are nonzero vector.

(1) 정사영 : $P_a b = \frac{\overbrace{b\cdot a}^{\text{inner product}}}{a\cdot a} a$

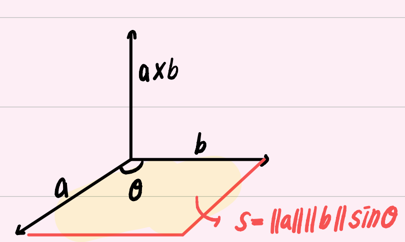
(2) length : $\|P_a b\| = \frac{|b\cdot a|}{\|a\|}$

1.3 벡터 외적

Def 1.9) Cross Product

끼인 각이 θ 인 a,b를 이웃변으로 하는 평행사변형의 넓이, 즉

$\|a\|\|b\|\sin\theta$ 를 크기로 하고 양의 방향을 갖는 벡터를 a,b의 외적.



Thm 1.14) Cross Product Property

(1) $a\times b = -b\times a$

(2) $(\alpha a)\times b = \alpha(a\times b) = a\times(\alpha b)$

(3) $a\times a = 0$

(4) $e_1\times e_2 = e_3$, $e_2\times e_3 = e_1$, $e_3\times e_1 = e_2$

(5) $a\times(b+c) = a\times b + a\times c$

Thm 1.15) 벡터 a,b,c를 이웃변으로 하는 평행육면체 부피는 $|a\times b\cdot c|$

Thm 1.16) $a\times b\cdot c = b\times c\cdot a = c\times a\cdot b$

Rmk) 내적·외적 동시에 계산 시 외적 먼저 계산함.

Def 1.10) Triple Scalar Product.

$a\times b\cdot c$ 를 벡터 a,b,c의 스칼라 3중적

Thm 1.17) Triple Scalar Product Property

(1) $a\times b\cdot a = a\times b\cdot b = 0$

(2) $a\times b\cdot c = a\cdot b\times c$

(3) $a\cdot b\times c = b\cdot c\times a = c\cdot a\times b$

Thm 1.18) $a=(a_1,a_2,a_3)$, $b=(b_1,b_2,b_3)$, $c=(c_1,c_2,c_3)$

(1) $a\times b = (a_2b_3-a_3b_3,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)$

(2) $a\times b\cdot c = (a_2b_3-a_3b_3)\cdot c_1 + (a_3b_1-a_1b_3)\cdot c_2 + (a_1b_2-a_2b_1)c_3$

※ EX3) $a=(1,1,-2)$, $b=(2,2,1)$ 에 동시에 수직인 unit vector?

$$a\times b = \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = (1+4, -(1+4), 2-2) = (5,-5,0)$$

는 a,b에 동시에 수직이고 length는 $\sqrt{25+25} = 5\sqrt{2}$

$\Rightarrow \pm\frac{a\times b}{5\sqrt{2}} = \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

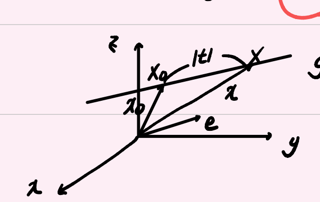
EX4) 벡터 a,b를 이웃변으로 하는 삼각형의 넓이? $\frac{1}{2}\|a\times b\|$

EX5) 세 점 P,Q,R을 잇는 △PQR의 넓이? put $a=\overrightarrow{PQ}$, $b=\overrightarrow{PR}$, 〃

1.4 공간의 직선과 평면

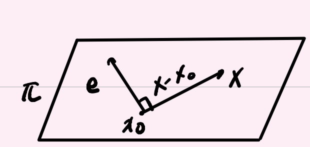
Thm 1.19) 한 점 x_0 지나고 단위벡터 e와 평행한 직선의 벡터 방정식

: $x = x_0 + te$, $t\in\mathbb{R}$



Thm 1.20) 한 점 x_0 지나고 단위벡터 e에 수직인 평면 π의 벡터방정식

: $(x-x_0)\cdot e = 0$



Ex8) 동일직선상에 있지 않는 영아닌 벡터 a,b,c 가 한 평면 위에 있을 때.

평면의 방정식? 두벡터 b-a, c-a 는 평행 X

(b-a) X (c-a) 가 주어진 평면에 직교하는 영 아닌 벡터 (=법벡터)

$\Rightarrow \quad (x-a)\cdot(b-a)\times(c-a) = 0$

1.5 선형변환

Def 1.11) Linear Transform

$L: \mathbb{R}^n \to \mathbb{R}^m$, L은 선형변환

$L(x_1, \cdots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, \ \cdots, \ a_{m1}x_1 + \cdots + a_{mn}x_n)$

Specially, m=1 인 경우 선형함수.

Def 1.12) determinant

Square matrix $A = [a_{ij}]$ 을 표현행렬로 하는 선형변환 L 각각의 길이, 넓이 or 부피가 $a(>0)$ 배로 늘어날 때,

define $|A| = \begin{cases} a & : \text{"+" oriented} \\ -a & : \text{"−" oriented} \end{cases}$

<Linear Algebra - Anton 11> Chapter 8 Linear Transformations

Def 1) If $T: V \to W$ is a ft from a vector space V to W,

then T is called a linear transformation from V to W

if (i) $T(ku) = kT(u)$ [Homogeneity property]

(ii) $T(u+v) = T(u) + T(v)$ [Additivity property]

In the special case where V = W,

the linear transformation T is called a linear operator on V.

Rmk) $T(k_1v_1 + k_2v_2 + \cdots + k_rv_r) = k_1T(v_1) + \cdots + k_rT(v_r)$

Thm 8.1.1) If $T: V \to W$ is a linear transformation, then

(a) $T(0) = 0$

(b) $T(u-v) = T(u) - T(v)$ for $\forall u, v \in V$

Thm 8.1.2) Let $T: V \to W$ be a linear transformation, where

V is finite dimensional.

If $S = \{v_1, v_2, \cdots, v_n\}$ (is a basis for V), then the image of any vector v in V

can be expressed as $T(v) = c_1T(v_1) + \cdots + c_nT(v_n)$

where $c_1, c_2, \cdots, c_n$ are coefficients required to express v

as a linear combination of vector in S.

(EX 10) Computing with Images of Basis Vectors.

Consider the basis $S = \{v_1, v_2, v_3\}$ for $\mathbb{R}^3$, where

$v_1 = (1,1,1), \quad v_2 = (1,1,0), \quad v_3 = (1,0,0)$

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation for which

$T(v_1) = (1,0), \quad T(v_2) = (2,-1), \quad T(v_3) = (4,3)$

Find a fomula for $T(x_1, x_2, x_3)$, and then use that fomula

to compute $T(2,-3,5)$.

(sol) We need to express $X = (x_1, x_2, x_3)$ as a linear combina-

-tion of v_1, v_2, and v_3.

$(x_1, x_2, x_3) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$$\begin{cases} c_1 + c_2 + c_3 = x_1 \\ c_1 + c_2 = x_2 \\ c_1 = x_3 \end{cases} \Rightarrow \quad c_1 = x_3, \ c_2 = x_2 - x_3, \ c_3 = x_1 - x_2$$

So $(x_1, x_2, x_3) = x_3(1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$

$= x_3v_1 + (x_2 - x_3)v_2 + (x_1 - x_2)v_3$

Thus $T(x_1, x_2, x_3) = x_3T(v_1) + (x_2 - x_3)T(v_2) + (x_1 - x_2)T(v_3)$

$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$

$= (4x_1 - 2x_2 - x_3, \ 3x_1 - 4x_2 + x_3)$

From this fomula, we obtain $T(2,-3,5) = (9, 23)$

- kernel and Range

Def 2) If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into 0 is called the kernel of T and is denoted by $\ker(T)$.
The set of all vectors in W that are images under T of at least one vector in V is called the range of T and is denoted by $R(T)$.

Thm 8.1.3) If $T: V \to W$ is a linear transformation, then
(a) The kernel of T is a subspace of V.
(b) The range of T is a subspace of W.

Def 3) Let $T: V \to W$ be a linear transformation.
If the range of T is finite-dimensional, then its dimension is called the rank of T; and if the kernel of T is finite-dimensional, then its dimension is called the nullity of T. The rank of T is denoted by $\operatorname{rank}(T)$ and the nullity of T by nullity of T by $\operatorname{nullity}(T)$.

Thm 8.1.4) Dimension Thm for linear transformations
If $T: V \to W$ is a linear transformation from an n-dimentional vector space V to W, then $\operatorname{rank}(T) + \operatorname{nullity}(T) = n$.
//

Thm 1.21 Determinent of 2x2 matrix
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc$