The Riemann Integral

If $f : [a,b] \to \mathbb{R}$: bdd. then lower and upper integrals of f on $[a,b]$: always $\exists$ and satisfy $\underline{\int_a^b} f \le \overline{\int_a^b} f$

Def) Let f be a bdd real-valued ft
on the closed bdd interval $[a,b]$.

If $\underline{\int_a^b} f = \overline{\int_a^b} f$. then f : Riemann integrable or integrable on $[a,b]$.

Def) Let f bdd real-valued ft on closed bdd interval $[a,b]$
The upper and lower integrals of f.
denoted $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ are defined by

$$\overline{\int_a^b} f = \inf \{ \mathcal{U}(P,f) : P \text{ is a partition of } [a,b] \}$$

$$\underline{\int_a^b} f = \sup \{ \mathcal{L}(P,f) : P \text{ is a partition of } [a,b] \}$$

Since the $\{\mathcal{U}(P,f)\}$ and $\{\mathcal{L}(P,f)\}$ are nonempty · bdd.
the lower & upper integral of f alway $\exists$.

(b) let $f: [0,1] \to \mathbb{R}$ be defined by

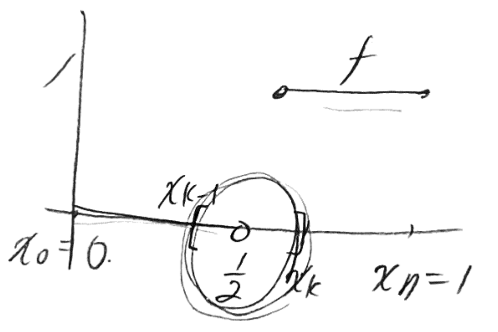
$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases} \to f \in R[0,1] \ \& \ \int_0^1 f = \frac{1}{2}.$$

(pb) Let $P = \{x_0, x_1, \dots, x_n\}$: partition of $[0,1]$

and let $k \in \{1, \dots, n\}$ be s.t. $x_{k-1} < \frac{1}{2} \le x_k$

If m_i and M_i denote infimum and supremum of f on $[x_{i-1}, x_i]$ respectively, then

$$m_i = \begin{cases} 0, & i = 1, \dots, k \\ 1, & i = k+1, \dots, n \end{cases} \quad \text{and} \quad M_i = \begin{cases} 0, & i = 1, \dots, k-1 \\ 1, & i = k, \dots, n. \end{cases}$$



Thus $L(P,f) = \sum_{i=k+1}^{n} \Delta x_i = 1 - x_k$

and $U(P,f) = \sum_{i=k}^{n} \Delta x_i = (1 - x_{k-1})$

Since $1 - x_k \le \frac{1}{2} < 1 - x_{k-1}$, $L(P,f) \le \frac{1}{2} \le U(P,f)$

for all partitions P of $[0,1]$.

Thus $\underline{\int_0^1} f \le \frac{1}{2} \le \overline{\int_0^1} f$.

Hence if $f \in R[0,1]$, then $\int_0^1 f = \frac{1}{2}$.

10) 체 F 위의 irreducible polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in F[x]$

단, $a_n \neq 0$ 에 대해 $f(x)$ 의 한 개의 root를 포함하는

F의 확대체가 존재함을 보이시오.

(pf) $f(x)$는 $F[x]$ 에서 기약다항식이므로 $F[x]/\langle f(x)\rangle$ 는 체.

Thm) 체 F에 대해 $p(x) \in F[x]$는 상수가 아닌 다항식이라 하자.

그러면 다음은 서로 동치.

① $p(x)$는 $F[x]$ 에서 기약

② $F[x]/\langle p(x)\rangle$ 는 체

F는 $F' = \{a + \langle f(x)\rangle \mid a \in F\}$ 와 환동형이고 $F[x]/\langle f(x)\rangle$ 는 F'을 포함

하므로 $F[x]/\langle f(x)\rangle$ 는 F를 포함하는 확대체로 생각할 수 있다.

그러면 $f(x+\langle f(x)\rangle) = (a_n + \langle f(x)\rangle)(x+\langle f(x)\rangle)^n +$

$\cdots + (a_1 + \langle f(x)\rangle)(x+\langle f(x)\rangle) + (a_0 + \langle f(x)\rangle)$

$= a_n x^n + \cdots + a_1 x + a_0 + \langle f(x)\rangle = \langle f(x)\rangle$

즉, $F[x]/\langle f(x)\rangle$ 는 $f(x)$의 근 $\neq x + \langle f(x)\rangle$를 포함하는

F 의 확대체 !