⇐) A, B are such sets.

⇒ $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Y는 A와 B의 union

⇒ $\overline{A} \cap Y = A$, $\overline{B} \cap Y = B$.

⇒ A and B are closed in Y, and so they are open in Y, and so they are open in Y as well. ∎

Lemma 23.2. C, D : separation of X

$Y \subset X$ a connected subspace

separation ? ① non-empty ② open ③ $U \cup V = X$ ④ $U \cap V = \emptyset$

Then, $Y \subset C$ or $Y \subset D$

separation 두번째 정의

(pf) Since C, D are open in X, $C \cap Y$ and $D \cap Y$ are open in Y.

And $(C \cap Y) \cup (D \cap Y) = Y$, and $(C \cap Y) \cap (D \cap Y) = \emptyset$

$C \cup D = X$ 인데, Y는 X의 subspace 이므로

$C \cap D = \emptyset$ 이므로 $\emptyset \cap Y = \emptyset$

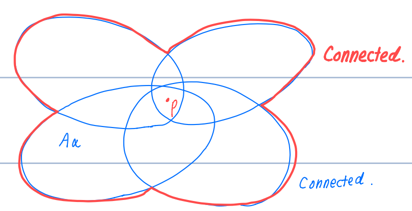
But Y is connected, so one of them is empty.

Therefore, $Y \subset C$ or $Y \subset D$ ∎

Thm 23.3. The union of a collection of connected subspaces of X that have a point in common is connected.

(pf) Let $\{A_\alpha\}$ be a collection of connected subspaces with $p \in A_\alpha$.

Assume that C and D form a separation of $Y = \bigcup A_\alpha$.



Without loss of generality, we assume $p \in C$. → C, D가 separation 이므로 점 p는 한 곳에만 포함되어야 함.

Since A_α is connected, $A_\alpha \subset C$ for all α by lemma 23.2.

Therefore, $Y = \bigcup A_\alpha \subset C$, a contradiction. ∎

Thm 23.4. $A \subset X$ a connected subspace

※ A가 Connected 이면 $\overline{A}$도 connected.
$(\because A \subset \underset{B}{\overline{A}} \subset \overline{A}$ 이라 보면 $\overline{A}$: Connected$)$

$A \subset B \subset \overline{A}$ ⇒ B is connected.

(pf) Assume that C and D form a separation of B.

By Lemma 23.2, we may assume $A \subset C$, implying $\overline{A} \subset \overline{C}$.

But $\bar{C} \cap D = \varnothing$, and so $B \cap D = \varnothing$.

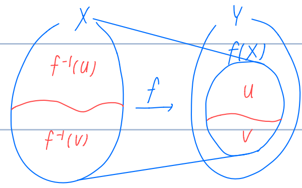
$B \subset \bar{A}$ 이고 $\bar{A} \subset \bar{C}$

$\Rightarrow B \subset \bar{C}$ 이므로 $\bar{C} \cap D = \varnothing \Rightarrow B \cap D = \varnothing$

B^c

This implies $D = \varnothing$, a contradiction.

nonempty set 이어야 separation.



Thm 23.5. $f: X \to Y$ continuous.

X is connected $\Rightarrow$ $f(X)$ is connected.

(pf) Assume for contradiction that $f(X)$ is not connected; let U, V be a separation of $f(X)$.

Now consider $f^{-1}(U)$ and $f^{-1}(V)$ in X.

nonempty set 이므로

- $f^{-1}(U)$, $f^{-1}(V)$ ② open ($\because f$ continuous), ① nonempty ($\because U, V \subset f(X)$)

- $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\varnothing) = \varnothing$ ④

$x \in$ $\Rightarrow x \in f^{-1}(U) \wedge x \in f^{-1}(V)$

- $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(f(X)) = X$ ③

$x \in$ $\Rightarrow x \in f^{-1}(U) \vee x \in f^{-1}(V)$

Therefore, the pair $f^{-1}(U)$ and $f^{-1}(V)$ is a separation of X, a contradiction. ∎

· Exercise §23

#2. #4. #7. #11

2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

3. Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

4. Show that if X is an infinite set, it is connected in the finite complement topology.

5. A space is ***totally disconnected*** if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects Bd A.

7. Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

11. Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

§ 23 Connected spaces Exercise.

2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

(pf) Using induction and [1. Thm 23.3] we see that $A(n) = A_1 \cup A_2 \cup \cdots \cup A_n$ is connected for all $n \geq 1$.

Since the spaces $A(n)$ have a point in common, namely any point of A_1, their union $\bigcup A(n) = \bigcup A_n$

is connected by [1. Thm 23.3] again. ∎

4. Show that if X is an infinite set, it is connected in the finite complement topology.

(pf) (Morten Poulsen) Suppose $\emptyset \subsetneq A \subsetneq X$ is open and closed.

Since A is open it follows that $X-A$ is finite.

Since A is closed it follows that $X-A$ is open, hence $X-(X-A) = A$ is finite.

Now, $X = A \cup (X-A)$ is finite, contradicting X is infinite.

Thus X and $\emptyset$ are the only subsets of X that are both open and closed, hence X is connected. ∎

7. Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

(pf) $\mathbb{R} = (-\infty, r) \cup [r, \infty)$ is a separation of $\mathbb{R}_\ell$ for any real number r.

It follows [1. Thm 23.1] that any subspace of $\mathbb{R}_\ell$ containing more than one point is disconnected

$\therefore$ $\mathbb{R}_\ell$ is totally disconnected. ∎

11. Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

(pf) Let $X = C \cup D$ be a separation of X.

Since fibres are connected, $p^{-1}(p(x)) \subset C$ for any $x \in C$ and

$p^{-1}(p(x)) \subset D$ for any $x \in D$ (1. Lemma 23.2)

Thus C and D are saturated open disjoint subspace of X and

therefore $p(C)$ and $p(D)$ are open disjoint subspace of Y.

In other words, $Y = p(C) \cup p(D)$ is a separation. $(\rightarrow\leftarrow)$

$\therefore$ X is connected ∎

§ 26. Compact Spaces.

(In analysis) $f: \boxed{[0,1]} \to \mathbb{R}$ continuous — §27 (정리) MVT, UCT (conti ⇒ uniformly conti)

In $\mathbb{R}_s$. 'closed + bounded' ⇔ 'compact'

Earlier definition of compactness came from the crucial point property of $[a,b]$;

'$\boxed{\text{every infinite subset has a limit point.}}$' → limit point compact $\underset{X}{\overset{O}{\rightleftarrows}}$ compact

↳ limit point compact = compact in $\mathbb{R}_s$ (아래 예제)

§28 $\boxed{\text{metric space, } \mathbb{R}_s}$

Def) $(X, \mathcal{T})$ a topological space

$\mathcal{A}$ 'open covering' of X : a collection of open subsets of X whose union is X.

→ X 에 있는 어떤 점도 $\mathcal{A}$에 있는 어떤 open set 하나에 의해서 포함 되어진다.
X 의 모든 점들을 포함하는 open set 들이 최소 하나는 있다.

$\mathcal{A}$의 subset 인데 여전히 union 하면 X를 cover 하는.
subcollection of $\mathcal{A}$ whose union is X.

X is said to be 'compact' if 'every' open covering $\mathcal{A}$ of X has a 'finite' subcovering of X.

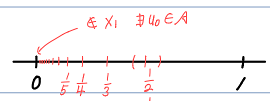
'임의의' open covering 에 대해 항상 'finite' 개의 subcovering 을 찾을 수 있다.

not compact if ∃ open covering $\mathcal{A}$ of X which does not have a finite subcovering of X.

→ finite subcover 로는 절대 X를 cover 할 수 없는 어떤 covering $\mathcal{A}$가 존재.

→ not compact 임을 보이기 쉬움.

Ex 1. $\{\frac{1}{n} \mid n=1,2,3,\cdots\}$ in $\mathbb{R}_s$: not compact



∵ ∃ open covering $\mathcal{A} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ which has no finite subcovering.

X 를 cover ! open subsets

finite 개로는 절대 X를 cover 불가.

㊟ limit point compact 개념으로 이해.

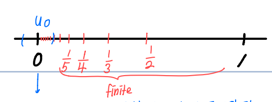
$X_1 = \{\frac{1}{n} \mid n=1,2,\cdots\}$
$X_2 = \{0\} \cup \{\frac{1}{n} \mid n=1,2,\cdots\}$ 라고 두면.

limit pt 는 $0 \notin X_1$, $0 \in X_2$ 이므로 X_1 : Not Compact 이다. X_2 : Compact

$\{0\} \cup \{\frac{1}{n} \mid n=1,2,3,\cdots\}$ in $\mathbb{R}_s$: compact

∵ Given any open covering $\mathcal{A}$, ∃ $U_0 \in \mathcal{A}$ contain 0.

Then U_0 contains all but finitely many of $\frac{1}{n}$'s.



n을 충분히 크게 해주면 U_0 안에 들어갈 것

→ U_0 안에 들어가지 않는 것들은 경우 finite개뿐!

Choose, for each $\frac{1}{n} \notin U_0$, an element of $\mathcal{A}$ containing $\frac{1}{n}$.

U_0 에 포함되지 않은 $\frac{1}{n}$ 들을 포함하기 위해서는 finite 개만 있으면 충분
U_0와 U_0 에 포함되지 않은 $\frac{1}{n}$ 이라는 점들을 포함하고 있는 finite 개의 open set 들을 다 union 하면 X_2 전체를 cover.

This collection along with U_0 is a finite subcovering.

㊟ 주의 X : Compact ?

$\mathcal{A} = \{X\}$ open covering, finite.

이건 특별한 open covering 이지 일반적인 open covering 이 아님.

∴ '어떤' open covering 이 주어지더라도 !!

Ex 2. $(0,1]$ in $\mathbb{R}_s$: Not compact (bounded , but not closed)

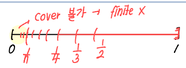
(열)

$\because$ $\exists$ open covering $A = \{ (\frac{1}{n}, 1] \mid n \in \mathbb{Z}_+ \}$ which has no finite subcovering.

smallest $\frac{1}{n}$

$\bigcup_{n=2} (\frac{1}{n}, 1] = (0,1]$ → open set 이면서 covering.

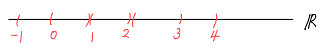
open subsets in $(0,1]$



$\mathbb{R}_s$: not compact (closed. but not bounded)

$\because$ $\exists$ open covering $A = \{ (n, n+2) \mid n \in \mathbb{Z} \}$ which has no finite subcovering.

(or. $A = \{ (-n, n) \mid n \in \mathbb{Z}_+ \}$



finite subcover 갖지 않는 open covering $(n, n+2)$.

Ex 3. $[0,1]$ in $\mathbb{R}_s$: compact (closed + bounded)

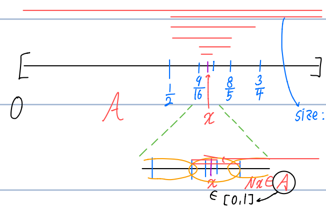
Proof in Analysis.

Assume that $[0,1]$ is not compact.

→ $\exists$ open covering A

$\nexists$ finite subcovering

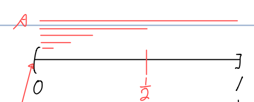
need infinitely many of A to cover



size : $\frac{1}{2}$씩 감소

finite 개로 절대 할수 없다고 했는데

x 한개로 표현되었다. (ㄱㄴ)



0 이 빠져있을 때 → x 가 0 이 될수 있는데,

0 이 포함되어 있지 않아서 이런 방식으로는 증명 불가.

$x = 0 \notin (0,1]$

$N_x \in A$? Not ensure.

not compact

· Exercise §26 #1

1. (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

(b) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

§ 26. Compact spaces Exercise.

1. (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

(b) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

(pf) (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Suppose $\mathcal{T}' \supset \mathcal{T}$.

If $(X, \mathcal{T}')$ is compact then $(X, \mathcal{T})$ is compact : Clear, since every open covering if $(X, \mathcal{T})$ is an open covering in $(X, \mathcal{T}')$

If $(X, \mathcal{T})$ is compact then $(X, \mathcal{T}')$ is in general not compact ;

Consider $[0,1]$ in standard topology and discrete topology. ∎

(b) If $(X, \mathcal{T})$ compact and $\mathcal{T}' \supset \mathcal{T}$ then the identity map $(X, \mathcal{T}') \to (X, \mathcal{T})$ is a bijective continuous map.

hence a homeomorphism. by Thm 26.6. This proves the result. ∎