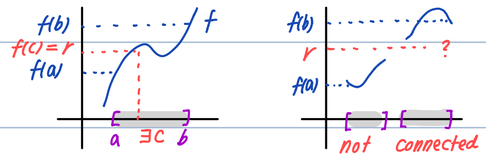
Chapter 3. Connectedness and Compactness

23. Connected space.

3 basic theorems about continuous functions $f : [a,b] \to \mathbb{R}$ in Calculus :

∘ Intermediate Value Theorem

$\forall r \in [f(a), f(b)],\ \exists c \in [a,b]$ s.t $f(c) = r$

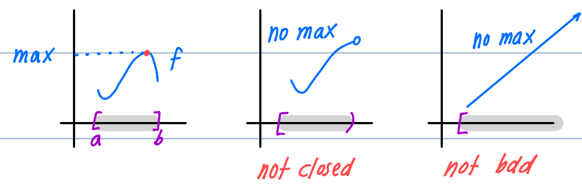


→ Connected 가 아니면
I-V-T 만족하지 않는 경우가 생긴다.
Connected 는 중요한 조건.

* use $[a,b]$: connected.

∘ Maximum Value Theorem

f has a maximum on $[a,b]$



* use $[a,b]$: closed + bdd

∘ Unifor Continuity Theorem.

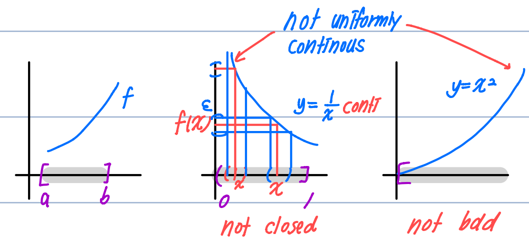
f : Conti at x

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t $|x_1 - x| < \delta \Rightarrow |f(x_1) - f(x)| < \varepsilon$

$\delta(x, \varepsilon)$

$\delta(\varepsilon)$

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.



* use $[a,b]$: closed + bdd

• f : Conti $\Rightarrow$ f : unifly conti

domain $[a,b] \to$ closed + bdd

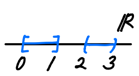
$\Rightarrow$ { 'connected'

'closed + bdd' ~ 'compact' (⇔ in $\mathbb{R}s$ and ⇐ in Topology)

metric

Hausdorff

§ 23. Connected Spaces.



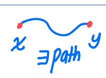
$X = [0,1] \cup (2,3)$ not connected in $\mathbb{R}s$

Why ? → 한 덩어리가 아니다.

덩어리들 정의 ? both open and closed ⇒ connected.

→ 연결되어 있지 않다.

정의 ?



⇒ path - connected .

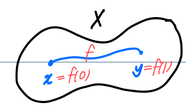
Def) (1) X 'connected' if there does not exist a 'separation' of X,

which is a pair U, V of ① nonempty ② open "closed" subsets s.t ③ $U \cup V = X$ and ④ $U \cap V = \emptyset$ disjoint.

(or, equivalently if $\emptyset, X$ are the only open and closed subsets.)

($U, X-U = V \rightarrow$ seperation)

(2) X 'path-connected' if every pair of points of X can be joined by a 'path' in X



where a path from x to y is a continuous map.

$f: [0,1] \rightarrow X$ s.t $f(0) = x$ and $f(1) = y$

Thm. X is path-connected $\overset{\not\Leftarrow}{\Rightarrow}$ X is connected.

① non-empty
② open
③ $U \cup V = X$
④ $U \cap V = \emptyset$

(pf) (귀류) Assume that X is not connected

Let U, V form a separation of X, and choose $x \in U$ and $y \in V$.

Since X is path-connected, $\exists$ a path $f: [0,1] \rightarrow X$ s.t $f(0) = x$, $f(1) = y$

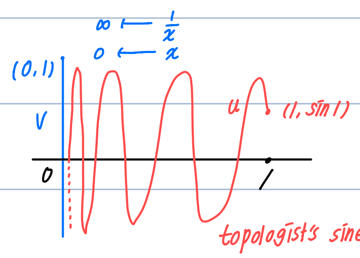
Since $[0,1]$ is connected, $f([0,1])$ is connected by Thm 23.5.

Now $f([0,1]) \cap U$ and $f([0,1]) \cap V$ are disjoint open subsets and there union is $f([0,1])$. → ②③④ 조건 만족.

Since $f([0,1])$ is connected, one of them must be empty.

But $x \in f([0,1]) \cap U$ and $y \in f([0,1]) \cap V$, a contradiction.

• The converse does not hold.



(Ex1) $X = \overset{U}{\{(x, \sin\frac{1}{x}) \mid 0 < x \le 1\}} \cup \overset{V}{(0 \times [-1,1])}$

$\boxed{\text{Connected}}$ because V is not open.

U open, not closed
V closed, not open

not path-connected because no path from $(0,1)$ to $(1, \sin 1)$.

(Ex) # 1

SEC 23.

(Ex) # 1

1. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$, what does connectedness of X in one topology imply about connectedness in the other?

(pf) Any seperation $X = U \cup V$ of $(X, \mathcal{T})$ is also a separation of $(X, \mathcal{T}')$.

This means that $(X, \mathcal{T})$ is disconnected $\Rightarrow$ $(X, \mathcal{T}')$ is disconnected

or, equivalently, $(X, \mathcal{T}')$ is connected $\Rightarrow$ $(X, \mathcal{T})$ is connected

when $\mathcal{T}' \supset \mathcal{T}$.