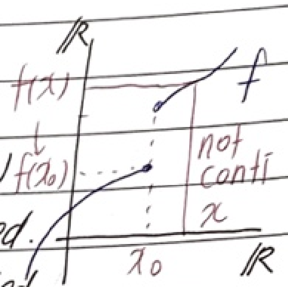
Chapter 2. Topological Spaces and Continuous Functions

§12 Topological spaces

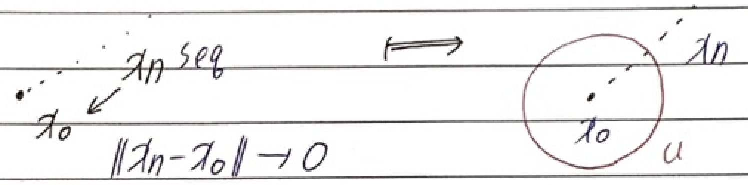
The definition of topological space was formulated:

- as broad as possible to include various models that were useful in mathematics, and
- narrow enough so that the standard theorems would hold for topological space in general



key point is the def of "Continuity":

- In Analysis ($\mathbb{R}$) we use 'limit' (or ε-δ def) ← 'metric' is needed.
- In Topology, we use 'open' ← 'metric' is not needed.



$\forall$ open $U \ni x_0$, $U \ni x_n$ for some n.

ε-nbd of x

$N_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$

$N_\varepsilon(x) \subset U$

interior pt.

Properties of open sets in $\mathbb{R}$ (in Analysis):

$(0,1)$ open, $[2,3)$ not open

(1) $\emptyset$ and $\mathbb{R}$ are open

(2) U_α is open for all $\alpha \in J$ ⇒ arbitrary union $\bigcup_{\alpha \in J} U_\alpha$ is open.

(3) U_i is open for all $i = 1, 2, \cdots n$ ⇒ finite intersection $\bigcap_{i=1}^{n} U_i$ is open.

⊛ $U_1 \cup U_2$, $\bigcup_{i=1}^{n} U_i$ finite $\cup$

$\bigcup_{i=1}^{\infty} U_i$ Countable $\cup$

$\bigcup_{\alpha \in J} U_\alpha$ arbitrary $\cup$

* Note that for open sets U_α, arbitrary intersection $\bigcap_{\alpha \in J} U_\alpha$ may not be open.

$\because \bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, \frac{1}{i}\right) = \underbrace{\{0\}}_{\text{not open}}$

$x \neq 0$, $\not\ni$; $\ni 0$

$-1 \quad -\frac{1}{2} \quad -\frac{1}{i} \quad 0 \quad \frac{1}{i} \quad \frac{1}{2} \quad 1 \quad \mathbb{R}$

$0 < x$, $\frac{1}{x} < i$

$\frac{1}{i} < x$

"open sets"

Def) $\mathcal{T}$ 'topology' on X : a collection of subsets of X s.t

(1) $\mathcal{T} \ni \emptyset, X$ (즉 open 으로 보겠다)

(2) $\mathcal{T} \ni$ arbitrary union of elements of $\mathcal{T}$ (or, $\bigcup_{\alpha \in J} U_\alpha$ for $U_\alpha \in \mathcal{T}$)

(3) $\mathcal{T} \ni$ finite intersection of elements of $\mathcal{T}$ (or, $\bigcap_{i=1}^{n} U_i$ for $U_i \in \mathcal{T}$)

Note that an element of $\mathcal{T}$ is called an 'openset'.

Topological space $(X, \mathcal{T})$: a pair of a set X and a topology $\mathcal{T} = \{$open sets$\}$.

(Ex 1) $X = \{a, b, c\}$ ← what is open? ③

a b c

① set

② Topology (given)

$\mathcal{T}$	Topology?	Condition	
$\{\emptyset, X\}$	Yes	(1)(2)(3)	$\{a\}$ open? No
$\{\emptyset, a, X\}$	Yes	(1)(2)(3)	$\{a\}$ open? Yes $\{b\}$? No
$\{\emptyset, a\}$	No	($\because$ (1) $\mathcal{T} \not\ni X$)	
$\{\emptyset, a, b, X\}$	No	($\because$ (2) $\mathcal{T} \not\ni \{a\} \cup \{b\}$)	$\{a, b\} \notin \mathcal{T}$ (1), (3)은 만족
$\{\emptyset, a, \{b, c\}, X\}$	Yes	(1)(2)(3)	$\{b\}$ open? No. $\{b, c\}$: open.

"largest" ← $\{\emptyset, a, b, c, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ Yes.

all subsets

$\{a\}$ open? Topology 에 따라 결정

"smallest" $\{\emptyset, X\}$ ← 2 open sets

(Ex 2) X any set

$\mathcal{T} = \{\emptyset, X\}$ ← trivial topology (smallest topology) only $\emptyset, X$ are open sets!

$\mathcal{T} = \{$all subsets of $X\}$ ← discrete topology (largest topology)

Any subset of X is open!

All elements of X are open!

no topology ⇒ open: X

(Ex 3) $X = \mathbb{R}$ as a set

$\mathcal{T} = \mathbb{R}_s$ 'Standard topology' on $\mathbb{R}$ ← in Analysis

- open sets : $\emptyset, \mathbb{R}, (1,2), (1,\infty), (1,2) \cup (3,5)$ $(a,b) \cup (c,d)$ open

- not open sets : $\{1\}, [1,2], [1,\infty), [1,2] \cup (3,5)$

(a,b)

$\mathcal{T} = \mathbb{R}_d$ 'discrete topology' on $\mathbb{R}$

- all subsets $(1,\infty), \{1\}, [1,2] \cup (3,5)$ are open.

All elements are open!

$\mathbb{R}_t$ 'trivial'

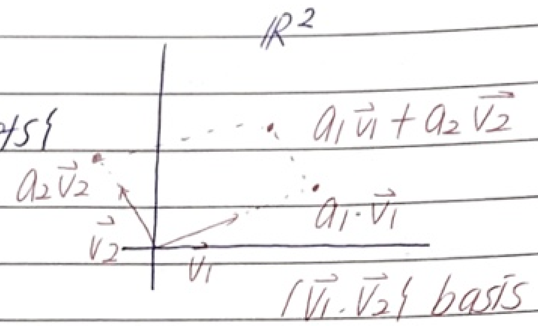
$\mathcal{T} = \mathbb{R}_t$ 'trivial topology' on $\mathbb{R}$

- only two subsets $\emptyset, \mathbb{R}$ are open.

* Be aware that open sets depend on their topologies!

SEC 12 (Excercises) 1, 3, 4

13. Basis for a Topology. = {open sets}



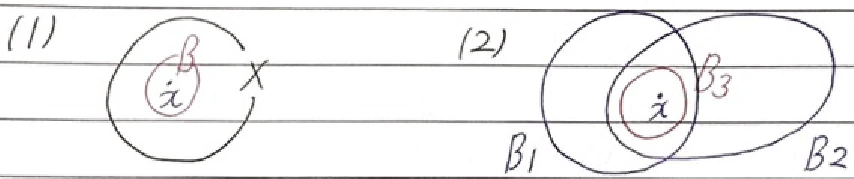
Def) X a set

$\mathcal{B}$ 'basis' for a topology on X:

a collection of subsets of X s.t.

(1) $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B$ ($\Leftarrow \bigcup B = X$)

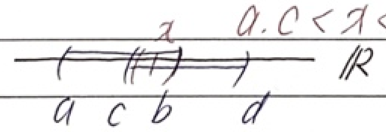
(2) $\forall x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$



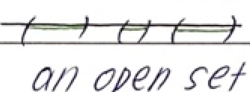
(Ex 1) $X = \mathbb{R}$

$\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R},\ a < b\}$: basis for the standard topology $\mathbb{R}_s$

- $\forall x \in \mathbb{R}$, $\exists (x-1, x+1) \in \mathcal{B}$ s.t. $x \in (x-1, x+1)$
- $\forall x \in (a,b) \cap (c,d)$, $\exists B_3 = (\max\{a,c\}, \min\{b,d\}) \in \mathcal{B}$ s.t. $x \in B_3 \subset (a,b) \cap (c,d)$



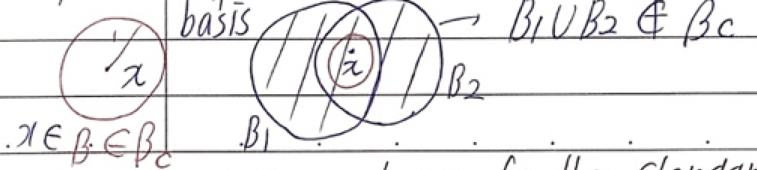
* An open set of $\mathbb{R}_s$ is a union of these basis elements.

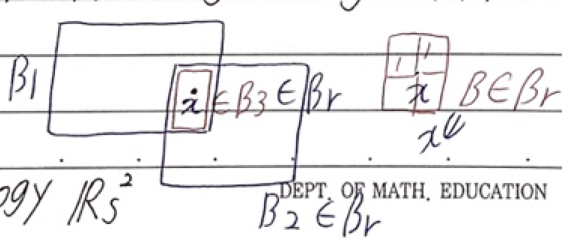


(Ex 2) $X = \mathbb{R}^2$

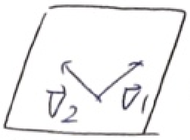
Topology? $\mathcal{B}_c$ = {circular regions (interiors)} $\mathcal{B}_r$ = {rectangular regions (int)}

basis

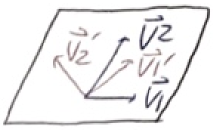




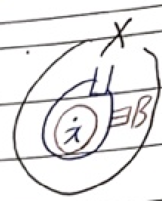
* Both are bases for the standard topology $\mathbb{R}_s^2$ that will be mentioned in Lemma 13.3



$\vec{v}_1, \vec{v}_2$ 로 만든 vector space ㄱ unique



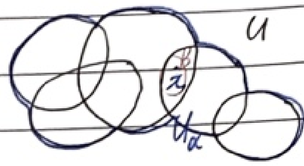
vector space를 span하는 basis는 여러개 찾을 수 있음



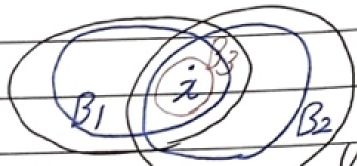
Topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ ← unique way.
$U \in \mathcal{T}$ (open in X) if $x \in U$, $\exists B \in \mathcal{B}$ s.t $x \in B \subset U$

Fact) Topology $\mathcal{T}$ generated by $\mathcal{B}$ is indeed a topology.

(proof) (1) $\mathcal{T} \ni \emptyset, X$
- $\emptyset$: no element, so trivial.
- X : $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t $x \in B$ by the 1st definition of basis and thus $x \in B \subset X$.



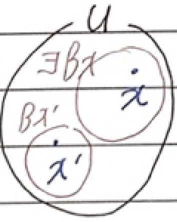
(2) $\mathcal{T} \ni U = \bigcup_{\alpha \in J} U_\alpha$ for $U_\alpha \in \mathcal{T}$
$\forall x \in U$, $\exists U_\alpha$ s.t $x \in U_\alpha$
Since $U_\alpha \in \mathcal{T}$, $\exists B \in \mathcal{B}$ s.t $x \in B \subset U_\alpha$ and thus $x \in B \subset U$
Therefore, $U \in \mathcal{T}$



(3) $\mathcal{T} \ni U = \bigcap_{i=1}^{n} U_i$ for $U_i \in \mathcal{T}$
First consider $U_1 \cap U_2$ and then $\forall x \in U_1 \cap U_2$.
$\exists B_1, B_2 \in \mathcal{B}$ s.t $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$.
$\exists B_3 \in \mathcal{B}$ s.t $x \in B_3 \subset B_1 \cap B_2$ by the 2nd def of basis.
and thus $x \in B_3 \subset U_1 \cap U_2$.
Therefore, $U_1 \cap U_2 \in \mathcal{T}$, and so $U \in \mathcal{T}$ by induction.

Lemma B.1 $\mathcal{T} = \{ \bigcup_{\alpha \in J} B_\alpha \mid B_\alpha \in \mathcal{B} \}$

(pf) $\supset$: trivial by def
$\subset$: $U \in \mathcal{T}$ 임을 보이자.



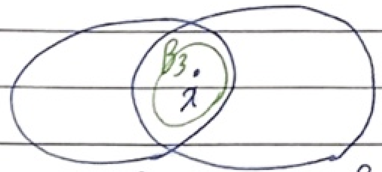
$$U = \overbrace{\bigcup_{x \in U} B_x}^{\subset U} \subset U$$
$$\underbrace{\bigcup_{x \in U} B_x}_{U_U}$$

Constructing a basis β for a topology $\mathcal{T}$ ← many ways

lemma 13.2 β is a collection of open sets of X s.t $\forall U \in \mathcal{T}$ and $\forall x \in U$. $B_1 \cap B_2$
$\exists B \in \beta$ s.t $x \in B \subset U$. ← $B_1 \cap B_2$ $B_1 \cap B_2$
Then β is a basis for $\mathcal{T}$.
B_3

(proof) (1) β is indeed a basis.
- $\forall x \in X$, $\exists B \in \beta$ s.t $x \in B$ since $X \in \mathcal{T}$.
- Consider any $x \in B_1 \cap B_2$ where $B_1, B_2 \in \beta$.

Since B_1 and B_2 are open, $B_1 \cap B_2$ is also open.
Therefore, $\exists B_3 \in \beta$ s.t $x \in B_3 \subset B_1 \cap B_2$.



B_1 B_2

(2) $\mathcal{T}$ is the topology generated by β.
Let $\mathcal{T}'$ be the topology generated by β.
We will show that $\mathcal{T} = \mathcal{T}'$
- $\mathcal{T} \subset \mathcal{T}'$: $\forall U \in \mathcal{T}$ and $\forall x \in U$, $\exists B \in \beta$ s.t $x \in B \subset U$.
  and thus $U \in \mathcal{T}'$
- $\mathcal{T} \supset \mathcal{T}'$: If $V \in \mathcal{T}'$, $V = \bigcup_{\alpha \in J} B_\alpha$ for some $B_\alpha \in \beta$ by Lemma 13.1
  Since all $B_\alpha \in \mathcal{T}$, $V \in \mathcal{T}$. ■

open sets

⟨Comparison of two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X⟩

$\mathcal{T} \subset \mathcal{T}'$: $\mathcal{T}'$ is larger (finer) than $\mathcal{T}$
$\mathcal{T}$ is smaller (coarser) than $\mathcal{T}'$.

(EX) $X = \{a, b, c\}$

$\{\emptyset, a, b, \{a,b\}, \{a,c\}, X\}$ — larger

$\subset$ $\supset$ comparable

$\{\emptyset, a, \{a,b\}, X\}$ — not comparable — $\{\emptyset, a, \{a,c\}, X\}$

$\cup$ $\subset$ $\cup$

$\{\emptyset, a, X\}$ $\{\emptyset, \{a,c\}, X\}$ — smaller

$\cup$ $\{\emptyset, X\}$ $\subset$

Lemma 13.3. $(X, \mathcal{T}, \mathcal{B})$, $(X, \mathcal{T}', \mathcal{B}')$: two topological spaces with bases

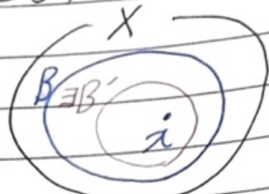
$\mathcal{T} \subset \mathcal{T}' \Leftrightarrow \forall x \in X$ and $\forall B \in \mathcal{B}$ containing x,

$\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$

(pf) ($\Rightarrow$) $\forall x \in X$, and $\forall B \in \mathcal{B}$ containing x, $B \in \mathcal{T}$,

and so $B \in \mathcal{T}'$ since $\mathcal{T} \subset \mathcal{T}'$.

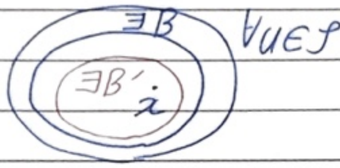
Therefore, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$



($\Leftarrow$) $\forall U \in \mathcal{T}$ and $\forall x \in U$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Then $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$, and so $x \in B' \subset U$

Therefore, $U \in \mathcal{T}'$

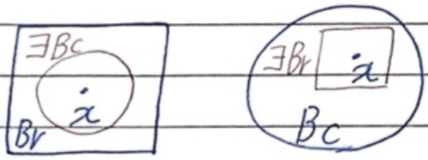


(EX) $X = \mathbb{R}^2$ $\quad \mathcal{B}_r \not\subset \mathcal{B}_c$, $\mathcal{B}_r \not\supset \mathcal{B}_c$.

$\mathcal{B}_c = \{$ circular regions $\} \rightarrow \mathcal{T}_c$: topology generated by $\mathcal{B}_c$

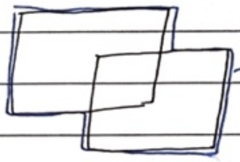
$\mathcal{B}_r = \{$ rectangular regions $\} \rightarrow \mathcal{T}_r$: topology generated by $\mathcal{B}_r$

Then $\mathcal{T}_c = \mathcal{T}_r$ (indeed, both are a standard topology $\mathbb{R}_s^2$)



"$\mathcal{T}_r \subset \mathcal{T}_c$" "$\mathcal{T}_c \subset \mathcal{T}_r$"

* Note that the set of all rectangular regions is not a topology



union is not rectangular anymore.

(EX) $X = \mathbb{R}$

$\mathcal{T}_t$: trivial topology $\{\emptyset, \mathbb{R}\}$

$\mathcal{T}_d$: discrete topology $\{$all subsets$\}$

$\mathcal{T}_s$: standard topology generated by $\beta_s = \{(a,b) \subset \mathbb{R} \mid a<b\}$

$\mathcal{T}_s \subset \mathcal{T}_l$: lower limit topology generated by $\beta_l = \{[a,b) \subset \mathbb{R} \mid a<b\}$

$(0,2)$ open? Yes

$$(0,2) = \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 2) \in \mathcal{T}_l$$

$\mathcal{T}_s \subset \mathcal{T}_K$: K-topology generated by $\beta_K = \{(a,b), (a,b)-K \subset \mathbb{R} \mid a<b\}$

where $K = \{\frac{1}{n} \mid n = 1, 2, \cdots\}$

$a<0<b$

$0 \quad \frac{1}{n} \cdots \frac{1}{4} \; \frac{1}{3} \; \frac{1}{2} \; 1 \leftarrow K$

$\mathcal{T}_s \supset \mathcal{T}_f$: finite complement topology $= \{U \subset \mathbb{R} \mid \mathbb{R} - U$ is finite or $\mathbb{R}\}$

$\emptyset, \mathbb{R}$

Then

$\mathcal{T}_d$

/ \

$\mathcal{T}_l \qquad \mathcal{T}_K$

\ /

$\mathcal{T}_s$

|

$\mathcal{T}_f$

|

$\mathcal{T}_t$

(1) $\mathcal{T}_l \supsetneq \mathcal{T}_s$

'$\supset$' Given $x \in (a,b) \in \beta_s$, $\exists [x,b) \in \beta_l$ s.t. $x \in [x,b) \subset (a,b)$

'$\not\subset$' For $0 \in [0,1) \in \beta_l$, $\nexists (a,b) \in \beta_s$ s.t. $0 \in (a,b) \subset [0,1)$.

(2) $\mathcal{T}_l$ and $\mathcal{T}_K$ are not comparable (lemma 13.4)

'$\not\supset$' For $0 \in (-1,1) - K \in \beta_K$, $\nexists [a,b) \in \beta_l$ s.t. $0 \in [a,b) \subset (-1,1) - K$

'$\not\subset$' For $2 \in [2,3) \in \beta_l$, $\nexists B_K = (a,b)$ or $(a,b) - K \in \beta_K$ s.t. $2 \in B_K \subset [2,3)$. $a<2$, $2<a$ $(-K)$

$*$ verify all the cases.

arb union $\quad$ finite intersection

$\mathcal{T} \supset \mathcal{B} \supset S$

① $\forall x \in X,\ \exists B \in \mathcal{B}$ s.t. $x \in B$ $(\Rightarrow \bigcup B = X)$

② $\forall x \in B_1 \cap B_2,\ \exists \underset{B_1 \cap B_2}{B_3} \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$

Def) S 'subbasis' for a topology on X : a collection of subsets of X whose union equals X.

$\mathcal{T} = \{$all unions of finite intersections of elements of $S\}$
'topology generated by S'

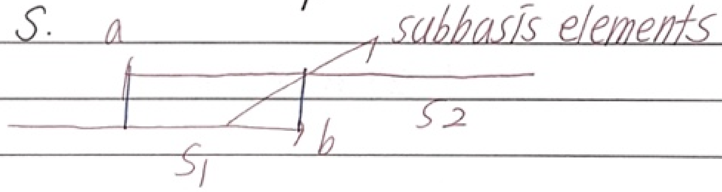
* Indeed. $\mathcal{B} = \{$all finite intersection of element of $S\}$ is a basis.

- Given $x \in X$, it belongs to an element of S and hence to an element of $\mathcal{B}$.
- For two elements $B_1 = S_1 \cap \cdots \cap S_m$ and $B_2 = S_1' \cap \cdots \cap S_n'$ of $\mathcal{B}$, $B_3 = \boxed{B_1 \cap B_2} = (S_1 \cap \cdots \cap S_m) \cap (S_1' \cap \cdots \cap S_n')$ is also a finite intersection of elements of S, so it belongs to $\mathcal{B}$.

Thus $\mathcal{T} = \{$all unions of elements of $\mathcal{B}\}$ is a topology by Lemma 13.1

(EX) $\mathbb{R}_s$: standard topology on $\mathbb{R}$ generated by a subbasis
$S = \{(a, \infty), (-\infty, b) \subset \mathbb{R} \mid a, b \in \mathbb{R}\}$

Here, we have a basis $\mathcal{B} = \{(a,b) \subset \mathbb{R} \mid a < b\}$ from the subbasis S.



SEC 13. (EX) #8

§14. Order Topology

simple order relation '<' : Comparability : $x \neq y \Rightarrow x < y$ or $y < x$

nonreflexivity : $x \nless x$

transitivity : $x<y,\ y<z \Rightarrow x<z$

X : a set with a simple order relation '<' and having at least two elements

$\beta = \{ (a,b),\ [a_0, b),\ (a, b_0] \subset X \mid a<b,$

a_0 : smallest elt (if any), b_0 : largest elt (if any)$\}$

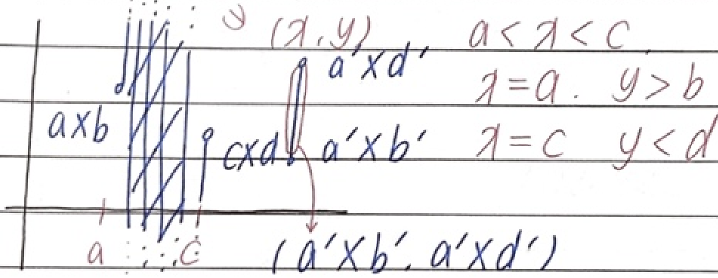
The topology generated by the basis β is called the 'order topology'

(Ex1) The standard topology $\mathbb{R}_s$ is just the order topology derived from the usual order on $\mathbb{R}$.

$\beta = \{(a,b) \subset \mathbb{R} \mid a<b\}$

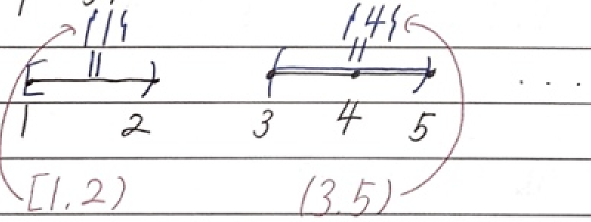
(Ex2) $\mathbb{R}\times\mathbb{R}$ in the dictionary order (no smallest or largest element).

$\beta = \{ (a\times b,\ c\times d) \subset \mathbb{R}\times\mathbb{R} \mid a<c$ or $(a=c$ and $b<d)\}$



(Ex3) $\mathbb{Z}_+ = \{$positive integers$\}$: ordered set with a smallest element $\{1\}$.

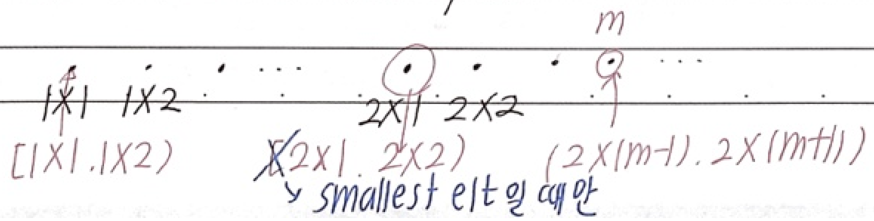
The topology on $\mathbb{Z}_+$ is the discrete topology, so every pt is open.



(Ex4) $X = \{1,2\}\times\mathbb{Z}_+$ in the dictionary order (1×1 is the smallest element)

Order : $1\times1,\ 1\times2,\ 1\times3,\ \cdots;\ 2\times1,\ 2\times2,\ 2\times3,\ \cdots$

The order topology on X is not discrete topology because $\{2\times1\}$ is not open.



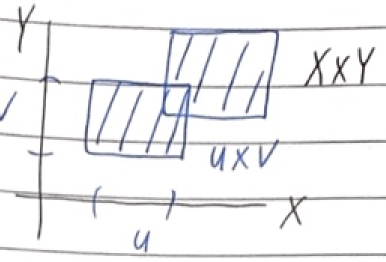
§15. Product Topology on $X \times Y$

X, Y : topological spaces

$\mathcal{B} = \{ U \times V \mid U$: open subsets of X, V : open subset of $Y \}$

: a basis

The topology generated by $\mathcal{B}$ is called the "product topology" on $X \times Y$

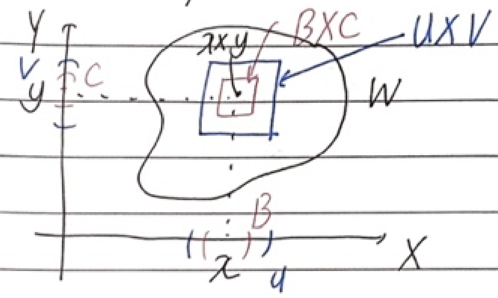


union 이 더이상 product 형태가 아니면
topology 성질 만족하지 않으므로,
union 정의할 필요 없는 basis 로 두자

같은 것

Thm 15.1. $\mathcal{B}, \mathcal{C}$: bases for the topologies of X, Y
Then $\mathcal{D} = \{ B \times C \mid B \in \mathcal{B}, C \in \mathcal{C} \}$ is a basis for the topology of $X \times Y$

(pf) Given open $W \subset X \times Y$ and $x \times y \in W$.
$\exists$ a basis element $U \times V$ s.t $x \times y \in U \times V \subset W$
where U is open in X and V is open in Y.

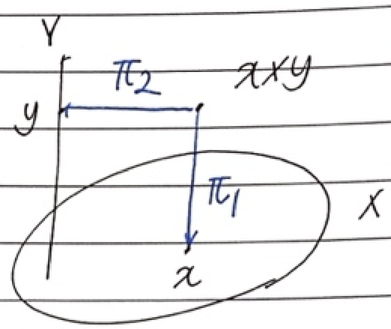


Now $\exists B \in \mathcal{B}$ s.t $x \in B \subset U$ and
$\exists C \in \mathcal{C}$ s.t $y \in C \subset V$.
Therefore, $x \times y \in B \times C \subset U \times V \subset W$ ∎

\- Projections of $X \times Y$

$\pi_1 : X \times Y \to X$ by $\pi_1(x, y) = x$

$\pi_2 : X \times Y \to Y$ by $\pi_2(x, y) = y$



SEC 16 Exercise 4. 6

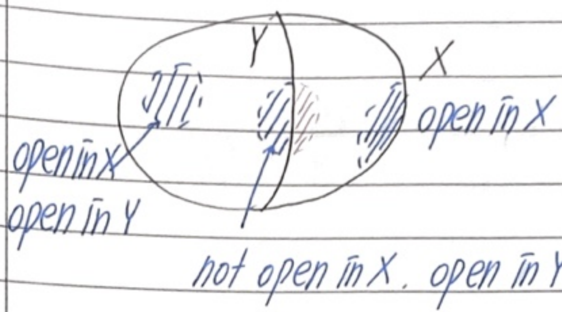
4. A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y.
Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

6. Show that the countable collection $\{ (a,b) \times (c,d) \mid a < b$ and $c < d$, and a, b, c, d are rational $\}$
is a basis for $\mathbb{R}^2$.

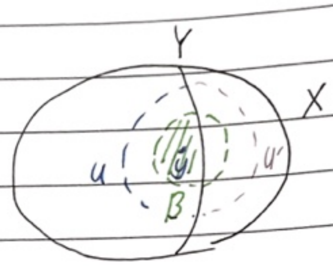
§16 Subspace Topology

Def) $Y \subset X$ a subspace (subset)
$(X, \mathcal{T}, \beta)$ a topological space
Then $(Y, \mathcal{T}_Y, \beta_Y)$ is a 'subspace topology'
where $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$ and $\beta_Y = \{B \cap Y \mid B \in \beta\}$

$U \in \mathcal{T}_Y \Rightarrow \exists U' \in \mathcal{T}$ s.t $U = U' \cap Y$
$U \cap Y \in \mathcal{T}_Y$. $U \in \mathcal{T}$



(1) $\mathcal{T}_Y$ is a topology itself
- $\emptyset = \emptyset \cap Y$, $Y = X \cap Y$ where $\emptyset, X \in \mathcal{T}$
- $\bigcup_{\alpha \in J} (U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y \in \mathcal{T}_Y$
- $\bigcap_{i=1}^{n} (U_i \cap Y) = (\bigcap_{i=1}^{n} U_i) \cap Y \in \mathcal{T}_Y$

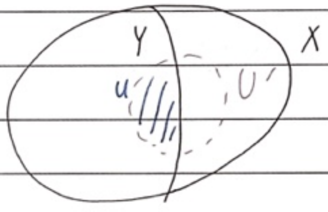


(2) β_Y is a basis for $\mathcal{T}_Y$
Given $y \in U \in \mathcal{T}_Y$, $\exists U' \in \mathcal{T}$ s.t $U = U' \cap Y$
Then $\exists B \in \beta$ s.t $y \in B \subset U'$, and so $y \in B \cap Y \subset U' \cap Y = U$

Note that $U \in \mathcal{T}$ is said to be 'open in X' and
$U' \in \mathcal{T}_Y$ is 'open in Y.'

Lemma 16.2 $Y \subset X$ a subspace
U is open in Y, and Y is open in X $\Rightarrow$ U is open in X

(pf) Since U is open in Y, $\exists U' \in \mathcal{T}$ s.t $U = U' \cap Y$
Since U' and Y are open in X, $U = U' \cap Y$ is open in X.



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(Ex) 1. $Y = [0,1) \cup \{2\} \subset \mathbb{R}_s$ $\quad$ [ ) • $\mathbb{R}$ (0 1 2)

open in Y?

$[0,1)$	Yes	$(= (-1,1) \cap Y)$
$(0,1)$	Yes	$(= (0,1) \cap Y)$
$[0, \frac{1}{2}]$	No	
$\{2\}$	Yes	$(= (\frac{3}{2}, 3) \cap Y)$
$(\frac{1}{2}, 1) \cup \{2\}$	Yes	$(= (\frac{1}{2}, 5) \cap Y)$

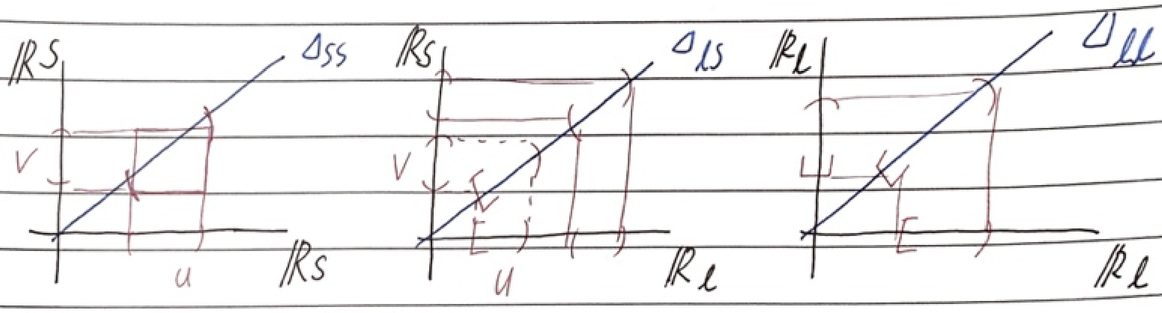
( ) $\quad$ [ )( ) $\quad$ $\mathbb{R}_s$ $\quad$ $\mathbb{R}_\ell$

(Ex) 2. (Exercise 8 in §16)

$\Delta = \{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ 'diagonal space' $\quad$ $X = \mathbb{R} \times \mathbb{R}$

Δ_{ss} : Subspace topology of $\mathbb{R}_s \times \mathbb{R}_s \to \Delta_{ss} = \mathbb{R}_s$

$\Delta_{\ell s}$: subspace topology of $\mathbb{R}_\ell \times \mathbb{R}_s \to \Delta_{\ell s} = \mathbb{R}_\ell$



Thm 16.3) A, B : subspaces of X, Y

Then the product topology on $A \times B$ of the subspace topologies of A and B is the subspace topology of $X \times Y$

(pf)

$$\begin{tikzcd} X, Y \arrow[r, "\text{Product}"] \arrow[d, "\text{subspaces}"'] \arrow[dr, phantom, "="] & X \times Y \arrow[d, "\text{subspace}"] \\ A, B \arrow[r, "\text{product}"'] & A \times B \end{tikzcd}$$

$U \times V$: general basis elt for $X \times Y$ where U is open in X and V is open in Y.

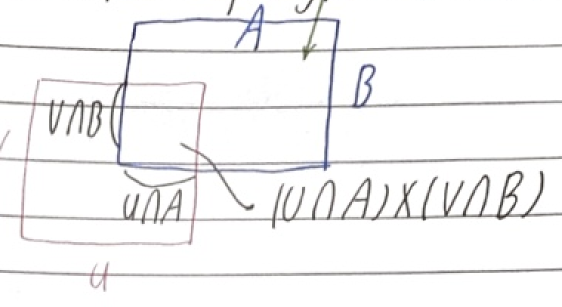
$X \times Y$
$\cup$
$A \times B$

Then $(U \times V) \cap (A \times B)$: general basis elt for the subspace topology on

$(U \cap A) \times (V \cap B)$: general basis elt for the product topology on $A \times B$

open × open
$U \cap A \times V \cap B$

These two bases for the subspace topology on $A \times B$ and for the product on $A \times B$ are the same because
$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$
Hence the topologies are the same.

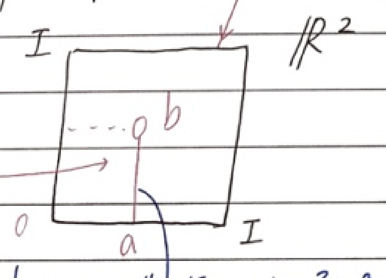


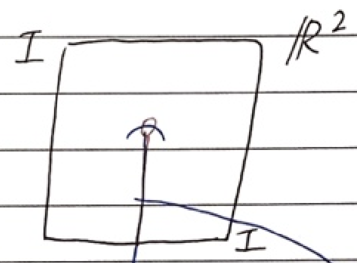
- Let X be an ordered set in the order topology. and Y be a subset of X.
The order relation on X, when restricted to Y makes Y into an ordered set.
However, the resulting order topology on Y need not be the same as the subspace topology on Y of X.

$$\begin{tikzcd} X \arrow[r, "\text{subspace}"] \arrow[d, "\text{order topology}"'] & Y \arrow[d, "\text{order topology}"] \arrow[dl, phantom, "\neq"] \\ X \arrow[r, "\text{subspace}"'] & Y \end{tikzcd}$$

(Ex) $I \times I \subset \mathbb{R}^2$: a subspace where $I = [0,1]$
Indeed, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ of the dictionary order topology on $\mathbb{R}^2$

$[a \times 0, a \times b)$





subspace 에 dic order를 준것

$\mathbb{R}^2$ 에 dic order를 주고 subspace 취한것.

open? No.

표현 불가

$(\sim, \sim)$.

$[0 \times 0, \sim)$

open? Yes

$[a \times 0, a \times b)$
$= (a \times (-1), a \times b) \cap I \times I$

§16 ※2, ※9