Thm 28.2 X a metrizable space

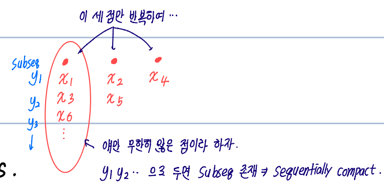
(1) X is compact

$\Leftrightarrow$ (2) X is limit point compact.

$\Leftrightarrow$ (3) X is sequentially compact.

(pf) (1) $\Rightarrow$ (2) Done by Thm 28.1

(2) $\Rightarrow$ (3) Let (x_n) be a seq of points of limit point compact X. (Given condition)

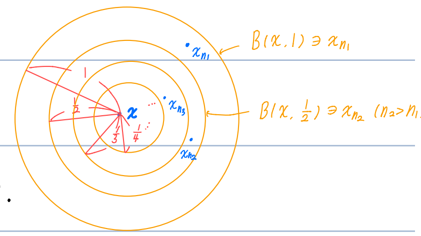


If $A = \{x_n \mid n \in \mathbb{Z}^+\}$ is finite, then $\exists\, x$ s.t $x = x_n$ for infinitely many n's.

Thus (x_n) has a convergent subsequence that is constant x's

If A is infinite, then A has a limit point x.

First choose n_1 so that $x_{n_1} \in B(x, 1)$.



Then suppose that the positive integer n_{i-1} is given.

Because $B(x, \frac{1}{i})$ intersects A in infinitely many points, choose $n_i > n_{i-1}$ s.t $x_{n_i} \in B(x, \frac{1}{i})$.

$> \cdots > n_2 > n_1$

Then the subsequence $x_{n_1}, x_{n_2}, \cdots, \cdots$ Converges to x.

(3) $\Rightarrow$ (1) (Step 1) We show that Lebesgue number lemma holds for sequentially compact X.

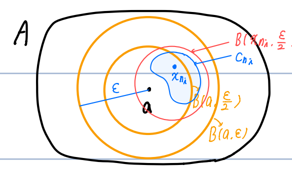
Let $\mathcal{A}$ be an open covering of X.

Assume that $\nexists\, \delta > 0$ s.t each set of diameter $< \delta$ has an element of $\mathcal{A}$ containing it.

For each $n \in \mathbb{Z}_+$, $\exists\, C_n$ of diameter $< \frac{1}{n}$, not contained in any element of $\mathcal{A}$.

Choose each $x_n \in C_n$, and then a subsequence (x_{n_i}) of (x_n) converges to a. (X가 sequentially compact 이므로)

Now $\exists\, A \in \mathcal{A}$ containing a, and choose $\varepsilon > 0$ s.t $B(a, \varepsilon) \subset A$ (open 이므로)



Some i satisfies both $\frac{1}{n_i} < \frac{\varepsilon}{2}$ and $B(x_{n_i}, a) < \frac{\varepsilon}{2}$.

Therefore, $C_{n_i} \subset B(x_{n_i}, \frac{\varepsilon}{2}) \subset B(a, \varepsilon) \subset A$, a contradiction.

$\rightarrow$ Satisfying Lebesgue number

(Step 2) We show that $\forall\, \varepsilon > 0$, $\exists$ a finite covering of X by open ε-balls.

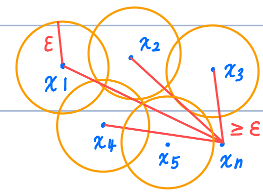
Assume that $\exists\, \varepsilon > 0$ s.t X cannot (모순 유도) be covered by finitely many ε-balls.

Choose any $x_1 \in X$, and then $B(x_1, \varepsilon) \neq X$ (otherwise an ε-ball covers X).

Choose $x_2 \in X - B(x_1, \varepsilon)$ and similarly $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \neq X$.

Given $x_1, \cdots, x_{n-1}$, choose $x_n \in X - B(x_1, \varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon)$ so that $d(x_n, x_i) \geq \varepsilon$ for $i = 1, \cdots, n-1$

Therefore, (x_n) cannot have a convergent subsequence, a contradiction.



→ 따라서 finite 개로 커버가 가능

(Step 3) We finally show that X is compact.

Given any open covering $\mathcal{A}$ of X, it has a Lebesgue number δ by Step 1.

By step 2, $\exists$ a finite covering of X by open $\frac{\delta}{3}$-balls. ← $\varepsilon = \frac{\delta}{3}$