Probability

Lecture outline . . .

- Probability
- Frequentist Probability
- Bayesian Probability

MOTIVATION

We need probability and decision making in the presence of uncertainty.

- What is the probability that a well is a success? – drill the well
- What is the probability that a valve has a crack? – replace the valve
- What is the probability that a seismic survey finds a reservoir? – acquire the seismic
- What is the probability that a reservoir seal will fail? – inject the CO2

Most of our decisions involve uncertainty:

- By quantifying probability, we can make better decisions.
- By communicating uncertainty our work is used for decision making!

PROBABILITY

A Measure that Honors Kolmogorov's 3 Axioms:

1. Probability of an event is a non-negative number.

$$\text{Prob}(A) \geq 0$$

2. Unit Measure, probability of the entire sample space is one (unity).

$$\text{Prob}(\Omega) = 1$$

3. Additivity of mutually exclusive events for unions.

$$\text{Prob}\left(\bigcup_{\boldsymbol{i=1}}^{\infty} \boldsymbol{A_i}\right) = \sum_{\boldsymbol{i=1}}^{\infty} \boldsymbol{Prob}(\boldsymbol{A_i})$$

e.g., probability of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ mutual exclusive events is $\boldsymbol{Prob}(\boldsymbol{A_1}) + \boldsymbol{Prob}(\boldsymbol{A_2})$

WHAT IS PROBABILITY

The 3 Probability Perspectives:

1. Long-term frequencies
   - Probability as ratio of outcomes
   - Requires repeated observations of an experiment

Frequentist Probability

1. Physical tendencies / propensities
   - Knowledge about the system
   - Could know the probability of coin toss without the experiment

Engineering & Science

2. Degrees of belief
   - Reflect our certainty about a result
   - Very flexible, assign probability to anything, updating with new information

Bayesian Probability

FREQUENTIST PROBABILITY

FREQUENTIST PROBABILITY

Measure of the likelihood that an event will occur. For random experiments and well-defined settings (such as coin tosses):

$$\text{Prob}(A) = \text{P}(A) = \lim_{n \to \infty} \left(\frac{n(A)}{n(\Omega)} \right)$$

frequentist approach to probability is the limit of relative frequency over a large number of trails.

where:

n(A) = number of times event A occurred
n(Ω) = number of trails

Example: Possibility of drilling a dry hole for the next well, encountering sandstone at a location ($\mathbf{u}_\alpha$), exceeding a rock porosity of 15% at a location ($\mathbf{u}_\alpha$).

'Frequentist probability is all about experiments and counting!'

PROBABILITY CONCEPTS VENN DIAGRAMS

Venn Diagrams are a tool to communicate probability

Samples ($i = 1, \dots, n$): individual outcomes of an experiment

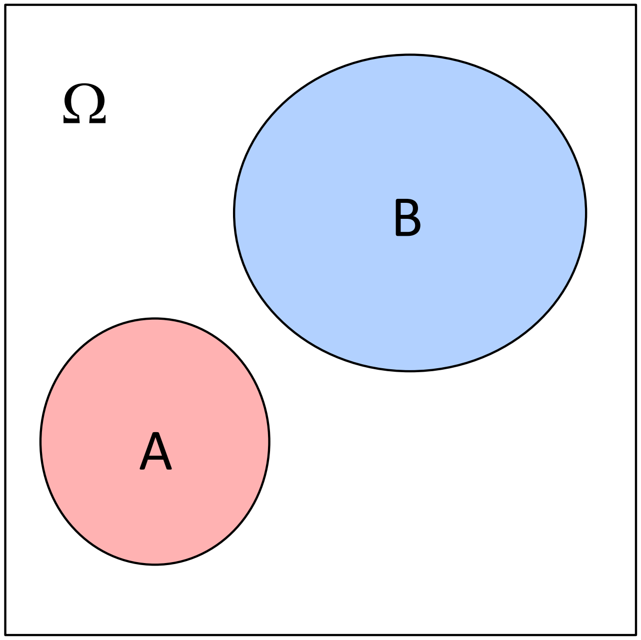
Event ($A, B, \dots$): Collection of simple events meeting a criterion (or set of criteria)

Occurrence of A: A has occurred if the outcome of experiment (sampling) belongs to it.

Sample Space (Ω): Collection of all possible events.

What do we learn from a Venn diagram?

- size of regions $\propto$ probability of occurrence
- proportion of $\boldsymbol{\Omega}$ = probablity
- overlap $\propto$ probability of joint occurrence
- excellent tool to visualize marginal, joint and conditional probability.



Venn Diagram – illustration of events and relations to each other.

PROBABILITY CONCEPTS VENN DIAGRAMS

Experiment:

- Facies determined from a set of well cores (N=3,000 measures at 1-foot increments)

Sample Space (Ω):

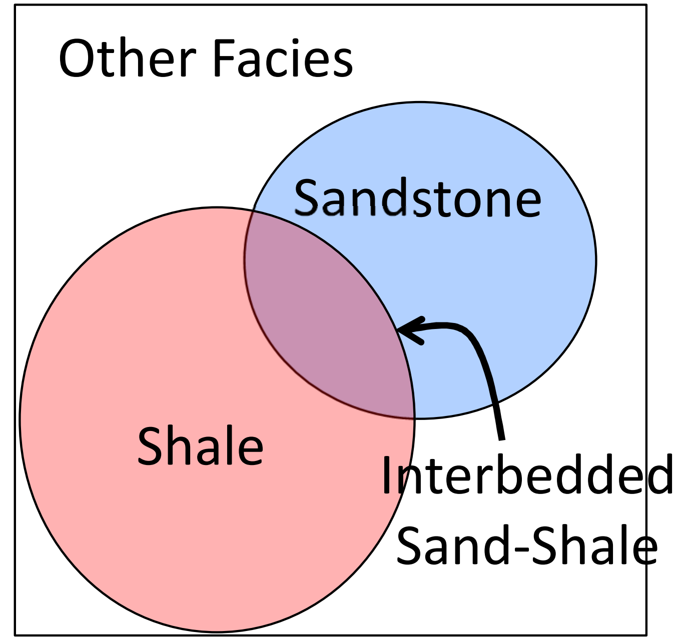
- Facies for the N=3,000 core measures

Event ($A, B, \ldots$):

- Facies = {Sandstone, Interbedded Sandstone and Shale, Shale and other facies}

Venn Diagram Tells Us About Probability:

- Prob{Other Facies} > Prob{Shale} > Prob{Sandstone} > Prob{Interbedded} = Prob{Shale and Sandstone}
- Prob{Sandstone and Shale given Sandstone } < Prob{Sandstone}

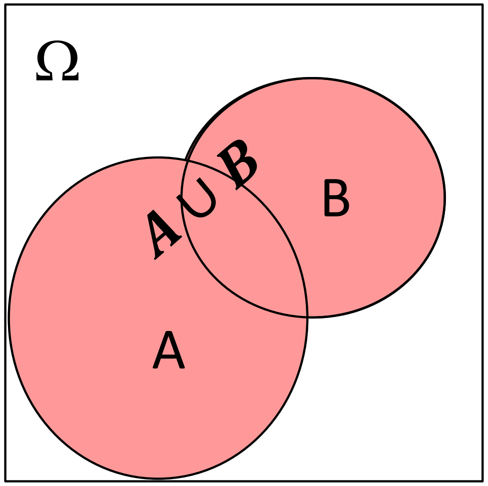


Venn Diagram – illustration of events and relations to each other.

Union of Events:

- All outcomes in the sample space that belong to either event A or B

$$\boldsymbol{A \cup B = \{x: x \in A \text{ or } x \in B\}}$$



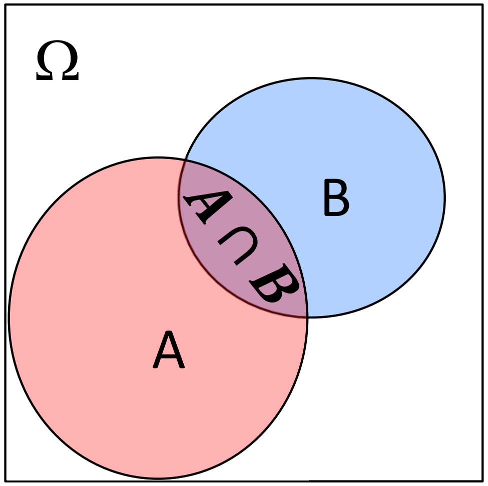
Venn Diagram – illustrating union.

Intersection of Events:

- All outcomes in the sample space that belong to both events A and B

$$\boldsymbol{A \cap B = \{x: x \in A \text{ and } x \in B\}}$$

- We will call this a joint probability later, $P(A, B)$

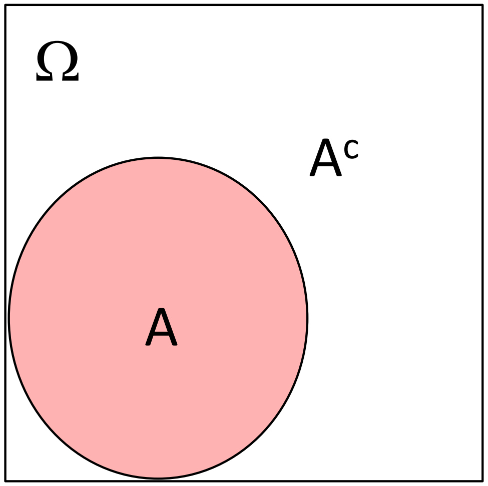


Venn Diagram – illustrating intersection.

PROBABILITY OPERATORS

Complementary Events: A^c

- All outcomes in the sample space that do not belong to A

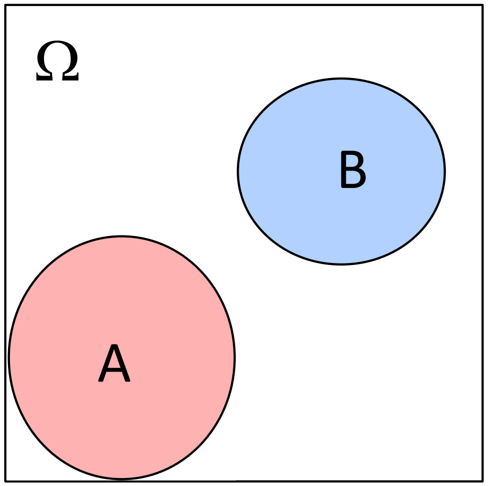


Venn Diagram – illustrating complementary events.

Mutually Exclusive Events:

- The events that do not intersect or do not have any common outcomes

$$A \cap B = \emptyset \rightarrow \text{Null Set}$$



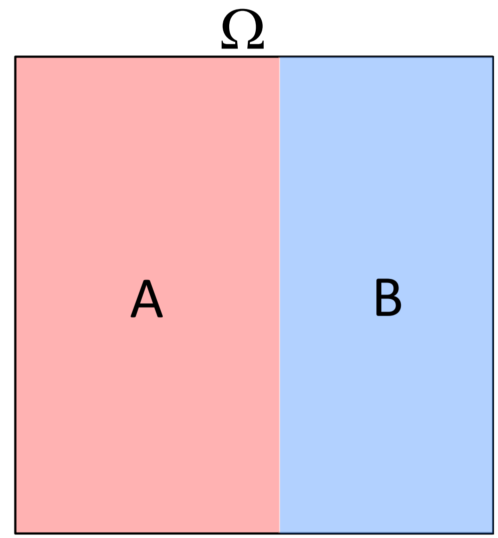
Venn Diagram – illustrating mutually exclusive.

Exhaustive, Mutually Exclusive Sequence of Events:

- The sequence of events whose union is equal to the sample space

$$A_1 \cup A_2 \cup \ \ldots \cup A_n = \Omega$$

- For example, all the samples are either A or B



Venn Diagram – illustrating exhaustive, mutually exclusive events.

PROBABILITY FROM A VENN DIAGRAM

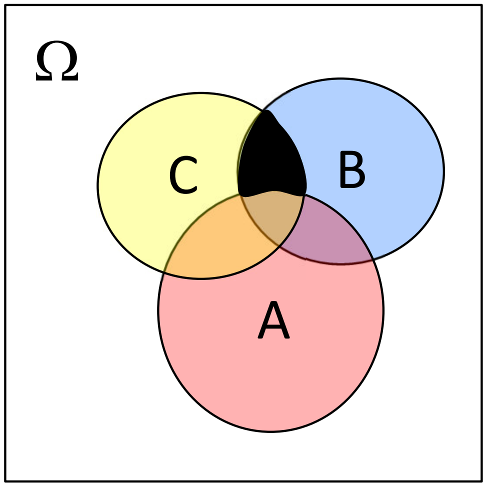
$$\mathrm{Prob}(A) = \mathrm{P}(A) = \frac{Area(A)}{Area(\Omega)}$$

where:

$\mathrm{Prob}(A) = \mathrm{P}(A) =$ area of A / total area
$\mathrm{Prob}(\Omega) = \mathrm{P}(\Omega) =$ area of Ω = probability of any possible outcome = 1.0

Example:
Define the cases:

- A: oil (A^C: dry hole)
- B: sandstone (B^C: shale)
- C: porosity $\geq$ 15% (C^C: porosity $<$ 15%)



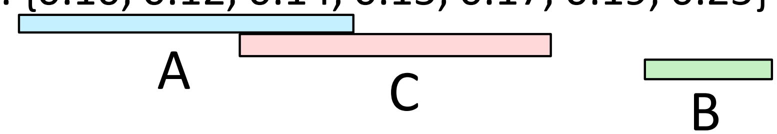
What is the probability of dry hole with sandstone and porosity $\geq$ 15%?

$\mathrm{Prob}(\boldsymbol{A^C} \cap \boldsymbol{B} \cap \boldsymbol{C})$ = $\mathrm{Area}(\boldsymbol{A^C} \cap \boldsymbol{B} \cap \boldsymbol{C})$ / $\mathrm{Area}(\Omega)$

FREQUENTIST PROBABILITY HANDS-ON

Example: A petrophysicist has measured porosity of 7 core samples from a carbonate formation in the laboratory as follows:

Sample Space: {0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25}



We would like to investigate the following events, find the samples for each event:

- Event A: Porosity values of less than 0.15, {0.10, 0.12, 0.14}
- Event B: Porosity values of greater than 0.20, {0.25}
- Event C: Porosity values of between 0.14 and 0.17 {0.14, 0.15, 0.17}

Union of Events:

$A \cup B =$ $\quad B \cup C =$ $\quad A \cup C =$

Intersection of Events:

$A \cap B =$ $\quad B \cap C =$ $\quad A \cap C =$

Complementary Events:

$A^c =$ $\quad B^c =$ $\quad C^c =$

All Events:

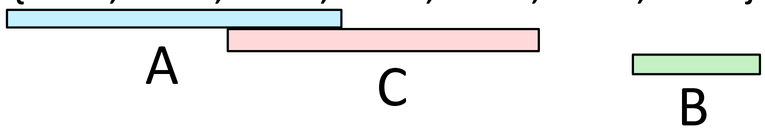
$A \cup B \cup C =$

Find the sets (group of samples) that satisfy these conditions.

FREQUENTIST PROBABILITY HANDS-ON

Example: A petrophysicist has measured porosity of 7 core samples from a carbonate formation in the laboratory as follows:

Sample Space: {0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25}



We would like to investigate the following events, find the samples for each event:

- Event A: Porosity values of less than 0.15, {0.10, 0.12, 0.14}
- Event B: Porosity values of greater than 0.20, {0.25}
- Event C: Porosity values of from 0.14 to 0.17 {0.14, 0.15, 0.17}

Union of Events:

$A \cup B = \{0.10, 0.12, 0.14, 0.25\}$ $B \cup C = \{0.14, 0.15, 0.17, 0.25\}$ $A \cup C = \{0.10, 0.12, 0.14, 0.15, 0.17\}$

Intersection of Events:

$A \cap B = \phi$ $B \cap C = \phi$ $A \cap C = \{0.14\}$

Complementary Events:

$A^c = \{0.15, 0.17, 0.19, 0.25\}$ $B^c = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.19\}$ $C^c = \{0.10, 0.12, 0.19, 0.25\}$

All Events:

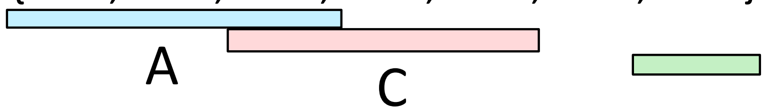
$A \cup B \cup \mathrm{C} = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.25\}$

Now calculate the probabilies for each condition.

FREQUENTIST PROBABILITY HANDS-ON

Example: A petrophysicist has measured porosity of 7 core samples from a carbonate formation in the laboratory as follows:

Sample Space: {0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25}



We would like to investigate the following events, calculate the probability for each event:

- Event A: Porosity values of less than 0.15, {0.10, 0.12, 0.14} — $P(A) = 3/7$
- Event B: Porosity values of greater than 0.20, {0.25} — $P(B) = 1/7$
- Event C: Porosity values of between 0.14 and 0.17 {0.14, 0.15, 0.17} — $P(C) = 3/7$

Union of Events:

$A \cup B = \{0.10, 0.12, 0.14, 0.25\}$ $P(A \cup B) = 4/7$

$B \cup C = \{0.14, 0.15, 0.17, 0.25\}$ $P(B \cup C) = 4/7$

$A \cup C = \{0.10, 0.12, 0.14, 0.15, 0.17\}$ $P(A \cup C) = 5/7$

Intersection of Events:

$A \cap B = \phi$, $P(A \cap B) = 0$

$B \cap C = \phi$, $P(B \cap C) = 0$

$A \cap C = \{0.14\}$, $P(A \cap C) = 1/7$

Complementary Events:

$A^c = \{0.15, 0.17, 0.19, 0.25\}$ P = 4/7

$B^c = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.19\}$ P = 6/7

$C^c = \{0.10, 0.12, 0.19, 0.25\}$ P = 4/7

All Events:

$A \cup B \cup C = \{0.10, 0.12, 0.14, 0.15, 0.17, 0.19, 0.25\} = \Omega$, $P(A \cup B \cup C) = 6/7$

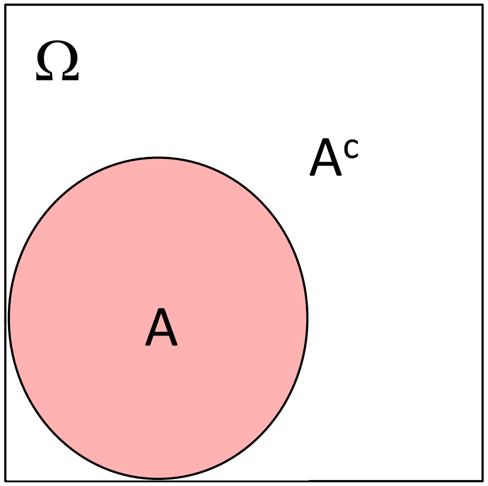
PROBABILITY DEFINITIONS

Non-negativity, Normalization:

- Fundamental probability constraints
  - Bounded $0 \leq P(A) \leq 1$
  - Closure $P(\Omega) = 1$
  - Null Sets $P(\phi) = 0$

Complimentary Events:

- Closure $P(A^c) + P(A) = 1$



Venn Diagram – illustrating complementary events.

PROBABILITY DEFINITIONS

The Addition Rule:

$$P(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$$

Must account for the intersection!

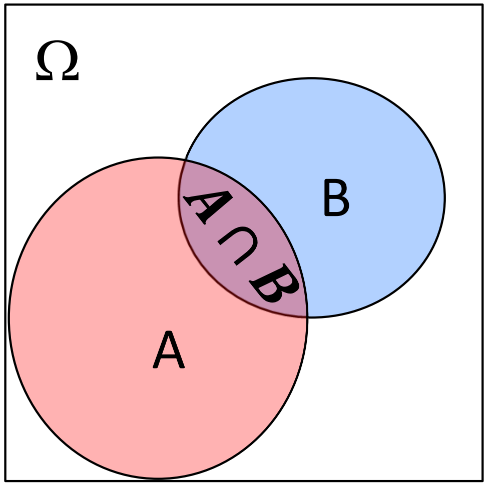
If mutually exclusive events:

$$A_i \cap A_j = \emptyset, \forall\, i \neq j$$

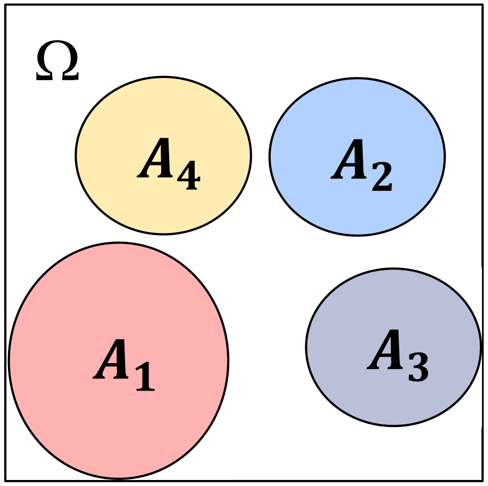
then,

$$\mathrm{P}\left(\bigcup_{i=1}^{k} \mathrm{A}_i\right) = \sum_{i=1}^{k} \mathrm{P}(\mathrm{A}_i)$$

no intersections to account for.



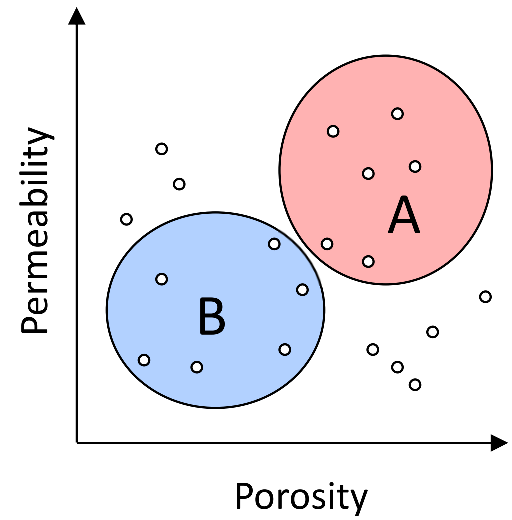
Venn Diagram – illustrating intersection.

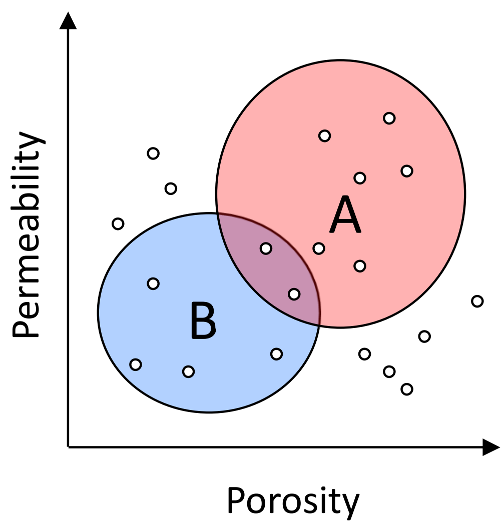


Venn Diagram – with mutually exclusive events.

Calculate the following probabilities for event A and B: Note
Event A: Sandstone and Event B: Shale

$$P(A) =$$
$$P(B) =$$
$$P(A \cap B) =$$
$$P(A \cup B) =$$





$$P(A) =$$
$$P(B) =$$
$$P(A \cap B) =$$
$$P(A \cup B) =$$

Hint, this is just counting the points!

PROBABILITY HANDS-ON

Calculate the following probabilities for event A and B: Note
Event A: Sandstone and Event B: Shale

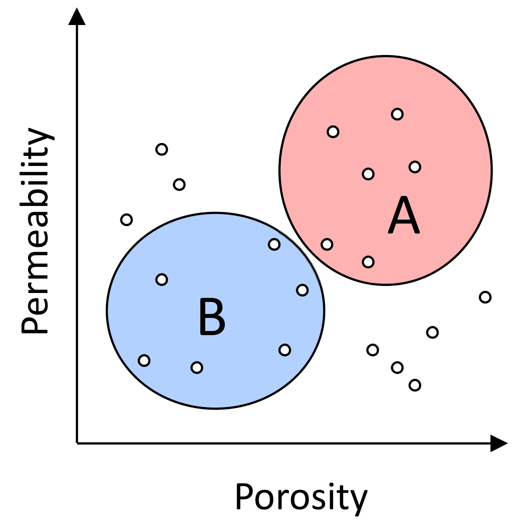
$\mathrm{P}(A) = {}^{6}/_{20} = 30\%$

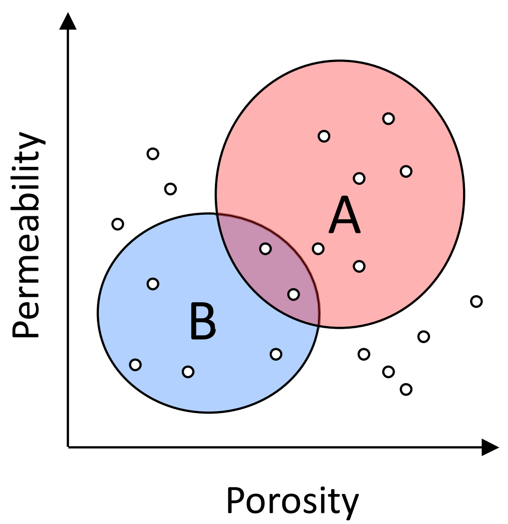
$\mathrm{P}(B) = {}^{6}/_{20} = 30\%$

$\mathrm{P}(A \cap B) = {}^{0}/_{20} = 0\%$

$\mathrm{P}(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$

$= 30\% + 30\% - 0\% = 60\%$





$\mathrm{P}(A) = {}^{8}/_{20} = 40\%$

$\mathrm{P}(B) = {}^{6}/_{20} = 30\%$

$\mathrm{P}(A \cap B) = {}^{2}/_{20} = 10\%$

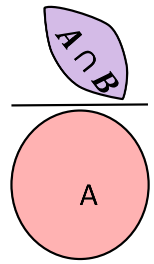
$\mathrm{P}(A \cup B) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P}(A \cap B)$

$= 40\% + 30\% - 10\% = 60\%$

PROBABILITY DEFINITIONS

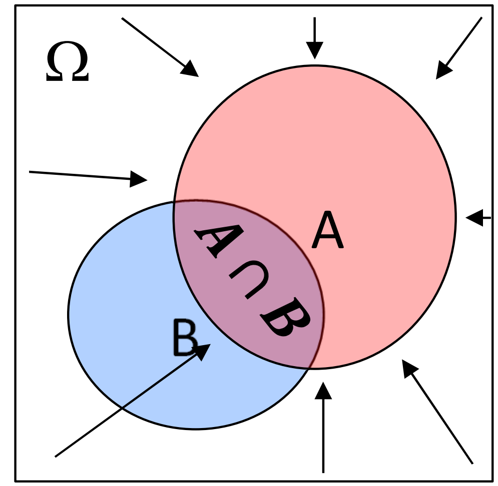
Probability of B given A occurred? $P(B \mid A)$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$



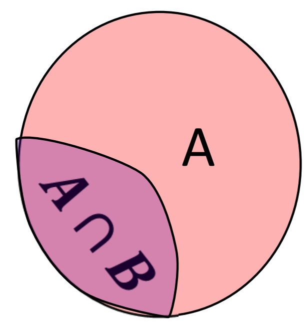
$P(A \cap B)$

$P(A)$



Conceptually we shrink space of possible outcomes.

A occurred so we shrink our space to only event A.



PROBABILITY DEFINITIONS

Now let's define three cases of probability and provide notation.

Marginal Probability: Probability of an event, irrespective of any other event

$$P(X), P(Y)$$

Conditional Probability: Probability of an event, given another event is already true.

$$P(X \; given \; Y), P(Y \; given \; X)$$

$$P(X \mid Y), P(Y|X)$$

Joint Probability: Probability of multiple events occurring together.

$$P(X \; and \; Y), P(Y \; and \; X)$$

$$P(X \; \cap \; Y), P(Y \cap X)$$

$$P(X, Y), P(Y, X)$$

PROBABILITY DEFINITIONS

General Form for Conditional Probability

$$P(C \mid B, A) = \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

Recall:

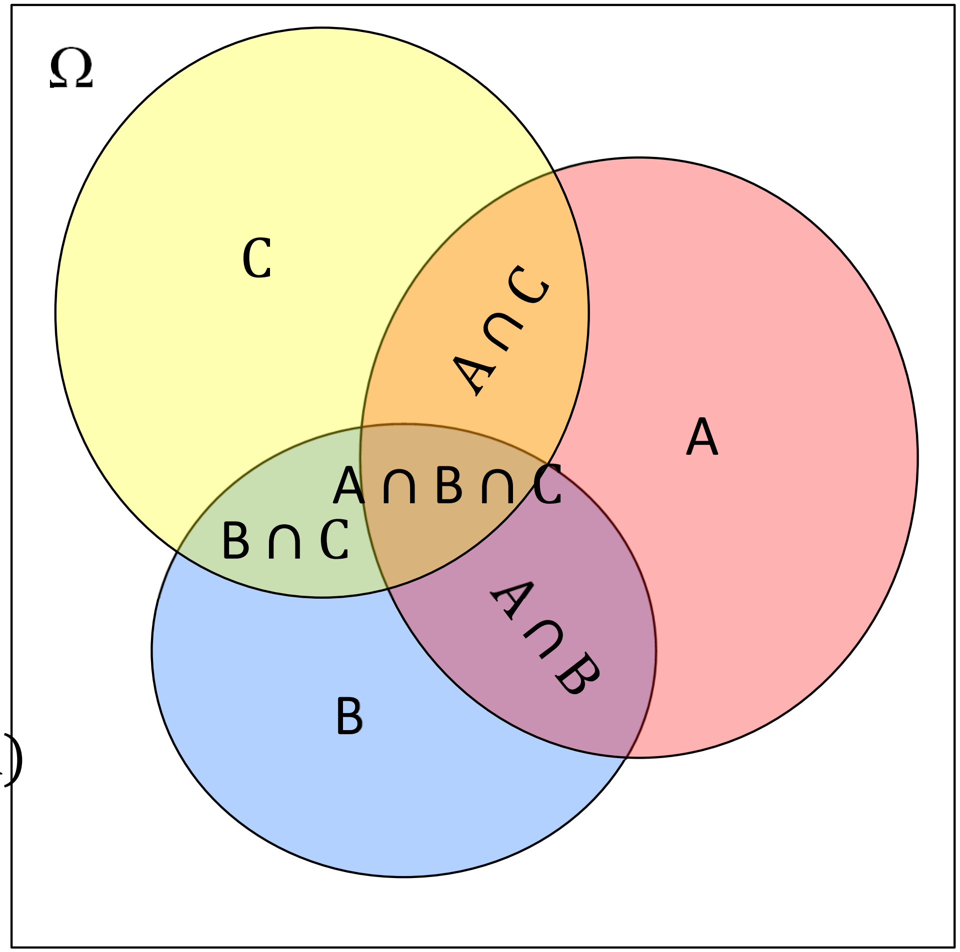
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Substitute:

$$P(C \mid B, A) = \frac{P(A \cap B \cap C)}{P(B|A)P(A)}$$

Reorganize:

$$P(A \cap B \cap C) = P(C \mid B, A)P(B|A)P(A)$$



General Form, Recursion of Conditionals

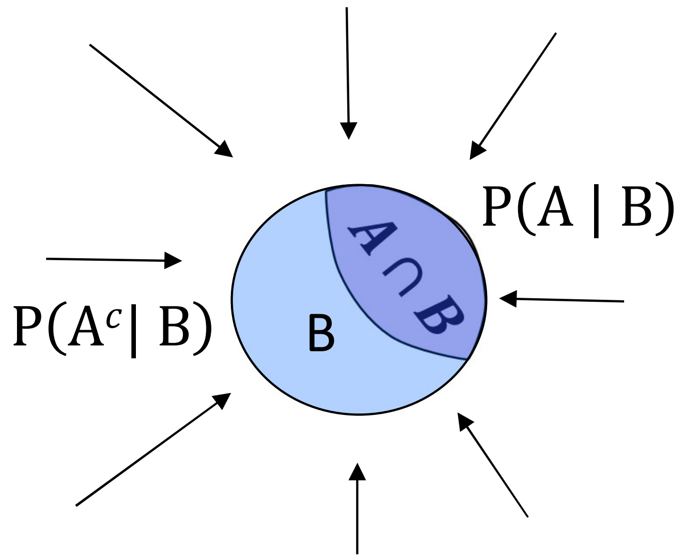
$$P(A_1 \cap \cdots \cap A_n) = P(A_n|A_{n-1}, \dots, A_1)P(A_{n-1}|A_{n-2}, \dots, A_1) \;\dots\; P(A_1)$$

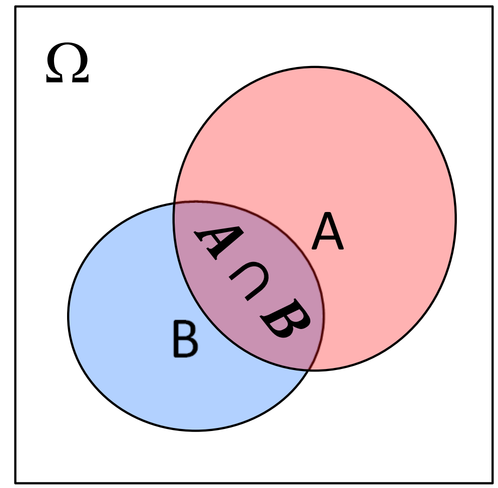
PROBABILITY DEFINITIONS

Closure with Conditional Probability

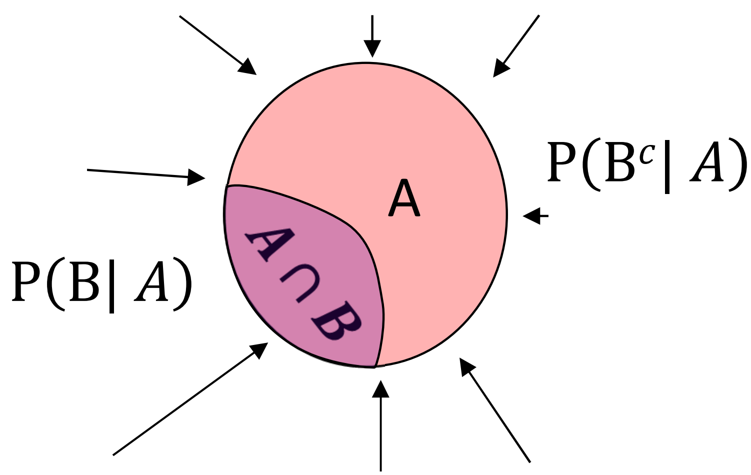
- Closure with conditional probabilities:

$$P(A \mid B) + P(A^c \mid B) = 1$$





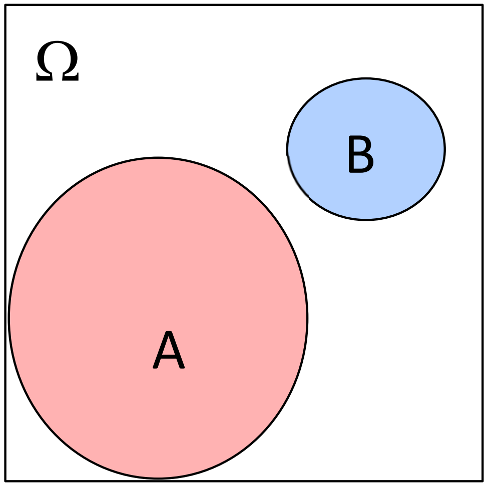
$$P(B \mid A) + P(B^c \mid A) = 1$$



Recall:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Case 1:



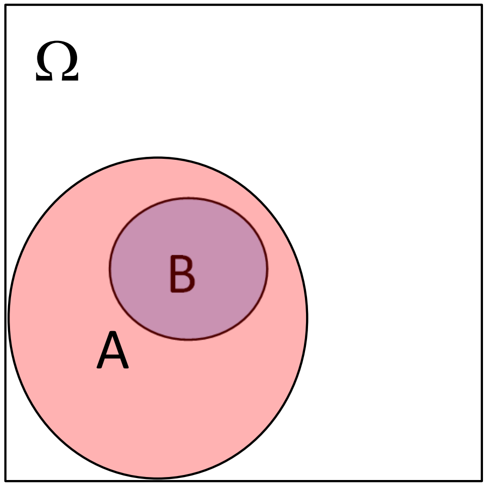
Venn Diagram – case 1.

For Case 1 calculate:

$$P(A \mid B) =$$

$$P(B \mid A) =$$

Case 2:



Venn Diagram – case 2.

For Case 2 calculate:

$$P(A \mid B) =$$

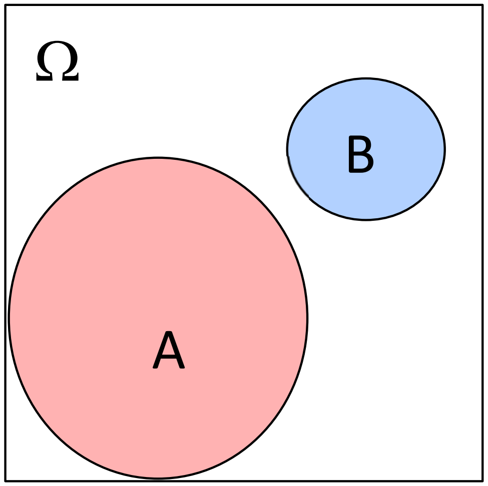
$$P(B \mid A) =$$

Michael Pyrcz, Associate Professor, The University of Texas at Austin

Recall:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Case 1:



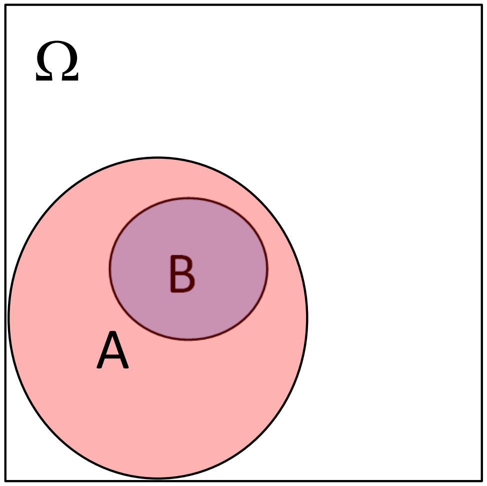
Venn Diagram – case 1.

For Case 1 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\emptyset}{P(B)} = \emptyset$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\emptyset}{P(A)} = \emptyset$$

Case 2:



Venn Diagram – case 2.

For Case 2 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = 1, \quad \text{since } P(A \cap B) = P(B)$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}, \quad \text{since } P(A \cap B) = P(B)$$

PROBABILITY HANDS-ON

Question: Calculate the following probabilities for events A and B:

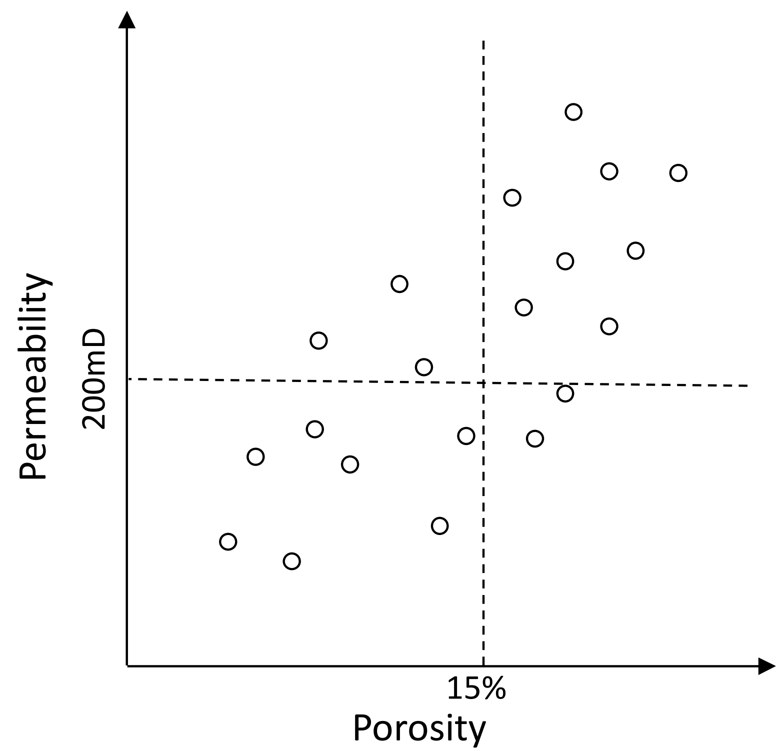
Event A: Porosity $> 15\%$
Event B: Permeability > 200 mD

For Case 1 calculate:

$$P(A \mid B) =$$

$$P(B \mid A) =$$

Bonus Question: How much information does event B tell you about event A and visa versa?



Hint, more point counting.

PROBABILITY HANDS-ON

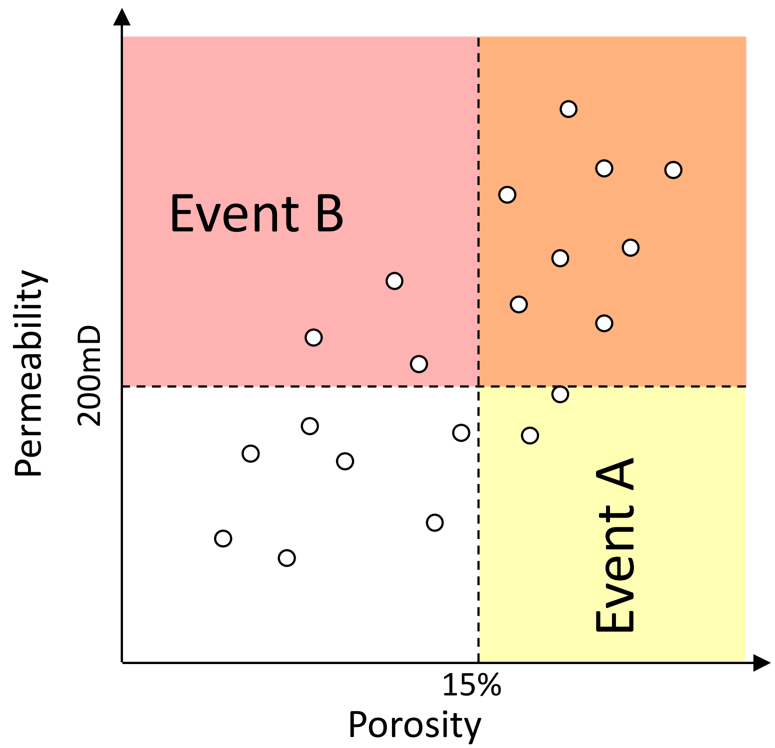
Question: Calculate the following probabilities for events A and B:

Event A: Porosity $> 15\%$
Event B: Permeability > 200 mD

For Case 1 calculate:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{8/20}{11/20} = 8/11$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{8/20}{10/20} = 8/10$$



Bonus Question: How much information does B tell you about A and visa versa?
$P(A) = 10/20$, $P(A|B) = 8/11$ Probability from 50% → 73%
$P(B) = 11/20$, $P(B|A) = 8/10$ Probability from 55% → 80%
We cannot work with A and B independently; they provide information about each other.

The Multiplication Rule:

$$\mathsf{P}(A \cap B) = \mathrm{P}(\mathrm{B}|\mathrm{A})\,\mathrm{P}(A)$$

We adjusted the definition of conditional probability.

If events A and B are independent:

$$\mathrm{P}(B|A) = \mathrm{P}(\mathrm{B})$$

knowing something about A does nothing to help predict B. Then by substitution:

$$\mathsf{P}(A \cap B) = \mathrm{P}(\mathrm{B})\,\mathrm{P}(A)$$

The general form given independence for all cases, $i = 1, \ldots, k$:

$$\mathsf{P}\left(\cap_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{A_i}\right) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{P}(\mathrm{A_i})$$

$$\text{e.g., } P(\mathrm{A_1} \cap \mathrm{A_2} \cap \mathrm{A_3}) = \mathrm{P}(\mathrm{A_1}\,)\,P(\mathrm{A_2})\,P(\mathrm{A_3})$$

PROBABILITY HANDS-ON

Given independence between fluid type and porosity:

Event A = Oil

Event B = Porosity > 10%

Given: $\mathrm{P(A)} = 30\%$ and $\mathrm{P(B)} = 50\%$

What is the P($A \cap B$)?

Given independence between fluid type, porosity and saturation:

Event A = Oil

Event B = Porosity > 10%

Event C = S_{oil} > 40%

Given: $\mathrm{P(A)} = 30\%, \mathrm{P(B)} = 50\%, \mathrm{P(C)} = 25\%$

What is the P($A \cap B \cap C$)?

Given independence between fluid type and porosity:

Event A = Oil

Event B = Porosity > 10%

Given: $\mathrm{P(A)} = 30\%$ and $\mathrm{P(B)} = 50\%$

What is the P($A \cap B$)?

P($A \cap B$) $= \mathrm{P(B)}\ \mathrm{P}(A)$ = 30% x 50% = 15%

Given independence between fluid type, porosity and saturation:

Event A = Oil

Given: $\mathrm{P(A)} = 30\%, \mathrm{P(B)} = 50\%, \mathrm{P(C)} = 10\%$

Event B = Porosity > 10%

Event C = S_{oil} > 40%

What is the P($A \cap B \cap C$)?

P($A \cap B \cap C$) = P(A) P(B) P(C) = 30% x 50% x 10% = 1.5%

PROBABILITY DEFINITIONS

Events A and B are independent if and only if:

$$\mathrm{P}(A \cap B) = \mathrm{P}(\mathrm{B})\mathrm{P}(A)$$

or

$$\mathrm{P}(\,A|B\,) = \mathrm{P}(A)$$

and

$$\mathrm{P}(\,B|A\,) = \mathrm{P}(\boldsymbol{B})$$

Recall the General Form: $\mathrm{P}\left(\bigcap_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{A_i}\right) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{k}} \mathrm{P}(\mathrm{A_i})$

Events A_1, A_2,…, A_n are independent if:

Then We Can Derive:

$$\mathrm{P}(\,\boldsymbol{A}|\boldsymbol{B}\,) = \frac{\boldsymbol{P}(\,\boldsymbol{A},\boldsymbol{B}\,)}{\boldsymbol{P}(\boldsymbol{B})} = \frac{\boldsymbol{P}(\boldsymbol{A})\ \cancel{\boldsymbol{P}(\boldsymbol{B})}}{\cancel{\boldsymbol{P}(\boldsymbol{B})}} = \mathbf{P}(\boldsymbol{A})$$

Example: Facies F1, F2 and F3 in 10 wells:

Position	Well 1	Well 2	Well 3	Well 4	Well 5	Well 6	Well 7	Well 8	Well 9	Well 10
Top	F3	F2	F2	F1	F1	F1	F2	F2	F1	F1
Middle	F1	F1	F1	F1	F2	F2	F1	F2	F2	F2
Bottom	F2	F2	F2	F3	F3	F3	F3	F3	F3	F2

Event A_1 = F1 is middle facies
Event A_2 = F3 is bottom facies

$\mathrm{P}(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = \mathrm{P}(\boldsymbol{A_1})\mathrm{P}(\boldsymbol{A_2})$ or

$\mathrm{P}(\boldsymbol{A_1}|\boldsymbol{A_2}) = \mathrm{P}(\boldsymbol{A_1})$ and $\mathrm{P}(\boldsymbol{A_2}|\boldsymbol{A_1}) = \mathrm{P}(\boldsymbol{A_2})$

Question: are events A1 and A2 independent?

PROBABILITY HANDS-ON

Example: Facies F1, F2 and F3 in 10 wells:

Position	Well 1	Well 2	Well 3	Well 4	Well 5	Well 6	Well 7	Well 8	Well 9	Well 10
Top	F3	F2	F2	F1	F1	F1	F2	F2	F1	F1
Middle	F1	F1	F1	F1	F2	F2	F1	F2	F2	F2
Bottom	F2	F2	F2	F3	F3	F3	F3	F3	F3	F2

Event A_1 = F1 is middle facies
Event A_2 = F3 is bottom facies

$P(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = P(\boldsymbol{A_1})P(\boldsymbol{A_2})$ or

$P(\boldsymbol{A_1}|\boldsymbol{A_2}) = P(\boldsymbol{A_1})$ and $P(\boldsymbol{A_2}|\boldsymbol{A_1}) = P(\boldsymbol{A_2})$

Question: are events A1 and A2 independent?

$$P(\boldsymbol{A_1}) = {}^{5}/_{10} = 50\%, P(\boldsymbol{A_2}) = {}^{6}/_{10} = 60\%, P(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = {}^{2}/_{10} = 20\%$$

$P(\boldsymbol{A_1})P(\boldsymbol{A_2}) = \mathbf{50\%} \cdot \mathbf{60\%} = \mathbf{30\%} \neq P(\boldsymbol{A_1} \cap \boldsymbol{A_2}) = {}^{2}/_{10} = 20\%$ Not independent.

Only need to show invalid for one way to demonstrate not independent.

BAYESIAN PROBABILITY

DERIVATION OF BAYES' THEOREM

Recall the Multiplication Rule:

$$P(B \cap A) = P(A|B)\,P(B)$$

$$P(A \cap B) = P(B|A)\,P(A)$$

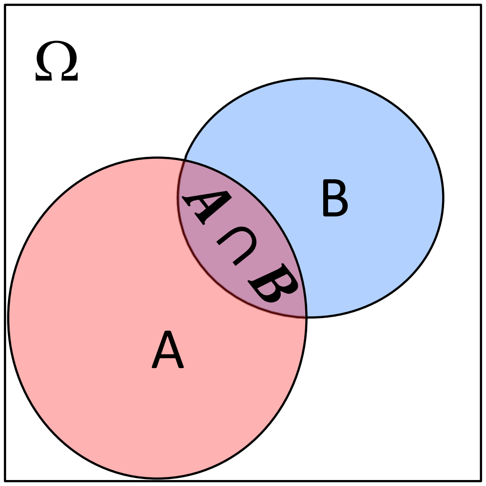
It follows that:

$$P(B \cap A) = P(A \cap B)$$

Therefore, we combine two product rules, substitute:

$$P(A|B)\,P(B) = P(B|A)\,P(A)$$

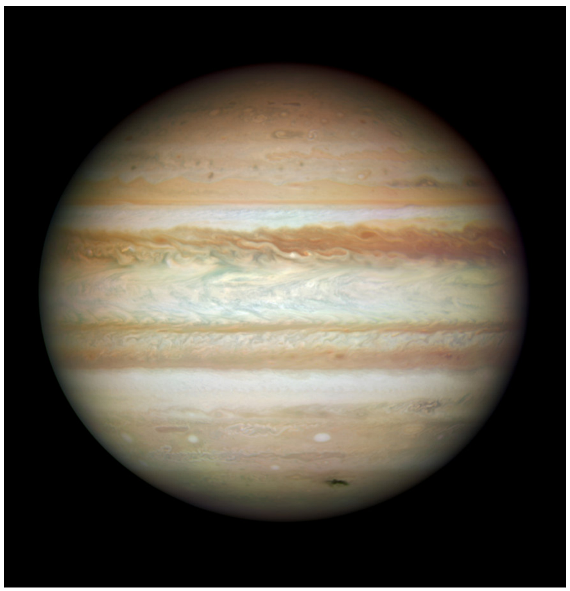
We get Bayes' Theorem!



Venn Diagram – illustrating intersection.

Bayesian Statistical Approaches:

- probabilities based on a degree of belief (expert experience) in an event
- updated as new information is available
- solve probability problems that we cannot use simple frequencies



Jupyter, image from
https://www.wikimedia.org

From Sivia (1996), What is the mass of Jupiter?

Frequentist: measure the mass of enough Jupiter-like planets from multiple solar systems.

Bayesian: form a prior probability and update with any available information.

PROBABILITY DEFINITIONS

Measure of the likelihood that an event will occur. For any occurrence:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayesian approach probability is interpreted as reasonable likelihood representing a state of knowledge or as quantification of a personal belief
Specify a prior and update with new information.

where:

$P(A)$ = prior
$P(B)$ = evidence
$P(B|A)$ = likelihood
$P(A|B)$ = posteriori

Example: Given a prior probability of 40%, a likelihood of 10% and evidence term of 20% the updated posteriori is 20%.

PROBABILITY DEFINITIONS

Bayesian Statistical Approaches:

- probabilities based on:
  - state of knowledge
  - degree of belief in an event

- utilize an assessment prior to data collection
- updated as new information is available
- solve probability problems that we cannot use simple frequencies

Advanced Concept on Uncertainty Modeling:

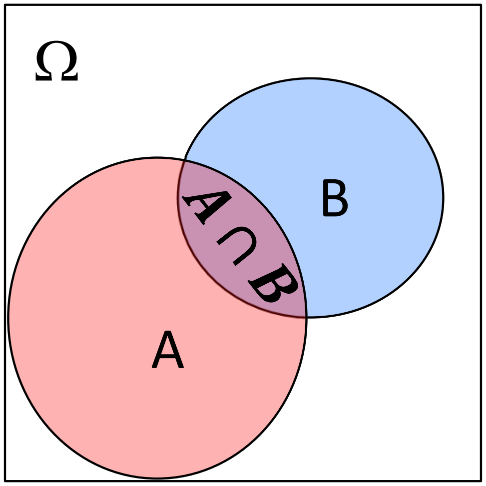
- Bayesian credibility intervals provide a more intuitive measure of uncertainty than Frequentist confidence intervals, more later…

PROBABILITY DEFINITIONS

Bayes' Theorem:

Make an easy adjustment and we get the popular form.

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$



Venn Diagram – illustrating intersection.

Observations:

1. We can get P(A | B) from P(B | A), as you will see this often comes in handy.
2. Each term is known as:

$$\text{Posterior} = \frac{\text{Likelihood x Prior}}{\text{Evidence}}$$

3. Prior should have no information from likelihood.
4. Evidence term is usually just a standardization to ensure closure.

Note: we got to Bayes' Theorem by fundamental frequentist approaches.

PROBABILITY DEFINITIONS

Bayes' Theorem:

A common example of Bayes' Theorem for modeling the subsurface.

Model Updating with a New Data Source:

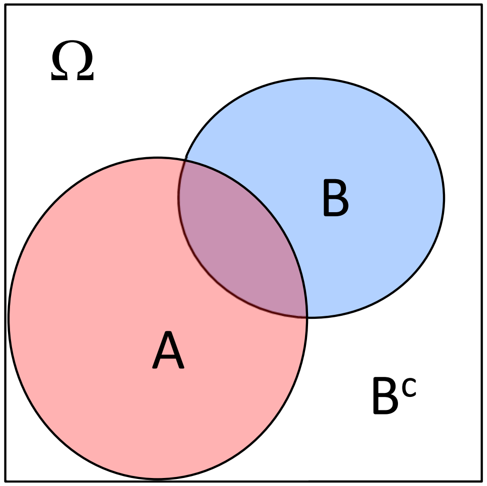
$$\underbrace{P(\text{Model} \mid \text{New Data})}_{\text{Posterior}} = \frac{\underbrace{P(\text{New Data} \mid \text{Model})}_{\text{Likelihood}}\ \underbrace{P(\text{Model})}_{\text{Prior}}}{\underbrace{P(\text{New Data})}_{\text{Evidence}}}$$

PROBABILITY DEFINITIONS

Bayes' Theorem:

Alternative form, symmetry:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} \qquad P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$



Venn Diagram – illustrating intersection.

Alternative form to calculate evidence term:

Given: $P(A) = \underbrace{P(A|B)\,P(B)}_{P(A \text{ and } B)} + \underbrace{P(A|B^c)\,P(B^c)}_{P(A \text{ and } B^c)}$

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$

Since evidence term is often not readily available, we derive it by probability summation (recall, *marginalization*) over all possible outcomes, {A, B} and {A, B^c}.

PROBABILITY DEFINITIONS

Bayesian approaches allow you to solve statistical problems that are not possible otherwise. Consider:

Event A	Event B
You have a disease	You test positive for the disease
There is fault compartmentalization	Geologist says there's a fault
Low permeability of a sample	The laboratory measure is low
A valve will fail	X-ray test is positive
You drill a dry well	Seismic AVO response looks poor

In all of these cases you need to calculate:

$$P\left(\text{Something is Happening} \mid \text{Looks like it's happening}\right) = \frac{P\left(\text{Looks like it's happening} \mid \text{Something is Happening}\right) P\left(\text{Something is Happening}\right)}{P\left(\text{Looks like it's happening}\right)}$$

Is Dr. Pyrcz's coin a fair coin?

Jupyter Notebook
Python Demonstration

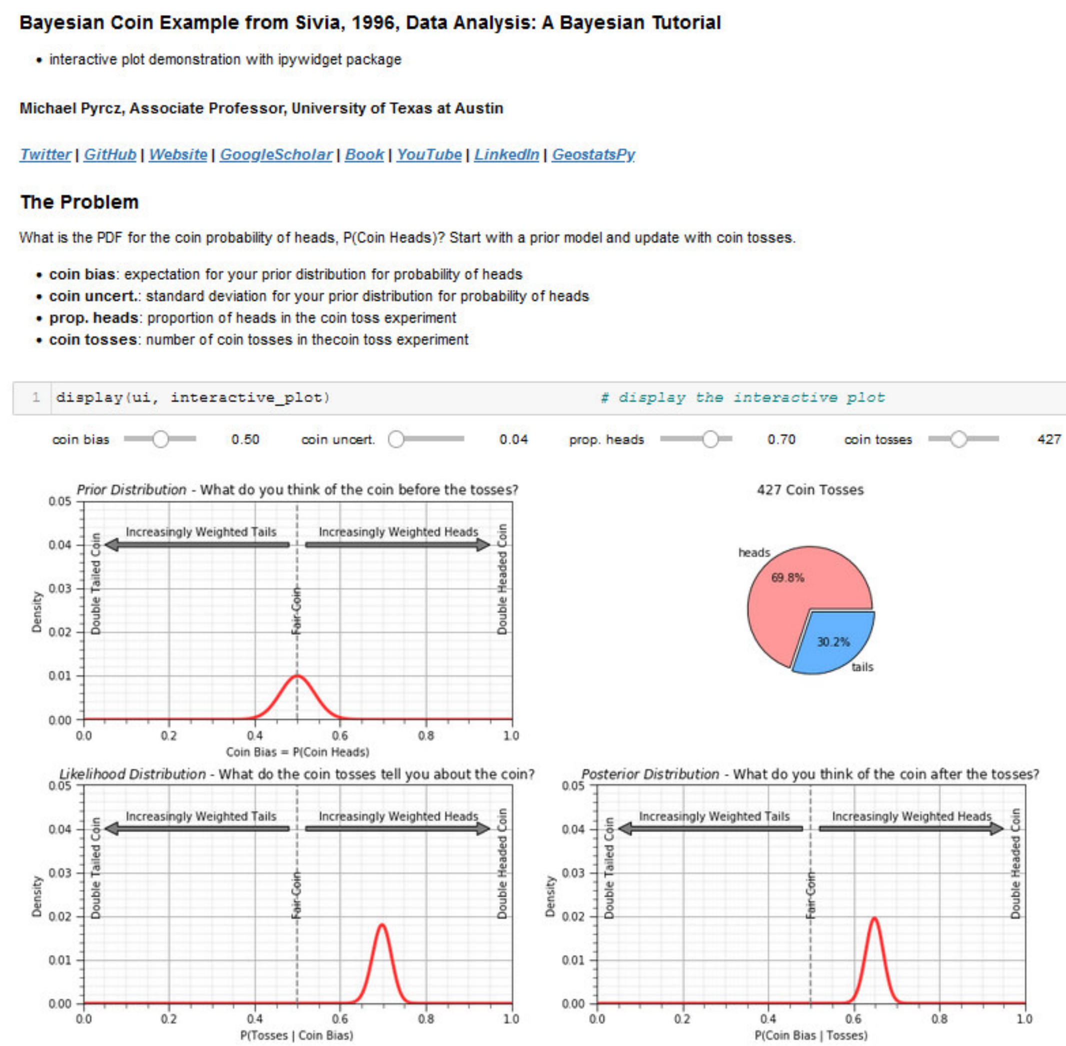
Things to try:

1. Try a naïve prior, I know nothing about Dr. Pyrcz's coin.

2. Try of very specific prior, I'm sure Dr. Pyrcz's coin is fair.

3. Try few and many coin tosses.

4. Contradiction between prior and likelihood.



The file is Interactive_Sivia_Coin_Toss.ipynb. An Excel version is available as Bayesian_Demo.xlsx.

Bayesian approaches allow you to solve statistical problems that are not possible otherwise. Consider:

True Positive Probability

Correct Detection Rate x Occurrence Rate

$$P(\text{Something is Happening} \mid \text{Looks like it's happening}) = \frac{P(\text{Looks like it's happening} \mid \text{Something is Happening})\, P(\text{Something is Happening})}{P(\text{Looks like it's happening})}$$

All Detection Probability (included true and false positives)

Often these terms are much easier to collect:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Let's try this out next.

PROBABILITY HANDS-ON

Example: Prior information at a site suggests a deepwater channel reservoir exists at a given location with probability of 60%. We consider further investigation with a 3D seismic survey.

3D seismic survey will indicate a channelized reservoir:
- is present with 90% probability if it really is present
- is not present with a probability 70% if it really is not

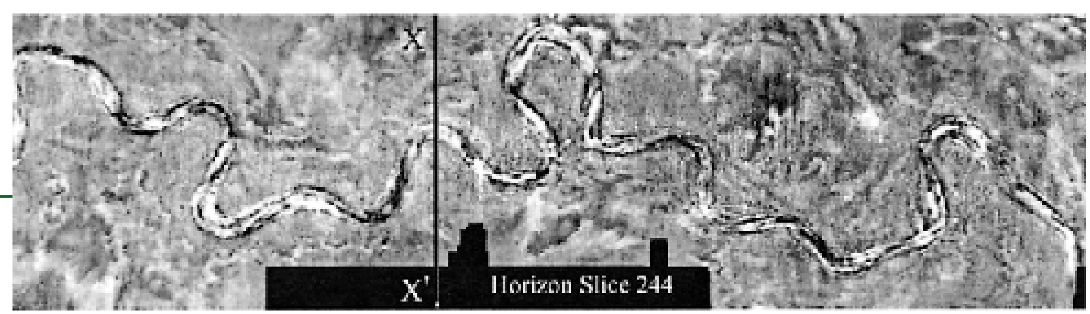
A=The deepwater channel is present
B =Seismic shows a deepwater channel
A^c =The deepwater channel not present
B^c =Seismic does not show a deepwater channel

$$P(A) = 0.6$$
$$P(B|A) = 0.9$$
$$P(B^c|A^c) = 0.7$$
$$P(A^c) =$$
$$P(B|A^c) =$$

Will a 3D seismic survey be useful?

Recall:
$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(Ac)}$$

Sesimic horizon of late Pleistocene leveed channels offshore Nigeria (Posamentier and Kolla, 2003).



PROBABILITY HANDS-ON

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$$P(A) = 0.6$$
$$P(B|A) = 0.9$$
$$P(B^c|A^c) = 0.7$$
$$P(A^c) = 1 - P(A) = 0.4$$
$$P(B|A^c) = 1 - P(B^c|A^c) = 0.3$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)} = \frac{\overbrace{(0.9)(0.6)}^{\text{True Positive}}}{\underbrace{(0.9)(0.6)}_{\text{True Positive}} + \underbrace{(0.3)(0.4)}_{\text{False Positive}}} = 82\%$$

PROBABILITY HANDS-ON

Example: One in every thousand blow out preventers (BOPs) has a serious crack. X-ray analysis has a 99% chance of detecting the crack if present. If the BOP does not have a crack, there is a 2% chance that the X-ray detects a crack. The rate of BOP cracks is 0.1%.

A BOP has been X-rayed and the result is positive. What is the probability that the BOP actually does have a crack?

Solution:
A = BOP has cracks   P(A|B) = ?
B = BOP tests positive
A^c = BOP does not have cracks
B^c = BOP did not test positive

P(A) = 0.001 – crack rate
P(B|A) = 0.99 – true positive
P(B|A^c) = 0.02 – false positive

Recall:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(Ac)}$$

Blow out preventer image from https://www.arescotx.com/photos-videos/what-is-a-blowout-preventer/

PROBABILITY HANDS-ON

Example: One in every thousand blow out preventers (BOPs) has a serious crack. X-ray analysis has a 99% chance of detecting the crack if present. If the BOP does not have a crack, there is a 2% chance that the X-ray detects a crack. The rate of BOP cracks is 0.1%.

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Solution:

A = BOP has cracks $\qquad$ $P(A|B) = ?$

B = BOP tests positive

A^c = BOP does not have cracks

B^c = BOP did not test positive

$P(A) = 0.001$ – crack rate

$P(A^c) = 0.999$ – not cracked rate

$P(B|A) = 0.99$ – true positive

$P(B|A^c) = 0.02$ – false positive

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)} = \frac{\overbrace{(0.99)(0.001)}^{\text{True Positive}}}{\underbrace{(0.99)(0.001)}_{\text{True Positive}} + \underbrace{(0.02)(0.999)}_{\text{False Positive}}} = 4.7\%$$

Probability of a crack in the BOP given a positive crack test is only 4.7%! Why? Cracks are very unlikely + high false positive rate (2%)!

PROBABILITY HANDS-ON

What did we learn?

- we can solve many general, important problems if we define the terms and use them consistently in Bayes' theorem
- use marginalization to solve for the evidence term
- combination of rare events and high false positive rates can make the conditional probability of an event given an indication of the event low!
- we can calculate the posterior and compare to the prior and use this to assess the value of information of a test!

$$P\left(\text{Something is Happening} \mid \text{Looks like it's happening}\right) = \frac{P\left(\text{Looks like it's happening} \mid \text{Something is Happening}\right) P\left(\text{Something is Happening}\right)}{P\left(\text{Looks like it's happening}\right)}$$

You have 3 machines making a product. They have different volumes and errors.

Machine 1	Machine 2	Machine 3
$P(X_1)$, 20% Production	$P(X_2)$, 30% Production	$P(X_3)$, 50% Production
$P(Y\|X_1)$, 5% Error Rate	$P(Y\|X_2)$, 3% Error Rate	$P(Y\|X_3)$, 1% Error Rate

Events - Y: Error, X_1: Machine 1, X_2: Machine 2, X_3: Machine 3

Example: Probability of an error in the product, $P(Y)$?

Hint: Calculate Marginal $P(Y) = \sum_{i=1}^{n} P(Y, X_i) = \sum_{i=1}^{n} P(Y|X_i)P(X_i)$
since exhaustive and mutually exclusive events.

Example https://en.wikipedia.org/wiki/Bayes%27_theorem

You have 3 machines making a product. They have different volumes and errors.

Machine 1	**Machine 2**	**Machine 3**
$P(X_1)$, 20% Production	$P(X_2)$, 30% Production	$P(X_3)$, 50% Production
$P(Y\|X_1)$, 5% Error Rate	$P(Y\|X_2)$, 3% Error Rate	$P(Y\|X_3)$, 1% Error Rate

Events - Y: Error, X_1: Machine 1, X_2: Machine 2, X_3: Machine 3

Example: Probability of an error in the product, $P(Y)$?

$$P(Y) = \sum_{i=1}^{n} P(Y, X_i) = \sum_{i=1}^{n} P(Y|X_i)P(X_i)$$

$$P(Y) = P(Y|X_1)P(X_1) + P(Y|X_2)P(X_2) + P(Y|X_3)P(X_3)$$

$$P(Y) = (0.20)(0.05) + (0.30)(0.03) + (0.50)(0.01)$$

$$P(Y) = 0.024 = 2.4\%$$

Example https://en.wikipedia.org/wiki/Bayes%27_theorem

You have 3 machines making a product. They have different volumes and errors.

Machine 1	Machine 2	Machine 3
$P(X_1)$, 20% Production	$P(X_2)$, 30% Production	$P(X_3)$, 50% Production
$P(Y\|X_1)$, 5% Error Rate	$P(Y\|X_2)$, 3% Error Rate	$P(Y\|X_3)$, 1% Error Rate

Events - Y: Error, X_1: Machine 1, X_2: Machine 2, X_3: Machine 3

Example: Probability product came each machine given an error is observed, $P(X_i|Y)$?

Note: From the previous slide: $P(Y) = 0.024 = 2.4\%$

Hint: calculate the conditional: $P(X_i|Y) = \frac{P(Y|X_i)P(X_i)}{P(Y)}$

Example https://en.wikipedia.org/wiki/Bayes%27_theorem

PROBABILITY HANDS-ON

You have 3 machines making a product. They have different volumes and errors.

Machine 1	Machine 2	Machine 3
$P(X_1)$, 20% Production	$P(X_2)$, 30% Production	$P(X_3)$, 50% Production
$P(Y\|X_1)$, 5% Error Rate	$P(Y\|X_2)$, 3% Error Rate	$P(Y\|X_3)$, 1% Error Rate

Events - Y: Error, X_1: Machine 1, X_2: Machine 2, X_3: Machine 3

Example: Probability product came each machine given an error is observed, $P(X_i|Y)$?

$$P(X_i|Y) = \frac{P(Y|X_i)P(X_i)}{P(Y)}$$

$$P(X_1|Y) = \frac{(0.05)(0.2)}{(0.024)} = 0.41$$

$$P(X_3|Y) = \frac{(0.01)(0.5)}{(0.024)} = 0.21$$

$$P(X_2|Y) = \frac{(0.03)(0.3)}{(0.024)} = 0.38$$

Example https://en.wikipedia.org/wiki/Bayes%27_theorem

PROBABILITY DEFINITIONS

There is an analytical solution for working with Gaussian parametric distributions for Bayesian updating (Sivia, 1996).

- Calculate the mean of the posterior from the prior and likelihood mean and variance.

$$\bar{x}_{\text{updated}} = \frac{\bar{x}_{\text{likelihood}}(\mathbf{u}) \cdot \sigma^2_{\text{prior}}(\mathbf{u}) + \bar{x}_{\text{prior}}(\mathbf{u}) \cdot \sigma^2_{\text{likelihood}}(\mathbf{u})}{[1 - \sigma^2_{\text{likelihood}}(\mathbf{u})][\sigma^2_{\text{prior}}(\mathbf{u}) - 1] + 1}$$

- Calculate the variance of the posterior form the prior and likelihood variances (no means, homoscedastic!).

$$\sigma^2_{\text{updated}}(\mathbf{u}) = \frac{\sigma^2_{\text{prior}}(\mathbf{u})\,\sigma^2_{\text{likelihood}}(\mathbf{u})}{[1 - \sigma^2_{\text{likelihood}}(\mathbf{u})][\sigma^2_{\text{prior}}(\mathbf{u}) - 1] + 1}$$

We will formalize mean (arithmetic average) and variance next lecture and the Gaussian parametric distribution later.

Bayesian probability, expanding beyond 2 mutually exclusive, exhaustive events.

General Form:

$$P(A_k \mid B) = \frac{P(B|A_k)\,P(A_k)}{P(B)}$$

if non-overlapping

$$A_i \bigcap A_j = \emptyset, \forall i, \forall j, i \neq j$$

and exhaustive

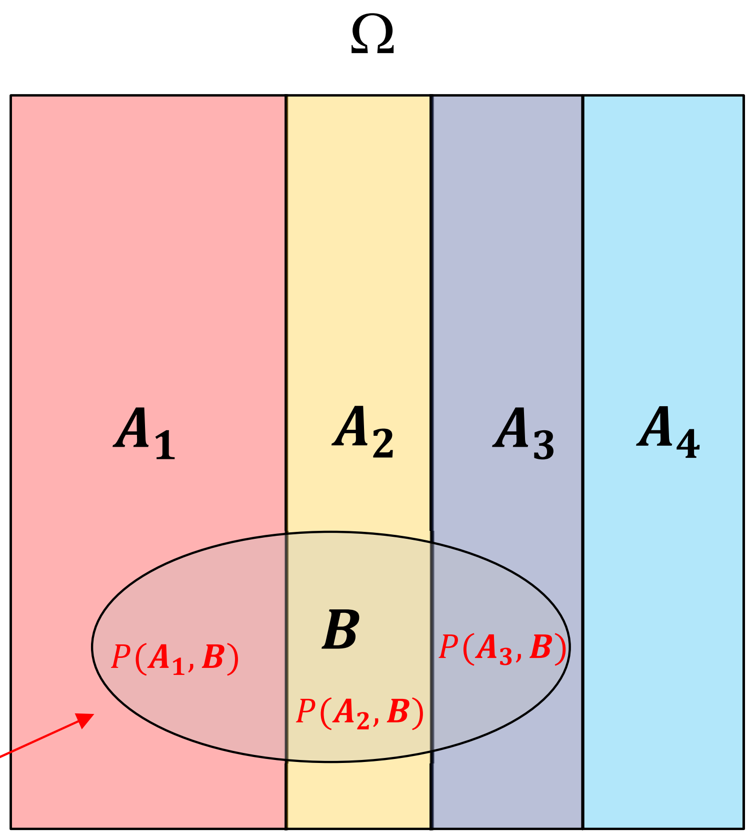
$$\bigcup_{k=1}^{K} A_k = \Omega$$

then:

$$P(B) = \sum_{k=1}^{K} P(B|A_k)\,P(A_k) = \sum_{k=1}^{K} P(B, A_k)$$

we substitute:

$$P(A_k \mid B) = \frac{P(B|A_k)\,P(A_k)}{\sum_{k=1}^{K} P(B, A_k)}$$



Venn Diagram – illustrating exhaustive, mutually exclusive series.

Careful, can't do this if not mutually exclusive and exhaustive events.

- More complicated to calculate evidence, $P(B)$

PROBABILITY

New Tools

Topic	Application to Subsurface Modeling
Frequentist Concepts	When sufficient observations are available use (long-run) counting to access the required probabilities. *Predict reservoir average porosity by pooling analogous fields.*
Independence Definition	Use the definitions of independence to check for independence in your data. *If independed from the outcome of interest, don't spend the money to collect the new data!*
Bayesian Concepts **Inversion of Conditionals**	Calculate a difficult to access conditional probability from an accessible one. Probability of event given indicator from probability indicator given event. *Calculate probability of sealing fault given indicator of sealing fault.*
Bayesian Concepts **Bayesian Updating**	Update prior belief with new information. *Calculate probability of exploration success rate given prior model and outcomes from exploration drilling program.*

Probability

Lecture outline . . .

- Probability
- Frequentist Probability
- Bayesian Probability