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# Refinement Type Directed Search for Meta-Interpretive Learning of Higher-Order Logic Programs

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# Abstract

The program synthesis problem within the Inductive Logic Programming (ILP) community has typically been seen as untyped. We consider the benefits of user provided types on background knowledge. Building on the Meta-Interpretive Learning (MIL) framework, we show that type checking is able to prune large parts of the hypothesis space of programs. The introduction of polymorphic type checking to the MIL approach to logic program synthesis is validated by strong theoretical and experimental results, showing a cubic reduction in the size of the search space and synthesis time, in terms of the number of typed background predicates. Additionally we are able to infer polymorphic types of synthesized clauses and of entire programs. The other advancement is in developing an approach to leveraging refinement types in ILP. Here we show that further pruning of the search space can be achieved, though the SMT solving used for refinement type checking comes at a significant cost timewise.

# Contents

1	Inti	roduction	4		
2	Rev	view of Program Synthesis	7		
	2.1	Program Synthesis as Proof Search	7		
	2.2	Dimensions of Program Synthesis	8		
		2.2.1 Synthesis Domains/Application Areas	9		
		2.2.2 Specifications			
		2.2.3 Paradigms and Types of Programs Learned	11		
3	Unt	typed Meta-Interpretive Learning	<b>15</b>		
	3.1	Logic Prerequisites	15		
	3.2	Meta-Interpretive Learning	17		
		3.2.1 Problem Statement	17		
		3.2.2 Meta-Interpreting and Metarules	18		
	3.3	$Metagol_{AI}$ : Untyped Abstracted MIL	19		
		3.3.1 Representation of Background Knowledge	20		
		3.3.2 The Algorithm	21		
4	Typed Meta-Interpretive Learning				
	4.1	Typing Definitions	24		
	4.2	Pruning the Search Space	26		
		4.2.1 A Worked Example	27		
		4.2.2 Granularity of types	28		
	4.3	Types Directing the Search	29		
	4.4	Showing Insufficiency of the Background Knowledge	31		
5	Syn	thesis with Polymorphic Types	<b>32</b>		
	5.1	Revised Problem Statement	32		
	5.2	$Metagol_{PT}$ : Type Checking through Unification	34		
		5.2.1 Derivation and General Types	34		
		5.2.2 The Algorithm	36		

		5.2.3	Forward Propagating Derivation Types	36
		5.2.4	Inferring the Most General Type	39
	5.3	Theor	etical Results	40
		5.3.1	Soundness	40
		5.3.2	Completeness	41
		5.3.3	Proportion of Relevant Predicates	43
	5.4	Exper	imental Results	44
		5.4.1	Derivation steps	45
		5.4.2	Experiment 1: Search Space Reduction	
		5.4.3	Experiment 2: Ratio Influence	
		5.4.4	Experiment 3: Simply Typed Droplasts	
6	Syn	thesis	with Refinement Types	<b>55</b>
	6.1	Repre	sentation of Refinements	55
	6.2		a Program Derivation to a Refinement	
		6.2.1	Backward Action on Refinements	
		6.2.2	High-level algorithm: $Metagol_{RT}$	58
		6.2.3	Tree-Shaped Grand Refinement	
	6.3	Grand	l Refinement Checking	
		6.3.1	SMT Solvers	
		6.3.2	Language of Refinements	
	6.4	Expre	ssiveness of Refinements	
		6.4.1	Z3 Sequence Theory	
		6.4.2	DTLIA: Quantifiers, Datatypes and Arithmetic	
	6.5		etical Results	66
	6.6		imental Results	
	0.0	6.6.1	Experiment 4: Droplasts with Sensible Predicates	
		6.6.2	Experiment 5: Droplasts with Additional Predicates	
_	~		•	
7	Cor	iclusio	n	73
Bi	blios	graphy		<b>76</b>

# Chapter 1

# Introduction

In the last decade there has been a great surge in the effective application of Artificial Intelligence techniques to a great number of practical problems, e.g. image classification [Deng et al., 2009], code completion [Raychev et al., 2014], as well as autonomous navigation [Bojarski et al., 2016]. Though these advances in AI technology have made previous instances of the above problems tractable, many of the techniques leveraged are not able to give a useful explanation of why decisions are being made (a prime example being artificial neural networks), and struggle to generalize over their entire input space [Gulwani et al., 2015]. For some problems, like in the case of autonomous driving cars, the ability to understand why certain decisions will be/were made could be of utmost importance with regard to safety and liability issues.

Comprehensibility The problem with the alluded to methods is that the generated programs do not necessarily have a description that allows humans to understand and effectively reason about the programs (e.g. the edge weights in neural networks do not give a ready interpretation of program behaviour). The focus of these approaches is on improving according to Michie [1988]'s weak criterion for machine learning, i.e. improving predictive accuracy. In contrast, programs described by programming languages do have semantics that are comprehensible to both machines and humans, which corresponds to Michie's criterion for strong machine learning. From this it follows that a fruitful approach to explainable AI might be to generate computer programs expressed in programming languages, i.e. code programs. Programming languages also strongly emphasize the composability of programs, motivating the usage of the term synthesis for the generation of code programs. Logic and functional languages in particular are good candidates for synthesis due to their high level of abstraction being associated with smaller

programs. Recent work on *ultra-strong* machine learning [Muggleton et al., 2018] has shown that learning logic relational programs can extend humans' capacity for understanding training data to a level beyond unassisted study.

**Inductive synthesis** Code program synthesis is not only useful in terms of explainable AI, it is also of interest for the sake of automating programming. Code synthesis holds promise as a tool for developers to become more efficient, but has also proven effective in empowering non-programmers in obtaining programs that fit their needs, e.g. learning spreadsheet transformations [Singh and Gulwani, 2012b]. For these users of program synthesis the synthesis problem is posed as finding a program that accounts for the examples that the user gives (in terms of inputs and outputs) and which is mostly likely to appropriately generalize from the examples [Gulwani, 2010. The paradigm that combines both automatic generation of code programs and learning from examples is known as *Inductive Programming* [Kitzelmann, 2010]. Versus general machine learning, inductive programming typically relies on very few examples to learn from, e.g. the single example f([1, 2, 3], [4, 5]) = [1, 2], [4] is enough to induce a program that drops the last element from input lists. For programmers this paradigm is useful for problems ranging from which constants to fill in for edge cases (see [Solar-Lezama, 2009]), up to the potential to discover novel efficient algorithms [Cropper and Muggleton, 2018].

Logic programs The above considerations motivate the study of Inductive Logic Programming (ILP), "a form of machine learning that uses logic programming to represent examples, background knowledge, and learned programs" [Cropper, 2017], an active field of study for almost thirty years [Muggleton, 1991]. The specification of an inductive logic problem is in terms of the examples the program needs to satisfy, along with background information in the form of asserted facts and available program fragments, and possibly rules about the structure of the program to be synthesized. A particularly powerful approach developed recently is that of the *Meta-Interpretive Learning* (MIL) [Muggleton et al., 2014a]. One of the major benefits of this approach (versus virtually all other existing synthesis systems) is that it is able do *predicate invention*, that is, it is able to construct helper predicates/functions.

The MIL framework essentially takes the user's problem description and sees it as a specification of a search space of possible programs to consider. The time required for synthesis is most influenced by how large the search space is that is traversed. The implementation of the MIL framework is

written in Prolog, a logic programming language that does not consider the types of its terms. The MIL algorithm in turn does not make use of the types of predicates during its search.

Contribution The thesis of this document is that types are a significant feature for the MIL approach to code program synthesis. Types provide an effective way of pruning out nonsensical programs in the search space, i.e. programs already rejected by type checking. Different gradations of types are considered, namely polymorphic types, i.e. types with type parameteres, and refinement types (with polymorphism), i.e. types with a proposition restricting inhabitants. The second system has strictly more accurate types than the first, but the type checking comes at a considerable cost. For polymorphic types we show significant benefits both theoretically and experimentally, showing a cubic reduction in the size of the search space and synthesis time, in terms of the number of typed background predicates. Additionally we are able to infer polymorphic types of synthesized clauses and of entire programs. For refinement typing the foremost contribution is in the introduction of refinement types to ILP, while the experimental results indicate more work is needed for refinement type checking to be truly effective.

Document organization This dissertation consists of seven chapters. The first chapter presents a brief introduction to the project. The other chapters are described as follows: chapter 2 contains a literature review of code synthesis approaches with a focus on types and ILP. Chapter 3 discusses in detail the existing MIL systems. Chapters 4 through 7 represent the novel contributions of this project. Chapter 4 discusses where MIL can benefit from the introduction of types. The subsequent chapter discusses how polymorpic type checking is introduced to the existing system. Chapter 6 deals with refinement types and with how Satisfiability Modulo Theories (SMT) solving is leveraged for type checking. Both chapter 5 and 6 have sections presenting theoretical and experimental results. The conclusion, chapter 7, considers the implications of the experimental results and relates this outcome to future work.

# Chapter 2

# Review of Program Synthesis

Following on from the arguments made in the introduction regarding the significance of program synthesis, this chapter opens with discussing the strong relation between program synthesis and the notion of proof search.

Subsequently the presentation switches to a literature review. Different aspects of the synthesis problem are highlighted, such as the differences in problem definitions, various application areas and multiple paradigms. We do not aim for comprehensiveness, instead we review a selection of disparate approaches in the literature based on contrasting features.

The focus is on code program synthesis approaches for synthesis of functional programs, usually based around typing information, and synthesis of logic programs, where we mainly look at the Inductive Logic Programming (ILP) approach. Methods outside of these areas are also touched upon.

## 2.1 Program Synthesis as Proof Search

There is a strong correspondence between program synthesis and automated theorem proving. In program synthesis there is a specification for which we try to find a program satisfying the specification. The user might provide helper functions to be used in the synthesized program. In automated theorem proving we state a proposition and want to find a proof for the validity of the formula. The user might provide lemmas that may be used in the proof.

As we will see later, many specifications can be expressed as a type that the synthesized program needs to satisfy. The connection between program synthesis and proof search is most eloquently expressed by the Curry-Howard correspondence [Sørensen and Urzyczyn, 2006]. This correspondence states that there is a direct mapping between programs (expressed in type theory)

and mathematical proofs. For every type theory, a formal system for describing (functional) programs with their types, there is a mapping between the types of the programs and logical formulae. From this correspondence it follows that type checking is the same as checking that a proof follows the rules of formal logical system (e.g. natural deduction).

This connection means that when we are interested in program synthesis we should be aware of the work that exists "on the other side" of the correspondence, namely in (semi-)automated theorem proving. On the side of theorem proving there is an interesting distinction made between systems that fully automate the construction of proofs [Bibel, 2013] and of so called proof assistants (e.g. Coq [2018], and Isabelle [Nipkow et al., 2002]), where software assists users in writing down proofs in a formal system. One form of assistance provided is the ability to do proof search. Whilst the ideal of theorem proving is full automation, for many non-trivial logical systems this is simply not tractable (in general, it is even undecidable to determine the truth of all propositions). Proof assistants are instead able to do proof search based on a (partial) proof, and lemmas already constructed by the user.

The idea of having the user guide proof automation is recognizable program synthesis as providing background knowledge, in the form of helper functions. How a user of a proof assistant asks the system to find a proof, and upon failure will try to provide additional guidance by providing additional lemmas (or working further on the proof) is also in clear relation to how users work with synthesis systems.

For this project we work in the setting of logic programs. Logic programming is at an interesting intersection of programming and theorem proving, as execution is performed based on SLD-resolution, the algorithm behind first-order theorem provers. For this document we will hence often rely on the terminology of proofs and proof search.

## 2.2 Dimensions of Program Synthesis

In this section we review several important features of program synthesis. Based on a survey of the literature (focused on type-based synthesis and approaches to ILP), we look at the domains and applications areas where synthesis has been successfully applied. Next we look at the different ways that the synthesis problem can be specified by the user. Following on we look at the types of programs that can be learned by different approaches. We finish on a discussion of the recognisable paradigms within the literature. We follow [Gulwani, 2010] in identifying the key aspects of an approach to the synthesis problem as capturing the user's intent in a specification, the space

of programs that is considered, and the approach to searching this space.

#### 2.2.1 Synthesis Domains/Application Areas

In this section we give an idea of the diversity of problems addressable by program synthesis, based on a selection of successful approaches in the literature. The review paper by [Gulwani, 2010] is great resource for entering upon the field of program synthesis, and unless otherwise annotated the reference for the below discussion is this paper.

Bitvector algorithms "typically describe some plausible yet unusual operation on integers or bit strings that could easily be programmed using either a longish fixed sequence of machine instructions or a loop, but the same thing can be done much more cleverly using just four or three or two carefully chosen instructions whose interactions are not at all obvious until explained or fathomed" [Warren, 2002]. In synthesis of bitvector algorithm initial approaches used straight-forward brute-force search [Massalin, 1987], while newer approaches are to leverage SMT solver reasoning [Jha et al., 2010].

Other systems are able to fill in holes left in programs. The Sketch system [Solar-Lezama, 2009] deals with finding the appropriate values for single value holes in imperative programs. The holes are typically boundary conditions, e.g. for loops. In template-based synthesis [Srivastava et al., 2013] the conditions on holes are less-restricted in that holes can be filled by arbitrary expressions. Templates are a way for the user to provide their insight to the synthesis system, by writing code or invariants with holes. Constraints are generated for these holes and SMT solvers are used to find solutions.

Sometimes we are able to give a very precise specification for algorithms. For example, we may be able to express the condition for properly implementing critical sections and shared variables for mutual exclusion algorithms. A synthesis system may then be able synthesis new concurrent algorithms by properly inserting lock acquiring and releasing statements. [Bar-David and Taubenfeld, 2003] More generally we might be able to very precisely describe how function behave. For functional programs a good place to assert such a specification is in the type of the program [Frankle et al., 2016]. In [Polikarpova et al., 2016] non-trivial algorithms over binary-search trees are synthesized from specifications expressed as refinement types. The synthesis technique derives from type checking rules for the programming language that is considered.

The are a couple of domains that are only well suited for synthesis from examples. We highlight string and matrix transformations. A string transformation is a mapping on strings. For example "tony.hoare@cs.ox.ac.uk" might

be mapped to "Hoare, T., UK". One of the applications of string transformation is to learn spreadsheet operations [Singh and Gulwani, 2012a]. String transformations are also an often considered problem in ILP, e.g. [Cropper et al., 2016]. Similarly one can try to learn matrix transformations, by giving input-output examples. These examples are typically larger than string transformations, making it a good benchmark test for synthesis systems [Wang et al., 2017].

Another avenue explored is inventing strategies for robots performing tasks. In [Cropper and Muggleton, 2016a] a very high-level strategy is learned for serving either tea or coffee for a table of cups. The system is provided with examples of good behaviour which are used to generate a higher-order logic program describing the actions that the robot should perform.

#### 2.2.2 Specifications

From the user's perspective one of the most important features of a synthesis system is which problems it is able to solve. In this subsection we present different ways of specifying the synthesis problem. We distinguish two main aspects to specification, namely how to specify the *goal* and how to provide background knowledge.

**Goals** The *goal* of a synthesis problem is a program that satisfies the specification. The are multiple ways to describe what is expected of the goal.

There is the programming-by-example specification, where the conditions on the goal are stated as input-output examples [Cropper and Muggleton, 2016b]. Examples are split in positive examples, ones that the synthesized program needs to satisfy, and negative examples (also called counter-examples) which should not be entailed by the program. A related specification of the goal is programming-by-demonstration [Lau et al., 2000]: in addition to providing input-output examples a (partial) trace is provided of the transformations that turned the input into the output.

As discussed in the previous section, there is the Sketch/templates approach to specification by writing programs with holes. In essence the program goal is already partially given by the user and only small fragments need to be filled in.

A very general approach is to say that any (set of) logical proposition(s) might be used as a specification. The previous section's mutual exclusion conditions would fall in this category. A more rigid framework follows from only allowing specifications over a function's arguments. By the Curry-Howard correspondence these propositions may also be stated as function typing [Frankle et al., 2016]. A recent development is that even examples

may be encoded in types [Osera and Zdancewic, 2015], yielding types as a very powerful specification vehicle.

Background knowledge There are systems that approach synthesis with only a specification. Such systems are forced to perform searches over very larges program spaces, the exhaustive search of small lambda-terms satisfying a type, by [Katayama, 2005], being such an example. More common is to accept guidance from the user, in which case we call the provided hints the background knowledge.

In inductive logic programming the user may provide background knowledge in the form of facts, defined Horn clauses (which may be higher-order) [Raedt, 2010]. In meta-interpretive learning [Muggleton et al., 2014b] in addition metarules are supplied. Metarules define structure for clauses which are *invented* by the synthesis system. In chapter 3 we will at these metarules in more detail.

In the setting where we are trying to synthesis functional programs from specification embedded in types background knowledge takes on the following forms. Type declarations informing the system of the data structures over which to operate. Type declarations for helper functions (with or without the actual definition of the functions). For encoding more precise properties refinement types are leveraged.

As a final example of background knowledge we highlight that the Sketch and template approaches to synthesis blur the line between providing goal specifications and background knowledge. Partial programs are more typically background knowledge, but are here used as the primary means of specification.

#### 2.2.3 Paradigms and Types of Programs Learned

In this subsection we look some of the paradigms for program synthesis. For the separate approaches we highlight the types of programs that have been synthesized by the method. The most straightforward approach to synthesis to enumerative the entire space of programs that your method is willing to consider, e.g. [Katayama, 2005] searches over all lambda-terms. We look at several more sophisticated methods. We will look at Inductive Logic Programming here as it will feature prominently in the remainder of the document.

Maintaining consistent programs One idea is to maintain the space of programs that are consistent with examples. With no examples the entire

space of programs works. By interatively adding examples one can start reducing this space of consistent programs. Version Space Algebra [Lau et al., 2000] is a powerful approach whereby the space of programs (version space) is maintained by a partial ordering of programs (usually by generality), which can be fully represented by the maxima and minima of the ordering. An update function is able to shrink these boundaries for each additional example considered.

Other approaches that are able to use a succinct representation of the space of solutions is the work by Gvero et al. [2013] where types are used to represent classes of expressions that are candidates for code completion queries in IDEs. In [Singh and Gulwani, 2012a], an exponentially sized space of consistent string transformation is maintained in polynomial space by a clever sharing of shared sub-expressions.

Constraint solving A very general approach to synthesis is to convert the problem to (logical) constraints. These constraints are written in a language of an off-the-shelf solver (in particular SMT solvers). The results of the solver are then used to construct program solutions.

In the Sketch/template approach to synthesis the holes in programs are surrounded by code that imposes conditions on the possibilities for such holes [Solar-Lezama, 2009] [Srivastava et al., 2013]. In Constraint-Based Synthesis of Datalog Programs [B et al., 2017] the derivation algorithm for (first-order) logic programming is encoded as logical constraints, with additionally that the predicates symbols are allowed to vary, again subject to constraints. To make problem tractable the constraints only encode derivations of a bounded length. Any solution found satisfying the constraints for partial derivations is then checked for being an actual solution by using the found program to build a derivation in Datalog itself. If the program does not work an additional constraint is generated excluding the program from being considered again.

Inductive logic programming We have that in ILP the norm is to learn untyped logic programs. Logic programs consist of Horn clauses, implications with a single atom in the consequent [Raedt, 2010]. These programs are commonly interpreted as either Prolog, or Datalog code. Prolog represents the SLD-resolution approach to program execution, and comes with features that make the language Turing-complete, whilst Datalog uses grounding of first-order logic to propositional logic to determine entailment (which is decidable). While ILP systems primarily are used to synthesize first-order programs, with an important feature being recursive programs, recent work makes invention of higher-order programs possible [Muggleton et al., 2015].

This document will further discuss the Meta-Interpretive Learning (MIL) framework as a particularly strong approach to ILP. A simplistic approach to typing in MIL was already considered in [Farquhar et al., 2015] in order to learn proof strategies. The system only support simple non-polymorphic types and hardcodes types in metarules.

Typing As already stated, the specification of functions is well addressed by types. There are a number of approaches in the literature on using types to synthesize functional programs. The norm is to synthesis programs according to an (existing) type system, with the guarantee that the programs type checks. Important features include being parametric polymorphism, algebraic data types, and structural recursion over these types. Modern systems are able to utilize refinement types as specification, but also direct the search [Polikarpova et al., 2016].

Type directed synthesis usually takes an approach that is very similar to theorem provers. As the program will need to conform to a type according to the rules of a type system, these rules are actually used to direct the search [Osera and Zdancewic, 2015]. The type specification is decomposed according the rules and non-deterministic choices are made when the premises of the rules need information that is not captured in the conclusion, i.e. the current program fragment to be proven.

It is interesting to note that type directed synthesis is quite flexible. In [Frankle et al., 2016] it is noted that examples can encoded in types, meaning that types are expressive enough to capture logical requirements on programs as well as inductive synthesis from examples. In [Gvero et al., 2013] types are used direct the generation of code completions in IDEs. Here types are viewed as set of consistent expressions all being candidates to be enumerated.

Arbitrary DSLs A more ambitious approach is to synthesis programs not restricted to a particular programming language. In work by [Wang et al., 2017] a domain expert provides a domain specific language (DSL) and the end-user provides examples. Along with the DSL concrete semantics (an evaluation function) and abstract semantics (a mapping to abstract values, e.g. value ranges) are provided. The domain experts chooses the appropriate abstractions made available to the system.

The approach uses finite tree automata (FTA) to encode abstract syntax trees (ASTs) of the DSL with predicates on the nodes encoding their abstract semantics. The FTA is used to maintain a set of programs whose abstract semantics are consistent with the examples. A "best" program is selected among the accepted programs and is checked against the examples. If un-

successful the tree automata is modified such that more nodes with abstract semantics become available and with the guarantee that the previously selected programs are no longer successful. This is iterated until a successful program is found. It is shown in the paper that this system is able to outperform multiple other purpose built systems. However, a severe limitation of this approach is that it does not handle DSLs with binders, e.g. lambda terms and let bindings.

# Chapter 3

# Untyped Meta-Interpretive Learning

This chapter contains an overview of the existing meta-interpretive learning (MIL) approach to one variant of the inductive logic programming problem. First, we state prerequisites before presenting a formal definition of the problem addressed by MIL. Subsequently the MIL framework is explained, along with the central role of metarules. Finally, the high-level algorithm is presented in the form of  $Metagol_{AI}$ , an implementation of MIL which incorporates a powerful extension to allow for higher-order abstractions.

# 3.1 Logic Prerequisites

We work within the framework of logic programming. A reader unfamiliar with this topic is referred to [Nienhuys-Cheng and De Wolf, 1997] for a comprehensive treatment. The primary logic programming features we will assume familiarity with are logical variables, unification on logical variables, and SLD-resolution (along with seeing a SLD-tree as a derivation/proof of a goal atom). We use the language of logic to introduce the requisite concepts. The definitions in this chapter mainly follow those of Cropper [2017]'s PhD thesis.

**First-order** A variable is a character string whose initial letter is uppercase. Function and predicate symbols are character strings whose initial letter is lowercase. The arity n of a predicate/function symbol p is the number of arguments it takes and is denoted as p/n. The predicate signature  $\mathcal{P}$  is the set of predicate symbols with arity greater than 0. A constant is a function symbol with arity zero. The constant signature  $\mathcal{C}$  is the set of constant sym-

bols. A variable which can be substituted by a constant or function symbol is called a *first-order*. The set of first-order variables is denoted as  $\mathcal{V}_1$ . A *term* is a variable, a constant symbol, or a function symbol of arity n immediately followed by a bracketed n-tuple of terms. A term which contains no variables is called *ground*. A formula  $p(t_1, \ldots, t_n)$ , where p is a predicate symbol of arity n and each  $t_i$  is a term, is called an *atom*. An atom is ground if all of its terms are ground. The  $\neg$  symbol is used for negation. A *literal* is an atom A or its negation  $\neg A$ .

Clauses A *clause* is a finite disjunction of literals. Each variable in a clause is implicitly universally quantified. A clause that contains no variables is ground. Clauses with at most one positive literal are called *Horn clauses*. A Horn clause with exactly one positive literal is called a *definite clause*:

**Definition 3.1.1.** A (first-order) definite clause is of the form:

$$A_0 \leftarrow A_1, \dots, A_m$$

where  $m \geq 0$  and each  $A_i$  is an atom of the form  $p(t_1, \ldots, t_n)$ , such that  $p \in \mathcal{P}$  and  $t_i \in \mathcal{C} \cup \mathcal{V}_1$ . The atom  $A_0$  is the *head* and the conjunction  $A_1, \ldots, A_m$  is the *body*.

A definite clause with no body literals is called a *fact*. A Horn clause with no head, i.e. no positive literal, is called a *qoal*.

**Higher-order** For higher-order logic, the quantification of first-order logic is extended to allow for quantifiers to range over predicate and function symbols. A variable which can be substituted by a predicate symbol is *higher-order*. The set of higher-order variables is denoted as  $V_2$ . A *higher-order* term is a higher-order variable or a predicate symbol. An atom which has at least one higher-order term is higher-order. A definite clause with at least one higher-order atom is higher-order:

**Definition 3.1.2.** A higher-order definite clause is of the form:

$$A_0 \leftarrow A_1, \ldots, A_m$$

where  $m \geq 0$  and each  $A_i$  is an atom of the form  $p(t_1, \ldots, t_n)$ , such that  $p \in \mathcal{P} \cup \mathcal{V}_2$  and  $t_i \in \mathcal{C} \cup \mathcal{P} \cup \mathcal{V}_1 \cup \mathcal{V}_2$ .

**Substitution** Given a formula with variables  $v_1, \ldots, v_n$ , simultaneously replacing the variables with terms  $t_1, \ldots, t_n$  is called a substitution. Such a substitution is denoted by  $\theta = \{v_1/t_1, \ldots, v_n/t_n\}$ . A substitution  $\theta$  unifies atoms A and B in the case  $A\theta = B\theta$ .

#### 3.2 Meta-Interpretive Learning

This section starts by formally introducing the problem addressed by MIL. To do so we first need the key MIL concept of metarules. The second part of this section gives an overview of meta-interpreting and how metarules lift this notion to meta-interpretive learning.

#### 3.2.1 Problem Statement

The user supplies a set of examples  $\mathcal{E}$  and background knowledge  $\mathcal{B}$ . All examples  $e \in \mathcal{E}$  are ground atoms over the same predicate name. The examples  $\mathcal{E} = (\mathcal{E}^+, \mathcal{E}^-)$  are divided into positive and negative examples. The background knowledge  $\mathcal{B} = \mathcal{D} \cup \mathcal{M}$  consists of definite clauses  $\mathcal{D}$ , representing program fragments, and metarules  $\mathcal{M}$ . The definite clauses also encode the facts (clauses without body) that are asserted.

**Definition 3.2.1.** A higher-order formula of the form

$$\exists \pi \forall \mu : A_0 \leftarrow A_1, \ldots, A_m$$

where  $m \geq 0$ ,  $\pi$  and  $\mu$  are disjoint sets of higher-order variables, is called a metarule. Each  $A_i$  is an atom of the form  $p(t_1, \ldots, t_n)$  such that  $p \in \mathcal{P} \cup \pi \cup \mu$  and each  $t_i \in \mathcal{C} \cup \mathcal{P} \cup \pi \cup \mu$ .

Metarules differ from higher-order definite clauses in that they allow existential quantification.

Table 3.1 lists common metarules, a selection of which will be used throughout this document. Note that we elide the quantifiers, e.g. the full definition of the Identity metarule is  $\exists P \exists Q \forall A \forall B \ P(A,B) \leftarrow Q(A,B)$ . When quantifiers are omitted, we always label universally quantified first-order variables with names from the start of the alphabet  $(A,B,C,\ldots)$  and existentially quantified higher-order variables with names starting from P on in the alphabet.

**Definition 3.2.2.** Given  $(\mathcal{B}, \mathcal{E}) = (\mathcal{B}, (\mathcal{E}^+, \mathcal{E}^-))$  as background knowledge and examples, respectively, a program H is a *consistent hypothesis*, denoted  $H \cup \mathcal{B} \models \mathcal{E}$ , if all positive examples are entailed by the program, and none of the negative examples are.

**Definition 3.2.3.** A  $MIL\ learner$  takes an input (B,E) and either outputs a definite program H that is a consistent hypothesis for the input, or terminates stating failure to find a program.

name	metarule
Identity	$P(A,B) \leftarrow Q(A,B)$
Precon	$P(A,B) \leftarrow Q(A), R(A,B)$
Curry	$P(A,B) \leftarrow P(A,B,R)$
Chain	$P(A,B) \leftarrow Q(A,C), R(C,B)$
Tailrec	$P(A,B) \leftarrow Q(A,C), P(A,B)$

Table 3.1: Common metarules, where variables A,B, and C are universally quantified and P, Q, and R are existentially quantified.

We consider MIL learners that have the additional guarantee that they return programs that are optimal in a textual complexity sense. The optimizing criterion used is the number of clauses in the program. The guarantee is that there is no other consistent hypothesis which has fewer clauses.

#### 3.2.2 Meta-Interpreting and Metarules

A Prolog meta-interpreter evaluates a Prolog-like language by unifying a goal with a head of one of the first-order clauses that it has available. The atoms in the body of the unified clause become new goals subject to the same procedure. For example, SLD-resolution on the goal ancestor(alice, charlie), given the definite clauses

```
parent(alice, bob)

parent(bob, charlie)

ancestor(A, B) \leftarrow parent(A, B)

ancestor(A, B) \leftarrow parent(C, B), ancestor(A, C)
```

yields by resolution with the last clause (unifying the goal with the head of this clause) that the goals required to prove become ancestor(alice, C) and parent(C, charlie). To prove the first of these two goals the first clause with ancestor as head is chosen (non-deterministically out of the two options). The two goals are now parent(alice, C) and parent(C, charlie). Unifying the first goal with the first fact fixes C to bob allowing both goals to be proven by the asserted facts.

The Meta-Interpretive Learning framework is a meta-interpreter that additionally tries to unify a goal with the head of a metarule, which is a higher-order clause. Upon selecting a metarule to prove the goal, the unification is saved in the form of a *meta-substitution*.

**Definition 3.2.4.** Let M be a metarule with the name x, C be a horn clause,  $\theta$  be a unifying substitution of M and C, and  $\Sigma \subseteq \theta$  be the substitutions where the variables are all existentially quantified in M, such that  $\Sigma = \{v_1/t_1, \ldots, v_n/t_n\}$ . Then a meta-substitution for M and C is an atom of the form:  $sub(x, [v_1, \ldots, v_n]\{v_1/t_1, \ldots, v_n/t_n\})$ , where the second argument is a list of logical variables with the appropriate substitution applied.

Saved meta-substitutions are reused for proving goals that are encountered later on, becoming available as unification targets like the definite clauses in the background knowledge. Upon completing the proof of all goals the meta-substitutions contain a description of the program. To obtain the program the saved substitutions are applied to the named metarules. Each such instantiated metarule corresponds to an invented clause of the program, with the predicate symbols being grounded through the substitution.

**Definition 3.2.5.** Let  $(\mathcal{B}, \mathcal{E})$  be a MIL input and H a hypothesis. Then a predicate p is an *invention* if it is in the predicate signature of H and not in the predicate signature of  $\mathcal{B} \cup \mathcal{E}$ .

Metarules form the heart of the MIL approach to learning. Metarules allow for introducing a strong bias to the hypothesis space. This is due to the clauses of programs in the hypothesis having to conform to the structure of the supplied metarules. Metarules hence give great control over the size of the search space, as well as how the search space is traversed. They also allow the user to specify which features they consider likely to be needed by the program. Examples are (tail-)recursion and higher-order clauses.

In recent work by Cropper and Muggleton [2016a] the MIL framework is expanded so as to allow background predicates and inventions with higher-order arguments.

# 3.3 Metagol $_{AI}$ : Untyped Abstracted MIL

The implementation of MIL that we consider as a basis in this document is the  $Metagol_{AI}$  system introduced by Cropper and Muggleton [2016a]. This section shows how to specify a problem instance in Prolog code and goes over a high-level version of the algorithm.

The improvement of the  $Metagol_{AI}$  versus the original Metagol implementation of MIL [Muggleton et al., 2014a] is in allowing higher-order predicates to be part of the program, in particular predicates that are termed *abstractions*. As shown in [Cropper and Muggleton, 2016a], learning higher-order inventions has benefits in terms of finding smaller programs, which leads to

reduced learning times, and can achieve high accuracy with a reduced number of examples.

**Definition 3.3.1.** A higher-order definition is a set of higher-order definite clauses with matching head predicates.

The following definition of map, operating over lists<sup>1</sup>, is an example of a higher-order definition:

$$map([], [], F) \leftarrow map([A|As], [B|Bs], F) \leftarrow F(A, B), map(As, Bs)$$

Any clause which takes an argument that is a predicate is termed an abstraction:

**Definition 3.3.2.** A higher-order definite clause of the form

$$\forall \tau \ p(s_1,\ldots,s_m) \leftarrow q(u_1,\ldots,u_n,v_1\ldots,v_o)$$

where o > 0,  $\tau \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$ ,  $p, q, v_1, \dots, v_o \in \mathcal{P}$ , and  $s_1, \dots, s_m, u_1, \dots, u_n \in \mathcal{V}_1$ , is called an *abstraction*.

The following clause, which increases each element of a list by one, is an example of an abstraction:

$$increment\_all(A, B) \leftarrow map(A, B, succ).$$

#### 3.3.1 Representation of Background Knowledge

For Prolog code we will use the **typewriter** font. Definite clauses in Prolog are very similar to the logic syntax, except the arrow  $(\leftarrow)$  is replaced by an ASCII version (:-) and every clause is terminated by a dot. For clauses with empty bodies the ersatz arrow is omitted.

The background knowledge is separated into three parts: primitive clauses, interpreted clauses and metarules. Primitive clauses are just standard Prolog definite clauses, as such they do not involve atoms whose predicate symbol are variable. Such clause definitions are added to the background knowledge by asserting that they are available as a primitive, e.g. id(X,X). is a definite clause, and by asserting prim(id/2) the predicate is added to the background knowledge.

<sup>&</sup>lt;sup>1</sup>A list in Prolog is either [], the empty list, or is a cons of a head H and a tail T, denoted as [H|T].

The interpreted clauses are Prolog clauses that do involve body atoms whose predicate symbols are existentially quantified, and in particular are arguments to the head of the clause. Because during the synthesis algorithm a higher-order argument might remain undetermined until the body is evaluated, the normal Prolog evaluation strategy is not sufficient (see the below algorithm). These clauses are asserted to be *interpreted*, e.g. interpreted(map/3).

Asserting metarule(Name, Subs, (Head: -Body)) adds a metarule with the name Name to the background knowledge. Subs is a list of the existential (higher-order) variables occurring in the metarule, Head is the head atom of the rule and Body is a list of the body atoms of the rule. Take the chain rule as an example:

metarule(chain, [P,Q,R], (P(A,B):-[Q(A,B),R(B,C)])).

#### 3.3.2 The Algorithm

Figure 3.1 contains the Prolog code for the abstracted MIL algorithm, which has support for higher-order abstraction.

Invocation The first clause, learn, is the invocation point of the algorithm. After the user has asserted their background knowledge, they run the algorithm by calling learn with a list of their positive and a list of their negative examples whereby, upon success, Prog gets instantiated to a program. The algorithm starts out with an empty program, the first argument to prove in the body of learn. If a program is found entailing the positive examples it is checked against the negative examples. Were one of the negative examples to be entailed, backtracking occurs and the search will continue for another program entailing the positive examples. When all the positive examples are entailed by the found program, and none of the negative examples are, the search successfully terminates.

Search The prove procedure is mainly for choosing the first goal in a list of goals to hand off to the prove\_aux procedure. The mutual recursion of prove and prove\_aux represents a left-most depth-first search over the goals that arise during synthesis. When a goal has been proven successfully by prove\_aux it may have changed the program under construction, hence the new program is passed along for proving the remaining Atoms goals. As the last disjuctive clause prove\_aux will always succeed with introducing a new invented clause, an additional mechanism, not shown in the code, of an limit on the number of inventions is used. The limit makes sure that the depth first search does not run off and keeps creating invented clauses for goals

```
learn(Pos,Neg,Prog):-
 prove(Pos,[],Prog),
 not(prove(Neg,Prog,Prog)).
prove([],Prog,Prog).
prove([Atom|Atoms],Prog1,Prog2):-
 prove_aux(Atom, Prog1, Prog3),
 prove(Atoms, Prog3, Prog2).
prove_aux(Atom,Prog,Prog):-
 prim(Atom),
 call(Atom).
prove_aux(Atom,Prog1,Prog2):-
 interpreted((Atom:-Body)),
 prove (Body, Prog1, Prog2).
prove_aux(Atom, Prog1, Prog2):-
 member(sub(Name,Subs),Prog1),
 metarule(Name, Subs, (Atom: -Body)),
 prove (Body, Prog1, Prog2).
prove_aux(Atom,Prog1,Prog2):-
 metarule(Name, Subs, (Atom: -Body)),
 prove(Body, [sub(Name, Subs)|Prog1], Prog2).
```

Figure 3.1: The Metagol<sub>AI</sub> algorithm.

that are difficult (or impossible) to prove. By re-running the algorithm with increasing invention limit all possible programs are considered.

**Additional proving rules** For the most important part of the algorithm, we consider each of the disjunctive bodies of prove\_aux separately:

• The first disjunct tries to prove an atom (whose predicate symbol might be a variable) by unifying the atom's predicate with a predicate asserted as background knowledge, considering all options based on the predicate's arity. When this unification succeeds the atom is unified with the head of the predicate, whereupon it is known that Atom represents a first-order atom which Prolog is able to evaluate. Evaluation is invoked by call(Atom). If successful the atom's predicate symbol will remain fixed for for any goals subsequently generated. If the call to call fails, or having tried all possible programs which included this particular choice for the atom causing Prolog to backtrack to this decision point, all remaining predicates in the background knowledge will be tried in

the same manner. If none lead to a successful program being found this disjunct of prove\_aux fails, causing the next disjunct to be tried.

- The second disjunct tries to prove the goal atom by unifying the atom with the head of one the head of interpreted background predicates. Again, if the predicate symbol is a variable it will become fixed upon successful unification. By unifying the head of an interpreted clause the corresponding variables in the Body change accordingly, thereby making the body of the interpreted clause the goals that need to be proved to be able to conclude the unified head.
- Upon failure to find a successful program by proving the atom by an interpreted predicate, the algorithm checks if it may make use of any of the invented clauses that it keeps track of in the form of metasubstitutions. For each meta-substitution that is already in the program the algorithm tries to unify the atom with a fresh head of the meta-substitution's metarule, under the restrictions of the variables that are already fixed in the substitution list Subs. If it succeeds, the body of the reused invented clause represents the goals that need to be proven.
- Finally, the last disjunct applies when all other disjuncts failed to lead to a successful program. Now a metarule is instantiated, with no conditions on the Substitutions for the existentially quantified predicate symbols. This means that Atom is used to create a new invention. The body atoms of the metarule become the goals to be proven and the invention is remembered by prepending a new meta-substitution to the existing program.

In overview: a depth-first search is conducted over partial programs, starting with the examples as goals. To prove each goal, first it is attempted to be proven by one of the primitive and interpreted predicates. In the first case Prolog is able to evaluate whether the atom holds, and in the second case the body of the higher-order predicate remain as goals to be proven by meta-interpretation. Otherwise the saved invented clauses are tried, again leading to additional goals to be proven. As a last resort an atom can be proven by creating a metarule-guided invention, where the body of the invention will likely contain goal atoms with predicate variables. If there are no more goals to prove a successful program has been found.

# Chapter 4

# Typed Meta-Interpretive Learning

Starting with this chapter, this document will focus on presenting novel contributions. This chapter highlights the potential benefits of adding typing to Meta-Interpretive Learning (MIL) before the subsequent chapters focus on tackling the search space reduction issue. After some preliminaries regarding types, three areas are discussed where types bring with them great potential. The first is the ability to prune away parts of the search space. Second, we consider the idea that types can provide guidance in choosing how to traverse the search space. The final highlighted feature is that it might be possible to inform the user that the background knowledge that they provided is insufficient for solving the synthesis problem.

## 4.1 Typing Definitions

As is usual in presenting logic, the definitions of the previous chapter did not include any type decorations. For our purposes a *type* is a set of values. A value that belongs to a type is called an *inhabitant*. Normally it is known for functions and predicates for which values a predicate or function makes sense. We will restrict the domain of predicates/functions to just these values. We will use such domain restrictions, along with co-domain restrictions, in defining the formal types of predicates and functions. As a general reference for refinement types with polymorphism see [Freeman and Pfenning, 1991].

**Polymorphic types** The types we consider are constructed from a collection of base types B, among others, the integers, int, and the Booleans, denoted bool. We also allow holes in our types, for which we use type vari

ables. The Cartesian product  $(\times)$  is used to construct types whose values are tuples of values from the supplied types. The arrow  $(\rightarrow)$  is used to construct function types from other types. We have that the grammar<sup>1</sup> for our types is stated as

$$T ::= b_i \mid X_i \mid T \times T \mid T \to T$$

where  $b_i \in B$  and  $X_j$  represents type variables. A type is *higher-order* if a function type appears in an argument position.

**Definition 4.1.1.** A predicate p, with arity n, is a function whose result type is Boolean. The *predicate type* of p is fixed by the types of its n arguments. This is denoted as:

$$p: T_1 \times \ldots \times T_n \to bool$$
 or  $p(a_1, \ldots, a_n): T_1 \times \ldots \times T_n \to bool$ 

where  $a_i : T_i$  are the formal arguments of p.

We will abbreviate the predicate typing  $p: T_1 \times \ldots \times T_n \to bool$  to the compacter  $p: [T_1, \ldots, T_n]$ . In what follows we will be lax in distinguishing between the type of an atom (always bool) and the type of the atom's predicate.

For function typing a similar definition holds, except that the co-domain is not fixed to be Boolean. For functions we do not use shorthand notation.

A type is *polymorphic* when it is parametrized by one or more types, i.e. it contains type variables. We always consider type variables to be universally quantified. For example, the polymorphic type of lists is list(X). Replacing the *type parameter* X with a type resolves the polymorphism, e.g. take X = int, then list(int) is the type of lists of integers, which is no longer polymorphic. A predicate is polymorphic when one of its argument types is.

**Refinement types** Given the above description of types we define the notion of refinement type, and refer to unrefined types as  $simple\ (polymorphic)$  types. A refinement type  $\{x:T\mid\varphi\}$  is the subset of the type T consisting of the values x that satisfy the formula  $\varphi$ . For this work we only allow refinements to occur on predicates.

In order to discuss the meaning of refinements on predicates, hereby a reminder regarding the interpretation of predicates: a predicate implements a relation, which identifies tuples, by mapping arguments included in the relation to  $\top$  (true) and otherwise returning  $\bot$  (false).

Let  $p(a_1, \ldots, a_n): T_1 \times \ldots \times T_n \to bool$  be a predicate with its type. A refinement  $\varphi$  for this type is a proposition that can mention any of the variables

<sup>&</sup>lt;sup>1</sup>This is in essence the type grammar of System-F [Girard, 1971].

 $a_i$ . The refined type is denoted as  $T_1 \times \ldots \times T_n \to bool\langle \varphi \rangle$ , with shorthand  $[T_1, \ldots, T_n]\langle \varphi \rangle$ . The semantics of this type is that  $\varphi$  denotes a necessary condition of any  $\top$  inhabitant of the Boolean result type. In conventional refinement type notation this corresponds to

$$bool\langle\varphi\rangle = \{b : bool \mid b = \bot \lor (b = \top \Rightarrow \varphi)\}$$

$$= \{b : bool \mid b = \bot \lor (b \neq \top \lor \varphi)\}$$

$$= \{b : bool \mid \neg\varphi \Rightarrow b = \bot\}.$$

Given a valuation for the arguments of p, the proposition  $\varphi$  has a truth value. When a valuation makes  $\varphi$  false, the values of the arguments to the predicate cannot be identified by the relation that the predicate is representing, as the predicate must return false for these arguments. Hence refinement are a way of restricting types to make them more accurate.

As an example consider the higher-order *map* predicate, as of it is a good candidate for a refinement type. The *map* relation guarantees that the list arguments are of the same length. This is encoded in a refinement type as follows:

$$map(A, B, F) : [list(T), list(S), [T, S]] \langle length(A) = length(B) \rangle$$

It is important to note that simple polymorphic types are precisely our refinement types with all refinements just being the *true* proposition (not imposing any restrictions). In this text we will elide types (and their refinements) when they are not of interest to the discussion at hand.

# 4.2 Pruning the Search Space

The MIL approach is surprisingly effective gives its simplicity. The algorithm is quite naive in regard to what possibilities for programs it is willing to consider. One major shortcoming, inherited from its logic programming origins, is its irreverence of types. Many of the logic programming languages do not concern themselves with types. In particular, Prolog does not have predicate types.

Consider having to prove a goal atom P([true, false, true], [1]), with P a predicate variable. We can immediately see that its type is [list(bool), list(int)]. Let us suppose that the given background predicates include the predicates succ: [int, int] (the successor relation on integers), tail: [list(X), list(X)] (the list tail relation), and map: [list(X), list(Y), [X, Y]] (the relation that maps another relation over elements). The metarules we consider for this example are Chain and Curry (see table 3.1). Clearly any predicate in the

background knowledge over non-list types need not be tried as a direct substitution for P. Also many inventions and interpreted predicates need not be tried, such as  $inv(A, B) \leftarrow tail(A, C), map(C, B, succ)$ , purely due to the typing of this clause being incompatible with the type of the example goal.

#### 4.2.1 A Worked Example

The work involved in the above example is considerable given how easy it is to see, based on typing, that it cannot succeed. First P will be unified with inv whereupon unification of the atoms assigns A = [true, false, true] and B =[1]. The body atoms tail([true, false, true], C) and map(C, [1], succ) become the new goals, being passed of to be proven by a recursive prove call. The atom tail([true, false, true], C) is proven in the first disjunct of prove\_aux, assigning C = [false, true]. Now the second goal map([false, true], 1, succ)is considered. None of the primitives predicates unify with the predicate name, hence the interpreter clause of prove\_aux is evaluated. Here the head of map is first checked for whether it happens to unify with base case over empty lists, which it does not. Subsequently the head of the inductive disjunct of map is unified, whereupon succ(false, 1) and map([true], [], succ) become the new goals. Again prove is called recursively, whereupon succ(false, 1)is selected to be proven by delegating the evaluation of the atom to Prolog itself. Prolog determines that this atom does not hold. It is only at this point that Metagol<sub>AI</sub> is able to decide that the invented clause  $inv(A, B) \leftarrow$ tail(A,C), map(C,B,succ) could not be used, leading the meta-interpreter to backtrack to the decision point of having to pick a way of proving the original atom P([true, false, true], [1]).

**Type Checked** In contrast, when we consider types, a unification attempt is enough to determine that the above work is unnecessary. The type annotations for the atoms are

$$P([true, false, true], [0]) : [list(bool), list(int)]$$

and

$$inv(A, B) : [list(int), list(int)] \leftarrow tail(A, C), map(C, B, succ).$$

Type checking corresponds to checking whether the type of inv can have zero or more of its variables unified such that it corresponds to P's type. Clearly the first arguments' types cause unification to fail, thereby rejecting the attempt to try to prove the goal atom with the invented clause.

This is just one example of where simple type checking helps considerably. Every time type checking can determine that one of the options considered by  $Metagol_{AI}$  cannot be successful exploration of a part of the search space can be skipped over, which is called *pruning*. When such parts of the search space contain multiple decisions points considered by the algorithm the benefit of type checking becomes considerable more significant.

#### 4.2.2 Granularity of types

Polymorphic types are already quite powerful in pruning parts of the search space. But suppose we make a simple adjustment to the above example. Instead of our goal's arguments including Booleans we change these values to integers, making the example goal P([1,0,1],[1]). A type check determines that the atom's type [list(int), list(int)] is unifiable with [list(int), list(int)], hence giving the go-ahead to attempt to prove the atom with inv.

Unnecessary work revisited Because polymorphic type checking is unable to rule out the invented clause the algorithm will be forced to do the same work as detailed above. When it reaches the atom succ(0,1) Prolog will instead determine that this atom does hold. Now even more work will be performed, namely the remaining goal, map([1], [], succ), will be passed off to prove\_aux, which again checks that this is not a primitive before checking the interpreted disjunctive clauses of map. Thanks to the first argument the base case does not apply, and because the second argument does not allow itself to be split into a head and tail the inductive case of map also does not apply. So in this case, there was not only more work in trying to prove additional goals before finally finding out that the invented clause cannot be used, but we also incurred a small cost for the unification attempt of the type checking.

Intuitively, when we look at the example goal and at the invented clause we are still able to immediately see that this clause cannot work. This is due to being able to reason about the lengths of the lists involved. Clearly all the tuples entailed by the invented clause have as property that the first argument has exactly one additional element versus its second argument.

**Refinement reasoning** The above noted relation on the arguments of inv is a property well captured by a refinement type and can be stated as:

$$inv(A, B) : [list(int), list(int)] \langle length(A) = length(B) + 1 \rangle$$

This type states that the predicate can only entail the arguments if the first list has a length exactly one longer than the second list. If we now return

to the type checking of P([1,0,1],[1]):[list(int),list(int)] against the type of inv we not only try to unify the polymorphic types, but we also check that the refinement is still satisfiable, i.e. does not preclude the arguments from being entailed. What happens is that the heads are speculatively unified, leading to the refinement becoming instantiated to length([1,0,1]) = length([1]) + 1. For this example checking the refinement can simply be done by expanding the definition of length, which directly derives the contradiction 3 = 2. The refinement type hence declares that these values are never part of the relation encoded by the inv clause. At this point the algorithm will give up on inv and will try the next option for proving the example atom. Refinement type checking is hence able to prune parts of the search space that simple polymorphic type checking is not able to.

There is, of course, the issue that reasoning over (instantiated) refinements is not entirely trivial. For the example the work needed to come to a contradiction was very simple. The main challenge for the usage of refinement types is in identifying refinements that suitably abstract from the predicates, and in finding algorithms that are able to very rapidly reason on the constraints specified by the refinements.

Composing refinements The above identified refinement for inv is not something a user would be able to supply to the system, as the clause in question is an invention. The body atoms on the other hand are background predicates which the user can annotate with appropriate refinements. The refinement for tail(A, C) could well be

$$length(A) = length(C) + 1$$

while map(C, B, F)'s refinement can be taken to be length(C) = length(B). The structure of the clause now indicates how these refinements compose, namely  $length(A) = length(C) + 1 \wedge length(C) = length(B)$ . Clearly, this approach to refinements is quite powerful in that is it able to capture sensible refinements even for invented clauses.

## 4.3 Types Directing the Search

The  $Metagol_{AI}$  algorithm uses a basic depth-first search procedure for determining the order in which goals are proven. A novel observation is that this search might be better steered by the information available in the goals that remain to be proven. The idea is that the ordering of the goals in the search needs to be guided by a heuristic function. Correspondingly the search

algorithm needs to be adjusted so as to implement the best-first search procedure. Best-first search keeps track of all the nodes left to explore, in our case the current goals left to prove, and selects among them the best node to explore/prove next (according to the heuristic). As it is non-obvious what the main features of interest are and how much weight they should have relative to each other, work is needed in determining good heuristic ordering functions.

The goals are the basic units whose proof order needs to be decided on, which, in the untyped case have quite limited information available. Obvious considerations are to prioritize those goals that are already entirely ground, including the predicate symbol(s). Amongst them the atoms with primitive predicate symbols would need to be sorted first as an inconsistency on such a goal would lead to immediate backtracking. Next one would sort on the number of variables in the goals, preferring atoms with fewer variables as these are less likely to incur non-determinism in their proof. Issues in ordering start to crop up when deciding on the ordering of atoms with different arities and whether the proportion of the number of ground arguments might not be a more useful feature than just the sheer number of them.

**Type Guidance** The introduction of types is interesting for heuristic search because it adds additional information to use for determining the ordering. As we saw in the section on pruning the search space, types of the predicates are often already known while the arguments of such predicates are still variables. During synthesis, e.g. of a new invention, there are also variables in the types, meaning that some types are more "complete" than others.

A heuristic in the typed environment could consider ordering goals with complete types (i.e. no variables in them) first, as these types will be able to rule out large parts of the background predicates. Subsequently it would be useful to consider how to order the predicates with incomplete types. To define a useful heuristic it would then be necessary to weight the type features relative to the non-type features.

Implementation The accompanying synthesis system implemented for this project has best-first search available as an option. There is a proof-ofconcept heuristic that looks at type information and the number of ground arguments. While the implementation is able to show some benefits in exploring less of the search space, the heuristic needs more thought before the value of heuristic search in MIL can be thoroughly evaluated. For now we defer work on heuristics.

# 4.4 Showing Insufficiency of the Background Knowledge

A third area where type annotated predicates might be useful is in determining that the predicates that the user supplied will never be able to prove the goals. Clearly it would be very beneficial for the user to be aware that the background knowledge that they are providing is insufficient for solving the problem. Ideally the synthesis algorithm is able to detect, no matter the size of the programs it will consider, whether there are any sensible programs at all in the hypothesis space.

Deciding on whether the background predicates can be composed is something that type checking is already able to reason about. Hence the idea is to leverage the user provided type annotations to check whether the background predicates compose at all.

One simple approach is to take the type of the examples and for each metarule try types for the body atoms. The types one would try are the ones on the background predicates. Simple types or refinement type checking might indicate that not a single sensible composition was found. Such a result would only prove that no single clause programs can be constructed for the given examples. If we relax the condition that the clause must match with the type of the examples we could show that no invention exist that is composed from just background predicates.

The above reasoning is not strong enough to make claims about the non-existence of programs that are only slightly more complex. The main issue is that the type of an invention does not have to be the same as the type of any of the background predicates. Determining what types can be generated for inventions would be one approach to trying to show insufficiency of the background knowledge.

In this thesis we limit ourselves to presenting the above argument for why types could be useful for showing insufficiency of the background knowledge, but will leave a solution to the problem for future work.

# Chapter 5

# Synthesis with Polymorphic Types

This chapter introduces polymorphic types to Meta-Interpretive Learning (MIL). The main motivation for the introduction of types is so that type checking is able to prune parts of the search space. For a motivating example please refer back to section 4.2.

We restate the problem that our system is able to address, noting the type annotations we expect the user to provide. We introduce the  $\mathrm{Metagol}_{PT}$  algorithm, which extends the  $\mathrm{Metagol}_{AI}$  algorithm with polymorphic type checking. We look at how the type annotated background knowledge, along with unification, can propagate types through to the newly derived goals. In the section that follows we look at inferring polymorphic types, i.e. generate the most general type of each clause, and of the program itself.

Towards the end of the chapter we argue for the algorithm's correctness in terms of being sound and complete (relative to  $Metagol_{AI}$ ). We close out the chapter by a theory result regarding the influence of types on the size of the search space, and perform experiments validating the work of introducing polymorphic types.

#### 5.1 Revised Problem Statement

We assume familiarity with the synthesis problem statement in section 3.2.1, as well as with the way users provide background knowledge to the Metagol<sub>AI</sub> algorithm, as discussed in section 3.3.1. Instead of reiterating the complete definitions we note the needed adjustments to these definitions.

In addition to supplying examples, the user now supplies a type for the examples, that is a single type that is consistent for all examples. For the back-

ground knowledge we stipulate that each atom that can become a goal within the MIL algorithm is annotated with a (polymorphic) type. For the primitive clauses only the head of the clause needs to be assigned a type. As an example, the (primitive predicate) head, which returns the first element of a list, was added to the untyped background knowledge as prim(head/2) (with the 2 signifying the predicate's arity). In the setting of typed synthesis predicates need to be asserted with their type, that is, prim(head: [list(X),X]) adds the head relation to the (typed) background knowledge.

The interpreted predicates and the metarules need types for their head atoms as well as for their body atoms. The variable names used for the types are shared in a definition of a clause, which is the main means of propagating type information.

For adding interpreted predicates to the untyped background knowledge, the definition of the interpreted map predicate was stated as a normal Prolog definition, plus an **interpreted** assertion:

```
map([],[],F).
map([A|S],[B|T],F):-F(A,B),map(S,T,F).
interpreted(map/3).
```

For the typed assertions of interpreted predicates, we need to annotate the atoms of the body of an interpreted clause in addition to the head atom. Following the notation used for metarules, the definition of the clause is moved into the interpreted assertion<sup>1</sup>:

Metarule definitions now include types on their atoms. For the untyped Chain metarule the assertion metarule(chain,[P,Q,R],(P(A,B):-[Q(A,C),R(C,B)])) sufficed. For adding the Chain metarule to the typed background knowledge the assertion becomes:

```
metarule(chain,[P,Q,R],

(P(A,B):[X,Y] := [Q(A,C):[X,Z],R(C,B):[Z,Y]]).
```

Definition 3.2.2, on consistent hypotheses, only needs adjustment in that the generated clauses of the program have simple types on their atoms.

<sup>&</sup>lt;sup>1</sup>Note that we make heavy use of Prolog's square bracket list notation here: to operate over values, as the data structure containing the multiple body atoms, and as a convenient syntax for predicate types.

**Definition 5.1.1.** A Typed MIL learner takes examples and background knowledge with polymorphic types and outputs a polymorphic typed definite program H that is a consistent hypothesis for the input.

# 5.2 Metagol<sub>PT</sub>: Type Checking through Unification

One of the major strengths of Prolog (and logic programming in general) is support for logical variables and unification on these variables. Unification plays an important role in most type checking and type inference algorithms (see, for example, [Kanellakis and Mitchell, 1989]). For our purposes we can reduce all type checking to unification.

Unification on types serves two purposes during synthesis. For type checking we want to show that a general (polymorphic) type can be instantiated to a more specific type, e.g. as in the case of head: [list(X), X], and proving an atom P([1],1):[list(int),int].

At the same type we often have type variables in atoms' types due to the exploratory nature of synthesis (e.g. the Z variable in the Chain metarule definition of the previous section will always be just a variable right after the metarule is used for inventing a clause). These type variables represent freedom in how to interpret the atom, in particular these variables represent that the type of the arguments is as of yet undetermined. Non-ground arguments (i.e. arguments with variables in them) will often initially have a variable for their type. For example, suppose we are trying to prove atom P([1],B):[list(int),X]. Unification with the head primitive on the predicate name and type fixes this atom's type to [list(int),int].

In either case, whether we are type checking or making types more specific, when unification succeeds we know we can proceed in our proof attempt.

#### 5.2.1 Derivation and General Types

The major change we introduce to the  $Metagol_{AI}$  algorithm is that each atom has both a  $Derivation\ Type\ (DT)$ , and a  $General\ Type\ (GT)$ , that is Atom:DT:GT. The derivation type is always an instance of the general type of the atom, which can be seen as instantiating any type parameters in the general type to correspond to the types of the values that the arguments have taken on in the derivation.

The derivation type is for keeping track of the type of the atom as it is used in the proof of the entailment for the user provided examples. The derivation type hence is as accurate as possible taking into account the *values* 

that the atom's arguments have taken on. The values an atom's arguments take on during the algorithm ultimately derive from the values of the user provided examples. As such the derivation type can be seen as deriving from the example goals given to the algorithm, which are the root of the derivation tree (implicitly) constructed by the algorithm. The algorithm will terminate when a successful complete derivation has been found for the example goals, at which point all arguments in the derivation tree will have been instantiated to values. Therefore the derivation types for atoms in a successful derivation are never polymorphic (contain no type parameters).

The general type, GT, on the other hand is not concerned with being accurate with regard to the values that the atom has taken on in the current derivation. Instead the general type sees the atom as the head atom of a definite clause and maintains the type that the atom's arguments may be instantiated with, i.e. its polymorphic type. The general type is hence determined by the constraints imposed on the arguments' types based on the types of the atoms that become goals to prove this atom. The general types can hence be seen as deriving from the leafs in the derivation, up through the subtree under this particular atom. In particular, the general type is not influenced by the example goals (and type) given to the algorithm. Upon a successful derivation being found, in general, it is not the case that the general type does not contain variables, instead the point of the general types (of the head atoms of inventions) is that they may be polymorphic.

An example As an example, suppose we have to prove P([1,2,3],2):DT:GT. Its derivation type will already be fixed as DT=[list(int),int], due to the algorithm being able to maintain fully accurate derivation types for arguments instantiated with values (given that the type of the example goals was provided and the background predicates had accurate types). As we have not yet proven the atom the general type will usually be  $GT = [X, Y].^2$  Suppose the invented clause the algorithm comes up with to prove the atom is inv(A,B):-tail(A,C), head(C,B). We have that the (general) type of the body atoms is tail:[list(X),list(X)] and head:[list(Y),Y]. The general type of the inv derives from these general types as being [list(X),X].

As part of proving the goal atom by this invention we unify the head of the invention with the goal to obtain inv([1,2,3],2):[list(int),int]: [list(X),X]. Now the body atoms of the invention need to proven, which are

<sup>&</sup>lt;sup>2</sup>In the case that the argument variables are shared with other atoms in a clause it might be that there are already constraints imposed on the general type (which would necessarily also be present in the derivation types). Such constraints make the type more specific than just two variables.

tail([1,2,3],C):[list(int),list(int)]:[list(X),list(X)] and atom head(C,2):[list(int),int]:[list(Y),Y]. The derivation types have the correct type for C imposed by the typing of the background predicates of tail and head forcing unification on C's type based on the type of the other arguments. Clearly the proof succeeds when C=2 is forced by proving the tail atom.

This example makes the utility of having a general type for inv apparent. If we would need to prove another goal such as Q([[1],[2]],[2]]: [list(list(int)),list(int)]:QGT, we are not restricted to just being able to retrieve the derivation type that was used in the proof for the previous goal. Instead we can check that this derivation type is also an instantiation of the general type of inv, thereby allowing reuse of the invented predicate.

#### 5.2.2 The Algorithm

The polymorphic type checking contribution to MIL of this thesis has resulted in the  $Metagol_{PT}$  algorithm in figure 5.1. The code in this figure is the  $Metagol_{AI}$  algorithm, except for the bold code which achieves type checking.

We have as main invariant of the algorithm that every goal provided as the first argument to prove\_aux has its derivation type DT instantiated to the most specific type that is consistent with how the atom was derived from the examples (using the program built up at to that point), and that GT is the most general type of the atom as is constrained by the typing of the background predicates and the invented clauses in the program. An atom's derivation type is always an instance of its general type, as can be easily checked.

#### 5.2.3 Forward Propagating Derivation Types

As part of specifying what problem the user would like to solve, the user supplies a type, Type, for the examples. In the learn clause of  $Metagol_{PT}$ , the Pos and Neg examples are of the form p(a, ..., z), just as in the untyped case. The first two lines of the new learn body make sure that the example atoms satisfy the above invariant, namely each example is mapped from p(a, ..., z) to p(a, ..., z):Type:GT, where Type is the user supplied type, and GT is a new entirely unconstrained type variable.

The prove clauses of  $Metagol_{PT}$  are taken unchanged from the untyped algorithm. Their main purpose is to implement (leftmost) depth-first search.

For this subsection we will focus on how the derivation type DT is used to prune parts of the search space and how the correct derivation type is assigned to new goals. For now it suffices to know that the difference between

```
learn(Pos,Neg,Type,Prog):-
 map(decorate_types(Type),Pos,PosTyped),
 map(decorate_types(Type),Neg,NegTyped),
 prove(PosTyped, [], Prog),
 not(prove(NegTyped,Prog,Prog)).
prove([],Prog,Prog).
prove([Atom|Atoms],Prog1,Prog2):-
 prove_aux(Atom, Prog1, Prog3),
 prove(Atoms, Prog3, Prog2).
prove_aux(Atom:DT:GT,Prog,Prog):-
 prim(Atom:DT),!,
 prim(Atom:GT),
 call(Atom).
prove_aux(Atom:DT:GT,Prog1,Prog2):-
 interpreted(Atom:DT:-BodyDT),
 interpreted(Atom:GT:-BodyGT),
 combine_types(BodyDT,BodyGT,Body),
 prove(Body,Prog1,Prog2).
prove_aux(Atom:DT:GT,Prog1,Prog2):-
 member(sub(Name, GTinv, Subs), Prog1),
 instance_of(DT,GTinv),
 GT=GTinv.
 metarule(Name, Subs, (Atom:DT:-BodyDT)),
 metarule(Name, Subs, (Atom: GT: -BodyGT)),
 combine_types(BodyDT,BodyGT,Body),
 prove (Body, Prog1, Prog2).
prove_aux(Atom:DT:GT,Prog1,Prog2):-
 metarule(Name, Subs, (Atom:DT:-BodyDT)),
 metarule(Name,Subs,(Atom:GT:-BodyGT)),
 combine_types(BodyDT,BodyGT,Body),
 prove(Body, [sub(Name, GT, Subs)|Prog1], Prog2).
```

Figure 5.1: The Metagol<sub>PT</sub> algorithm.

the body atoms BodyDT, the body atoms with derivation type, and the body atoms Body in the prove\_aux clauses in 5.1 is that Body's atoms additionally have their general type set.

In discussing the prove\_aux clause disjuncts we maintain the invariant that the goal atoms, i.e the first argument to the clause and the Body atoms, have the derivation type DT instantiated to the most accurate known type.

Disjuncts In the first disjunct of prove\_aux we check whether the atom can be proved by one of the primitive background predicates. An atom's predicate will now be matched against primitives, not only on arity, but also on the DT type. A primitive can only be chosen when the derivation type is an instance of the primitive's type. For example, take the goal P([1,2],B) with derivation type [list(int),int]. When considering the tail:[list(X),list(X)] predicate, unification of the types fails due to int and list(X) not being unifiable. A predicate such as head:[list(X),X] does pass the unification test due to X=int equalizing the types. Hence accurate derivation type checking occurs at the prim(Atom:DT) line, and failure to unify the types means that Prolog's evaluation will not be invoked for the atom.

The second disjunct matches against the type annotated interpreted background clauses. The assertion <code>interpreted(Atom:DT:-BodyDT)</code> says to find a background clause definition such that the head of the clause successfully unifies both the <code>Atom</code>, containing a predicate name and values, as well as the derivation type <code>DT</code> with the clause's head and type. When the head unifies successfully, the definition of the interpreted clause makes sure that types of the body atoms are appropriately unified as well. Upon successful selection of a background predicate, the general type of each atom is appended to their derivation type.

The third disjunct tries to reuse an invented clause of the program. Here we use that the polymorphic type of (the head of) the clause is saved along with the meta-substitution and the name of the metarule used for the invention. The <code>instance\_of(DT,GTinv)</code> goal asserts that the derivation type is an instance of <code>GTinv</code>. Note that <code>DT</code> is not unified with <code>GTinv</code> itself, which would make the type to specific, instead <code>DT</code> is unified with a copy of <code>GTinv</code>. We create a new instance of the invented clause, making sure that the same meta-substitutions constraints are imposed, and generate body atoms with the appropriate derivation types.

The final disjunct deals with the option of inventing a new clause. The only major change to note: in unifying the head of the metarule with the atom we need to prove, the derivation type (and the general polymorphic

#### 5.2.4 Inferring the Most General Type

Having looked at how goals get the correct derivation type assigned, we now look at how the general type can be determined for atoms. Note that determining the general type serves two purposes. First, we need the general type of an invention if we are to reuse an invention wherever we can, e.g. an invention inv(A,B):-tail(A,C),head(C,B) derived to prove P([1,2],2) has derivation type [list(int),int] when invented for these values. Instead we want to know the more general polymorphic type, [list(X),X], such that the invention can be reused, e.g. to prove P([[],[1]],[1]): [list(list(int)),list(int)]. Second, which follows directly from knowing the general type of inventions, is that we can give a general type to the entire program, i.e. to the head of the clause used for the examples. This means we can also generalize from the example type that we have been given.

All of the background knowledge (primitive and interpreted predicates, and metarules) is already annotated with general types (and become more accurate for goal derivation types by making them more specific through unification). This is the property exploited in the algorithm, and why we see unification on the background knowledge twice, once for the derivation types and once for the general types.

Disjuncts revisited For the first disjunct it should be clear that unifying GT with the general type of the already selected primitive keeps track of the general type of the atom. For the interpreted clauses of the second disjunct this holds true as well, except that body atoms of the interpreted clause become goals which can restrict the general type during their proof. We have that background knowledge is only annotated with a single type, which is why we unify twice, once we get body atoms with derivation types and once we get body atoms with general types. The combine\_types predicate combines these lists of singly typed atoms, of the form BodyAtom:DTy and BodyAtom:GTy, into atoms with both a derivation and general type, e.g. BodyAtom:DTy:GTy.

For the third disjunct we already explained how the general type of an invention is used to type check. To keep track of the general type, note that the general type of the (head of the) invention must be the general type of the atom, which is asserted by the equality. This equality also makes sure that the general type among all usages of the invention remain consistent for this one general type. The metarule instantiation with the general type GT

makes sure constraints imposed by a new goal are directly reflected in the general type of the invention.

The final disjunct, handling new inventions, is similar in reasoning to the previous disjunct. The one thing to note is that because the new invention is saved as part of the program, the general type for the clause is stored with it. This is the general polymorphic type that will be shared amongst all uses of the invention<sup>3</sup>.

#### 5.3 Theoretical Results

To argue the correctness of the  $Metagol_{PT}$  algorithm we establish soundness, i.e. the programs found by the algorithm are correct for the examples, and relative completeness, that is, every program found by the  $Metagol_{AI}$  algorithm will also be found by  $Metagol_{PT}$ . Related to the completeness result, we also briefly look at how sound pruning impacts predictive accuracy. As a separate result we characterize how types on predicates can bound the size of the search space.

#### 5.3.1 Soundness

**Definition 5.3.1.** An inductive synthesis algorithm is *sound* if every program returned by the algorithm is a consistent hypothesis (definition 3.2.2).

To establish soundness of  $Metagol_{PT}$ , we make the following assumption: the  $Metagol_{AI}$  algorithm is sound. Though we assume it here, it is not too hard to be convinced that this assumption is true. In essence the meta-interpreter extends the proof procedure of SLD-resolution with additional higher-order rules, and at the same time maintains the proof steps in the derivation of the entailment of the examples that an implementation of SLD-resolution does not itself maintain.

**Proposition 5.3.1** Given that the Metagol<sub>AI</sub> algorithm is sound, the Metagol<sub>PT</sub> algorithm is sound.

*Proof.* Assume the precondition.

Note that in logic/constraint programming we have that adding more constraints to clauses can only reduce the number of solutions found. For the Metagol<sub>PT</sub> algorithm to succeed the same derivation that the Metagol<sub>AI</sub> algorithm establishes must be found, i.e. the derivation on atoms without

<sup>&</sup>lt;sup>3</sup>An alternative simple approach would be to modify the definition of meta-subtitutions to include types on the predicate names, forgoing the need to store the type separately.

their types is exactly a Metagol<sub>AI</sub> derivation. This is due to the Metagol<sub>PT</sub> algorithm only adding constraints to the Metagol<sub>AI</sub> algorithm.

Clearly the type checking of  $\mathrm{Metagol}_{PT}$  only imposes additional conditions on the proof of atoms, hence the derivations found by  $\mathrm{Metagol}_{PT}$  must be a subset of the derivations found by  $\mathrm{Metagol}_{AI}$ . Conclude that any returned program by  $\mathrm{Metagol}_{AI}$  must as well be a consistent hypothesis.  $\square$ 

The only thing to remark with regard to the additional constraints imposed by  $\text{Metagol}_{PT}$  is that they are not as simple as just additional atoms in bodies. The head atoms gain some freedom in that they have additional variables in their types, and some of the body atoms in the clauses are modified instead of just added. These changes do not matter, as in the end the algorithm can only succeed when the atoms (without their types) form a proper  $\text{Metagol}_{AI}$  proof.

#### 5.3.2 Completeness

For our completeness result we assume that the background knowledge is typable by the polymorphic types introduced in section 4.1. We also assume that the provided background knowledge has correct types, i.e. the types might be too general, but they are not inaccurate. For completeness relative to the  $\text{Metagol}_{AI}$  algorithm we show that when the  $\text{Metagol}_{AI}$  algorithm is able to find a program, then  $\text{Metagol}_{PT}$  must find the same program (syntactically identical).

Remember, from section 3.3.2, that the search procedure of the Metagol<sub>AI</sub> (and hence of the Metagol<sub>PT</sub>) algorithm is leftmost depth-first.

**Definition 5.3.2.** A procedure that discards parts of a search space performs *inconsistency pruning* when the parts of the search space pruned away never contain any successful nodes.

**Proposition 5.3.2** Let A be a depth-first search algorithm. If an algorithm A' is algorithm A except that it does additional inconsistency pruning, then when algorithm A finds its first successful node s, algorithm A' will also find node s as its first successful node.

*Proof.* Assume the stated relationship between A and A' and that algorithm A has found its first successful node s. Suppose that either A' does not find a successful node, or finds a node t. Due to sound pruning we have that A' cannot have pruned away the part of the space containing s, and must come across it eventually. Hence A' must find node t. Because the traversal order of the nodes is the same (modulo unsuccessful nodes being left out) it follows that t = s.

First note that a program with a type check error cannot be a consistent hypothesis (given at least one positive example). That  $\operatorname{Metagol}_{PT}$  adds inconsistency pruning relative to  $\operatorname{Metagol}_{AI}$  follows from that a type checking failure leads to pruning and that any program with a type check error cannot be made to type check by adding additional clauses to the program (i.e. continuing the search).

That  $Metagol_{PT}$ 's type checking through unification correctly implements type checking can be established by induction on the size of the derivation tree constructed during the algorithm<sup>4</sup>. The induction hypothesis is that the values are always inhabitants of the derivation types. When the algorithm tries to prove a goal with a primitive/interpreted/invented predicate and the typing of the predicate cannot be instantiated to the derivation type of goal, that predicate will be rejected due to a type error. This rejection is sound due to the predicate's type indicating that the predicate cannot successfully the values of this type. Otherwise, the predicate may be used to continue the search, at which point it is easy to show that the values in the new goals are again inhabitants of their type.

Using the above result we can immediately conclude the below proposition, by instantiating the algorithms A and A' by  $\text{Metagol}_{AI}$  and  $\text{Metagol}_{PT}$ , respectively.

**Proposition 5.3.3** The (simply typable) programs found by the Metagol<sub>AI</sub> algorithm and the Metagol<sub>PT</sub> algorithm are exactly the same syntactically.

We will use this result in justifying why it is not worthwhile to experimentally investigate the difference in accuracy of  $Metagol_{PT}$  and  $Metagol_{AI}$ . The above result only holds when the algorithms are allowed to run arbitrarily long, long enough to find the (first) successful program. In practice a timeout is used. As we will see in the experimental work, the untyped system is considerably slower than the typed system. A consequence is that the typed system can find programs for which the untyped system will use more time than is allowed by a timeout. In such cases the predictive accuracy of the typed system will vastly outperform the untyped system. In experiments where both systems are able to find the program the predictive accuracy is not impacted by type checking.

<sup>&</sup>lt;sup>4</sup>Note that we plan to make the presentation of the type system formal such that this proof, amongst others, can be proven formally.

#### 5.3.3 Proportion of Relevant Predicates

For the remainder of the theoretical examination we focus on a particular language class of logic programs. The metarules in table 3.1, e.g. containing the Chain rule  $P(A,B) \leftarrow Q(A,C), R(C,B)$ , focus on restricting the structure of the class of programs where each clause has a head atom and two body atoms. Each atom's predicate has exactly arity two. This class is known as  $H_2^2$ .

The number of programs in this class is given as  $O(|M|^n(p^3)^n)$ , where |M| is the number of metarules, n is the number of clauses and p is the number of predicate symbols [Lin et al., 2014]. The  $p^3$  term is due to that all three of the predicates in a clause being chosen independently from each other. If higher-order abstractions, each with  $k \geq 1$  higher-order variables are considered, this result is updated to  $O(|M|^n p^{(2+k)n})$  [Cropper and Muggleton, 2016a].

We will suppose that the predicates in the background knowledge are annotated with polymorphic types. The improvement of taking types into consideration is due to predicate typing having to match up, meaning that predicates in a clause usually cannot be chosen independently from each other. Only a portion of the background predicates' types will line up.

Given p predicates with types, fixing one of the three predicates in a  $H_2^2$  clause restricts the choice of the two other predicates of the clause. It hence makes sense to consider the maximum number of predicates that remain as possible choices for any predicate of a  $H_2^2$  clause, after either (or both) of the other predicates have been selected. This is determined per instance of the background knowledge.

Let  $\overline{p} \leq p$  be the worst case number of predicates that remain as choices for any of the predicates in a  $H_2^2$  clause. This value is determined by an exhaustive search over the three predicate places in a clause, filling in any one predicate and checking how many of the predicates can still be substituted for the remaining predicate variables.

**Definition 5.3.3.** Given p typed predicates, with  $\overline{p} \leq p$  an upper bound on the number of typed predicates that can be filled in any of the predicate variables of a  $H_2^2$  clause, given that another predicate of the clause has already been selected,  $t = \overline{p}/p$  is the worst case proportional constant.

The ratio will always be between 0 and 1, with lower values indicating a greater reduction in the search space. The proportional constant is a convenient value to work with due to it abstracting away the number of predicates in the program. The following result characterizes the reduction in the hypothesis space of programs given that predicates are properly annotated with types.

**Proposition 5.3.4** Given p typed predicates, and the worst case proportional constant t for the predicates, the hypothesis space  $H_2^2$  is reduced by a factor of  $t^{3n}$ , where n is the number of clauses, versus the untyped hypothesis space.

The p term in  $O(|M|^n(p^3)^n)$  is the (maximum) number of predicates that can be filled in for any predicate variables in a untyped (unabstracted) clause. For the typed case we know that this maximum is  $\overline{p}$ , and hence we substitute  $\overline{p}$  for p in the size bound:

$$|M|^{n}(\overline{p}^{3})^{n} = |M|^{n}((tp)^{3})^{n}$$

$$= |M|^{n}(t^{3n})(p^{3n})$$

$$= (t^{3n})(|M|^{n}(p^{3})^{n}))$$

In case of the abstracted search space we have that the reduction factor is  $t^{(2+k)n}$ , using the exact same reasoning. These results imply that using types to prune the search space leads to a considerable reduction in effort.

#### 5.4 Experimental Results

There are two main concerns when evaluating synthesis systems: the speed with which they are able to find programs and, in an inductive setting, the predictive accuracy of the found programs for the relation that is being learned.

As shown in the previous section, when the search algorithm is (leftmost) depth-first search, the first encountered program that correctly entails the examples is always the same. It follows that the accuracy of the found programs (barring timeouts, which we will not consider in this section) is not affected by polymorphic type checking based pruning, justifying the decision of only performing experiments that check the impact on the size of the search space and the impact on the time needed for synthesis.

We perform three experiments to evaluate the benefit of polymorphic types to MIL. First, the Search Space Reduction experiment checks that the inclusion of irrelevant background predicates has negative effects for the untyped framework, though the typed system is able to ignore them. Second, the Ratio Influence checks that the implementation is able to come close to the theoretical result regarding better than linear influence of the ratio on the search space. Finally, we do a statistical experiment, Simply Typed Droplasts, on the synthesis of the droplasts program, checking for a time speedup and a reduction in the number of derivation steps.

For simplicity's sake we will compare  $Metagol_{PT}$  to the untyped system by just disabling  $Metagol_{PT}$ 's type checking<sup>5</sup>. Note that this means that there is some additional overhead versus  $Metagol_{AI}$ , which does not keep track of types at all. The number of proof steps is not impacted by this overhead, only the time needed is (though the impact should be quite small).

The Prolog implementation used for running the experiments is SWI-Prolog.

#### 5.4.1 Derivation steps

In determining how efficient program synthesis is, we propose to keep track of the number of decisions made for traversing the program search space. That is, a decision is a choice made to construct part of a potential program. A decision in the MIL framework corresponds to

- trying to prove an atom with a primitive predicate,
- trying to prove an interpreted higher-order predicate, by expanding the body,
- trying an existing invented predicate/clause,
- or choosing to apply a metarule, creating an invention.

Figure 5.2 indicates with bold lines where in the Metagol<sub>PT</sub> algorithm the (global) decision counter is increased. Note, that for our purposes, a decision is made *after* type-checking. For the untyped algorithm the checks occur at the same place (modulo additional type checking lines).

Due to the correspondence of synthesis with constructing proofs, in particular derivations for the positive examples, the decisions will also be called proof steps or derivation steps.

#### 5.4.2 Experiment 1: Search Space Reduction

In this experiment we verify, via deterministic tests, that 1) adding typed predicates that do not compose w.r.t. the input type are ignored by the typed system, and 2) that the difference in time and proof steps corresponds to the theoretical results in proposition 5.3.4.

<sup>&</sup>lt;sup>5</sup>Time constraints caused us to decide that proper experiments generating both typed and untyped experiments will have to be deferred to future work.

```
prove_aux(Atom:DT:GT,Prog,Prog):-
 prim(Atom:DT),!,
 prim(Atom:GT),
 increase_counter,
 call(Atom).
prove_aux(Atom:DT:GT,Prog1,Prog2):-
 interpreted(Atom:DT:-BodyDT),
 interpreted(Atom:GT:-BodyGT),
 combine_types(BodyDT,BodyGT,Body),
 increase_counter,
 prove(Body,Prog1,Prog2).
prove_aux(Atom:DT:GT,Prog1,Prog2):-
 member(sub(Name,GTinv,Subs),Prog1),
 instance_of(DT,GTinv),
 GT=GTinv,
 metarule(Name,Subs,(Atom:DT:-BodyDT)),
 metarule(Name,Subs,(Atom:GT:-BodyGT)),
 combine_types(BodyDT,BodyGT,Body),
 increase_counter,
 prove(Body,Prog1,Prog2).
prove_aux(Atom:DT:GT,Prog1,Prog2):-
 metarule(Name,Subs,(Atom:DT:-BodyDT)),
 metarule(Name,Subs,(Atom:GT:-BodyGT)),
 combine_types(BodyDT,BodyGT,Body),
 increase_counter,
 prove(Body, [sub(Name,GT,Subs)|Prog1],Prog2).
```

Figure 5.2: Metagol<sub>PT</sub>'s prove\_aux annotated with decision counter.

Null Hypothesis 5.4.1 Metagol<sub>PT</sub> with type checking enabled is not able to prune the search space relative to Metagol<sub>AI</sub> (i.e. Metagol<sub>PT</sub> with type checking disabled).

Null Hypothesis 5.4.2 Metagol<sub>PT</sub> with type checking enabled is not able to learn faster than Metagol<sub>AI</sub>.

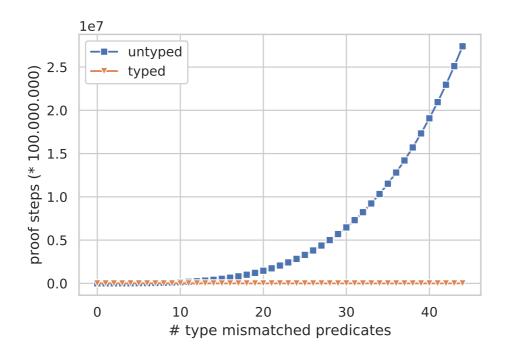
**Setup** We ask the system to prove a single positive example p(0,1), of type [int, int]. In order to strictly control the search space we only provide the chain metarule. As we want to traverse the entire search space we will only introduce predicates that cannot contribute to a successful program.

The predicate(s) in the background knowledge need to be general enough to always succeed in unifying with goal atom (thereby making sure that the largest possible search space is traversed). The following predicate was chosen due to it being well-behaved when the search space is traversed depth first:

$$to\_zero(X,0)$$
.

We add one instance of this predicate with type [int, int], thereby allowing it to be used by the typed system. To test how the system handles completely irrelevant background predicates we iteratively add additional instances of this predicate, but now with type [bottom, bottom], where bottom is a dummy type. Predicates with both bottom and int types cannot occur in the body of the Chain metarule.

**Result** The graph on top in figure 5.3 shows the number proof steps that were needed to traverse the search space, when the number of clauses in the largest program considered is restricted to three. The graph on the bottom in this figure shows the time spent on traversing the search space in both the typed and the untyped system.



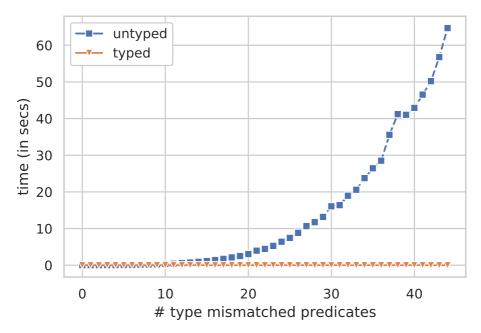


Figure 5.3: Number of proof steps and time for increasing number of mismatched predicates. Max clauses is three. Typed proof steps are constant at 83 and the time required for typed synthesis is (approximately) constant at 0.014 seconds.

The number of steps required by the typed system remain constant at 83. Correspondingly, the time for the typed program is also as near to constant, never needing more than 0.014 seconds. The untyped program on the other hand traverses bigger and bigger search spaces with increasing number of (irrelevant) background predicates. These graphs are strong evidence for rejecting hypothesis 5.4.1, regarding not being able to further prune the search space, and for rejecting hypothesis 5.4.2, regarding performance not being affected relative to the untyped system.

Additional Analysis It is interesting to note that the graph for the proof steps (and time as well), for a maximum of three clauses, is best explained by a polynomial of degree four. This deviates from the known theory result regarding the size of the search space being  $O(|M|^n(p^3)^n)$ , where |M| = 1, n = 3, and p is varied in the experiment. Based on the theory one would expect a polynomials of degree 9 to best explain the experimental data. The experiment was repeated to try to better understand this observation: the number of clauses was limited to four, which resulted in a polynomial of degree five best fitting the data, while a degree 12 polynomial would be expected according to the theory.

Actually it is not hard to see that the theory result does not take into account certain conditions on the predicates imposed by the MIL framework. It only represents the size of the search space traversed by a completely naive brute-force algorithm. One clear example is that the predicates in the head are restricted to the predicate of the examples and to invented symbols. It follows that at least one of the three predicates in a clause is restricted to n, instead of p. This observation allows us to replace the  $p^3$  in the size expression with  $p^2$  (given  $p \gg n$ ). This restricted size bound still does not fully explain our lower degree polynomials, meaning that we need to refine the theory to accurately capture the hypothesis space that is actually considered by the MIL framework.

#### 5.4.3 Experiment 2: Ratio Influence

For this experiment we consider the worst case proportional constant  $t = \overline{p}/p$  as a varying ratio, where p is the total number of background knowledge predicates, of which at most  $\overline{p}$  of which can be used for any predicate variable. Via a deterministic test we determine the influence of the ratio of matching types predicates versus all background predicates on the search space that

 $<sup>^6\</sup>mathrm{By}$  running linear regression and determining that terms of higher degree always are given trivially sized coefficients

the typed system,  $Metagol_{PT}$ , explores. Again, we are interested in the size of the search space, hence we do not try to find a successful program.

Null Hypothesis 5.4.3 Ratio unimportant The ratio t does not influence the size of the search space explored.

**Setup** We take as basis the previous experiment. Again, we take the  $to\_zero(X,0)$  predicate, though this time we insert 25 of them in the background knowledge. We vary the types of the predicates such that different ratios are achieved, e.g. 1/25, 2/25, etc. The predicates that will be allowed by type checking will have type [int, int] and the other predicates will have type [bottom, bottom]. We restrict the number of clauses to three.

**Result** Figure 5.4 has a plot showing how the time necessary for traversing the search space is influenced by the ratio of type matching versus all predicates. The second plot shows the same behaviour but for the number of proof steps by the untyped and simply typed systems.

There is enough evidence to *reject* the hypothesis that the ratio of correctly typed background predicates *does not* matter. Clearly the ratio has significant influence on the time taken and the number of decisions made by the algorithm. Note that the untyped system will not care about the ratio of the matching typed predicates as it will not consider this feature at all, hence leading to its constant behaviour.

The plots are again best explained by degree four polynomials, again demonstrating the need for better fitting theory. The time plot is interesting in particular for that it reveals that the  $Metagol_{PT}$  with type checking turned off, which is how this test was conducted for the untyped results, apparently has overhead from the types that are carried around, even more so than the typed system, as clearly demonstrated by the case when the ratio is 1.0. This is likely due to an increased cost for making placeholder variables for the types that are not being checked, which appears to have non-trivial cost in SWI-Prolog.

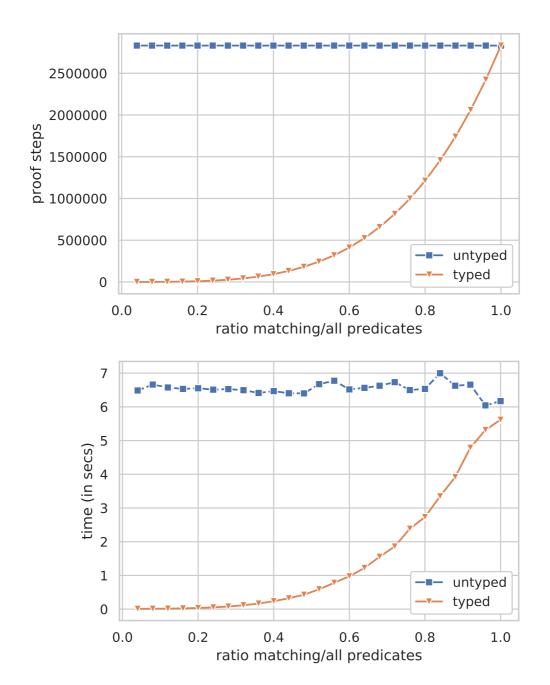


Figure 5.4: Number of proof steps and time for increasing ratio of type matching predicates out of all predicates.

#### 5.4.4 Experiment 3: Simply Typed Droplasts

As a final experiment we take a popular exercise from the literature [Kitzelmann, 2007], namely the droplasts program. This polymorphic program takes a list of lists and drops the last element from each of the inner lists. The program learned is:

```
droplasts(A,B):-map(A,B,droplasts_1).
droplasts_1(A,B):-reverse(A,C),droplasts_2(C,B).
droplasts_2(A,B):-tail(A,C),reverse(C,B).
```

Null Hypothesis 5.4.4 Metagol<sub>PT</sub> is not able to improve on the untyped system, in the time required and the number of proof steps needed, in a (semi-)realistic setting.

Setup This experiment is conducted stochastically. We generate small random input examples (outer and inner lists (of integers) of length between 2 and 5)<sup>7</sup> and run them through a reference implementation to obtain correct positive examples. As we are interested in the effect of type checking on the search space based on the background predicates, we fix the number of (positive) examples generated to three.

We provide the synthesis system with predicates for a list concat relation (appending elements at the back), the tail relation, the reverse relation and the two-place identity relation, all with appropriate polymorphic types. The type of the examples is set to [list(list(int)), list(list(int))]. We give it the chain metarule, along with metarules for abstraction to the following predicates. The higher-order predicates made available are the map, the reduceback and the filter relations.

For the experiment we add additional typed predicates, which for sake of execution time we take to be simple, forgoing excessive costs associated with non-determinism and resolution<sup>8</sup>. The additional predicates only match on particular values for their arguments, where the values are randomly chosen from a small distribution. The types of predicates are correct for the values of the argument and are either: bool, nat, int, list(int), list(list(int)), or list(list(X)).

 $<sup>^{7}</sup>$ The rather limited lengths are chosen so that we can reuse most of this experiment when we consider the refinement experiment for the following chapter

<sup>&</sup>lt;sup>8</sup>The typed system would for the most part not incur these costs, hence adding an additional cost for resolution and non-determinism would mostly be a tool to manipulate the results in your favor.

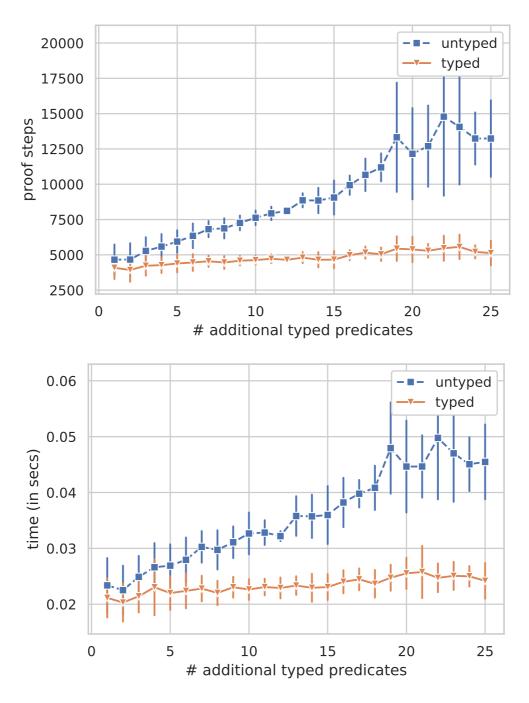


Figure 5.5: Average number of proof steps and time for increasing number of typed background predicates. Standard deviation is depicted by bars.

For each number of additional background predicates we run ten trials, for each trial generating random positive examples and predicates. We average over the trials and calculate the standard deviation of the sample.

**Result** The plots in figure 5.5 depict the average time and number of proof steps required by the untyped and typed systems. The standard deviations are included as bars.

These graphs are reasonable evidence for concluding that the typed system is able to outperform the untyped system in a non-toy synthesis example, thereby rejecting hypothesis 5.4.4.

Analysis The advantage of type checking observed in the previous, deterministic experiments is less pronounced in this experiment. The main reason for this is that while in the previous tests the untyped program needed to explore the entire space of possible programs, i.e. with each background predicate at each possible position in all clauses, now the untyped system is able to discard parts of its search space based on the *values* in the background predicates not unifying.

# Chapter 6

# Synthesis with Refinement Types

Based on the success of polymorphic type checking in making synthesis more efficient, it is natural to consider whether more expressive types can be leveraged for further improvement. The extension to polymorphic types we will consider in this chapter is refinement types, i.e. polymorphic types with an additional proposition restricting inhabitants of the type (see section 4.1 for a definition). We will often use the term *simple types* to refer to polymorphic types without refinements.

First we discuss how the user specifies refinement types for the background knowledge. As the MIL algorithm conceptually only needs minor adjustments, we next introduce the adapted algorithm  $\text{Metagol}_{RT}$ . We show how a refinement proposition representing the entire program can be obtained from the proof structure maintained by the algorithm.

Refinement checking is accomplished by proving (un)satisfiability by a SMT solver. We discuss the levels of refinement expressivity available, and how there is a tradeoff to be made. The chapter closes with theory and experimental results.

#### 6.1 Representation of Refinements

The reader is referred to section 4.1 for the definition and some of the syntax used for refinement types. We assume familiarity with the problem statement for program synthesis using polymorphic typed predicates, as well as with the way users provide their background knowledge to the Metagol<sub>PT</sub> algorithm, as can be found in section 5.1.

The user supplied positive and negative examples are unchanged versus

the simply typed case: the user provides a single consistent (non-refinement) type for all the examples. As in the case of polymorphic type checking, we need the user to supply refinement types for the background knowledge. The refinement types on the predicates should be such that they are consistent with the predicates, i.e. for any predicate  $p(A_0, \ldots, A_k) : [T_0, \ldots, T_k] \langle \varphi \rangle$ , which gets its arguments instantiated to  $a_0, \ldots, a_k$ , it is never the case that  $p(a_0, \ldots, a_k)$  holds while  $\varphi[a_0/A_0, \ldots, a_k/A_k]$  is false. Note that the user is not otherwise restricted, they can choose to use the most general refinement (true) or to make it as precise as they like (even as far as the refinement completely characterizing the predicate).

For specifying refinements in Prolog we require a way to refer to the arguments. Instead of just annotating the predicate's name with a type we also write out its formal arguments when asserting that a predicate belongs to the background knowledge. The primitive predicate assertions gain an additional argument for the refinement on the simple type. The simple type itself is appended to the atom representing the predicate's name and formal arguments. For example  $prim(tail(A,B):[list(X),list(X)],\langle length(A) = length(B)+1\rangle)$ , is the predicate representing the tail relation with the refinement stating the length property that always holds of its arguments. Note that the refinement bracket syntax  $(\langle \ldots \rangle)$  made its way into our Prolog notation. For now we hide the actual syntax used in writing down refinements by this notation and comeback to the refinement specification language in section 6.4.

The interpreted predicates gain an additional assertion for the specification of the refinement type. We will need to keep track of structure of the program derived, and hence will keep track of the existentially quantified predicate names in the interpreted clause. As an example, the assertion of the map predicate being an interpreted background predicate becomes:

```
\label{eq:interpreted} \begin{split} & \operatorname{interpreted}([\mathsf{F}], \operatorname{map}([], [], \mathsf{F}) : [\operatorname{list}(\mathsf{X}), \operatorname{list}(\mathsf{Y}), [\mathsf{X}, \mathsf{Y}]] : - \ []) \, . \\ & \operatorname{interpreted}([\mathsf{F}], \operatorname{map}([\mathsf{A}|\mathsf{S}], [\mathsf{B}|\mathsf{T}], \mathsf{F}) : [\operatorname{list}(\mathsf{X}), \operatorname{list}(\mathsf{Y}), [\mathsf{X}, \mathsf{Y}]] : - \ [\mathsf{F}(\mathsf{A}, \mathsf{B}) : [\mathsf{X}, \mathsf{Y}], \operatorname{map}(\mathsf{S}, \mathsf{T}, \mathsf{F}) : [\operatorname{list}(\mathsf{X}), \operatorname{list}(\mathsf{Y}), [\mathsf{X}, \mathsf{Y}]]]) \, . \\ & \operatorname{interpreted\_ref}(\operatorname{map}(\mathsf{C}, \mathsf{D}, \mathsf{F}), \langle \operatorname{length}(C) = \operatorname{length}(D) \rangle) \end{split}
```

The metarules will not need to keep track of refinements (directly) and hence remain unchanged versus the simple type problem statement, i.e. all atoms are decorated with simple types. The definition of consistent hypotheses, and that of MIL learner are essentially unchanged: the programs learned are still simply typed, and the only adjustment needed is that the supplied background predicates are refinement typed.

# 6.2 From a Program Derivation to a Refinement

This section starts out with observing how refinements on predicates give rise to a refinement over a program. Next we present how refinement type checking can be integrated at a high level into the MIL algorithm. This highlevel algorithm will delegate the type checking to a subroutine, hiding the complexity. We subsequently look how this type checking subroutine is able to construct a single proposition that needs to be checked for satisfiability.

#### 6.2.1 Backward Action on Refinements

We now look at how the validity of substituting the head of a definite clause with its body gives rise to inferring refinements. Suppose we are in a position of needing to prove that an atom Q(X,Y) is an inhabitant of its type  $[T_X, T_Y]\langle \varphi_Q \rangle$ , where  $\varphi_Q$  might be a unknown refinement and X and Y might both be variables (and not yet concrete terms) of as of yet undetermined type.

**Substitution of refinements** Suppose that after applying the chain metarule that Q is assigned the body  $Q(X,Y) \leftarrow R(X,A), S(A,Y)$  with the following typing (again involving unknowns):

$$R(X, A) : [T_X, T_A] \langle \varphi_R \rangle$$
  
 $S(A, B) : [T_A, T_Y] \langle \varphi_S \rangle$ 

If the refinement of Q(X,Y) was unknown we now know that  $\varphi_R \wedge \varphi_S$  is an accurate substitution for  $\varphi_Q$  as the just invented definite clause would be the only way of deriving Q(X,Y). To see that this is an accurate substitution observe that any occurrence of Q(X,Y) in the program may be replaced by the newly constructed body. If  $\varphi_Q$  is not unknown it is the case that there is already another body assigned to Q(X,Y), which means we are adding a disjunctive clause. The refinement  $\varphi_Q$  needs to be updated to  $\varphi_Q' = \varphi_Q \vee (\varphi_R \wedge \varphi_S)$ .

**Backward action** From the above description it follow that the types in the body of an invention have a kind of *backward action* with regard to the type of the predicate that is being invented. This backward action leads to additional named (argument) variables occurring in refinement types, names

not present in a predicate head's arguments. Therefore a context needs to be maintained for any such existentially qualified variables. The significance of this backward action is that is leads to a grand refinement for the entire program, i.e. a single proposition whose satisfiability determines whether the program is still consistent. For example, it could be that R(X, A) has to be invented leading to an adjustment of its refinement type, which would in turn change Q(X,Y)'s type. In the same way Q(X,Y) could occur in the body of the clause invented for predicate of the examples. Hence we have that the entire program's refinement type is influenced by any adjustment to a type due to filling in a definition for a predicate.

Note that for interpreted predicates we have that the same property holds: the refinement for the head of the interpreted clause can be made more accurate by conjuncting it with the refinements of the predicates that occur in the clause's body.

#### 6.2.2 High-level algorithm: Metagol<sub>RT</sub>

Figure 6.1 contains the code for  $Metagol_{RT}$ , the refinement type checking Meta-Interpretive Learning algorithm. The algorithm is, in essence, the  $Metagol_{PT}$  algorithm from the previous chapter with additional pruning. The code in bold notes all the changes made to implement refinement checking. The same four disjunctive clauses remain and fulfill the same tasks.

Each disjunct maintains in its Prog arguments what the currently derived program is. For the first disjunct this is done by unifying the selected predicate name with one of the variables already in Prog, hence the change in the program Prog needs no further work. The other disjuncts explicitly keep track of the structure of the proof constructed by the algorithm. For invented clauses a meta-substitution is already kept track of. Uses of higher-order background predicates and of invented clauses are now also saved.

The check\_refinement predicate achieves pruning by deriving the proposition representing the constraints imposed on the entire program by the refinements. The stored proof of the program gives rise to the grand refinement, the proposition whose satisfiability determines whether the program being considered is still a viable option, or else is already inconsistent. When the grand refinement is satisfiable the call to check\_refinement holds for the supplied program and the search continues. If the refinement of the supplied program is proven unsatisfiable, the proof search procedure of Prolog starts backtracking, discarding (at least) the last choice made for the program.

When considering just filling in the existential variables from a clause body we have that an inconsistent refinement will not become consistent upon filling in more variables in the clause. This monotonicity property

```
learn(Pos,Neg,Type,Prog):-
  map(decorate_types(Type),Pos,PosTyped),
  map(decorate_types(Type), Neg, NegTyped),
  prove(PosTyped,[],Prog),
  not(prove(NegTyped,Prog,Prog)).
prove([],Prog,Prog).
prove([Atom|Atoms],Prog1,Prog2):-
  prove_aux(Atom, Prog1, Prog3),
  prove(Atoms, Prog3, Prog2).
prove_aux(Atom:DT:GT,Prog,Prog):-
  prim(Atom:DT),!,
  prim(Atom:GT),
  check_refinement(Prog),
  call(Atom).
prove_aux(Atom:DT:GT,Prog1,Prog3):-
  interpreted(Subs,Atom:DT:-BodyDT),
  interpreted(Subs,Atom:GT:-BodyGT),
  combine_types(BodyDT,BodyGT,Body),
  Prog2=[inter(Atom,DT,GT,Subs)|Prog1]
  check_refinement(Prog2),
  prove(Body, Prog2, Prog3).
prove_aux(Atom:DT:GT,Prog1,Prog3):-
  member(sub(Name,GTinv,Subs),Prog1),
  check_unifies_with(GTinv,DT),
  GT=GTinv,
  metarule(Name,Subs,(Atom:DT:-BodyDT)),
  metarule(Name, Subs, (Atom: GT:-BodyGT)),
  combine_types(BodyDT,BodyGT,Body),
  Prog2=[inv(Name,Atom,DT,GT,Subs)|Prog1],
  check_refinement(Prog2),
  prove(Body, Prog2, Prog3).
prove_aux(Atom:DT:GT,Prog1,Prog3):-
  metarule(Name, Subs, (Atom:DT:-BodyDT)),
  metarule(Name, Subs, (Atom: GT:-BodyGT)),
  combine_types(BodyDT,BodyGT,Body),
  Prog2=[sub(Name, Atom, DT, GT, Subs)|Prog1],
  prove(Body, Prog2, Prog3).
```

Figure 6.1: Metagol<sub>RT</sub>: refinement type checking MIL.

carries over to the grand refinement of the entire program, i.e. once proving unsatisfiable subsequent additions to the program cannot make the grand refinement satisfiable. This observation is enough to guarantee sound pruning of the search space.

#### 6.2.3 Tree-Shaped Grand Refinement

As noted in the previous section the algorithm maintains the structure of the program constructed in the form of how the invented and interpreted clauses are used. This structural information encodes where predicates occur in the program and over which variables they operate.

Tree-shape derivation The Meta-Interpretive Learning approach to synthesis extends Prolog's backward-chaining algorithm with additional methods for proving atoms. The backward-chaining algorithm come downs to the idea that a goal atom is proven by unifying the goal with the head of a definite clause leading to the body atoms of this clause becoming the goals. Hence a goal atom has child goal atoms, which in turn have child goal atoms, etc., until a goal atom is an asserted fact (i.e. has not body to prove). This means that the proof of a goal atom forms a tree of goals.

The derivation maintained in the  $Metagol_{RT}$  algorithm maintains this structure in Prog, but just for the proof steps that involve interpreted and invented clauses. For usages of interpreted predicates we store inter(Atom,DT,GT,Subs), where Atom's predicate identifies the interpreted clause used. The inter(...) atom encodes the information needed to reconstruct goal body atoms, instantiating known predicate variables with the substitution Subs. For invented clauses we have that sub(Name,...) and inv(Name,...) use Name to identify the metarule used, thereby giving access to the body goals. The leafs of this proof tree are the atoms that are either proved by a primitive, meaning that the chosen primitive will be stored in the Sub substitution of the parent goal, or are atoms who are yet to be proven (whose predicate symbol might be a variable). Hence the information stored in Prog is sufficient to recreate the proof tree with known predicate variables resolved and leaving unknown predicates as variables.

Constructing the Grand Refinement As follows from section 6.2.1, the grand refinement can be directly derived from this proof structure. We explain a traversal of the derivation tree whereby the grand refinement is constructed and at the same time a *variable context* is maintained. The variable context is the set of typed variables that occur in the derivation,

along with the values that they are assigned to (in case an argument variable has already been assigned a value).

The details of the algorithm for converting a derivation to its grand refinement are in figure 6.2. The basics are that two mutually recursive clauses build up the grand refinement bottom up. Leafs, i.e. those atoms whose predicate is a variable or a primitive, are directly convertible, see the first two disjuncts of atom\_to\_refinement. In the case of interpreted predicates the body refinement is first derived, and this refinement is conjuncted with the refinement that was supplied for the interpreted clause. For every atom the arguments are added to the context. In body\_to\_refinement a list of atoms is a clause body, which represents a conjunction of atoms, hence the main task of this clause is to collect contexts and conjunct refinements.

The grand refinement is derived by supplying atom\_to\_refinement with the example goal that algorithm is currently trying to prove, along with the current program derivation. As a result we obtain a variable context, containing variable names with typing (and possibly values), and a single large proposition: the grand refinement.

#### 6.3 Grand Refinement Checking

The only issue not dealt with in the previous sections is that of establishing whether the grand refinement is satisfiable. Satisfiability of the refinement is defined as there being an assignment of the variables in the context such that the grand refinement is true.

This section explores the possibilities that Satisfiability Modulo Theories (SMT) provide as a framework for specifying refinements. In this framework the grand refinement corresponds to a SMT logic formula. SMT solvers will be used to try to prove unsatisfiability (also called inconsistency) of this formula.

#### 6.3.1 SMT Solvers

The main motivation for choosing to explore the effectiveness of refinement types in pruning the search space is that SMT solvers have proven to be very efficient in solving satisfiability problems.

Satisfiability Modulo Theories (SMT) are theories for stating logic problems regarding a set of formulas having a satisfying model. The formulas are expressed in a suitably restricted logic. The satisfiability problem is encoded in a language supported by SMT solvers. A common standard, with widespread support, is the SMTLIB (2.0) language [Barrett et al., 2010].

```
atom_to_refinement(Pred(Args):Ty,Prog,[],\langle true \rangle):-
  var(Pred).
atom_to_refinement(Pred(Args):Type,Prog,Context,Refinement):-
 prim(Atom:Type,Refinement),
  typed_args_to_context(Args,Type,Context).
atom_to_refinement(Pred(Args):Type,Prog,Context,Refinement):-
 member(inter(Pred(Args), Type, GT, Subs), Prog),
  interpreted(Subs,Pred(Args):Type:-Body),
  interpreted_ref(Pred(_),InterRef),
 body_to_refinement(Body,Prog,BodyContext,BodyRef),
  typed_args_to_context(Args,Type,InterCtx),
  append(BodyContext,InterCtx,Context),
 Refinement=and(BodyRef,InterRef).
atom_to_refinement(Pred(Args):Type,Prog,Context,Refinement):-
  (member(app(Name, Pred(Args), Type, GT, Subs), Prog);
  (member(sub(Name, Pred(Args), Type, GT, Subs), Prog)),
 metarule(Name, Subs, (Pred(Args): Type: -Body)),
 body_to_refinement(Body,Prog,BodyContext,Refinement),
  typed_args_to_context(Args,Type,InterCtx),
  append(BodyContext,InterCtx,Context).
body_to_refinement([],Prog,Context,\langle true \rangle).
body_to_refinement([Atom|Atoms],Prog,Context,Refinement):-
  atom_to_refinement(Atom,Prog,Ctx1,Ref1),
 body_to_refinement(Atoms,Prog,Ctx2,Ref2),
  append(Ctx1,Ctx2,Context),
 Refinement=and(Ref1,Ref2).
```

Figure 6.2: Prolog code for converting a derivation to a Grand Refinement.

SMT solvers are special purpose programs utilizing the best available algorithms to prove satisfiability, often relying on heuristics. Forerunners in performance and support of new logics are the Z3 solver [De Moura and Bjørner, 2008], and the CVC4 solver [Barrett et al., 2011].

#### 6.3.2 Language of Refinements

Up till this point we have relied on mathematical syntax for specifying refinements. The only requirement we have had on refinements is that they are propositions that may mention the arguments of a predicate, where we left the availability of certain functions and notations out of scope.

With our choice for solving satisfiability of the grand refinement fixed, we can use this decision to guide our specification of refinements. As any language we choose needs to be translated to a format that the SMT solvers are able to work with, the simplest solution is to take the syntax of SMTLIB.

Variables in refinements The main feature on top of SMTLIB that we need is to be able to correctly keep track of the named variables in the refinements. The chosen solution is to move from a refinement being a single string to list of strings and terms. An example for such a refinement is for the map predicate:

The first thing to notice is the Lisp-like syntax of the refinement, and that the SMTLIB language has support for functions, e.g. length. The significance of the interspersing of variable names is that a variable occurring multiple times in the program derivation (and hence also in the grand refinement), are identified with one another. This also makes it easy to translate such a single derivation variable to just one SMT variable. The SMT translation of the grand refinement just has this single variable name in place of all the occurrences of the variable.

**Encoded problem** The translation of the grand refinement to a SMT problem now proceeds as follows. First the context variables are considered, which are the same logical variables that occur in the refinement. For each variable there is a declaration of a new SMT variable with a new name, with typing as it occurs in the derivation. If the variable has a known value in the derivation a SMT equality assertion is generated for the variable with this value. Note that there may occur higher-order variables in predicate

arguments, but these will never occur as variables in refinements and hence are filtered out.

For the grand refinement we have that each single refinement occurring in it can now be "folded down" from a list of strings and Prolog variables to a single string. The names chosen for the SMT variables are used as substitutions for the variables that occur in the refinements, where upon string concatenation becomes available. The structure of the grand refinement itself, involving disjunctions and conjunctions over refinements, can now be made into a single string by replacing nodes such as and(ref1,ref2) by RefStr from the following code:

The set of formulas given to the SMT solver are thus: the variable declarations, assertions for values of variables (as far as they are known), and the translation of the grand refinement type.

#### 6.4 Expressiveness of Refinements

Having chosen the syntax of the type refinements, this section looks at the available choices in regards the logic theories that SMT solvers support. As has been the theme of this document we identified higher-order predicates as promising, with as argument that many standard list functions/predicates have simple refinements. We therefore focus on SMT logic theories that are able to reason over lists.

#### 6.4.1 Z3 Sequence Theory

The first candidate is the Z3 Sequence [N. Bjørner and Veanes., 2012] theory. This theory is able to reason over lists of bounded length and includes such function symbols as: seq.concat, seq.len, seq.indexof, seq.extract, etc. The theory is already undecidable, but it is the smallest theory that we identified that includes list reasoning.

In this theory it is very easy to write refinements using just the set of available function symbols. As our main example, the map predicate's refinement assertion looks very familiar:

Beyond its undecidability, the main issue with this theory is that it lacks expressiveness. For example, we might want a refinement on a sort predicate that states that all the elements of the input and the output lists are the same. Such a refinement is not possible in this theory.

#### 6.4.2 DTLIA: Quantifiers, Datatypes and Arithmetic

Lists are the prime example of algebraic datatypes (ADTs). In the ADT formulation of lists there is only the empty list constructor and the cons list constructor for an element and a second list. The only operations available are to pattern match on these two constructors. Embracing algebraic datatypes leads more possibilities beyond just lists, e.q. option types and record types. The ADT formulation of lists comes with no functions predefined, that is, notions such as length will have to be user-defined. Many important functions on lists are recursive, e.g. the length function. When reasoning over lengths it is often useful to also compare lengths. This means we also need some basic arithmetic notions.

Recently there has been work on SMT solvers to support algebraic data types, the logic fragment of which is called DT [Reynolds and Blanchette, 2015]. This theory on its own is decidable, though with quantifiers it is not. In other recent work progress<sup>1</sup> has been made on supporting recursive functions on these data types [Reynolds et al., 2016]. SMT solvers translate recursive functions definitions (on ADTs) to quantified formula, therefore we have to relinquish decidability. The need to support simple arithmetic is satisfied by the Linear Integer Arithmetic (LIA) theory. The combination of theories DT (with quantifiers over Uninterpreted Functions) and LIA is the theory (UF)DTLIA. Only the CVC4 solver implements this theory.

The complexity of the recursive function definitions can be hidden from the user for standard definitions, e.g. dt\_length in the following:

Behind the scenes the dt\_length predicate is able to insert the appropriate SMTLIB function definition in the set of formulas handed to the SMT solver. The below code shows such a definition. Note that SMTLIB does not support the polymorphism of this example (the example uses Y as a type parameter), but that the generated code must actually generate a separate function definition based on the types in each instance of the function being in the grand refinement.

 $<sup>^1{\</sup>rm The}$  work is so recent that I discovered a unsoundness issue in the implementation. The issue was promptly fixed: https://github.com/CVC4/CVC4/issues/2133

For user-defined recursive SMT functions the effort required to integrate with the system becomes significant. The implementation of this system is quite non-trivial while the details are rather uninteresting. The document will not look further into the matter.

The *DTLIA* is our preferred theory in terms of expressiveness. Recursive functions give enough power to express subset inclusions on lists, hence it becomes possible to express the property of the **sort** predicate that the elements are permuted. It is interesting to note that the recursive functions are so powerful that they essentially allow us to fully encode (first-order) Prolog predicates in the SMT logic. Hence users get the option to choose, on a sliding scale, between very accurate refinements and very coarse refinements. The understanding is that these more complicated refinements might take more time to reason over, though could also be useful in detecting inconsistency earlier.

#### 6.5 Theoretical Results

The results from the previous chapter can be directly lifted to the setting of refinement types. The reason for why the propositions can be directly restated and their proofs only slightly modified is that the refinement types have the same soundness properties for the synthesis algorithm as simple types. In particular the main feature is the inconsistency pruning of the search space.

**Proposition 6.5.1** The programs found by the Metagol<sub>AI</sub> algorithm (which can be assigned a polymorphic types with refinements) and the Metagol<sub>RT</sub> algorithm are exactly the same.

The proof follows again by that a depth-first search algorithm is used to traverse the program hypothesis space and that programs are still encountered in the same order, just that the some inconsistent programs have been skipped over by the  $\mathrm{Metagol}_{RT}$  algorithm due to type checking. For a more precise argument see section 5.3.

The reduction of the space of  $H_2^2$  programs also directly applicable to the refinement type system. The refinements are for the most part an improvement with regard to the granularity of the types, i.e. they will further restrict the possibilities for choices of predicates in a clause. This means that the result regarding the worst case proportional constant applies but that refinements in the background knowledge will shrink this worst case ratio further.

#### 6.6 Experimental Results

We present experiments which check whether there is sufficient evidence to claim that the current implementation of refinement type checking has significant advantages, i.e. whether we are justified in *rejecting* the following:

Null Hypothesis 6.6.1 Refinement type checking cannot reduce the number of proof steps compared to non-refinement polymorphic type checking.

As in the previous chapter we focus on the droplasts program. This program takes a list of lists and drops the last element from each of the inner lists. We perform statistical experiments to evaluate whether refinement type checking has any benefits versus only simple types checking (and by extension versus the untyped system).

In the first experiment we add predicates with polymorphic types that compose well, but whose refinement types lead to being able to decide that some combinations of predicates in a clause will not work. This test is primarily to show a difference between simple type checking and refinement type checking. The second experiment tries to be a bit more general and includes predicates that are less sensible, though allow for a bigger difference between the untyped case and the typed cases.

# 6.6.1 Experiment 4: Droplasts with Sensible Predicates

The program we are synthesizing is droplasts. We perform a statistical experiment where we have random background knowledge containing predicates with refinement types and are iteratively adding additional refinement typed predicates to check how the refinement type checking system behaves when the composition of refinements rules out certain programs.

**Setup** We follow the setup of the simply typed droplasts experiment of section 5.4.4. We generate small random input examples, outer and inner

lists (of integers) of length between 2 and 5, and run them through a reference implementation to obtain correct positive examples. We choose this length restriction for the examples, because the SMT solver needs more time for larger lists. Adding additional examples would make the found programs more accurate, but would come at the cost of even longer synthesis times (due to the SMT solver being invoked more often). As we are interested in the effect of type checking on the search space we fix the number of (positive) examples generated to three.

We provide the synthesis system with the following predicates:

- concat: [A:list(T),B:T,C:list(T)]  $\langle length(A) + 1 = length(C) \rangle$ The relation that appends an element at the back, where length is the length function over ADT lists definable in SMT-LIB.
- tail: [A:list(X),B:list(X)]  $\langle A=cons(\_,B)\rangle$ The relation taking of the head of a list, with accurate refinement representation.
- reverse: [A:list(X),B:list(X)]  $\langle rev(A,B) \rangle$ The reverse relation on lists. The refinement is a SMT-LIB quadratic time predicate function checking whether the arguments are each others' reversal.
- map: [A:list(X),B:list(Y),F:[X,Y]]  $\langle length(A) = length(B) \rangle$ The higher-order map relation with a refinement concerning lengths.
- reduceback: [list(T),list(T),[list(T),T,list(T)]] \( \langle true \rangle \)

  The reduceback relation which essentially replaces cons in a list with the function parameter. The current refinement system cannot assign a more useful refinement.
- filter: [A:list(T),B:list(T),F:[T]]  $\langle length(B) \leq length(A) \rangle$ The filter relation whereby element can only be retained from the first list if the F predicate holds for the element.

As part of the experiment we add random background predicates. These predicates drop an element at a fixed index from a list. The refinement for these predicates encode that lists of lower length are unaffected and lists of length equal to this index, or higher, have their lengths reduced by one.

The SMT solver's timeout has been set to 30 milliseconds, which still gives enough time for the solver to prove some problem instances unsat.

We use the CVC4 SMT solver, for which we specify the logic DTLIA, with options for finite model finding and inductive reasoning enabled.

For each number of additional background predicates we run ten trials, for each trial generating positive examples and random predicates. We average over the trials and calculate the standard deviation of the sample.

**Result** The plots in figure 6.3 depict the average time and number of proof steps required by the untyped, simply typed and refinement type systems. The standard deviations are included as bars.

The amount of time necessary for refinement checking is an issue made obvious by the first graph. The proportion of type spent in Prolog running the Metagol code is trivial compared to the time spent waiting for the SMT solver. We are aware that this means that the current implementation does not afford a lot of practical benefit, instead we should approach the work on refinement types as a proof of concept.

For the plot on the right we have that the refinement system is able to cut down on the search space explored relative to the untyped and simply typed systems, though with considerable variance<sup>2</sup>. Note that because the types for the most part compose that the difference between the simply typed and untyped systems is small, again showing that worst case proportional constant is significant in improving on the untyped system. Given these graphs we are inclined to reject the hypothesis that refinement type checking does not have any advantages, though we do so with the knowledge that the amount of data generated is actually rather limited and a sudden spike in variance. Another provision is that improvement in search space reduction comes at a very severe cost in regard to execution time.

As a sanity check of the implementations we use the soundness results from this, and the previous, chapter. By soundness we know that when a system prunes part of the search space they do so only when that part of the search space cannot yield a successful program. Hence the number of proof steps for the  $\text{Metagol}_{PT}$  should always be at most the number of proof steps of the  $\text{Metagol}_{AI}$  system, and the  $\text{Metagol}_{RT}$  system should always need at most the number of proof steps of the  $\text{Metagol}_{PT}$  system. We can confirm this for each separate trial that was run, giving some confidence in the correctness of the implementation.

<sup>&</sup>lt;sup>2</sup>The sudden increase of variance is a sign that this experiment is not particularly well designed. In future work other long running experiments will be used to evaluate the work.

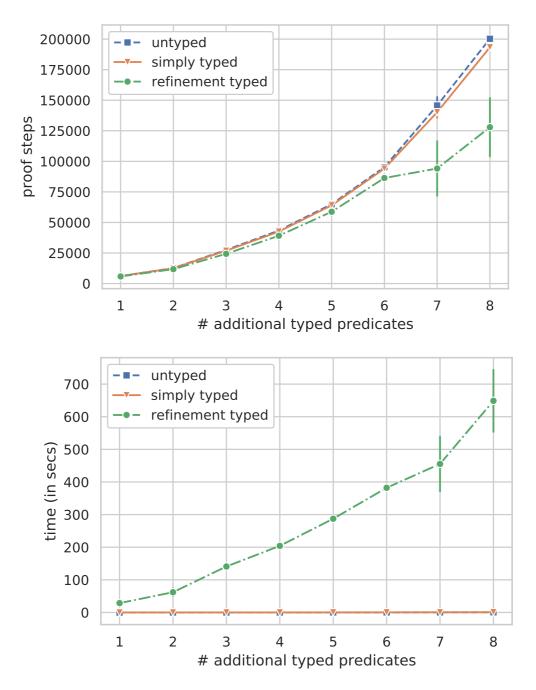


Figure 6.3: Average number of proof steps and time for increasing number of typed background predicates. Standard error is depicted by bars.

# 6.6.2 Experiment 5: Droplasts with Additional Predicates

In order to distinguish more between the untyped and the simply typed cases in the previous experiment an (even) longer running experiment was conducted.

**Setup** We take an identical setup to the previous experiment, though now remove the identity predicate from the base background knowledge and add the following refinement typed predicates:

```
\begin{split} & \texttt{dumb0([0],[]):[list(nat),list(X)]} \langle true \rangle \, . \\ & \texttt{dumb1(W,0):[list(nat),int]} \langle false \rangle \text{:-findall(K,(between(3,4,K)),W)} \, . \\ & \texttt{dumb2(W,0):[list(int),int]} \langle false \rangle \text{:-findall(K,(between(2,7,K)),W)} \, . \end{split}
```

The first and the second predicate should usually be ruled out due to polymorphic types and the second and third have refinements that state that they cannot be used in a program.

**Result** The plots in figure 6.4 depict the average time and number of proof steps required by the untyped, simply typed and refinement type systems. The standard deviations are included as bars.

As is unsurprising, the time results show that our refinement checking is very slow, and has very high variance as well. The plot of the number of proofs steps to the right shows that while there is evidence for concluding that the refinement type checking system can improve in regard to the untyped system, the simply typed and refinement typed systems are quite close in the size of the search space traversed. This experiment is not able to show significant improvements for the refinement type checking. A better considered experiment is needed, which we have to defer to future work.

The data does show that the refinement type checking system is able to do with strictly fewer proof steps than the (simple) polymorphic typed system and the untyped system. We can cautiously use this as evidence that the refinement typed system is able to improve on the untyped (and simply typed) system, though we have to keep in mind that this is solely for a minor reduction in the size of the search space and that it comes at a very large expense in terms of runtime.

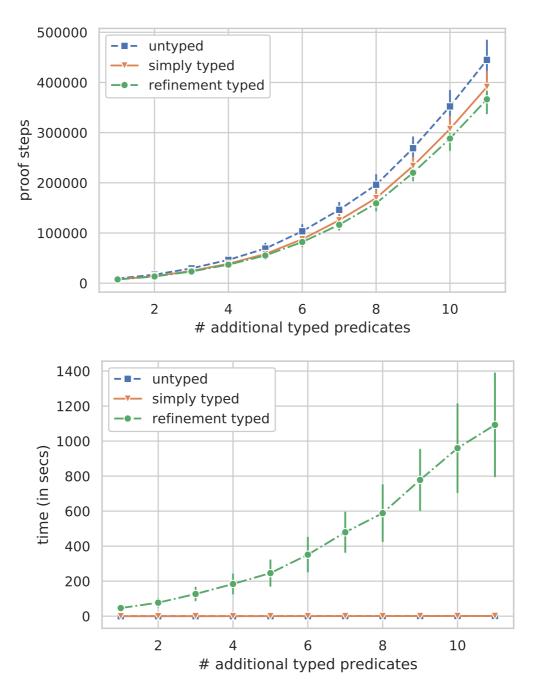


Figure 6.4: Average number of proof steps and time for increasing number of typed background predicates. Standard error is depicted by bars.

# Chapter 7

### Conclusion

Logic program synthesis by Meta-Interpretive Learning (MIL) benefits significantly, in terms of search space explored and time needed, from type checking. The difference in performance between the untyped  $Metagol_{AI}$  algorithm and the same algorithm with simple type checking,  $Metagol_{PT}$ , is impressive. On the other hand, the value of the refinement types checking introduced in this document is foremost theoretical: there is only limited experimental improvement, but the work does represent an advancement in terms of bringing refinement typing to Inductive Logic Programming (ILP).

#### Summary and evaluation

In building on the MIL framework, the basic synthesis algorithm has shown why it is such a convincing approach to ILP: it is a simple, succinct, and highly adaptable extension of the algorithm used for proving atoms in logic programming. The  $Metagol_{AI}$  variant of the framework was essential in allowing the system to express higher-order programs. It is this work on higher-order programs that made it possible to consider interesting refinement type properties. The major deficiency we identified with the system was its naiveté in not taking typing into account.

The addition of polymorphic types to the MIL algorithm is a very good fit. The syntax for types has been chosen such that it is easy to leverage the powerful unification of Prolog, which is entirely responsible for type checking. In experimental work we have shown that the introduction of simple type checking is very effective, showing significant improvements in both shrinking the search space explored and the time required for synthesis. There is (up to) a cubic reduction in the size of the search space and synthesis time, in terms of the number of typed background predicates. As annotating background knowledge with polymorphic types is only a small burden for users, this

result provides a strong argument for taking polymorphic type checking into consideration for future ILP systems.

We presented the theory that introduces refinement types to ILP. Refinements on types allow for more accurate type checking. The cost of checking the refinements is however not insignificant. We leverage SMT solving, which in the current approach incurs such overhead that timewise the refinement checking has a severe detrimental effect. The overhead is attributable to the logics considered not being as performant as hoped, and to the shear number of invocation of the solver. The experimental work shows that some additional pruning of the search space is achieved, though more work is needed to make refinement type checking in ILP worthwhile.

#### Future work

**Theory** The experimental work on polymorphic types (section 5.4) showed (and partially explained) a discrepancy between the theory of the size of hypothesis spaces and the search space over programs considered by MIL algorithms. A better characterization of the search spaces considered by MIL would be useful in predicting the impact of algorithmic changes.

The work regarding polymorphic type checking itself can be made more convincing by presenting formal proofs of soundness and completeness. In the future a formal type system should be introduced for this purpose.

**Justifying refinements** While the polymorphic type checking approach is quite satisfactory, the refinement types work can be improved upon in a number of areas. Following on from the experimental work, better experiments are needed to justify refinement type checking as a worthwhile approach to further pruning the search space.

In addition, as the main aim of refinement type checking is in improving performance, the most direct way of addressing this issue is by making use of more performant theories for the SMT solvers. The work is mainly in identifying logics that are expressive enough to state useful properties, while requiring much less time to prove (un)satisfiability. Analysis of the number of invocations of the SMT solvers in the experimental data of section 6.4 could be used to express the performance needed of SMT solvers for the current approach, which involves many SMT invocations, to be a sensible way forward.

**Reduction to SMT** The current approach asks the SMT solver to solve an entirely new problem every time it is invoked, but when a previous refinement check was satisfiable it is usually the case that only additional assertions need

to be added. This structural property could be leveraged to make solvers more efficient by allowing them to reuse work performed for the previous checks. A more ambitious approach (briefly considered for this project) is to completely encode the ILP synthesis problem as a set of constraints for a SMT solver. The prospect of integrating the checking of refinements into such a single SMT problem is especially enticing.

Directing the search In section 4.3 we already explored the possibility of types being able to guide the traversal of the search space, which given a well-considered heuristic would be able to further prune the search space. Such further pruning might also have implications for the viability of the current refinement type checking approach as this could significantly reduce the number of invocations of the SMT solver.

Functional metarules The MIL approach to synthesis is especially powerful in that it is able to invent program clauses. The main reason for this capability are the metarules. It appears that introducing rules for structural inventions might be applicable outside of ILP. Introduction of metarules to the setting of functional programs would a major new avenue to explore. Existing type system-based approaches could be extended, thereby introducing the (almost) unique feature of invention of helper clauses/functions to the field of functional program synthesis.

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