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SYNTHESIS OF CONTINUOUS-VALUED LOGIC FUNCTIONS DEFINED IN TABULAR FORM

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The choice table provides one of the techniques for the representation of functions in continuous-valued logic [1]. The need to synthesize functions from choice tables arise in the design of hybrid [2] and analog [3] computers, and also in other applications of continuous-valued logic that are surveyed in [4]. The structure of the original table is determined by the external specification of the device or unit being designed.

Algorithms are available for the synthesis of continuous-valued logic functions from choice tables of a special form, for instance, from ordered choice tables [3]. It is noted in [4] that a general algorithm to synthesize a continuous-valued logic function from an arbitrary choice table is still unknown.

In the present article, we derive a criterion that decides whether a given choice table defines some continuous-valued logic function and construct a simple algorithm to synthesize the function from the table.

BASIC CONCEPTS AND DEFINITIONS

Following [4], we define the algebra of continuous-valued logic as the quasi-Boolean algebra $\Delta = (C, \wedge, \vee, \bar{\ })$, where $C = [A, B]$ is a continuous interval in the set of real numbers with a linear order relation $\leq$ defined in the usual way; $M = (A + B)/2$ is the midpoint of this interval, called the median. The basic operations of conjunction $\wedge$, disjunction $\vee$, and negation $\bar{\ }$ are defined for any $x, y \in C$ in the following way:

$$x \wedge y = \min(x, y), \quad x \vee y = \max(x, y), \quad \bar{x} = 2M - x. \tag{1}$$

The interval C is treated as the set of feasible values of the degree of truth of logical variables; A is the degree of truth of an absolutely false statement, and B is the degree of truth of an absolutely true statement. The basic operations (1) can be used to define additional operations that are used in traditional logic: implication, equivalence, exclusive disjunction, etc. The conjunction sign is omitted in the formulas of continuous-valued logic throughout the rest of the article.

A continuous-valued logic function (CLF) is any function f: $C^n \to C$ formed as a superposition of a finite number of basic operations applied to independent variables $x_1, x_2, \ldots, x_n \in C$.

The definitions of basic operations (1) imply that a CLF always takes the value of one of its arguments or its negation. Moreover, the value of the function is completely determined by the ordering of the arguments and their negations in the relation $\leq$. Thus, an arbitrary CLF is uniquely representable by a table that lists all the possible orderings of the arguments and their negations and shows the function value for each ordering. Such tables are called choice tables [4]. Table 1 is the choice table for the function $f = x_1 \vee x_2\bar{x}_2$.

The quasi-Boolean algebra Δ is a distributive lattice with pseudocomplementation [2]; most laws of binary logic [4] hold in this algebra: commutativity, associativity, distributivity, de Morgan, Kleene, absorption, double negation, idempotency of elements. The validity of these laws can be established, for instance, using choice tables. A special feature of continuous logic is that the law of excluded middle and the law of contradiction do not hold: $x \wedge \bar{x} \neq B$, $x \wedge \bar{x} \neq A$.

Disjunctive normal form (DNF) and conjunctive normal (CNF) are used as the standard forms of CLF [4]. Unlike similar forms of functions in binary logic, elementary conjuncts (disjuncts) may contain both the argument x_i and its negation

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TABLE 1

Serial number	Region	Function value
1	$x_1 \le x_2 \le \bar{x}_2 \le \bar{x}_1$	x_2
2	$x_1 \le \bar{x}_2 \le x_2 \le \bar{x}_1$	$\bar{x}_2$
3	$\bar{x}_1 \le x_2 \le \bar{x}_2 \le x_1$	x_1
4	$\bar{x}_1 \le \bar{x}_2 \le x_2 \le x_1$	x_1
5	$x_2 \le x_1 \le \bar{x}_1 \le \bar{x}_2$	x_1
6	$x_2 \le \bar{x}_1 \le x_1 \le \bar{x}_2$	x_1
7	$\bar{x}_2 \le x_1 \le \bar{x}_1 \le x_2$	x_1
8	$\bar{x}_2 \le \bar{x}_1 \le x_1 \le x_2$	x_1

$\bar{x}_i$. It follows from the basic laws of continuous-valued logic that every CLF is representable in disjunctive (conjunctive) normal form.

Note that the definition of the algebra of continuous-valued logic as a quasi-Boolean algebra Δ is quite common at present [2-4], although this is not the only possible definition. Thus, McNaughton's infinite-valued logic [5] uses two basic operations of negation and implication, and the result of the implication operation in general is not identical with the value of one of the arguments or the negation of an argument.

TABULAR REPRESENTATION OF FUNCTIONS IN CONTINUOUS-VALUED LOGIC

As we have noted previously, any CLF is uniquely representable by a corresponding choice table. There are a total of $(2n)!$ orderings of n arguments and their negations. However, the values of the variable x_i and its negation $\bar{x}_i$ are symmetrical about the median M, i.e., $x_i \le x_j$ if and only if $\bar{x}_j \le \bar{x}_i$. Thus, the total number of rows in a choice table of a function of n arguments [2] is $L = 2^n n!$. For each ordering, the function may independently take one of $2n$ possible values. There are thus $N = (2n)^L$ different choice tables that define a function of n arguments.

The number of different CLF of one, two, and three arguments has been estimated in [2, 4]. Table 2 compares these estimates with the number of different choice tables for the corresponding number of arguments. The count in [2, 4] also includes the functions that take the values A and B, and the number of choice tables was accordingly calculated from the formula $N = (2n + 2)^L$. We see from Table 2 that only a small number of functions defined by choice tables are functions of continuous-valued logic.

To avoid multiple-level indices, we introduce the following representation of choice tables. Let $x = (x_1, x_2, \ldots, x_n)$ be the set of arguments, and let variables x_i with indices i from $n + 1$ to $2n$ represent negations of arguments: $x_i = \bar{x}_{i-n}$ for $i = \overline{n + 1, m}$, where $m = 2n$. The choice table T is treated as a set of rows $T = \{t\}$, $|T| = L$; $t = (p\ \alpha)$ is a row of the table, $p = (i_1, i_2, \ldots, i_m)$ is the sequence of argument indices in the particular ordering, α is the index of the argument that represents the function value. Thus, $f(x) = x_\alpha$ for $x \in D_t$, where the region $D_t \subseteq C^n$ is defined by the inequalities $x_{i_1} \le x_{i_2} \le \ldots \le x_{i_m}$.

We say that row t of choice table T overlaps row t' if the index subsets Φ_t and $\tilde{\Phi}_{t'}$ satisfy the inclusion $\Phi_t \subseteq \tilde{\Phi}_{t'}$, where

$$
\begin{aligned}
&t = ((i_1, \ldots, i_{k-1}, a, i_{k+1}, \ldots, i_m), a),\\
&t' = ((j_1, \ldots, j_{l-1}, b, j_{l+1}, \ldots, j_m), b),\\
&\Phi_t = \{a, i_{k+1}, \ldots, i_m\}\ (x_a \leqslant x_i \text{ for } \forall i \in \Phi_t,\\
&\tilde{\Phi}_{t'} = \Phi_{t'} \backslash \{b\} = \{j_{l+1}, \ldots, j_m\}.
\end{aligned}
\tag{2}
$$

Rows t and t' of choice table T are called overlapping if one of them overlaps the other.

THEOREM 1. A choice table that defines a function of continuous-valued logic does not contain overlapping rows.

TABLE 2

Number of arguments	Number of CLF	Number of choice tables
1	6	8
2	84	1679616
3	43918	$\sim 223 \cdot 10^{41}$

Proof. Take an arbitrary function of continuous-valued logic f. Without loss of generality we may assume that f is represented in disjunctive normal form:

$$f(x) = \bigvee_{q} K_q, \tag{3}$$

where $K_q = \bigwedge_{i \in I_q} x_i$ are elementary conjuncts; I_q is the set of indices i of the variables x_i entering conjunct q.

Construct the choice table T for the function f. Assume that in table T some row t overlaps row t', i.e., $\Phi_t \subseteq \bar{\Phi}_{t'}$ by (2). Assume that the sets of arguments $x \in D_t$ and $x' \in D_{t'}$ satisfy rows t and t' of table T respectively in the sense of strict inequalities.

Since $f(x) = x_\alpha$, then by definition of disjunction as the maximum argument value, at least one of the conjuncts in DNF (3) takes the value x_α on x. Suppose that this conjunct is K_u:

$$K_u = \bigwedge_{i \in I_u} x_i, \; K_u(x) = x_\alpha.$$

Then by definition of conjunction as the minimum argument value, we have $I_u \subseteq \Phi_t$, and from $\Phi_t \subseteq \bar{\Phi}_{t'}$ we conclude that

$$I_u \subseteq \bar{\Phi}_{t'}. \tag{4}$$

Consider the value of the conjunct K_u on x'. Since $f(x') = x_b$, we have $K_u(x') \le x_b$, and thus there exists $j \in I_u$ such that $x_j \le x_b$, and so $j \notin \bar{\Phi}_{t'}$. Hence

$$I_u \not\subseteq \bar{\Phi}_{t'}. \tag{5}$$

Comparing (4) and (5), we obtain a contradiction, which proves that a choice table defining a CLF cannot contain overlapping rows.

SYNTHESIS OF CONTINUOUS-VALUED LOGIC FUNCTIONS FROM CHOICE TABLES

Given is a choice table T without overlapping rows. Let us construct an algorithm that synthesizes a CLF from the table T.

The maximum constituent of row t in table T is the conjunct

$$\Phi_t = \bigwedge_{i \in \Phi_t} x_i, \; (x_\alpha <= x_i \;\; \text{for} \;\; \forall i \in \Phi_i). \tag{6}$$

From definition (6) we have the following properties of constituents: for any set of arguments x satisfying row t ($x \in D_t$) we have

$$\varphi_t(x) = x_\alpha. \tag{7}$$

TABLE 3

Serial number	Region	F_2	F_2	f_1	f_2	f_2^1	f_2^2	g_2^1	g_2^2	g_2^3
1	$x_1 \leq x_2 \leq \bar{x}_2 \leq \bar{x}_1$	x_2	x_2	x_2	$\bar{x}_1$		x_2			x_2
2	$x_1 \leq \bar{x}_2 \leq x_2 \leq \bar{x}_1$	x_2	$\bar{x}_1$	x_2	$\bar{x}_1$	$\bar{x}_1$		$\bar{x}_1$		
3	$\bar{x}_1 \leq x_2 \leq \bar{x}_2 \leq x_1$	$\bar{x}_2$	x_2	$\bar{x}_2$	x_2	x_2				x_2
4	$\bar{x}_1 \leq \bar{x}_2 \leq x_2 \leq x_1$	$\bar{x}_2$	$\bar{x}_2$	$\bar{x}_2$	$\bar{x}_2$	$\bar{x}_2$				$\bar{x}_2$
5	$x_2 \leq x_1 \leq \bar{x}_1 \leq \bar{x}_2$	x_1	x_1	x_1	$\bar{x}_1$		x_1			x_1
6	$x_2 \leq \bar{x}_1 \leq x_1 \leq \bar{x}_2$	x_1	$\bar{x}_1$	x_1	$\bar{x}_1$	$\bar{x}_1$				$\bar{x}_1$
7	$\bar{x}_2 \leq x_1 \leq \bar{x}_1 \leq x_2$	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_1$	$\bar{x}_1$		$\bar{x}_2$		$\bar{x}_2$	
8	$\bar{x}_2 \leq \bar{x}_1 \leq x_1 \leq x_2$	$\bar{x}_1$	$\bar{x}_1$	$\bar{x}_1$	$\bar{x}_1$	$\bar{x}_1$				$\bar{x}_1$

LEMMA 1. For two nonoverlapping rows t and t' in choice table T and the set of arguments $\boldsymbol{x}'$ satisfying $t'(\boldsymbol{x}' \in D_{t'})$, we have the inequality

$$\varphi_t(\boldsymbol{x}') \leq \varphi_{t'}(\boldsymbol{x}'). \tag{8}$$

Proof. Assume the contrary. Let $\varphi_t(\boldsymbol{x}') > \varphi_{t'}(\boldsymbol{x}')$. By the property of maximum constituents (7), $\varphi_t(\boldsymbol{x}') = x_b$. Then $x_b < \bigwedge_{i \in \Phi_t} x_i$, or by definition of conjunction as the minimum argument value

$$x_b < x_i \quad \text{for} \quad \forall i \in \Phi_t. \tag{9}$$

By definition (2), the set $\bar{\Phi}_{t'}$ consists of argument indices j such that

$$x_b \leq x_j \quad \text{for} \quad \forall j \in \bar{\Phi}_{t'}. \tag{10}$$

Then, comparing (9) and (10), we have $\Phi_t \subseteq \bar{\Phi}_{t'}$. Row t of table T thus overlaps row t'. A contradiction. Q.E.D.

THEOREM 2. An arbitrary choice table T without overlapping rows defines a function of continuous-valued logic that is equal to the disjunction of the maximum constituents of the rows in the table.

Proof. By definition of the maximum constituent (6), for any choice table T we can construct the function

$$f(\boldsymbol{x}) = \bigvee_{t \in T} \varphi_t. \tag{11}$$

Since table T is without overlapping rows, the constituent φ_t has properties (7) and (8). Consider the value of the function f on an arbitrary set of argument $\boldsymbol{x}' \in C^n$. Every set of arguments always satisfies one of the rows of table T. Assume that the set $\boldsymbol{x}'$ satisfies the row $t'(\boldsymbol{x}' \in D_{t'})$. Then $\varphi_{t'}(\boldsymbol{x}') = x_b$ and for $\forall t \in T$, $t \neq t'$: $\varphi_t(\boldsymbol{x}') \leq x_b$. Therefore, by definition of disjunction as the maximum argument value,

$$f(\boldsymbol{x}') = \bigvee_{t \in T} \varphi_t = x_b.$$

Thus, the value of the function f defined by (11) is identical with the value defined by the choice table T. Since the set of arguments $\boldsymbol{x}'$ is arbitrary, the values are identical for $\forall \boldsymbol{x}' \in C^n$. Q.E.D.

We can similarly define the minimum constituents of the rows of table T:

$$\psi_t = \bigvee_{i \in \Psi_t} x_i, \text{ where } \Psi = \{i_1, \ldots, i_{k-1}, a\} \quad (x_i \leq x_a \quad \text{for} \quad \forall i \in \Psi). \tag{12}$$

Then Theorem 2 can be restated in dual form that defines CLF as a conjunction of the minimum constituents of the rows of a table:

$$f(x) = \bigwedge_{t \in T} \psi_t. \tag{13}$$

THEOREM 3. A choice table defines a function of continuous-valued logic if and only if it does not contain overlapping rows.

The theorem is a direct corollary of Theorems 1 and 2.

By Theorem 2, the CLF synthesis algorithm consists of two sequential steps.

Step 1. Construct the maximum constituents (6) (minimum constituents (12)) of the rows of the table.

Step 2. Construct the DNF (CNF) of the function as a disjunction (11) (conjunction (13)) of the maximum (minimum) constituents.

The average size of a constituent is n ($n = m/2$), and the overall complexity of the algorithm is of order $o(L \cdot n)$. The function can be minimized by the methods proposed in [2]. The size of the DNF (CNF) constructed by the proposed procedure can be reduced by parallelizing the execution of steps 1 and 2 of the algorithm by means of the law of absorption:

$$x \vee (xy) = x, \quad x(x \vee y) = x.$$

Direct search of overlapping rows (2) in the choice table requires a check for each of $L^2 - L$ pairs of different rows. However, by definition of constituents and disjunctive (conjunctive) normal forms, the algorithm synthesizes functions also for tables that contain overlapping rows. In this case, the values of the resulting function on sets of arguments corresponding to overlapping rows are different from the values shown in the choice table.

This property can be exploited for practical function synthesis. Without first checking the choice table T for the presence of overlapping rows, synthesize the DNF (CNF) f of a continuous-valued logic function. Then construct the choice table T' of the function f. If T and T' are identical, then the original table defines the CLF f. Otherwise, the function defined by table T is not a function of continuous-valued logic. Analytical representation of such functions requires hybrid logic [3] or predicate choice algebra [4].

In predicate choice algebra, the proposed algorithm constructs a sequence of CLFs. To this end, we repeat the algorithm for all the rows of the table on which the function value differs from the initial value. Each CLF defines an initial function on a subset without overlapping rows. Issues of optimization by practically meaningful partitioning criteria are linked with general questions of minimization of functions of predicate choice algebra and fall outside the scope of our study.

EXAMPLES OF FUNCTION SYNTHESIS

The two-argument functions F_1 and F_2 are defined by choice tables T_1 and T_2, respectively. Here

$$T_1 = \{((1, 2, 4, 3), 2),\ ((1, 4, 2, 3), 2),\ ((3, 2, 4, 1), 4),\ ((3, 4, 2, 1), 4),$$
$$((2, 1, 3, 4), 1),\ ((2, 3, 1, 4), 1),\ ((4, 1, 3, 2), 3),\ ((4, 3, 1, 2), 3)\};$$
$$T_2 = \{((1, 2, 4, 3), 2),\ ((1, 4, 2, 3), 3),\ ((3, 2, 4, 1), 2),\ ((3, 4, 2, 1), 4),$$
$$((2, 1, 3, 4), 1),\ ((2, 3, 1, 4), 3),\ ((4, 1, 3, 2), 4),\ ((4, 3, 1, 2), 3)\}.$$

Table 3 is a visual representation of F_1 and F_2. Let us synthesize these functions.

The maximum constituents of F_1 have the following form:

$$\varphi_1 = x_2\bar{x}_2\bar{x}_1, \quad \varphi_2 = x_2\bar{x}_1, \quad \varphi_3 = \bar{x}_2x_1, \quad \varphi_4 = \bar{x}_2x_2x_1,$$
$$\varphi_5 = x_1\bar{x}_1\bar{x}_2, \quad \varphi_6 = x_1\bar{x}_2, \quad \varphi_7 = \bar{x}_1x_2, \quad \varphi_8 = \bar{x}_1x_1x_2.$$

By commutativity of conjunction, φ_7 equals φ_2; φ_6 equals φ_3; φ_2 absorbs φ_1 and φ_8; φ_3 absorbs φ_4 and φ_5. We thus obtain the DNF

$$f_1 = \varphi_2 \vee \varphi_3 = x_1 \bar{x}_2 \vee \bar{x}_1 x_2.$$

The table of the function f_1 is identical with the table of the function F_1 (see Table 3). Thus, F_1 is a function of continuous logic, and f_1 is its disjunctive normal form. The absence of overlapping rows in table T_1 can be established also by a direct check.

Construct the maximum constituents of the function F_2:

$$\varphi_1 = x_2\bar{x}_2\bar{x}_1, \quad \varphi_2 = \bar{x}_1, \quad \varphi_3 = x_2\bar{x}_2x_1, \quad \varphi_4 = \bar{x}_2x_2x_1,$$
$$\varphi_5 = x_1\bar{x}_1\bar{x}_2, \quad \varphi_6 = \bar{x}_1x_1\bar{x}_2, \quad \varphi_7 = \bar{x}_2x_1\bar{x}_1x_2, \quad \varphi_8 = \bar{x}_1x_1x_2.$$

The constituent φ_3 equals φ_4; φ_5 equals φ_6; φ_2 absorbs φ_1, φ_5, φ_7, φ_8. Construct the DNF

$$f_2 = \varphi_2 \vee \varphi_3 = \bar{x}_1 \vee x_1x_2\bar{x}_2.$$

The table of the function f_2 is different from the table of F_2 in regions 1, 5, 7 (see Table 3). Indeed, row 2 of table T_2 overlaps rows 1, 5, 7, and row 8 overlaps row 7:

$$\Phi_2 \subseteq \Phi_1, \quad \Phi_2 \subseteq \Phi_5, \quad \Phi_2 \subseteq \Phi_7, \quad \Phi_8 \subseteq \Phi_7,$$

where $\Phi_2 = \{3\}$, $\Phi_8 = \{3, 1, 2\}$, $\Phi_1 = \{4, 3\}$, $\Phi_5 = \{3, 4\}$, $\Phi_7 = \{1, 3, 2\}$.

Thus, F_2 is not a function of continuous-valued logic. To represent F_2 in predicate choice algebra, we can use one of the following forms:

$$F_2 = \begin{cases} f_2^1 = \bar{x}_1 \vee x_1x_2\bar{x}_2, & x \in D_2, D_3, D_4, D_6, D_8, \\ f_2^2 = x_2\bar{x}_2x_1 \vee x_1\bar{x}_1x_2, & x \in D_1, D_5, D_7; \end{cases} \tag{14}$$

$$F_2 = \begin{cases} g_2^1 = \bar{x}_1, & x \in D_2, \\ g_2^2 = \bar{x}_2, & x \in D_7, \\ g_2^3 = x_1\bar{x}_1 \vee x_2\bar{x}_2, & x \in D_1, D_3, D_4, D_5, D_6, D_8. \end{cases} \tag{15}$$

Representations (14) and (15) of function F_2 correspond to different partitions of table T_2 into subsets of nonoverlapping rows.

REFERENCES

1. S. A. Ginzburg, Continuous-Valued Mathematical Logic and Representation of Functions [in Russian], Energiya, Moscow (1968).
2. P. N. Shimbirev, Hybrid Continuous-Logic Devices [in Russian], Energoatomizdat, Moscow (1990).
3. T. M. Zolotova, F. I. Kerbnikov, and M. A. Rozenblat, Redundancy of Analog Automation Devices [in Russian], Energiya, Moscow (1975).
4. L. I. Volgin and V. I. Levin, Continuous-Valued Logic. Theory and Application [in Russian], Tallin (1991).
5. R. McNaughton, "Theorem on infinite-valued propositional logic," Kibern. Sb. IL, Moscow, No. 3, 59-78 (1961).