$\Delta CoVaR$ Introduction and Estimation

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Outline

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- Definition of $CoVaR$
- Properties of $CoVaR$
- Implementation of $\Delta CoVaR$ Estimation
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1. Introduction

The most commonly used measure of risk in financial institutions:

- **VaR (Value at Risk)**
  - Focuses on the risk of an individual institution in isolation
  - $q\% - VaR_i$ is the maximum loss of institution i at the $q\%$-confidence level

The risk measures of individual institutions do not necessarily reflect their contribution to overall systemic risk.

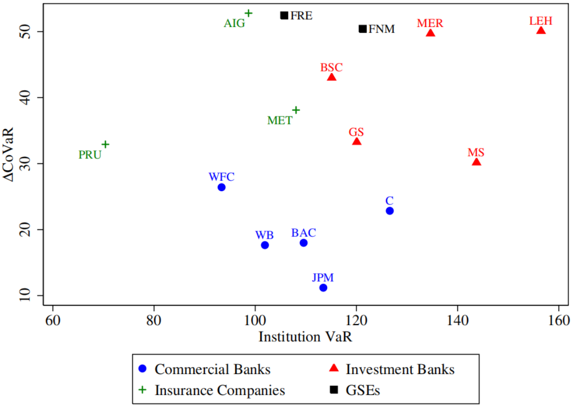
Reasons:

- Individually systemic
- Systemic as part of a herd

- **Systemic risk**
  - A time-series dimension
  - Build in times of low volatility, and materialize during crises

1. Introduction

In this research paper, the authors propose a new simplified form of systemic risk contribution measure called $\Delta CoVaR$. This new measure captures tail dependency and includes negative spillover dynamics in times of crisis.



VaR and $\Delta CoVaR$ relationship is very weak.

2. Definition of $CoVaR$

Recall that VaR_q^i is implicitly defined as the q% quantile:

$$Pr\left(X^i \leq VaR_q^i\right) = q\%$$

- X^i is the loss of institution i for which the VaR_q^i is defined
- VaR_q^i is typically a positive number when $q > 50$
- More risk corresponds to a greater VaR_q^i
- Define X^i as the "return loss"

2. Definition of $CoVaR$

We denote by $CoVaR_q^{j|i}$ the VaR of institution j (or the financial system) conditional on some event $\mathbb{C}\mathbb{C}\ (X^j)$, $X^i = VaR_q^i$, of institution i. That is, $\mathrm{CoVaR}_q^{j|i}$ is implicitly defined by the $q\%$ - quantile of the conditional probability distribution:

$$Pr\left(X^j \leq CoVaR_q^{j|\mathbb{C}(X^i)} \mid \mathbb{C}\ (X^i)\right) = q\%$$

We denote institution i's contribution to by

$$\Delta CoVaR_q^{j|i} = CoVaR_q^{j|X^i=VaR_q^i} - CoVaR_q^{j|X^i=Median^i}$$

2. Definition of $CoVaR$

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- $\Delta CoVaR$
  - Captures the increase in $CoVaR$ as one shifts the conditioning event from the median return of institution i to the adverse VaR_q^i(with equality).
  - Measures the “tail dependency” between two random variables.
  - Is related to the correlation coefficient, while $CoVaR$ corresponds to a conditional variance.

$\Delta CoVaR_q^{j|i}$ is a statistical tail-dependency measure and does not necessarily correctly capture externalities or spillover effects.

Reasons:

- The externalities are typically not fully observable in equilibrium
- Also captures the common exposure to exogenous aggregate macroeconomic risk factors

3. Properties of $CoVaR$

- It can be used for any tail measures to analyse the interdependent risk
- Cloning property
- It does not differentiate between the causality and common factor effect
- Tail distribution
- Directionality

4. $\Delta CoVaR$ Estimation - Introduction

To perform the $\Delta CoVaR$ calculation procedure, the first challenge we meet is to get a ***time-varying VaR dataset*** of an institution.

There are several approaches to do that:

- Rather than using nonparametric or semiparametric methods, which are likely to give a somewhat rough estimation in this case, because it is hard for these kinds of models to give a convincing result in a limited samples, since it is always difficult to capture the current risk conditions using some techniques like rolling window, the results are always centralized. And maybe other issues such as choosing hyperparameters like window size and so on are involved as well.

- The author introduces a technique called ***linear factor structure*** in the appendix A. Collaborated with ***quantile regression***, this ***parametric method*** can give a rather more accurate estimation, which can be used to compute VaR precisely at different time as well as $CoVaR$.
  By the way, the idea of linear factor structure resembles ARMA-GARCH model to a certain extent, there are many similarities between them.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

- **Linear Factor structure of Losses**
  - We use the linear factor structure shown below, where losses $\mathbb{X}_t^i$ is also introduced as a direct determinant of the volatility:

$$\mathbb{X}_t^j = \phi_0 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\ \phi_2 + \left(\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5\right)\Delta\mathbb{Z}_t^j$$

- **Interpretation of Linear Factor structure model**
  - Our analysis relies on publicly available data and focuses on return losses to market equity, $\mathbb{X}_t^i = -\frac{\Delta\mathbb{V}_t^i}{\mathbb{V}_{t-1}^i} = -\frac{\mathbb{V}_t^i - \mathbb{V}_{t-1}^i}{\mathbb{V}_{t-1}^i}$.
  - $\mathbb{M}_{t-1}$ is a vector of one-period lagged state variables.
  - The error term series $\left\{\Delta\mathbb{Z}_t^j\right\}$ is assumed to be *i.i.d.* with zero mean and unit variance, i.e., $\mathbb{E}\left[\Delta\mathbb{Z}_t^j \middle| \mathbb{M}_{t-1}, \mathbb{X}_t^i\right] = 0$; $\mathbb{Var}\left[\Delta\mathbb{Z}_t^j \middle| \mathbb{M}_{t-1}, \mathbb{X}_t^i\right] = 1$. By the way, $\Delta\mathbb{Z}_t^j$ can be interpreted as the latent risk factors exposed to institution j at time t, and in this case, refers to the system.

Many variants can be derived from the original version of linear factor structure. We will use the variants to estimate the $\Delta CoVaR$ and VaR.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

Conventional method – two-step OLS

- We rewrite the formula as follows:

$$\mathbb{X}_t^j = \phi_0 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\ \phi_2 + \left(\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5\right)\Delta\mathbb{Z}_t^j$$

- The coefficients ϕ_0, $\overrightarrow{\boldsymbol{\phi_1}}$ and ϕ_2 could be estimated consistently via OLS of $\mathbb{X}_t^j$ on $\mathbb{M}_{t-1}$ and $\mathbb{X}_t^i$. The predicted value of such an OLS regression would be the mean of $\mathbb{X}_t^j$ conditional on $\mathbb{M}_{t-1}$ and $\mathbb{X}_t^i$, i.e., $\mathbb{E}\left[\mathbb{X}_t^j \middle| \mathbb{M}_{t-1}, \mathbb{X}_t^i\right] = \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\widehat{\phi_2}$.
- In order to compute the $CoVaR$ and VaR from OLS regressions, one would have to rely on the residuals from the last regression to estimate ϕ_3, $\overrightarrow{\boldsymbol{\phi_4}}$ and ϕ_5 and then make distributional assumptions about $\Delta\mathbb{Z}$ from $\mathbb{X}_t^j = \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\widehat{\phi_2} + \left(\widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\widehat{\phi_5}\right)\Delta\mathbb{Z}_t^j$.

There are two main issues, one is that two-step procedure cannot always guarantee the accuracy of the result; the other one is that estimations rely heavily on the distributional assumption of $\Delta\mathbb{Z}$ (But it is not always bad!).

4. $\Delta CoVaR$ Estimation - Linear Factor structure

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To overcome the weaknesses of the conventional method. The author introduces two solutions to improve that.
In order to make our estimations more accurate, we are going to carry out both of the two procedures, and compare them from several aspects.

- **MLE (Maximum Likelihood Estimation)**
  - One is via maximum likelihood estimation when distributional assumptions about error term $\Delta\mathbb{Z}$ (also call residual) are made.
  - Generally speaking, this method avoids sequential parameterizations, thus would be somewhat more accurate than the conventional one.
  - Besides, it is more flexible than ***quantile regressions***, since we can make many assumptions in terms of the distribution of the error term. Especially, ***heavy-tailed distributions*** like NIG (Normal Inverse Gaussian) are preferred in risk management area.
  - However, the computational cost would be very high when some complicated distributions are involved.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

MLE Implementation Details

- Let $\boldsymbol{\Theta} = \left(\phi_0, \overrightarrow{\boldsymbol{\phi_1}}, \phi_2, \phi_3, \overrightarrow{\boldsymbol{\phi_4}}, \phi_5\right)^T$.
- Then, the log-likelihood function of linear factor structure is given by:

$$l(\boldsymbol{\Theta}) = \sum_{t=1}^{n} \log\left[\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)\right].$$

Proof. To perform maximum-likelihood estimation, we analyze the conditional joint distribution $\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_1^j, \cdots, \Delta\mathbb{Z}_n^j | \mathbb{X}_1^j, \cdots, \mathbb{X}_n^j, \mathbb{M}_0, \cdots, \mathbb{M}_{t-1}, \mathbb{X}_1^i, \cdots, \mathbb{X}_n^j\right)$, where $\boldsymbol{\Theta}$ is the parameter vector. Using the property of $i.i.d$ assumption of $\Delta\mathbb{Z}$, and notice the filtration feature. We can separate this probability density function into several parts, then obtain:
$\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_1^j, \cdots, \Delta\mathbb{Z}_n^j | \mathbb{X}_1^j, \cdots, \mathbb{X}_n^j, \mathbb{M}_0, \cdots, \mathbb{M}_{t-1}, \mathbb{X}_1^i, \cdots, \mathbb{X}_n^j\right) = \prod_{t=1}^{n} \mathbb{f}\left(\boldsymbol{\Theta}; \mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)$
Then, we can obtain the log-likelihood function $l(\boldsymbol{\Theta}) = \sum_{t=1}^{n} \log\left[\mathbb{f}\left(\boldsymbol{\Theta}; \mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)\right]$.

We will then carry out MLE procedure by assuming different distributions on error term series $\left\{\Delta\mathbb{Z}_t^j\right\}$. Besides, however, in practice, the $\left\{\Delta\mathbb{Z}_t^j\right\}$s' zero mean and unit variance assumption does not always perform well.
To give more flexibility to this model, we are going to use MLE to estimate the mean and variance of $\left\{\Delta\mathbb{Z}_t^j\right\}$ at the same time, to get a better estimate the parameters.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

- **MLE Implementation Details - Gaussian case - Get LL function**
  - If $\left\{\Delta\mathbb{Z}_t^j\right\}$ follows a Gaussian distribution, and based on zero mean, unit variance assumption, we can get $\Delta\mathbb{Z}_t^j \sim \mathbb{N}(\mu,\ \sigma)$. Then conditioning on filtration $\mathcal{F}_t$, we can deduce that $\left[\Delta\mathbb{Z}_t^j \mid \mathcal{F}_t\right] = \left(\frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5}\right) \sim$ $\mathbb{N}(\mu,\sigma)$, i.e. $\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right) = \frac{1}{\sigma\sqrt{2\pi}}\mathbb{e}^{-\frac{1}{2\sigma^2}\cdot\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5}\right)^2}$
  - Thus, the log-likelihood function of linear factor structure is given by:

$$l_{\mathbb{N}}(\boldsymbol{\Theta}) = -n\log(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5}\right)^2$$

Proof. $l_{\mathbb{N}}(\boldsymbol{\Theta}) = \sum_{t=1}^{n}\log\left[\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)\right] = \sum_{t=1}^{n}\log\left[\frac{1}{\sigma\sqrt{2\pi}}\mathbb{e}^{-\frac{1}{2\sigma^2}\cdot\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5}\right)^2}\right]$

$$= \sum_{t=1}^{n}\left[-\log(\sigma) - \frac{1}{2}\log(2\pi) - \frac{1}{2\sigma^2}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5}\right)^2\right]$$

$$= -n\log(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\ \phi_5}\right)^2$$

4. $\Delta CoVaR$ Estimation - Linear Factor structure

MLE Implementation Details - Gaussian case – Dist Transformation

- Once we get the predicted values of the coefficients via NLopt, we can deduce as follows, and here comes the problem of distribution transformation.

$$
\begin{aligned}
[\mathbb{X}_t^j \mid \mathcal{F}_t] &= \left[\widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\widehat{\phi_2} + \left(\widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\widehat{\phi_5}\right)\Delta\mathbb{Z}_t^j \middle| \mathcal{F}_t\right] \\
&= \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\widehat{\phi_2} + \left(\widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\widehat{\phi_5}\right)[\Delta\mathbb{Z}_t^j \mid \mathcal{F}_t] \\
&= \widehat{loc} + \widehat{scale}[\Delta\mathbb{Z}_t^j \mid \mathcal{F}_t]
\end{aligned}
$$

  where $\widehat{loc} = \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_1}} + \mathbb{X}_t^i\widehat{\phi_2}$ and $\widehat{scale} = \widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\widehat{\phi_5}$

- Now, we should refer to the conclusions of linear transformation on characteristic functions to determine the distribution (on the next page).
- If $\left\{\Delta\mathbb{Z}_t^j\right\}$ follows a standard Gaussian distribution, i.e. $[\Delta\mathbb{Z}_t^j \mid \mathcal{F}_t] \sim \mathbb{N}(\hat{\mu}, \hat{\sigma}^2)$. Then, $[\mathbb{X}_t^j \mid \mathcal{F}_t] \sim \mathbb{N}\left[\widehat{loc} + \hat{\mu}\cdot\widehat{scale},\ \left(\hat{\sigma}\cdot\widehat{scale}\right)^2\right]$.

Note: NLopt is a free/open-source library for nonlinear optimization, providing a common interface for a number of different free optimization routines available online as well as original implementations of various other algorithms.

Reference: https://nlopt.readthedocs.io/en/latest/

4. $\Delta CoVaR$ Estimation - Linear Factor structure

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- **MLE Implementation Details - Gaussian case – Dist Transformation**
  - For a real number u, the Gaussian ***characteristic function*** is:

    $\varphi(u,\ \mu,\ \sigma) = \mathbb{e}^{iu\mu - \frac{1}{2}u^2\sigma^2}$

  The location-scale-based parameterization implemented in SciPy is the same as the definition above, where $loc = \mu,\ scale = \sigma$.

  - Consider a linear transformation: $\mathcal{L}(x) = \mathcal{A}x + \mathcal{B}$, where $\mathcal{A} \neq 0$.
  - For a random variable $\xi \sim \mathbb{N}(\mu, \sigma^2)$, the linear transformation of it $\mathcal{L}(\xi) \sim \mathbb{N}[\mathcal{A}\mu + \mathcal{B},\ (\mathcal{A}\sigma)^2]$.

    Proof. $\varphi_{\mathcal{L}(\xi)}(u, \tilde{\mu}, \tilde{\sigma}) = \varphi_{\mathcal{A}\xi+\mathcal{B}}(u) \coloneqq \mathbb{E}\left[\mathbb{e}^{iu(\mathcal{A}\xi+\mathcal{B})}\right] = \mathbb{e}^{iu\mathcal{B}}\mathbb{E}\left[\mathbb{e}^{i(\mathcal{A}u)\xi}\right]$

    $\Rightarrow\ \mathbb{E}\left[\mathbb{e}^{i(\mathcal{A}u)\xi}\right] = \varphi_{\xi}(\mathcal{A}u, \mu, \sigma) = \mathbb{e}^{i(\mathcal{A}u)\mu - \frac{1}{2}(\mathcal{A}u)^2\sigma^2} = \mathbb{e}^{iu(\mathcal{A}\mu) - \frac{1}{2}u^2(\mathcal{A}\sigma)^2}$

    $\Rightarrow\ \varphi_{\mathcal{L}(\xi)}(u, \tilde{\mu}, \tilde{\sigma}) = \mathbb{e}^{iu\mathcal{B}}\mathbb{E}\left[\mathbb{e}^{i(\mathcal{A}u)\xi}\right] = \mathbb{e}^{iu\mathcal{B}}\mathbb{e}^{iu(\mathcal{A}\mu) - \frac{1}{2}u^2(\mathcal{A}\sigma)^2} = \mathbb{e}^{iu(\mathcal{A}\mu+\mathcal{B}) - \frac{1}{2}u^2(\mathcal{A}\sigma)^2}$

    $\Rightarrow\ \tilde{\mu} = \mathcal{A}\mu + \mathcal{B},\quad \tilde{\sigma} = \mathcal{A}\sigma.$

    $\Rightarrow\ \boldsymbol{\widetilde{loc}} = \tilde{\mu} = \boldsymbol{\mathcal{A}loc + \mathcal{B}},\quad \boldsymbol{\widetilde{scale}} = \tilde{\sigma} = \boldsymbol{\mathcal{A} \cdot scale}.$

4. $\Delta CoVaR$ Estimation - Linear Factor structure

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- **MLE Implementation Details - Gaussian case – Python code**

scipy.stats.norm = <scipy.stats._continuous_distns.norm_gen object>

- A normal continuous random variable.
- The *location (loc)* keyword specifies the mean. The *scale (scale)* keyword specifies the standard deviation.
- As an instance of the rv_continuous class, norm object inherits from it a collection of generic methods, and completes them with details specific for this particular distribution.
- For a real number x, the probability density function for norm is:

$$\mathbb{f}(x,\ \mu,\ \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The location-scale-based parameterization implemented in SciPy is the same as the definition above, where $loc = \mu,\ scale = \sigma$.

Reference:
https://scipy.github.io/devdocs/reference/generated/scipy.stats.norm.html

4. $\Delta CoVaR$ Estimation - Linear Factor structure

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- **Quantile Regression**
  - The other one is via ***quantile regressions***, which incorporate estimates of the conditional mean and the conditional variance to produce conditional quantiles.

  - It can be performed without the distributional assumptions of error term $\Delta\mathbb{Z}$, which are needed for estimations via OLS and MLE.
  - Meanwhile, the issue of two-step parameterizations is avoided.
  - Thus, this method is just a kind of regression, so the computational cost is very low. We can get the results really quickly.

  - But it is just because of the simplicity, which makes it hard to carry out further development.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

Quantile Regression Implementation Details

- When we do some modification on the original version of the Linear Factor structure, we can obtain the formula as follows:

$$\mathbb{X}_t^j = \left(\phi_0 + \phi_3 \Delta\mathbb{Z}_t^j\right) + \mathbb{M}_{t-1}\left(\overrightarrow{\boldsymbol{\phi_1}} + \overrightarrow{\boldsymbol{\phi_4}}\Delta\mathbb{Z}_t^j\right) + \mathbb{X}_t^i\left(\phi_2 + \phi_5 \Delta\mathbb{Z}_t^j\right)$$

- Then use the idea of quantile regressions to modify this model a little bit. It follows immediately that the inverse c.d.f. of $\mathbb{X}_t^j$ is:

$$F_{\mathbb{X}_t^j}^{-1}(q|\mathbb{M}_{t-1}, \mathbb{X}_t^i) = \left(\phi_0 + \phi_3 F_{\Delta\mathbb{Z}_t^j}^{-1}\right) + \mathbb{M}_{t-1}\left(\overrightarrow{\boldsymbol{\phi_1}} + \overrightarrow{\boldsymbol{\phi_4}} F_{\Delta\mathbb{Z}_t^j}^{-1}\right) + \mathbb{X}_t^i\left(\phi_2 + \phi_5 F_{\Delta\mathbb{Z}_t^j}^{-1}\right)$$

  where we denote the c.d.f. of $\Delta\mathbb{Z}_t^j$ by $F_{\Delta\mathbb{Z}_t^j}(\cdot)$, and its inverse c.d.f. by $F_{\Delta\mathbb{Z}_t^j}^{-1}(\cdot)$, for different quantiles $q \in (0, 100)$.

- Finally, we rewrite this formula as the following form:

$$F_{\mathbb{X}_t^j}^{-1}(q|\mathbb{M}_{t-1}, \mathbb{X}_t^i) = \alpha_q + \mathbb{M}_{t-1}\gamma_q + \mathbb{X}_t^i\beta_q$$

- Thus, using this form, we can directly carry out quantile regressions regardless of the distribution of $\Delta\mathbb{Z}_t^j$.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

Quantile Regression Implementation Details

- In the last page, we eventually obtain the formula as follows:

$$F^{-1}_{\mathbb{X}_t^j}(q|\mathbb{M}_{t-1}, \mathbb{X}_t^i) = \alpha_q + \mathbb{M}_{t-1}\gamma_q + \mathbb{X}_t^i\beta_q$$

- According to the author, we can directly use this formula to derive the another two to estimate the VaR and $CoVaR$.
- For VaR, we have: $\mathbb{X}_t^i = \alpha_q^i + \gamma_q^i\mathbb{M}_{t-1} + \varepsilon_{q,t}^i$

$$\Rightarrow \quad VaR_{q,t}^i = \hat{\alpha}_q^i + \hat{\gamma}_q^i\mathbb{M}_{t-1}$$

- For $CoVaR$, we have:

$$\mathbb{X}_t^{j|i} = \alpha_q^{j|i} + \gamma_q^{j|i}\mathbb{M}_{t-1} + \beta_q^{j|i}\mathbb{X}_t^i + \varepsilon_{q,t}^{j|i}$$

$$\Rightarrow \quad CoVaR_{q,t}^i = \hat{\alpha}_q^{j|i} + \hat{\gamma}_q^{j|i}\mathbb{M}_{t-1} + \hat{\beta}_q^{j|i}VaR_{q,t}^i$$

- Finally, for $\Delta CoVaR$, we have two ways to compute:

$$\Delta CoVaR_{q,t}^i = CoVaR_{q,t}^i - CoVaR_{50,t}^i \; *$$

$$\Delta CoVaR_{q,t}^i = \hat{\beta}_q^{j|i}\left(VaR_{q,t}^i - VaR_{50,t}^i\right)$$

4. $\Delta CoVaR$ Estimation - Linear Factor structure

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- **Note on Linear Factor structure – 1. Prediction Issues.**
  - If we want to make some predictions on $CoVaR$, there is not doubt that the original version and the variants introduced before of Linear Factor structure do not work, since estimations involve current information.
  - In other words, if we want to use Linear Factor structure to make predictions, we should modify this model a little bit.

  - We still use the linear factor structure, where losses $\mathbb{X}_t^i$ is also introduced as a direct determinant of the volatility. But here, we substitute the original losses $\mathbb{X}_t^i$ to one-period lagged series, i.e.,
  $\mathbb{X}_t^j = \phi_0 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} + \mathbb{X}_{t-1}^i\ \phi_2 + \left(\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_{t-1}^i\ \phi_5\right)\Delta\mathbb{Z}_t^j$

  - In my opinion, there may not be some severe problems after doing that. After all, $\mathbb{X}_{t-1}^i$ is indeed closely related to $\mathbb{X}_t^i$.

Furtherly, the author expands this idea to propose a more convincing theory, which is called Forward $-\Delta CoVaR$.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

- **Note on Linear Factor structure – 1. Prediction Issues.**
  - According to the author, for Forward $-\Delta CoVaR$, we have the following formula to calculate:
$$\Delta CoVaR_{q,t}^{i} = a + c\mathbb{M}_{t-h} + b\mathbb{X}_{t-h}^{i} + \eta_t^i$$
$$\Rightarrow \quad \Delta_h^{Fwd} CoVaR_{q,t}^{i} = \hat{a} + \hat{c}\mathbb{M}_{t-h} + \hat{b}\mathbb{X}_{t-h}^{i}$$
where $\mathbb{X}_{t-h}^{i}$ are the vector of characteristics for institution i, $\mathbb{M}_{t-h}$ is the vector of macro state variables lagged h quarters. More specifically, for a forecast horizon $h = 1, 4, 8$ quarters, we run regressions.

  - Importantly, there is one something we should keep in mind.

Any tail risk measure, estimated at a high frequency, is by its very nature imprecise.

We are not going to do some further work in the report, but it is a good complement of $\Delta CoVaR$ to indicate the risk.

4. $\Delta CoVaR$ Estimation - Linear Factor structure

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- **Note on Linear Factor structure – 2. Criticism.**
  - From my perspective, the calculation of VaR using quantile regression introduced by the author is not effective enough.

$$\mathbb{X}_t^i = \alpha_q^i + \gamma_q^i \mathbb{M}_{t-1} + \varepsilon_{q,t}^i$$
$$\Rightarrow \quad VaR_{q,t}^i = \hat{\alpha}_q^i + \hat{\gamma}_q^i \mathbb{M}_{t-1}$$

  - In this formula, losses of a institution are only the matter of state variables, i.e., market conditions.
  - However, the losses are also closely related to its own state one-period ago. So, referring to the idea of AR(Auto Regressive) model, we can add a one-period lagged variable to this model. Then we can obtain:

$$\mathbb{X}_t^j = \alpha_q^i + \gamma_q^i \mathbb{M}_{t-1} + \beta_q^i \mathbb{X}_{t-1}^i + \varepsilon_{q,t}^i$$
$$\Rightarrow \quad VaR_{q,t}^i = \hat{\alpha}_q^i + \hat{\gamma}_q^i \mathbb{M}_{t-1} + \hat{\beta}_q^i \mathbb{X}_{t-1}^i$$

5. Empirical Study - Introduction

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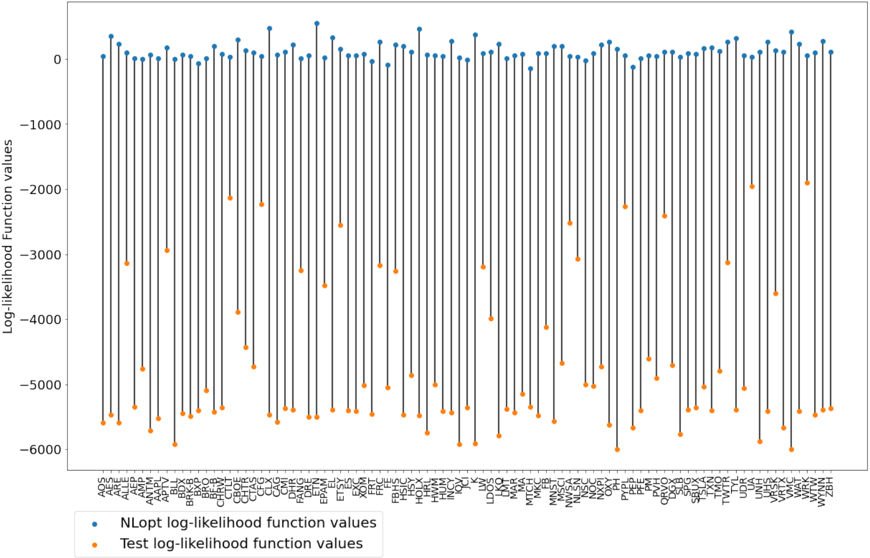
- **Coding Environment**
  - Python3 + JupyterLab + Windows10
  - Main Packages: Numpy, pandas, scipy, NLopt, matplotlib
- **Data Sources**

#1. _updated_A_snp_tickers_sectors.csv (Last change was on April 21, 2022)
It is found at https://en.wikipedia.org/wiki/List_of_S%26P_500_companies
#2. _updated_B_DTB3.csv (First state variable, Second state variable) (Last change was on April 21, 2022)
It is found at https://fred.stlouisfed.org/series/DTB3
#3. _updated_B_DLTBOARD.csv (Second state variable - first period) (Last change was on June 30, 2000)
It is found at https://fred.stlouisfed.org/series/DLTBOARD
#4. _updated_C_USTREASURY-LONGTERMRATES.csv (Second state variable - second period) (Last change was on January 31, 2022)
It is found at https://data.nasdaq.com/data/USTREASURY/LONGTERMRATES-treasury-long-term-rates
#5. _updated_B_TEDRATE.csv (Third state variable) (Last change was on January 21, 2022)
It is found at https://fred.stlouisfed.org/series/TEDRATE
#6. _updated_B_DBAA.csv (Fourth state variable) (Last change was on April 21, 2022)
It is found at https://fred.stlouisfed.org/series/DBAA
#7. _updated_B_DGS10.csv (Fourth state variable) (Last change was on April 21, 2022)
It is found at https://fred.stlouisfed.org/series/DGS10

5. Empirical Study – Summary of state variables

	3MO_Yield_Change	Term_Spread_Change	TED_Spread	Credit_Spread_Change	Market_Return	Excess_Return	Equity_Volatility
mean	-0.114919	0.033065	40.405595	-0.033317	0.218303	0.190356	1.031359
std	8.236839	2.602870	41.010937	1.709793	2.518130	3.662225	0.733573
min	-80.000000	-14.600000	6.600000	-7.400000	-18.195465	-18.888800	0.233922
25%	-1.000000	-1.400000	20.200000	-0.800000	-0.838042	-1.454589	0.623337
75%	2.000000	1.600000	42.600000	0.600000	1.470086	1.742171	1.189720
max	69.000000	11.600000	412.000000	18.600000	13.106071	19.366352	5.960092

5. Empirical Study - MLE Algorithm Robustness Testing



5. Empirical Study – 99%$CoVaR$ computational methods comparison

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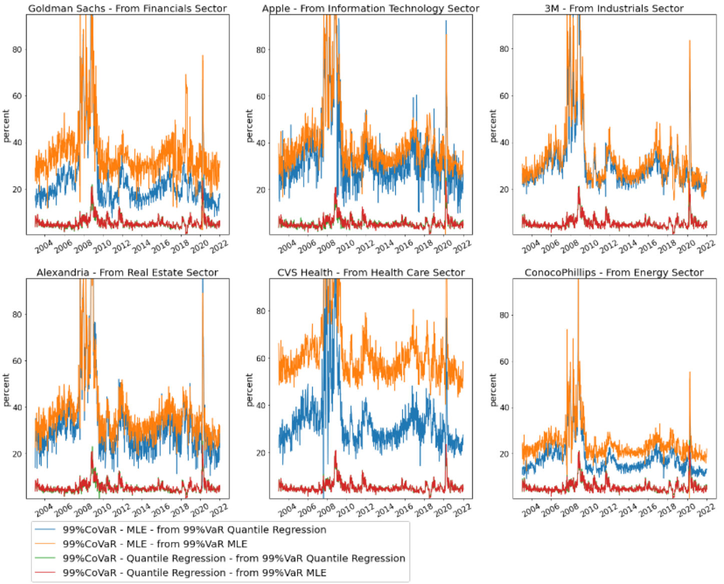
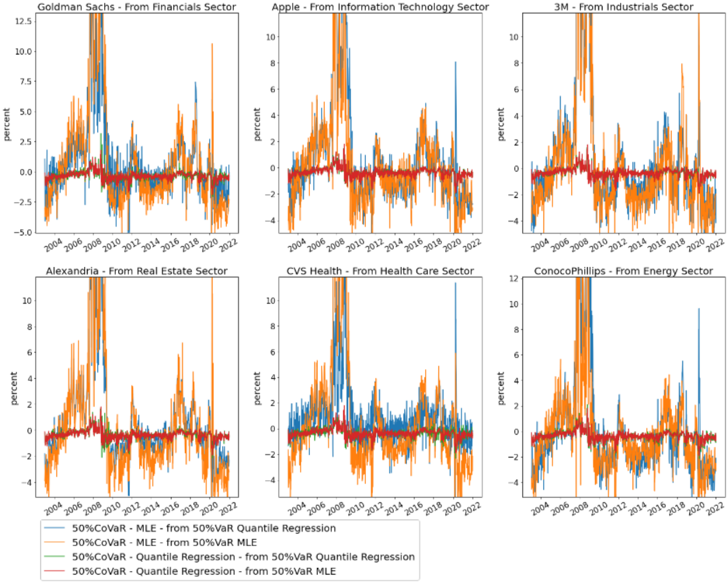
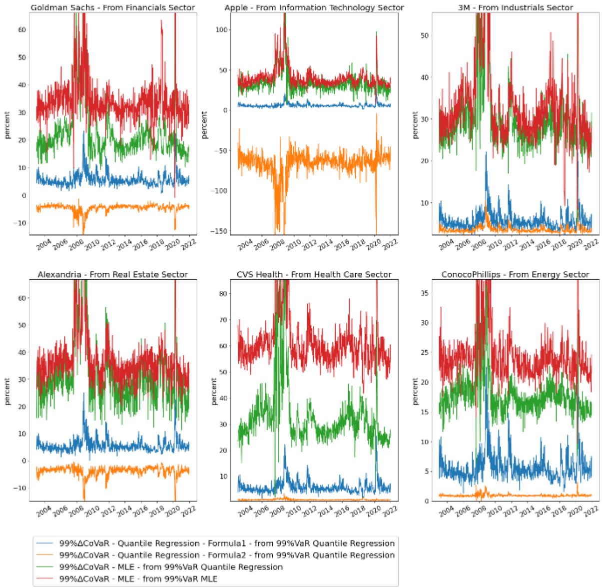


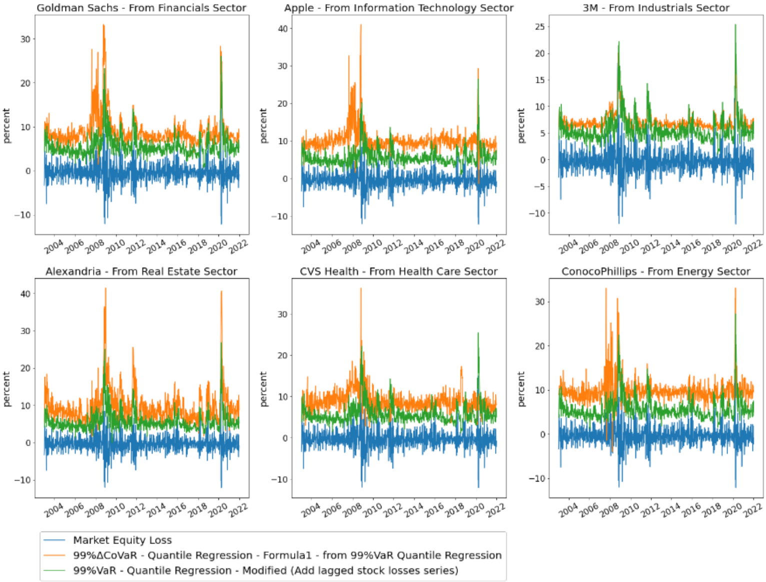
5. Empirical Study – 50%$CoVaR$ computational methods comparison



5. Empirical Study – 99%$\Delta CoVaR$ computational methods comparison



5. Empirical Study – 99%$\Delta CoVaR$ and 99%VaR results



5. Empirical Study – 99%VaR Quantile regression —— take Goldman Sachs as an example

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QuantReg Regression Results

Dep. Variable:	GS	Pseudo R-squared:	0.3845
Model:	QuantReg	Bandwidth:	2.374
Method:	Least Squares	Sparsity:	107.9
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	5.0480	0.634	7.966	0.000	3.804	6.291
3MO_Yield_Change	-0.2192	0.068	-3.236	0.001	-0.352	-0.086
Term_Spread_Change	0.2333	0.328	0.711	0.477	-0.411	0.877
TED_Spread	0.0318	0.021	1.544	0.123	-0.009	0.072
Credit_Spread_Change	0.1241	0.649	0.191	0.848	-1.149	1.397
Market_Return	-0.1551	0.422	-0.368	0.713	-0.983	0.673
Excess_Return	0.0871	0.119	0.735	0.462	-0.145	0.320
Equity_Volatility	2.7503	1.175	2.341	0.019	0.445	5.056
stock_lagged	-0.1353	0.160	-0.845	0.399	-0.450	0.179

5. Empirical Study – 99%*CoVaR* Quantile regression —— take Goldman Sachs as an example

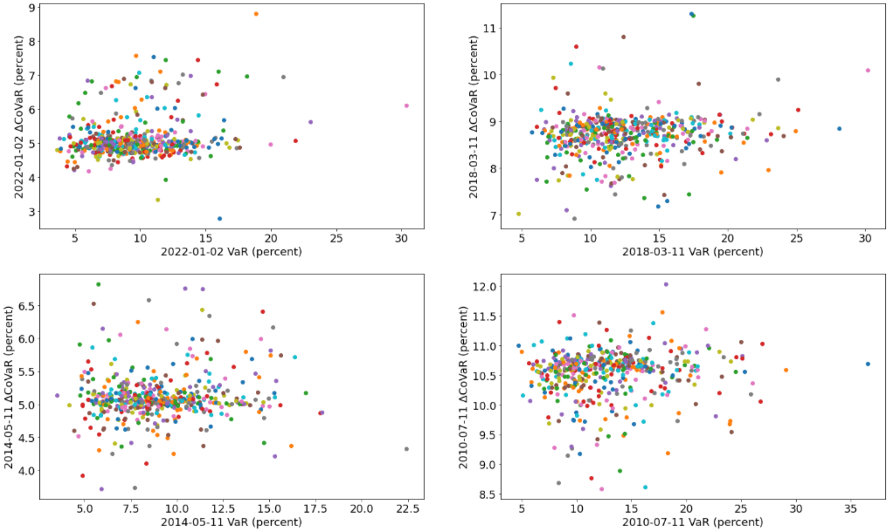
Stony Brook University

QuantReg Regression Results

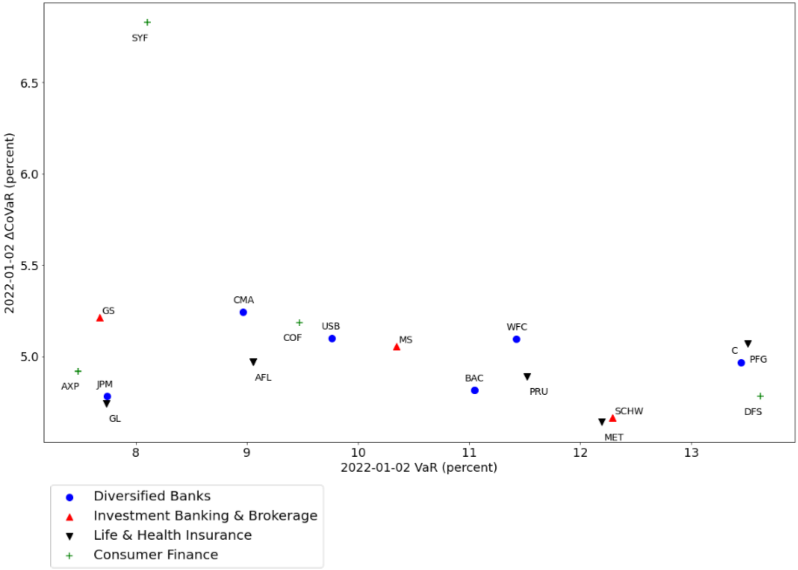
Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3558
Model:	QuantReg	Bandwidth:	1.403
Method:	Least Squares	Sparsity:	50.78
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	5.0712	2.136	2.374	0.018	0.879	9.263
3MO_Yield_Change	-0.1609	0.089	-1.812	0.070	-0.335	0.013
Term_Spread_Change	0.2053	0.144	1.429	0.153	-0.077	0.487
TED_Spread	0.0153	0.015	1.004	0.315	-0.015	0.045
Credit_Spread_Change	0.0248	0.156	0.159	0.874	-0.281	0.331
Market_Return	-0.4239	0.108	-3.911	0.000	-0.637	-0.211
Excess_Return	-0.1022	0.090	-1.138	0.256	-0.279	0.074
Equity_Volatility	4.2378	1.233	3.437	0.001	1.819	6.657
GS_VaR_0.99	-0.4983	0.422	-1.182	0.237	-1.326	0.329

5. Empirical Study – 99%$\Delta CoVaR$ and 99%VaR scatter plot at four times



5. Empirical Study – cross-sectional 99%$\Delta CoVaR$ and 99%VaR scatter plot



Further Work

- **Calculate Forward** $-\,\Delta \boldsymbol{CoVaR}$ **as a complement of** $\Delta \boldsymbol{CoVaR}$

- **Fit NIG (Normal Inverse Gaussian) distribution in MLE**

Reference

- Adrian, T., and Brunnermeier, M. K. (2016), "CoVaR," American Economic Review, 106 (7), 1705-1741. DOI: 10.1257/aer.20120555.
- Smaga, P. (2014), "The concept of systemic risk," Systemic Risk Centre Special Paper, (5).