$CoVaR$

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1. Introduction

Due to the financial crisis, the financial system caused a considerable part of the loss of economic value, which in turn spilled over to the real economy. The expansion of distress leads to the emergence of systemic risk. Therefore, there has been extensive research conducted by financial experts regarding the issue of systemic risk. A systemic risk is defined by the European Central Bank as a risk that adversely affects the functioning of a financial system to the extent that growth and welfare are materially affected (Smaga 2014).

The most common tool taken into consideration for measuring risk management in financial institutions, and which was also mentioned in our class, is called Value at Risk (VaR), which has the purpose of assessing the level of financial risk associated with a particular investment decision. However, VaR is a measure specific to individual institutions, so it cannot capture the risks incurred by dynamic joint ventures between institutions, such as a dynamic relationship between two financial institutions. In order to address this challenge, a measure of systemic risk that is able to quantify the contribution of systemic risk of one financial institution to other institutions has been proposed. Besides the cross-sectional dimension, the time-series dimension of systemic risk also exists. It is common for systemic risks to accumulate in times of low volatility, then become apparent during times of crises. An appropriate measure would capture the accumulation of systemic risks during these periods. In other words, high-frequency risk measures that are primarily based on price movements are prone to bias (Adrian and Brunnermeier 2016).

A new measure of contributions to systemic risk is proposed in Adrian and Brunnermeier (2016), called the $\Delta CoVaR$. Compared to the previous measure, this measure captures tail dependence and includes negative spillover dynamics during periods of crisis. As regards an institution's $CoVaR$, it is defined as the VaR of the whole financial system as if it were in a particular state of distress or the median state. By taking into account the difference between $CoVaR$ when a financial institution is in distress as opposed to $CoVaR$ when it is in a "normal" state, called $\Delta CoVaR$, it can be used to identify an institution's contribution to overall systemic risk in a non-causal manner. $\Delta CoVaR$ is a statistical measure of tail dependence and is therefore best considered as a valuable reduced-form analytical tool for capturing (tail) comovements (Adrian and Brunnermeier 2016).

2. $CoVaR$ Methodology

2.1 Definition of $\Delta CoVaR$

Adrian and Brunnermeier (2016) recall that VaR_q^i is implicitly defined as the q% quantile:

$$Pr(X^i \leq VaR_q^i) = q\%,$$

where X^i is the loss of institution i for which the VaR is defined. By definition, VaR is generally a positive number when $q > 50$, corresponding to the generally accepted sign convention. This implies that increased risk is associated with greater VaR. Adrian and Brunnermeier (2016) define X^i as the "return loss".

Definition 1 *We denote by* $CoVaR_q^{j|\mathbb{C}(X^i)}$ *the* VaR *of institution* j *(or the financial system) conditional on some event* $\mathbb{C}(X^i)$ *of institution* i. *That is,* $CoVaR_q^{j|\mathbb{C}(X^i)}$ *is implicitly defined by the* q%*-quantile of the conditional probability distribution:*

$$\Pr\left(X^j \middle| \mathbb{C}(X^i) \leq CoVaR_q^{j|\mathbb{C}(X^i)}\right) = q\%$$

We denote institution i's contribution to j by

$$\Delta CoVaR_q^{j|i} = CoVaR_q^{j|X^i=VaR_q^i} - CoVaR_q^{j|X^i=VaR_{50}^i}$$

$\Delta CoVaR$ is the difference between the change in $CoVaR$ as the conditioning event shifts from the median return of institution i to its adverse VaR (with equality). It is a measure of "tail dependency" between two random return variables. The position of $\Delta CoVaR$ here is a measure of the correlation coefficient for jointly normally distributed random variables, while $CoVaR$ symbolizes the conditional variances. In other words, conditional analysis reduces variance by itself, while conditional analysis involving adverse events increases expected loss (Adrian and Brunnermeier 2016).

$\Delta CoVaR_q^{j|i}$ is a measure of statistical tail-dependency, and it may not accurately represent externalities or spillover effects. In the first place, because other institutions may reposition themselves to lessen the impact of externalities, externalities are not always fully evident when the system is in equilibrium. For the second reason, exogenous aggregate macroeconomic risk factors are reflected in $\Delta CoVaR_q^{j|i}$ (Adrian and Brunnermeier 2016).

2.2 Properties of $\Delta CoVaR$

Any Tail Measure. $CoVaR$ has the surprising property that it can analyze interdependent risks using any tail measure. Despite the fact that Adrian and Brunnermeier (2016) only apply the $CoVaR$ measure to financial systems and financial institutions, this measure is well suited for analyzing the systemic risks specific to financial equity markets.

Cloning Property. $CoVaR$ measurements are identical when sectors are divided into smaller clones, as clonal properties indicate.

Causality. The $\Delta CoVaR$ method does not distinguish between causality and the impact of common factors. There are several benefits to this property since it enables users to identify risks which are "systemic as part of the herd" even if no direct causal relationship exists. An example would be if all sectors were exposed to the same factors and one sector experienced distress, it is unlikely that this would lead to a systemic crisis. On the other hand, if these sectors are under stress because of similar factors, $\Delta CoVaR$ will capture the risk of "systemic as part of the herd".

Tail Distribution. A compelling feature of $CoVaR$ is its tail distribution since $CoVaR$ emphasizes the extreme tail distribution more than traditional VaR, and it reflects the shift to higher moments.

Directionality. $CoVaR$ is directional. This means that the $CoVaR$ of the system conditioned on institution i does not have the same value as the $CoVaR$ of institution i conditioned on the system.

3. ΔCoVaR Estimation

To perform the $\Delta CoVaR$ calculation procedure, the first challenge we meet is to get a time-varying VaR dataset of an institution. There are several approaches to do that:

Rather than using nonparametric or semiparametric methods, which are likely to give a somewhat rough estimation in this case, because it is hard for these kinds of models to give a convincing result in very limited samples, since it is always difficult to capture the current risk conditions using some techniques like rolling window, the results are always centralized. And maybe other issues such as choosing hyperparameters like window size and so on are involved as well.

Adrian and Brunnermeier (2016) introduce a technique called linear factor structure in the appendix A. Collaborated with quantile regression, this parametric method can give a rather more accurate estimation, which can be used to compute VaR precisely at different time as well as $CoVaR$.

By the way, the idea of linear factor structure resembles ARMA-GARCH model to a certain extent, there are many similarities between them.

We use the **linear factor structure** shown below, where losses $\mathbb{X}_t^i$ is also introduced as a direct determinant of the volatility:

$$\mathbb{X}_t^j = \phi_0 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} + \mathbb{X}_t^i \ \phi_2 + \left(\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i \ \phi_5\right)\Delta\mathbb{Z}_t^j$$

- Our analysis relies on publicly available data and focuses on return losses to market equity, $\mathbb{X}_t^i = -\frac{\Delta\mathbb{V}_t^i}{\mathbb{V}_{t-1}^i} = -\frac{\mathbb{V}_t^i - \mathbb{V}_{t-1}^i}{\mathbb{V}_{t-1}^i}$.
- $\mathbb{M}_{t-1}$ is a vector of one-period lagged state variables.
- The error term series $\{\Delta\mathbb{Z}_t^j\}$ is assumed to be i.i.d. with zero mean and unit variance, i.e., $\mathbb{E}\left[\Delta\mathbb{Z}_t^j \middle| \mathbb{M}_{t-1}, \mathbb{X}_t^i\right] = 0$; $\mathbb{Var}\left[\Delta\mathbb{Z}_t^j \middle| \mathbb{M}_{t-1}, \mathbb{X}_t^i\right] = 1$. By the way, $\Delta\mathbb{Z}_t^j$ can be interpreted as the latent risk factors exposed to institution j at time t, and in this case, refers to the system.

Many variants can be derived from the original version of the linear factor structure. We will use the variants to estimate the $\Delta CoVaR$ and VaR.

3.1 Conventional Method – Two-Step OLS

We rewrite the formula as follows:

$$\mathbb{X}_t^j = \phi_0 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} + \mathbb{X}_t^i \ \phi_2 + \left(\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i \ \phi_5\right)\Delta\mathbb{Z}_t^j$$

The coefficients ϕ_0, $\overrightarrow{\boldsymbol{\phi_1}}$ and ϕ_2 could be estimated consistently via OLS of $\mathbb{X}_t^j$ on $\mathbb{M}_{t-1}$ and $\mathbb{X}_t^i$. The predicted value of such an OLS regression would be the mean of $\mathbb{X}_t^j$ conditional on $\mathbb{M}_{t-1}$ and $\mathbb{X}_t^i$, i.e., $\mathbb{E}\left[\mathbb{X}_t^j \middle| \mathbb{M}_{t-1}, \mathbb{X}_t^i\right] = \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_1}}} + \mathbb{X}_t^i\widehat{\phi_2}$.

In order to compute the $CoVaR$ and VaR from OLS regressions, one would have to rely on the residuals from the last regression to estimate ϕ_3, $\overrightarrow{\boldsymbol{\phi_4}}$ and ϕ_5 and then make distributional assumptions about $\Delta\mathbb{Z}$ from $\mathbb{X}_t^j = \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_1}}} + \mathbb{X}_t^i\widehat{\phi_2} + \left(\widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_4}}} + \mathbb{X}_t^i\widehat{\phi_5}\right)\Delta\mathbb{Z}_t^j$.

There are two main issues, one is that two-step procedure cannot always guarantee the accuracy of the result; the other one is that estimations rely heavily on the distributional assumption of $\Delta\mathbb{Z}$ (But it is not always bad!). To overcome the weaknesses of the conventional method. The author introduces two solutions to improve that. In order to make our estimations more accurate, we are going to carry out both of the two procedures, and compare them from several aspects.

3.2 MLE (Maximum Likelihood Estimation) Method

One solution is via maximum likelihood estimation when distributional assumptions about error term $\Delta\mathbb{Z}$ (also call residual) are made. Generally speaking, this method avoids sequential parameterizations, thus would be somewhat more accurate than the conventional one. Besides, it is more flexible than quantile regressions, since we can make many assumptions in terms of the distribution of the error term. Especially, heavy-tailed distributions like NIG (Normal Inverse Gaussian) are preferred in the risk management area. However, the computational cost would be very high when some complicated distributions are involved. Next, some MLE implementation details will be involved.
For a real number u, the Gaussian characteristic function is:

$$\varphi(u,\ \mu,\ \sigma) = \mathbb{e}^{iu\mu-\frac{1}{2}u^2\sigma^2}$$

The location-scale-based parameterization implemented in SciPy is the same as the definition above, where $loc=\mu,\ scale=\sigma.$
Consider a linear transformation: $\mathcal{L}(x) = \mathcal{A}x + \mathcal{B}$, where $\mathcal{A} \neq 0$.

Theorem 1 *For a random variable* $\xi \sim \mathbb{N}(\mu,\ \sigma^2)$*, the linear transformation of it*

$\mathcal{L}(\xi) \sim \mathbb{N}[\mathcal{A}\mu + \mathcal{B},\ \ (\mathcal{A}\sigma)^2]$.

Proof. $\varphi_{\mathcal{L}(\xi)}(u, \tilde{\mu},\ \tilde{\sigma}) = \varphi_{\mathcal{A}\xi+\mathcal{B}}(u) \coloneqq \mathbb{E}\left[\mathbb{e}^{iu(\mathcal{A}\xi+\mathcal{B})}\right] = \mathbb{e}^{iu\mathcal{B}}\mathbb{E}\left[\mathbb{e}^{i(\mathcal{A}u)\xi}\right]$

$$\Rightarrow\quad \mathbb{E}\left[\mathbb{e}^{i(\mathcal{A}u)\xi}\right] = \varphi_\xi(\mathcal{A}u, \mu, \sigma) = \mathbb{e}^{i(\mathcal{A}u)\mu-\frac{1}{2}(\mathcal{A}u)^2\sigma^2} = \mathbb{e}^{iu(\mathcal{A}\mu)-\frac{1}{2}u^2(\mathcal{A}\sigma)^2}$$

$$\Rightarrow\quad \varphi_{\mathcal{L}(\xi)}(u, \tilde{\mu},\ \tilde{\sigma}) = \mathbb{e}^{iu\mathcal{B}}\mathbb{E}\left[\mathbb{e}^{i(\mathcal{A}u)\xi}\right] = \mathbb{e}^{iu\mathcal{B}}\mathbb{e}^{iu(\mathcal{A}\mu)-\frac{1}{2}u^2(\mathcal{A}\sigma)^2}$$

$$= \mathbb{e}^{iu(\mathcal{A}\mu+\mathcal{B})-\frac{1}{2}u^2(\mathcal{A}\sigma)^2}$$

$$\Rightarrow\quad \tilde{\mu} = \mathcal{A}\mu + \mathcal{B},\quad \tilde{\sigma} = \mathcal{A}\sigma.$$

$$\Rightarrow\quad \widetilde{\boldsymbol{loc}} = \tilde{\mu} = \boldsymbol{\mathcal{A}loc} + \boldsymbol{\mathcal{B}},\quad \widetilde{\boldsymbol{scale}} = \tilde{\sigma} = \boldsymbol{\mathcal{A}} \cdot \boldsymbol{scale}.$$

■

We will use **Theorem 1** to derive the distribution of $\mathbb{X}_t^j$ later.

Let parameter vector $\boldsymbol{\Theta} = \left(\phi_0, \overrightarrow{\boldsymbol{\phi_1}}, \phi_2, \phi_3, \overrightarrow{\boldsymbol{\phi_4}}, \phi_5\right)^T$.

Theorem2 *The log-likelihood function of linear factor structure is given by:*

$$l(\boldsymbol{\Theta}) = \sum_{t=1}^{n} \log\left[\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)\right].$$

Proof. To perform maximum-likelihood estimation, we analyze the conditional joint distribution $\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_1^j, \cdots, \Delta\mathbb{Z}_n^j | \mathbb{X}_1^j, \cdots, \mathbb{X}_n^j, \mathbb{M}_0, \cdots, \mathbb{M}_{t-1}, \mathbb{X}_1^i, \cdots, \mathbb{X}_n^j\right)$, where $\boldsymbol{\Theta}$ is the parameter vector.

Using the property of $i.i.d$ assumption of $\Delta\mathbb{Z}$, and notice the filtration feature. We can separate this probability density function into several parts, then obtain:

$$\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_1^j, \cdots, \Delta\mathbb{Z}_n^j | \mathbb{X}_1^j, \cdots, \mathbb{X}_n^j, \mathbb{M}_0, \cdots, \mathbb{M}_{t-1}, \mathbb{X}_1^i, \cdots, \mathbb{X}_n^j\right) = \prod_{t=1}^{n} \mathbb{f}\left(\boldsymbol{\Theta}; \mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)$$

Then, we can obtain the log-likelihood function

$$l(\boldsymbol{\Theta}) = \sum_{t=1}^{n} \log\left[\mathbb{f}\left(\boldsymbol{\Theta}; \mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)\right].$$

■

We will then carry out MLE procedure by assuming different distributions on error term series $\left\{\Delta\mathbb{Z}_t^j\right\}$. Besides, however, in practice, the $\left\{\Delta\mathbb{Z}_t^j\right\}$s' zero mean and unit variance assumption does not always perform well.

To give more flexibility to this model, we are going to use MLE to estimate the mean and variance of $\left\{\Delta\mathbb{Z}_t^j\right\}$ at the same time, to get a better estimate the parameters. If $\left\{\Delta\mathbb{Z}_t^j\right\}$ follows a Gaussian distribution, and based on zero mean, unit variance assumption, we can get $\Delta\mathbb{Z}_t^j \sim \mathbb{N}(\mu, \ \sigma)$. Then conditioning on filtration $\mathcal{F}_t$, we can deduce that $\left[\Delta\mathbb{Z}_t^j | \ \mathcal{F}_t\right] = \left(\frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i \ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i \ \phi_5}\right) \sim \mathbb{N}(\mu, \sigma)$, i.e.,

$$\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{e}^{-\frac{1}{2\sigma^2}\cdot\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i \ \phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i \ \phi_5}\right)^2}.$$

Theorem 3 *The log-likelihood function of linear factor structure is given by:*

$$l_{\mathbb{N}}(\boldsymbol{\Theta}) =- nlog(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\,\phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\,\phi_5}\right)^2$$

Proof. $l_{\mathbb{N}}(\boldsymbol{\Theta}) = \sum_{t=1}^{n}\log\left[\mathbb{f}\left(\boldsymbol{\Theta}; \Delta\mathbb{Z}_t^j | \mathbb{X}_t^j, \mathbb{M}_{t-1}, \mathbb{X}_t^i\right)\right]$

$$= \sum_{t=1}^{n}\log\left[\frac{1}{\sigma\sqrt{2\pi}}\mathbb{e}^{-\frac{1}{2\sigma^2}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\,\phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\,\phi_5}\right)^2}\right]$$

$$= \sum_{t=1}^{n}\left[-log(\sigma) - \frac{1}{2}\log(2\pi) - \frac{1}{2\sigma^2}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\,\phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\,\phi_5}\right)^2\right]$$

$$=- nlog(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\left(\mu - \frac{\mathbb{X}_t^j - \phi_0 - \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} - \mathbb{X}_t^i\,\phi_2}{\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_t^i\,\phi_5}\right)^2$$

■

Once we get the predicted values of the coefficients via some suitable nonlinear optimization packages, we can deduce as follows, and here comes the problem of distribution transformation.

$$[\mathbb{X}_t^j |\, \mathcal{F}_t] = \left[\widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_1}}} + \mathbb{X}_t^i\widehat{\phi_2} + \left(\widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_4}}} + \mathbb{X}_t^i\widehat{\phi_5}\right)\Delta\mathbb{Z}_t^j | \mathcal{F}_t\right]$$

$$= \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_1}}} + \mathbb{X}_t^i\widehat{\phi_2} + \left(\widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_4}}} + \mathbb{X}_t^i\widehat{\phi_5}\right)\left[\Delta\mathbb{Z}_t^j |\, \mathcal{F}_t\right]$$

$$= \widehat{loc} + \widehat{scale}[\Delta\mathbb{Z}_t^j |\, \mathcal{F}_t]$$

where $\widehat{loc} = \widehat{\phi_0} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_1}}} + \mathbb{X}_t^i\widehat{\phi_2}$ and $\widehat{scale} = \widehat{\phi_3} + \mathbb{M}_{t-1}\widehat{\overrightarrow{\boldsymbol{\phi_4}}} + \mathbb{X}_t^i\widehat{\phi_5}$.

Now, we can refer to **Theorem1** (linear transformation on characteristic functions) to determine the distribution. If $\{\Delta\mathbb{Z}_t^j\}$ follows a standard Gaussian distribution, i.e., $[\Delta\mathbb{Z}_t^j |\, \mathcal{F}_t] \sim \mathbb{N}(\hat{\mu},\ \hat{\sigma}^2)$. Then, $[\mathbb{X}_t^j |\, \mathcal{F}_t] \sim \mathbb{N}\left[\widehat{loc} + \hat{\mu}\cdot\widehat{scale},\ \left(\hat{\sigma}\cdot\widehat{scale}\right)^2\right]$.

Once getting the formula, combining with some suitable packages in Python, we can quickly write our own code to implement the idea of MLE to estimate the $\Delta CoVaR$ and VaR.

3.3 Quantile Regression Method

The other solution is via quantile regressions, which incorporate estimates of the conditional mean and the conditional variance to produce conditional quantiles. It can be performed without the distributional assumptions of error term $\Delta\mathbb{Z}$, which are needed for estimations via OLS and MLE. Meanwhile, the issue of two-step parameterizations is avoided. Thus, this method is just a kind of regression, so the computational cost is very low. We can get the results really quickly. But it is just because of the simplicity, which makes it hard to carry out further development. Next, some quantile regression implementation details will be involved. When we do some modifications to the original version of the Linear Factor structure, we can obtain the formula as follows:

$$\mathbb{X}_t^j = \left(\phi_0 + \phi_3 \Delta\mathbb{Z}_t^j\right) + \mathbb{M}_{t-1}\left(\overrightarrow{\boldsymbol{\phi_1}} + \overrightarrow{\boldsymbol{\phi_4}}\Delta\mathbb{Z}_t^j\right) + \mathbb{X}_t^i\left(\phi_2 + \phi_5 \Delta\mathbb{Z}_t^j\right)$$

Then use the idea of quantile regressions to modify this model a little bit. It follows immediately that the inverse c.d.f. of $\mathbb{X}_t^j$ is:

$$F_{\mathbb{X}_t^j}^{-1}(q|\mathbb{M}_{t-1}, \mathbb{X}_t^i) = \left(\phi_0 + \phi_3 F_{\Delta\mathbb{Z}_t^j}^{-1}\right) + \mathbb{M}_{t-1}\left(\overrightarrow{\boldsymbol{\phi_1}} + \overrightarrow{\boldsymbol{\phi_4}} F_{\Delta\mathbb{Z}_t^j}^{-1}\right) + \mathbb{X}_t^i\left(\phi_2 + \phi_5 F_{\Delta\mathbb{Z}_t^j}^{-1}\right)$$

where we denote the c.d.f. of $\Delta\mathbb{Z}_t^j$ by $F_{\Delta\mathbb{Z}_t^j}(\cdot)$, and its inverse c.d.f. by $F_{\Delta\mathbb{Z}_t^j}^{-1}(\cdot)$, for different quantiles $q \in (0,\ 100)$.

Finally, we rewrite this formula as the following form:

$$F_{\mathbb{X}_t^j}^{-1}(q|\mathbb{M}_{t-1}, \mathbb{X}_t^i) = \alpha_q + \mathbb{M}_{t-1}\gamma_q + \mathbb{X}_t^i\beta_q$$

Thus, using this form, we can directly carry out quantile regressions regardless of the distribution of $\Delta\mathbb{Z}_t^j$. According to the author, we can directly use this formula to derive the another two to estimate the VaR and $CoVaR$.

For VaR, we have:

$$\mathbb{X}_t^i = \alpha_q^i + \gamma_q^i \mathbb{M}_{t-1} + \varepsilon_{q,t}^i$$

$$\Rightarrow \quad VaR_{q,t}^i = \hat{\alpha}_q^i + \hat{\gamma}_q^i \mathbb{M}_{t-1}$$

For $CoVaR$, we have:

$$\mathbb{X}_t^{j|i} = \alpha_q^{j|i} + \gamma_q^{j|i}\mathbb{M}_{t-1} + \beta_q^{j|i}\mathbb{X}_t^i + \varepsilon_{q,t}^{j|i}$$

$$\Rightarrow \quad CoVaR_{q,t}^i = \widehat{\alpha}_q^{j|i} + \widehat{\gamma}_q^{j|i}\mathbb{M}_{t-1} + \widehat{\beta}_q^{j|i}VaR_{q,t}^i$$

Finally, for $\Delta CoVaR$, we have two ways to compute:

$$\Delta CoVaR_{q,t}^i = CoVaR_{q,t}^i - CoVaR_{50,t}^i \quad * \text{— Formula 1}$$

$$\Delta CoVaR_{q,t}^i = \widehat{\beta}_q^{j|i}\left(VaR_{q,t}^i - VaR_{50,t}^i\right) \quad \text{— Formula 2}$$

We can see there are two ways to compute $\Delta CoVaR$. Generally, Formula 1 is regarded as the more reliable way to reflect the systemic condition, so we mainly use it in the empirical study section.

3.4 Note on Linear Factor Structure — Prediction Issues

If we want to make some predictions on $CoVaR$, there is no doubt that the original version and the variants introduced before of Linear Factor structure do not work, since estimations involve current information. In other words, if we want to use the Linear Factor structure to make predictions, we should modify this model a little bit. We still use the linear factor structure, where losses $\mathbb{X}_t^i$ is also introduced as a direct determinant of the volatility. But here, we substitute the original losses $\mathbb{X}_t^i$ to one-period lagged series, i.e., $\mathbb{X}_t^j = \phi_0 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_1}} + \mathbb{X}_{t-1}^i \ \phi_2 + (\phi_3 + \mathbb{M}_{t-1}\overrightarrow{\boldsymbol{\phi_4}} + \mathbb{X}_{t-1}^i \ \phi_5)\Delta\mathbb{Z}_t^j$.

Generally, there should not be some severe problems after doing that. After all, $\mathbb{X}_{t-1}^i$ is indeed closely related to $\mathbb{X}_t^i$. Furtherly, Adrian and Brunnermeier (2016) expand this idea to propose a more convincing theory, which is called $\mathbf{Forward} - \boldsymbol{\Delta CoVaR}$.

According to Adrian and Brunnermeier (2016), for $\text{Forward} - \Delta CoVaR$, we have the following formula to calculate:

$$\Delta CoVaR_{q,t}^i = a + c\mathbb{M}_{t-h} + b\mathbb{X}_{t-h}^i + \eta_t^i$$

$$\Rightarrow \quad \Delta_h^{Fwd}CoVaR_{q,t}^i = \widehat{a} + \widehat{c}\mathbb{M}_{t-h} + \widehat{b}\mathbb{X}_{t-h}^i$$

where $\mathbb{X}_{t-h}^i$ are the vector of characteristics for institution i, $\mathbb{M}_{t-h}$ is the vector of macro state variables lagged-h quarters. More specifically, for a forecast horizon $h = 1,\ 4,\ 8$ quarters, we run regressions.

Importantly, according to Adrian and Brunnermeier (2016), there is one something we should keep in mind. Any tail risk measure, estimated at a high frequency, is by its very nature imprecise. We are not going to do some further work in the report, but it is a good complement of $\Delta CoVaR$ to indicate the risk.

3.5 Note on Linear Factor Structure — A Criticism

From my perspective, the calculation of VaR using quantile regression introduced by Adrian and Brunnermeier (2016) is not effective enough.

$$\mathbb{X}_t^i = \alpha_q^i + \gamma_q^i \mathbb{M}_{t-1} + \varepsilon_{q,t}^i \;\; \Rightarrow \;\; VaR_{q,t}^i = \hat{\alpha}_q^i + \hat{\gamma}_q^i \mathbb{M}_{t-1}$$

In this formula, losses of an institution are only a matter of state variables, i.e., market conditions. However, the losses are also closely related to its own state one period ago. So, referring to the idea of AR(Auto Regressive) model, we can add a one-period lagged variable to this model. Then we can obtain:

$$\mathbb{X}_t^j = \alpha_q^i + \gamma_q^i \mathbb{M}_{t-1} + \beta_q^i \mathbb{X}_{t-1}^i + \varepsilon_{q,t}^i \;\; \Rightarrow \;\; VaR_{q,t}^i = \hat{\alpha}_q^i + \hat{\gamma}_q^i \mathbb{M}_{t-1} + \hat{\beta}_q^i \mathbb{X}_{t-1}^i.$$

4. Empirical Study

In this section, first, we downloaded the data from some websites (which are listed in the Appendix), and did many works about data clean to get the inputs of our estimation model. Second, we analyzed what is the best method among all of the methods the authors proposed. Finally, we used this optimized model to calculate the $\Delta CoVaR$ and VaR, and replicated some of the authors' research following their steps. Specifically, the time period we analyzed was from 2002-12-31 to 2021-12-31.

4.1 Data Clean

After the data clean, we got the state variables and losses series of stocks in S&P 500 and market, which is measured by S&P 500 index. Importantly, they are all sampled with a weekly frequency by convention. After some unit transformation, we can see the summary of state variables, which is very closed to what the authors presented.

[Table 1]

4.2 MLE Algorithm Robustness Testing

The implementation of the MLE method is a little bit difficult, since there are totally 20 parameters to get nonlinear optimized. It is obviously a high dimension problem, where the convergence can not always be guaranteed. So before starting our research, we have to test the robustness of the algorithm.
To do this, what we did was to output the fitted values of parameters and used them to calculate the log-likelihood function values. Then, compared with which are calculated using random parameters. Finally, we found all of the 493 samples passed the test. Here is a simple visualization of the part of results.

[Figure 1]

4.3 VaR Computational Methods Comparison

In this part, we calculated the VaR from different methods. They are the nonparametric method, semiparametric method, MLE method, and two versions of quantile regression, one is what the author propose and the other one is modified by adding an one period lagged stock series.

[Figure 2]

From the results, we found that MLE has a most volatile performance, and is likely to overestimate the risk. The values computed by the nonparametric method behave like a pairwise function, which is very rough. A similar conclusion can be applied to the semiparametric method. Finally, the results of quantile regression that are modified and not are very close, maybe the modified one has a more conservative estimate of risk. We choose this method to perform the following computations.

4.4 $CoVaR$ Computational Methods Comparison

In this part, similarly, we calculated the $99\% CoVaR$ and $50\% CoVaR$ from four different methods, which are listed in the figure.

[Figure 3]

[Figure 4]

From the results, we can draw the conclusion that MLE is also likely to overestimate the risk in this case, while quantile regressions behave similarly, regardless of the input. It is partially because the weight of these coefficients is not very large.

4.5 $\Delta CoVaR$ Computational Methods Comparison

From the calculation above, we could immediately get the values of $\Delta CoVaR$.

[Figure 5]

We can see from the Figure 5, all of the curves behave strangely except for the quantile regression using the first formula give a moderate estimate. Thus, we choose this method to perform the following computations.

4.6 $\Delta CoVaR$ and VaR Results

From the methods we chose in previous parts, we replicated the authors' figures.

[Figure 6]

We can see that this figure resembles the authors' to a certain extent. Next, we generated these 99% $CoVaR$ and 99% VaR regression results of the stocks above. We can see that the pseudo – R square is not very large, thus the results are a little bit far from satisfactory.

[Table 2, 3, 4]

[Table 5, 6, 7]

[Table 8, 9, 10]

[Table 11, 12, 13]

4.7 Sector Analysis

As what was done by the authors, we got the 99% $CoVaR$ and 99% VaR scatter plot to see the relationship between them at four times.

[Figure 7]

We finally got a rough plot for cross-sectional analysis.

[Figure 8]

We can conclude that the 99% $CoVaR$ and 99% VaR from stocks of different sections behave very differently.

References

Adrian, T., and Brunnermeier, M. K. (2016), "CoVaR," *American Economic Review*, 106 (7), 1705-1741. DOI: 10.1257/aer.20120555.

Smaga, P. (2014), "The concept of systemic risk," *Systemic Risk Centre Special Paper*, (5).

Appendices

A Data Sources and Introduction of State Variables

All of the macroeconomic data can be obtained from the links listed below:
(i) _updated_A_snp_tickers_sectors.csv
It is found at https://en.wikipedia.org/wiki/List_of_S%26P_500_companies
Last change was on April 21, 2022.

(ii) _updated_B_DTB3.csv (First state variable, Second state variable)
It is found at https://fred.stlouisfed.org/series/DTB3
Last change was on April 21, 2022.

(iii) _updated_B_DLTBOARD.csv (Second state variable - first period)
It is found at https://fred.stlouisfed.org/series/DLTBOARD
Last change was on June 30, 2000.

(iv)_updated_C_USTREASURY-LONGTERMRATES.csv (Second state variable - second period)
It is found at https://data.nasdaq.com/data/USTREASURY/LONGTERMRATES-treasury-long-term-rates
Last change was on January 31, 2022.

(v) _updated_B_TEDRATE.csv (Third state variable)
It is found at https://fred.stlouisfed.org/series/TEDRATE
Last change was on January 21, 2022.

(vi) _updated_B_DBAA.csv (Fourth state variable)
It is found at https://fred.stlouisfed.org/series/DBAA
Last change was on April 21, 2022.

(vii) _updated_B_DGS10.csv (Fourth state variable)
It is found at https://fred.stlouisfed.org/series/DGS10
Last change was on April 21, 2022.

The state variables are: (The spreads and spread changes are expressed in weekly basis points, and returns are in weekly percent)
(i) Three-month yield Change
The author use the change in the three-month Treasury bill rate because the author finds that the change, not the level, is most significant in explaining the tails of financial sector market-valued asset returns

(ii) Term spread Change
Measured by the yield spread between the long term bond composite and the three-month bill rate

(iii) TED spread
Defined as the difference between the three-month Libor rate and the three-month secondary market bill rate

(iv) Credit spread Change
Between Moody's Baa-rated bonds and the ten year Treasury rate

(v) Market Return

(vi) Real Estate Excess Return
Measured by real estate sector return in excess of the market financial sector return

(vii) Equity Volatility
It can be computed as the 22 day rolling standard deviation of the daily CRSP equity market return

B Figures and tables

	3MO_Yield_Change	Term_Spread_Change	TED_Spread	Credit_Spread_Change	Market_Return	Excess_Return	Equity_Volatility
mean	-0.114919	0.033065	40.405595	-0.033317	0.218303	0.190356	1.031359
std	8.236839	2.602870	41.010937	1.709793	2.518130	3.662225	0.733573
min	-80.000000	-14.600000	6.600000	-7.400000	-18.195465	-18.888800	0.233922
25%	-1.000000	-1.400000	20.200000	-0.800000	-0.838042	-1.454589	0.623337
75%	2.000000	1.600000	42.600000	0.600000	1.470086	1.742171	1.189720
max	69.000000	11.600000	412.000000	18.600000	13.106071	19.366352	5.960092

[Table 1]
Summary of state variables

QuantReg Regression Results

Dep. Variable:	GS	Pseudo R-squared:	0.3845
Model:	QuantReg	Bandwidth:	2.374
Method:	Least Squares	Sparsity:	107.9
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	5.0480	0.634	7.966	0.000	3.804	6.291
3MO_Yield_Change	-0.2192	0.068	-3.236	0.001	-0.352	-0.086
Term_Spread_Change	0.2333	0.328	0.711	0.477	-0.411	0.877
TED_Spread	0.0318	0.021	1.544	0.123	-0.009	0.072
Credit_Spread_Change	0.1241	0.649	0.191	0.848	-1.149	1.397
Market_Return	-0.1551	0.422	-0.368	0.713	-0.983	0.673
Excess_Return	0.0871	0.119	0.735	0.462	-0.145	0.320
Equity_Volatility	2.7503	1.175	2.341	0.019	0.445	5.056
stock_lagged	-0.1353	0.160	-0.845	0.399	-0.450	0.179

[Table 2]

99%VaR Quantile regression of Goldman Sachs

QuantReg Regression Results

Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3558
Model:	QuantReg	Bandwidth:	1.403
Method:	Least Squares	Sparsity:	50.78
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	5.0712	2.136	2.374	0.018	0.879	9.263
3MO_Yield_Change	-0.1609	0.089	-1.812	0.070	-0.335	0.013
Term_Spread_Change	0.2053	0.144	1.429	0.153	-0.077	0.487
TED_Spread	0.0153	0.015	1.004	0.315	-0.015	0.045
Credit_Spread_Change	0.0248	0.156	0.159	0.874	-0.281	0.331
Market_Return	-0.4239	0.108	-3.911	0.000	-0.637	-0.211
Excess_Return	-0.1022	0.090	-1.138	0.256	-0.279	0.074
Equity_Volatility	4.2378	1.233	3.437	0.001	1.819	6.657
GS_VaR_0.99	-0.4983	0.422	-1.182	0.237	-1.326	0.329

[Table 3]

99%$CoVaR$ Quantile regression of Goldman Sachs

QuantReg Regression Results

Dep. Variable:	AAPL	Pseudo R-squared:	0.1940
Model:	QuantReg	Bandwidth:	2.789
Method:	Least Squares	Sparsity:	169.2
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	9.2568	1.817	5.094	0.000	5.691	12.823
3MO_Yield_Change	-0.1647	0.092	-1.795	0.073	-0.345	0.015
Term_Spread_Change	-0.1709	0.415	-0.412	0.680	-0.984	0.643
TED_Spread	0.0494	0.030	1.666	0.096	-0.009	0.108
Credit_Spread_Change	0.5221	0.588	0.889	0.374	-0.631	1.675
Market_Return	-0.3802	0.429	-0.887	0.375	-1.221	0.461
Excess_Return	-0.1185	0.107	-1.111	0.267	-0.328	0.091
Equity_Volatility	-0.9962	1.987	-0.501	0.616	-4.896	2.904
stock_lagged	-0.0147	0.219	-0.067	0.947	-0.445	0.416

[Table 4]

99% VaR Quantile regression of Apple

QuantReg Regression Results

Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3627
Model:	QuantReg	Bandwidth:	1.385
Method:	Least Squares	Sparsity:	53.43
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	59.9508	46.034	1.302	0.193	-30.386	150.287
3MO_Yield_Change	-1.0529	0.828	-1.272	0.204	-2.677	0.571
Term_Spread_Change	-0.9453	0.873	-1.083	0.279	-2.658	0.767
TED_Spread	0.2976	0.253	1.178	0.239	-0.198	0.794
Credit_Spread_Change	3.3478	2.648	1.264	0.207	-1.850	8.545
Market_Return	-2.6027	1.852	-1.406	0.160	-6.236	1.031
Excess_Return	-0.8442	0.612	-1.379	0.168	-2.045	0.357
Equity_Volatility	-3.6388	5.022	-0.725	0.469	-13.494	6.216
AAPL_VaR_0.99	-6.1277	4.986	-1.229	0.219	-15.912	3.657

The condition number is large, 9.7e+03. This might indicate that there are strong multicollinearity or other numerical problems.

[Table 5]

99% $CoVaR$ Quantile regression of Apple

Dep. Variable:	MMM	Pseudo R-squared:	0.1480
Model:	QuantReg	Bandwidth:	1.639
Method:	Least Squares	Sparsity:	119.9
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	5.1777	0.673	7.691	0.000	3.857	6.499
3MO_Yield_Change	0.0280	0.182	0.154	0.878	-0.329	0.385
Term_Spread_Change	-0.0961	0.251	-0.383	0.702	-0.589	0.397
TED_Spread	0.0085	0.025	0.340	0.734	-0.041	0.058
Credit_Spread_Change	-0.0522	0.301	-0.173	0.862	-0.643	0.539
Market_Return	-0.3612	0.245	-1.476	0.140	-0.841	0.119
Excess_Return	-0.0138	0.138	-0.100	0.920	-0.284	0.257
Equity_Volatility	1.8123	0.713	2.540	0.011	0.412	3.212
stock_lagged	-0.0864	0.160	-0.539	0.590	-0.401	0.228

[Table 6]
99%VaR Quantile regression of 3M

QuantReg Regression Results

Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3557
Model:	QuantReg	Bandwidth:	1.415
Method:	Least Squares	Sparsity:	56.70
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	0.1764	6.881	0.026	0.980	-13.326	13.679
3MO_Yield_Change	-0.0671	0.051	-1.313	0.189	-0.167	0.033
Term_Spread_Change	0.1554	0.149	1.042	0.298	-0.137	0.448
TED_Spread	-0.0051	0.015	-0.345	0.730	-0.034	0.024
Credit_Spread_Change	0.0167	0.172	0.098	0.922	-0.320	0.354
Market_Return	-0.2576	0.437	-0.589	0.556	-1.116	0.600
Excess_Return	-0.1090	0.062	-1.770	0.077	-0.230	0.012
Equity_Volatility	2.0576	2.451	0.839	0.401	-2.753	6.868
MMM_VaR_0.99	0.4666	1.331	0.350	0.726	-2.146	3.079

The condition number is large, 1.82e+03. This might indicate that there are strong multicollinearity or other numerical problems.

[Table 7]
99%$CoVaR$ Quantile regression of 3M

QuantReg Regression Results

Dep. Variable:	ARE	Pseudo R-squared:	0.4125
Model:	QuantReg	Bandwidth:	2.629
Method:	Least Squares	Sparsity:	102.6
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	3.8848	0.883	4.399	0.000	2.152	5.618
3MO_Yield_Change	0.0108	0.050	0.215	0.830	-0.087	0.109
Term_Spread_Change	0.5247	0.203	2.587	0.010	0.127	0.923
TED_Spread	-0.0167	0.008	-2.105	0.036	-0.032	-0.001
Credit_Spread_Change	-0.2922	0.370	-0.789	0.430	-1.019	0.434
Market_Return	-0.7840	0.283	-2.768	0.006	-1.340	-0.228
Excess_Return	-0.1949	0.269	-0.724	0.469	-0.723	0.333
Equity_Volatility	6.3685	0.944	6.746	0.000	4.516	8.221
stock_lagged	-0.3145	0.235	-1.341	0.180	-0.775	0.146

[Table 8]

99%VaR Quantile regression of Alexandria

QuantReg Regression Results

Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3689
Model:	QuantReg	Bandwidth:	1.401
Method:	Least Squares	Sparsity:	51.46
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	4.1222	1.918	2.149	0.032	0.358	7.886
3MO_Yield_Change	-0.0458	0.030	-1.531	0.126	-0.105	0.013
Term_Spread_Change	0.2858	0.328	0.872	0.383	-0.357	0.929
TED_Spread	-0.0084	0.012	-0.677	0.498	-0.033	0.016
Credit_Spread_Change	-0.0687	0.165	-0.418	0.676	-0.392	0.254
Market_Return	-0.6066	0.207	-2.930	0.003	-1.013	-0.200
Excess_Return	-0.1383	0.083	-1.664	0.096	-0.301	0.025
Equity_Volatility	5.2605	2.942	1.788	0.074	-0.513	11.034
ARE_VaR_0.99	-0.3835	0.473	-0.810	0.418	-1.312	0.545

[Table 9]

99%$CoVaR$ Quantile regression of Alexandria

QuantReg Regression Results

Dep. Variable:	CVS	Pseudo R-squared:	0.08584
Model:	QuantReg	Bandwidth:	2.185
Method:	Least Squares	Sparsity:	116.1
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	7.3797	0.781	9.454	0.000	5.848	8.912
3MO_Yield_Change	0.1346	0.161	0.835	0.404	-0.182	0.451
Term_Spread_Change	-0.6955	0.195	-3.564	0.000	-1.078	-0.313
TED_Spread	0.0457	0.012	3.671	0.000	0.021	0.070
Credit_Spread_Change	-0.1717	0.249	-0.689	0.491	-0.660	0.317
Market_Return	-0.2800	0.243	-1.152	0.250	-0.757	0.197
Excess_Return	0.3629	0.102	3.555	0.000	0.163	0.563
Equity_Volatility	-0.4699	0.674	-0.697	0.486	-1.792	0.852
stock_lagged	-0.1939	0.230	-0.843	0.400	-0.646	0.258

[Table 10]

99%VaR Quantile regression of CVS Health

QuantReg Regression Results

Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3545
Model:	QuantReg	Bandwidth:	1.398
Method:	Least Squares	Sparsity:	61.17
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	1.9818	2.385	0.831	0.406	-2.699	6.663
3MO_Yield_Change	-0.0786	0.055	-1.422	0.155	-0.187	0.030
Term_Spread_Change	0.1832	0.280	0.655	0.513	-0.366	0.732
TED_Spread	-0.0045	0.017	-0.267	0.790	-0.038	0.029
Credit_Spread_Change	-0.0244	0.189	-0.129	0.897	-0.396	0.347
Market_Return	-0.3747	0.147	-2.557	0.011	-0.662	-0.087
Excess_Return	-0.1514	0.135	-1.118	0.264	-0.417	0.114
Equity_Volatility	2.7957	0.326	8.583	0.000	2.156	3.435
CVS_VaR_0.99	0.1032	0.325	0.317	0.751	-0.536	0.742

[Table 11]

99%$CoVaR$ Quantile regression of CVS Health

QuantReg Regression Results

Dep. Variable:	COP	Pseudo R-squared:	0.2890
Model:	QuantReg	Bandwidth:	2.370
Method:	Least Squares	Sparsity:	125.6
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	8.5687	0.690	12.414	0.000	7.214	9.923
3MO_Yield_Change	-0.2165	0.202	-1.070	0.285	-0.614	0.180
Term_Spread_Change	-0.1238	0.285	-0.434	0.664	-0.684	0.436
TED_Spread	0.0037	0.023	0.165	0.869	-0.041	0.048
Credit_Spread_Change	0.7432	0.262	2.841	0.005	0.230	1.257
Market_Return	-0.0830	0.302	-0.275	0.783	-0.675	0.509
Excess_Return	-0.0296	0.127	-0.234	0.815	-0.278	0.219
Equity_Volatility	1.2098	0.681	1.776	0.076	-0.127	2.547
stock_lagged	-0.1853	0.137	-1.348	0.178	-0.455	0.084

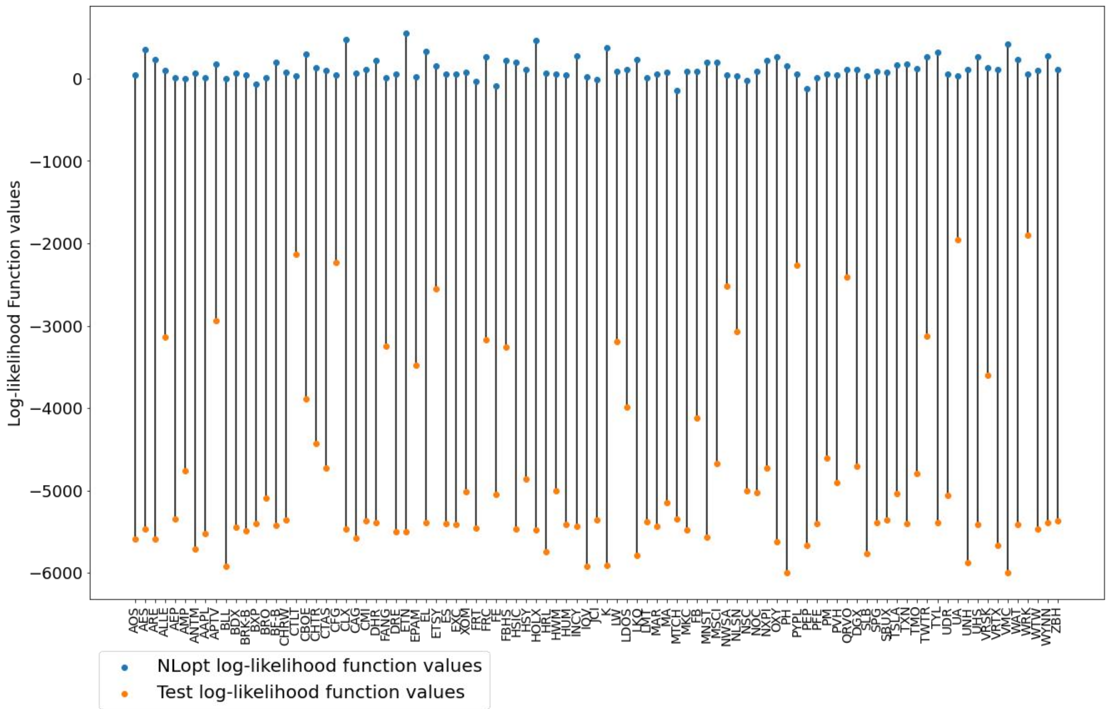
[Table 12]
99%VaR Quantile regression of ConocoPhillips

QuantReg Regression Results

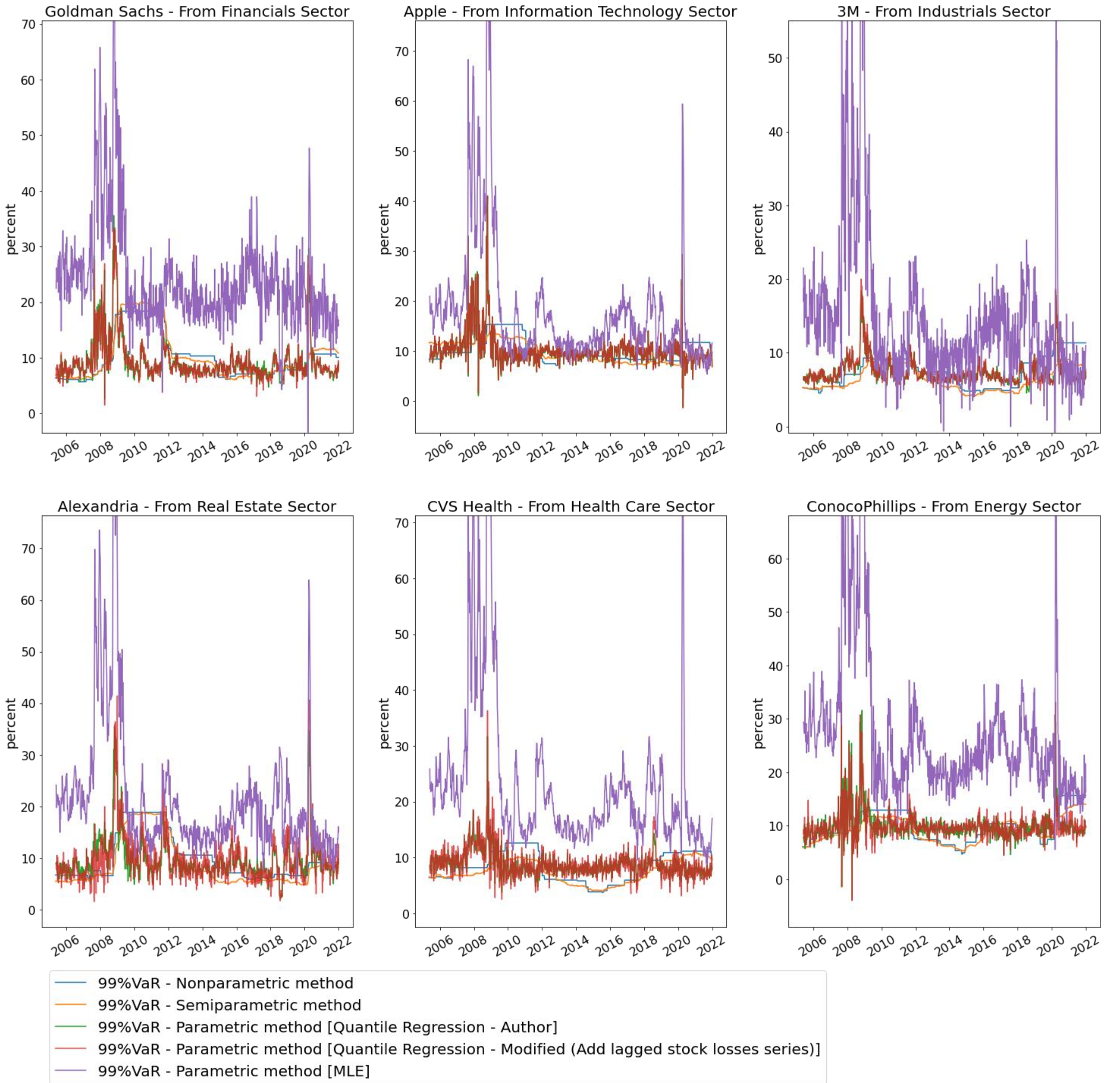
Dep. Variable:	Market_Losses	Pseudo R-squared:	0.3556
Model:	QuantReg	Bandwidth:	1.450
Method:	Least Squares	Sparsity:	60.93
Date:	Fri, 06 May 2022	No. Observations:	991
Time:	20:10:24	Df Residuals:	982
		Df Model:	8

	coef	std err	t	P>|t|	[0.025	0.975]
const	1.8341	3.788	0.484	0.628	-5.598	9.267
3MO_Yield_Change	-0.0329	0.103	-0.321	0.748	-0.234	0.168
Term_Spread_Change	0.0765	0.128	0.596	0.552	-0.176	0.329
TED_Spread	-0.0032	0.012	-0.275	0.783	-0.026	0.020
Credit_Spread_Change	-0.0311	0.389	-0.080	0.936	-0.795	0.733
Market_Return	-0.4099	0.140	-2.927	0.004	-0.685	-0.135
Excess_Return	-0.0770	0.064	-1.202	0.230	-0.203	0.049
Equity_Volatility	2.8947	0.568	5.093	0.000	1.779	4.010
COP_VaR_0.99	0.0893	0.442	0.202	0.840	-0.778	0.957

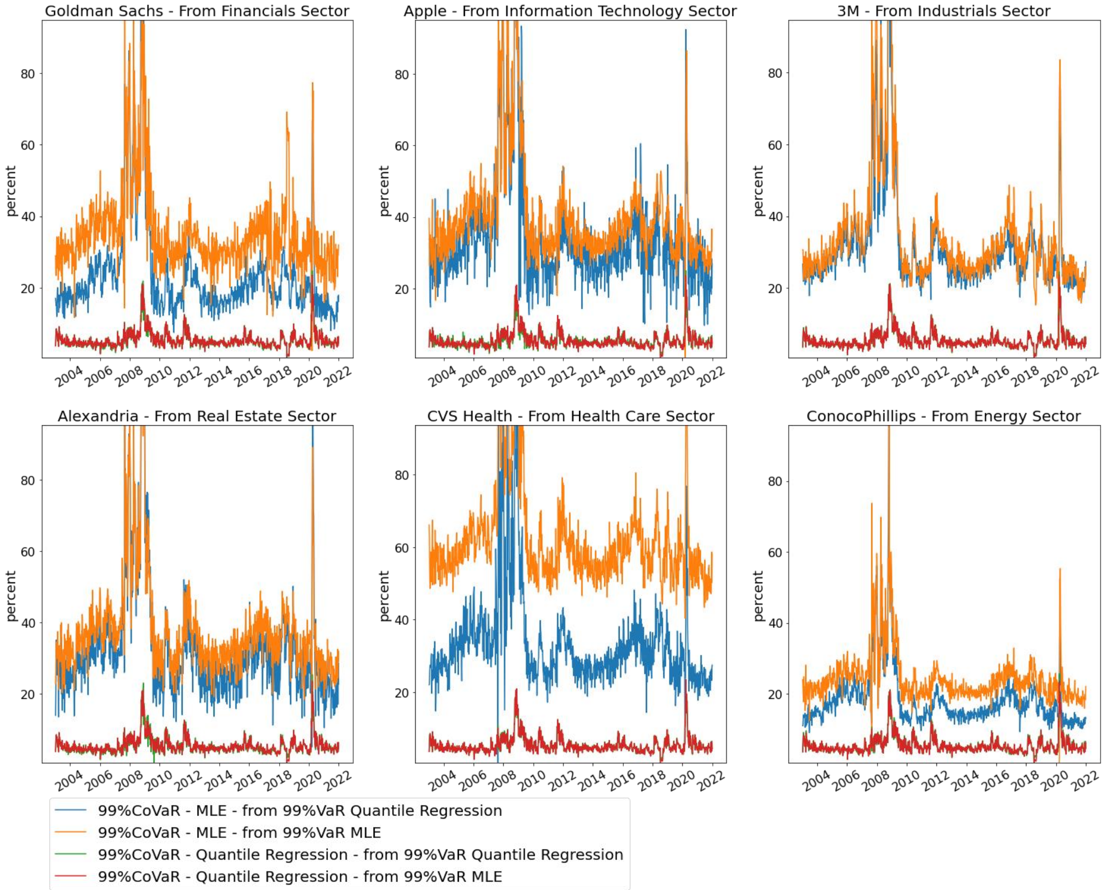
[Table 13]
99%$CoVaR$ Quantile regression of ConocoPhillips



[Figure 1]
MLE Algorithm Robustness Testing

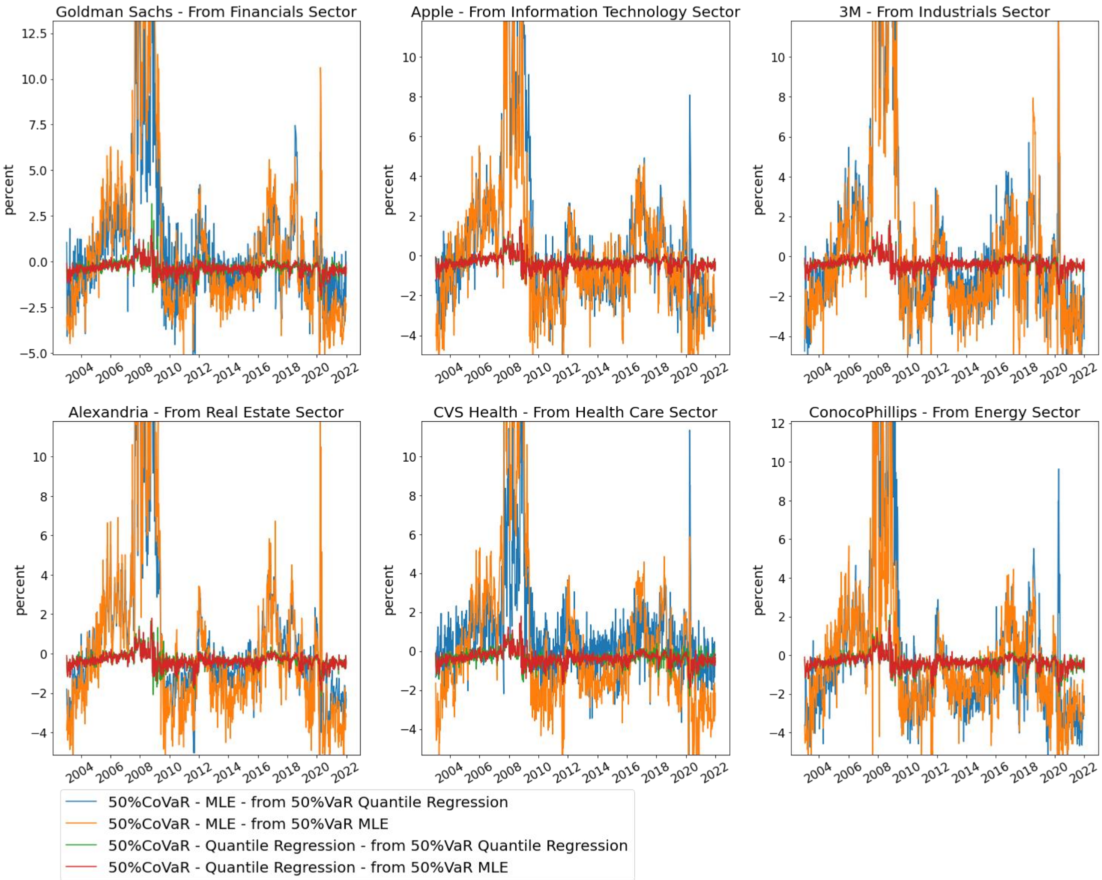


[Figure 2]
99%VaR computational methods comparison



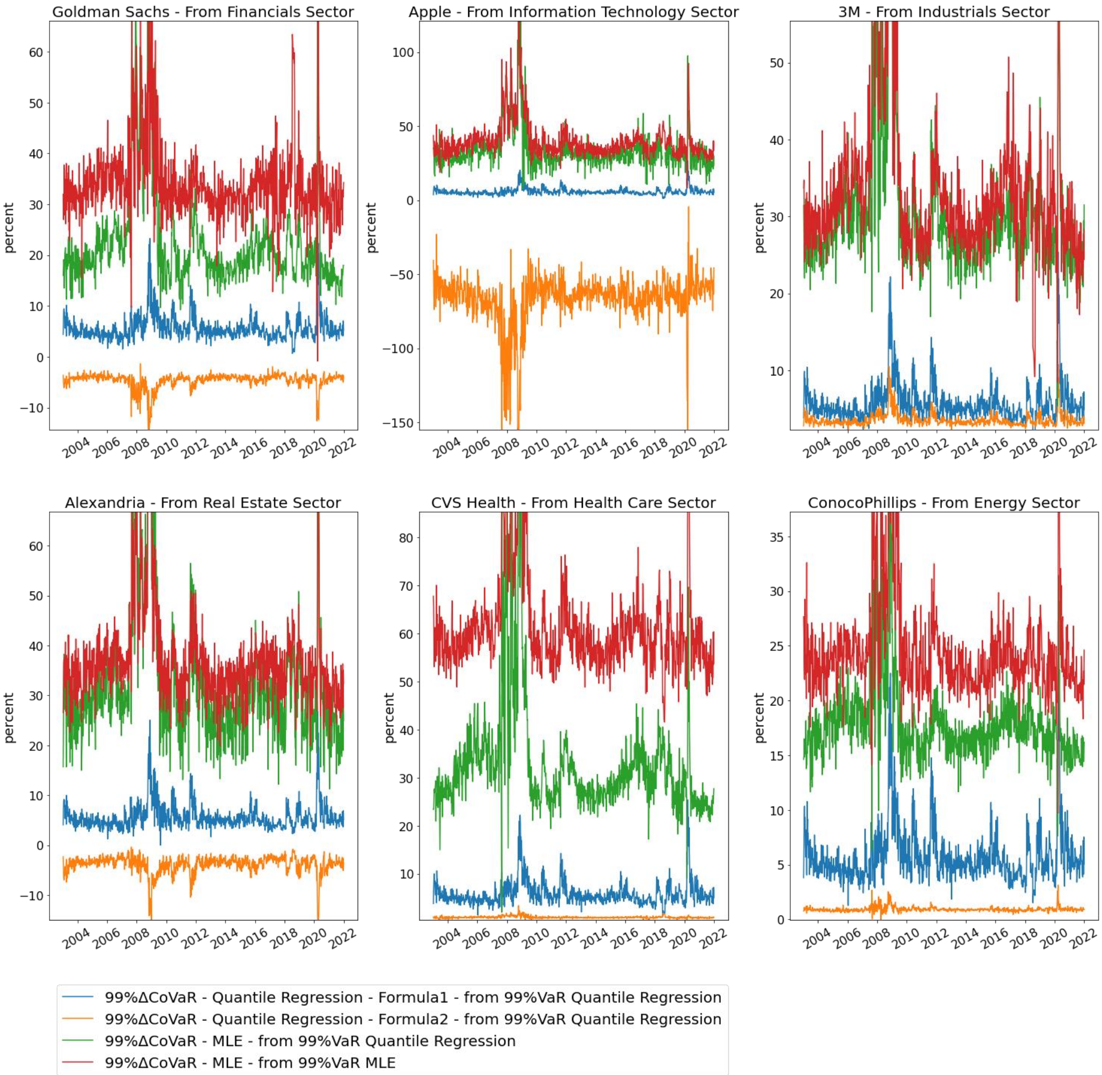
[Figure 3]

99%$CoVaR$ computational methods comparison



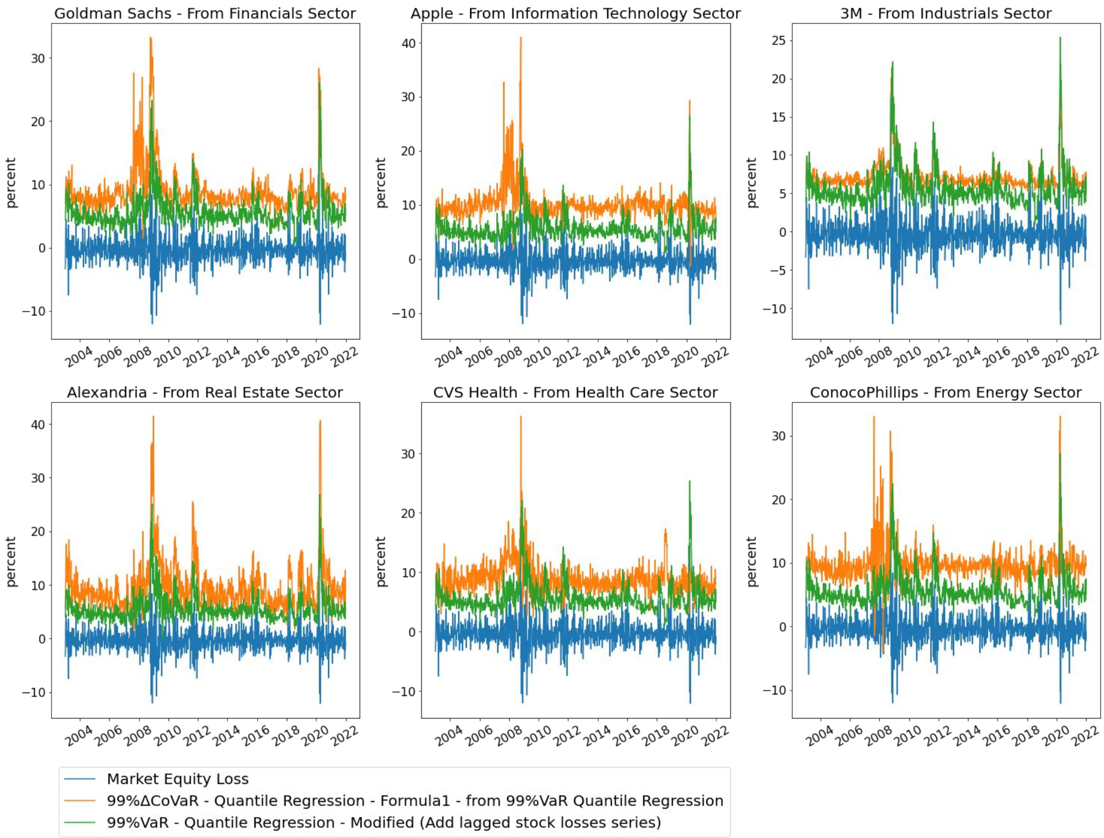
[Figure 4]

50%$CoVaR$ computational methods comparison



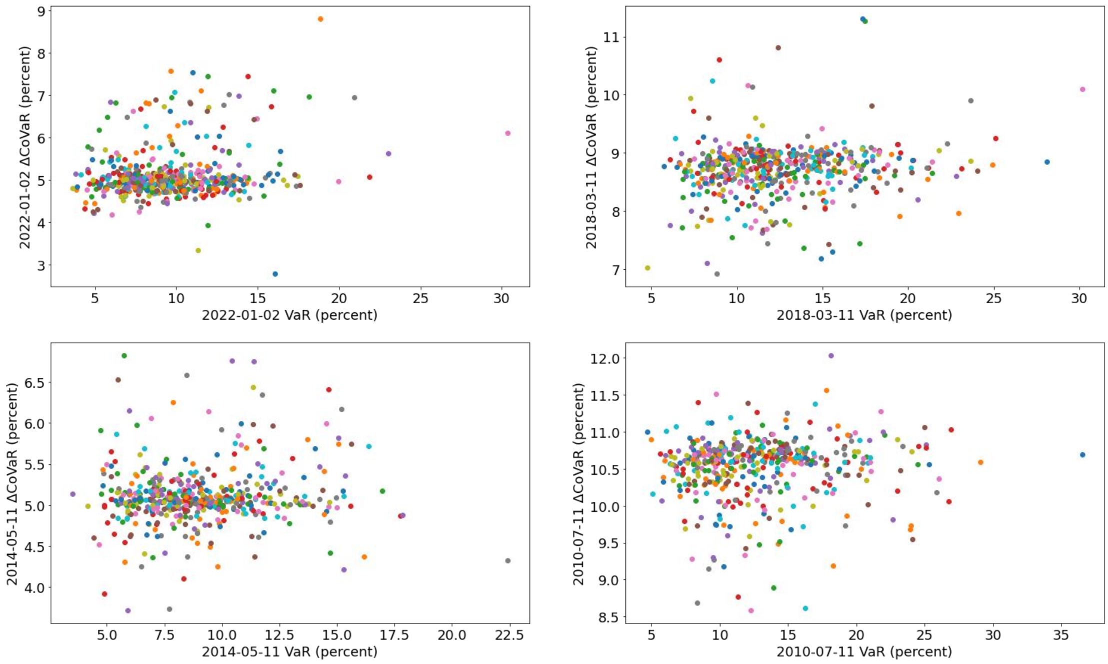
[Figure 5]

$99\%\Delta CoVaR$ computational methods comparison



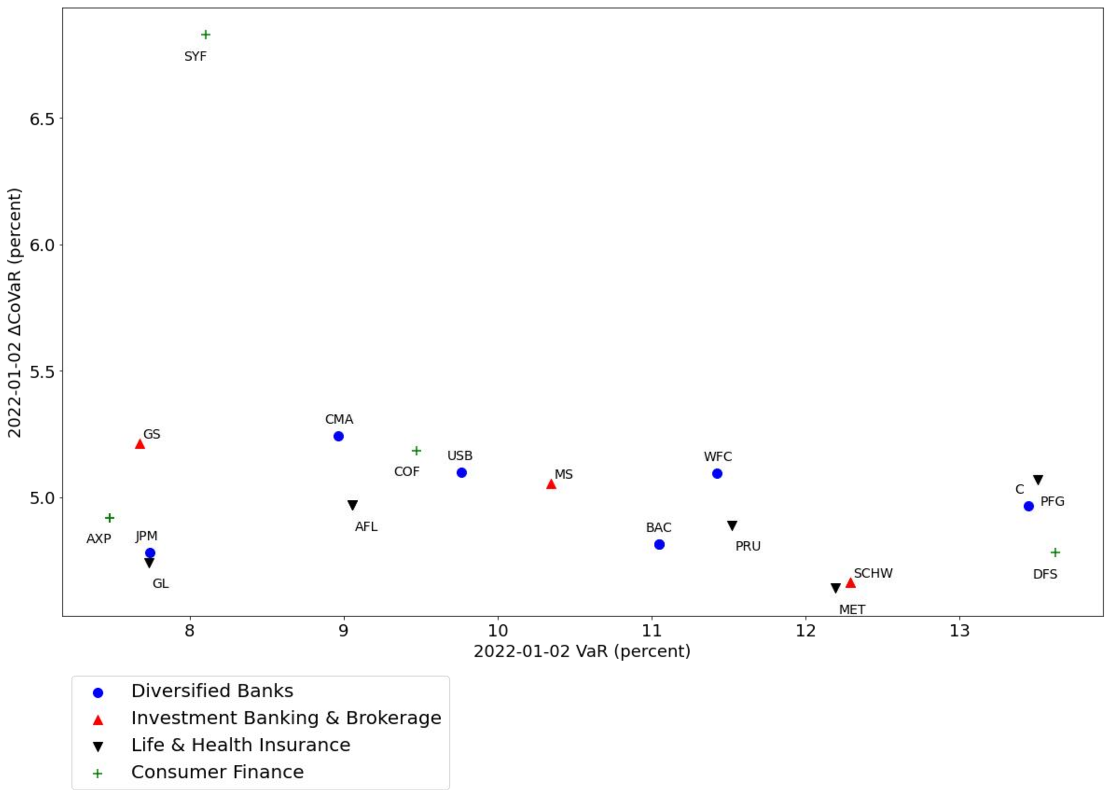
[Figure 6]

$99\%\Delta CoVaR$ and $99\%VaR$ results



[Figure 7]

99%$CoVaR$ and 99%VaR scatter plot at four times



[Figure 8]

cross-sectional 99%$CoVaR$ and 99%VaR scatter plot