Calculation of basic rocket orbits

D. Ng
Level 5 Laboratory, Computational Physics,
School of Physics, University of Bristol, UK.
dn16018@my.bristol.ac.uk
(Dated: May 4, 2018)

A simulation of a rocket orbiting around one and two stationary bodies was computed sequentially by means of the iterative 4th-order Runge-Kutta method.

I. INTRODUCTION

In computational physics, the 4th-order Runge-Kutta is a common method used to find numerical solutions for many differential equations which don't have an analytical solution. The method expresses a 2nd order linear differential equations into a set of first order ordinary equations.

One of the general first order ordinary equations may be expressed as

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t) \tag{1}$$

where the derivative of a vector over time is a function of the vector and time itself. For every time step h that it computed, the variables of equation (1) above can be expressed as an iterative equation which is useful for modern computation modelling devices since they tend to append consecutive values onto a list of data over time. The Runge-Kutta method can be iterated with the following equation[1]

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{h}{6}[\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4] \tag{2}$$

$$t_{i+1} = t_i + h \tag{3}$$

For every appended unit of time with time-step h, it can be seen that every consecutive $\mathbf{x}$ value is a function of its previous value and the standard Runge-Kutta coefficients $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}_4$. The method essentially compartmentalize every interval between $\mathbf{x}_{i+1}$ and $\mathbf{x}_i$ into 4 increments. The differential at each of these 4 steps are then evaluated as opposed to the standard Euler method which only evaluates differentials at 2 different steps.

As shown in FIG. 1. the Runge-Kutta coefficients $\mathbf{k}_1$ and $\mathbf{k}_4$ are calculated at the beginning and end of the interval respectively whereas $\mathbf{k}_2$ and $\mathbf{k}_3$ are based in the midpoint of the interval. The calculations of the coefficients can be summarized as[2]

$$\mathbf{k}_1 = \mathbf{f}(t_i, \mathbf{x}_i + h\mathbf{k}_0) \tag{4}$$

$$\mathbf{k}_2 = f\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{h\mathbf{k}_1}{2}\right) \tag{5}$$

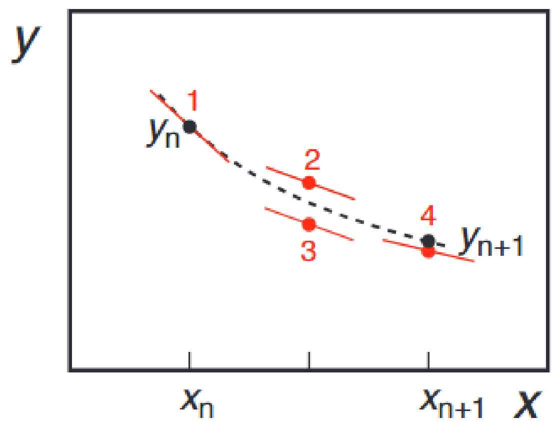

FIG. 1. The position between the interval for which the Runge-Kutta coefficients are computed.

$$\mathbf{k}_3 = f\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{h\mathbf{k}_2}{2}\right) \tag{6}$$

$$\mathbf{k}_4 = f(t_i + h, \mathbf{x}_i + h\mathbf{k}_3) \tag{7}$$

It can be seen from equations (2-7) that the calculations of the variables are all coupled together since

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i, h\sum_{n=1}^{4} \alpha\mathbf{k}_{\mathrm{n}}) \tag{8}$$

and that

$$\mathbf{k}_{\mathrm{n}} = f(t_i, h, \mathbf{x}_{\mathrm{i}}, \mathbf{k}_{\mathrm{n\text{-}1}}) \tag{9}$$

where $\mathbf{k}_0 = 0$.

II PROBLEM A

II.1 The Problem

In part A the objective was to simulate a bound orbit of a moving body orbiting within the confines of a stable gravitational potential imposed by a body with a bigger mass. In this scenario the central body can be modelled with a huge mass such that it renders the mass of the satellite negligible. Although Newton's law of Gravitation has been superseded by Einstein's Theory of General Relativity, it is nevertheless a good approximation given that the velocity of bodies in motion are not comparable to that of light.[3] For the dynamics between 2 bodies 1 and 2, Newton's Law of Gravitation can be written as

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12} \tag{10}$$

where $\mathbf{F}_{21}$ is the force imposed onto body 2 due to body 1. In order to make it work with the Runge-Kutta method, equation (10) can be expressed as [4]

$$\ddot{\mathbf{r}} = -\frac{MG}{|\mathbf{r}|^3}\mathbf{r} \tag{11}$$

where M is the mass of the central body, G is the gravitational constant and $\mathbf{r}$ the position of the satellite with respect to the centre of the central body, as illustrated by FIG. 2. It can be seen that as a result the mass of the satellite cancels out in equation (11). Equation (11) can then be expressed in forms

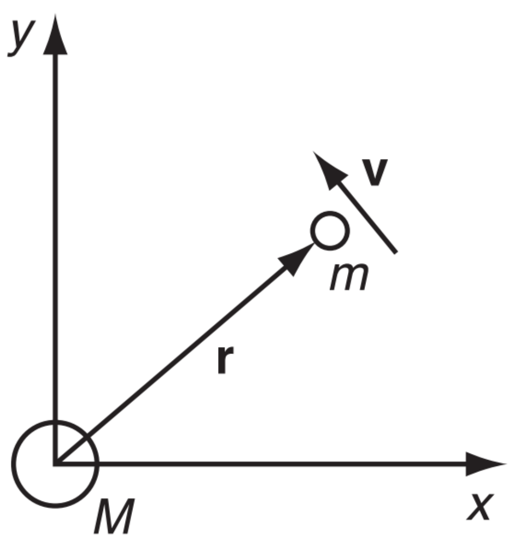

FIG. 2. The free body diagram for a rocket orbiting around a stationary body.

of their individual components

$$\frac{dx}{dt} = f_1(t, x, y, v_x, v_y) = v_x \tag{12}$$

$$\frac{dy}{dt} = f_2(t, x, y, v_x, v_y) = v_y \tag{13}$$

$$\frac{dv_x}{dt} = f_3(t, x, y, v_x, v_y) = \frac{-GMx}{(x^2+y^2)^{3/2}} \tag{14}$$

$$\frac{dv_y}{dt} = f_4(t, x, y, v_x, v_y) = \frac{-GMy}{(x^2+y^2)^{3/2}} \tag{15}$$

Each component of the variables are depended on 4 different Runge-Kutta coefficients, this meant that there will be 16 k-values that needs to be computed for each iteration. As there are 4 orders of differentials, this meant that the error associated with this method will be to the power of 4 of the time-step, h.

II.2 Results

A simulation class was created and used in Python for the rest of the exercise. The fields of the class were first initialized, where all the variables can be defined within just one line. This allows for an easier parameters check when it comes to running and configuring the program. The use of a class also allows any arbitrary simulation to be created and ran with a different configuration of parameters. The positions, velocities and energies were first initialized as lists with no specified dimensions. This way is better as they are values that will be appended, where the dimensions of their lists can only be known with hindsight. Starting the empty lists with say N dimensions can potentially end up with either more or less list elements than needed.

The equations f_1 and f_2 are defined to simply return their velocity component counterparts, while f_3 and f_4 are defined to return the iterative values of the individual components of the gravitational force. It was clear at that point that f_1 and f_2 will remain the same throughout the whole exercise, while f_3 and f_4 will require some changes since there will be more bodies involved. Functions for computing the kinetic, potential and total energy was also defined within the simulation class which are themselves functions of the positions and velocities. The potential energy has to be defined relatively with the distance between the two bodies. The distance between them is simply the radius of the orbit, since the central body is defined as stationary in the origin, hence a function that calculates the radius of the orbit was created.

A function defining the crash condition of the orbiting body was also defined as a boolean. If at any point the radius of the orbit is less than the radius of the central body, the crash condition becomes true. While running the program within a loop, for every iteration of the Runge-Kutta the crash condition is constantly being checked and returned. As a consequence the loop necessarily breaks if the crash condition becomes true, stopping the simulation altogether.

The initial motivation to use a class as opposed to a conventional method was partly so that any simulation can be arbitrarily defined by the user. However, upon finishing the creation of the program and implementing the flexibility of the user input, it was realised that it is very hard for a user to create a working orbit himself unless the specifics of parameters were known. Nevertheless it allows the creator of the program to easily configure the parameters of the simulation and define them in just a single line, which helps immensely with writing the report

The Runge-Kutta body of the code was looped and several graphs were created. A distance graph was created in 2D to show the plane trajectory of the path of the orbit and another one was created in 3D to show the time-evolution of the orbit, with z-axis defined as the time. An energy graph was also created which contains the kinetic, potential and total energy of the orbiter, as shown by FIG. 3-6.

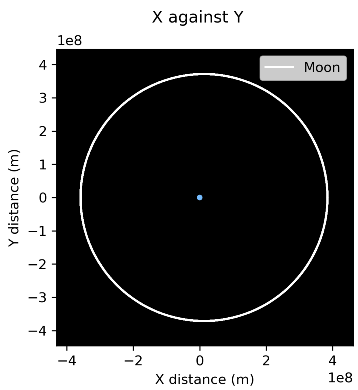

FIG. 3. The trajectory of the moon in a circular orbit going around the Earth.

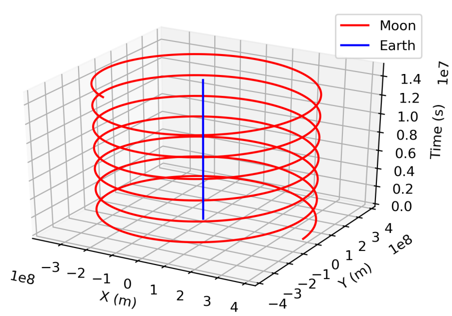

FIG. 4. The time evolution of the trajectory of the Moon in a circular orbit going around the Earth.

II.3 Discussion

With an initial velocity of 1000ms^{-1} and 1300ms^{-1} in the positive y-direction, a circular and an elliptical orbit were obtained respectively. In both cases the total energies are conserved, since they are constant over time. Their energies also look sensible, with the absolute value of the potential energy higher than the kinetic energy, and a negative total energy, which is the condition required to be in a bound orbit. Both the kinetic and potential energy are the mirror image of each other, which means that at every point in space and time they

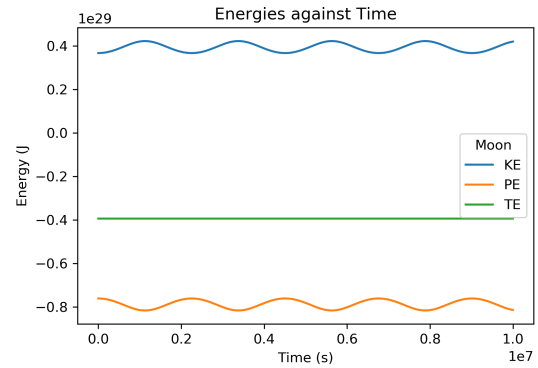

FIG. 5. The energies of the Moon orbiting around the Earth in a circular orbit.

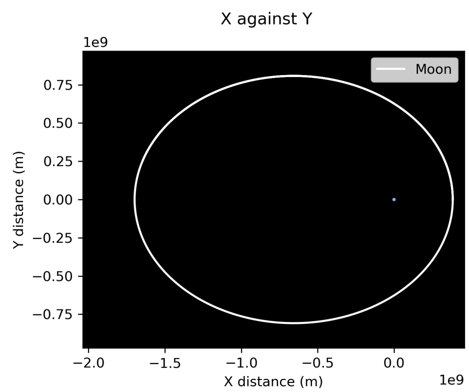

FIG. 6. The trajectory of the Moon in an elliptical orbit going around the Earth.

are in a stable orbit since their oscillatory changes cancel each other out to give a constant total energy. The energies of the orbiter around an elliptical orbit is correct as we can see from the energy plot that there's an oscillatory spike in both kinetic and potential energy from time to time. This is due to the fact that the closer the orbiter it is to the central body, the faster it travels and the smaller its radius orbit will be. Since kinetic energy is proportional to the square of velocity and potential energy is inversely proportional to the radius of the orbit, it is evident that the spikes happens when it orbits closer to the central body. After plotting the time-step h against the total energy of the orbit, as shown on FIG. 9, it can be seen the the total energy of the orbit is accurately conserved regardless of the time-step used. It is interesting that the total energy of the system is independent of the time-step, atleast for the range of values of h that has been investigated.

Note that the fact that there is an oscillatory nature in the time evolution of both the kinetic and potential energy of the body shown in the circular orbit above shows how the radius of the orbit is still not absolutely constant. To investigate this further we can explore a low Earth orbit of a real life scenario, which is the orbit trajectory of the International Space Station, as shown in FIG. 10-12. The ISS has a very low Earth orbit

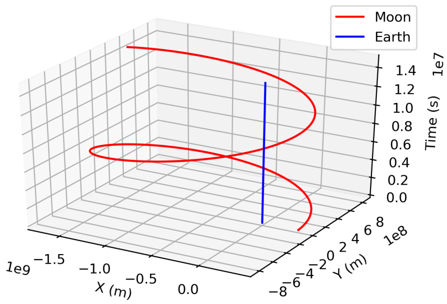

FIG. 7. The time evolution of the trajectory of the Moon in an elliptical orbit going around the Earth.

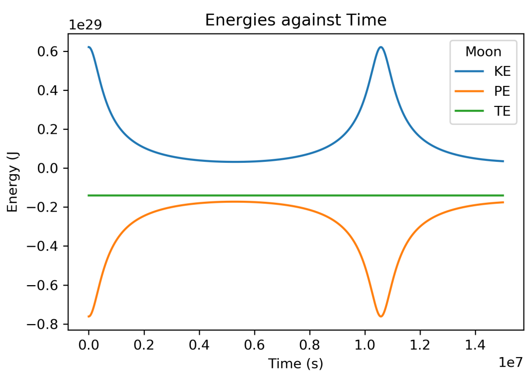

FIG. 8. The energies of the Moon orbiting around the Earth in an elliptical orbit.

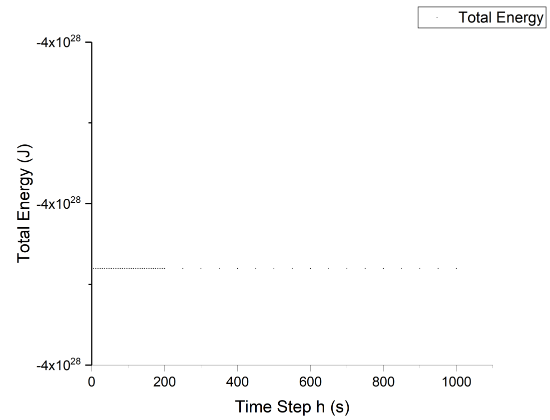

FIG. 9. A plot of time-step against total energy of the orbit.

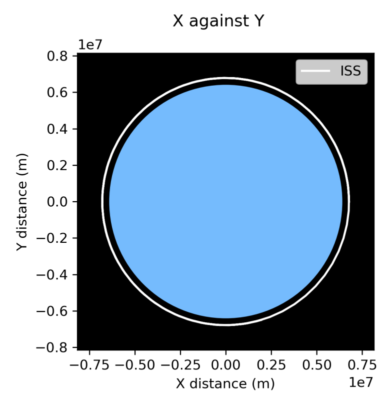

FIG. 10. The trajectory of the ISS in a circular orbit going around the Earth.

which hovers just 435km above the Earth's surface. It can be seen on FIG. 11 that the oscillatory nature of the kinetic and potential energies become imperceptible. This confirms that the closer it is the satellite is orbiting around the Earth, the more stable the orbit will be.

III. PROBLEM (B)

III.1 The problem.

The objective for this part was to send a rocket around the Moon to take a photo and then back to Earth. The computational procedure of this part is similar to that of part A, except that f_3 and f_4 needs to be changed in order to include the forces from the moon's contribution, expressed below

$$f_3 = \frac{dv_x}{dt} = -\frac{-GM_e x}{(x^2+y^2)^{3/2}} - \frac{GM_m(x-d)}{((x-d)^2+y^2)^{3/2}} \quad (16)$$

$$f_4 = \frac{dv_y}{dt} = -\frac{-GM_e x}{(x^2+y^2)^{3/2}} - \frac{GM_m y}{((x-d)^2+y^2)^{3/2}} \quad (17)$$

III.2 Results

For this part of the experiment it took a numerous amount of trial and error in order to get the exact parameters for the rocket to be gliding through in a lowest energy stable orbit. Some changes had to be made to the fields of the class to allow more variables to be added in i.e. the mass of the moon. In addition there is also an improvement to the crash condition of the orbit where this time the radius of Moon was also taken into account. Again the distance and energy graphs are plotted and shown in FIG. (13-14) below.

III.3 Discussion

It can be seen from FIG. 13 that the gravitational potential for the rocket with respect to the Earth is stronger than with respect to the Moon since the rocket has to travel very far away from the Earth before it falls into the orbit of the moon. This implies that the gravitational pull from the moon onto the rocket is significantly lower than from the Earth's. A reasonable starting point for establishing the starting height was to

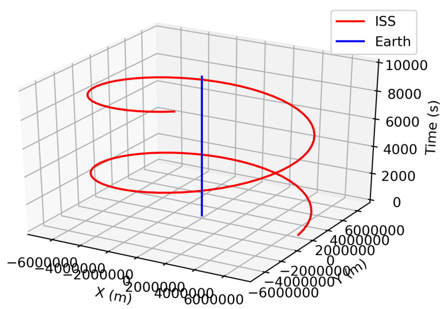

FIG. 11. The time evolution of the trajectory of the ISS in a circular orbit going around the Earth.

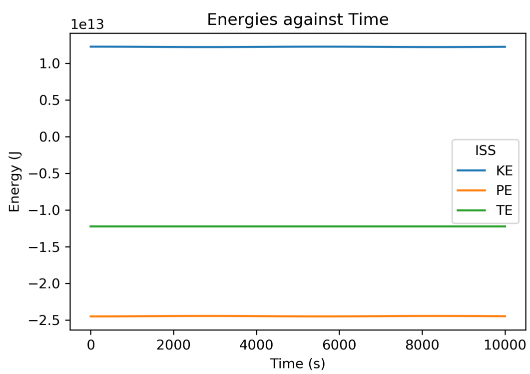

FIG. 12. The energies of the ISS orbiting around the Earth in a circular orbit.

use the range of height above the surface of the Earth given by the exercise manual, in this case it is between $6500km$ and $7000km$. Through many trial and error with a combination of running the program over looping a range of values of height and initial velocity, it was found that starting at the height of 6629km above the surface of the Earth to be optimal. Since the velocity of the rocket was only restricted to the y-direction, it was also optimal to start on the left side of the Earth, since it will then gain momentum in the y-direction by the said velocity.

By comparing the radius of the orbit with respect to the moon with the radius of the moon itself, taking their difference will give the distance of closest approach between those two. A loop was then ran for a range of velocity in between the velocity that gave the minimum energy orbit. The minimum distance between the rocket and the moon was then obtained to be 1928.1974434 metre above the moon surface at a velocity of 7677.3891000000021 m/s.

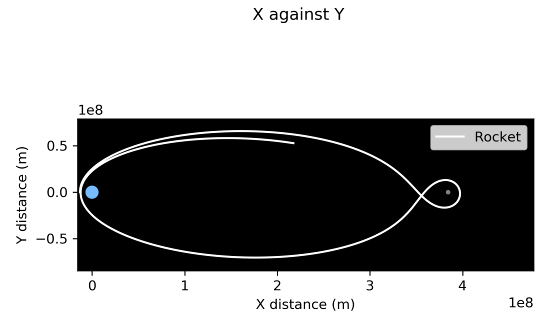

FIG. 13. The trajectory of a rocket in an orbit going around the Earth and the Moon.

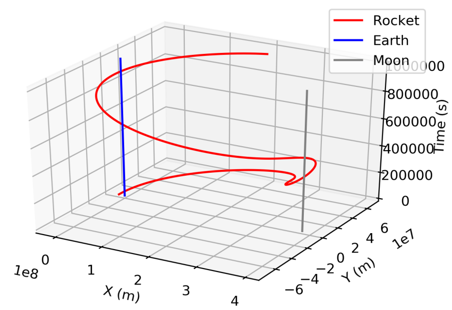

FIG. 14. The time evolution of the trajectory of a rocket in an orbit going around the Earth and the Moon.

REFERENCES

[1] S. Hanna. *Level 5 Computational Physics & Level 6 PHYS30009 Intro to Computational Physics - Lecture 6*. (December 11, 2017) **p. 1**.

[2] S. Hanna. *Level 5 Computational Physics & Level 6 PHYS30009 Intro to Computational Physics - Lecture 5*. (November 27, 2017) **p. 4**.

[3] L. Fang and Y. Chu, *From Newton's Laws to Einstein's Theory of Relativity*. Science Press. (1987)

[4] S. Hanna. *Level 5 Laboratory: Computational Physics - Exercise 4*. (2018)

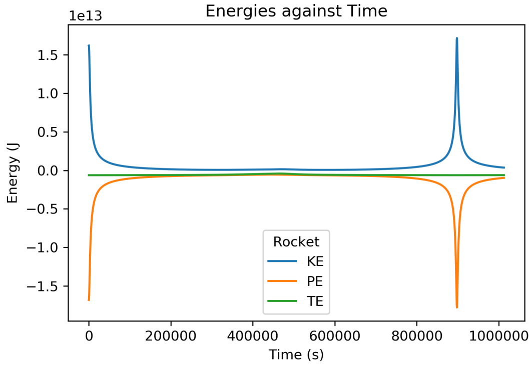

FIG. 15. The energies of a rocket orbiting around the Earth and the Moon.