# On the investigation of free-fall with fixed or varying drag

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## I. INTRODUCTION

In mathematics and physics, there exists many differential equations for which a closed form solution does not exist, meaning that they could not be solved analytically. In such cases, a numerical procedure is often sought out so that numerical analysis can be performed on these differential equation. The Euler method, first introduced by Leonard Euler in his book *Institutionum calculi integralis*[1], allows an ordinary differential equation to be solved iteratively providing that the values of initial condition.

The Euler method works by first replacing the ratio of infinitesimal differences of a differential in an ODE by a ratio of finite differences[2]

$$\frac{dy}{dx} \rightarrow \frac{\Delta y}{\Delta x} \tag{1}$$

A continuous function $y = f(x)$ is then approximated for $x \geqslant 0$, satisfying the first order differential equation[3]

$$\frac{dy}{dx} = f(x, y) \tag{2}$$

on $x > 0$, where the initial conditions is given with values $(x_0, y_0)$ such that

$$y(x_0) = y_0 \tag{3}$$

For a finite value of interval composed of $\Delta x$, its derivative is then computed incrementally and a new $y$ value is re-evaluated at each increment, summarised by

$$y_{n+1} = y_n + \Delta x f(x_n, y_n) \quad ; \quad x_{n+1} = x_n + \Delta x \tag{4}$$

where $\Delta y = \Delta x f(x_n, y_n)$, as shown on Figure 1.

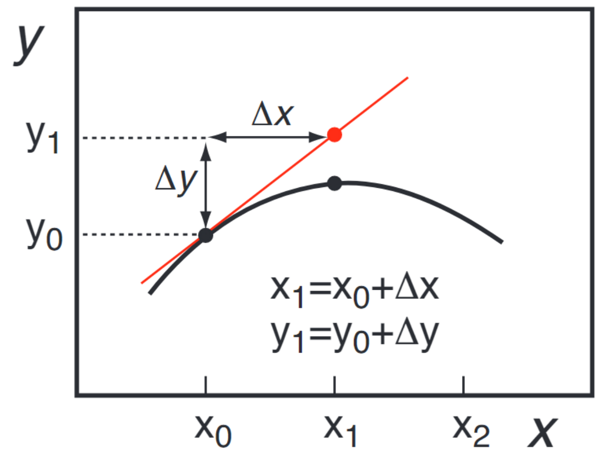


FIG. 1. FIG. 1. Computing the new $x$ and $y$ values with one iteration using the Euler method.

## II PROBLEM A

### II.1 The Problem

For an object falling in the air, its motion can be described by Newton's 2nd Law, whereby the net force on the body is equal to its weight due to gravitational acceleration plus the air resistance

$$m\mathbf{a} = \mathbf{W} + \mathbf{F}_{drag} \tag{5}$$

The whole system can then be modelled by a 2nd order differential equation separated into two 1st order differential equations, summarised by

$$m\frac{dv_y}{dt} = -mg - k|v_y|v_y \tag{6}$$

and

$$\frac{dy}{dt} = v_y \tag{7}$$

By applying Equation (4) onto equation (6) and (7), we find the iterative formula

$$v_{y,n+1} = v_{y_n} - \Delta t\left(g + \frac{k}{m}|v_{y,n}|v_{y,n}\right) \tag{8}$$

$$y_{n+1} = y_n + \Delta t v_{y,n} \tag{9}$$

$$\frac{dy}{dt} = v_y \tag{10}$$

where the air resistance term is included which is proportional to the square of the velocity acting in the opposite direction

$$\mathbf{F}_{drag} = -kv\mathbf{v} \tag{11}$$

The constant $k$ is described by

$$k = \frac{C_d \rho_0 A}{2} \tag{12}$$

with $C_d$ being the drag coefficient, $A$ being the cross sectional area of the falling body and $\rho_0$ as the air density.

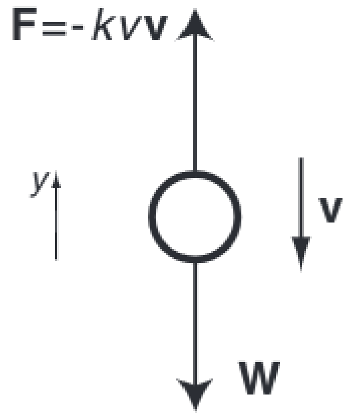


FIG. 2. The free body diagram for a free falling body showing the forces acting on it.

### II.2 Results

A set of default values were ran for part A, as shown in the table 1 below. These default values were designated to that

| Constants | Default Value |
|---|---|
| g | $9.81\text{ms}^{-1}$ |
| $\rho_0$ | $1.2\text{kgm}^{-3}$ |
| $C_d$ | 0.47 |
| A | $\pi/100\ \text{m}^2$ |
| M | 10 kg |
| dt | 1 s |
| $y_0$ | 1000 m |

TABLE I. : The default values assigned for the parameters on the program.

of a sphere with a radius of 0.1m and hence it has a surface area of $\pi/100\text{m}^2$. The drag coefficient, $C_d$, has a value of $1.2\text{kgm}^{-3}$ for a sphere and the acceleration due to gravity, g, was taken as a constant with a value of $9.81\text{ms}^{-1}$. The starting height was taken as 1km and the sphere was assumed to be released with a zero initial velocity. Every iterated value of distance, time and velocity is being appended into their own corresponding lists.

To stop the calculation when the object reaches the floor, a conditional while loop was used to keep up the iteration of the algorithm until one of the values in the distance lists is less than or equal to zero. The velocity and distance is then plotted against time and they were compared with the analytical method. Its absolute error is then calculated by taking the absolute value of the difference between both results, as shown in figure (2-5).

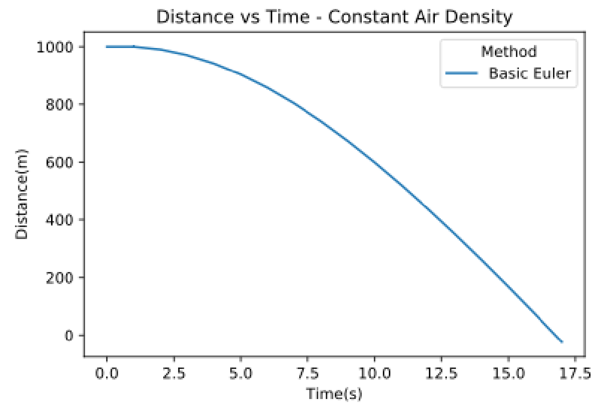


FIG. 3. The distance-time graph for a free falling sphere using the Euler and Analytical method.

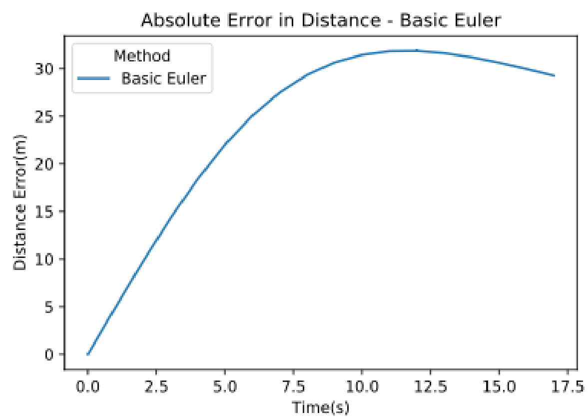


FIG. 4. The absolute error of the distance-time graph for a free falling sphere using the Euler method, when compared to the Analytical method.

### II.3 Discussion

For the basic Euler method with constant air density, it took the projectile approximately 17s to reach the floor from a height of 1000m with a final and maximum velocity of -96.7$\text{ms}^{-1}$. It was not clear whether the projectile had reached terminal velocity since the distance used was too small. For now, the errors for both velocity and distance seem large for a $\Delta t$ of 1s but it cannot be ascertained until it is compared with different step sizes, which will be explored later. Theoretically, the errors should decrease as $\Delta t$ become smaller due to the way the iterative formula of the Euler method is defined.

For a large $\Delta t$ and a small height, it is obvious that the resolution of the graph is lost, as there are not enough data point to extract any significant meaning out of the graph due to the huge amount information that is lost. $\Delta t$ is chosen as 1s here to simply check the sensibility of the graphs. In this case the data seem sensible, as a gradual negative gradient is observed in the distance-time graph and a positive gradient(increasing velocity) is observed in the velocity-time graph.

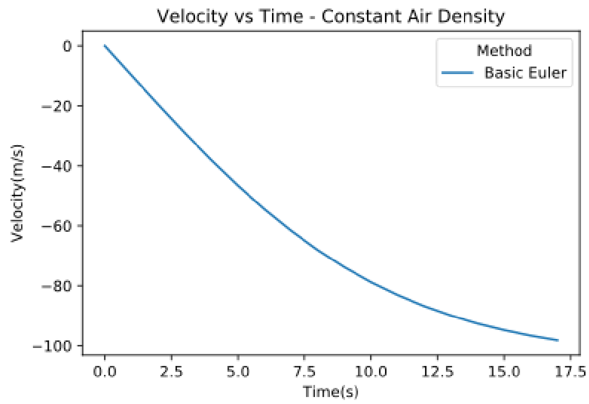


FIG. 5. The velocity-time graph for a free falling sphere using the Euler and Analytical method.

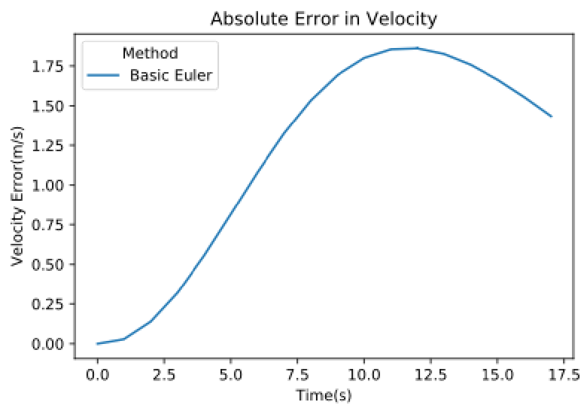


FIG. 6. The absolute error of the velocity-time graph for the Euler method, when compared to the Analytical method.

## III. PROBLEM (B)

### III.1 The problem.

The close form solution for equation (6) can be obtained analytically as

$$y = y_0 - \frac{m}{2k}\log_e\left[\cosh^2\left(\sqrt{\frac{kg}{m}}.t\right)\right] \tag{13}$$

$$v_y = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}.t\right) \tag{14}$$

again equation (4) can be applied on equation 13 & 14 so that they can be computed numerically.

### III.2 Results

The same default values were used for this case as before and the same analysis was done except this time it was not able to obtain an absolute error because there is no other method to compare to. However the precision of the method was explored with a different value of $\Delta t$ as it is believed that taking the limit of $\Delta t$ to a very small value can lead to a more precise answer. The distance-time and velocity-time graphs are shown in figure (6-7) respectively.

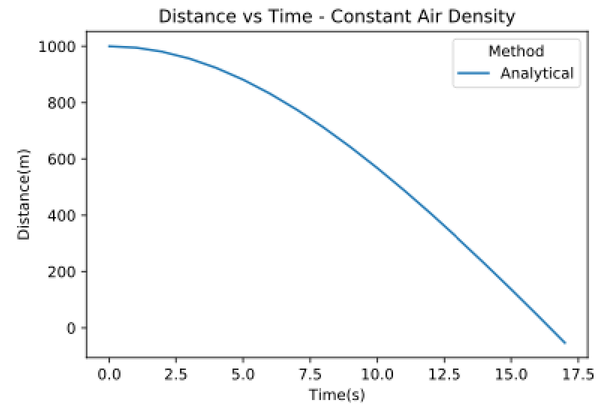


FIG. 7. The distance-time graph for a free falling sphere using the Euler and Analytical method.

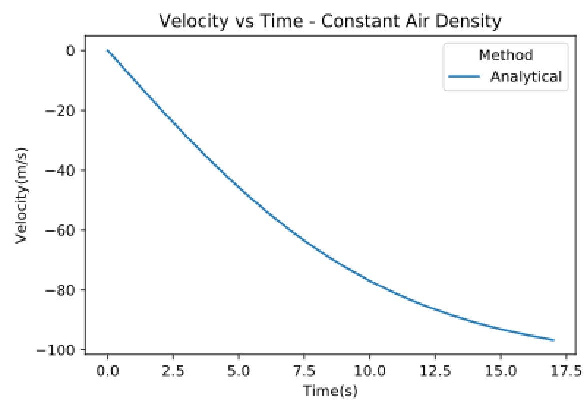


FIG. 8. The velocity-time graph for a free falling sphere using the Euler and Analytical method.

### III.3 Discussion

For the analytical method with constant air density, it took the projectile approximately 17s to reach the floor from a height of 1000m with a final and maximum velocity of -96.7ms$^{-1}$. Again its terminal velocity can only be assumed as its final velocity before it reaches the ground. Again the data seem sensible, with a gradual negative gradient being observed in the distance-time graph and a positive gradient(increasing velocity) is observed in the velocity-time graph.

## IV. PROBLEM (C)

### IV.1 The problem.

The Euler method is part (A) is now modified. By using the Euler method itself, the gradient at mid-point between each time-step is computed. The modified Euler method now re-expresses Equation (4) as

$$y_{\text{mid}} = y_n + \frac{\Delta x}{2} f(x_n, y_n) \quad ; \quad x_{\text{mid}} = x_n + \frac{\Delta x}{2} \tag{15}$$

$$y_{n+1} = y_n + \Delta x f(x_{\text{mid}}, y_{\text{mid}}) \quad ; \quad x_{n+1} = x_n + \Delta x \tag{16}$$

The modified Euler method allows a more accurate approximation of the function y itself.

### IV.2 Results

#### IV.2.1 Comparing all 3 methods with the same parameters.

The same default values are used for this part to explore errors of both basic and modified Euler method. The graphs are shown in figure (9-12)

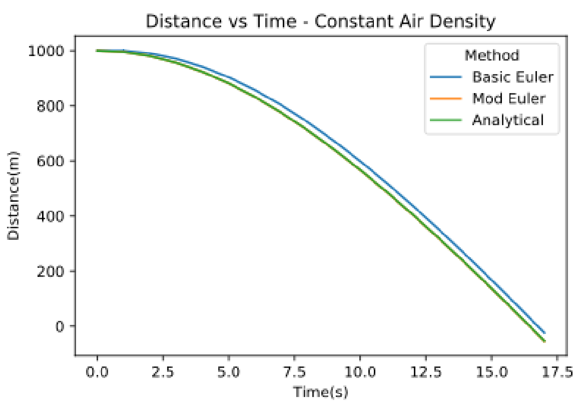


FIG. 9. The distance-time graph for a free falling sphere using the Euler, Analytical and Mod Euler method.

### IV.3 Discussion

#### IV.3.1 Comparing all 3 methods with the same parameters.

Evidently, we can tell from Figure 10 and Figure 12 that for the same default parameters used, the error from the modified Euler method is significantly lower than the error from the basic Euler method by a wide margin. This confirms our theoretical prediction from equation (15) and (16) that indeed the modified Euler method allows a more accurate approximation of the function.

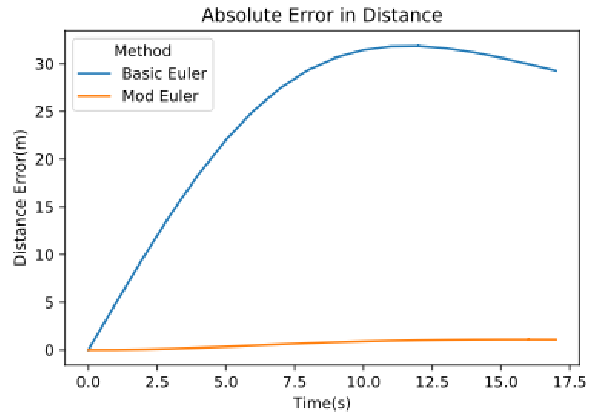


FIG. 10. The absolute error of the distance-time graph for a free falling sphere using the Euler and Mod Euler method, when compared to the Analytical method.

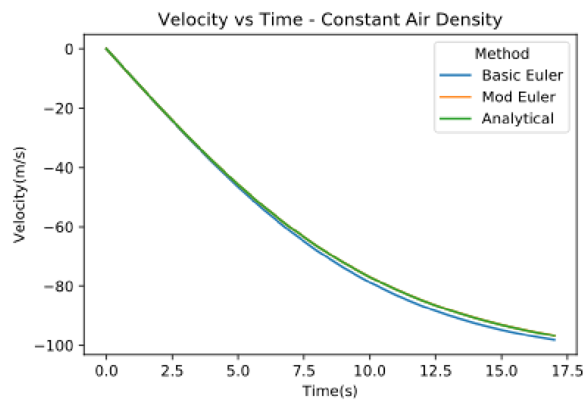


FIG. 11. The velocity-time graph for a free falling sphere using the Euler, Analytical and Mod Euler method.

## V. PROBLEM D

### V.1 THE PROBLEM

The air density is now made varying with the altitude, where the its variation is a modelled as an exponential decay, described by

$$\rho(y) = \rho_0 \exp(-y/h) \tag{17}$$

where h takes the value of 7.64km. A function was also introduced in the program such that the value of sound barrier is given for a different height value, since the speed of sound varies with respect to the altitude above sea level. Together the maximum velocity of the projectile reached at a height can be compared with the speed of sound at that height, hence evaluate on whether the projectile had broken the sound barrier. The parameters used to simulate Baumgartner's jump are shown in Table (2) below

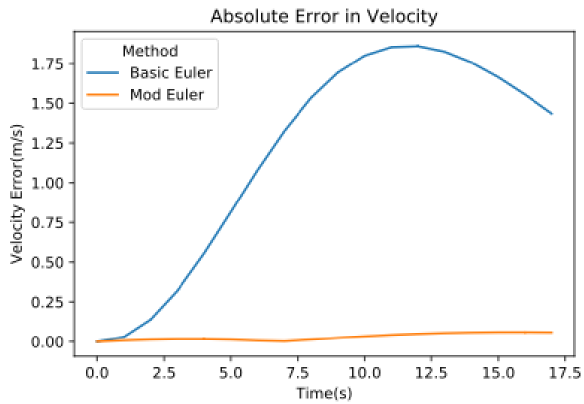


FIG. 12. The absolute error of the velocity-time graph for a free falling sphere using the Euler and Mod Euler method, when compared to the Analytical method.

| Constants | Default Value |
|---|---|
| g | $9.81ms^{-1}$ |
| $\rho_0$ | $1.2kgm^{-3}$ |
| $C_d$ | 1.2 |
| A | 0.3 m |
| M | 90 kg |
| dt | 0.1 s |
| $y_0$ | 39045 m |

TABLE II. : The values assigned for the parameters for the simulation of Baumgartner's jump.

### V.2 RESULT

Assuming Baumgartner did not use a parachute, he reaches the floor after approximately 235.8s with a velocity of -64.9ms$^{-1}$ from a height of 39045m. He attained a maximum velocity of -378.5ms$^{-1}$ at an altitude of 27185m, which has a sound barrier of approximately 294.9ms$^{-1}$. This means that he successfully broke the sound barrier.

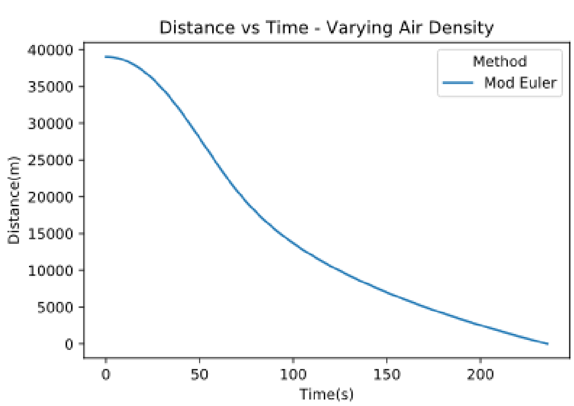


FIG. 13. The distance-time graph for Baumgartner's jump simulation using the modified Euler method.

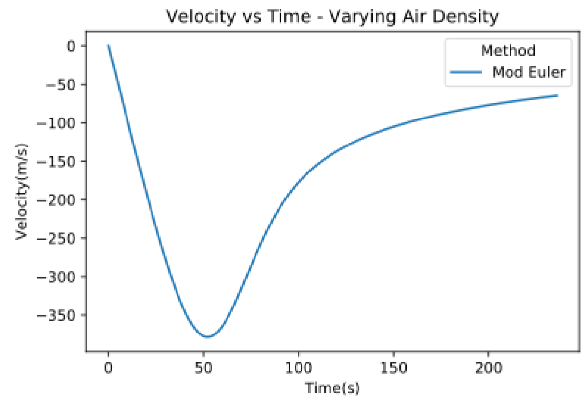


FIG. 14. The velocity-time graph for Baumgartner's jump simulation using the modified Euler method.

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[3] D. Greenspan (2014), *Numerical Solution of Ordinary Differential Equations for Classical, Relativistic and Nano Systems*. Wiley-VCH. (January 13, 2006) **p. 20-21**.