Notes on Paul Wilmott Introduces Quantitative Finance

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0.1 Chapter 1: Products and Markets: Equities, Commodities, Exchange Rates, Forwards and Futures

THE TIME VALUE OF MONEY

Two types of interest:

- Simple interest - When the interest you received is based only on the amount you initially invest
- Compound interest - When you also get interest on your interest.

Two forms of compound interest: discretely compounded and continuously compounded.

Invest \$1 in bank at a discrete interest rate r paid once per annum. At the end of year 1 bank account will contain $1 \times (1+r)$. After 2 years and n years I will have $(1+r)^2$ and $(1+r)^n$. For $r = 10\%$, for first and second year I will have \$1.10 and \$1.21.

Suppose I receive m interest payments at a rate of r/m per annum. After 1 year I'll have

$$\left(1+\frac{r}{m}\right)^m \tag{1}$$

Imagine interest payments come at increasingly frequent intervals but at an increasingly smaller interest rate. Taking limit $m \to 100$ leads to a rate of interest that is paid continuously, giving money amount in bank after one year if interest is continuously compounded

$$\begin{aligned}\left(1+\frac{r}{m}\right)^m &= e^{\ln(1+\frac{r}{m})^m} \\ &= e^{m\ln(1+\frac{r}{m})} \\ &\sim e^r\end{aligned} \tag{2}$$

After time t this amounts to

$$\left(1+\frac{r}{m}\right)^{mt} \sim e^{rt} \tag{3}$$

This can also be derived by a differential equation. For an amount $M(t)$ in the bank at time t, checking bank account at time t and slightly later $t+dt$, the amount will have increased by

$$M(t+dt) - M(t) \approx \frac{dM}{dt}dt + ... \tag{4}$$

where RHS is just a Taylor series expansion. But the interest I receive must be $\propto M(t), r,$ and dt. Hence

$$\begin{aligned}\frac{dM}{dt}dt &= rM(t)dt \\ \implies \frac{dM}{dt} &= rM(t)\end{aligned} \tag{5}$$

with solution

$$M(t) = M(0)e^{rt} \tag{6}$$

relating money value I have now to the value in the future. Conversely, if I know I'll get \$1 at time T in the future, its value at an earlier time t is $e^{-r(T-t)}$. This relates future cashflows to present by multiplying this factor e.g. for $r = 0.05$, the present value ($t = 0$) of \$1000000 to be received in two years ($T = 2$) is \$1000000 $\times e^{-0.05*2}$ = \$904837.

FIXED-INCOME SECURITIES

Two types of interest payments exist: **fixed** and **floating**. **Coupon-bearing** bonds pay out a known amount every six months or year, etc. This is the **coupon** and would often be a fixed rate of interest. At the end of your fixed term you get a final coupon and return of the **principal**, the amount on which the interest was calculated. **Interest rate swaps** are an exchange of a fixed rate of interest for a floating rate of interest. Governments and companies issue bonds as a form of borrowing. The less creditworthy the issuer, the higher the interest that they will have to pay out. Bonds are actively traded, with prices that continually fluctuate.

INFLATION PROOF BONDS

UK inflation is measured by **Retail Price Index (RPI)**. This index is

- a measure of year-on-year inflation using a 'basket' of goods and services including mortgage interest payments.
- published monthly.
- roughly speaking, the amounts of the coupon and principal are scaled with the increase in the RPI over the period from the issue of the bond to the time of the payment.

US inflation is measured by **Consumer Price Index (CPI)**.

FORWARDS AND FUTURES

A **forward contract** is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price.

- No money changes hands until the **delivery date** or **maturity** of the contract.
- It's an absolute obligation to buy the asset at the delivery date.
- The asset can be a stock, a commodity or a currency.
- The **delivery price** is the amount that's paid for the asset at the delivery date.
  - Set at the time the forward contract is entered into.
  - At an amount that gives the forward contract a value of zero initially.
  - As maturity approaches the particular forward contract value we hold will change in value, from initially zero to the difference between the underlying asset and the delivery price at maturity.
- The **forward price** of different maturities are the delivery prices for forward contracts of the quoted maturities, should we enter into such a contract now.

A **futures contract** is very similar to forward contract.

- Usually traded through an exchange.
- Profit or loss from futures position is calculated everyday and the value change is paid from one party to the other, hence there is a gradual payment of funds from initiation until maturity.
- Because you settle the change in value on a daily basis, the value of a futures contract at any time during its life is zero.
- Prices vary day to day, but at maturity must be the same as the asset you're buying.
- Provided interest rates are known in advance, forward prices and futures prices of the same maturity must be identical.

Forwards and futures have two main uses:

- Speculation
  - In believing market will rise, you can benefit by entering into a forward/futures contract.
  - Money will exchange hands at maturity/every day in your favour.
- Hedging i.e. avoidance of risk

- If you are expecting to get paid in yen in six months' time but your expenses are all in dollars, you can enter into a futures contract to guarantee an exchange rate for your yen income amount.
- You're locked in this dollar/yen exchange rate. The lack of exposure to fluctuations means you won't benefit if yen appreciates.

The **no-arbitrage** principle: Consider a forward contract that obliges us to pay $\$F$ at time T to receive the underlying asset. The **spot price** $\$S(t)$ is the asset price at present time t for which we could get immediate delivery of the asset. At maturity, we'll pay $ and receive the asset, then worth $\$S(T)$, a value that remains unknown until time T which determines the profit/loss amount $S(T) - F$. By entering into a special portfolio of trades now we can eliminate all randomness in the future.

1. Enter into the forward contract, which costs nothing up front but exposes us to the uncertainty in the asset value at maturity.
2. Simultaneously sell the asset a.k.a **going short** i.e. when you sell something you don't own with some timing restrictions.
3. Still with net position of zero, we now have $S(t)$ amount of cash from the sale of asset, a forward contract, and a short asset position $-S(t)$. Put cash in bank to receive interest.
4. At maturity we pay F and receive asset $S(T)$. The bank account amount with interest is now $S(t)e^{r(T-t)}$, with a net position at maturity of $S(t)e^{r(T-t)} - F$.
5. The no-arbitrage principle states that a portfolio started with zero worth end up with a predictable amount, which should also be zero. Hence we've a relationship between spot price and forward price.

$$F = S(t)e^{r(T-t)} \tag{7}$$

Table 1.1 Cashflows in a hedged portfolio of asset and forward.

Holding	Worth today (t)	Worth at maturity (T)
Forward	0	$S(T) - F$
$-$Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Total	0	$S(t)e^{r(T-t)} - F$