Notes on Paul Wilmott Introduces Quantitative Finance

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0.1 Chapter 1: Products and Markets: Equities, Commodities, Exchange Rates, Forwards and Futures

THE TIME VALUE OF MONEY

Two types of interest:

- Simple interest - When the interest you received is based only on the amount you initially invest
- Compound interest - When you also get interest on your interest.

Two forms of compound interest: discretely compounded and continuously compounded.

Invest \$1 in bank at a discrete interest rate r paid once per annum. At the end of year 1 bank account will contain $1 \times (1 + r)$. After 2 years and n years I will have $(1 + r)^2$ and $(1 + r)^n$. For $r = 10\%$, for first and second year I will have \$1.10 and \$1.21.

Suppose I receive m interest payments at a rate of r/m per annum. After 1 year I'll have

$$\left(1+\frac{r}{m}\right)^m \tag{1}$$

Imagine interest payments come at increasingly frequent intervals but at an increasingly smaller interest rate. Taking limit $m \to 100$ leads to a rate of interest that is paid continuously, giving money amount in bank after one year if interest is continuously compounded

$$\begin{aligned}\left(1+\frac{r}{m}\right)^m &= e^{\ln(1+\frac{r}{m})^m} \\ &= e^{m\ln(1+\frac{r}{m})} \\ &\sim e^r\end{aligned} \tag{2}$$

After time t this amounts to

$$\left(1+\frac{r}{m}\right)^{mt} \sim e^{rt} \tag{3}$$

This can also be derived by a differential equation. For an amount $M(t)$ in the bank at time t, checking bank account at time t and slightly later $t + dt$, the amount will have increased by

$$M(t+dt) - M(t) \approx \frac{dM}{dt}dt + ... \tag{4}$$

where RHS is just a Taylor series expansion. But the interest I receive must be $\propto M(t), r$, and dt. Hence

$$\begin{aligned}\frac{dM}{dt}dt &= rM(t)dt \\ \implies \frac{dM}{dt} &= rM(t)\end{aligned} \tag{5}$$

with solution

$$M(t) = M(0)e^{rt} \tag{6}$$

relating money value I have now to the value in the future. Conversely, if I know I'll get \$1 at time T in the future, its value at an earlier time t is $e^{-r(T-t)}$. This relates future cashflows to present by multiplying this factor e.g. for $r = 0.05$, the present value ($t = 0$) of \$1000000 to be received in two years ($T = 2$) is \$1000000 $\times e^{-0.05*2}$ = \$904837.

FIXED-INCOME SECURITIES

Two types of interest payments exist: **fixed** and **floating**. **Coupon-bearing** bonds pay out a known amount every six months or year, etc. This is the **coupon** and would often be a fixed rate of interest. At the end of your fixed term you get a final coupon and return of the **principal**, the amount on which the interest was calculated. **Interest rate swaps** are an exchange of a fixed rate of interest for a floating rate of interest. Governments and companies issue bonds as a form of borrowing. The less creditworthy the issuer, the higher the interest that they will have to pay out. Bonds are actively traded, with prices that continually fluctuate.

<u>INFLATION PROOF BONDS</u>

UK inflation is measured by **Retail Price Index (RPI)**. This index is

- a measure of year-on-year inflation using a 'basket' of goods and services including mortgage interest payments.
- published monthly.
- roughly speaking, the amounts of the coupon and principal are scaled with the increase in the RPI over the period from the issue of the bond to the time of the payment.

US inflation is measured by **Consumer Price Index (CPI)**.

<u>FORWARDS AND FUTURES</u>

A **forward contract** is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price.

- No money changes hands until the **delivery date** or **maturity** of the contract.
- It's an absolute obligation to buy the asset at the delivery date.
- The asset can be a stock, a commodity or a currency.
- The **delivery price** is the amount that's paid for the asset at the delivery date.
  - Set at the time the forward contract is entered into.
  - At an amount that gives the forward contract a value of zero initially.
  - As maturity approaches the particular forward contract value we hold will change in value, from initially zero to the difference between the underlying asset and the delivery price at maturity.
- The **forward price** of different maturities are the delivery prices for forward contracts of the quoted maturities, should we enter into such a contract now.

A **futures contract** is very similar to forward contract.

- Usually traded through an exchange, are very liquid instruments and have lots of rules and regulations surrounding them.
- Profit or loss from futures position is calculated everyday and the value change is paid from one party to the other, hence there is a gradual payment of funds from initiation until maturity.
- Because you settle the change in value on a daily basis, the value of a futures contract at any time during its life is zero.
- Prices vary day to day, but at maturity must be the same as the asset you're buying.
- Provided interest rates are known in advance, forward prices and futures prices of the same maturity must be identical.

Forwards and futures have two main uses:

- Speculation
  - In believing market will rise, you can benefit by entering into a forward/futures contract.
  - Money will exchange hands at maturity/every day in your favour.
- Hedging i.e. avoidance of risk

– If you are expecting to get paid in yen in six months' time but your expenses are all in dollars, you can enter into a futures contract to guarantee an exchange rate for your yen income amount.

– You're locked in this dollar/yen exchange rate. The lack of exposure to fluctuations means you won't benefit if yen appreciates.

The **no-arbitrage** principle: Consider a forward contract that obliges us to pay $\$F$ at time T to receive the underlying asset. The **spot price** $\$S(t)$ is the asset price at present time t for which we could get immediate delivery of the asset. At maturity, we'll pay \$ and receive the asset, then worth $\$S(T)$, a value that remains unknown until time T which determines the profit/loss amount $S(T) - F$. By entering into a special portfolio of trades now we can eliminate all randomness in the future.

1. Enter into the forward contract, which costs nothing up front but exposes us to the uncertainty in the asset value at maturity.

2. Simultaneously sell the asset a.k.a **going short** i.e. when you sell something you don't own with some timing restrictions.

3. Still with net position of zero, we now have $S(t)$ amount of cash from the sale of asset, a forward contract, and a short asset position $-S(t)$. Put cash in bank to receive interest.

4. At maturity we pay F and receive asset $S(T)$. The bank account amount with interest is now $S(t)e^{r(T-t)}$, with a net position at maturity of $S(t)e^{r(T-t)} - F$.

Table 1.1 Cashflows in a hedged portfolio of asset and forward.

Holding	Worth today (t)	Worth at maturity (T)
Forward	0	$S(T) - F$
$-$Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Total	0	$S(t)e^{r(T-t)} - F$

5. The no-arbitrage principle states that a portfolio started with zero worth end up with a predictable amount, which should also be zero. Hence we've a relationship between spot price and forward price.

$$F = S(t)e^{r(T-t)} \tag{7}$$

6. If $F < S(t)e^{r(T-t)}$, a riskless arbitrage opportunity can be exploited by entering into the same deals.

7. At maturity you will have $S(t)e^{r(T-t)}$ in the bank, a short asset and a long forward. The asset position cancels when you hand over the amount F, leaving you with a profit of $S(t)e^{r(T-t)} - F$.

8. If $F > S(t)e^{r(T-t)}$, simply enter into the opposite positions i.e. going short the forward in order to make a riskless profit.

MORE ABOUT FUTURES

The nature of futures contracts:

Available assets

A futures contract will specify

- the asset which is being invested in
- the quantity of asset that must be delivered
- the quality of the commodities, usually comes in a variety of grades i.e. oil, sugar, orange juice, wheat, etc. futures contracts lay down rules for precisely what grade of oil, sugar, etc. may be delivered.

Delivery and settlement

There may be some leeway in the precise delivery date

- Most futures contracts are closed out before delivery, with the trader taking the opposite position before maturity.
- If position is not closed, then asset is delivered.
- When the asset is another financial contract settlement is usually made in cash.

Margin

- **Marking to market** - The changes in value of futures contracts are settled each day.
- Exchanges insist on traders depositing a sum of money in a **margin account** to cover changes in their positions value.
- As the position is marked to market daily, money is deposited or withdrawn from this margin account.

Two types of margin: **Initial margin**, **Maintenance margin**
The initial amount is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks, etc.) must be deposited.

Commodity futures
Futures on commodities don't necessarily obey the no-arbitrage law due to storage. In practice the future price will be higher than the theoretical no-storage-cost amount since the holder of the futures contract must compensate the holder of the commodity for his storage costs. Often the people holding the commodity benefit from it in some way. The benefit from holding the commodity is commonly measured in terms of the **convenience yield** c:

$$F = S(t)e^{(r+s-c)(T-t)} \tag{8}$$

Whenever $F < S(t)e^{r(T-t)}$ the market is said to be in **backwardation**, otherwise if $F > S(t)e^{r(T-t)}$ the market is in **contango**.

FX futures
Modifying the result from no-arbitrage to allow for interest received on the foreign currency r_f, we get

$$F = S(T)e^{(r-r_f)(T-t)}. \tag{9}$$

Index futures
Futures contracts on stock indices are settled in cash, with dividends playing a similar role to that of a foreign interest rate on FX futures. So

$$F = S(t)e^{(r-q)(T-t)} \tag{10}$$

0.2 Chapter 2: Derivatives

OPTIONS
The holder of future or forward contracts is obliged to trade at the maturity of the contract, unless the position is closed before maturity. Otherwise, the holder must take possession of the asset of the contract e.g commodity, currency regardless of whether its price has risen or fallen. The simplest **option** gives the holder the *right* to trade in the future at a previously agreed price but takes away the obligation. Specifically, a **call option** is the right to buy an asset for an agreed amount at a specified time in the future.
Example: A call option on Microsoft stock

- The holder has the right to purchase one Microsoft stock for \$25 in one months's time, with \$24.5 current price.
- The price \$25 is called the **exercise/strike price**.

- The **expiry** or **expiration date** is the date on which we must **exercise** our option, should we choose to.
- The **underlying asset** is the stock the option is based on.

We would exercise the option at expiry if the stock S is above the strike E and not if it's below. At expiry it's worth

$$\max(S - E, 0), \tag{11}$$

where the function of the underlying asset is called the **payoff function**, with 'max' representing optionality.

A **put option** is the right to *sell* a particular asset for an agreed amount a a specified time in the future. The holder of a put option wants the stock price to fall so that he can sell the asset for more than it's worth. The payoff function for a put option is

$$\max(E - S, 0), \tag{12}$$

with option only being exercised if the stock falls below the strike price.
The higher the strike the lower the value of a call option but the higher the value of the puts. Since the call allows you to buy the underlying for the strike, so that the lower the strike price the more this right is worth. The opposite is true for a put since it allows you to sell the underlying for the strike price. Also, the longer the time to maturity, the higher the value of the call. As the time to expiry decrease, as there is less and less time for the underlying to move, so the option value must converge to the payoff function.
Calls and puts have a non-linear dependence on they underlying asset. This contrasts with futures which have a linear dependence on the underlying. Calls and puts are the two simplest forms of options and are oftenr referred to as **vanilla**.

DEFINITION OF COMMON TERMS

- **Premium** - The amount paid for the contract initially.
- **Underlying (asset)** - The financial instrument on which the option value depends. Stocks, commodities, currencies and indices are going to be denoted by S. The option payoff is defined as some function of the underlying asset at expiry.
- **Strike (price)** or **exercise price** - The amount for which the underlying can be bought (call) or sold (put). This will be denoted by E. This definition only really applies to the simple calls and puts. For more complicated contracts, this definition is extended.
- **Expiration (date)** or **expiry (date)** - Date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights, denoted by T.
- **Intrinsic value** - The payoff that would be received if the underlying is at its current level when the option expires.
- **Time value** - Any value that the option has above its intrinsic value. The uncertainty surrounding the future value of the underlying asset means that the option value is generally different from the intrinsic value.
- **In the money** - An option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike.
- **Out of the money** - An option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike.
- **At the money** - A call or put with a strike that is close to the current asset level.
- **Long position** - A positive amount of quantity, or a positive exposure to a quantity.
- **Short position** - A negative amount of a quantity, or a negative exposure to a quantity. Many assets can be sold short, with some constraints on the length of time before they must be bought back.

PAYOFF DIAGRAMS

A **payoff diagram** plots the value of an option at expiry as a function of the underlying. At expiry the option is worth a known amount. For a call and put option the contract is worth max(S-E, 0) and max(E-S, 0) respectively, represented by the bold lines below.

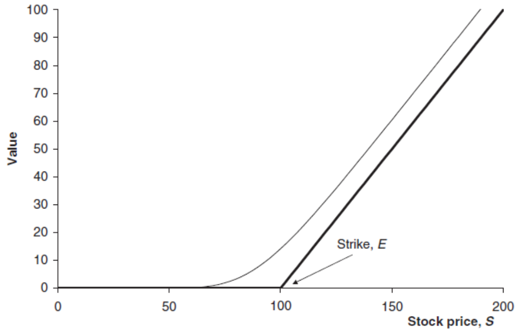

Figure 2.5 Payoff diagram for a call option.

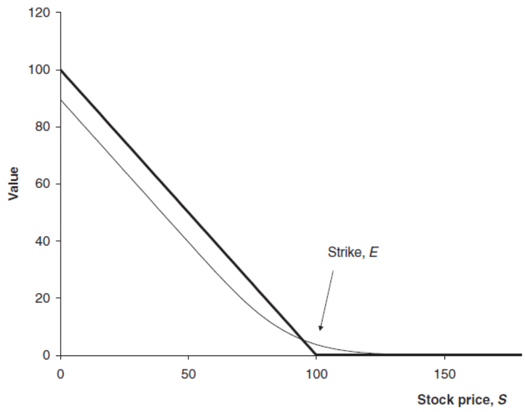

Figure 2.8 Payoff diagram for a put option.

Other representations of value

The payoff diagrams above only shows the money worth of your option contract at expiry. It makes no allowance for how much premium you had to pay for the option. In a **profit diagram** for a call option, we adjust for the original cost of the option by subtracting from the payoff the premium originally paid for the call option.

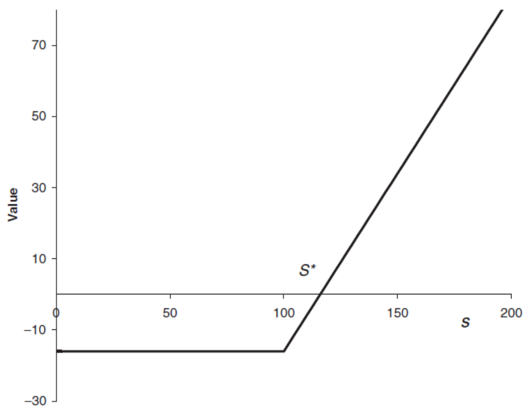

Figure 2.11 Profit diagram for a call option.

This figure is helpful because it shows how far into the money the asset must be at expiry before the option becomes profitable. Asset value S^* is the point which divides profit from loss; if the asset at expiry is above S^* then the contract has made a profit and vice versa. The profit diagram takes no account of the time value of money. The premium is paid up front but the payoff, if any, is only received at expiry. To be consistent one should either discount the payoff by multiplying by $r^{-r(T-t)}$ to value everything at the present, or multiply the premium by $e^{r(T-t)}$ to value all cashflows at expiry.

WRITING OPTIONS

The **writer** of an option is the person who promises to deliver the underlying asset, if the option is a call, or buy it, if the option is a put. The writer is the person who receives the premium, and is liable if the option is exercised. The option holder can sell the option on to someone else to close his position. The purchaser of the option hands over a premium in return for special rights, and uncertain outcome. The writer receives a guaranteed payment up front, but then has obligations in the future.

MARGIN

Buying an option

- Downside: initial premium
- Upside: may be unlimited

Writing an option

- Downside: could be huge
- Upside: Limited

To cover the risk of default in the event of an unfavourable outcome, the **clearing houses** that register and settle options insist on the deposit of a margin by the writers of options.

MARKET CONVENTIONS

Often simpler option contracts bought and sold through exchanges are standardised to follow conventions. Simple calls and puts come in series, referring to the strike and expiry dates. Typically a stock has three choices of expiries trading at any time. Having standardised contracts traded through an exchange promotes liquidity of the instruments. **Over the counter (OTC)** contracts are an agreement between two parties, often brough together by an intermediary. The agreed terms are flexible, without needing to follow any conventions.

THE VALUE OF THE OPTION BEFORE EXPIRY

How much is an option contract worth *now*, before expiry? How much would you pay for a contract, a piece of paper, giving you rights in the future? What is clear is that the contract value before expiry will depend on how high the asset price is today and how long there is before expiry. The longer the time to expiry, the more time there is for the asset to rise or fall.

Let $V(S,t)$ be a function of the value of the underlying asset S at time t which represents the value of the option contract. At expiry date $t = T$, the value of the contract at expiry function is just the payoff function, which we know from before. For a call option it's

$$V(S,T) = \max(S - E, 0). \tag{13}$$

The fine lines in Figure 2.5 and 2.8 are the values of the contracts $V(S,t)$ at *some time before expiry*, plotted against S.

FACTORS AFFECTING DERIVATIVE PRICES

The underlying value asset S and time to expiry t are **variables** of the options price. The interest rate and strike price are examples of **parameters** of the options price. The interest rate affects the option value via the time value of money since the payoff is received in the future. The higher the strike in a call, the lower the value of the call. The **volatility** is an important parameter which impacts an option's value. It is a measure of fluctuation in the asset price i.e. a measure of randomness. The technical definition of volatility is the 'annualized standard deviation of the asset returns'.

SPECULATION AND GEARING

A dramatic move in the underlying that leads to an option expires in the money may lead to a large profit relative the amount of investment.

Example: Today's date is 14th April and the price of Wilmott Inc. stock is \$666. The cost of a 680 call option with expiry 22nd August is \$39. I expect the stock to rise significantly between now and August, how can I profit if I am right?
Buy the stock: Suppose I buy the stock for \$666. And suppose that by the middle of August the stock has risen to \$730. I will have made a profit of \$64 per stock. More importantly my investment will have risen by

$$\frac{730-666}{666} \times 100 = 9.6\%. \tag{14}$$

Buy the call: If I buy the call option for \$39, then at expiry I can exercise the call, paying \$680 to receive something worth \$730. I have paid \$39 and I get back \$50. This is a profit of \$11 per option, but in percentage terms I have made

$$\frac{\text{value of asset at expiry} - \text{strike} - \text{cost of call}}{\text{cost of call}} \times 100 = \frac{730-680-39}{39} \times 100 = 28\%. \tag{15}$$

This is an example of **gearing** or **leverage**. The out-of-the-money option has a high gearing, a possible high payoff for a small investment. The downside of this leverage is that the call option is more likely than not to expire completely worthless and you will lose all of your investment. If Wilmott Inc. remains at \$666 then the stock investment has the same value but the call option experiences a 100% loss.

For highly leveraged contracts, the buyer is very likely to lose but at the risk of only a small amount. But the writer is risking a large loss in order to make a probable small profit. The writer is likely to think twice about such a deal unless he can offset his risk by buying other contracts. This offsetting of risk by buying other related contracts is called **hedging**.

EARLY EXERCISE

The simple options described above are examples of **European options** because exercise is only permitted at expiry. Some contracts allow the holder to exercise at any time before expiry, and these are called **American options**. American options give the holder more rights than their European equivalent and can therefore be more valuable, and they can never be less valuable. The main point of interest with American-style contracts is deciding when to exercise. **Bermudan options** allow exercise on specified dates, or in specified periods.

PUT-CALL PARITY

Imagine buying a European call option with a strike of E and an expiry of T and writing a European put option with the same strike and expiry. With a present date of t, the payoff you receive at T for the call and put will look like the lines in first and second plot of Figure 2.14 respectively. The payoff for the put is negative, since writing the option leads to liability for the payoff. The portfolio payoff for the two options is the sum of individual payoffs i.e.

$$\max(S(T)-E,0) - \max(E-S(T),0) = S(T)-E, \tag{16}$$

If I buy the asset today it will cost me $S(t)$ and be worth $S(T)$ at expiry. It's unclear what the value of $S(T)$ will be but to guarantee to get that amount you'd have to buy the asset. To lock in a payment of E at time T involves a cash flow of $Ee^{-r(T-t)}$ at time t. The conclusion is that the portfolio of a long call and a short put gives exactly thet same payoff as a long asset, short cash position. The equality of these cashflows is independent of the future behaviour of the stock and is model independent:

$$C - P = S - Ee^{-r(T-t)}, \tag{17}$$

where C and P are today's values of the call and put respectively. This relationship holds at any time up to expiry and is known as **put-call parity**. For European options, longing a call and shorting a put with the same strike is equivalent to longing a forward contract with a forward price equivalent to the options' strike price. If this relationship did not hold for whatever reason there would be riskless arbitrage opportunities to be exploited to make money.

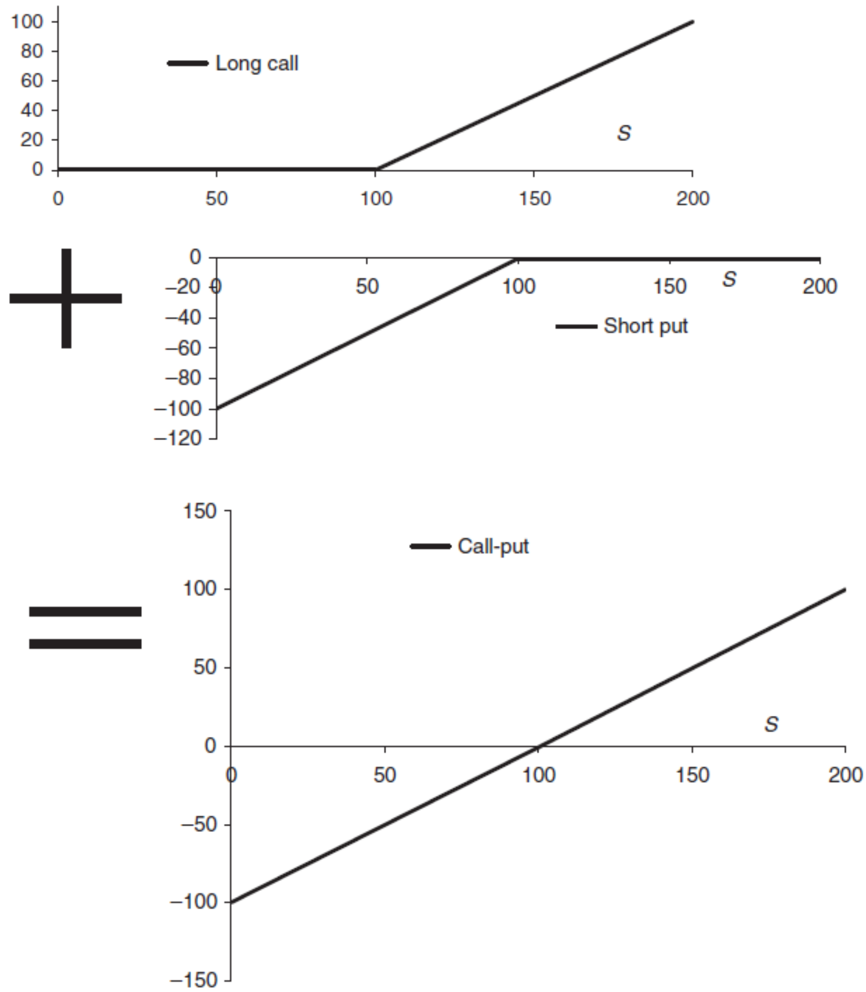

Figure 2.14 Schematic diagram showing put-call parity.

Table 2.1 Cashflows in a hedged portfolio of options and asset.

Holding	Worth today (t)	Worth at expiry (T)
Call	C	$\max(S(T) - E, 0)$
−Put	$-P$	$-\max(E - S(T), 0)$
−Stock	$-S(t)$	$-S(T)$
Cash	$Ee^{-r(T-t)}$	E
Total	$C - P - S(t) + Ee^{-r(T-t)}$	0

Table 2.1 shows the cashflows in the perfectly hedged portfolio. In this table I have set up the cashflows to have a guaranteed value of zero at expiry.

BINARIES OR DIGITALS

The **binary** or **digital options** have a payoff at expiry that is discontinuous in the underlying asset price.