Notes on Paul Wilmott Introduces Quantitative Finance

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0.1 Chapter 1: Products and Markets: Equities, Commodities, Exchange Rates, Forwards and Futures

THE TIME VALUE OF MONEY

Two types of interest:

- Simple interest - When the interest you received is based only on the amount you initially invest
- Compound interest - When you also get interest on your interest.

Two forms of compound interest: discretely compounded and continuously compounded.

Invest \$1 in bank at a discrete interest rate r paid once per annum. At the end of year 1 bank account will contain $1 \times (1+r)$. After 2 years and n years I will have $(1+r)^2$ and $(1+r)^n$. For $r = 10\%$, for first and second year I will have \$1.10 and \$1.21.

Suppose I receive m interest payments at a rate of r/m per annum. After 1 year I'll have

$$\left(1+\frac{r}{m}\right)^m \tag{1}$$

Imagine interest payments come at increasingly frequent intervals but at an increasingly smaller interest rate. Taking limit $m \to 100$ leads to a rate of interest that is paid continuously, giving money amount in bank after one year if interest is continuously compounded

$$\begin{aligned}\left(1+\frac{r}{m}\right)^m &= e^{\ln(1+\frac{r}{m})^m} \\ &= e^{m\ln(1+\frac{r}{m})} \\ &\sim e^r\end{aligned} \tag{2}$$

After time t this amounts to

$$\left(1+\frac{r}{m}\right)^{mt} \sim e^{rt} \tag{3}$$

This can also be derived by a differential equation. For an amount $M(t)$ in the bank at time t, checking bank account at time t and slightly later $t + dt$, the amount will have increased by

$$M(t+dt) - M(t) \approx \frac{dM}{dt}dt + ... \tag{4}$$

where RHS is just a Taylor series expansion. But the interest I receive must be $\propto M(t), r$, and dt. Hence

$$\begin{aligned}\frac{dM}{dt}dt &= rM(t)dt \\ \implies \frac{dM}{dt} &= rM(t)\end{aligned} \tag{5}$$

with solution

$$M(t) = M(0)e^{rt} \tag{6}$$

relating money value I have now to the value in the future. Conversely, if I know I'll get \$1 at time T in the future, its value at an earlier time t is $e^{-r(T-t)}$. This relates future cashflows to present by multiplying this factor e.g. for $r = 0.05$, the present value ($t = 0$) of \$1000000 to be received in two years ($T = 2$) is \$1000000 $\times e^{-0.05*2}$ = \$904837.

FIXED-INCOME SECURITIES

Two types of interest payments exist: **fixed** and **floating**. **Coupon-bearing** bonds pay out a known amount every six months or year, etc. This is the **coupon** and would often be a fixed rate of interest. At the end of your fixed term you get a final coupon and return of the **principal**, the amount on which the interest was calculated. **Interest rate swaps** are an exchange of a fixed rate of interest for a floating rate of interest. Governments and companies issue bonds as a form of borrowing. The less creditworthy the issuer, the higher the interest that they will have to pay out. Bonds are actively traded, with prices that continually fluctuate.

INFLATION PROOF BONDS

UK inflation is measured by **Retail Price Index (RPI)**. This index is

- a measure of year-on-year inflation using a 'basket' of goods and services including mortgage interest payments.
- published monthly.
- roughly speaking, the amounts of the coupon and principal are scaled with the increase in the RPI over the period from the issue of the bond to the time of the payment.

US inflation is measured by **Consumer Price Index (CPI)**.

FORWARDS AND FUTURES

A **forward contract** is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price.

- No money changes hands until the **delivery date** or **maturity** of the contract.
- It's an absolute obligation to buy the asset at the delivery date.
- The asset can be a stock, a commodity or a currency.
- The **delivery price** is the amount that's paid for the asset at the delivery date.
  - Set at the time the forward contract is entered into.
  - At an amount that gives the forward contract a value of zero initially.
  - As maturity approaches the particular forward contract value we hold will change in value, from initially zero to the difference between the underlying asset and the delivery price at maturity.
- The **forward price** of different maturities are the delivery prices for forward contracts of the quoted maturities, should we enter into such a contract now.

A **futures contract** is very similar to forward contract. Futures are derivative financial contracts that obligate the parties to transact an asset at a predetermined future date and price. The buyer must purchase or the seller must sell the underlying asset at the set price, regardless of the current market price at the expiration date.

- Usually traded through an exchange, are very liquid instruments and have lots of rules and regulations surrounding them.
- Profit or loss from futures position is calculated everyday and the value change is paid from one party to the other, hence there is a gradual payment of funds from initiation until maturity.
- Because you settle the change in value on a daily basis, the value of a futures contract at any time during its life is zero.
- Prices vary day to day, but at maturity must be the same as the asset you're buying.
- Provided interest rates are known in advance, forward prices and futures prices of the same maturity must be identical.

Forwards and futures have two main uses:

- Speculation
  - In believing market will rise, you can benefit by entering into a forward/futures contract.

  - Money will exchange hands at maturity/every day in your favour.

- Hedging i.e. avoidance of risk
  - If you are expecting to get paid in yen in six months' time but your expenses are all in dollars, you can enter into a futures contract to guarantee an exchange rate for your yen income amount.
  - You're locked in this dollar/yen exchange rate. The lack of exposure to fluctuations means you won't benefit if yen appreciates.

The **no-arbitrage** principle: Consider a forward contract that obliges us to pay \$$F$ at time T to receive the underlying asset. The **spot price** \$$S(t)$ is the asset price at present time t for which we could get immediate delivery of the asset. At maturity, we'll pay \$ and receive the asset, then worth \$$S(T)$, a value that remains unknown until time T which determines the profit/loss amount $S(T) - F$. By entering into a special portfolio of trades now we can eliminate all randomness in the future.

1. Enter into the forward contract, which costs nothing up front but exposes us to the uncertainty in the asset value at maturity.
2. Simultaneously sell the asset a.k.a **going short** i.e. when you sell something you don't own with some timing restrictions.
3. Still with net position of zero, we now have $S(t)$ amount of cash from the sale of asset, a forward contract, and a short asset position $-S(t)$. Put cash in bank to receive interest.
4. At maturity we pay F and receive asset $S(T)$. The bank account amount with interest is now $S(t)e^{r(T-t)}$, with a net position at maturity of $S(t)e^{r(T-t)} - F$.

Table 1.1 Cashflows in a hedged portfolio of asset and forward.

Holding	Worth today (t)	Worth at maturity (T)
Forward	0	$S(T) - F$
$-$Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Total	0	$S(t)e^{r(T-t)} - F$

5. The no-arbitrage principle states that a portfolio started with zero worth end up with a predictable amount, which should also be zero. Hence we've a relationship between spot price and forward price.

$$F = S(t)e^{r(T-t)} \tag{7}$$

6. If $F < S(t)e^{r(T-t)}$, a riskless arbitrage opportunity can be exploited by entering into the same deals.
7. At maturity you will have $S(t)e^{r(T-t)}$ in the bank, a short asset and a long forward. The asset position cancels when you hand over the amount F, leaving you with a profit of $S(t)e^{r(T-t)} - F$.
8. If $F > S(t)e^{r(T-t)}$, simply enter into the opposite positions i.e. going short the forward in order to make a riskless profit.

MORE ABOUT FUTURES

The nature of futures contracts:

Available assets

A futures contract will specify

- the asset which is being invested in
- the quantity of asset that must be delivered
- the quality of the commodities, usually comes in a variety of grades i.e. oil, sugar, orange juice, wheat, etc. futures contracts lay down rules for precisely what grade of oil, sugar, etc. may be delivered.

Delivery and settlement
There may be some leeway in the precise delivery date

- Most futures contracts are closed out before delivery, with the trader taking the opposite position before maturity.
- If position is not closed, then asset is delivered.
- When the asset is another financial contract settlement is usually made in cash.

Margin

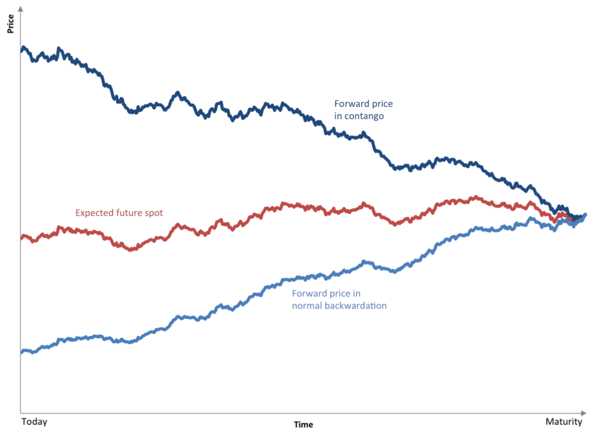
- **Marking to market** - The changes in value of futures contracts are settled each day.
- Exchanges insist on traders depositing a sum of money in a **margin account** to cover changes in their positions value.
- As the position is marked to market daily, money is deposited or withdrawn from this margin account.

Two types of margin: **Initial margin**, **Maintenance margin**
The initial amount is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks, etc.) must be deposited.

Commodity futures
Futures on commodities don't necessarily obey the no-arbitrage law due to storage. In practice the future price will be higher than the theoretical no-storage-cost amount since the holder of the futures contract must compensate the holder of the commodity for his storage costs. Often the people holding the commodity benefit from it in some way. The benefit from holding the commodity is commonly measured in terms of the **convenience yield** c:

$$F = S(t)e^{(r+s-c)(T-t)} \tag{8}$$

Whenever $F < S(t)e^{r(T-t)}$ the market is said to be in **backwardation**, otherwise if $F > S(t)e^{r(T-t)}$ the market is in **contango**.



FX futures
Modifying the result from no-arbitrage to allow for interest received on the foreign currency r_f, we get

$$F = S(T)e^{(r-r_f)(T-t)}. \tag{9}$$

Index futures
Futures contracts on stock indices are settled in cash, with dividends playing a similar role to that of a foreign interest rate on FX futures. So

$$F = S(t)e^{(r-q)(T-t)} \tag{10}$$

0.2 Chapter 2: Derivatives

OPTIONS

The holder of future or forward contracts is obliged to trade at the maturity of the contract, unless the position is closed before maturity. Otherwise, the holder must take possession of the asset of the contract e.g commodity, currency regardless of whether its price has risen or fallen. The simplest **option** gives the holder the *right* to trade in the future at a previously agreed price but takes away the obligation. Specifically, a **call option** is the right to buy an asset for an agreed amount at a specified time in the future.
Example: A call option on Microsoft stock

- The holder has the right to purchase one Microsoft stock for \$25 in one month's time, with \$24.5 current price.
- The price \$25 is called the **exercise/strike price**.
- The **expiry** or **expiration date** is the date on which we must **exercise** our option, should we choose to.
- The **underlying asset** is the stock the option is based on.

We would exercise the option at expiry if the stock S is above the strike E and not if it's below. At expiry it's worth

$$\max(S - E, 0), \tag{11}$$

where the function of the underlying asset is called the **payoff function**, with 'max' representing optionality.

A **put option** is the right to *sell* a particular asset for an agreed amount a a specified time in the future. The holder of a put option wants the stock price to fall so that he can sell the asset for more than it's worth. The payoff function for a put option is

$$\max(E - S, 0), \tag{12}$$

with option only being exercised if the stock falls below the strike price.
The higher the strike the lower the value of a call option but the higher the value of the puts. Since the call allows you to buy the underlying for the strike, so that the lower the strike price the more this right is worth. The opposite is true for a put since it allows you to sell the underlying for the strike price. Also, the longer the time to maturity, the higher the value of the call. As the time to expiry decrease, as there is less and less time for the underlying to move, so the option value must converge to the payoff function.
Calls and puts have a non-linear dependence on they underlying asset. This contrasts with futures which have a linear dependence on the underlying. Calls and puts are the two simplest forms of options and are oftenr referred to as **vanilla**.

DEFINITION OF COMMON TERMS

- **Premium -** The amount paid for the contract initially.
- **Underlying (asset) -** The financial instrument on which the option value depends. Stocks, commodities, currencies and indices are going to be denoted by S. The option payoff is defined as some function of the underlying asset at expiry.
- **Strike (price)** or **exercise price -** The amount for which the underlying can be bought (call) or sold (put). This will be denoted by E. This definition only really applies to the simple calls and puts. For more complicated contracts, this definition is extended.
- **Expiration (date)** or **expiry (date) -** Date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights, denoted by T.
- **Intrinsic value -** The payoff that would be received if the underlying is at its current level when the option expires.

- **Time value** - Any value that the option has above its intrinsic value. The uncertainty surrounding the future value of the underlying asset means that the option value is generally different from the intrinsic value.
- **In the money** - An option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike.
- **Out of the money** - An option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike.
- **At the money** - A call or put with a strike that is close to the current asset level.
- **Long position** - A positive amount of quantity, or a positive exposure to a quantity.
- **Short position** - A negative amount of a quantity, or a negative exposure to a quantity. Many assets can be sold short, with some constraints on the length of time before they must be bought back.

PAYOFF DIAGRAMS

A **payoff diagram** plots the value of an option at expiry as a function of the underlying. At expiry the option is worth a known amount. For a call and put option the contract is worth max(S-E, 0) and max(E-S, 0) respectively, represented by the bold lines below.

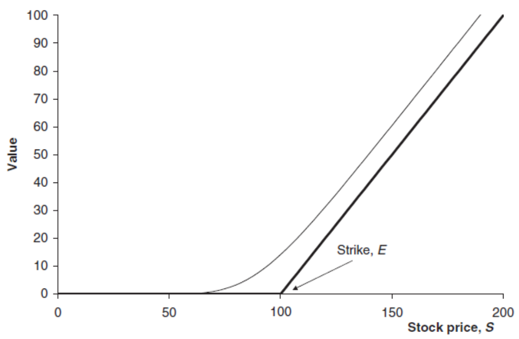


Figure 2.5 Payoff diagram for a call option.

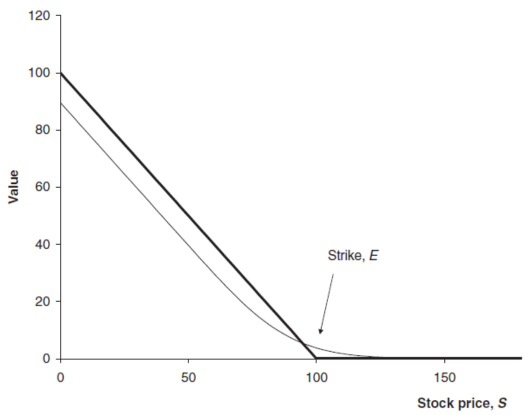


Figure 2.8 Payoff diagram for a put option.

Other representations of value

The payoff diagrams above only shows the money worth of your option contract at expiry. It makes no allowance for how much premium you had to pay for the option. In a **profit diagram** for a call option, we adjust for the original cost of the option by subtracting from the payoff the premium originally paid for the call option.

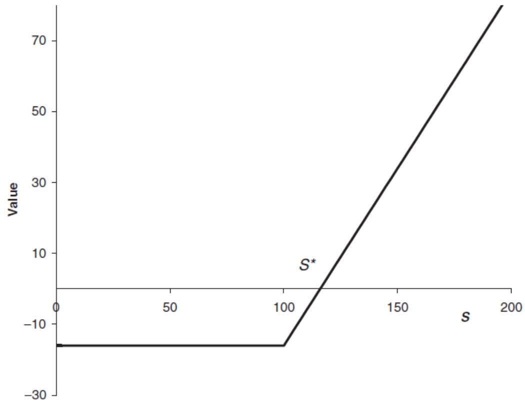


Figure 2.11 Profit diagram for a call option.

This figure is helpful because it shows how far into the money the asset must be at expiry before the option becomes profitable. Asset value S^* is the point which divides profit from loss; if the asset at expiry is above S^* then the contract has made a profit and vice versa. The profit diagram takes no account of the time value of money. The premium is paid up front but the payoff, if any, is only received at expiry. To be consistent one should either discount the payoff by multiplying by $r^{-r(T-t)}$ to value everything at the present, or multiply the premium by $e^{r(T-t)}$ to value all cashflows at expiry.

WRITING OPTIONS

The **writer** of an option is the person who promises to deliver the underlying asset, if the option is a call, or buy it, if the option is a put. The writer is the person who receives the premium, and is liable if the option is exercised. The option holder can sell the option on to someone else to close his position. The purchaser of the option hands over a premium in return for special rights, and uncertain outcome. The writer receives a guaranteed payment up front, but then has obligations in the future.

MARGIN

Buying an option

- Downside: initial premium
- Upside: may be unlimited

Writing an option

- Downside: could be huge
- Upside: Limited

To cover the risk of default in the event of an unfavourable outcome, the **clearing houses** that register and settle options insist on the deposit of a margin by the writers of options.

MARKET CONVENTIONS

Often simpler option contracts bought and sold through exchanges are standardised to follow conventions. Simple calls and puts come in series, referring to the strike and expiry dates. Typically a stock has three choices of expiries trading at any time. Having standardised contracts traded through an exchange promotes liquidity of the instruments. **Over the counter (OTC)** contracts are an agreement between two parties, often brough together by an intermediary. The agreed terms are flexible, without needing to follow any conventions.

THE VALUE OF THE OPTION BEFORE EXPIRY

How much is an option contract worth *now*, before expiry? How much would you pay for a contract, a piece of paper, giving you rights in the future? What is clear is that the contract value before expiry will depend on how high the asset price is today and how long there is before expiry. The longer the time to expiry, the more time there is for the asset to rise or fall.

Let $V(S,t)$ be a function of the value of the underlying asset S at time t which represents the value of the option contract. At expiry date $t = T$, the value of the contract at expiry function is just the payoff function, which we know from before. For a call option it's

$$V(S,T) = \max(S - E, 0). \tag{13}$$

The fine lines in Figure 2.5 and 2.8 are the values of the contracts $V(S,t)$ at *some time before expiry*, plotted against S.

FACTORS AFFECTING DERIVATIVE PRICES

The underlying value asset S and time to expiry t are **variables** of the options price. The interest rate and strike price are examples of **parameters** of the options price. The interest rate affects the option value via the time value of money since the payoff is received in the future. The higher the strike in a call, the lower the value of the call. The **volatility** is an important parameter which impacts an option's value. It is a measure of fluctuation in the asset price i.e. a measure of randomness. The technical definition of volatility is the 'annualized standard deviation of the asset returns'.

SPECULATION AND GEARING

A dramatic move in the underlying that leads to an option expires in the money may lead to a large profit relative the amount of investment.

Example: Today's date is 14th April and the price of Wilmott Inc. stock is \$666. The cost of a 680 call option with expiry 22nd August is \$39. I expect the stock to rise significantly between now and August, how can I profit if I am right?
Buy the stock: Suppose I buy the stock for \$666. And suppose that by the middle of August the stock has risen to \$730. I will have made a profit of \$64 per stock. More importantly my investment will have risen by

$$\frac{730 - 666}{666} \times 100 = 9.6\%. \tag{14}$$

Buy the call: If I buy the call option for \$39, then at expiry I can exercise the call, paying \$680 to receive something worth \$730. I have paid \$39 and I get back \$50. This is a profit of \$11 per option, but in percentage terms I have made

$$\frac{\text{value of asset at expiry} - \text{strike} - \text{cost of call}}{\text{cost of call}} \times 100 = \frac{730 - 680 - 39}{39} \times 100 = 28\%. \tag{15}$$

This is an example of **gearing** or **leverage**. The out-of-the-money option has a high gearing, a possible high payoff for a small investment. The downside of this leverage is that the call option is more likely than not to expire completely worthless and you will lose all of your investment. If Wilmott Inc. remains at \$666 then the stock investment has the same value but the call option experiences a 100% loss.

For highly leveraged contracts, the buyer is very likely to lose but at the risk of only a small amount. But the writer is risking a large loss in order to make a probable small profit. The writer is likely to think twice about such a deal unless he can offset his risk by buying other contracts. This offsetting of risk by buying other related contracts is called **hedging**.

EARLY EXERCISE

The simple options described above are examples of **European options** because exercise is only permitted at expiry. Some contracts allow the holder to exercise at any time before expiry, and these are called **American options**. American options give the holder more rights than their European equivalent and can therefore be more valuable, and they can never be less valuable. The main point of interest with American-style contracts is deciding when to exercise. Most stock options are traded American-style while most index options are traded European-style. **Bermudan options** allow exercise on specified dates, or in specified periods.

PUT-CALL PARITY

Imagine buying a European call option with a strike of E and an expiry of T and writing a European put option with the same strike and expiry. With a present date of t, the payoff you receive at T for the call and put will look like the lines in first and second plot of Figure 2.14 respectively. The payoff for the put is negative, since writing the option leads to liability for the payoff. The portfolio payoff for the two options is the sum of individual payoffs i.e.

$$\max(S(T) - E, 0) - \max(E - S(T), 0) = S(T) - E, \tag{16}$$

If I buy the asset today it will cost me $S(t)$ and be worth $S(T)$ at expiry. It's unclear what the value of $S(T)$ will be but to guarantee to get that amount you'd have to buy the asset. To lock in a payment of E at time T involves a cash flow of $Ee^{-r(T-t)}$ at time t. The conclusion is that the portfolio of a long call and a short put gives exactly thet same payoff as a long asset, short cash position. The equality of these cashflows is independent of the future behaviour of the stock and is model independent:

$$C - P = S - Ee^{-r(T-t)}, \tag{17}$$

where C and P are today's values of the call and put respectively. This relationship holds at any time up to expiry and is known as **put-call parity**. For European options, longing a call and shorting a put with the same strike is equivalent to longing a forward contract with a forward price equivalent to the options' strike price. If this relationship did not hold for whatever reason there would be riskless arbitrage opportunities to be exploited to make money.

Table 2.1 shows the cashflows in the perfectly hedged portfolio. In this table I have set up the cashflows to have a guaranteed value of zero at expiry.

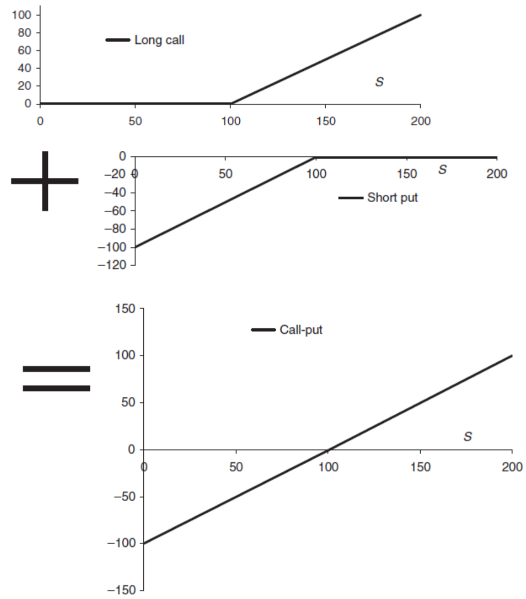


Figure 2.14 Schematic diagram showing put-call parity.

Table 2.1 Cashflows in a hedged portfolio of options and asset.

Holding	Worth today (t)	Worth at expiry (T)
Call	C	$\max(S(T) - E, 0)$
−Put	$-P$	$-\max(E - S(T), 0)$
−Stock	$-S(t)$	$-S(T)$
Cash	$Ee^{-r(T-t)}$	E
Total	$C - P - S(t) + Ee^{-r(T-t)}$	0

BINARIES OR DIGITALS

The **binary** or **digital options** have a payoff at expiry that is discontinuous in the underlying asset price. Examples of payoff diagram for a binary call and a binary put are shown below. This contract pays \$1 at expiry, time T, if the asset price is then greater than the exercise price E.

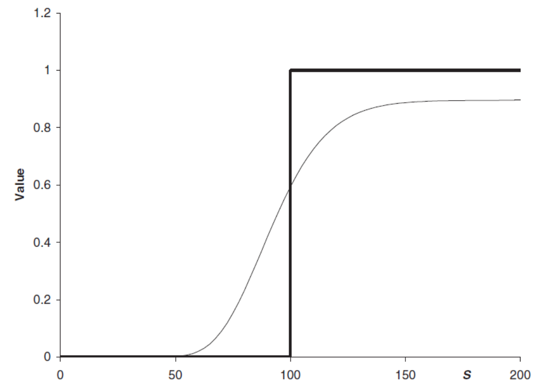


Figure 2.15 Payoff diagram for a binary call option.

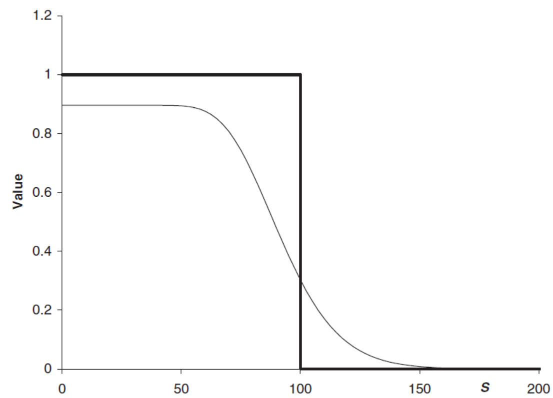


Figure 2.16 Payoff diagram for a binary put option.

The holder of a binary put receives \$1 if the asset is below E at expiry. The binary put would be bought by someone expecting a modest fall in the asset price. If you hold both a binary call and a binary put with the same strikes and expiries, you will always get \$1 regardless of the level of the underlying at expiry i.e.

$$\text{Binary call} + \text{Binary put} = e^{-r(t-t)} \tag{18}$$

This is also known as the **binary put-call parity relationship**.

BULL AND BEAR SPREADS

The bull and bear spreads is an example of **portfolio of options** or an **option strategy**.

- Suppose I buy one call option with a strike of 100 and write another with a strike of 120, both with the same expiry. The resulting portfolio has a payoff as shown in Figure 2.17.
- The payoff is continuous and is zero below 100, 20 above 120 and linear in between. This strategy is called a **bull spread** or **call spread** since it benefits if the market is rising.
- For a bull spread made up of calls with strikes E_1 and E_2 where $E_2 > E_1$, the payoff is

  $$\frac{1}{E_2 - E_1}(\max(S - E_1, 0) - \max(S - E_2, 0)). \tag{19}$$

  Here the payoff is scaled to 1. Similarly, Figure 2.18 shows the payoff of a **bear spread** or a **put spread** i.e. benefit from a falling market, where if I write a put option with strike 100 and buy a put with strike 120.

- A strategy involving options of the same type (i.e. calls or puts) is called a **spread**.

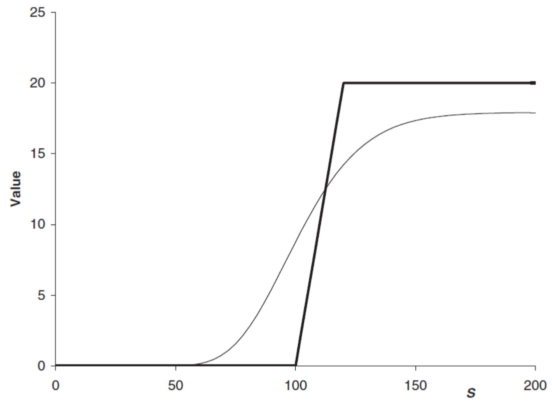


Figure 2.17 Payoff diagram for a bull spread.

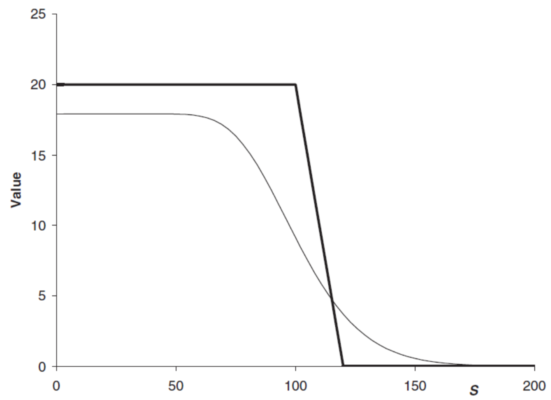


Figure 2.18 Payoff diagram for a bear spread.

STRADDLES AND STRANGLES

The **straddle** consists of a call and a put with the same strike, with a payoff diagram shown in Figure 2.19. Such a position is usually bought at the money by someone who expects the underlying to either rise or fall, but not to remain at the same level. The straddle would be sold by someone with the opposite view, someone who expects the underlying price to remain stable. This is an expensive strategy as money is spent on both premium, hoping for an explosive move on the price of the underlying. This is equivalent to being long volatility i.e. betting on increasing volatility since the options are priced with a given level of volatility and having bought two options meant that profits are only made if they are extra volatile.

The **strangle** consists of a call and a put with different strikes, and can be either an **out-of-the-money strangle**(out of the money put and call) or an **in-the-money strangle**. The payoff for an out-of-the money strangle is shown in Figure 2.22. Here the motivated buyer tend to expect an even larger move in the underlying one way or another. The contract is usually bought when the asset is around the middle of the two strikes and is cheaper than a straddle. This cheapness means that the gearing for the out-of-the-money strangle is higher than that for the straddle. The downside is that there is a much greater range over which the strangle has no payoff at expiry, for the straddle there is only the one point at which there is no payoff.

The straddles and strangles are called **volatility trades** since the contracts are bought or sold based on speculations on the direction of volatility. Straddles and strangles are rarely held until expiry. A strategy involving options of different types i.e. both calls and puts, is called a **combination**.

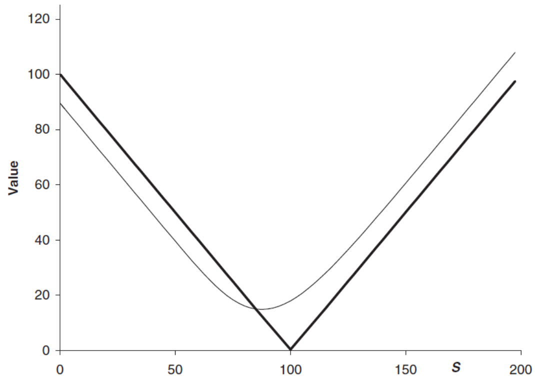


Figure 2.19 Payoff diagram for a straddle.

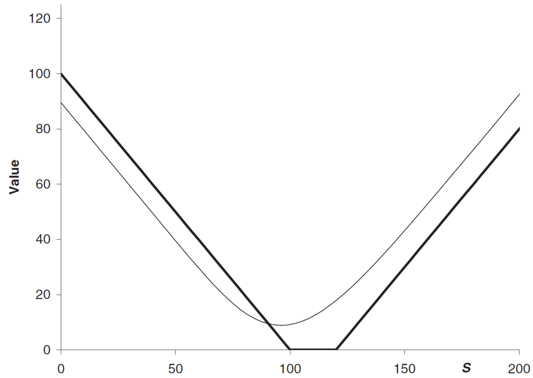


Figure 2.22 Payoff diagram for a strangle.

RISK REVERSAL

The **risk reversal** is a combination of a long call, with a strike above the current spot, and a short put, with a strike below the current spot at the same expiry. Its payoff is shown in Figure 2.23. This strategy

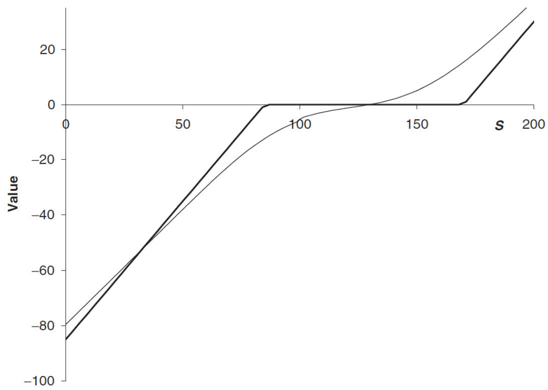


Figure 2.23 Payoff diagram for a risk reversal.

protects against unfavorable price movements in the underlying position but limits the profits that can be made on that position. If an investor is long a stock, they could create a short risk reversal to hedge their position by buying a put option and selling a call option. If an investor is short an underlying asset, the investor hedges the position with a long risk reversal by purchasing a call option and writing a put option on the underlying instrument. If the price of the underlying asset rises, the call option will become more valuable, offsetting the loss on the short position. If the price drops, the trader will profit on their short position in the underlying, but only down to the strike price of the written put.

BUTTERFLIES AND CONDORS

A **butterfly spread** involves the purchase and sale of three options with different strikes. For example, buying a call of 90, writing two calls struck at 100 and buying a 110 call gives the payoff in Figure 2.24 i.e. maximum payoff is 10. This is a cheap position that is typically entered into if the underlying is believed to not go anywhere. The **condor** is like a butterfly except that four strikes, and four call options, are used. The payoff is shown in Figure 2.25.

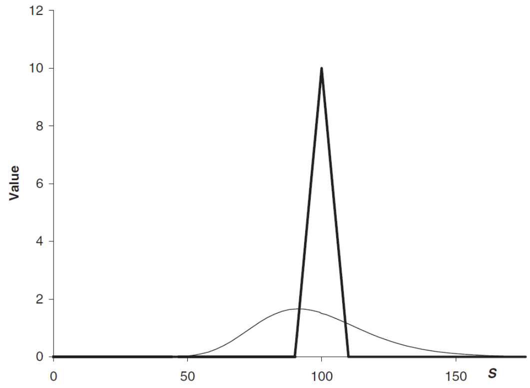


Figure 2.24 Payoff diagram for a butterfly spread.

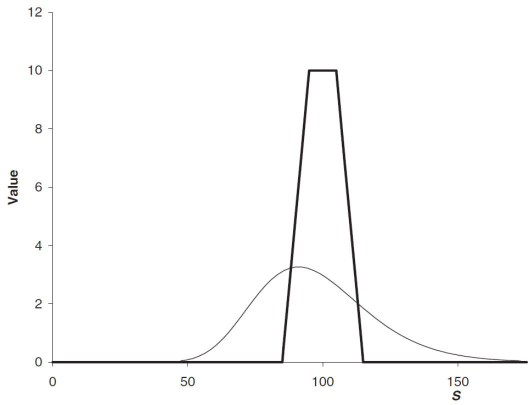


Figure 2.25 Payoff diagram for a condor.

LEAPS AND FLEX

Long-term equity anticipation securities(LEAPS) are longer-dated exchange-traded calls and puts. They are standardized so that they expire in January each year and are available with expiries up to three years. They come with three strikes, corresponding to at the money and 20% in and out of the money with respect to the underlying asset price when issued. The **FLexible EXchange-traded options(FLEX)** allow for a degree of customisation in the expiry date (up to five years), the strike price and the exercise style.

WARRANTS

Warrants are call options issued by a company on its own equity.

- Usually have a longer lifespan
- On exercise the company issues new stock to the warrant holder
- On exercise the holder of a traded option receives stock that has already been issued.
- Exercise is usually allowed any time before expiry, but after an initial waiting period.
- The typical lifespan of a warrant is five or more years. Occasionally **perpetual warrants** are issued, these have no maturity.

CONVERTIBLE BONDS

Convertible bonds(CBs) pay a stream of coupons with a final repayment of principal at maturity, but they can be converted into the underlying stock before expiry. On conversion rights to future coupons are lost. If the stock price is low then there is little incentive to convert to the stock, the coupon stream is more valuable. In this case the CB behaves like a bond. If the stock price is high then conversion is likely and the CB responds to the movement in the asset. Because the CB can be converted into the asset, its value has to be at least the value of the asset. This makes CBs similar to American options; early exercise and conversion are mathematically the same. As a hybrid security, the price of a convertible bond is especially sensitive to changes in interest rates, the price of the underlying stock, and the issuer's credit rating.

OVER THE COUNTER OPTIONS

Over the counter (OTC) options are sold privately from one counterparty to another. A **term sheet** specifies the precise details of an OTC contract. Figure 2.27 shows a term sheet for an OTC put option.

Preliminary and Indicative
For Discussion Purposes Only

Over-the-counter Option linked to the S&P500 Index

Option Type	European put option, with contingent premium feature
Option Seller	XXXX
Option Buyer	[dealing name to be advised]
Notional Amount	USD 20MM
Trade Date	[]
Expiration Date	[]
Underlying Index	S&P500
Settlement	Cash settlement
Cash Settlement Date	5 business days after the Expiration Date
Cash Settlement Amount	Calculated as per the following formula: #Contracts * max[0, S&Pstrike – S&Pfinal] where #Contracts = Notional Amount / S&Pinitial This is the same as a conventional put option: **S&Pstrike** will be equal **to 95% of the closing price on the Trade Date** **S&Pfinal** will be the level of the Underlying Index at the valuation time on the Expiration Date **S&Pinitial** is the level of the Underlying Index at the time of execution
Initial Premium Amount	[2%] of Notional Amount
Initial Premium Payment Date	5 business days after Trade Date
Additional Premium Amounts	[1.43%] of Notional Amount per Trigger Level
Additional Premium Payment Dates	The Additional Premium Amounts shall be due only if the Underlying Index at any time from and including the Trade Date and to and including the Expiration Date is equal to or greater than any of the Trigger Levels.
Trigger Levels	103%, 106% and 109% of **S&P500initial**
Documentation	ISDA
Governing law	New York

This indicative term sheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 2.27 Term sheet for an OTC 'put.'

In this OTC

- The holder gets a put option on S&P 500, but cheaper than a vanilla put option.
- This contract is cheap because part of the premium does not have to be paid until and unless the underlying index trades above a specific level.
- Each time a new level is reached an extra payment is triggered. This feature means that the contract is not vanilla, and makes the pricing more complicated.
- Quantities in square brackets will be set at the time that the deal is struck.

0.3 Chapter 3: The Binomial Model

INTRODUCTION

The binomial model may be thought of as being either a genuine model for the behaviour of equities, or, alternatively as a numerical method for the solution of the Black-Scholes equation;in this case, very similar to an explicit finite-difference method. It's a teaching aid to explain delta hedging, risk elimination and risk-neutral valuation.
Cons:

- Poor model for stock price behaviour. It says the stock can either go up or down by a known amount which is unrealistic. Results that follow from this model hinge on there being only two prices for the stock tomorrow and will break upon introduction of a third state.
- It's an old numerical scheme that predates modern numerical methods.

EQUITIES CAN GO DOWN AS WELL AS UP

Scenario: We will have a stock, and a call option on that stock expiring tomorrow. The stock can either rise or fall by a known amount between today and tomorrow. Interest rates are zero. There is a certain probability p of the stock rising and $1-p$ of the stock falling. Let's introduce the call option on the stock which has a strike of \$100 and expires tomorrow. If the stock price rises to 101, the option's payoff will just be $101 - 100 = 1$. If the stock falls to 99 tomorrow, the payoff is then zero, since the option has expired out of the money. There is a 0.6 probability of getting 1 and a 0.4 probability of getting zero. Interest rates are zero. What is the option worth today?

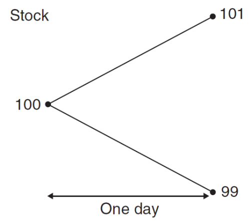


Figure 3.1 The stock can rise or fall over the next day, only two future prices are possible.

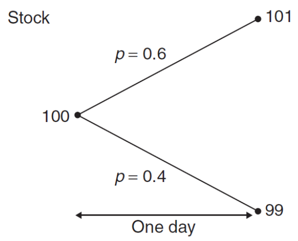


Figure 3.2 Probabilities associated with the future stock prices.

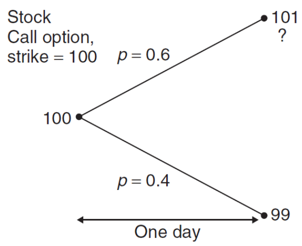


Figure 3.3 What is the option payoff if the stock rises?

It's $\frac{1}{2}$. To see this we must construct a portfolio consisting of one option and short $\frac{1}{2}$ of the underlying stock, shown in Figure 3.7. If the stock rises to 101 then this portfolio is worth $1 - \frac{1}{2} \times 101$; the one

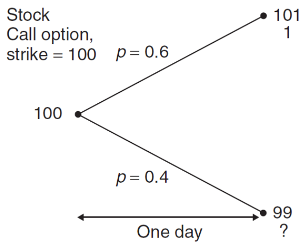


Figure 3.4 What is the option payoff if the stock falls?

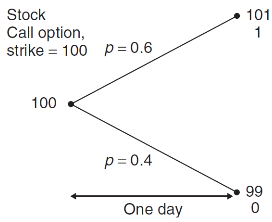


Figure 3.5 Now we know the option values in both 'states of the world.'

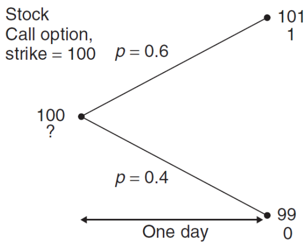


Figure 3.6 What is the option worth today?

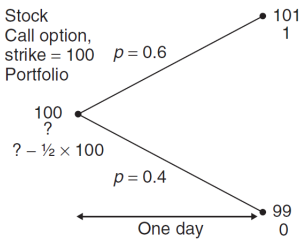


Figure 3.7 Long one option, short half of the stock.

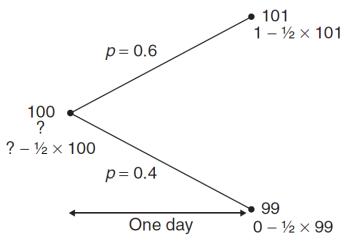


Figure 3.8 The portfolio values at expiration.

being from the option payoff and the $-\frac{1}{2} \times 101$ being from a short $(-)$ position $(\frac{1}{2})$ in the stock (now worth 101). If the stock falls to 99 then this portfolio is worth $0 - \frac{1}{2} \times 99$; the zero being from the option payoff and the $-\frac{1}{2} \times 99$ being from a short $(-)$ position $(\frac{1}{2})$ in the stock (now worth 99), shown in Figure

3.8. In either case, tomorrow, at expiration, the portfolio takes the value $-\frac{99}{2}$. If the portfolio is worth $-\frac{99}{2}$ tomorrow regardless of whether the stock rises or falls and interest rates are zero, then it must also be worth $-\frac{99}{2}$ today. This is a perfectly risk-free portfolio and an example of **no arbitrage**: There are two ways to ensure that we have $\frac{99}{2}$ tomorrow. 1) Buy one option and sell one half of the stock. 2) Put the money under the mattress. Both 'portfolios' must be worth the same today. Therefore

$$\begin{aligned}\text{the option value} - \frac{1}{2} \times 100 &= -\frac{1}{2} \times 99 \\ \text{the option value} &\implies \frac{1}{2}\end{aligned} \tag{20}$$

WHY SHOULD THIS "THEORETICAL PRICE" BE THE "MARKET PRICE"?

If the theoretical price and the market price are not the same, then there is risk-free money to be made. If the option costs less than 0.5 simply buy it and hedge to make a profit. If it is worth more than 0.5 in the market then sell it and hedge, and make a guaranteed profit. In practice this arbitrage opportunity will make the option price converge to 0.5 due to supply and demand.

The expected payoff is 0.6 for this option. If you're willing to pay 0.6 or more, then you're **risk seeking**. If you're paying 0.55, the expected return would be $(0.6 - 0.55)/0.55 \approx 9\%$. and the option writer stands to guarantee a profit of 0.05.

HOW DID I KNOW TO SELL $\frac{1}{2}$ OF THE STOCK FOR HEDGING?

Let Δ = quantity of stock that must be sold for hedging. Starting with one option, $-\Delta$ of the stock gives a portfolio value of $(? - \Delta \times 100)$. Tomorrow the portfolio is worth $(1 - \Delta \times 100)$ or $(0 - \Delta \times 100)$ if the stock rises or falls respectively. Equating these two gives

$$1 - \Delta \times 101 = 0 - \Delta \times 99 \tag{21}$$

$$\implies \Delta = 0.5 \tag{22}$$

Another example: Stock price is 100, can rise to 103 or fall to 98. Value a call option with a strike price of 100. Interest rates are zero. The portfolio value is $(? - \Delta \times 100)$. Tomorrow the portfolio is either worth $(3 - \Delta 103)$ or $(0 - \Delta 98)$. Hence

$$\Delta = \frac{3-0}{103-98} = \frac{3}{5} = 0.6. \tag{23}$$

The portfolio value tomorrow is then -0.6×98. With zero interest rate, the portfolio value today must equal the risk-free portfolio value tomorrow:

$$? - 0.6 \times 100 = -0.6 \times 98 \implies ? = 1.2 \tag{24}$$

Delta hedging means choosing Δ such that the portfolio value does not depend on the direction of the stock. Generalising this we have

$$\Delta = \frac{\text{Range of option payoffs}}{\text{Range of stock prices}}. \tag{25}$$

Where Δ can be thought as the sensitivity of the option to changes in the stock.

HOW DOES THIS CHANGE IF INTEREST RATES ARE NON-ZERO?

We delta hedge as before to construct risk-free portfolio with the same delta. Then present value that back in time, by multiplying a discount factor. Example: Same as first example, but now $r = 0.1$. The discount factor for going back one day is

$$\frac{1}{1 + 0.1/252} = 0.9996. \tag{26}$$

The portfolio value today must be the present value of the portfolio value tomorrow

$$\begin{aligned}? - 0.5 \times 100 &= -0.5 \times 99 \times 0.9996. \\ ? &\implies 0.51963\end{aligned} \tag{27}$$

THE REAL AND RISK-NEUTRAL WORLDS

Some properties of the real world:

- We know all about delta hedging and risk elimination.
- We are very sensitive to risk, and expect greater return for taking risk.
- It turns out that only the two stock prices matter for option pricing, not the probabilities.

The **risk-neutral world** is one where people don't care about risk, and has the following properties:

- We don't care about risk, and don't expect any extra return for taking unnecessary risk.
- We don't ever need statistics for estimating probabilities of events happening.
- We believe that everything is priced using simple expectations.

Suppose in a risk-neutral world a stock is currently worth \$100 and could rise/fall to \$101/\$99. If the stock is correctly priced using expectations, the probabilities of the stock price rising or falling should just be 50/50 due to symmetry. On this risk-neutral world the **risk-neutral probabilities** p' is calculated from

$$p' \times 101 + (1 - p') \times 99 = 100 \tag{28}$$

This gives $p' = 0.5$. The calculation however is wrong to only use simple expectations for pricing with no allowance made for risk. The real probabilities are still 60% and 40% and hence p' is not real. In the same world, the call option will be valued by simple expectations with no regard to risk with

$$0.5 \times 1 + 0.5 \times 0 = 0.5 \tag{29}$$

This is called **risk-neutral expectation**.
When interest rates are non-zero we must perform exactly the same operations, but whenever we equate values at different times we must allow for present valuing. With r = 0.1 we calculate the risk-neutral probabilities from

$$\begin{aligned} 0.9996 \times (p' \times 101 + (1 - p') \times 99) = 100. \\ \implies p' = 0.51984 \end{aligned} \tag{30}$$

The expected payoff is

$$0.51984 \times 1 + (1 - 0.51984) \times 0 = 0.51984 \tag{31}$$

Its present(option) value is

$$0.9996 \times 0.51984 = 0.51963 \tag{32}$$

The risk-neutral probability p' that we have just calculated (the 0.5 in the first example) is not real, it does not exist, it is a mathematical construct.

AND NOW USING SYMBOLS

During a time step δt, assume a binomial modelled asset which initially has the value S can either rise to a value $u \times S$ or fall to a value $v \times S$ with $0 < v < 1 < u$ and probability of rise and fall are p and $1 - p$ respectively. The three constants u, v and p are chosen to give the binomial walk the same characteristics as the asset we are modelling.

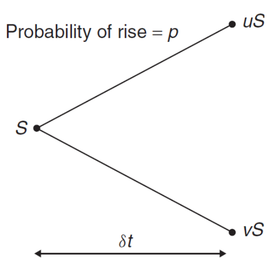


Figure 3.20 The model, using symbols.

Let μ be the drift of the asset and σ the volatility. The drift is the average rate at which the asset rises and the volatility is a measure of its randomness. Let the expressions of u, v and p for now be

$$u = 1 + \sigma\sqrt{\delta t} \tag{33}$$

$$v = 1 - \sigma\sqrt{\delta t} \tag{34}$$

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma} \tag{35}$$

Average asset change

The expected asset price after one time step is

$$\begin{aligned} puS + (1-p)vS &= \left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(1+\sigma\sqrt{\delta t})S + \left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(1-\sigma\sqrt{\delta t})S \\ &= S + S\mu\delta t \\ &= (1+\mu\delta t)S \end{aligned} \tag{36}$$

Hence the expected change in asset is $\mu S\delta t$ and the expected **return** is $\mu\delta t$.

Standard deviation of asset price change

$$\begin{aligned} \frac{\sum(x_i - \bar{x})^2}{n-1} &= p(uS - (1+\mu\delta t)S)^2 + (1-p)(vS - (1+\mu\delta t)S)^2 \\ &= S^2\Big(p(u-1-\mu\delta t)^2 + (1-p)(v-1-\mu\delta t)^2\Big) \\ &= S^2\left(\left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(1+\sigma\sqrt{\delta t}-1-\mu\delta t)^2 + \left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(1-\sigma\sqrt{\delta t}-1-\mu\delta t)^2\right) \\ &= S^2\left(\left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(\sigma\sqrt{\delta t}-\mu\delta t)^2 + \left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(\sigma\sqrt{\delta t}+\mu\delta t)^2\right) \\ &= S^2\left(\left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(\sigma^2\delta t - 2\mu\sigma\delta t^{\frac{3}{2}} + \mu^2\delta t^2) + \left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(\sigma^2\delta t + 2\mu\sigma\delta t^{\frac{3}{2}} + \mu^2\delta t^2)\right) \\ &= S^2(\sigma^2\delta t - \mu^2\delta t^2) \end{aligned} \tag{37}$$

Hence the standard deviation of asset changes and returns are approximately $S\sigma\sqrt{\delta t}$ and $\sigma\sqrt{\delta t}$.

AN EQUATION FOR THE VALUE OF AN OPTION

At time t a portfolio consisting of one option and a short position in a quantity Δ of the underlying has value

$$\Gamma = V - \Delta S \tag{38}$$

At time $t + \delta t$ the asset can rise or fall with values of either V^+ or V^-. Hence the portfolio value is either $V^+ - \Delta uS$ or $V^- - \Delta vS$.

Hedging

Having the freedom to choose Δ, we can make the value of this portfolio the same whether the asset rises or falls. This is ensured if we make

$$\begin{aligned} V^+ - \Delta uS &= V^- - \Delta vS \\ \implies \Delta &= \frac{V^+ - V^-}{(u-v)S} \end{aligned} \tag{39}$$

Substituting in Δ, the portfolio value value if the stock rises or falls is then respectively

$$\begin{aligned} V^+ - \Delta uS &= V^+ - \frac{u(V^+ - V^-)}{(u-v)} \\ V^- - \Delta vS &= V^- - \frac{v(V^+ - V^-)}{(u-v)} \end{aligned} \tag{40}$$

No arbitrage

Let the original portfolio value plus its change in value be

$$\Pi + \delta\Pi \tag{41}$$

where $\delta\Pi$ is the interest earned at the risk-free rate,

$$\delta\Pi = r\Pi\delta t \tag{42}$$

Hence

$$\begin{aligned}\Pi + \delta\Pi &= \Pi + r\Pi\delta t \\ &= \Pi(1 + r\delta t) \\ &= V^- - \frac{v(V^+ - V^-)}{(u - v)}\end{aligned} \tag{43}$$

$$\begin{aligned}\Pi &= V - \Delta S \\ &= V - \frac{(V^+ - V^-)}{(u - v)}\end{aligned} \tag{44}$$

$$\begin{aligned}\implies \Pi(1 + r\delta t) &= V^- - \frac{v(V^+ - V^-)}{(u - v)} \\ (1 + r\delta t)\left(V - \frac{(V^+ - V^-)}{(u - v)}\right) &= \frac{V^-(u - v) - v(V^+ - V^-)}{(u - v)} \\ (1 + r\delta t)V - (1 + r\delta t)\frac{(V^+ - V^-)}{(u - v)} &= \frac{uV^- - vV^- - vV^+ + vV^-}{(u - v)} \\ (1 + r\delta t)V &= (1 + r\delta t)\frac{(V^+ - V^-)}{(u - v)} + \frac{uV^- - vV^+}{(u - v)}\end{aligned} \tag{45}$$

This is an equation for V given V^+, and V, the option values at the next time step, and the parameters u and v describing the random walk of the asset. Using $u = 1 + \sigma\sqrt{\delta t}$ and $v = 1 - \sigma\sqrt{\delta t}$, this can be rewritten as

$$\begin{aligned}V &= \frac{1}{1 + r\delta t}\left(\frac{(1 + r\delta t)(V^+ - V^-)}{1 + \sigma\sqrt{\delta t} - 1 + \sigma\sqrt{\delta t}} + \frac{(1 + \sigma\sqrt{\delta t})V^- - (1 - \sigma\sqrt{\delta t})V^+}{1 + \sigma\sqrt{\delta t} - 1 + \sigma\sqrt{\delta t}}\right) \\ &= \frac{1}{1 + r\delta t}\left(\frac{V^+ - V^- + V^+ r\delta t - V^- r\delta t}{2\sigma\sqrt{\delta t}} + \frac{V^- + V^-\sigma\sqrt{\delta t} - V^+ + V^+\sigma\sqrt{\delta t}}{2\sigma\sqrt{\delta t}}\right) \\ &= \frac{1}{1 + r\delta t}\left(\frac{V^+ r\delta t - V^- r\delta t + V^-\sigma\sqrt{\delta t} + V^+\sigma\sqrt{\delta t}}{2\sigma\sqrt{\delta t}}\right) \\ &= \frac{1}{1 + r\delta t}\left(\frac{V^+ r\sqrt{\delta t} - V^- r\sqrt{\delta t} + V^-\sigma + V^+\sigma}{2\sigma}\right) \\ &= \frac{1}{1 + r\delta t}\left[\left(\frac{r\sqrt{\delta t} + \sigma}{2\sigma}\right)V^+ + \left(\frac{\sigma - r\sqrt{\delta t}}{2\sigma}\right)V^-\right] \\ &= \frac{1}{1 + r\delta t}\left[\left(\frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}\right)V^+ + \left(\frac{1}{2} - \frac{r\sqrt{\delta t}}{2\sigma}\right)V^-\right] \\ &= \frac{1}{1 + r\delta t}(p'V^+ + (1 - p')V^-)\end{aligned} \tag{46}$$

where

$$p' = \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma} \tag{47}$$

If only the expression contained p, the real probability of a stock rise, then this expression would be the expected value at the next time step. But since p did not appear in Equation (46), the probability of a rise or fall is irrelevant as far as option pricing is concerned. Comparing the the expression for p' with the expression for the actual probability p:

$$\begin{aligned}p' &= \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma} \\ p &= \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\end{aligned} \tag{48}$$

The expressions differ by either interest rate r or drift μ. p' is the **risk-neutral probability**. It's like the real probability, but the real probability if the drift rate were r instead of μ. Observe that the risk-free interest rate plays two roles in option valuation. It's used once for discounting to give present value, and it's used as the drift rate in the risk-neutral asset price random walk. Interpreting p' as a probability, Eq.(3.3) is the statement that the option value at any time is the present value of the risk-neutral expected value at any later time. Under the risk neutrality assumption, today's fair price of a derivative is equal to the expected value of its future payoff discounted by the risk free rate.

THE BINOMIAL TREE

The binomial tree model, allows a stock with value S to move up to uS or down to vS after next time step. After two time steps, the asset will be either u^2S (up-up), uvS (up-down) or v^2S(down-down). Extending this random walk until expiry results in a **binomial tree** structure, shown in Fig 3.22. The

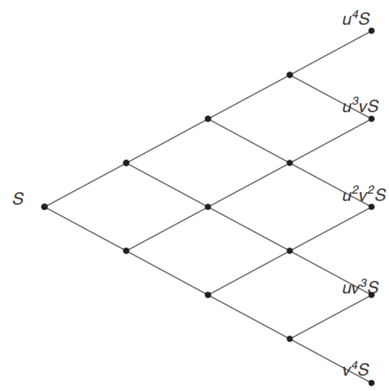


Figure 3.22 The binomial tree: a schematic version.

top and bottom branches of the tree at expiry can only be reached by one path each, either all up or all down moves. Whereas there will be several paths possible for each of the intermediate values at expiry. Therefore the intermediate values are more likely to be reached than the end values if one were doing a simulation. The binomial tree therefore contains within it an approximation to the probability density function for the lognormal random walk.

THE ASSET PRICE DISTRIBUTION

The probability of reaching a particular node in the binomial tree depends on the number if distinct paths to that node and the probabilities of the up and down moves. Since up and down moves are approximately equally likely and since there are more paths to the interior prices than to the two extremes we will find that the probability distribution of future prices is roughly bell shaped. In Figure 3.23 is shown the number of paths to each node after four time steps and the probability of getting to each. In Figure 3.24 this is interpreted as probability density functions at a sequence of times.

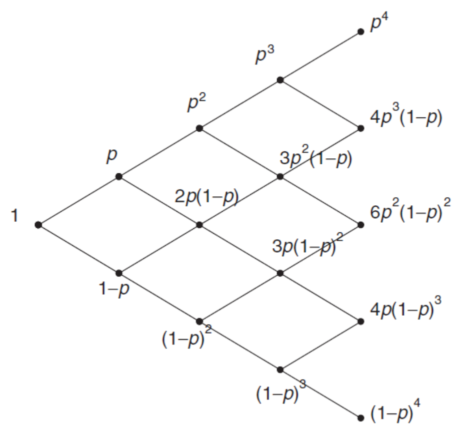


Figure 3.23 Counting paths.

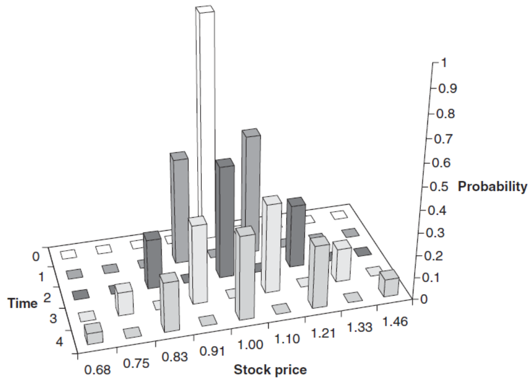


Figure 3.24 The probability distribution of future asset prices.

VALUING BACK DOWN THE TREE

We know V^+ and V^- at expiry, time T, because we know the option value as a function of the asset i.e. the payoff function. Knowing the value of the option at expiry we can find the option value at time $T - \delta t$ for all values of S on the tree and all the other values of S of the previous time steps until the beginning. Here $S = 100, \delta t = 1/12, r = 0.1$ and $\sigma = 0.2$. The option is a European call with a strike of 100 and four months to expiration. Using these numbers we have $u = 1.0604$, $v = 0.9431$ and $p' = 0.5567$. As an example, after one time step the asset takes either the value $100 \times 1.0604 = 106.04$ or $100 \times 0.9431 = 94.31$. Working back from expiry, the option value at the time step before expiry when $S = 119.22$ is given by

$$e^{-0.1 \times 0.0833}(0.5567 \times 26.42 + (1 - 0.5567) \times 12.44) = 20.05. \tag{49}$$

Working right back down the tree to the present time, the option value when the asset is 100 is 6.13.

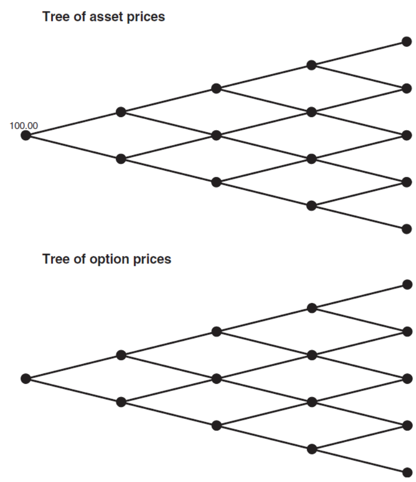


Figure 3.25 The two trees, asset and option.

THE GREEKS

The greeks are defined as derivatives of the option value with respect to various variables and parameters. From thet binomial model the option's delta is defined by

$$\frac{V^+ - V^-}{(u - v)S}. \tag{50}$$

The delta uses the option value at the two points marked 'D' together with today's asset price and the parameters u and v. In the limit as the time step approaches zero, the delta becomes

$$\frac{\partial V}{\partial S} \tag{51}$$

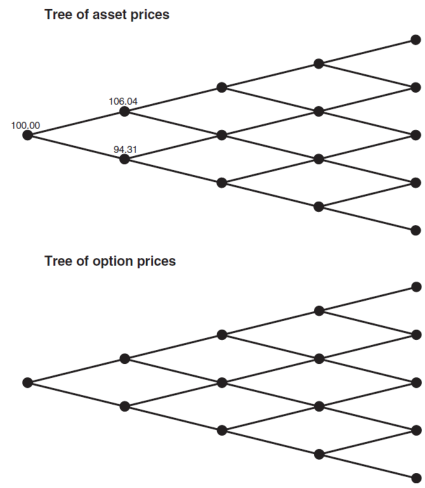


Figure 3.26 Start building up the stock-price tree.

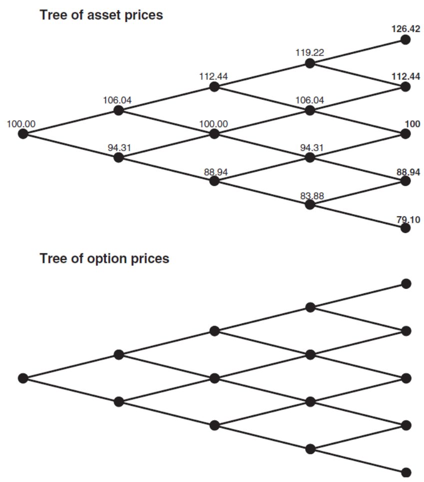


Figure 3.27 The finished stock tree.

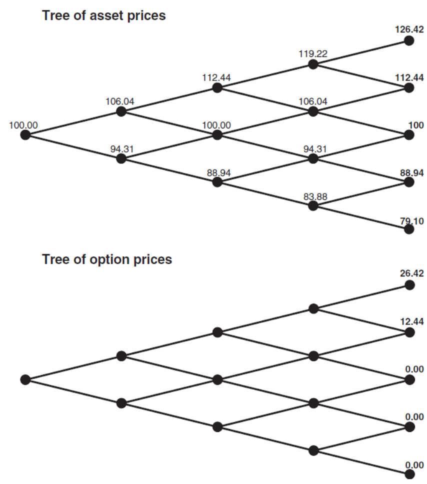


Figure 3.28 The option payoff.

The gamma of the option is defined as a derivative of the option with respect to the underlying, representing the sensitivity of the delta to the asset

$$\frac{\partial^2 V}{\partial S^2}. \tag{52}$$

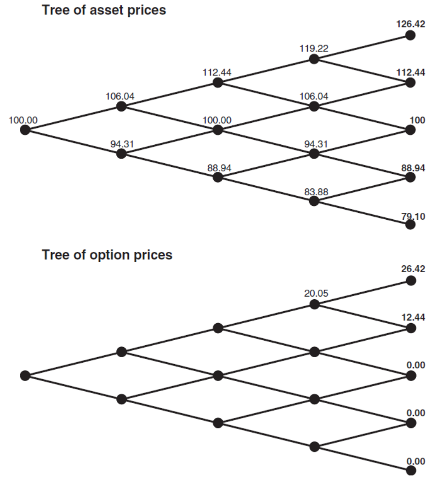


Figure 3.29 Work backwards one 'node' at a time.

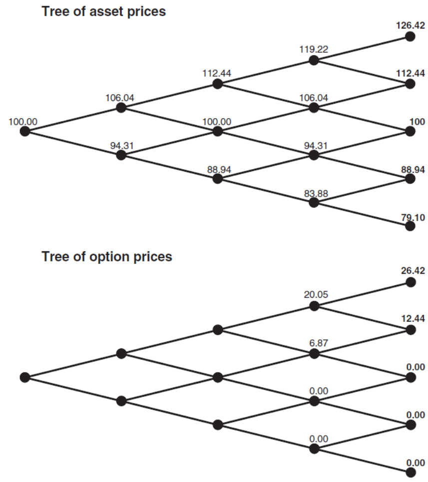


Figure 3.30 First time step completed.

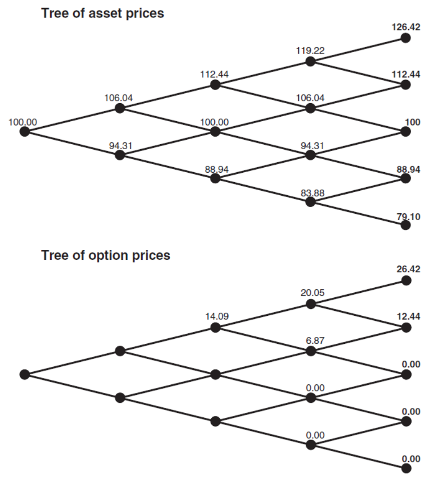


Figure 3.31 Starting on next time step.

It is a measure of how much we must rehedge at next time step. We can calculate the delta at points marked with a D in Figure 3.34 from the option value one time step further in the future. The gamma is then just the change in the delta from one of these to the other divided by the distance between them.

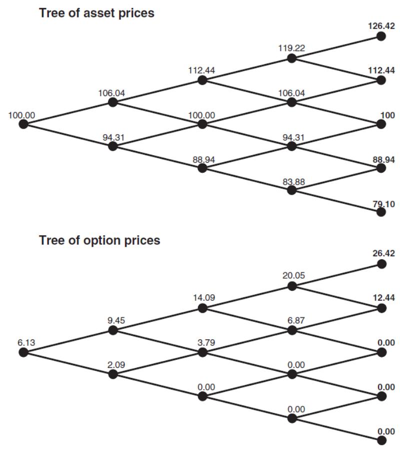


Figure 3.32 The finished option-price tree. Today's option price is therefore 6.13.

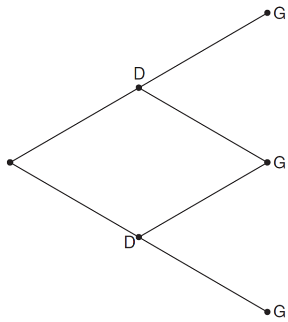


Figure 3.34 Calculating the delta and gamma.

This calculation uses the points marked 'G' in Figure 3.34.
The theta of the option is the sensitivity of the option price to time, assuming that the asset price does not change. An obvious choice for the discrete-time definition of theta is to interpolate between V^+ and V to find a theoretical option value had the asset not changed and use this to estimate

$$\frac{\partial V}{\partial t}, \tag{53}$$

which results in

$$\frac{\frac{1}{2}(V^+ + V^-) - V}{\delta t}. \tag{54}$$

The vega is the sensitivity of the option value to the volatility

$$\frac{\partial V}{\partial \sigma}. \tag{55}$$

Suppose we want to find the option value and vega when the volatility is 20%. The most efficient way to do this is to calculate the option price twice, using a binomial tree, with two different values of .Calculate the option value using a volatility of $\sigma \pm \epsilon$, for a small number ϵ; call the values you find $V_\pm$. The option value is approximated by the average value

$$V = \frac{1}{2}(V_+ + V_-) \tag{56}$$

and the vega is approximated by

$$\frac{V_+ - V_-}{2\epsilon}. \tag{57}$$

<u>**EARLY EXERCISE**</u>

We use the same binomial tree with the same u, v and p with a slight difference in V. We must ensure that there are no arbitrage opportunities at any of the nodes. Let S_j^n be the asset price at the nth time step, at the node j from the bottom, $0 \le j \le n$. In our lognormal world we have

$$S_j^n = Su^j v^{n-j} \tag{58}$$

where S is the current asset price. Let V_j^n be the option value at the same node. Our ultimate goal is to find V_0^0 knowing the payoff, i.e. knowing V_j^M for all $0 \le j \le M$ where M is the number of time steps. Here arbitrage is possible if the theoretical option value goes below the payoff at any time and it's time to exercise. If we find that

$$\frac{V_{j+1}^{n+1} - V_j^{n+1}}{u-v} + \frac{1}{1+r\delta t}\frac{uV_j^{n+1} - vV_{j+1}^{n+1}}{u-v} \ge \text{Payoff}(S_j^n) \tag{59}$$

then we use this as our new value. But if

$$\frac{V_{j+1}^{n+1} - V_j^{n+1}}{u-v} + \frac{1}{1+r\delta t}\frac{uV_j^{n+1} - vV_{j+1}^{n+1}}{u-v} < \text{Payoff}(S_j^n) \tag{60}$$

then we should exercise, giving us a better value of

$$V_j^n = \text{Payoff}(S_j^n). \tag{61}$$

Putting two together we get

$$V_j^n = \max\left(\frac{V_{j+1}^{n+1} - V_j^{n+1}}{u-v} + \frac{1}{1+r\delta t}\frac{uV_j^{n+1} - vV_{j+1}^{n+1}}{u-v}, \text{Payoff}(S_j^n)\right) \tag{62}$$

THE CONTINUOUS TIME LIMIT

The binomial model is a discrete-time model that can lead to the Black-Scholes equation which is in continuous time. Examine

$$V = \frac{1}{1+r\delta t}(p'V^+ + (1-p')V^-) \tag{63}$$

as $\delta t \to 0$, we have

$$\begin{aligned} u &\sim 1 + \sigma\sqrt{\delta t}, \\ v &\sim 1 - \sigma\sqrt{\delta t} \end{aligned} \tag{64}$$

Next we write

$$\begin{aligned} V &= V(S,t) \\ V^+ &= V(uS, t+\delta t) \\ V^- &= V(vS, t+\delta t) \end{aligned} \tag{65}$$

Expanding these expressions in Taylor series

$$V(S+\delta S, t+\delta t) \approx V(S,t) + \delta t\frac{\partial V}{\partial t} + \delta S\frac{\partial V}{\partial S} + \frac{1}{2}\delta S^2\frac{\partial^2 V}{\partial S^2} + ... \tag{66}$$

we get

$$\begin{aligned} V^+ &= V(uS, t+\delta t) \\ &= V(S + S(u-1), t+\delta t) \\ &= V(S, t+\delta t) + \frac{\partial V(S,t+\delta t)}{\partial S}S(u-1) + \frac{1}{2}\frac{\partial^2 V(S,t+\delta t)}{\partial S^2}S^2(u-1)^2 + O(u^3) \end{aligned} \tag{67}$$

$$\begin{aligned} V^- &= V(vS, t+\delta t) \\ &= V(S + S(v-1), t+\delta t) \\ &= V(S, t+\delta t) + \frac{\partial V(S,t+\delta t)}{\partial S}S(v-1) + \frac{1}{2}\frac{\partial^2 V(S,t+\delta t)}{\partial S^2}S^2(v-1)^2 + O(v^3) \end{aligned} \tag{68}$$

$$
\begin{aligned}
p'V^+ + (1-p')V^- &= p'\left(V + \frac{\partial V}{\partial S}S(u-1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3)\right) \\
&\quad + (1-p')\left(V + \frac{\partial V}{\partial S}S(v-1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(v-1)^2 + O(v^3)\right) \\
&= p'V + p'\frac{\partial V}{\partial S}S(u-1) + p'\frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + V + \frac{\partial V}{\partial S}S(v-1) \\
&\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(v-1)^2 - p'V - p'\frac{\partial V}{\partial S}S(v-1) - p'\frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(v-1)^2 \\
&= V + \frac{\partial V}{\partial S}S\left(p'(u-1) + (v-1) - p'(v-1)\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3) \\
&= V + \frac{\partial V}{\partial S}S\left(p'(u-1) + (1-p')(v-1)\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3)
\end{aligned}
\tag{69}
$$

For the binomial random walk to have the correct drift over a time period of δt we need

$$
\begin{aligned}
&puS + (1-p)vS = SE\left[e^{(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma\phi\sqrt{\delta t}}\right] = Se^{\mu\delta t} \\
\implies\ & pu + (1-p)v = e^{\mu\delta t}
\end{aligned}
\tag{70}
$$

Putting it all together, we have

$$
V = \frac{1}{1 + r\delta t}(p'V^+ + (1-p')V^-) \tag{71}
$$

$$
\begin{aligned}
\implies V(1 + r\delta t) &= V + \frac{\partial V}{\partial S}S\left(p'(u-1) + (1-p')(v-1)\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3) \\
&= V + \frac{\partial V}{\partial S}S(p'u - p' + v - 1 - p'v + p') + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3) \\
&= V + \frac{\partial V}{\partial S}S(p'u + (1-p')v - 1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3) \\
&= V + \frac{\partial V}{\partial S}S(e^{\mu\delta t} - 1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(u-1)^2 + O(u^3) \\
&= V + \frac{\partial V}{\partial S}Sr\delta t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2\delta t + O(u^3)
\end{aligned}
\tag{72}
$$

We end up with

$$
V(S,0) + r\delta t V(S,0) = V(S,\delta t) + Sr\delta t\frac{\partial V(S,\delta t)}{\partial S} + \frac{1}{2}S^2\sigma^2\delta t\frac{\partial^2 V(S,\delta t)}{\partial S^2} \tag{73}
$$

$$
\begin{aligned}
\implies 0 &= V(S,\delta t) - V(S,0) + Sr\delta t\frac{\partial V(S,\delta t)}{\partial S} + \frac{1}{2}S^2\sigma^2\delta t\frac{\partial^2 V(S,\delta t)}{\partial S^2} - r\delta t V(S,0) \\
&= \frac{V(S,\delta t) - V(S,0)}{\delta t} + Sr\frac{\partial V(S,\delta t)}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V(S,\delta t)}{\partial S^2} - rV(S,0)
\end{aligned}
\tag{74}
$$

Taking the limit of $\delta t \to 0$, we finally arrive at the Black-Scholes equation

$$
0 = \frac{\partial V(S,0)}{\partial t} + Sr\frac{\partial V(S,0)}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V(S,0)}{\partial S^2} - rV(S,0) \tag{75}
$$

$$
\implies \frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{76}
$$

0.4 Chapter 4: The Random Behaviour of Assets

THE POPULAR FORMS OF "ANALYSIS"
Three forms of analysis:
Fundamental Analysis

- Trying to determine the 'correct' worth of a company by an in-depth study of balance sheets, management teams, patent applications, competitors, lawsuits, etc.
- Two difficulties:
  - Difficult, and all the most important stuff can be hidden 'off balance sheet.'
  - 'The market can stay irrational longer than you can stay solvent' (Keynes). In other words, even if you have the perfect model for the value of a firm it doesn't mean you can make money. You have to find some mispricing and then hope that the rest of the world starts to see your point of view. And this may never happen.

Technical Analysis

- You don't care anything about the company other than the information contained within its stock price history.
- You draw trendlines, look for specific patterns in the share price and make predictions accordingly.

Quantitative Analysis

- Treating financial quantities such as stock prices or interest rates as random, and then choosing the best models for that randomness.

WHY WE NEED A MODEL FOR RANDOMNESS: JENSEN'S INEQUALITY
Jensen's inequality allows us to see the importance of randomness in option theory. Suppose a stock worth 100 today could equally likely be 50 or 150 in one year's time. How can we value an option on this stock, a call option with a strike of 100 expiring in one year? First scenario: We say that that we

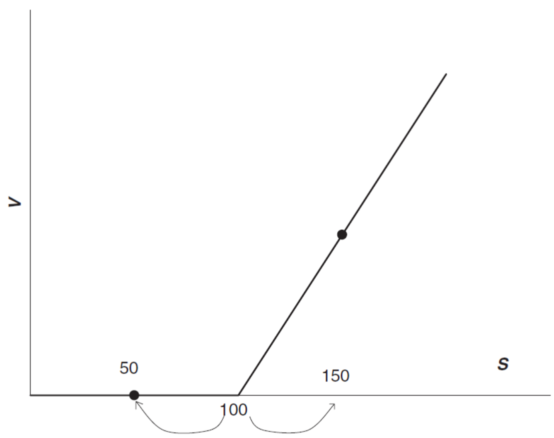


Figure 4.1 Future scenarios.

expect the stock price to be at 100 in one year, this being the average of the possible future values. The payoff for the call option would then be 0, since it is exactly at the money. And the present value of this is zero. But you'd expect the value to be greater than zero, since half the time there is some payoff.
Second scenario: Look at the two possible payoffs and then calculate that expectation. If the stock falls to 50 then the payoff is zero, if it rises to 150 then the payoff is 50. The average payoff is therefore 25, which we could present value to give us some idea of the option's value. This calculation illustrates that the order in which we do the payoff calculation and the expectation matters. In this example we had

$$\text{Payoff}(\text{Expected}[\text{stock price}]) = 0 \tag{77}$$

whereas

$$\text{Expected}[\text{Payoff}(\text{stock price})] = 25 \tag{78}$$

This is an example of Jensen's inequality. For a convex function $f(S)$(payoff function of a call) of a random variable S(stock price),

$$E[f(S)] \geq f(E[S]) \tag{79}$$

Let

$$S = \bar{S} + \epsilon \tag{80}$$

where $\bar{S} = E[S]$, so the $E[\epsilon] = 0$. Then

$$\begin{aligned} E[f(S)] &= E\Big[f(\bar{S}+\epsilon)\Big] \\ &= E\Big[f(\bar{S}) + \epsilon f'(\bar{S}) + \frac{1}{2}\epsilon^2 f''(\bar{S}) + ...\Big] \\ &\approx f(\bar{S}) + \frac{1}{2} f''(\bar{S}) E\big[\epsilon^2\big] \\ &= f(E[s]) + \frac{1}{2} f''(E[S]) E\big[\epsilon^2\big]. \end{aligned} \tag{81}$$

So $E[f(S)]$ is greater than $f(E[S])$ by approximately

$$\frac{1}{2} f''(E[S]) E\big[\epsilon^2\big]. \tag{82}$$

This shows the importance of two concepts:

- $f''(E[S])$: The **convexity** of an option. As a rule this adds value to an option. It also means that any intuition we may get from linear contracts (forwards and futures) might not be helpful with non-linear instruments such as options.
- $E[\epsilon^2]$: Randomness in the underlying, and its variance. As stated above, modeling randomness is the key to modeling options.

SIMILARITIES BETWEEN EQUITIES, CURRENCIES, COMMODITIES AND INDICES

In investments we look for **return**, meaning the percentage growth in the value of an asset, together with accumulated dividends, over some period:

$$\text{Return} = \frac{\text{Change in value of the asset + accumulated cashflows}}{\text{Original value of the asset}} \tag{83}$$

When it comes to modelling assets, the returns are more important than actual stock price. For two stocks A and B that are worth respectively \$100 and \$1000 - both growing on average by \$10 per annum - will both have an absolute growth of \$10 and percentage growth of 10% and 1% respectively. Hence if we have \$1000 to invest we would be better off investing in ten of asset A than one of asset B.

EXAMINING RETURNS

Figure 4.2 shows a typical plot of the quoted price of Perez Companc. It has a general upward trend that's not guaranteed. any mathematical model of a financial asset must acknowledge the randomness and have a probabilistic foundation. Remembering that the returns are more important to us than the absolute level of the asset price, I show in Figure 4.3 how to calculate returns on a spreadsheet. Let S_i be the ith day asset value. The return from day i to $i+1$ is given by (ignoring dividends)

$$\frac{S_{i+1} - S_i}{S_i} = R_i \tag{84}$$

Figure 4.4 shows the daily return for Perez Companc. It looks like 'noise' and will be modelled as such. The mean of the returns distribution is

$$\bar{R} = \frac{1}{M}\sum_{i=1}^{M} R_i \tag{85}$$

with sample standard deviation as

$$\sqrt{\frac{1}{M-1}\sum_{i=1}^{M}(R_i - \bar{R})^2} \tag{86}$$

where M is the number of returns in the sample (one fewer than the number of asset prices). From the data in this example we find that the mean is 0.002916 and the standard deviation is 0.024521.

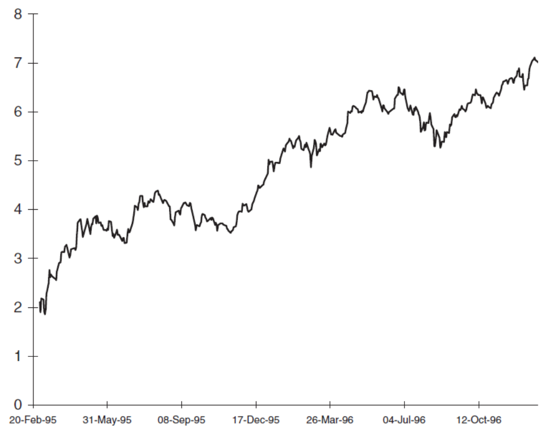


Figure 4.2 Perez Companc from February 1995 to November 1996.

Date	Perez	Return				
01-Mar-95	2.11			Average return	0.002916	
02-Mar-95	1.90	-0.1		Standard deviation	0.024521	
03-Mar-95	2.18	0.149906				
06-Mar-95	2.16	-0.01081				
07-Mar-95	1.91	-0.11258	=AVERAGE(C3:C463)			
08-Mar-95	1.86	-0.02985				
09-Mar-95	1.97	0.061538				
10-Mar-95	2.27	0.15		=STDEVP(C3:C463)		
13-Mar-95	2.49	0.099874				
14-Mar-95	2.76	0.108565				
15-Mar-95	2.61	-0.05426				
16-Mar-95	2.67	0.021858				
17-Mar-95	2.64	-0.0107				
20-Mar-95	2.60	-0.01622	=(B13-B12)/B12			
21-Mar-95	2.59	-0.00275				
22-Mar-95	2.59	-0.00275				
23-Mar-95	2.55	-0.01232				
24-Mar-95	2.73	0.069307				
27-Mar-95	2.91	0.064815				
28-Mar-95	2.92	0.002899				
29-Mar-95	2.92	0				
30-Mar-95	3.12	0.069364				
31-Mar-95	3.14	0.005405				
03-Apr-95	3.13	-0.00269				
04-Apr-95	3.24	0.037736				
05-Apr-95	3.25	0.002597				
06-Apr-95	3.28	0.007772				
07-Apr-95	3.21	-0.02057				
10-Apr-95	3.02	-0.06037				
11-Apr-95	3.08	0.019553				
12-Apr-95	3.19	0.035616				
17-Apr-95	3.21	0.007936				
18-Apr-95	3.17	-0.01312				
19-Apr-95	3.24	0.021277				

Figure 4.3 Spreadsheet for calculating asset returns.

Figure 4.5 shows the frequency distribution of daily returns for Perez Companc, scaled and translated to have a zero mean, a standard deviation of one and an area under the curve of one. On the same plot is drawn the probability density function for the standardised Normal distribution function

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}, \tag{87}$$

where ϕ is a standardised Normal variable. If we believe that the empirical returns are close enough to Normal for this to be a good approximation, then we can model the returns as a random variable drawn from a Normal distribution with a known, constant, non-zero mean and a known, non-zero standard deviation

$$\begin{aligned} R_i &= \frac{S_{i+1} - S_i}{S_i} \\ &= \text{mean} + \text{standard deviation} \times \phi \end{aligned} \tag{88}$$

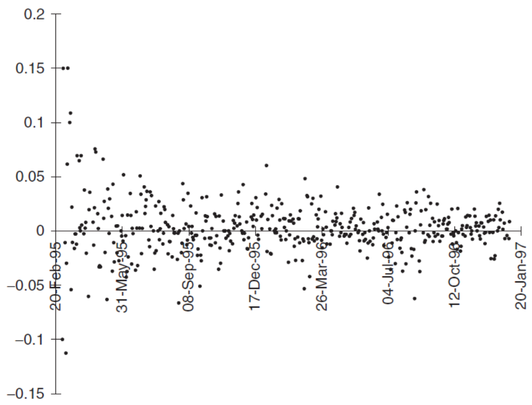


Figure 4.4 Daily returns of Perez Companc.

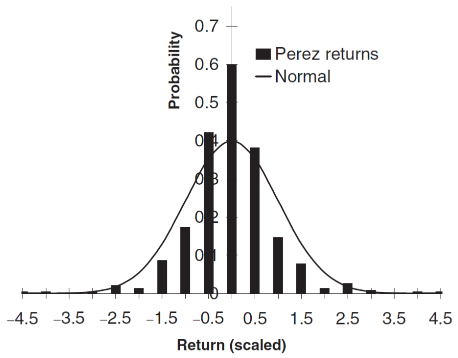


Figure 4.5 Normalized frequency distribution of Perez Companc and the standardized Normal distribution.

TIMESCALES

The time series of the mean and standard deviation of the returns scale with the time step size δt i.e. the larger the time between sampling the more the asset will have moved in the meantime, *on average*. Let mean $= \mu\ \delta t$, for some μ that's assumed to be constant. Here μ represents the annualised average return or the drift. Ignoring randomness for the moment, our model is simply

$$\begin{aligned} \frac{S_{i+1} - S_i}{S_i} &= \mu\delta t \\ \implies S_{i+1} &= S_i(1 + \mu\delta t) \end{aligned} \tag{89}$$

If the asset begins at S_0 at time $t = 0$ then after one time step $t = \delta t$ and

$$S_1 = S_0(1 + \mu\delta t). \tag{90}$$

After two time steps $t = 2\delta t$

$$\begin{aligned} S_2 &= S_1(1 + \mu\delta t) \\ &= S_0(1 + \mu\delta t)^2 \end{aligned} \tag{91}$$

and after M time steps $t = M\delta t = T$ and

$$\begin{aligned} S_M &= S_0(1 + \mu\delta t)^M \\ &= S_0 e^{M\ln(1+\mu\delta t)} \\ &\approx S_0 e^{\mu M\delta t} \\ &= S_0 e^{\mu T}. \end{aligned} \tag{92}$$

In the limit as $\delta t \to 0$ with the total time T fixed, this approximation becomes exact. This result is important for two reasons:

1. In the absence of randomness the asset exhibits exponential growth.
2. If the mean of the returns distribution was chosen to scale with any other power of δt, it would have resulted infinite values of the asset.

Under the square root of standard deviation

$$\sqrt{\frac{1}{M-1}\sum_{i=1}^{M}(R_i-\bar{R})^2} \tag{93}$$

there are $M = T/\delta t$ number of terms. In order for the standard deviation to remain finite as we let $\delta t \to 0$, the individual terms in the expression must each be of $O(\delta t)$. Since each term is a square of a return, the standard deviation of the asset return over a time step δt must be $O(\delta t^{1/2})$:

$$\text{standard deviation} = \sigma \delta t^{1/2}, \tag{94}$$

where parameter σ measures the amount of randomness, the larger it is the more uncertain is the return. It represents the annualised standard deviation of asset returns. Putting these scalings explicitly into our asset return model

$$\begin{aligned} R_i = \frac{S_{i+1}-S_i}{S_i} &= \mu\delta t + \sigma\phi\delta t^{1/2} \\ \implies S_{i+1}-S_i &= \mu S_i \delta t + \sigma S_i \phi \delta t^{1/2} \end{aligned} \tag{95}$$

LHS is the asset price change from time step i to time step $i+1$. RHS is the "model", think of it as a **random walk** of the asset price. We know exactly where the asset price is today but tomorrow's value is unknown. It is distributed about today's value according to (4.5), shown in Figure 4.7.

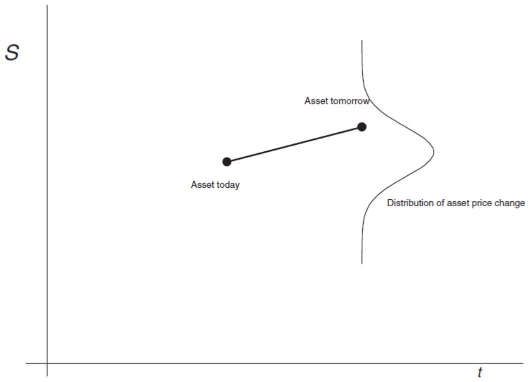


Figure 4.7 A representation of the random walk.

The drift

The parameter μ is called the **drift rate**, the **expected return** or the **growth rate** of the asset. Statistically it is very hard to measure since the mean scales with the usually small parameter δt. It can be estimated by

$$\mu = \frac{1}{M\delta t}\sum_{i=1}^{M} R_i. \tag{96}$$

The unit of time that is usually used is the year, in which case μ is quoted as an annualised growth rate.

The volatility

The parameter σ is called the volatility of the asset. It can be estimated by

$$\sqrt{\frac{1}{(M-1)\delta t}\sum_{i=1}^{M}(R_i-\bar{R})^2}. \tag{97}$$

The drift is not apparent over short timescales for which the volatility dominates. Over long timescales i.e. decades, the drift becomes important. Figure 4.8 is a realized path of the logarithm of an asset,

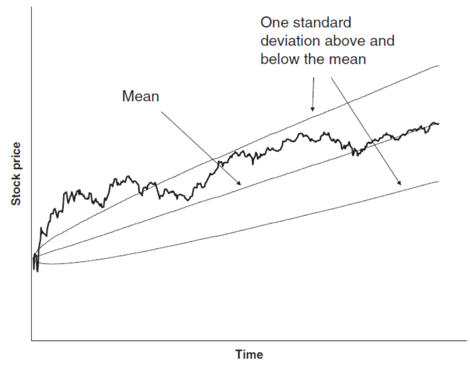


Figure 4.8 Path of the logarithm of an asset, its expected path and one standard deviation above and below.

together with its expected path and a 'confidence interval.' In this example the confidence interval represents one standard deviation. With the assumption of Normality this means that 68% of the time the asset should be within this range. The mean path is growing linearly in time and the confidence interval grows like the square root of time. Thus over short timescales the volatility dominates.

ESTIMATING VOLATILITY

The most common estimate of volatility is simply

$$\sqrt{\frac{1}{(M-1)\delta t}\sum_{i=1}^{M}(R_i-\bar{R})^2}. \tag{98}$$

If δt is sufficiently small the mean return R term can be ignored. For small δt

$$\sqrt{\frac{1}{(M-1)\delta t}\sum_{i=1}^{M}(\log S(t_i)-\log S(t_{i-1}))^2}. \tag{99}$$

can also be used, where $S(t_i)$ is the closing price on day t_i. Volatility changes inevitably over time due to changing economic circumstances, seasonality, etc. Typically you would use daily closing prices to work out daily returns and then use the past 10, 30, 100, ... daily returns in the formula above. Or you could use returns over longer or shorter periods. Since all returns are equally weighted, while they are in the estimate of volatility, any large return will stay in the estimate of vol until the 10 (or 30 or 100) days have past.

THE WIENER PROCESS

Continuous time limit of

$$\begin{aligned} R_i = \frac{S_{i+1}-S_i}{S_i} &= \mu\delta t + \sigma\phi\delta t^{1/2} \\ \implies S_{i+1}-S_i &= \mu S_i\delta t + \sigma S_i\phi\delta t^{1/2} \end{aligned} \tag{100}$$

In the limit of $\delta t = 0$, δt becomes dt but $\delta t^{1/2}$ does not become $dt^{1/2}$. In the zero-time step limit any random $dt^{1/2}$ will dominate any deterministic dt term. Yet the factor in front of $dt^{1/2}$ has a mean of zero, so maybe it does not outweigh the drift after all. It turns out that since the variance of the random term is $O(\delta t)$ we can make sensible continuous-time limit of our discrete-time model.

Let $\phi\delta t^{1/2} = dX$, where dX is a random variable, drawn from a Normal distribution with mean zero and variance dt:

$$E[dX] = 0 \quad \text{and} \quad E[dX^2] = dt. \tag{101}$$

This is not exactly what it is, but it is close enough to give the right idea. This is called a **Wiener process**. The important point is that we can build up a continuous-time theory using Wiener processes instead of Normal distributions and discrete time.

THE WIDELY ACCEPTED MODEL FOR EQUITIES, COMMODITIES AND INDICES

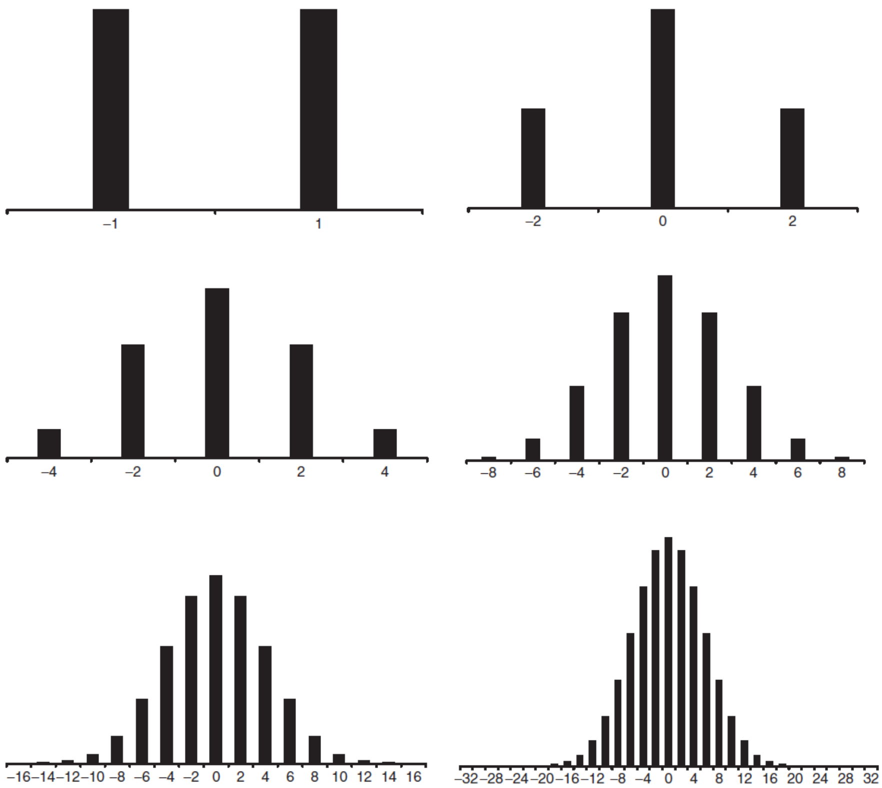
Our asset price model in the continuous-time limit, using the Wiener process notation, can be written as

$$dS = \mu S dt + \sigma S dX \tag{102}$$

This is a **stochastic differential equation** – a continuous-time model of an asset price.

Why do we like the Normal distribution?

Toss one coin, heads you win one dollar, tails you lose one dollar. Now toss two coins, same rules, for each head you get one dollar, but lose one for each tail. The sequence of figures below shows the probability density function of winnings/losses after an increasing number of tosses, which look more and more like a Normal distribution. This is a demonstration of **Central Limit Theorem**. Let $X_1, X_2, ...$



be a sequence of independent identically distributed (i.i.d) random variables with finite means m and finite non-zero variances s^2 then the sum

$$S_n = X_1 + X_2 + ... + X_n \tag{103}$$

in the limit as $n \to \infty$ is distributed Normally with mean nm and variance ns^2. Or if we rescale,

$$S'_n = \frac{X_1 + X_2 + ... + X_n - nm}{\sqrt{n}s} \tag{104}$$

tends to the standardised Normal distribution. In other words, if we add up enough i.i.d random variables (with finite mean and standard deviation) we end up with something that's Normally distributed.

0.5 Chapter 5: Elementary Stochastic Calculus

A MOTIVATING EXAMPLE

Toss a coin. Every time you throw a head I give you \$1, every time you throw a tail you give me \$1. Figure 5.1 shows how much money you have after six tosses. In this experiment the sequence was

THHTHT, and we finished even. If I use R_i to mean the random amount, either \$1 or -\$1, you make on the ith toss then we have.

$$E[R_i] = 0, E[R_i^2] = 1 \text{ and } E[R_iR_j] = 0 \tag{105}$$

Let S_i be the total amount of money you have won up to and including the ith toss so that

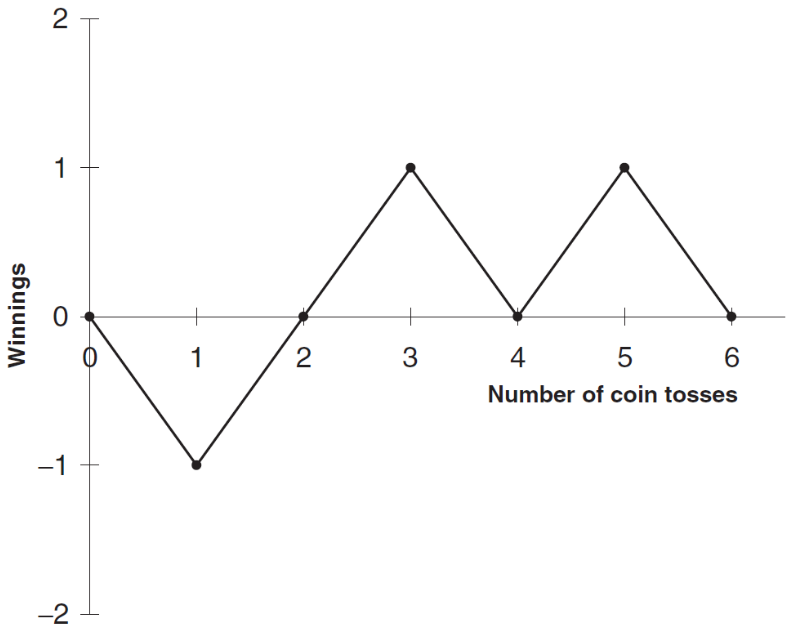


Figure 5.1 The outcome of a coin-tossing experiment.

$$S_i = \sum_{j=1}^{i} R_j. \tag{106}$$

Later on it will be useful if we have $S_0 = 0$, i.e. you start with no money. If we now calculate expectations of S_i it does matter what information we have. If we calculate expectations of future events before the experiment has even begun then

$$E[S_i] = 0 \text{ and } E[S_i^2] = E[R_1^2 + 2R_1R_2 + ...] = i. \tag{107}$$

On the other hand, suppose there have been five tosses already, can I use this information and what can we say about expectations for the sixth toss? This is the conditional expectation. The expectation of S_6 conditional upon the previous five tosses gives

$$E[S_6|R_1, ..., R_5] = S_5 \tag{108}$$

<u>THE MARKOV PROPERTY</u>

This result is special, the expected value of the random variable S_i conditional upon all of the past events only depends on the previous value S_{i1}. This is the **Markov property**. Note that it doesn't have to be the case that the expected value of the random variable S_i is the same as the previous value. This can be generalised to say that given information about S_j for some values of $1 \leq j < i$ then the only information that *is* of use to us in estimating S_i is the value of S_j for the largest j for which we have information.

<u>THE MARTINGALE PROPERTY</u>

You know how much money you have won after the fifth toss. Your expected winnings after the sixth toss, and indeed after any number of tosses if we keep playing, is just the amount you already hold. That is, the conditional expectation of your winnings at any time in the future is just the amount you already hold:

$$E[S_i|S_j, j < i] = S_j. \tag{109}$$

This is called the **martingale property**.

QUADRATIC VARIATION

The **quadratic variation** of the random walk is defined by

$$\sum_{j=1}^{i}(S_j - S_{j-1})^2. \tag{110}$$

Because you either win or lose an amount \$1 after each toss, $|S_j - S_{j-1}| = 1$. Thus the quadratic variation variation is always i:

$$\sum_{j=1}^{i}(S_j - S_{j-1})^2 = \sum_{j=1}^{i}|S_j - S_{j-1}| = i \tag{111}$$

BROWNIAN MOTION

Redefining previous rules of coin tossing experiment:

1. Restrict time allowed for the six tosses to a period t, so each toss will take a time $t/6$.
2. The size of the bet will not be \$1 but $t/6$.

Still Markovian and Martingale, its quadratic variation measured over the whole experiment is set up such that it's just the time taken for the experiment.

$$\sum_{j=1}^{6}(S_j - S_{j-1})^2 = 6 \times \left(\sqrt{\frac{t}{6}}\right)^2 = t \tag{112}$$

Changing the rules again to now have n tosses in the allowed time t, with an amount $\sqrt{t/n}$ riding on each row. Again, the Markov and martingale properties are retained and the quadratic variation is still

$$\sum_{j=1}^{n}(S_j - S_{j-1})^2 = n \times \left(\sqrt{\frac{t}{n}}\right)^2 = t. \tag{113}$$

Making n larger and larger, decreasing time between tosses, with a smaller amount for each bet. The new scalings are chosen so that the time step decreases like n^{-1} but the bet size only decreases by $n^{-1/2}$. Figure 5.2 shows a series of experiments, each lasting for a time 1, with increasing number of tosses per experiment. In the limit $n = \infty$, the resulting random walk stays finite. It has an expectation,

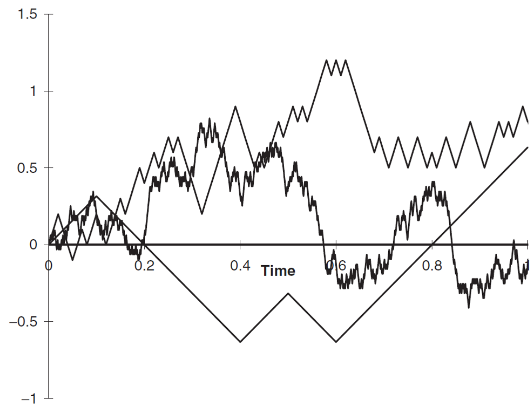


Figure 5.2 A series of coin-tossing experiments, the limit of which is Brownian motion.

conditional on a starting value of zero, of $E[S(t)] = 0$ and a variance $E[S(t)^2] = t$. Denoting the amount won or the value of the random variable after a time t as $S(t)$. The Brownian motion, denoted by $X(t)$, is the limiting process for this random walk as the time steps go to zero.

Important properties of Brownian motion:

- **Finiteness:** Any other scaling of the bet size or 'increments' with time step would have resulted in either a random walk going to infinity in a finite time, or a limit in which there was no motion at all. It is important that the increment scales with the square root of the time step.
- **Continuity:** The paths are continuous, there are no discontinuities. Brownian motion is the continuous-time limit of our discrete time random walk.
- **Markov:** The conditional distribution of $X(t)$ given information up until $\tau < t$ depends only on $X(\tau)$.
- **Martingale:** Given information up until $\tau < t$ the conditional expectation of $X(t)$ is $X(\tau)$.
- **Quadratic variation:** If we divide up the time 0 to t in a partition with $n+1$ partition points $t_i = it/n$ then
$$\sum_{j=1}^{n}(X(t_j) - X(t_{j-1}))^2 \to t. \text{ (almost surely)} \tag{114}$$
- **Normality:** Over finite time increments t_{i-1} to t_i, $X(t_i) - X(t_{i-1})$ is Normally distributed with mean zero and variance $t_i - t_{i-1}$.

STOCHASTIC INTEGRATION

Define a **stochastic integral** by

$$W(t) = \int_0^t f(\tau)dX(\tau) = \lim_{n\to\infty}\sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1})) \tag{115}$$

with

$$t_j = \frac{jt}{n}. \tag{116}$$

Crucially, note that the function $f(t)$ which I am integrating is evaluated in the summation at the left-hand point t_{j-1}. It will be crucially important that each function evaluation does not know about the random increment that multiplies it i.e. the integration is **non anticipatory**.

STOCHASTIC DIFFERENTIAL EQUATIONS

It's common to use the shorthand notation

$$dW = f(t)dX \tag{117}$$

which comes from differentiating

$$W(t) = \int_0^t f(\tau)dX(\tau) \tag{118}$$

Think of dX as being an increment in X, i.e. a Normal random variable with mean zero and standard deviation $dt^{1/2}$. Extending the idea, the stochastic differential equation

$$dW = g(t)dt + f(t)dX \tag{119}$$

is simply the shorthand for

$$W(t) = \int_0^t g(\tau)d\tau + \int_0^t f(\tau)dX(\tau). \tag{120}$$

THE MEAN SQUARE LIMIT

Examine the quantity

$$E\left[\left(\sum_{j=1}^{n}(X(t_j) - X(t_{j-1}))^2 - t\right)^2\right] \tag{121}$$

where

$$t_j = \frac{jt}{n}. \tag{122}$$

This can be expanded as

$$E\left[\sum_{j=1}^{n}(X(t_j)-X(t_{j-1})^4+2\sum_{i=1}^{n}\sum_{j<i}(X(t_i)-X(t_{i-1}))^2(X(t_j)-X(t_{j-1}))^2-2t\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2+t^2\right] \tag{123}$$

Since $X(t_j)-X(t_{j-1})$ is Normally distributed with mean zero and variance t/n we have

$$E[(X(t_j)-X(t_{j-1}))^2]=\frac{t}{n} \tag{124}$$

and

$$E[(X(t_j)-X(t_{j-1}))^4]=\frac{3t^2}{n^2} \tag{125}$$

Thus

$$\begin{aligned} E\left[\left(\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2-t\right)^2\right] &= n\frac{3t^2}{n^2}+n(n-1)\frac{t^2}{n^2}-2tn\frac{t}{n}+t^2 \\ &= O\left(\frac{1}{n}\right) \end{aligned} \tag{126}$$

As $n\to\infty$ this tends to zero. We therefore say that

$$\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2=t \tag{127}$$

in the 'mean square limit'. This is often written as

$$\int_0^t (dX)^2=t. \tag{128}$$

FUNCTIONS OF STOCHASTIC VARIABLES AND ITO'S LEMMA

Figure 5.3 shows a realisation of a Brownian motion $X(t)$ and the function $F(X)=X^2$. The ordinary

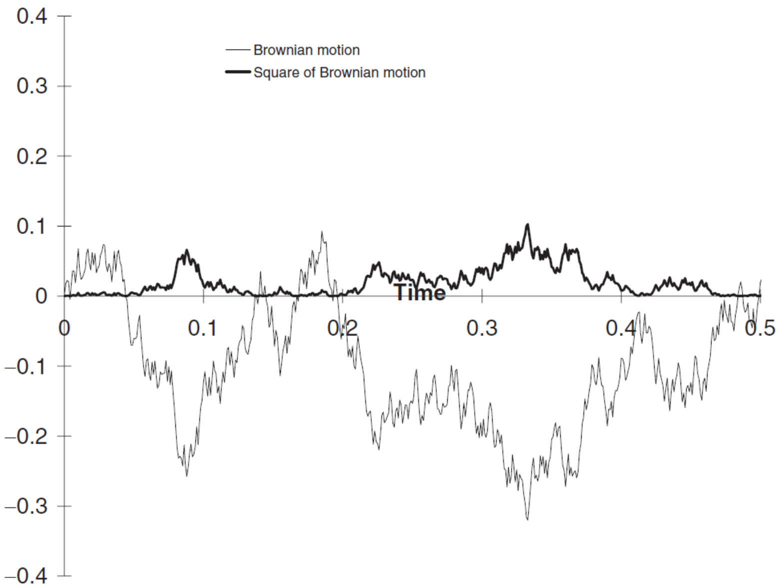


Figure 5.3 A realization of a Brownian motion and its square.

rules of calculus do not generally hold in a stochastic environment i.e. if $F=X^2$ it is not necessarily true that $dF=2XdX$. Introducing a timescale h

$$\frac{\delta t}{n}=h \tag{129}$$

so small where the function $F(X(t+h))$ can be approximated by a Taylor series:

$$F(X(t+h)) - F(X(t)) = (X(t+h) - X(t))\frac{dF(X(t))}{dX} + \frac{1}{2}(X(t+h) - X(t))^2\frac{d^2F(X(t))}{dX^2} + ... \tag{130}$$

from this it follows that

$$\begin{aligned}
&(F(X(t+h)) - F(X(t))) + (F(X(t+2h)) - F(X(t+h))) + ... + (F(X(t+nh)) - F(X(t+(n-1)h))) \\
&= (X(t+h) - X(t))\frac{dF(X(t))}{dX} + \frac{1}{2}(X(t+h) - X(t))^2\frac{d^2F(X(t))}{dX^2} + ... \\
&+(X(t+2h) - X(t+h))\frac{dF(X(t+h))}{dX} + \frac{1}{2}(X(t+2h) - X(t+h))^2\frac{d^2F(X(t+h))}{dX^2} + ... \\
&= \sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))\frac{dF(X(t+(j-1)h))}{dX} \\
&+\frac{1}{2}\sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))^2\frac{d^2F(X(t+(j-1)h))}{dX^2} \\
&= \sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))\frac{dF(X(t+(j-1)h))}{dX} \\
&+\frac{1}{2}\frac{d^2F(X(t))}{dX^2}\sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))^2 + ...
\end{aligned} \tag{131}$$

where the approximation

$$\frac{d^2F(X(t+(j-1)h))}{dX^2} = \frac{d^2F(X(t))}{dX^2} \tag{132}$$

is used. Now observe that the first line is just

$$\begin{aligned}
&(F(X(t+h)) - F(X(t))) + (F(X(t+2h)) - F(X(t+h))) + ... + (F(X(t+nh)) - F(X(t+(n-1)h))) \\
&= F(X(t+nh)) - F(X(t+(n-1)h)) + F(X(t+(n-1)h)) - ... - F(X(t+2h)) + F(X(t+2h)) \\
&-F(X(t+h)) + F(X(t+h)) - F(X(t)) \\
&= F(X(t+nh)) - F(X(t)) \\
&= F(X(t+\delta t)) - F(X(t))
\end{aligned} \tag{133}$$

By using what we previously had

$$\begin{aligned}
W(t) &= \int_0^t f(\tau)dX(\tau) \\
&= \lim_{n\to\infty}\sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1})) \\
&= \lim_{n\to\infty}\sum_{j=1}^{n} f((j-1)t/n)(X(jt/n) - X((j-1)t/n))
\end{aligned} \tag{134}$$

we see that by definition the second line is just

$$\begin{aligned}
&\sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))\frac{dF(X(t+(j-1)h))}{dX} \\
&= \sum_{j=1}^{n}(X(t+j\delta t/n) - X(t+(j-1)\delta t/n))\frac{dF(X(t+(j-1)\delta t/n)))}{dX} \\
&= \int_t^{t+\delta t}\frac{dF}{dX}dX
\end{aligned}$$

and the last is

$$\frac{1}{2}\frac{d^2F(X(t))}{dX^2}\sum_{j=1}^{n}(X(t+jh)-X(t+(j-1)h))^2 = \frac{1}{2}\frac{d^2F(X(t))}{dX^2}\sum_{j=1}^{n}(X(t+j\delta t/n)-X(t+(j-1)\delta t/n))^2$$
$$= \frac{1}{2}\frac{d^2F(X(t))}{dX^2}\delta t$$
$$= \frac{1}{2}\int_t^{t+\delta t}\frac{d^2F(X(\tau))}{dX^2}d\tau \tag{135}$$

Thus we have

$$F(X(t+\delta t))-F(X(t)) = \int_t^{t+\delta t}\frac{dF(X(\tau))}{dX}dX(\tau)+\frac{1}{2}\int_t^{t+\delta t}\frac{d^2F(X(\tau))}{dX^2}d\tau \tag{136}$$

Extending this result over longer timescales, from zero up to t, over which F does vary substantially to get

$$F(X(t)) = F(X(0))+\int_0^t\frac{dF(X(\tau))}{dX}dX(\tau)+\frac{1}{2}\int_0^t\frac{d^2F(X(\tau))}{dX^2}d\tau \tag{137}$$

which is the integral version of **Ito's Lemma**, usually written as

$$dF = \frac{dF}{dX}dX+\frac{1}{2}\frac{d^2F}{dX^2}dt \tag{138}$$

If $F = X^2$ then

$$\frac{dF}{dX} = 2X \text{ and } \frac{d^2F}{dX^2} = 2 \tag{139}$$

hence Ito's lemma tells us that the stochastic differential equation that F satisfies is

$$dF = 2XdX+dt. \tag{140}$$

This is not what we would get if X were a deterministic variable. In integrated form

$$X^2 = F(X) = F(0)+\int_0^t 2XdX+\int_0^t 1d\tau = \int_0^t 2XdX+t. \tag{141}$$

Therefore,

$$\int_0^t XdX = \frac{1}{2}X^2-\frac{1}{2}t. \tag{142}$$

INTERPRETATION OF ITO'S LEMMA

In figure 5.4 we see at the top a realisation of a stock price, just a basic lognormal random walk. Below this is the value of an option on this stock. Notice that both plots have a direction to them (both are rising overall) and both have a random element (the bouncing around of the values). Both look stochastic and we know that the stock price satisfies a stochastic differential equation

$$dS = \mu Sdt+\sigma SdX \tag{143}$$

so maybe the option value (call it $V(S,t)$) also satisfies a stochastic differential equation

$$dV = \underline{\quad\quad}dt+\underline{\quad\quad}dX. \tag{144}$$

Ito will tell us the underlined variables.

ITO AND TAYLOR

If we were to do a naive Taylor series expansion of F, completely disregarding the nature of X, and treating dX as a small increment in X, we would get

$$F(X+dX) = F(X)+\frac{dF}{dX}dX+\frac{1}{2}\frac{d^2F}{dX^2}dX^2 \tag{145}$$

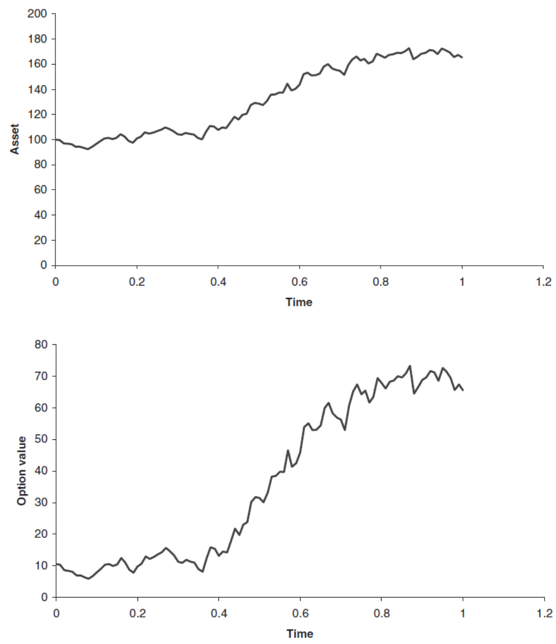


Figure 5.4 A realization of a stock price and the value of an option on that stock.

ignoring higher-order terms. We could argue that $F(X+dX)-F(X)$ was just the 'change in' F and so

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dX^2 \tag{146}$$

This is very similar to Ito's Lemma

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt \tag{147}$$

with the only difference being that there is a dX^2 instead of dt. However, since in a sense

$$\int_0^t (dX)^2 = t \tag{148}$$

I could perhaps write

$$dX^2 = dt. \tag{149}$$

This is technically incorrect, but it's a good rule of thumb for differentiating a function of a random variable.
To generalise slightly, suppose my stochastic differential equation is

$$dS = a(S)dt + b(S)dX, \tag{150}$$

say, for some functions $a(S)$ and $b(S)$. Here dX is the usual Brownian increment. Now if I have a function of S, $V(S)$, what stochastic differential equation does it satisfy? The answer is

$$dV = \frac{dV}{dS}dS + \frac{1}{2}b^2\frac{d^2V}{dS^2}dt \tag{151}$$

We could derive this properly or just cheat by using Taylor series with $dX2 = dt$. I could, if I wanted, substitute for dS from $dS = a(S)dt + b(S)dX$ to get an equation for dV in terms of the pure Brownian motion X:

$$dX = \left(a(S)\frac{dV}{dS} + \frac{1}{2}b(S)^2\frac{d^2V}{dS^2}\right)dt + b(S)\frac{dV}{dS}dX. \tag{152}$$

Intuition behind $\mathbf{dX^2 = dt}$

We shouldn't really think of dX^2 as being the square of a single Normally distributed random variable, mean zero, variance dt. No, we should think of it as the sum of squares of lots and lots (an infinite number) of independent and identically distributed Normal variables, each one having mean zero and a very, very small (infinitesimal) variance. What happens when you add together lots of i.i.d. variables? In this case we get a quantity with a mean of dt and a variance which goes rapidly to zero as the 'lots' approach 'infinity.'

ITO IN HIGHER DIMENSIONS

In financial problems we often have functions of one stochastic variable S and a deterministic variable t, time: $V(S,t)$. If

$$dS = a(S,t)dt + b(S,t)dX, \tag{153}$$

then the increment dV is given by

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}dt \tag{154}$$

Occasionally, we have a function of two, or more, random variables, and time as well: $V(S_1, S_2, t)$. An example would be the value of an option to buy the more valuable out of Nike and Reebok. Let the behaviour of S_1 and S_2 have the general form

$$dS_1 = a_1(S_1, S_2, t)dt + b_1(S_1, S_2, t)dX_1 \tag{155}$$

and

$$dS_2 = a_2(S_1, S_2, t)dt + b_2(S_1, S_2, t)dX_2 \tag{156}$$

The two Brownian increments dX_1 and dX_2 can be thought of as being Normally distributed with variance dt, and are correlated. Let ρ be the correlation between these two random variables, which can be a function of S_1, S_2 and t but must satisfy

$$-1 \le \rho \le 1 \tag{157}$$

The 'rules of thumb' are:

$$dX_1^2 = dt, dX_2^2 = dt \text{ and } dX_1X_2 = \rho dt. \tag{158}$$

Ito's lemma becomes

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1 + \frac{\partial V}{\partial S_2}dS_2 + \frac{1}{2}b_1^2\frac{\partial^2 V}{\partial S_1^2}dt + \rho b_1 b_2\frac{\partial^2 V}{\partial S_1 \partial S_2}dt + \frac{1}{2}b_2^2\frac{\partial^2 V}{\partial S_2^2}dt. \tag{159}$$

SOME PERTINENT EXAMPLES

Brownian Motion with drift

Example 1: A simple Brownian motion with a drift:

$$dS = \mu dt + \sigma dX \tag{160}$$

is shown in Fig. 5.5. Note that S has gone negative. Such a random walk would therefore not be a good model for many financial quantities, such as interest rates or equity prices. This stochastic differential equation can be integrated exactly to get

$$S(t) = S(0) + \mu t + \sigma(X(t) - X(0)) \tag{161}$$

Example 2: A similar Brownian motion with a drift and randomness scales with S:

$$dS = \mu S dt + \sigma S dX \tag{162}$$

shown in Fig. 5.6.

If S starts out positive it can never go negative; the closer that S gets to zero the smaller the increments dS. For this reason I have had to start the simulation with a non-zero value for S. This property of this random walk is clearly seen if we examine the function $F(S) = \log S$ using Ito's lemma. From Ito's we have

$$\begin{aligned} dF &= \frac{dF}{dS}dS + \frac{1}{2}\sigma^2 S^2\frac{d^2F}{dS^2}dt \\ &= \frac{1}{S}dS - \frac{1}{2}\sigma^2 S^2\frac{1}{S^2}dt \\ &= \frac{1}{S}(\mu S dt + \sigma S dX) - \frac{1}{2}\sigma^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX. \end{aligned} \tag{163}$$

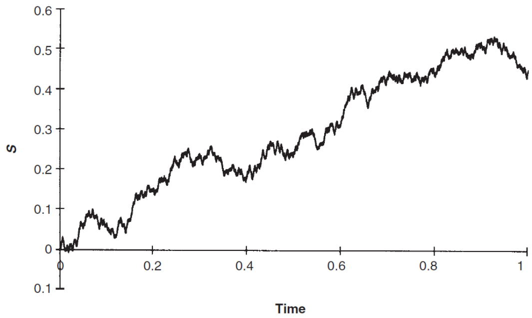


Figure 5.5 A realization of $dS = \mu\, dt + \sigma\, dX$.

This shows us that $\log S$ can range between minus and plus infinity but cannot reach these limits in a finite time, therefore S cannot reach zero or infinity in a finite time. The integral form of this stochastic differential equation follows simply from the stochastic differential equation for $\log S$:

$$\begin{aligned} dF &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX \\ \implies \int_0^t \frac{1}{S} dS &= \int_0^t (\mu - \frac{1}{2}\sigma^2)d\tau + \int_0^t \sigma dX \\ \implies \log(S(t) - S(0)) &= (\mu - \frac{1}{2}\sigma^2)t + \sigma(X(t) - X(0)) \\ \implies S(t) &= S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma(X(t) - X(0))} \end{aligned} \tag{164}$$

The stochastic differential equation $dS = Sdt + SdX$ is important for modelling many asset classes. If

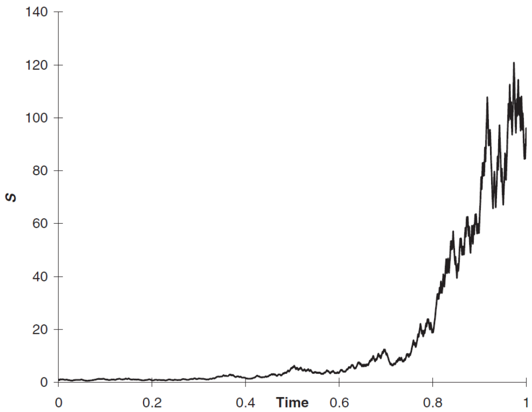


Figure 5.6 A realization of $dS = \mu S\, dt + \sigma S\, dX$.

we have some function $V(S, t)$ then from Ito it follows that

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt. \tag{165}$$

Example 3: A mean-reverting random walk

$$dS = (\nu - \mu S)dt + \sigma dX \tag{166}$$

is shown in Fig 5.8. If S is large, the negative coefficient in front of dt means that S will move down on average, if S is small it rises on average. There is still no incentive for S to stay positive in this random walk. With r instead of S this random walk is the Vasicek model for the short-term interest rate. Mean-reverting models are used for modelling a random variable that 'isn't going anywhere' i.e. interest rates.

Example 4: Another mean-reverting random walk

$$dS = (\nu - \mu S)dt + \sigma S^{1/2} dX \tag{167}$$

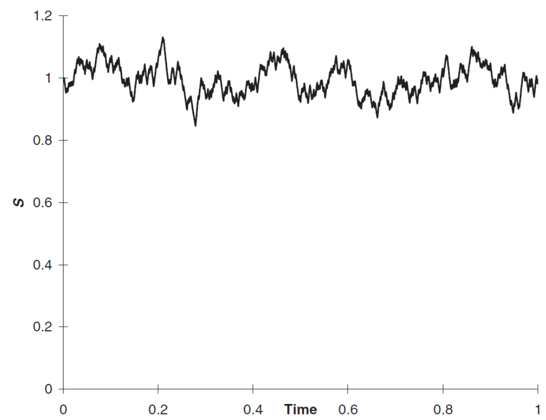


Figure 5.8 A realization of $dS = (\nu - \mu S)\,dt + \sigma\, dX$.

Now if S ever gets close to zero the randomness decreases, perhaps this will stop S from going negative? By Ito's lemma, we have

$$\begin{aligned} dS &= a(S,t)dt + b(S,t)dX \\ dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}dt. \end{aligned} \tag{168}$$

Let $F = S^{1/2}$ such that

$$\frac{dF}{dS} = \frac{1}{2}S^{-1/2} \text{ and } \frac{\partial^2 F}{\partial S^2} = -\frac{1}{4}S^{-3/2} \tag{169}$$

F then satisfies the stochastic differential equation

$$\begin{aligned} dF &= \frac{\partial F}{\partial S}dS + \frac{1}{2}b^2\frac{\partial^2 F}{\partial S^2}dt \\ &= \frac{1}{2}S^{-1/2}\left[(\nu - \mu S)dt + \sigma S^{1/2}dX\right] + \frac{1}{2}(\sigma S^{1/2})^2\left(-\frac{1}{4}S^{-3/2}\right)dt \\ &= \frac{1}{2}S^{-1/2}\nu dt - \frac{1}{2}\mu S^{1/2}dt + \frac{1}{2}\sigma dX - \frac{1}{8}\sigma S^{-1/2}dt \\ &= \frac{1}{2F}\nu dt - \frac{1}{2}\mu F dt + \frac{1}{2}\sigma dX - \frac{1}{8F}\sigma dt \\ &= \left(\frac{4\nu - \sigma^2}{8F} - \frac{1}{2}\mu F\right)dt + \frac{1}{2}\sigma dX. \end{aligned} \tag{170}$$

The original stochastic differential equation with a variable coefficient in front of the random term is now turned into a stochastic differential equation with a constant random term, making the drift term more complicated. In particular, the drift is now singular at $F = S = 0$. Something special is happening at $S = 0$.

Instead of examining $F(S) = S^{1/2}$, can I find a function $F(S)$ such that its stochastic differential equation has a zero drift term? For this I will need

$$(\nu - \mu S)\frac{dF}{dS} + \frac{1}{2}\sigma^2 S\frac{d^2F}{dS^2} = 0. \tag{171}$$

Integrating this

$$\begin{aligned} 0 &= \frac{2}{\sigma^2}\left(\frac{\nu}{S} - \mu\right)\frac{dF}{dS} + \frac{d}{dS}\left(\frac{dF}{dS}\right) \\ &= \frac{2}{\sigma^2}\left(\frac{\nu}{S} - \mu\right)e^{\int \frac{2}{\sigma^2}(\frac{\nu}{S}-\mu)dS}\frac{dF}{dS} + e^{\int \frac{2}{\sigma^2}(\frac{\nu}{S}-\mu)dS}\frac{d}{dS}\left(\frac{dF}{dS}\right) \\ &= \frac{d}{dS}\left[e^{\int \frac{2}{\sigma^2}(\frac{\nu}{S}-\mu)dS}\right]\frac{dF}{dS} + e^{\int \frac{2}{\sigma^2}(\frac{\nu}{S}-\mu)dS}\frac{d}{dS}\left(\frac{dF}{dS}\right) \\ &= \frac{d}{dS}\left[e^{\int \frac{2}{\sigma^2}(\frac{\nu}{S}-\mu)dS}\frac{dF}{dS}\right] \end{aligned} \tag{172}$$

To integrate the exponent of the exponential

$$\begin{aligned}
\exp\left(\int \frac{2}{\sigma^2}\left(\frac{\nu}{S}-\mu\right) dS\right) &= C_1 \exp\left(\frac{2}{\sigma^2}\left(\nu \log(S)-\mu S\right)\right) \\
&= C_1 \exp\left(\frac{2\nu}{\sigma^2}\log(S)\right)\exp\left(-\frac{2\mu S}{\sigma^2}\right) \\
&= C_1 S^{\frac{2\nu}{\sigma^2}} e^{-\frac{2\mu S}{\sigma^2}}
\end{aligned} \tag{173}$$

Hence we obtain

$$\begin{aligned}
0 &= \frac{d}{dS}\left[C_1 S^{\frac{2\nu}{\sigma^2}} e^{-\frac{2\mu S}{\sigma^2}} \frac{dF}{dS}\right] \\
\implies C_2 &= C_1 S^{\frac{2\nu}{\sigma^2}} e^{-\frac{2\mu S}{\sigma^2}} \frac{dF}{dS} \\
\implies \frac{dF}{dS} &= A S^{-\frac{2\nu}{\sigma^2}} e^{\frac{2\mu S}{\sigma^2}}
\end{aligned} \tag{174}$$

for any constant $A = C_2/C_1$. Observe that if

$$\frac{2\nu}{\sigma^2} \geq 1, \tag{175}$$

$\frac{dF}{dS}$ cannot be integrated at $S = 0$ because we have $\frac{1}{S^n}$ where $ngeq1$. This makes the origin **non attainable**. In other words, if the parameter is sufficiently large it forces the random walk to stay away from zero. This is the Cox, Ingersoll & Ross model for the short-term interest rate.

SUMMARY

If we think of S as the value of an asset for which we have a stochastic differential equation, a 'model,' then we can handle functions of the asset, and ultimately value contracts such as options.

0.6 Chapter 6: The Black-Scholes Model

Introduction

The basic building blocks of derivatives theory are delta hedging and no arbitrage. This chapter includes:

- stochastic differential equation model for equities and exploit the correlation between this asset and an option on this asset to make a perfectly risk-free portfolio.
- using no arbitrage to equate returns on all risk-free portfolios to the risk-free interest rate, the so-called 'no free lunch' argument

A VERY SPECIAL PORTFOLIO

The value of a call option is clearly going to be a function of various parameters in the contract, such as the strike price E and the time to expiry $T - t$, T is the date of expiry, and t is the current time. The value will also depend on properties of the asset itself, such as its price, its drift and its volatility, as well as the risk-free rate of interest. The option value is written as

$$V(S, t; \sigma, \mu; E, T; r). \tag{176}$$

- S and t are variables
- σ and μ are parameters associated with the asset price
- E and T are parameters associated with the details of the particular contract
- r is a parameter associated with the currency in which the asset is quoted

One simple observation is that a call option will rise in value if the underlying asset rises, and will fall if the asset falls. This is clear since a call has a larger payoff the greater the value of the underlying at expiry. This is an example of correlation between two financial instruments, in this case the correlation is positive. A put and the underlying have a negative correlation. We can exploit these correlations to construct a very special portfolio.

Use Π to denote the value of a portfolio of one long option position and a short position in some quantity δ, delta, of the underlying:

$$\Pi = V(S,t) - \Delta S. \tag{177}$$

$V(S,t)$ is the option and $-\Delta S$ is the short asset position. The quantity Δ will for the moment be some constant quantity of our choosing. We will assume that the underlying follows a lognormal random walk

$$dS = \mu S dt + \sigma S dX. \tag{178}$$

The change in the portfolio value from t to $t + dt$ is due partly to the change in the option value and partly to the change in the underlying:

$$d\Pi = dV - \Delta dS. \tag{179}$$

Notice that δ has not changed during the time step; we have not anticipated the change in S. From Ito we have

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \tag{180}$$

Thus the portfolio changes by

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta dS \tag{181}$$

This says that our special portfolio takes different values depending on what the asset does over the next time step. In the binomial model there were two different values that the portfolio could take, represented by the up and down movements of the asset. In the Black–Scholes model there's a whole spectrum of possible values represented by the dS terms ... so the dS terms represent the risk in the portfolio. And just as in the binomial model we're going to make these terms disappear.

ELIMINATION OF RISK: DELTA HEDGING

In

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta dS \tag{182}$$

the deterministic terms are those with the dt, and the random terms are those with the dS. Pretending for the moment that we know V and its derivatives then we know everything about the right-hand side except for the value of dS. And this quantity we can never know in advance. Theoretically(and almost in practice), the risk in our portfolio i.e. the random terms

$$\left(\frac{\partial V}{\partial S} - \Delta\right) dS \tag{183}$$

can be reduced or even eliminated by carefully choosing Δ. Choosing $\frac{\partial V}{\partial S} = \Delta$ reduces the randomness to zero.

Any reduction in randomness is generally termed **hedging**, whether that randomness is due to fluctuations in the stock market or the outcome of a horse race. The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and its underlying), is generally called **delta hedging**. Delta hedging is an example of a **dynamic hedging** strategy. From one time step to the next the quantity $\frac{\partial V}{\partial S}$ changes, since it is, like V, a function of the ever-changing variables S and t. This means that the perfect hedge must be continually rebalanced.

NO ARBITRAGE

With risk eliminated above, our portfolio value changes by the amount

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \tag{184}$$

This is a *riskless* change. If we have a completely risk-free change $d\Pi$ in the portfolio value Π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi dt \tag{185}$$

This is an example of the **no-arbitrage** principle.

Consider in turn what might happen if the return on the portfolio were, first, greater and, second, less than the risk-free rate. If we were guaranteed to get a return of greater than r from the delta-hedged portfolio then what we could do is borrow from the bank, paying interest at the rate r, invest in the risk-free option/stock portfolio and make a profit. If, on the other hand, the return were less than the risk-free rate we should go short the option, delta hedge it, and invest the cash in the bank. Either way, we make a riskless profit in excess of the risk-free rate of interest. At this point we say that, all things being equal, the action of investors buying and selling to exploit the arbitrage opportunity will cause the market price of the option to move in the direction that eliminates the arbitrage.

THE BLACK-SCHOLES EQUATION

By substituting

$$\Pi = V(S,t) - \Delta S, \tag{186}$$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{187}$$

and

$$\frac{\partial V}{\partial S} = \Delta \tag{188}$$

into

$$d\Pi = r\Pi dt \tag{189}$$

we obtain

$$\begin{aligned} \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt &= r\left(V - \frac{\partial V}{\partial S} S \right) dt \\ \implies \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \end{aligned} \tag{190}$$

This is the **Black-Scholes equation**, which is a **linear parabolic partial differential equation**. Almost all PDEs in finance are almost always linear, meaning that if you have two solutions of the equation then the sum of these is itself also a solution. Financial equations are also usually parabolic, meaning that they are related to the heat or diffusion equation of mechanics.

Why is there no drift rate μ? Any dependence on the drift dropped out at the same time as we eliminated the dS component of the portfolio. The economic argument for this is that since we can perfectly hedge the option with the underlying we should not be rewarded for taking unnecessary risk; only the risk-free rate of return is in the equation. This means that if you and I agree on the volatility of an asset we will agree on the value of its derivatives *even if we have differing estimates of the drift.*

Another way of looking at the hedging argument is to ask what happens if we hold a portfolio consisting of just the stock, in a quantity Δ, and cash. If Δ is the partial derivative of some option value then such a portfolio will yield an amount at expiry that is simply that option's payoff. In other words, we can use the same Black–Scholes argument to **replicate** an option just by buying and selling the underlying asset. This leads to the idea of a **complete market**. In a complete market an option can be replicated with the underlying, thus making options redundant. Why buy an option when you can get the same payoff by trading in the asset? Many things conspire to make markets incomplete such as transaction costs.

Slopes, gradients, etc.

The Black–Scholes partial differential equation is a relationship between the option value, the gradient in the S and t directions and the gradient of the gradient in the S direction. Imagine you're at expiry of a call option. At that time do you know the option value as a function of the underlying asset S? Yes, of course, it's just the payoff function

$$\max(S - E, 0). \tag{191}$$

Meaning you know the term rV in the BS equation. Do you know the slope of the option value in the S direction at expiry? You certainly do. It's zero for $S < E$ and one for $S > E$. Now you know the

$rS\frac{\partial V}{\partial S}$ term of the BSE. Mathematically, this is represented by the Heaviside function, $\mathcal{H}(.)$, zero when its argument is negative and one when it is positive. So

$$\frac{\partial V}{\partial S} = \mathcal{H}(S - E) \tag{192}$$

The slope of the slope in the S direction is thus

$$\frac{\partial^2 V}{\partial S^2} = 0. \tag{193}$$

So we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \times 0 + rS\mathcal{H}(S - E) - r\max(S - E, 0) = 0. \tag{194}$$

This is an equation for $\frac{\partial V}{\partial t}$. For example, if $S < E$ we have

$$\frac{\partial V}{\partial t} + rS \times 0 - r\max(S - E, 0) = 0. \tag{195}$$

$$\implies \frac{\partial V}{\partial t} = 0 \tag{196}$$

If $S > E$ we have

$$\frac{\partial V}{\partial t} = -rS + rS - RE = -rE \tag{197}$$

Significance: If we know $\frac{\partial V}{\partial t}$ then we know the slope of the option value in the t direction. If we know this slope then we can find the option value at the time just before expiry. If we are at time $T - \delta t$, where δt is small, then the option value will be approximately

$$V = 0 \text{ for } S < E \tag{198}$$

and

$$V = S - E + rE\delta t \text{ for } S > E. \tag{199}$$

See how we have found the option value one time step before expiry? We can keep repeating this procedure over and over, working backwards in time until we get to the present. And as the time step gets smaller, so this approximation to the option value gets more accurate.

THE BLACK-SCHOLES ASSUMPTIONS