Notes on Paul Wilmott Introduces Quantitative Finance

Darren Ng

July 2021

0.1 Chapter 1: Products and Markets: Equities, Commodities, Exchange Rates, Forwards and Futures

THE TIME VALUE OF MONEY

Two types of interest:

- Simple interest - When the interest you received is based only on the amount you initially invest
- Compound interest - When you also get interest on your interest.

Two forms of compound interest: discretely compounded and continuously compounded.

Invest \$1 in bank at a discrete interest rate r paid once per annum. At the end of year 1 bank account will contain $1 \times (1+r)$. After 2 years and n years I will have $(1+r)^2$ and $(1+r)^n$. For $r = 10\%$, for first and second year I will have \$1.10 and \$1.21.

Suppose I receive m interest payments at a rate of r/m per annum. After 1 year I'll have

$$\left(1+\frac{r}{m}\right)^m \tag{1}$$

Imagine interest payments come at increasingly frequent intervals but at an increasingly smaller interest rate. Taking limit $m \to 100$ leads to a rate of interest that is paid continuously, giving money amount in bank after one year if interest is continuously compounded

$$\begin{aligned}\left(1+\frac{r}{m}\right)^m &= e^{\ln(1+\frac{r}{m})^m} \\ &= e^{m\ln(1+\frac{r}{m})} \\ &\sim e^r\end{aligned} \tag{2}$$

After time t this amounts to

$$\left(1+\frac{r}{m}\right)^{mt} \sim e^{rt} \tag{3}$$

This can also be derived by a differential equation. For an amount $M(t)$ in the bank at time t, checking bank account at time t and slightly later $t + dt$, the amount will have increased by

$$M(t+dt) - M(t) \approx \frac{dM}{dt}dt + ... \tag{4}$$

where RHS is just a Taylor series expansion. But the interest I receive must be $\propto M(t), r$, and dt. Hence

$$\begin{aligned}\frac{dM}{dt}dt &= rM(t)dt \\ \implies \frac{dM}{dt} &= rM(t)\end{aligned} \tag{5}$$

with solution

$$M(t) = M(0)e^{rt} \tag{6}$$

relating money value I have now to the value in the future. Conversely, if I know I'll get \$1 at time T in the future, its value at an earlier time t is $e^{-r(T-t)}$. This relates future cashflows to present by multiplying this factor e.g. for $r = 0.05$, the present value ($t = 0$) of \$1000000 to be received in two years ($T = 2$) is \$1000000 $\times e^{-0.05*2}$ = \$904837.

FIXED-INCOME SECURITIES

Two types of interest payments exist: **fixed** and **floating**. **Coupon-bearing** bonds pay out a known amount every six months or year, etc. This is the **coupon** and would often be a fixed rate of interest. At the end of your fixed term you get a final coupon and return of the **principal**, the amount on which the interest was calculated. **Interest rate swaps** are an exchange of a fixed rate of interest for a floating rate of interest. Governments and companies issue bonds as a form of borrowing. The less creditworthy the issuer, the higher the interest that they will have to pay out. Bonds are actively traded, with prices that continually fluctuate.

INFLATION PROOF BONDS

UK inflation is measured by **Retail Price Index (RPI)**. This index is

- a measure of year-on-year inflation using a 'basket' of goods and services including mortgage interest payments.
- published monthly.
- roughly speaking, the amounts of the coupon and principal are scaled with the increase in the RPI over the period from the issue of the bond to the time of the payment.

US inflation is measured by **Consumer Price Index (CPI)**.

FORWARDS AND FUTURES

A **forward contract** is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price.

- No money changes hands until the **delivery date** or **maturity** of the contract.
- It's an absolute obligation to buy the asset at the delivery date.
- The asset can be a stock, a commodity or a currency.
- The **delivery price** is the amount that's paid for the asset at the delivery date.
  - Set at the time the forward contract is entered into.
  - At an amount that gives the forward contract a value of zero initially.
  - As maturity approaches the particular forward contract value we hold will change in value, from initially zero to the difference between the underlying asset and the delivery price at maturity.
- The **forward price** of different maturities are the delivery prices for forward contracts of the quoted maturities, should we enter into such a contract now.

A **futures contract** is very similar to forward contract. Futures are derivative financial contracts that obligate the parties to transact an asset at a predetermined future date and price. The buyer must purchase or the seller must sell the underlying asset at the set price, regardless of the current market price at the expiration date.

- Usually traded through an exchange, are very liquid instruments and have lots of rules and regulations surrounding them.
- Profit or loss from futures position is calculated everyday and the value change is paid from one party to the other, hence there is a gradual payment of funds from initiation until maturity.
- Because you settle the change in value on a daily basis, the value of a futures contract at any time during its life is zero.
- Prices vary day to day, but at maturity must be the same as the asset you're buying.
- Provided interest rates are known in advance, forward prices and futures prices of the same maturity must be identical.

Forwards and futures have two main uses:

- Speculation
  - In believing market will rise, you can benefit by entering into a forward/futures contract.

– Money will exchange hands at maturity/every day in your favour.

- Hedging i.e. avoidance of risk
  - If you are expecting to get paid in yen in six months' time but your expenses are all in dollars, you can enter into a futures contract to guarantee an exchange rate for your yen income amount.
  - You're locked in this dollar/yen exchange rate. The lack of exposure to fluctuations means you won't benefit if yen appreciates.

The **no-arbitrage** principle: Consider a forward contract that obliges us to pay $\$F$ at time T to receive the underlying asset. The **spot price** $\$S(t)$ is the asset price at present time t for which we could get immediate delivery of the asset. At maturity, we'll pay \$ and receive the asset, then worth $\$S(T)$, a value that remains unknown until time T which determines the profit/loss amount $S(T) - F$. By entering into a special portfolio of trades now we can eliminate all randomness in the future.

1. Enter into the forward contract, which costs nothing up front but exposes us to the uncertainty in the asset value at maturity.
2. Simultaneously sell the asset a.k.a **going short** i.e. when you sell something you don't own with some timing restrictions.
3. Still with net position of zero, we now have $S(t)$ amount of cash from the sale of asset, a forward contract, and a short asset position $-S(t)$. Put cash in bank to receive interest.
4. At maturity we pay F and receive asset $S(T)$. The bank account amount with interest is now $S(t)e^{r(T-t)}$, with a net position at maturity of $S(t)e^{r(T-t)} - F$.

Table 1.1 Cashflows in a hedged portfolio of asset and forward.

Holding	Worth today (t)	Worth at maturity (T)
Forward	0	$S(T) - F$
−Stock	$-S(t)$	$-S(T)$
Cash	$S(t)$	$S(t)e^{r(T-t)}$
Total	0	$S(t)e^{r(T-t)} - F$

5. The no-arbitrage principle states that a portfolio started with zero worth end up with a predictable amount, which should also be zero. Hence we've a relationship between spot price and forward price.

$$F = S(t)e^{r(T-t)} \tag{7}$$

6. If $F < S(t)e^{r(T-t)}$, a riskless arbitrage opportunity can be exploited by entering into the same deals.
7. At maturity you will have $S(t)e^{r(T-t)}$ in the bank, a short asset and a long forward. The asset position cancels when you hand over the amount F, leaving you with a profit of $S(t)e^{r(T-t)} - F$.
8. If $F > S(t)e^{r(T-t)}$, simply enter into the opposite positions i.e. going short the forward in order to make a riskless profit.

MORE ABOUT FUTURES

The nature of futures contracts:

Available assets

A futures contract will specify

- the asset which is being invested in
- the quantity of asset that must be delivered
- the quality of the commodities, usually comes in a variety of grades i.e. oil, sugar, orange juice, wheat, etc. futures contracts lay down rules for precisely what grade of oil, sugar, etc. may be delivered.

Delivery and settlement
There may be some leeway in the precise delivery date

- Most futures contracts are closed out before delivery, with the trader taking the opposite position before maturity.
- If position is not closed, then asset is delivered.
- When the asset is another financial contract settlement is usually made in cash.

Margin

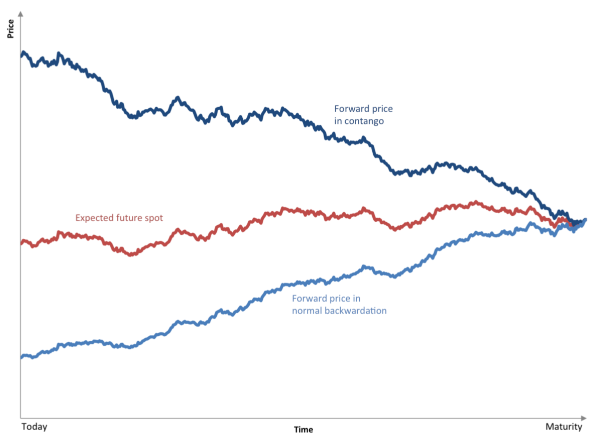
- **Marking to market** - The changes in value of futures contracts are settled each day.
- Exchanges insist on traders depositing a sum of money in a **margin account** to cover changes in their positions value.
- As the position is marked to market daily, money is deposited or withdrawn from this margin account.

Two types of margin: **Initial margin**, **Maintenance margin**
The initial amount is the amount deposited at the initiation of the contract. The total amount held as margin must stay above a prescribed maintenance margin. If it ever falls below this level then more money (or equivalent in bonds, stocks, etc.) must be deposited.

Commodity futures
Futures on commodities don't necessarily obey the no-arbitrage law due to storage. In practice the future price will be higher than the theoretical no-storage-cost amount since the holder of the futures contract must compensate the holder of the commodity for his storage costs. Often the people holding the commodity benefit from it in some way. The benefit from holding the commodity is commonly measured in terms of the **convenience yield** c:

$$F = S(t)e^{(r+s-c)(T-t)} \tag{8}$$

Whenever $F < S(t)e^{r(T-t)}$ the market is said to be in **backwardation**, otherwise if $F > S(t)e^{r(T-t)}$ the market is in **contango**.



FX futures
Modifying the result from no-arbitrage to allow for interest received on the foreign currency r_f, we get

$$F = S(T)e^{(r-r_f)(T-t)}. \tag{9}$$

Index futures
Futures contracts on stock indices are settled in cash, with dividends playing a similar role to that of a foreign interest rate on FX futures. So

$$F = S(t)e^{(r-q)(T-t)} \tag{10}$$

0.2 Chapter 2: Derivatives

OPTIONS

The holder of future or forward contracts is obliged to trade at the maturity of the contract, unless the position is closed before maturity. Otherwise, the holder must take possession of the asset of the contract e.g commodity, currency regardless of whether its price has risen or fallen. The simplest **option** gives the holder the *right* to trade in the future at a previously agreed price but takes away the obligation. Specifically, a **call option** is the right to buy an asset for an agreed amount at a specified time in the future.
Example: A call option on Microsoft stock

- The holder has the right to purchase one Microsoft stock for \$25 in one month's time, with \$24.5 current price.
- The price \$25 is called the **exercise/strike price**.
- The **expiry** or **expiration date** is the date on which we must **exercise** our option, should we choose to.
- The **underlying asset** is the stock the option is based on.

We would exercise the option at expiry if the stock S is above the strike E and not if it's below. At expiry it's worth

$$\max(S - E, 0), \tag{11}$$

where the function of the underlying asset is called the **payoff function**, with 'max' representing optionality.

A **put option** is the right to *sell* a particular asset for an agreed amount a a specified time in the future. The holder of a put option wants the stock price to fall so that he can sell the asset for more than it's worth. The payoff function for a put option is

$$\max(E - S, 0), \tag{12}$$

with option only being exercised if the stock falls below the strike price.
The higher the strike the lower the value of a call option but the higher the value of the puts. Since the call allows you to buy the underlying for the strike, so that the lower the strike price the more this right is worth. The opposite is true for a put since it allows you to sell the underlying for the strike price. Also, the longer the time to maturity, the higher the value of the call. As the time to expiry decrease, as there is less and less time for the underlying to move, so the option value must converge to the payoff function.
Calls and puts have a non-linear dependence on they underlying asset. This contrasts with futures which have a linear dependence on the underlying. Calls and puts are the two simplest forms of options and are oftenr referred to as **vanilla**.

DEFINITION OF COMMON TERMS

- **Premium** - The amount paid for the contract initially.
- **Underlying (asset)** - The financial instrument on which the option value depends. Stocks, commodities, currencies and indices are going to be denoted by S. The option payoff is defined as some function of the underlying asset at expiry.
- **Strike (price)** or **exercise price** - The amount for which the underlying can be bought (call) or sold (put). This will be denoted by E. This definition only really applies to the simple calls and puts. For more complicated contracts, this definition is extended.
- **Expiration (date)** or **expiry (date)** - Date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights, denoted by T.
- **Intrinsic value** - The payoff that would be received if the underlying is at its current level when the option expires.

- **Time value** - Any value that the option has above its intrinsic value. The uncertainty surrounding the future value of the underlying asset means that the option value is generally different from the intrinsic value.
- **In the money** - An option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike.
- **Out of the money** - An option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike.
- **At the money** - A call or put with a strike that is close to the current asset level.
- **Long position** - A positive amount of quantity, or a positive exposure to a quantity.
- **Short position** - A negative amount of a quantity, or a negative exposure to a quantity. Many assets can be sold short, with some constraints on the length of time before they must be bought back.

PAYOFF DIAGRAMS

A **payoff diagram** plots the value of an option at expiry as a function of the underlying. At expiry the option is worth a known amount. For a call and put option the contract is worth max(S-E, 0) and max(E-S, 0) respectively, represented by the bold lines below.

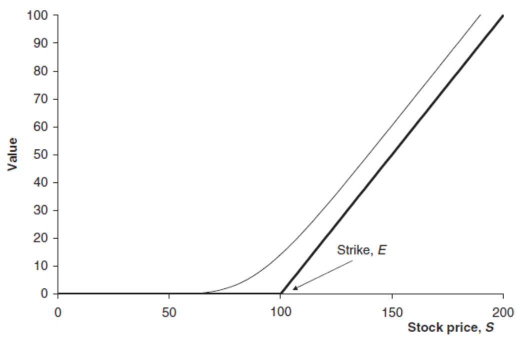


Figure 2.5 Payoff diagram for a call option.

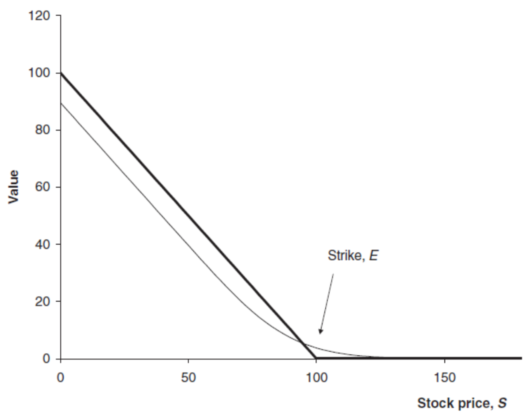


Figure 2.8 Payoff diagram for a put option.

Other representations of value

The payoff diagrams above only shows the money worth of your option contract at expiry. It makes no allowance for how much premium you had to pay for the option. In a **profit diagram** for a call option, we adjust for the original cost of the option by subtracting from the payoff the premium originally paid for the call option.

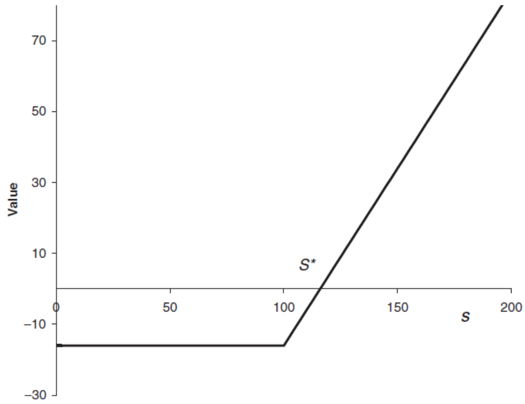


Figure 2.11 Profit diagram for a call option.

This figure is helpful because it shows how far into the money the asset must be at expiry before the option becomes profitable. Asset value S^* is the point which divides profit from loss; if the asset at expiry is above S^* then the contract has made a profit and vice versa. The profit diagram takes no account of the time value of money. The premium is paid up front but the payoff, if any, is only received at expiry. To be consistent one should either discount the payoff by multiplying by $r^{-r(T-t)}$ to value everything at the present, or multiply the premium by $e^{r(T-t)}$ to value all cashflows at expiry.

WRITING OPTIONS

The **writer** of an option is the person who promises to deliver the underlying asset, if the option is a call, or buy it, if the option is a put. The writer is the person who receives the premium, and is liable if the option is exercised. The option holder can sell the option on to someone else to close his position. The purchaser of the option hands over a premium in return for special rights, and uncertain outcome. The writer receives a guaranteed payment up front, but then has obligations in the future.

MARGIN

Buying an option

- Downside: initial premium
- Upside: may be unlimited

Writing an option

- Downside: could be huge
- Upside: Limited

To cover the risk of default in the event of an unfavourable outcome, the **clearing houses** that register and settle options insist on the deposit of a margin by the writers of options.

MARKET CONVENTIONS

Often simpler option contracts bought and sold through exchanges are standardised to follow conventions. Simple calls and puts come in series, referring to the strike and expiry dates. Typically a stock has three choices of expiries trading at any time. Having standardised contracts traded through an exchange promotes liquidity of the instruments. **Over the counter (OTC)** contracts are an agreement between two parties, often brough together by an intermediary. The agreed terms are flexible, without needing to follow any conventions.

THE VALUE OF THE OPTION BEFORE EXPIRY

How much is an option contract worth *now*, before expiry? How much would you pay for a contract, a piece of paper, giving you rights in the future? What is clear is that the contract value before expiry will depend on how high the asset price is today and how long there is before expiry. The longer the time to expiry, the more time there is for the asset to rise or fall.

Let $V(S,t)$ be a function of the value of the underlying asset S at time t which represents the value of the option contract. At expiry date $t = T$, the value of the contract at expiry function is just the payoff function, which we know from before. For a call option it's

$$V(S,T) = \max(S - E, 0). \tag{13}$$

The fine lines in Figure 2.5 and 2.8 are the values of the contracts $V(S,t)$ at *some time before expiry*, plotted against S.

FACTORS AFFECTING DERIVATIVE PRICES

The underlying value asset S and time to expiry t are **variables** of the options price. The interest rate and strike price are examples of **parameters** of the options price. The interest rate affects the option value via the time value of money since the payoff is received in the future. The higher the strike in a call, the lower the value of the call. The **volatility** is an important parameter which impacts an option's value. It is a measure of fluctuation in the asset price i.e. a measure of randomness. The technical definition of volatility is the 'annualized standard deviation of the asset returns'.

SPECULATION AND GEARING

A dramatic move in the underlying that leads to an option expires in the money may lead to a large profit relative the amount of investment.

Example: Today's date is 14th April and the price of Wilmott Inc. stock is \$666. The cost of a 680 call option with expiry 22nd August is \$39. I expect the stock to rise significantly between now and August, how can I profit if I am right?
Buy the stock: Suppose I buy the stock for \$666. And suppose that by the middle of August the stock has risen to \$730. I will have made a profit of \$64 per stock. More importantly my investment will have risen by

$$\frac{730 - 666}{666} \times 100 = 9.6\%. \tag{14}$$

Buy the call: If I buy the call option for \$39, then at expiry I can exercise the call, paying \$680 to receive something worth \$730. I have paid \$39 and I get back \$50. This is a profit of \$11 per option, but in percentage terms I have made

$$\frac{\text{value of asset at expiry} - \text{strike} - \text{cost of call}}{\text{cost of call}} \times 100 = \frac{730 - 680 - 39}{39} \times 100 = 28\%. \tag{15}$$

This is an example of **gearing** or **leverage**. The out-of-the-money option has a high gearing, a possible high payoff for a small investment. The downside of this leverage is that the call option is more likely than not to expire completely worthless and you will lose all of your investment. If Wilmott Inc. remains at \$666 then the stock investment has the same value but the call option experiences a 100% loss.

For highly leveraged contracts, the buyer is very likely to lose but at the risk of only a small amount. But the writer is risking a large loss in order to make a probable small profit. The writer is likely to think twice about such a deal unless he can offset his risk by buying other contracts. This offsetting of risk by buying other related contracts is called **hedging**.

EARLY EXERCISE

The simple options described above are examples of **European options** because exercise is only permitted at expiry. Some contracts allow the holder to exercise at any time before expiry, and these are called **American options**. American options give the holder more rights than their European equivalent and can therefore be more valuable, and they can never be less valuable. The main point of interest with American-style contracts is deciding when to exercise. Most stock options are traded American-style while most index options are traded European-style. **Bermudan options** allow exercise on specified dates, or in specified periods.

PUT-CALL PARITY

Imagine buying a European call option with a strike of E and an expiry of T and writing a European put option with the same strike and expiry. With a present date of t, the payoff you receive at T for the call and put will look like the lines in first and second plot of Figure 2.14 respectively. The payoff for the put is negative, since writing the option leads to liability for the payoff. The portfolio payoff for the two options is the sum of individual payoffs i.e.

$$\max(S(T) - E, 0) - \max(E - S(T), 0) = S(T) - E, \tag{16}$$

If I buy the asset today it will cost me $S(t)$ and be worth $S(T)$ at expiry. It's unclear what the value of $S(T)$ will be but to guarantee to get that amount you'd have to buy the asset. To lock in a payment of E at time T involves a cash flow of $Ee^{-r(T-t)}$ at time t. The conclusion is that the portfolio of a long call and a short put gives exactly thet same payoff as a long asset, short cash position. The equality of these cashflows is independent of the future behaviour of the stock and is model independent:

$$C - P = S - Ee^{-r(T-t)}, \tag{17}$$

where C and P are today's values of the call and put respectively. This relationship holds at any time up to expiry and is known as **put-call parity**. For European options, longing a call and shorting a put with the same strike is equivalent to longing a forward contract with a forward price equivalent to the options' strike price. If this relationship did not hold for whatever reason there would be riskless arbitrage opportunities to be exploited to make money.

Table 2.1 shows the cashflows in the perfectly hedged portfolio. In this table I have set up the cashflows to have a guaranteed value of zero at expiry.

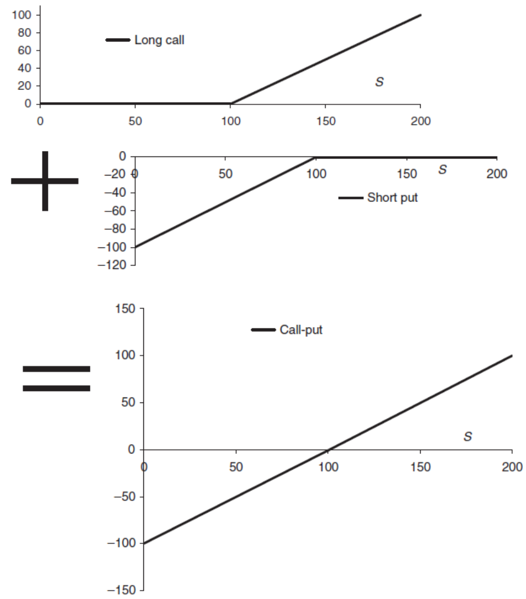


Figure 2.14 Schematic diagram showing put-call parity.

Table 2.1 Cashflows in a hedged portfolio of options and asset.

Holding	Worth today (t)	Worth at expiry (T)
Call	C	$\max(S(T) - E, 0)$
−Put	$-P$	$-\max(E - S(T), 0)$
−Stock	$-S(t)$	$-S(T)$
Cash	$Ee^{-r(T-t)}$	E
Total	$C - P - S(t) + Ee^{-r(T-t)}$	0

BINARIES OR DIGITALS

The **binary** or **digital options** have a payoff at expiry that is discontinuous in the underlying asset price. Examples of payoff diagram for a binary call and a binary put are shown below. This contract pays \$1 at expiry, time T, if the asset price is then greater than the exercise price E.

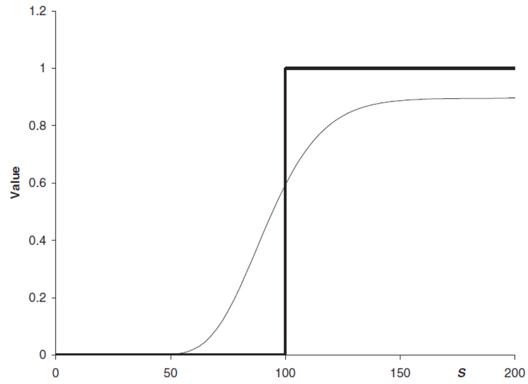


Figure 2.15 Payoff diagram for a binary call option.

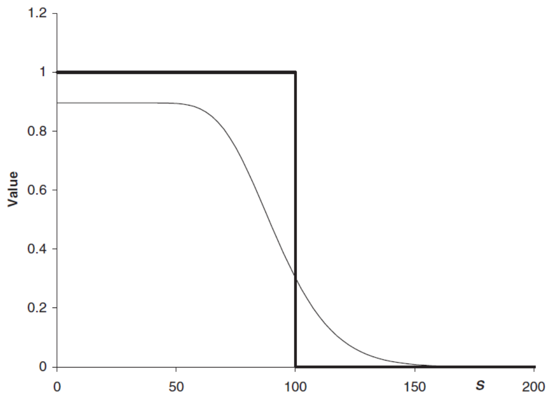


Figure 2.16 Payoff diagram for a binary put option.

The holder of a binary put receives \$1 if the asset is below E at expiry. The binary put would be bought by someone expecting a modest fall in the asset price. If you hold both a binary call and a binary put with the same strikes and expiries, you will always get \$1 regardless of the level of the underlying at expiry i.e.

$$\text{Binary call} + \text{Binary put} = e^{-r(t-t)} \tag{18}$$

This is also known as the **binary put-call parity relationship**.

BULL AND BEAR SPREADS

The bull and bear spreads is an example of **portfolio of options** or an **option strategy**.

- Suppose I buy one call option with a strike of 100 and write another with a strike of 120, both with the same expiry. The resulting portfolio has a payoff as shown in Figure 2.17.
- The payoff is continuous and is zero below 100, 20 above 120 and linear in between. This strategy is called a **bull spread** or **call spread** since it benefits if the market is rising.
- For a bull spread made up of calls with strikes E_1 and E_2 where $E_2 > E_1$, the payoff is

$$\frac{1}{E_2 - E_1}(\max(S - E_1, 0) - \max(S - E_2, 0)). \tag{19}$$

  Here the payoff is scaled to 1. Similarly, Figure 2.18 shows the payoff of a **bear spread** or a **put spread** i.e. benefit from a falling market, where if I write a put option with strike 100 and buy a put with strike 120.

- A strategy involving options of the same type (i.e. calls or puts) is called a **spread**.

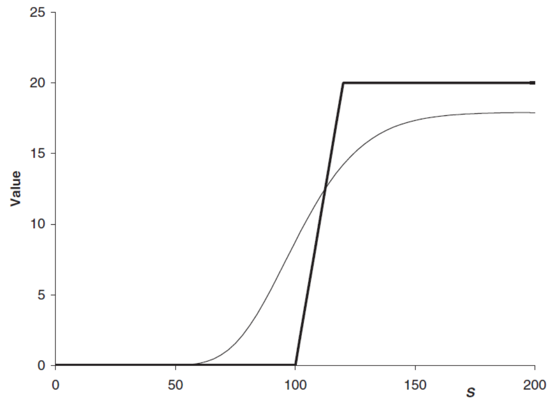


Figure 2.17 Payoff diagram for a bull spread.

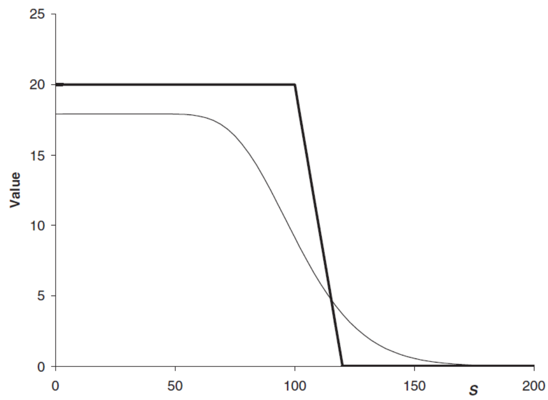


Figure 2.18 Payoff diagram for a bear spread.

STRADDLES AND STRANGLES

The **straddle** consists of a call and a put with the same strike, with a payoff diagram shown in Figure 2.19. Such a position is usually bought at the money by someone who expects the underlying to either rise or fall, but not to remain at the same level. The straddle would be sold by someone with the opposite view, someone who expects the underlying price to remain stable. This is an expensive strategy as money is spent on both premium, hoping for an explosive move on the price of the underlying. This is equivalent to being long volatility i.e. betting on increasing volatility since the options are priced with a given level of volatility and having bought two options meant that profits are only made if they are extra volatile.

The **strangle** consists of a call and a put with different strikes, and can be either an **out-of-the-money strangle**(out of the money put and call) or an **in-the-money strangle**. The payoff for an out-of-the money strangle is shown in Figure 2.22. Here the motivated buyer tend to expect an even larger move in the underlying one way or another. The contract is usually bought when the asset is around the middle of the two strikes and is cheaper than a straddle. This cheapness means that the gearing for the out-of-the-money strangle is higher than that for the straddle. The downside is that there is a much greater range over which the strangle has no payoff at expiry, for the straddle there is only the one point at which there is no payoff.

The straddles and strangles are called **volatility trades** since the contracts are bought or sold based on speculations on the direction of volatility. Straddles and strangles are rarely held until expiry. A strategy involving options of different types i.e. both calls and puts, is called a **combination**.

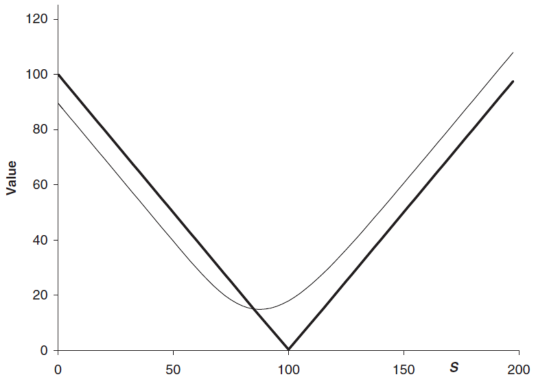


Figure 2.19 Payoff diagram for a straddle.

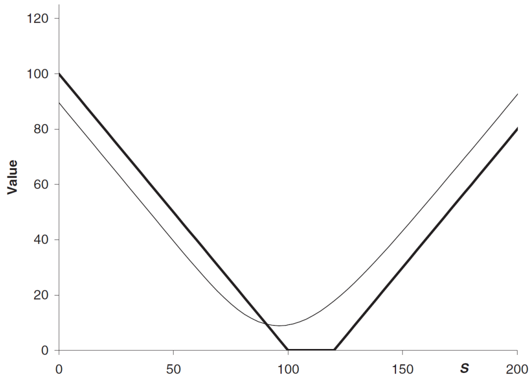


Figure 2.22 Payoff diagram for a strangle.

RISK REVERSAL

The **risk reversal** is a combination of a long call, with a strike above the current spot, and a short put, with a strike below the current spot at the same expiry. Its payoff is shown in Figure 2.23. This strategy

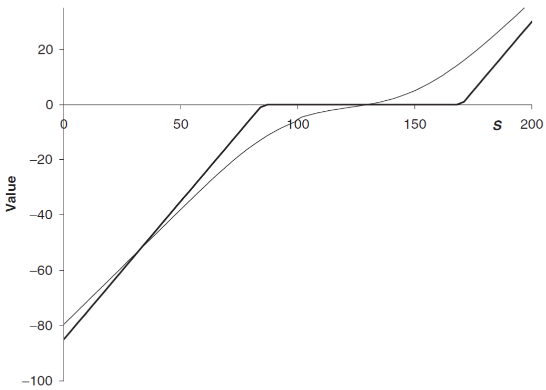


Figure 2.23 Payoff diagram for a risk reversal.

protects against unfavorable price movements in the underlying position but limits the profits that can be made on that position. If an investor is long a stock, they could create a short risk reversal to hedge their position by buying a put option and selling a call option. If an investor is short an underlying asset, the investor hedges the position with a long risk reversal by purchasing a call option and writing a put option on the underlying instrument. If the price of the underlying asset rises, the call option will become more valuable, offsetting the loss on the short position. If the price drops, the trader will profit on their short position in the underlying, but only down to the strike price of the written put.

BUTTERFLIES AND CONDORS

A **butterfly spread** involves the purchase and sale of three options with different strikes. For example, buying a call of 90, writing two calls struck at 100 and buying a 110 call gives the payoff in Figure 2.24 i.e. maximum payoff is 10. This is a cheap position that is typically entered into if the underlying is believed to not go anywhere. The **condor** is like a butterfly except that four strikes, and four call options, are used. The payoff is shown in Figure 2.25.

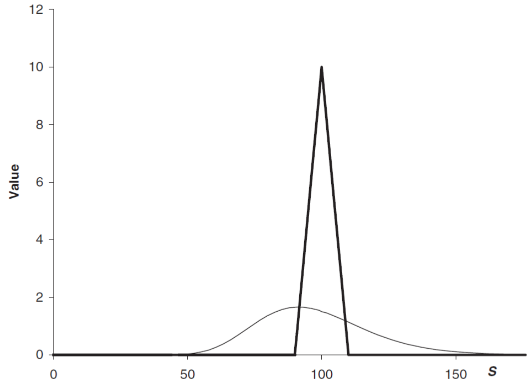


Figure 2.24 Payoff diagram for a butterfly spread.

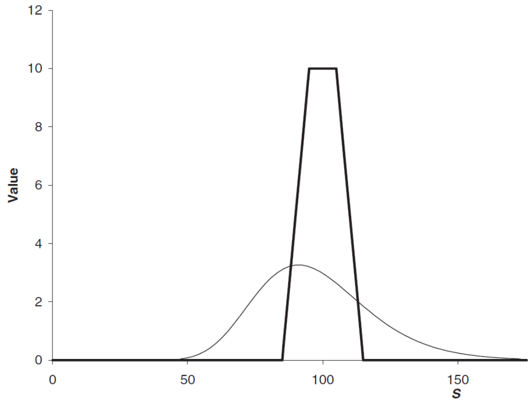


Figure 2.25 Payoff diagram for a condor.

LEAPS AND FLEX

Long-term equity anticipation securities(LEAPS) are longer-dated exchange-traded calls and puts. They are standardized so that they expire in January each year and are available with expiries up to three years. They come with three strikes, corresponding to at the money and 20% in and out of the money with respect to the underlying asset price when issued. The **FLexible EXchange-traded options(FLEX)** allow for a degree of customisation in the expiry date (up to five years), the strike price and the exercise style.

WARRANTS

Warrants are call options issued by a company on its own equity.

- Usually have a longer lifespan
- On exercise the company issues new stock to the warrant holder
- On exercise the holder of a traded option receives stock that has already been issued.
- Exercise is usually allowed any time before expiry, but after an initial waiting period.
- The typical lifespan of a warrant is five or more years. Occasionally **perpetual warrants** are issued, these have no maturity.

CONVERTIBLE BONDS

Convertible bonds(CBs) pay a stream of coupons with a final repayment of principal at maturity, but they can be converted into the underlying stock before expiry. On conversion rights to future coupons are lost. If the stock price is low then there is little incentive to convert to the stock, the coupon stream is more valuable. In this case the CB behaves like a bond. If the stock price is high then conversion is likely and the CB responds to the movement in the asset. Because the CB can be converted into the asset, its value has to be at least the value of the asset. This makes CBs similar to American options; early exercise and conversion are mathematically the same. As a hybrid security, the price of a convertible bond is especially sensitive to changes in interest rates, the price of the underlying stock, and the issuer's credit rating.

OVER THE COUNTER OPTIONS

Over the counter (OTC) options are sold privately from one counterparty to another. A **term sheet** specifies the precise details of an OTC contract. Figure 2.27 shows a term sheet for an OTC put option.

Preliminary and Indicative
For Discussion Purposes Only

Over-the-counter Option linked to the S&P500 Index

Option Type	European put option, with contingent premium feature
Option Seller	XXXX
Option Buyer	[dealing name to be advised]
Notional Amount	USD 20MM
Trade Date	[]
Expiration Date	[]
Underlying Index	S&P500
Settlement	Cash settlement
Cash Settlement Date	5 business days after the Expiration Date
Cash Settlement Amount	Calculated as per the following formula: #Contracts * max[0, S&Pstrike – S&Pfinal] where #Contracts = Notional Amount / S&Pinitial This is the same as a conventional put option: **S&Pstrike** will be equal **to 95% of the closing price on the Trade Date** **S&Pfinal** will be the level of the Underlying Index at the valuation time on the Expiration Date **S&Pinitial** is the level of the Underlying Index at the time of execution
Initial Premium Amount	[2%] of Notional Amount
Initial Premium Payment Date	5 business days after Trade Date
Additional Premium Amounts	[1.43%] of Notional Amount per Trigger Level
Additional Premium Payment Dates	The Additional Premium Amounts shall be due only if the Underlying Index at any time from and including the Trade Date and to and including the Expiration Date is equal to or greater than any of the Trigger Levels.
Trigger Levels	103%, 106% and 109% of **S&P500initial**
Documentation	ISDA
Governing law	New York

This indicative term sheet is neither an offer to buy or sell securities or an OTC derivative product which includes options, swaps, forwards and structured notes having similar features to OTC derivative transactions, nor a solicitation to buy or sell securities or an OTC derivative product. The proposal contained in the foregoing is not a complete description of the terms of a particular transaction and is subject to change without limitation.

Figure 2.27 Term sheet for an OTC 'put.'

In this OTC

- The holder gets a put option on S&P 500, but cheaper than a vanilla put option.
- This contract is cheap because part of the premium does not have to be paid until and unless the underlying index trades above a specific level.
- Each time a new level is reached an extra payment is triggered. This feature means that the contract is not vanilla, and makes the pricing more complicated.
- Quantities in square brackets will be set at the time that the deal is struck.

0.3 Chapter 3: The Binomial Model

INTRODUCTION

The binomial model may be thought of as being either a genuine model for the behaviour of equities, or, alternatively as a numerical method for the solution of the Black-Scholes equation;in this case, very similar to an explicit finite-difference method. It's a teaching aid to explain delta hedging, risk elimination and risk-neutral valuation.
Cons:

- Poor model for stock price behaviour. It says the stock can either go up or down by a known amount which is unrealistic. Results that follow from this model hinge on there being only two prices for the stock tomorrow and will break upon introduction of a third state.
- It's an old numerical scheme that predates modern numerical methods.

EQUITIES CAN GO DOWN AS WELL AS UP

Scenario: We will have a stock, and a call option on that stock expiring tomorrow. The stock can either rise or fall by a known amount between today and tomorrow. Interest rates are zero. There is a certain probability p of the stock rising and $1-p$ of the stock falling. Let's introduce the call option on the stock which has a strike of \$100 and expires tomorrow. If the stock price rises to 101, the option's payoff will just be $101-100=1$. If the stock falls to 99 tomorrow, the payoff is then zero, since the option has expired out of the money. There is a 0.6 probability of getting 1 and a 0.4 probability of getting zero. Interest rates are zero. What is the option worth today?

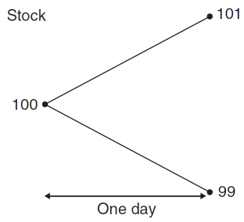


Figure 3.1 The stock can rise or fall over the next day, only two future prices are possible.

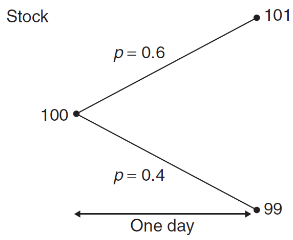


Figure 3.2 Probabilities associated with the future stock prices.

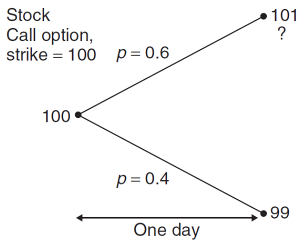


Figure 3.3 What is the option payoff if the stock rises?

It's $\frac{1}{2}$. To see this we must construct a portfolio consisting of one option and short $\frac{1}{2}$ of the underlying stock, shown in Figure 3.7. If the stock rises to 101 then this portfolio is worth $1-\frac{1}{2}\times 101$; the one

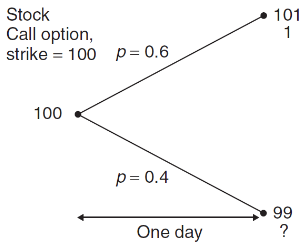


Figure 3.4 What is the option payoff if the stock falls?

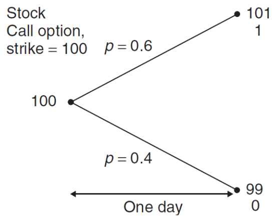


Figure 3.5 Now we know the option values in both 'states of the world.'

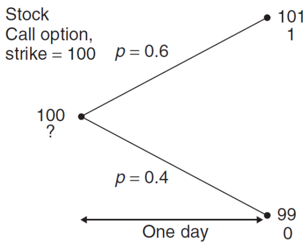


Figure 3.6 What is the option worth today?

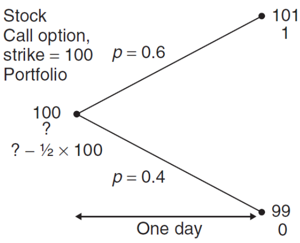


Figure 3.7 Long one option, short half of the stock.

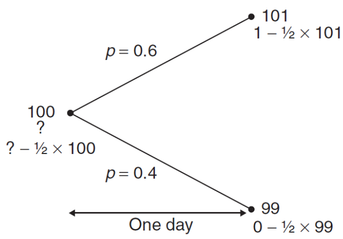


Figure 3.8 The portfolio values at expiration.

being from the option payoff and the $-\frac{1}{2} \times 101$ being from a short $(-)$ position $(\frac{1}{2})$ in the stock (now worth 101). If the stock falls to 99 then this portfolio is worth $0 - \frac{1}{2} \times 99$; the zero being from the option payoff and the $-\frac{1}{2} \times 99$ being from a short $(-)$ position $(\frac{1}{2})$ in the stock (now worth 99), shown in Figure

3.8. In either case, tomorrow, at expiration, the portfolio takes the value $-\frac{99}{2}$. If the portfolio is worth $-\frac{99}{2}$ tomorrow regardless of whether the stock rises or falls and interest rates are zero, then it must also be worth $-\frac{99}{2}$ today. This is a perfectly risk-free portfolio and an example of **no arbitrage**: There are two ways to ensure that we have $\frac{99}{2}$ tomorrow. 1) Buy one option and sell one half of the stock. 2) Put the money under the mattress. Both 'portfolios' must be worth the same today. Therefore

$$\begin{aligned} \text{the option value} - \frac{1}{2} \times 100 &= -\frac{1}{2} \times 99 \\ \text{the option value} &\implies \frac{1}{2} \end{aligned} \tag{20}$$

WHY SHOULD THIS "THEORETICAL PRICE" BE THE "MARKET PRICE"?

If the theoretical price and the market price are not the same, then there is risk-free money to be made. If the option costs less than 0.5 simply buy it and hedge to make a profit. If it is worth more than 0.5 in the market then sell it and hedge, and make a guaranteed profit. In practice this arbitrage opportunity will make the option price converge to 0.5 due to supply and demand.

The expected payoff is 0.6 for this option. If you're willing to pay 0.6 or more, then you're **risk seeking**. If you're paying 0.55, the expected return would be $(0.6 - 0.55)/0.55 \approx 9\%$. and the option writer stands to guarantee a profit of 0.05.

HOW DID I KNOW TO SELL $\frac{1}{2}$ OF THE STOCK FOR HEDGING?

Let Δ = quantity of stock that must be sold for hedging. Starting with one option, $-\Delta$ of the stock gives a portfolio value of $(? - \Delta \times 100)$. Tomorrow the portfolio is worth $(1 - \Delta \times 100)$ or $(0 - \Delta \times 100)$ if the stock rises or falls respectively. Equating these two gives

$$1 - \Delta \times 101 = 0 - \Delta \times 99 \tag{21}$$

$$\implies \Delta = 0.5 \tag{22}$$

Another example: Stock price is 100, can rise to 103 or fall to 98. Value a call option with a strike price of 100. Interest rates are zero. The portfolio value is $(? - \Delta \times 100)$. Tomorrow the portfolio is either worth $(3 - \Delta 103)$ or $(0 - \Delta 98)$. Hence

$$\Delta = \frac{3-0}{103-98} = \frac{3}{5} = 0.6. \tag{23}$$

The portfolio value tomorrow is then -0.6×98. With zero interest rate, the portfolio value today must equal the risk-free portfolio value tomorrow:

$$? - 0.6 \times 100 = -0.6 \times 98 \implies ? = 1.2 \tag{24}$$

Delta hedging means choosing Δ such that the portfolio value does not depend on the direction of the stock. Generalising this we have

$$\Delta = \frac{\text{Range of option payoffs}}{\text{Range of stock prices}}. \tag{25}$$

Where Δ can be thought as the sensitivity of the option to changes in the stock.

HOW DOES THIS CHANGE IF INTEREST RATES ARE NON-ZERO?

We delta hedge as before to construct risk-free portfolio with the same delta. Then present value that back in time, by multiplying a discount factor. Example: Same as first example, but now $r = 0.1$. The discount factor for going back one day is

$$\frac{1}{1 + 0.1/252} = 0.9996. \tag{26}$$

The portfolio value today must be the present value of the portfolio value tomorrow

$$\begin{aligned} ? - 0.5 \times 100 &= -0.5 \times 99 \times 0.9996. \\ ? &\implies 0.51963 \end{aligned} \tag{27}$$

THE REAL AND RISK-NEUTRAL WORLDS

Some properties of the real world:

- We know all about delta hedging and risk elimination.
- We are very sensitive to risk, and expect greater return for taking risk.
- It turns out that only the two stock prices matter for option pricing, not the probabilities.

The **risk-neutral world** is one where people don't care about risk, and has the following properties:

- We don't care about risk, and don't expect any extra return for taking unnecessary risk.
- We don't ever need statistics for estimating probabilities of events happening.
- We believe that everything is priced using simple expectations.

Suppose in a risk-neutral world a stock is currently worth \$100 and could rise/fall to \$101/\$99. If the stock is correctly priced using expectations, the probabilities of the stock price rising or falling should just be 50/50 due to symmetry. On this risk-neutral world the **risk-neutral probabilities** p' is calculated from

$$p' \times 101 + (1 - p') \times 99 = 100 \tag{28}$$

This gives $p' = 0.5$. The calculation however is wrong to only use simple expectations for pricing with no allowance made for risk. The real probabilities are still 60% and 40% and hence p' is not real. In the same world, the call option will be valued by simple expectations with no regard to risk with

$$0.5 \times 1 + 0.5 \times 0 = 0.5 \tag{29}$$

This is called **risk-neutral expectation**.
When interest rates are non-zero we must perform exactly the same operations, but whenever we equate values at different times we must allow for present valuing. With r = 0.1 we calculate the risk-neutral probabilities from

$$\begin{aligned} 0.9996 \times (p' \times 101 + (1 - p') \times 99) = 100. \\ \implies p' = 0.51984 \end{aligned} \tag{30}$$

The expected payoff is

$$0.51984 \times 1 + (1 - 0.51984) \times 0 = 0.51984 \tag{31}$$

Its present(option) value is

$$0.9996 \times 0.51984 = 0.51963 \tag{32}$$

The risk-neutral probability p' that we have just calculated (the 0.5 in the first example) is not real, it does not exist, it is a mathematical construct.
<u>AND NOW USING SYMBOLS</u>