

1 Simple affine transformations in 3D

1.1 Translate

by Δ (x, y, z)

$$\begin{bmatrix} x' \\ y' \\ z' \\ - \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (1)$$

1.2 Scale

about origin by S (x, y, z)

$$\begin{bmatrix} x' \\ y' \\ z' \\ - \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (2)$$

1.3 Rotate

about O_z by θ

$$\begin{bmatrix} x' \\ y' \\ z' \\ - \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (3)$$

about O_x by θ

$$\begin{bmatrix} x' \\ y' \\ z' \\ - \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (4)$$

about O_y by θ

$$\begin{bmatrix} x' \\ y' \\ z' \\ - \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (5)$$

2 Axonometric projections

$T_{axonometric_{plane}} = T_{rot_1} * T_{rot_2} * T_{orto_{plane}}$
e.g.

$$T_{axonometric_{z=0}} = T_{rot_y} * T_{rot_x} * T_{orto_{z=0}} \quad (6)$$

$$T_{axonometric_{y=0}} = T_{rot_x} * T_{rot_z} * T_{orto_{y=0}} \quad (7)$$

$$T_{axonometric_{x=0}} = T_{rot_z} * T_{rot_y} * T_{orto_{x=0}} \quad (8)$$

where $T_{rot_{axis}}$ is one of matrices from Section 1.3
and $T_{orto_{plane}}$ is orthographic projection onto a plane

$$\text{e.g. } T_{orto_{z=0}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{so we get } T_{axonometric_{z=0}} = \begin{bmatrix} \cos \theta & \sin \theta * \cos \phi & 0 & 0 \\ 0 & \cos \phi & 0 & 0 \\ \sin \theta & -\cos \theta * \sin \phi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where θ and ϕ are rotation angles around O_x and O_y accordingly
trimetric, isometric and dimetric have different ratios of distortion coefficients
(K_x, K_y, K_z)

where

$$K_{xx=0} = \sqrt{\sin^2 \psi - \cos^2 \psi * \sin^2 \theta}$$

$$K_{yx=0} = \sqrt{\cos^2 \psi + \sin^2 \psi * \sin^2 \theta}$$

$$K_{zx=0} = \sqrt{\cos^2 \theta}$$

$$K_{xy=0} = \sqrt{\cos^2 \psi}$$

$$K_{yy=0} = \sqrt{\sin^2 \phi - \cos^2 \phi * \sin^2 \psi}$$

$$K_{zy=0} = \sqrt{\cos^2 \phi + \sin^2 \phi * \sin^2 \psi}$$

$$K_{xz=0} = \sqrt{\cos^2 \theta + \sin^2 \theta * \sin^2 \phi}$$

$$K_{yz=0} = \sqrt{\cos^2 \phi}$$

$$K_{zz=0} = \sqrt{\sin^2 \theta - \cos^2 \theta * \sin^2 \phi}$$

2.1 Isometric

Isometric projections are commonly used in technical drawings and used to be used in some computer game graphics. In an isometric projection the three axes appear 120° drawings and used to from each other and are equally foreshortened. It can be achieved by rotating an object 45° in the plane of the screen and $\sim 35.3^\circ (\arctan(1/\sqrt{2}))$ through the horizontal axis.

$$K_x = K_y = K_z$$

2.2 Dimetric

$$K_x = K_y \neq K_z$$

$$K_y = K_z \neq K_x$$

$$K_z = K_x \neq K_y$$

right angle dimetric projection: $K_x = K_z = 0.94; K_y = 0.47$

$$K_x = K_z = 1; K_y = 0.5$$

$$\begin{bmatrix} x' \\ y' \\ - \\ - \end{bmatrix} = T_{dimetric \text{ right angle}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad (9)$$

$$T_{dimetric \text{ right angle}} = \begin{bmatrix} \cos 7^\circ 10' & \sin 7^\circ 10' * \cos 41^\circ 25' & 0 & 0 \\ 0 & \cos 41^\circ 25' & 0 & 0 \\ \sin 7^\circ 10' & -\cos 7^\circ 10' * \sin 41^\circ 25' & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.99 & 0.094 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0.125 & 0.655 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (10)$$