Exercise Sheet 9

Ex 1

a) $$T(n) = \begin{cases} a & \text{for } n = 1 \\ c + T\left(\frac{n}{2}\right) & \text{else} \end{cases}$$

Assumption $T(n) \in \mathcal{O}(\log(n)) \Rightarrow T(n) \leq c' \cdot \log_2(n)$

Case $n > 1$:

Basis $n = 2$

$$T(2) = c + T\left(\frac{2}{2}\right) = c + a \in \mathcal{O}(\log(2)) \quad \checkmark$$

($\frac{2}{2} = 1$, $T(1) = a$)

Inductive Step $\frac{n}{2} \longmapsto n$

$$T(n) = c + T\left(\frac{n}{2}\right)$$

$$\overset{IA}{\leq} c + c' \cdot \log_2\left(\frac{n}{2}\right)$$

$$= c + c' \cdot \log_2(n) - c' \underbrace{\log_2(2)}_{1}$$

$$= c - c' + c' \cdot \log_2(n)$$

$$\leq c' \cdot \log_2(n) \qquad c' > 1 \quad \checkmark$$

$T(n) \in \mathcal{O}(\log n)$ for $n > 1$

Case $n = 1$

$T(n) = a \in \mathcal{O}(1) \in \mathcal{O}(\log n)$ ✓

b) $T(n) = \begin{cases} a & \text{for } n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n^3 & \text{else} \end{cases}$

Case $n > 1$

$a = 2; \; b = 2; \; f(n) = n^3$

~~with $a = b \Rightarrow$ case 2 Master theorem~~

$f(n) \in \Omega\left(n^{\log_b(a+\varepsilon)}\right)$ mit $\varepsilon = \varepsilon > 0$

$f(n) \in \Omega\left(n^{(\log_2(2+\varepsilon))}\right) = \Omega\left(n^3\right)$

Case 3 $\Rightarrow$ check regularity condition

$a f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ $\quad 0 \leq c \leq 1$, $n > n_0$

$2 \cdot \left(\frac{n}{2}\right)^3 \leq c \cdot n^3$

$n^3 \leq \frac{1}{4} c \cdot n^3$ $\quad c \geq \frac{3}{4}$ ✓

$T(n) \in \Theta(f(n)) = \Theta\left(n^3\right)$

case $n = 1$

$T(n) = a \in \Theta(1) \in \Theta\left(n^3\right)$

Therefore

$T(n) \in \Theta\left(n^3\right)$

2.

$$T(n) = \begin{cases} 1 & \text{for } n = 1 \\ 4T\left(\frac{n}{2}\right) + n^2 & \text{else} \end{cases}$$

case $n > 1$:

$a = 4$; $b = 2$; $f(n) = n^2$

$f(n) \in \Theta(n^{\log_b a}) = \Theta(n^2)$

$\Rightarrow$ case 2

$T(n) \in \Theta(n^{\log_b a} \cdot \log_2 n)$

$T(n) \in \Theta(n^2 \cdot \log_2(n))$

case $n = 1$

$T(n) \in O(1) \in O(n^2 \log_2(n))$

$\Rightarrow$

$T(n) \in \Theta(n^2 \log_2 n)$

3. $T(n) = \begin{cases} a & \text{for } n \leq 2 \\ T(n) + a & \text{else} \end{cases}$

Case $n > 2$

Substitution $n = 2^m$

$S(m)$ is $T(2^m)$

$S(m) = T(2^m) = T\left(\sqrt{2^m}\right) + a$

$= T\left(2^{\frac{m}{2}}\right) + a$

$S(m) = S\left(\frac{m}{2}\right) + a$

$a' = 1 \; ; \; b' = 2 \; ; \; f(m) = a$

$f(m) \in \Theta(1) \in m^{\log_{b'} a'} = m^{\log_2 1}$, $\log_2 1 = 0$

$\Rightarrow$ case 2

$S(m) \in \Theta(m^{\log_{b'} a'} \log_2 m)$, $m^{\log_{b'} a'} = m^1$

$S(m) \in \Theta(\log_2 m)$

Rücksubstitution $m = \log_2 n$

$T(n) \in \Theta(\log_2(\log_2 n))$

case $n \leq 2$

$a \in O(1) \in O(\log_2(\log_2(n)))$

Therefore

$T(n) = \Theta(\log_2(\log_2(n)))$