Ex 11

1. Given is a (a, b) - tree, with

$a = 4$
$b = 9$

Where

c_i are the costs of the ith operation &
Φ_i is the potential of the tree, which is equal to the number of nodes in the tree with $> a$ and $< b = \{5, 6, 7, 8\}$ children

Good introduction

To show, that the costs of n insert / remove operations are $O(n)$

Assumption:

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B \qquad (1)$$

We will show that (1) holds for an insert and remove operation

Case Insert:

From observation, we have to consider three cases:

a) the node where the insertion takes place has b children
b) the node where the insertion takes place has b-1 children
c) the node where the insertion takes place has less then b-1 children

a) after insertion has $b+1$ children
$\Rightarrow$ a split up takes place
$\Rightarrow$ the potential increases by 2
The recursion goes up through all parent nodes to the stop node, wehre no further spliting is required. Eeach of those splitting increases the potential by 2.
If the stop node was of degree $b-1$ and is now b, the potential is decreased by one.
If the stop node was of degree a and has now a degree of $a+1$, the potential is increased by one.
Let m be the number of split operations
costs

$$c_i \leq A \cdot m + B$$

and

$$\varphi_i = \varphi_{i-1} + 2m \pm 1$$

$$m = \frac{1}{2}(\varphi_i - \varphi_{i-1} \mp 1) = \frac{1}{2}(\varphi_i - \varphi_{i-1}) \mp \frac{1}{2}$$

therefore

$$c_i \leq \underbrace{A\frac{1}{2}}_{A'}(\varphi_i - \varphi_{i-1}) \underbrace{+ A\frac{1}{2} + B}_{B'}$$

$$c_i \leq A'(\varphi_i - \varphi_{i-1}) + B'$$

In this case, the assumption (1) holds.

OK

b) From Observation, no splitup is required. The potential decreases by 1

$$\varphi_i = \varphi_{i-1} - 1$$

$$\varphi_i - \varphi_{i-1} = -1$$

therfore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B$$

$$= -A + B$$

$$= A' + B$$ OK ✓

c) no split-up, the potential doesn't change

$$\varphi_i = \varphi_i - 1 \quad \Rightarrow \varphi_i - \varphi_{i-1} = 0$$

therfore

$$a_i \leq A(\varphi_i - \varphi_{i-1}) + B$$

~~B~~ OK ✓

Case Remove

From observation, we have do consider four cases:

a) the node where the remove takes place has 'a' children and a neighbor node with at least a+1 children

b) the node where the remove takes place has 'a' children and no neighbor nodes with at least a+1 children

c) the node where the remove takes place has more then a+1 children

d) the node when the remove takes place has exactly a + 1 children

a) ~~the po~~ Borrowing a node from neighbor

$$\varphi_i = \begin{cases} \varphi_{i-1}+1 & \text{if neighbor node has exactly } b \text{ children} \\ \varphi_{i-1}-1 & \text{if neighbor node has exactly } a+1 \text{ children} \\ \varphi_{i-1} & \text{else} \end{cases}$$

therefore $\varphi_i - \varphi_{i-1} = \{-1, 0, 1\}$

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B$$

End

OK

b) Merging with neighbor node
Both nodes have "a" children, the remaining node has b-1 children. ⇒ The potential rises by one, Each merging recursion increases the potential further by one
If the stopnode was of degree a+1 and now has degree a, the potential decreases by on, If the stopnode had a degree of b and now has a degree of b-1, the potential increases by one.
Let m be the number of merge operations.

Costs:

$$c_i \leq A \cdot m + B$$

and

$$\varphi_i = \varphi_{i-1} + m \pm 1$$

$$m = \varphi_i - \varphi_{i-1} \mp 1$$

therefore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) \underbrace{+ A + B}_{B'}$$

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B'$$ ✓ OK

c) the potential doesn't change

$$\varphi_i = \varphi_{i-1}$$

therefore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B$$ ✓ OK

d) the potential decreases by one

$$\varphi_i = \varphi_{i-1} - 1$$

$$\varphi_i - \varphi_{i-1} = -1$$

therefor

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B$$

$$= -A + B$$

$$= A' + B$$ ✓ OK

Conclusion:
For all cases $c_i \leq A(\varphi_i - \varphi_{i-1}) + B$ holds
The costs of n remove or insert operations is

$$\sum_{i=0}^{n} c_i \in \Theta(n)$$ OK

proof lecture slice 53