

Ex 11

1. Given is a (a, b) -tree, with
 $a = 4$
 $b = 9$

Where

c_i are the costs of the operation &
 Φ_i is the potential of the tree, which
is equal to the number of
nodes in the tree with $>a$
and $\leq b = \{5, 6, 7, 8\}$ children

To show, that the costs of n
insert/remove operations are $O(n)$

Assumption:

$$c_i \leq A(\Phi_{i+1} - \Phi_{i-1}) + B \quad (1)$$

We will show that (1) holds for
an insert and remove operation

Case Insert:

From observation, we have to consider
three cases:

- the node where the insertion takes place has b children
- the node where the insertion takes place has $b-1$ children
- the node where the insertion takes place has less than $b-1$ children

a) after insertion has $b+1$ children
 \Rightarrow a split up takes place
 \Rightarrow the potential increases by 2
 The recursion goes up through all parent nodes to the stop node, where no further splitting is required. Each of those splitting increases the potential by 2.

If the stop node was of degree $b-1$ and is now b , the potential is decreased by one.

If the stop node was of degree a and has now a degree of $a+1$, the potential is increased by one.

Let m be the number of split operations costs

$$C_i \leq A \cdot m + B$$

and

$$\varphi_i = \varphi_{i-1} + 2m + 1$$

$$m = \frac{1}{2}(\varphi_i - \varphi_{i-1} - 1) = \frac{1}{2}(\varphi_i - \varphi_{i-1}) - \frac{1}{2}$$

therefore

$$C_i \leq A \underbrace{\frac{1}{2}(\varphi_i - \varphi_{i-1})}_{A'} + \underbrace{A \frac{1}{2}}_{B'} + B$$

$$C_i \leq A'(\varphi_i - \varphi_{i-1}) + B'$$

In this case, the assumption (1) holds.

b) From observation, no split up is required. The potential decreases by 1

$$\varphi_i = \varphi_{i-1} - 1$$

$$\varphi_i - \varphi_{i-1} = -1$$

Therefore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B$$

$$= -A + B$$

$$= A' + B$$

✓

c) no split-up, the potential doesn't change

$$\varphi_i = \varphi_{i-1} \Rightarrow \varphi_i - \varphi_{i-1} = 0$$

Therefore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B$$

~~etc B~~

✓

Case Remove

From observation we have to consider four cases:

a) the node where the remove takes place has 'd' children and a neighbor node with at least $a+1$ children

b) the node where the remove takes place has 'd' children and no neighbor nodes with at least $a+1$ children

- c) the node where the remove takes place has more than $a+1$ children
- d) the node when the remove takes place has exactly $a+1$ children

a) ~~the po~~ Borrowing a node from neighbor

$$q_i = \begin{cases} q_{i-1} & \text{if neighbor node has exactly } \\ & b \text{ children} \\ q_{i-1}-1 & \text{if neighbor node has exactly } \\ & a+1 \text{ children} \\ q_{i-1} & \text{else} \end{cases}$$

Therefore $q_i - q_{i-1} = \{-1, 0, 1\}$

$$c_i \leq A(q_i - q_{i-1}) + B$$

End ✓

b)

Merging with neighbor node
 Both nodes have "a" children.
 The remaining node has $b-1$ children. \Rightarrow The potential rises by one.
 Each merging recursion increases the potential further by one
 If the stopnode was of degree $a+1$ and now has degree a , the potential decreases by one. If the stopnode had a degree of b and now has a degree of $b-1$, the potential increases by one.

Let m be the number of merge operations.

Costs:

$$c_i \leq A \cdot m + B$$

- and

$$\varphi_i = \varphi_{i-1} + m \pm 1$$

$$q_i \text{ or } \varphi_i - \varphi_{i-1} \mp 1$$

therefore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + \underbrace{A \cdot + B}_{B'}$$

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B' \quad \checkmark$$

a) the potential doesn't change

$$\varphi_i = \varphi_{i-1}$$

therefore

$$c_i \leq A(\varphi_i - \varphi_{i-1}) + B \quad \checkmark$$

b) the potential decreases by one

$$\varphi_i = \varphi_{i-1} - 1$$

$$\varphi_i - \varphi_{i-1} = -1$$

therefore

$$\begin{aligned} c_i &\leq A(\varphi_i - \varphi_{i-1}) + B \\ &= -A + B \\ &= A' + B \quad \checkmark \end{aligned}$$

Conclusion:

For all cases $c_i \leq A(\varphi_i - \varphi_{i-1}) + B$ holds

The costs of n remove or insert operations is

$$\sum_{i=0}^n c_i \in O(n)$$