Exercise Sheet 3

Ex 1

a) $\log_2 n = O(n)$

Statement:

$\log_2 n \leq C \cdot n \quad \forall n \geq n_0 \quad n, n_0 \in \mathbb{N}$

Base Case $C = 1 \quad n = n_0 = 2$

$\log_2 2 \leq 1 \cdot 2$

$1 \leq 2$ ✓

Inductive Step $n \longmapsto n_0 + 1$ with $C = 1$

$\log_2 (n_0 + 1) \leq 1 \cdot (n_0 + 1)$

$n_0 + 1 \leq 2^{n_0 + 1}$ ✓

$\Rightarrow \log_2 (n) = O(n)$

b) $\log_2 n = \Omega(n)$

Statement:

$\log_2 n \geq C \cdot n \quad n \geq n_0 \quad n \in \mathbb{N}$

$\Rightarrow n \geq 2^{C \cdot n} \quad C > 0 \quad C \in \mathbb{R}^+$

Base Case $n_0 = 1$:

$\log_2 1 \geq C \cdot 1$

$1 \geq 2^{1 \cdot C}$ ↯ $\Rightarrow$ This is only true for $C = 0$ which is exluded in the statement $\Rightarrow$ Base Case $n_0 = 1$ violates the statement!

Inductive Step $n \longmapsto n_0 + 1$

$2^{C(n_0+1)} = 2^C \, 2^{C \cdot n_0} = (2^{n_0+1})^C \leq n_0 + 1$ ↯

$\Rightarrow$ for $C \geq 1$ not true (does not hold for $C \geq 1$)

$\Rightarrow$ for $0 < C < 1$, for some value of n_0, the growth of n_0 will be dominant

$\Rightarrow \log_2 n$ is not $\Omega(n) = C \cdot n$: for any C, there exists a $n > n_0$ which doesn't hold $\Omega(n) \leq \log_2 n$

Ex. 2.

$$\log_b n = \frac{\log_a n}{\log_a b}$$

for $b > 1 \Rightarrow \log_a b$ is constant for a given b

$$O(n) = C \cdot n \geq \frac{\log_a n}{\log_a b}$$

Because $O(n)$ has a constant C, we can ~~have~~ add a constant on the other side of the inequation

if $b = 1$

$b^y = n \Leftrightarrow \log_b n = y$

if $b = 1$

$1^y = n \Leftrightarrow \log_1 n = y$

$\Rightarrow \log_1 n$ has infinite solutions

$\Rightarrow \log_1 n$ is comonly not defined

$$\log_1 n = \frac{\log_a n}{\underbrace{\log_a 1}_{\text{is } 0}}$$

$\Rightarrow$ division by 0

if $b < 1$

- b^y decreases with an increasing of y if $b < 1$
- b^y increases with an increasing of y if $b > 1$

$\Rightarrow$ The proportionality of the log is changed, if $b < 1$

if $b(n)$: $b = \sqrt{n}$

Statement: $\log_{\sqrt{n}} n \leq C \cdot n$

$$\frac{\log n}{\underbrace{\log \sqrt{n}}} \leq C \cdot n$$

$\rightarrow$ Denominator is not constant

$\Rightarrow$ Basis changes the runtime not constantly $\Rightarrow$ It is dependent from n

Ex 3

$f_1 : \sqrt{n}$

$f_2 : n \cdot \log_{10}(n)$

$f_3 : n \cdot \log_2(n^2)$

$f_4 : n^2$

$f_5 : n^2 \cdot \log_2(n^2)$

$f_i = \Theta(f_{i+1})$:

$i=1$ $\quad \lim\limits_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim\limits_{n\to\infty} \frac{\cancel{\sqrt{n}}\ 1}{\frac{n}{\sqrt{n}} \cdot \log_{10} n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n} \cdot \log_{10} n} = 0$ f

$i=2$ $\quad \lim\limits_{n\to\infty} \frac{f_2(n)}{f_3(n)} = \lim\limits_{n\to\infty} \frac{\cancel{n} \cdot \log_{10}(n)}{\cancel{n} \cdot \log_2(n^2)} = \cancel{\lim\limits_{n\to\infty} \frac{\log_{10}(n)}{2 \cdot \log_{10}(n)}}$

$= \lim\limits_{n\to\infty} \frac{\log_{10}(n)}{\frac{\log_{10}(n^2)}{\log_{10}(2)}} = \lim\limits_{n\to\infty} \frac{\cancel{\log_{10}(n)} \cdot \log_{10}(2)}{2 \cdot \cancel{\log_{10}(n)}} = \begin{cases} >0 \\ <\infty \end{cases}$ ✓

$i=3$ $\quad \lim\limits_{n\to\infty} \frac{f_3(n)}{f_4(n)} = \lim\limits_{n\to\infty} \frac{\cancel{n} \cdot \log_2(n^2)}{n^{\cancel{2}}} = \lim\limits_{n\to\infty} \frac{2 \cdot \log_2(n)}{n} = \lim\limits_{n\to\infty} \frac{2\,\frac{1}{n \cdot \ln(2)}}{1} = 0$ f

Hopital

$i=4$ $\quad \lim\limits_{n\to\infty} \frac{f_4(n)}{f(n)} = \lim\limits_{n\to\infty} \frac{\cancel{n^2}\ 1}{\cancel{n^2} \log_2(n^2)} = \lim\limits_{n\to\infty} \frac{1}{2 \cdot \log_2(n)} = 0$ f

$f_i = \Theta(f_i + 1)$ does not hold for $i = \{1, 3, 4\}$