Exercise Sheet 3

Ex 1

a) $\log_2 n = \mathcal{O}(n)$

Statement:

$\log_2 n \leq C \cdot n \quad \forall n \geq n_0 \quad n, n_0 \in \mathbb{N}$

Base Case $C = 1 \quad n = n_0 = 2$

$\log_2 2 \leq 1 \cdot 2$

$1 \leq 2$ ✓

Inductive Step $n \longmapsto n_0 + 1$ with $C = 1$

$\log_2 (n_0 + 1) \leq 1 \cdot (n_0 + 1)$

$n_0 + 1 \leq 2^{n_0 + 1}$ ✓

$\Rightarrow \log_2 (n) = \mathcal{O}(n)$ OK

b) $\log_2 n = \Omega(n)$

Statement: Please mention that you use an inverse proof.

$\log_2 n \geq C \cdot n \qquad n \geq n_0 \quad n \in \mathbb{N}$

$\Rightarrow n \geq 2^{C \cdot n} \qquad C > 0 \quad C \in \mathbb{R}^+$

Base Case $n_0 = 1$:

$\log_2 1 \geq C \cdot 1$

$1 \geq 2^{1 \cdot C}$ $f \Rightarrow$ This is only true for $C = 0$ which is exluded in the statement $\Rightarrow$ Base Case $n_0 = 1$ violates the statement!

Inductive Step $n \longmapsto n_0 + 1$

$2^{C(n_0 + 1)} = 2^C \, 2^{C \cdot n_0} = \left(2^{n_0 + 1}\right)^C \leq n_0 + 1$ f

$\Rightarrow$ for $C \geq 1$ not true (does not hold for $C \geq 1$)

$\Rightarrow$ for $0 < C < 1$, for some value of n_0, the growth of n_0 will be dominant

$\Rightarrow \log_2 n$ is not $\Omega(n) = C \cdot n$: for any C, there exists a $n > n_0$ which doesn't hold $\Omega(n) \leq \log_2 n$ OK

Ex. 2.

$$\log_b n = \frac{\log_a n}{\log_a b}$$

for $b > 1 \Rightarrow \log_a b$ is constant for a given b

You could also show this for $\Theta(\log_b n) = \Theta(\log n)$ which would suit the "constant factor" statement better.

$$O(n) = C \cdot n \geq \frac{\log_a n}{\log_a b}$$

Because $O(n)$ has a constant C, we can ~~have~~ add a constant on the other side of the inequation

$\log_b n = \log n / \log b \leq C * \log n$
$\Rightarrow C_1 = C * \log b$
$\Rightarrow \log_b n \leq C_1 * \log n$

if $b = 1$

$b^y = n \Leftrightarrow \log_b n = y$

if $b = 1$

$1^y = n \Leftrightarrow \log_1 n = y$

$\Rightarrow \log_1 n$ has infinite solutions

$\Rightarrow \log_1 n$ is comonly not defined

$$\log_1 n = \frac{\log_a n}{\underbrace{\log_a 1}_{\text{is } 0}}$$

$\Rightarrow$ division by 0 OK

if $b < 1$

- b^y decreases with an increasing of y if $b < 1$
- b^y increases with an increasing of y if $b > 1$

$\Rightarrow$ The proportionality of the log is changed, if $b < 1$ Correct

if $b(n)$: $b = \sqrt{n}$

Statement: $\log_{\sqrt{n}} n \leq C \cdot n$

$$\frac{\log n}{\log \sqrt{n}} \leq C \cdot n$$

$\rightarrow$ Denominator is not constant

$\Rightarrow$ Basis changes the runtime not constantly $\Rightarrow$ It is dependent from n

OK

Fachhochschule Nordwestschweiz

Ex 3

f_1 : $\sqrt{n}$
f_2 : $n \cdot \log_{10}(n)$
f_3 : $n \cdot \log_2(n^2)$
f_4 : n^2
f_5 : $n^2 \cdot \log_2(n^2)$ OK

~~$f_i = \Theta(f_{i+1})$:~~

"Justify your decisions, particularly for the i cases where fi = Θ(fi+1) does not hold."
"justify your decisions" was meant for the O(n) cases too.
=> Just add a conclusion to each term.

i=1

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\ 1}{\cancel{n}\cdot \log_{10} n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}\cdot \log_{10} n} = 0$$ f

=> f1 in O(f2)

i=2

$$\lim_{n\to\infty} \frac{f_2(n)}{f_3(n)} = \lim_{n\to\infty} \frac{\cancel{n}\cdot \log_{10}(n)}{\cancel{n}\cdot \log_2(n^2)} = \cancel{\lim_{n\to\infty} \frac{\log_{10}(n)}{2\cdot \log_{10}(n)}}$$

$$= \lim_{n\to\infty} \frac{\log_{10}(n)}{\frac{\log_{10}(n^2)}{\log_{10}(2)}} = \lim_{n\to\infty} \frac{\cancel{\log_{10}(n)}\cdot \log_{10}(2)}{2\cdot \cancel{\log_{10}(n)}} = \begin{cases} >0 \\ <\infty \end{cases}$$

=> f2 in Θ(f3)

i=3

$$\lim_{n\to\infty} \frac{f_3(n)}{f_4(n)} = \lim_{n\to\infty} \frac{\cancel{n}\cdot \log_2(n^2)}{n^{\cancel{2}}} = \lim_{n\to\infty} \frac{2\cdot \log_2(n)}{n} \underset{\text{Hopital}}{=} \lim_{n\to\infty} \frac{2\frac{1}{n\cdot \ln(2)}}{1} = 0$$ f

=> f3 in O(f4)

i=4

$$\lim_{n\to\infty} \frac{f_4(n)}{f(n)} = \lim_{n\to\infty} \frac{\cancel{n^2}\ 1}{\cancel{n^2}\log_2(n^2)} = \lim_{n\to\infty} \frac{1}{2\cdot \log_2(n)} = 0$$ f

=> f4 in O(f5)

$f_i = \Theta(f_i+1)$ does not hold for $i = \{1, 3, 4\}$