# Exercise Sheet 9

## Ex 1

a)
$$T(n) = \begin{cases} a & \text{for } n=1 \\ c + T\left(\frac{n}{2}\right) & \text{else} \end{cases}$$

Assumption $T(n) \in O(\log(n)) \Rightarrow T(n) \le c' \cdot \log_2(n)$

Case $n > 1$:

Basis $n=2$

$$T(2) = c + \underbrace{T\left(\tfrac{2}{2}\right)}_{a} = c + a \in O(\log(2))$$ ✓

OK

Inductive Step $\frac{n}{2} \longmapsto n$

$$T(n) = c + T\left(\frac{n}{2}\right)$$
$$\overset{IA}{\le} c + c' \cdot \log_2\left(\frac{n}{2}\right)$$
$$= c + c' \cdot \log_2(n) - c' \underbrace{\log_2(2)}_{1}$$
$$= c - c' + c' \cdot \log_2(n)$$
$$\le c' \cdot \log_2(n) \qquad c' > 1$$ ✓

$T(n) \in O(\log n)$ for $n > 1$

OK

Case $n = 1$

$$T(n) = a \in O(1) \in O(\log n)$$ ✓

b) $T(n) = \begin{cases} a & \text{for } n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n^3 & \text{else} \end{cases}$

Case $n > 1$

$a = 2$ ; $b = 2$ ; $f(n) = n^3$ OK

~~with $a = b \Rightarrow$ case 2 ~~Mathe~~ Master theorem~~

$f(n) \in \Omega\left(n^{\log_b (a + \varepsilon)}\right)$ mit $\cancel{6} = \varepsilon > 0$

$f(n) \in \Omega\left(n^{\log_2 (2 + 6)}\right) = \Omega\left(n^3\right)$

Case 3 $\Rightarrow$ check regularity condition

$a\, f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ $\quad 0 \leq c < 1$

$n > n_0$

$2 \cdot \left(\frac{n}{2}\right)^3 \leq c \cdot n^3$

1/4* $n^3 \leq \frac{\cancel{1}}{4} c \cdot n^3$ $\quad c \geq \frac{3}{\cancel{4}}$ 1/4 ✓

$T(n) \in \Theta(f(n)) = \Theta(n^3)$

case $n = 1$

$T(n) = a \in O(1) \in O(n^3)$

Therefore

$T(n) \in \Theta(n^3)$

2.

$$T(n) = \begin{cases} 1 & \text{for } n = 1 \\ 4T\left(\frac{n}{2}\right) + n^2 & \text{else} \end{cases}$$

case $n > 1$:

$a = 4$; $b = 2$; $f(n) = n^2$

$$f(n) \in \Theta\left(n^{\log_b a}\right) = \Theta\left(n^2\right)$$

$\Rightarrow$ case 2

$$T(n) \cancel{=} \in \Theta\left(\cancel{n^{\log}} \; n^{\log_b a} \cdot \log_2 n\right)$$

$$T(n) \cancel{=} \in \Theta\left(n^2 \cdot \log_2(n)\right)$$

OK

case $n = 1$

$$T(n) \in O(1) \in O\left(n^2 \log_2(n)\right)$$

$\Rightarrow$

$$T(n) \in \Theta\left(n^2 \log_2 n\right)$$

OK

3.

$$T(n) = \begin{cases} a & \text{for } n \leq 2 \\ T(\sqrt{n}) + a & \text{else} \end{cases}$$

case $n > 2$

Substitution $n = 2^m$ ~~dafür~~

$S(m)$ is $T(2^m)$

$$S(m) = T(2^m) = T(\sqrt{2^m}) + a$$

$$= T(2^{\frac{m}{2}}) + a$$

$$S(m) = S\left(\frac{m}{2}\right) + a$$

$a' = 1$ ; $b' = 2$ ; $f(m) = a$ OK

$$f(m) \in \Theta(1) \in m^{\log_{b'} a'} = m^{\underbrace{\log_2 1}_{0}}$$

$\Rightarrow$ case 2

$$S(m) \in \Theta(\underbrace{m^{\log_{b'} a'}}_{1} \log_2 m)$$

$$S(m) \in \Theta(\log_2 m)$$

Rücksubstitution $m = \log_2 n$

$$T(n) \in \Theta(\log_2(\log_2 n))$$ OK

case $n \leq 2$

$$a \in \Theta(1) \in O(\log_2(\log_2(n)))$$

Therefore

$$T(n) = \Theta(\log_2(\log_2(n)))$$ OK