A Linear Time Algorithm for Finding Minimum Spanning Tree Replacement Edges

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Abstract

Given an undirected, weighted graph, the minimum spanning tree (MST) is a tree that connects all of the vertices of the graph with minimum sum of edge weights. In real world applications, network designers often seek to quickly find a replacement edge for each edge in the MST. For example, when a traffic accident closes a road in a transportation network, or a line goes down in a communication network, the replacement edge may reconnect the MST at lowest cost. In the paper, we consider the case of finding the lowest cost replacement edge for each edge of the MST. A previous algorithm by Tarjan takes $O(m\alpha(n,m))$ time, where $\alpha(n,m)$ is the inverse Ackermann's function. Our algorithm is the first that runs in O(n+m) time and O(n+m) space. Moreover, it is easy to implement and our experimental study demonstrates fast performance on several types of graphs. The most vital edge is the graph edge whose removal causes the largest increase in weight of the minimum spanning tree. Our algorithm also finds the most vital edge in linear time.

1. Introduction

Let G = (V, E) be an undirected, weighted graph on n = |V| vertices and m = |E| edges, with weight function w(e) for each edge $e \in E$. A minimum spanning tree T = MST(G) is a subset of n - 1 edges with the minimal sum of weights that connects the n vertices.

In real world applications that use MST, often these edges represent roadways, transmission lines, and communication channels. When an edge deteriorates, for example, a traffic accident shuts a road or a link goes down, we wish to quickly find its *replacement edge* to maintain the MST. The replacement edge is the lightest weight edge that reconnects the MST. For example, Cattaneo et al. [2] maintain a minimum spanning tree for the graph of the Internet Autonomous Systems using dynamic graphs. Edges may be inserted or deleted, and a deletion of an MST edge triggers an expensive operation to find a replacement edge of lightest weight that reconnects the MST in $O(m \log n)$ time from the non-tree edges, or $O(m + n \log n)$ time when a cache is used to store partial results from previous delete operations. In the paper, we consider the case of finding all minimum spanning tree replacement edges. Our algorithm is the first that runs in O(n + m) time and O(n + m) space, which is an asymptotic improvement over the previous best algorithm due to Tarjan [16] that runs in $O(m\alpha(n, m))$, where $\alpha(n, m)$ is the inverse Ackermann's function.

The main results of this paper are the first linear-time algorithm for finding all replacement edges in the minimum spanning tree. Our linear time and space algorithm is an asymptotic improvement from all prior algorithms, uses only a tree data structure, alleviates the need to use least common ancestor (LCA) algorithms, and is easy to implement.

2. Linear-Time Algorithm

Given an undirected, weighted graph G = (V, E), a minimum spanning tree T, and the remaining non-tree edges E - T, sorted by weight from lowest to highest weight, Algorithm 1 gives the minimum spanning tree replacement edges algorithm. Since replacement edges are found immediately after computing an MST, we can re-use the sorted edges from Kruskal's [12] MST algorithm.

Our approach is to root the MST at an arbitrary vertex v_r . Using a depth-first search (DFS) traversal from the root, the algorithm visits each vertex and sets its parent P[v] to the traversal order. The parent of the root v_r , $P[v_r]$, is set to v_r . A counter is used to record the first time each vertex $v \in V$ is visited (IN[v]), and when the traversal finishes with vertex v (OUT[v]). Figure 1 shows the depth-first traversal of the MST tree T.

Algorithm 1 Linear Time MST Replacement Edges

```
Input: Graph G, MST edges labeled, and sorted list of non-MST edges
 1: procedure PATHLABEL(s, t, e)
       if IN[s] < IN[t] < OUT[s] then
                                                                                                                                           \triangleright s is ancestor of t
 2:
          return
 3:
       if IN[t] < IN[s] < OUT[t] then
                                                                                                                                            \triangleright t is ancestor of s
 4:
          PLAN \leftarrow ANC, k_1 \leftarrow IN[t], k_2 \leftarrow IN[s]
 5:
 6:
          if IN[s] < IN[t] then
 7:
             PLAN \leftarrow LEFT, k_1 \leftarrow OUT[s], k_2 \leftarrow IN[t]
                                                                                                                                                  \triangleright s is left of t
 8:
 9:
             PLAN \leftarrow RIGHT, k_1 \leftarrow OUT[t], k_2 \leftarrow IN[s]
                                                                                                                                                \triangleright s is right of t
10:
       i \leftarrow 0
11:
        \hat{v} \leftarrow s
12:
        while k_1 < k_2 do
                                                                                                                      \triangleright Detecting when below LCA(s, t)
13:
          if \hat{v}.next = P[\hat{v}] then
                                                                                                   \triangleright If true, inspect the MST edge from \hat{v} to parent
14:
             \hat{e} \leftarrow \langle \hat{v}, P[\hat{v}] \rangle
                                                                                                                                                      \triangleright \hat{e} \in MST
15:
             if R_{\hat{e}} = \emptyset then
                                                                                                          > Replacement edge has not been found yet
16:
                R_{\hat{e}} \leftarrow e

    ⊳ Set the replacement edge

17:
                L[i] \leftarrow \hat{v}
                                                                                                               \triangleright Add \hat{v} to list for updating next pointer
18:
                i \leftarrow i + 1
19:
          \hat{v} \leftarrow \hat{v}.\mathsf{next}
20:
          switch PLAN do
21:
22:
             case ANC
23:
                k_2 \leftarrow \text{IN}[\hat{v}]
             case LEFT
24:
                k_1 \leftarrow \text{OUT}[\hat{v}]
25:
             case RIGHT
26:
                k_2 \leftarrow \text{IN}[\hat{v}]
27:
       for j \leftarrow 0, i-1 do
                                                                               ▶ Update the next pointer of each vertex adjacent from below...
28:
          L[j].next \leftarrow \hat{v}
                                                                                           ⊳ to a tree edge with newly assigned replacement edge
29:
     procedure FINDREPLACEMENTEDGES
       Root the MST T at arbitrary vertex v_r and store parents in P[.]
31:
                                                                                                                               ⊳ root's parent points to root
32:
        Run DFS on T, setting IN[v] and OUT[v] to the sequence when v is first and last visited, respectively.
33:
       for all vertices v \in V do
34:
          v.\mathsf{next} \leftarrow P[v]
35:
       for k \leftarrow 1, m-n+1 do
                                                                                                        \triangleright Scan the m-n+1 sorted non-MST edges
36:
37:
           \langle v_i, v_j \rangle \leftarrow e_k
          PATHLABEL(v_i, v_j, \langle v_i, v_j \rangle)
38:
          PATHLABEL(v_i, v_i, \langle v_i, v_i \rangle)
39:
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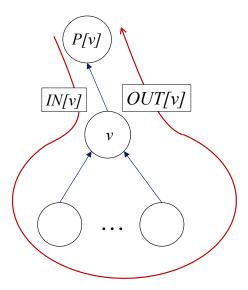


Figure 1. Depth-first traversal of the minimum spanning tree tree T, setting each parent P[v], and IN[v], and OUT[v] to the counter when vertex v is first and last visited, respectively.

By definition, the MST replacement edge for a given MST edge e is the edge from the remainder set E-T of lightest weight that reconnects the MST T when edge e is removed. Each of the m-n+1 edges in E-T induces a cycle with the MST edges in T. Hence, the remainder set induces m-n+1 unique cycles, one for each non-MST tree edge. For any given MST tree edge, there is a subset of cycles that include the edge, and the cycle with the lightest weight non-MST tree edge represents its replacement edge.

Since the m-n+1 remaining edges are sorted by weight, we can scan these edges from lowest to highest, and the corresponding cycles with participating tree edges. As we inspect each cycle, any tree edge without a replacement edge set is assigned the current edge the first time a cycle includes that tree edge. In practice, we may terminate the scanning of remaining edges once n-1-k replacement edges are identified, where k is the number of bridges in G. (Since removing a bridge edge disconnects the graph G, bridge edges will always be included in the MST, and will not have a replacement edge.) Counting the number of bridges in the graph G takes O(n+m) time.

For each remaining edge $\langle s,t \rangle$, if vertex s is an descendant of t, (if and only if $\mathrm{IN}[t] < \mathrm{IN}[s] < \mathrm{OUT}[s] < \mathrm{OUT}[t]$), we make a single PATHLABEL call for the edges from s up to t. Otherwise, two calls are made to PATHLABEL, corresponding to inspecting the left and right paths of the cycle from s and t, respectively, that would meet at the least common ancestor (LCA) of s and t in the tree. We assume w.l.o.g. that s is visited in the depth-first seach traversal before t. Let's call $z = \mathrm{LCA}[s,t]$. It is useful to use z in describing the approach, yet we never actually need to find the LCA z. We know $\mathrm{IN}[z] < \mathrm{IN}[s] < \mathrm{OUT}[s] < \mathrm{IN}[t] < \mathrm{OUT}[t] < \mathrm{OUT}[t]$ by definition of the depth-first traversal. As illustrated in Figure 2, consider a vertex w that lies on the path from s to the root v_r . Vertex w must either lie on the path from s to the LCA t (where t out t or t

2.1. Proof of correctness

Claim 1 The lowest weight non-MST edge that induces a cycle containing an MST edge e is the replacement for e. This follows from the Cut Property.

Claim 2 Algorithm 1 traverses the cycle induced by an non-MST edge from descendent to ancestor.

Proof of claim 2 First observe that the parent is set for each vertex in DFS order so that the traversal carried out by lines 13–19 follows a single path from descendent to ancestor. For each $\langle s,t\rangle$ edge s may be the ancestor of t or vice versa, or there is a least common ancestor (LCA) between s and t. The lines 2–10 always sets the starting vertex in the traversal of the cycle so that it proceeds from descendent to ancestor as follows.

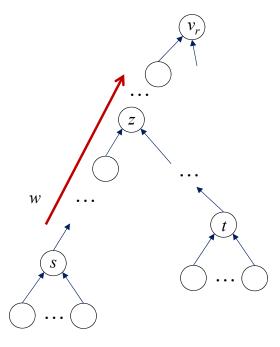


Figure 2. The PathLabel algorithm detects when vertex w on the path from s to the root v_r is an ancestor of the vertex z = LCA[s, t], without determining z.

If s is the ancestor of t then no traversal is made because line 2 returns. If t is the ancestor of s, then the traversal begins with s at line 12 and each next pointer is a parent that leads to t. Otherwise, there is an LCA and from lines 38–39 each branch is traversed from s and t up to the LCA.

The next pointers are re-assigned at lines 28-29 to the last vertex in the traversal which must be an ancestor of the starting vertex. Any subsequent call to the next pointer at line 20 again leads to an ancestor.

Claim 3 Algorithm 1 traverses only the exact edges in a cycle induced by a non-MST edge.

Proof of claim 3 We prove this by loop invariance for a single cycle. Let $\langle s, t \rangle$ be a non-MST edge and denote the cycle it induces by $s, v_i, v_{i+1}, \dots, t, s$.

The loop invariant is each vertex at the start of the while loop at lines 13–27 is an endpoint of an MST edge in the cycle induced by $\langle s, t \rangle$.

The base step at line 12 holds trivially since the starting vertex is s.

The inductive step maintains the loop invariant as follows. At each iteration the next vertex is set and by Claim 2 this must be a predecessor in the path from descendent to ancestor, thus every iteration produces the sequence v_i, v_{i+1}, \ldots, t where t is an ancestor or LCA.

Termination of the loop is determined by new values for either k_1 or k_2 between lines 21–27. If the case was that t was the ancestor of s, then k_2 decreases in value as the next pointers approach t. Otherwise there is an LCA and if s is older, meaning it precedes t in the ancestry tree, then it is in the *left* branch and k_1 increases in value as next pointers approach t, otherwise we have the *right* branch and similarly the loop ends as next pointers move towards the other endpoint.

Theorem 1 Given an undirected, weighted graph G = (V, E), then Algorithm 1 correctly finds all minimum cost replacement edges in the Minimum Spanning Tree of G.

Proof: First observe that all non-MST edges are processed in ascending order by weight between lines 36–39. Then the $\langle s,t\rangle$ edge that induces the first cycle to contain an MST edge must be the replacement edge for that MST edge following Claim 1 and the order of processing. This is carried out by line 16, hence each MST edge gets the first non-MST edge that induces a cycle containing it.

It follows from Claim 3 and the loop over all non-MST edges at lines 36–39 that all MST edges in a cycle will get a replacement edge.

At the end of a cycle, the next pointers are set to the ancestor or LCA so that any edge from this cycle cannot be traversed again.

2.2. Complexity analysis

Claim 4 Algorithm 1 traverses at most 2m = O(m) next pointers.

Proof of claim 4 To count the number of next pointers traversed, we partition the calls into two groups: when next pointers coincide with parent pointers, and when next pointers are in a compressed path. For the first group, since there are n-1 tree edges, a next pointer is called twice (in and out of the tree edge) before the path compression in PATHLABEL in lines 28-29 of Algorithm 1. Hence, at most 2(n-1) next pointers are traversed in this group. In the second group, we have a next pointer to start a compressed path in PATHLABEL for each of the two endpoints of the E-T remaining edges. While a traversal up the ancestry of the MST in lines 13-19 may traverse other compressed path next pointers as the traversal potentially alternates with the first group of next pointers, we have already counted these with the first group. Hence, there are at most 2(m-n+1) next pointers traversed in this group. The maximum number of next pointers traversed is the sum 2(n-1)+2(m-n+1)=2m=O(m).

Theorem 2 Given an undirected, weighted graph G = (V, E), it is possible to find all minimum cost replacement edges in the Minimum Spanning Tree of G in O(n+m) time and O(n+m) space.

Proof: Algorithm 1 accomplishes this as follows. Let T be the Minimum Spanning Tree of G. Initializing all values in the parent array P takes O(n) time. There are n-1 edges in T thus DFS on T at line 33 takes O(n) time. There are at most $m-n+1 \leq O(m)$ non-MST edges. Then applying Claims 3 and 4 on all non-MST edges leads to O(m) time. All data structures are simple arrays, taking O(n) space, and all non-MST edges take O(m) space. Therefore it takes O(n+m) time and O(n+m) space as claimed.

2.3. Related Work

Spira and Pan [13] presented an $O(n^2)$ algorithm to update the MST when new vertices are added, and could find all replacement edges in $O(n^3)$ time. Chin and Houck [4] improved this bound to $O(n^2)$ using a more efficient approach to insert and delete vertices from the graph. Tarjan [16] gave an $O(m\alpha(n,m))$ time algorithm using path compression, where $\alpha(n,m)$ is the inverse Ackermann's function. Kooshesh and Crawford [11] proposed an algorithm that computes the replacement for every edge in the minimum spanning tree that runs in $O(\max(C_{\rm mst}, n \log n))$, where $C_{\rm mst}$ is the cost of computing a minimum spanning tree of G' = (V, E - T). Their approach is based on computing efficiently the possible replacement edges from the remaining edge set.

In 1994, Katajainen and Träff [10] designed a parallel algorithm that runs in $O(\log n)$ time and O(m) space using m processors on a MINIMUM CRCW PRAM machine. Their approach uses path product and path labelling techniques. However, this approach does not improve the sequential running time.

Das and Loui [5] solved a similar problem of *node replacement* for deleted vertices in the MST, that runs in $O(m \log n)$ sequential time, or $O(m\alpha(m,n))$ when the edges E are pre-sorted by weight; and a parallel algorithm that takes $O(\log^2 n)$ time and m processors on a CREW PRAM.

3. Experimental Study

The minimum spanning tree replacement edge algorithm is both easy to implement and has fast sequential running time. In this section, we perform an empirical study using graphs from the 10th DIMACS Implementation Challenge on Graph Partitioning and Graph Clustering [1].

The first category of graphs uses Delaunay triangularization of random points in the plane [6]. Table 1 gives the size of each graph, and the edge weights are uniformly random.

The second category of graphs is a synthetic real-world network using the Kronecker generator R-MAT [3] and the same parameters A = 0.57, B = 0.19, C = 0.19, D = 0.05 selected in the DIMACS Challenge and the Graph500 Benchmark. We modify the generator so that the graph is connected, and assign uniformly random edge weights. For these R-MAT graphs

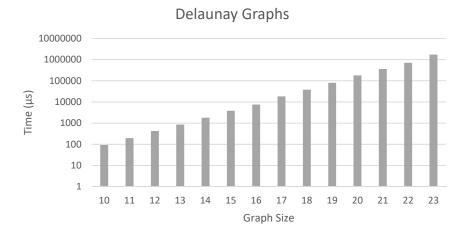


Figure 3. Execution time to find the minimum spanning tree replacement edges in the Delaunay graphs.

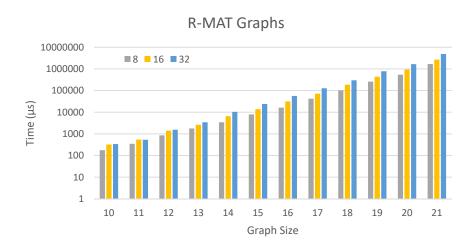


Figure 4. Execution time to find the minimum spanning tree replacement edges in the R-MAT graphs.

on n vertices, we varied the number of edges using an edge factor of 8, 16, and 32, where the number m of edges is n times this edge factor. For instance, an edge factor of 8 implies that m = 8n.

We test our implementation on an Intel Xeon CPU E5-2680 with clock speed of 2.70GHz, using GNU C compiler (gcc) version 9.1.0 and optimization level 3 ("-03"). The performance results are given in Figures 3 and 4, for the Delaunay triangularization and R-MAT graphs, respectively. Each plot is on a log-log scale. The horizontal axis labelled "Graph Size" is the logarithm in base 2 of the number of vertices and the vertical axis is the running time in microseconds (μ s). Clearly, the algorithm runs in fast linear time with respect to the graph size.

4. Most Vital Edge Algorithm

The most vital edge of a connected, weighted graph G is the edge whose removal causes the largest increase in the weight of the minimum spanning tree [8]. When the graph contains bridges (which can be found in linear time [15]), the most vital edge is undefined. Hsu et al. [7] designed algorithms to find the most vital edge in $O(m \log m)$ and $O(n^2)$ time. Iwano and Katoh [9] improve this with $O(m+n\log n)$ and $O(m\alpha(m,n))$ time algorithms. Suraweera et al. [14] prove that the most vital edge is in the minimum spanning tree. Hence, once Algorithm 1 finds all replacement edges of the minimum spanning tree, the most vital edge takes O(n) time by simply finding the tree edge with maximum difference in weight from its replacement edge. Thus, our approach will also find the most vital edge in O(n+m) time, and is the first linear algorithm for finding the most vital edge of the minimum spanning tree.

Graph	n	m
delaunay_n10	1024	3056
delaunay_n11	2048	6127
delaunay_n12	4096	12264
delaunay_n13	8192	24547
delaunay_n14	16384	49122
delaunay_n15	32768	98274
delaunay_n16	65536	196575
delaunay_n17	131072	393176
delaunay_n18	262144	786396
delaunay_n19	524289	1572954
delaunay_n20	1048577	3145817
delaunay_n21	2097153	6291539
delaunay_n22	4194305	12583000
delaunay_n23	8388609	25165915

Table 1. Number of vertices and edges in the Delaunay Graphs from the 10th DIMACS Implementation Challenge.

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