Advance Control Engineering 2 - Assignment 2

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Please answer to the following problems and questions.
Some rules

– Reports can be done in live script.

– Assignment can be develop in groups of maximum three.

– Simulink and any other support shall be provided.

– Questions and clarifications are well welcome.

1 Control of a flexible shaft

According to the derived model of the flexible shaft available in the lecture reading. Implement a state feedback with integrator control, in discrete time domain, which is able to

– damp the internal dynamic of the system

– track a constant speed set point with zero error at steady state

– compensate a load step

Moreover

– implement a *Simscape* model of the flexible shaft mathematical model (see video lecture).

– implement a load estimator, supposing that the dynamic of the load is much slower than the natural dynamic of the flexible shaft.

– design the state feedback gain, as well as, the observer gains using for both the steady state linear quadratic regulator method.

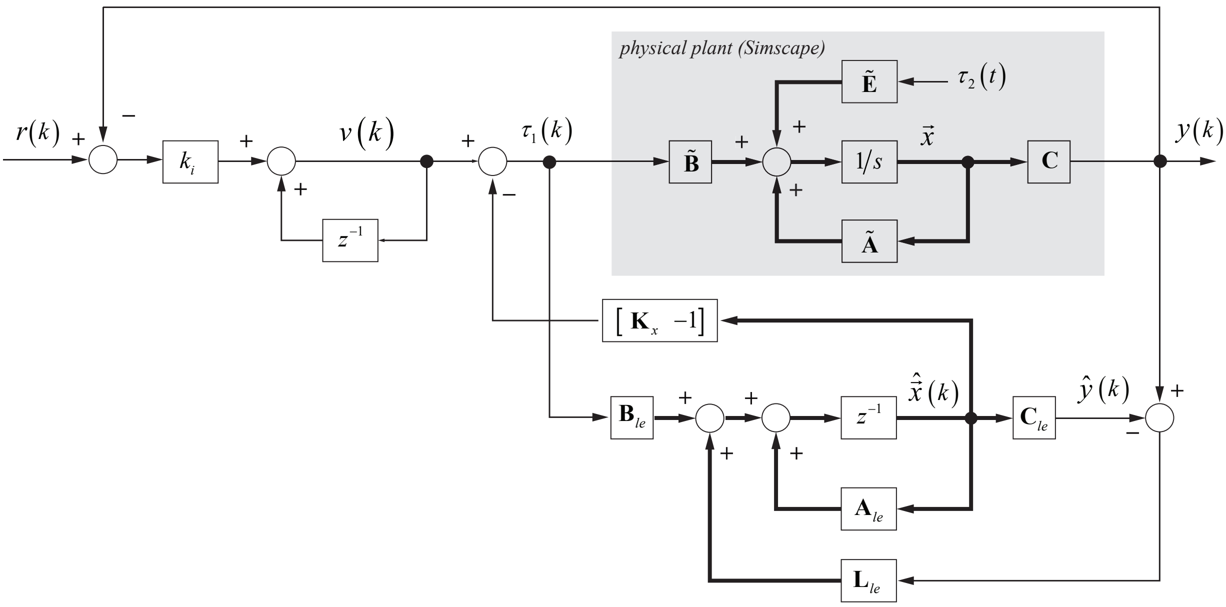

Figure 1: Description of the control strategy with integration and compensation of the load.

1.1 Parameters

$J_1 = 2.5\,\mathrm{kg\,m^2}$	$J_2 = 25\,\mathrm{kg\,m^2}$	$b_\theta = 0.0\,\mathrm{N\,m\,s\,rad^{-1}}$
$k_\theta = 100\,\mathrm{N\,m\,s\,rad^{-1}}$	$b = 0.0\,\mathrm{N\,m\,s\,rad^{-1}}$	$\tau_2 = 10.0\,\mathrm{N\,m}$

Table 1: Flexible shaft data.

2 Control of a Segway

Implement linear quadratic regulator with final condition constraint, according to the structure of Figure 2, of the plant shown in Figure 3. Consider a proper initial condition different from the origin.

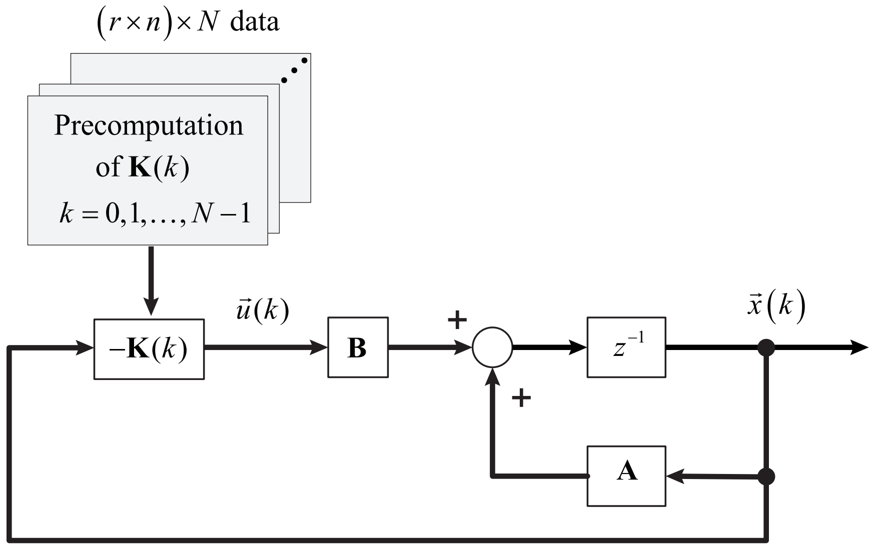

Figure 2: Example of finite time optimal regulator.

2.1 Model derivation

According Figure 3 the system is governed (control input) by the torque $\tau_m(t)$.

The generalized coordinates which describe the system are

- The wheel angle $\alpha(t)$
- The bar angle $\vartheta(t)$

Both, the wheel and the bar of respectively of mass m_1 and m_2 and inertia J_1 and J_2 are subjected to viscosity force described by the coefficient b_1 and b_2 respectively.

To derive the dynamical equations which govern the system, we apply the Euler-Lagrange equations.

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i} = Q \quad i = 1, 2, \ldots, n \tag{2.1}$$

where the Lagrange function $\mathscr{L}$ is

$$\mathscr{L} = E_{kin} - E_{pot} \tag{2.2}$$

To derive the total kinetic energy we first write the vector speed for both masses which are subject to move

Let $\vec{v}_1$ be the speed of the wheel. It can be expressed as follows

$$\vec{v}_1 = r_1\dot{\alpha}\vec{i} \tag{2.3}$$

Let $\vec{v}_2$ be the speed of the bar. It can be expressed as follows

$$\vec{v}_2 = \left(r_1\dot{\alpha} + r_2\dot{\vartheta}\cos\vartheta\right)\vec{i} - r_2\dot{\vartheta}\sin\vartheta\vec{j} \tag{2.4}$$

The total kinetic energy can be written as follows

$$E_{kin} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}J_1\dot{\alpha}^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}J_2\dot{\vartheta}^2 \tag{2.5}$$

where

$$v_1^2 = \left(r_1\dot{\alpha}\right)^2 \tag{2.6}$$

and

$$v_2^2 = \big(r_1\dot\alpha + r_2\dot\vartheta\cos\vartheta\big)^2 + \big(r_2\dot\vartheta\sin\vartheta\big)^2 \tag{2.7}$$

The potential energy can be expressed as follows

$$E_{pot} = m_2 g r_2 \cos\vartheta. \tag{2.8}$$

The resulting Lagrangian is

$$\mathscr{L} = \frac{1}{2}\Big[\big(m_1+m_2\big)r_1^2 + J_1\Big]\dot\alpha^2 + \frac{1}{2}\Big[m_2 r_2^2 + J_2\Big]\dot\vartheta^2 + m_2 r_2\big(r_1\dot\alpha\dot\vartheta - g\big) \tag{2.9}$$

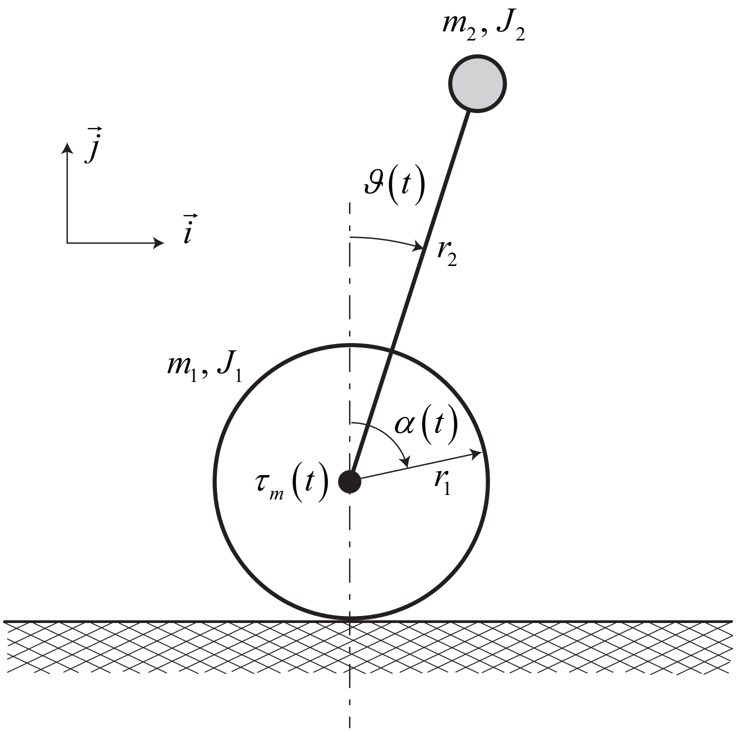

Figure 3: Segway.

Now, suppose to apply Eq. (2.1) for the variables α and ϑ and we obtain the following dynamic equations

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot\alpha} - \frac{\partial\mathscr{L}}{\partial\alpha} = \tau_m - b_1\dot\alpha - b_2\big(\dot\alpha + \dot\vartheta\big) \tag{2.10}$$

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot\vartheta} - \frac{\partial\mathscr{L}}{\partial\vartheta} = \tau_m - b_1\dot\alpha \tag{2.11}$$

which results in following set of equations

$$\begin{cases} \Big[\big(m_1+m_2\big)r_1^2 + J_1\Big]\ddot\alpha + m_2 r_1 r_2\Big[\ddot\vartheta\cos\vartheta - \dot\vartheta^2\sin\vartheta\Big] + \big(b_1+b_2\big)\dot\alpha + b_2\dot\vartheta - \tau_m = 0 \\ \big(m_2 r_2^2 + J_2\big)\ddot\vartheta + m_2 r_1 r_2\Big[\ddot\alpha\cos\vartheta + g/r_1\sin\vartheta\Big] + b_2\dot\alpha + b_2\dot\vartheta + \tau_m = 0 \end{cases} \tag{2.12}$$

solving the system of Eq. (2.12) respect to $\ddot{\alpha}$ and $\ddot{\vartheta}$ we obtain the following system

$$\begin{cases} \dot{\alpha} = \omega_1 \\ \dot{\omega}_1 = \dfrac{2}{\mathscr{D}}\Bigg[m_2 r_1 r_2 \cos\vartheta\Big(\tau_m + b_2\omega_1 + b_2\omega_2 + g m_2 r_2 \sin\vartheta\Big) - 2\Big(J_2 + m_2 r_2^2\Big) \\ \qquad\qquad \Big(-\tau_m + b_1\omega_1 + b_2\omega_1 + b_2\omega_2 - m_2 r_1 r_2 \omega_2^2 \sin\vartheta\Big)\Bigg] \\ \dot{\vartheta} = \omega_2 \\ \dot{\omega}_2 = -\dfrac{2}{\mathscr{D}}\Bigg[\Big[J_1 + \big(m_1 + m_2\big) r_1^2\Big]\Big(\tau_m + b_2\omega_1 + b_2\omega_2 + g m_2 r_2 \sin\vartheta\Big) + \\ \qquad\qquad m_2 r_1 r_2 \cos\vartheta\Big(\tau_m - b_1\omega_1 - b_2\omega_1 - b_2\omega_2 + m_2 r_1 r_2 \omega_2^2 \sin\vartheta\Big)\Bigg] \end{cases} \tag{2.13}$$

where

$$\mathscr{D} = 2J_1\big(J_2 + m_2 r_2^2\big) + r_1^2\Big[2J_2\big(m_1 + m_2\big) + m_2\big(2m_1 + m_2\big) r_2^2\Big] - m_2^2 r_1^2 r_2^2 \cos 2\vartheta \tag{2.14}$$

Let's now to derive the linearized model around the equilibrium point

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \tau_m^0 = 0 \tag{2.15}$$

Lets to define the transition function as well

$$\begin{cases} f_1\big(\alpha, \omega_1, \vartheta, \omega_2\big) = \omega_1 \\ f_2\big(\alpha, \omega_1, \vartheta, \omega_2\big) = \dfrac{2}{\mathscr{D}}\Bigg[m_2 r_1 r_2 \cos\vartheta\Big(\tau_m + b_2\omega_1 + b_2\omega_2 + g m_2 r_2 \sin\vartheta\Big) \\ \qquad\qquad - 2\Big(J_2 + m_2 r_2^2\Big)\Big(-\tau_m + b_1\omega_1 + b_2\omega_1 + b_2\omega_2 - m_2 r_1 r_2 \omega_2^2 \sin\vartheta\Big)\Bigg] \\ f_3\big(\alpha, \omega_1, \vartheta, \omega_2\big) = \omega_2 \\ f_4\big(\alpha, \omega_1, \vartheta, \omega_2\big) = -\dfrac{2}{\mathscr{D}}\Bigg[\Big[J_1 + \big(m_1 + m_2\big) r_1^2\Big]\Big(\tau_m + b_2\omega_1 + b_2\omega_2 + g m_2 r_2 \sin\vartheta\Big) + \\ \qquad\qquad m_2 r_1 r_2 \cos\vartheta\Big(\tau_m - b_1\omega_1 - b_2\omega_1 - b_2\omega_2 + m_2 r_1 r_2 \omega_2^2 \sin\vartheta\Big)\Bigg] \end{cases} \tag{2.16}$$

and we calculate the following terms[1]

$$\left.\frac{\partial f_1}{\partial \omega_1}\right|_{\vec{x}_0,\tau_m^0} = 1, \qquad \left.\frac{\partial f_3}{\partial \omega_2}\right|_{\vec{x}_0,\tau_m^0} = 1 \tag{2.17}$$

$$\left.\frac{\partial f_2}{\partial \omega_1}\right|_{\vec{x}_0,\tau_m^0} = \frac{1}{\mathscr{D}_0}\Big[2b_2m_2r_1r_2 - 2\big(b_1+b_2\big)\big(J_2+m_2r_2^2\big)\Big] \tag{2.18}$$

$$\left.\frac{\partial f_2}{\partial \vartheta}\right|_{\vec{x}_0,\tau_m^0} = \frac{1}{\mathscr{D}_0}\Big[2gm_2^2r_1r_2^2\Big] \tag{2.19}$$

$$\left.\frac{\partial f_2}{\partial \omega_2}\right|_{\vec{x}_0,\tau_m^0} = \frac{1}{\mathscr{D}_0}\Big[2b_2\Big(m_2r_1r_2 - J_2 - m_2r_2^2\Big)\Big] \tag{2.20}$$

$$\left.\frac{\partial f_4}{\partial \omega_1}\right|_{\vec{x}_0,\tau_m^0} = \frac{1}{\mathscr{D}_0}\Big[-2b_2\Big[J_1\big(m_1+m_2\big)r_1^2\Big] + 2\big(b_1+b_2\big)m_2r_1r_2\Big] \tag{2.21}$$

$$\left.\frac{\partial f_4}{\partial \vartheta}\right|_{\vec{x}_0,\tau_m^0} = \frac{1}{\mathscr{D}_0}\Big[-2gm_2r_2\Big[J_1 + \big(m_1+m_2\big)r_1^2\Big]\Big] \tag{2.22}$$

$$\left.\frac{\partial f_4}{\partial \omega_2}\right|_{\vec{x}_0,\tau_m^0} = \frac{1}{\mathscr{D}_0}\Big[-2\Big[b_2\Big(J_1 + \big(m_1+m_2\big)r_1^2\Big) - b_2m_2r_1r_2\Big]\Big] \tag{2.23}$$

$$\left.\frac{\partial f_2}{\partial \tau_m}\right|_{\vec{x}_0,\tau_m^0} = \frac{2\Big(J_2 + m_2r_1r_2 + m_2r_2^2\Big)}{\mathscr{D}_0} \tag{2.24}$$

$$\left.\frac{\partial f_4}{\partial \tau_m}\right|_{\vec{x}_0,\tau_m^0} = \frac{-2\Big(J_1 + \big(m_1+m_2\big)r_1^2 + m_2r_2r_2\Big)}{\mathscr{D}_0} \tag{2.25}$$

where

$$\mathscr{D}_0 = -m_2^2r_1^2r_2^2 + 2J_1\big(J_2+m_2r_2^2\big) + r_1^2\Big[2J_2\big(m_1+m_2\big) + m_2\big(2m_1+m_2\big)r_2^2\Big] \tag{2.26}$$

Using the results from Eq. (2.17) to Eq. (2.25) you can build the equivalent linear model around the equilibrium point

$$\begin{aligned}\dot{\vec{x}}(t) &= \tilde{\mathbf{A}}\vec{x}(t) + \tilde{\mathbf{B}}\tau_m(t)\\ \vec{y}(t) &= \mathbf{C}\vec{x}(t)\end{aligned} \tag{2.27}$$

[1] remind that

$$\left.\frac{\partial}{\partial \vec{x}}\vec{f}(\vec{x})\right|_{\vec{x}_0,\vec{u}_0} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix}_{\vec{x}_0,\vec{u}_0} = \tilde{\mathbf{A}}, \qquad \left.\frac{\partial}{\partial \vec{u}}\vec{f}(\vec{x})\right|_{\vec{x}_0,\vec{u}_0} = \begin{bmatrix}\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n}\end{bmatrix}_{\vec{x}_0,\vec{u}_0} = \tilde{\mathbf{B}}$$

where

$$\vec{x}(t) = \begin{bmatrix} \alpha(t) \\ \omega_1(t) \\ \vartheta(t) \\ \omega_2(t) \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.28}$$

2.2 Parameters

- $m_1 = 100\,\mathrm{kg}$
- $m_2 = 25\,\mathrm{kg}$
- $J_1 = 0.25\,\mathrm{kg\,m^2}$
- $J_2 = 25\,\mathrm{kg\,m^2}$
- $r_1 = 0.35\,\mathrm{m}$
- $r_2 = 1.75\,\mathrm{m}$
- $b_1 = 0\,\mathrm{N\,m\,s\,rad^{-1}}$
- $b_2 = 0\,\mathrm{N\,m\,s\,rad^{-1}}$
- $g = 9.81\,\mathrm{m\,s^{-2}}$

3 Additional questions

- Find the length of the sides of a parallelepiped: x, y and z such that the volume $f(x, y, z) = xyz$ is maximum when its surface is 1, $2xy + 2xz + 2yz = 1$.
- Write in a explicit way (symbolic) the steady state Riccati equation for the linear quadratic optimal problem considering $\mathbf{A} = \begin{bmatrix} 1 & t_s \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ t_s \end{bmatrix}$ and matrices $\mathbf{Q}$ and $\mathbf{R}$ as for your design.
- Write in pseudo-code language a program which implements an integrator, with setting of the initial condition, to be implemented into a microcontroller.