Effects of eddy currents in transformer windings

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Synopsis

The effects of eddy currents in transformer windings are considered, and a method is derived for calculating the variation of winding resistance and leakage inductance with frequency for transformers with single-layer, multilayer and sectionalised windings. The method consists in dividing the winding into portions, calculating the d.c. resistances and d.c. leakage inductances of each of these portions, and then multiplying the d.c. values by appropriate factors to obtain the corresponding a.c. values. These a.c. values are then referred to, say, the primary winding and summed to give the total winding resistance and leakage inductance of the transformer. Formulas are derived and quoted for calculating the d.c. resistances and leakage inductances of the winding portions. Theoretical expressions are derived for the variation with frequency etc. of the factors by which the d.c. values must be multiplied to obtain the corresponding a.c. values. These expressions are presented in the form of graphs, permitting the factors to be read as required.

List of symbols

MKS units are used throughout the paper.

a = breadth of a conductor (Fig. 7)
B = magnetic flux density
b = winding breadth (Fig. 7)
$D = 2\alpha h \tanh \dfrac{\alpha h}{2}$
F_R = ratio of a.c. value of R_W to d.c. value
F_L = ratio of a.c. value of L_W to d.c. value
$F_{L\infty}$ = value of F_L for a portion with an infinite number of layers
g = height of intersection gap (Fig. 7)
H = magnetic field strength
h = height of a conductor (Fig. 7)
I = total current in a conductor
i = current element
J = current density in a conductor
J_h = current density at the top of a conductor
L_g = leakage inductance due to an intersection gap
L_l = leakage inductance
L_{up} = leakage inductance due to pth interlayer gap
L_U = leakage inductance due to all interlayer gaps in a portion
L_W = leakage inductance due to flux cutting the conductors
l_T = mean turn length
$M = \alpha h \coth \alpha h$
$M_{1/2}$ = value of M with h replaced by $h/2$
m = number of whole layers in a winding portion
N = number of turns
N_l = number of turns per layer
N_p = number of turns in a winding portion (i.e. mN_l or $(m+\frac{1}{2})N_l$)
$n = N_2/N_1$
P, Q = constants of integration
p = positive integer $\leqslant m$
R_l = leakage resistance
R_W = resistance of a winding portion
s = magnetic path length
U = total height of all interlayer gaps in a winding portion
u = height of an interlayer gap
V_h = voltage across a layer at the tops of the conductors
V_i = induced voltage across a layer at the tops of the conductors
V_{ip} = induced voltage in each of the layers between the pth layer and the position of zero m.m.f. due to the flux crossing pth layer
V_l = total voltage across the ends of a layer
V_r = resistive voltage drop across a layer at the tops of the conductors
W_{up} = magnetic energy stored in pth interlayer gap
W_g = magnetic energy stored in an intersection gap
x = length measured in direction of winding height (Fig. 7)
$Z_W = R_W + j\omega L_W$
$\alpha = \sqrt{\dfrac{j\omega\mu_0\eta}{\rho}}$
δ = insulation thickness
$\eta = \dfrac{N_l a}{b}$
θ = temperature
ρ = resistivity
ϕ = magnetic flux
ϕ_a, ϕ_b, ϕ_c = parts of the flux cutting the winding
ϕ_p = flux crossing pth layer
ϕ_t = total flux cutting the winding
ω = angular frequency

Subscripts etc.

0 = d.c. value of—
′ = real part of—
″ = imaginary part of—

1 Introduction

The performance of a transformer can be calculated from its equivalent circuit, and methods are available for calculating the values of the equivalent-circuit elements from a knowledge of the geometry of the transformer and of the materials used in its construction. These methods usually give the d.c. values of the elements, but the variation with frequency of the shunt inductance and shunt loss can be obtained from a knowledge of the complex permeability of the core. The frequency variation of the winding capacitances (and their loss) is often negligible, but can be obtained from a knowledge of the complex permittivities of the dielectric materials involved. The other elements in the equivalent circuit are the winding resistance and leakage inductance. These elements will vary with frequency because of the effects of the eddy currents induced in the windings, and it is the purpose of the paper to produce a method for calculating the values of these elements at any frequency.

The effects of eddy currents on the effective resistance of coils has been considered by several authors in the past.[1,2,3] It is, in principle, possible to apply these results to transformer windings, although most of the papers do not consider

Paper 5058 J, first received 20th January and in revised form 12th April 1966
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transformer windings specifically. Hence it can be quite difficult to apply the results to, for example, a sectionalised transformer winding. None of these papers considers the effects of eddy currents on the leakage inductance of transformers.

It is obvious that the problem of eddy currents in transformer windings is physically related to the problem of eddy currents in slot-wound-armature conductors. This latter problem has also been treated by authors in the past,[4,5] and probably the results given for armature conductors have been applied to transformer windings. However, it does not seem to be easy to apply these results in such a way as to obtain accurate answers for transformer windings. The usefulness of the results derived for armature conductors is further reduced by the fact that they are intended for use at power frequencies, and the designer of a radio-frequency transformer will often find that a useful function is not plotted to a sufficiently high value of the frequency-dependent variable. It was decided not to try to modify the results for slot-wound-armature conductors, but instead to derive results for transformers directly, using the same type of mathematical approach as was used to derive the results for armature conductors. This mathematical approach has the advantage that it enables the frequency dependence of the winding resistance and leakage inductance to be derived simultaneously, from the real and imaginary parts of a complex function.

The terms 'leakage resistance' and 'leakage impedance' are used in the paper. These terms are defined by analogy with 'leakage inductance', being, respectively, equal to the effective winding resistance and the vector sum of the effective winding resistance and the leakage reactance.

2 Basic principles

Consider one layer of any general transformer winding, as shown in Fig. 1. The leakage flux crossing the winding

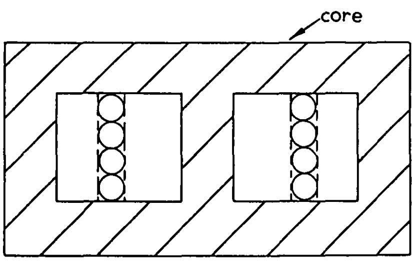

Fig. 1
One layer of a general transformer winding

space via this layer will induce eddy currents in the conductors of the layer, thereby producing an increase in the impedance of the layer. The leakage flux in the layer will also result in the storing of magnetic energy. This stored magnetic energy is that associated with the leakage inductance of the transformer. Thus the leakage flux in the layer determines the a.c. winding resistance and leakage inductance which are associated with that layer.

Hence the following important principle is established:

> When considering the leakage impedance due to a particular layer it is only necessary to consider the other layers of the winding insofar as they affect the flux in the layer being considered.

Consider the transformer winding shown in Fig. 2. The m.m.f. diagram shows that the m.m.f. acting at any point across the winding space can be considered to be caused by the total current between that point and an adjacent position of zero m.m.f. This is not a special property of the winding shown in Fig. 2, but it is true for all transformer windings. Hence another important principle is established:

> The leakage-flux distribution across any layer depends only on the current in that layer and the total current between the layer and an adjacent position of zero m.m.f.

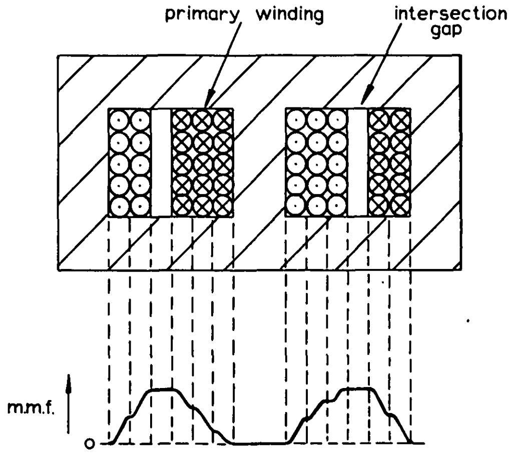

Fig. 2
A typical transformer, with associated m.m.f. diagram
○ ⊗ Indicates current directions

It is clear from the two principles established that, when considering the leakage impedance due to a particular layer, the only other layers which need be considered are those which lie between that layer and an adjacent position of zero m.m.f. Hence, for the purpose of calculating the leakage impedance, it is permissible to consider the winding space to consist of a number of separate parts, each part containing one position of zero m.m.f. Such parts will be termed 'winding portions'. The leakage impedances due to these portions can be summed to give the total leakage impedance referred to, say, the primary winding. It should, however, be noted that the calculated leakage impedance of each portion will be referred to the current flowing in that portion (since the calculation involves equating I^2R_l and $\frac{1}{2}I^2L_l$ to the dissipated and stored energy, respectively). Thus the calculated leakage impedances of portions forming part of the secondary winding will have to be divided by the square of the transformer turns ratio to refer them to the primary winding. An intersection gap can be considered to be part of either of the adjacent portions. The leakage impedances due to these gaps will, however, be referred automatically to the primary winding, if the gaps are considered to be part of portions which themselves form part of the primary winding.

3 Frequency-independent components of the leakage resistance and leakage inductance

It follows from the principles established in Section 2 that the leakage impedance due to the primary of the transformer shown in Fig. 2 can be calculated by considering only the portion consisting of the primary winding and the intersection gap. The relevant diagram is shown in Fig. 3, where the circular conductors have been replaced by square conductors with the same cross-sectional area. This has been done in order to make the mathematics more manageable. In addition, the mathematical treatment neglects the curvature of the conductors by considering a mean turn length. The errors introduced by these assumptions were thought unlikely to be large for typical communication transformers. Fig. 3 also shows the d.c. m.m.f. diagram. Increasing the frequency will alter the current distribution across each conductor, but the total net current flow in each conductor will remain unaltered. Hence increasing the frequency will alter the m.m.f. acting on the conductors, but will leave the m.m.f.

acting on the intersection gap and the interlayer gaps unaltered. Thus the leakage inductance due to the gaps will be frequency independent.

For a winding of the type shown in Fig. 2, the leakage

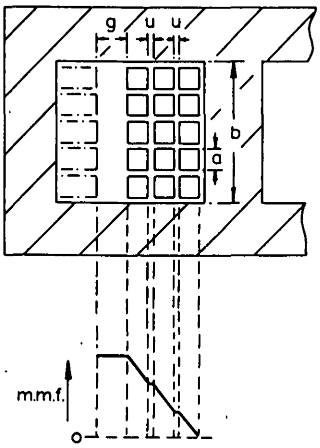

Fig. 3

Primary winding and intersection gap, with associated d.c. m.m.f. diagram

Current in each conductor $= I_1$
Number of primary turns $= N_1$
Number of turns per layer $= N_l$
Number of layers $= m$
Mean turn length $= l_T$

inductance due to the intersection gap can easily be shown (by considering the stored energy) to be given by

$$L_g = \frac{\mu_0 N_1^2 l_T g}{b} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

The leakage inductance due to the interlayer gaps may be derived as follows: The field strength in the pth interlayer gap (counting from the position of zero m.m.f.) is given by

$$H = \frac{pN_l I_1}{b}$$

(This formula assumes that the field strength is constant across the breadth of the winding and that the magnetic path length is b. These assumptions, which are made several times in the course of the derivation, are justifiable for typical communication transformers.[6])

Hence the stored energy in the pth gap is given by

$$W_{up} = \frac{\mu_0 p^2 N_l^2 I_1^2}{2b^2} ubl_T \text{ (since stored magnetic energy per unit volume} = \tfrac{1}{2}BH)$$

Therefore

$$L_{up} = \frac{\mu_0 N_l^2 u l_T}{b} p^2$$

Hence the total leakage inductance due to the interlayer gaps is given by

$$L_U = \frac{\mu_0 N_l^2 u l_T}{b} \sum_{p=1}^{p=m-1} p^2$$

Let the sum of interlayer gaps be denoted by U, where $U = (m - 1)u$.

Since $N_1 = mN_l$, we have the expression

$$L_U = \left(\frac{\mu_0 N_1^2 l_T U}{3b}\right)\left(1 - \frac{1}{2m}\right) \quad . \; . \; . \; . \; . \; . \quad (2)$$

If the transformer has sectionalised primary and secondary windings it is still always possible to divide the windings into portions, with one position of zero m.m.f. associated with each portion. Consider, for example, the transformer shown in Fig. 4. Here the primary winding has been placed between the two halves of the secondary winding. The m.m.f. diagram shows that the complete winding can be split up into four portions, i.e. AB, CB, CD, and ED. It will be observed that the intersection gaps have been referred to the primary

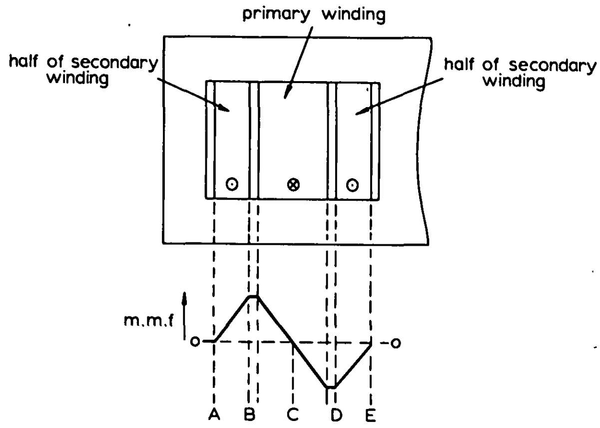

Fig. 4

Transformer with sectionalised secondary winding, with associated m.m.f. diagram

⊙ ⊗ Indicates current directions

portions for the sake of convenience. From the symmetry of the diagram it is clear that the total leakage impedance will be twice that due to portions AB and CB. If the primary contains an even number of layers, splitting it will produce two portions, each with an integral number of layers. It will then be possible to calculate the frequency-independent components of leakage inductance by applying eqns. 1 and 2 to each of the portions. When applied to a portion, eqns. 1 and 2 may be written as follows:

$$L_g = \frac{\mu_0 N_p^2 l_T g}{b} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$L_U = \frac{\mu_0 N_p^2 l_T U}{3b}\left(1 - \frac{1}{2m}\right) \quad . \; . \; . \; . \; . \; . \quad (4)$$

g, U and m all refer to the portion being considered.

If the primary contains an odd number of layers, splitting it will produce two portions, each having an associated layer of half conductors. These layers of half conductors will, for convenience, be termed 'half layers'. A winding portion containing a half layer is shown in Fig. 5. Since the formulas

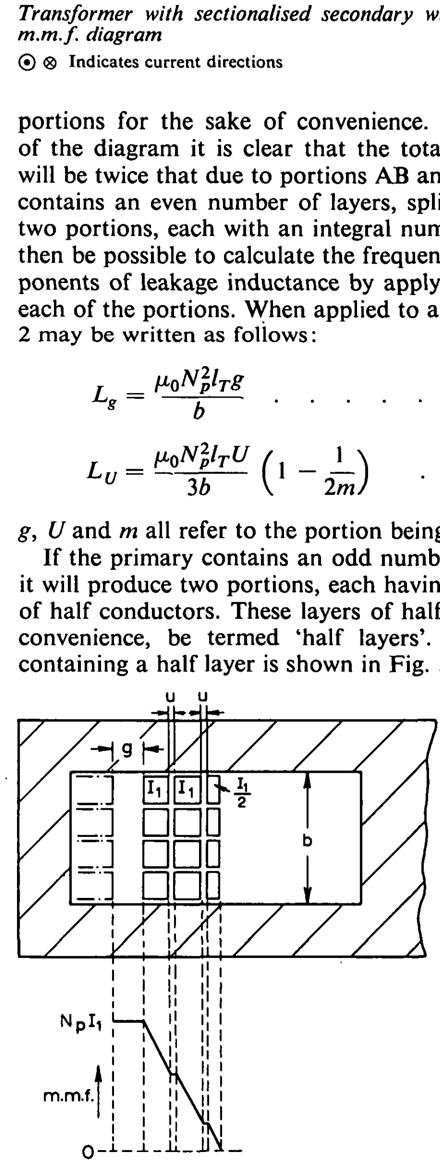

Fig. 5

Winding portion containing a half layer, with associated m.m.f. diagram

Number of layers in portion $= (m + \frac{1}{2})$
Number of interlayer gaps $= m$ (the gap between the half layer and the adjacent full layer being regarded just as any other interlayer gap)
N_p = number of turns in portion $= (m + \frac{1}{2})N_l$
N_l = number of turns per layer

for L_g and L_U have only been proved for portions containing integral numbers of layers, it will be necessary to derive new formulas for portions containing half layers.

These formulas may be derived from stored-energy consideration in a manner analogous to that used for deriving eqns. 3 and 4. It can thus be shown that eqn. 3 may be used irrespective of whether or not the portion contains a half layer of conductors, provided that half-layer turns are considered as half turns.

However, the equation for L_U for portions containing a half layer is as follows:

$$L_U = \frac{\mu_0 N_p^2 l_T U}{3b}\left(\frac{2m-1}{2m+1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It can be seen that eqn. 5 is not same as eqn. 4, and hence it is important to choose the correct equation, depending on whether there are m or $(m + \frac{1}{2})$ layers (where m is an integral number).

There are, of course, ways of sectionalising transformer windings other than that shown in Fig. 4 (Figs. 6*a* and 6*b*).

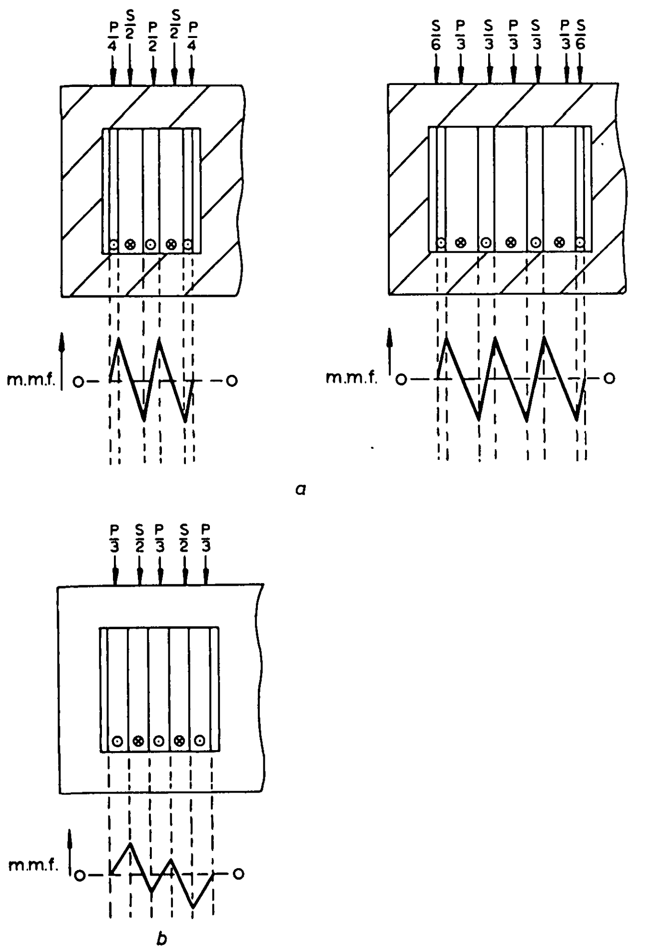

Fig. 6

a Two optimum sectionalising arrangements
b Nonoptimum sectionalising arrangement
⊙ ⊗ Indicates current directions

However, to minimise leakage inductance it is necessary to arrange all the positive and negative peaks of the m.m.f. diagram to be of equal height (as can be deduced from stored-energy considerations). For such windings the positions of zero m.m.f. will occur either at the end or the middle of each section. Hence, when the winding is divided into portions, the splitting of sections containing an odd number of layers will produce portions containing half layers, but other fractions of layers will not occur. If the winding is sectionalised so that all the positive and negative peaks of m.m.f. are not equal, splitting the winding into portions may necessitate layers being split into unequal parts. It would, of course, be possible to calculate L_U for windings with $(m + \frac{1}{3})$ layers etc. However, this will not be done here, as there seems to be no reason for adopting a nonoptimum method of sectionalising.

4 Frequency-dependent components of the leakage resistance and leakage inductance

It has already been explained that, for the purpose of calculating leakage impedance, it is permissible to consider each portion of the winding separately. Fig. 7 shows a

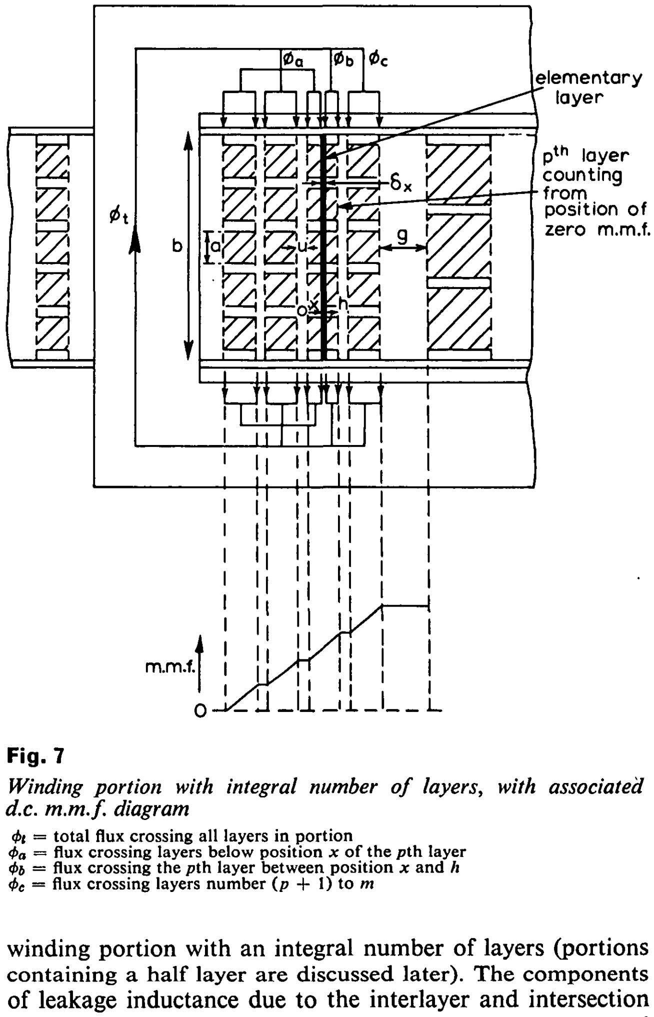

Fig. 7

Winding portion with integral number of layers, with associated d.c. m.m.f. diagram

ϕ_t = total flux crossing all layers in portion
ϕ_a = flux crossing layers below position x of the pth layer
ϕ_b = flux crossing the pth layer between position x and h
ϕ_c = flux crossing layers number $(p + 1)$ to m

winding portion with an integral number of layers (portions containing a half layer are discussed later). The components of leakage inductance due to the interlayer and intersection gaps have been calculated in Section 3. Hence these gaps and the magnetic flux associated with them will be neglected in this Section, and only the flux crossing the winding space via the layers of conductors, i.e. the 'layer flux', will be considered. By considering the pth layer of conductors and an elementary layer at position x of the pth layer (see Fig. 7), it is possible to show that the current density at the tops of the conductors (i.e. at the edge where the m.m.f. is greatest) is given by

$$J_h = \frac{N_l I}{\eta b h}\left\{M + \left(\frac{p-1}{2}\right)D\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and that the total flux in the pth layer is given by

$$\phi_p = \frac{\mu_0 l_T N_l I}{b\alpha^2 h}\{(p - \tfrac{1}{2})D\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $M = \alpha h \coth \alpha h$, $D = 2\alpha h \tanh \dfrac{\alpha h}{2}$ and $\alpha = \sqrt{\dfrac{j\omega\mu_0\eta}{\rho}}$.

Now the total voltage V across a general portion can be obtained by summing the resistive-voltage drops and induced voltages at tops of the conductors. These resistive-voltage drops can be calculated using eqn. 6, and the induced voltages can be calculated using eqn. 7. Hence

$$V = \frac{\rho l_T N_l^2 I}{\eta b h}\left\{mM + \frac{m(m^2-1)D}{3}\right\}$$

Where m = number of layers.

So the component of leakage impedance due to the flux cutting the conductors Z_W is given by

$$Z_w = \frac{\rho l_T N_l^2}{\eta b h}\left\{mM + \frac{m(m^2-1)D}{3}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

A full derivation of this equation is given in the Appendix.

Now the d.c. winding resistance $R_{w0} = \dfrac{m\rho N_l^2 l_T}{\eta b h}$. Hence the a.c. winding resistance R_W is given by

$$R_W = R_{w0}\left\{M' + \frac{(m^2-1)D'}{3}\right\}$$

where M' and D' are the real parts of M and D, respectively, or

$$R_w = F_R R_{w0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{where } F_R = M' + \frac{(m^2-1)D'}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The d.c. leakage inductance L_{w0} due to the flux cutting the conductors can be derived from the stored magnetic energy, as calculated from the m.m.f. diagram in the classical manner. Consideration of Fig. 7 shows that the magnetic energy stored in all the layers is the same as if there were no gaps between the layers. Thus it is easy to deduce

$$L_{w0} = \frac{\mu_0 m^3 N_l^2 l_T h}{3b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Let L_w be the a.c. leakage inductance due to the flux cutting the conductors; then, from eqn. 8,

$$\omega L_w = \frac{m\rho l_T N_l^2}{\eta b h}\left\{M'' + \frac{(m^2-1)D''}{3}\right\}$$

where M'' and D'' are the imaginary parts of M and D, respectively. Therefore, from eqn. 11,

$$L_w = L_{w0}\left\{\frac{3M'' + (m^2-1)D''}{m^2|\alpha^2 h^2|}\right\}$$

$$\text{or} \quad L_w = F_L L_{w0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\text{where } F_L = \frac{3M'' + (m^2-1)D''}{m^2|\alpha^2 h^2|} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

F_R has been calculated for up to four layers from eqn. 10. The calculated curves of F_R against $|\alpha^2 h^2|$ are shown in Fig. 8. M' and D' have been plotted against $|\alpha^2 h^2|$ in Fig. 9, so that using these graphs, together with eqn. 10, F_R can be calculated for any number of layers as required.

F_L has been calculated for one and two layers using eqn. 13. It was realised by inspection of eqn. 13 that for three or more layers the calculated value of F_L would be very close to the value obtained for an infinite number of layers. Hence eqn. 13 was used to calculate the value of F_L for an infinite number of layers, giving

$$F_{L\infty} = \frac{D''}{|\alpha^2 h^2|} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The calculated curves of F_L against $|\alpha^2 h^2|$ are shown in Fig. 10.

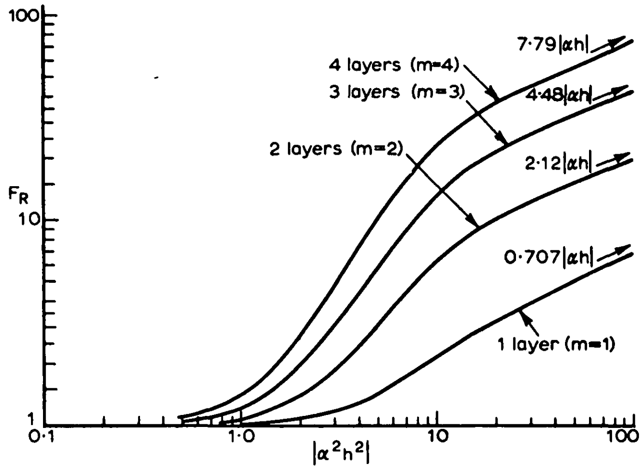

Fig. 8

F_R as a function of $|\alpha^2 h^2|$ for winding portions with integral numbers of layers

For $m > 4$, use eqn. 10

viz. $F_R = M' + \frac{(m^2-1)}{3}D'$; M' and D' are given in Fig. 9

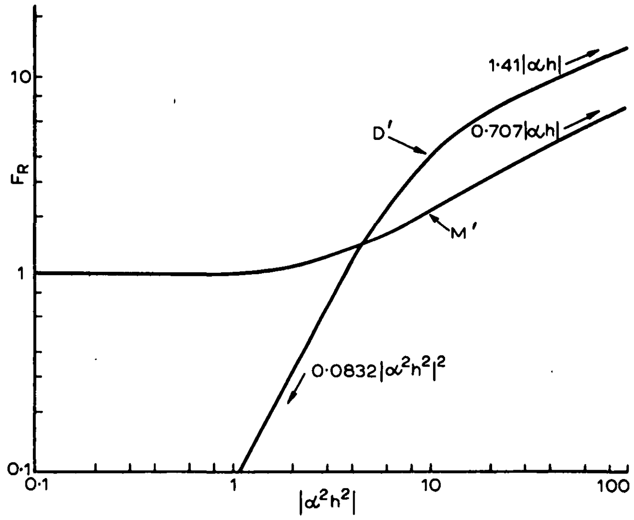

Fig. 9

M' and D' as functions of $|\alpha^2 h^2|$

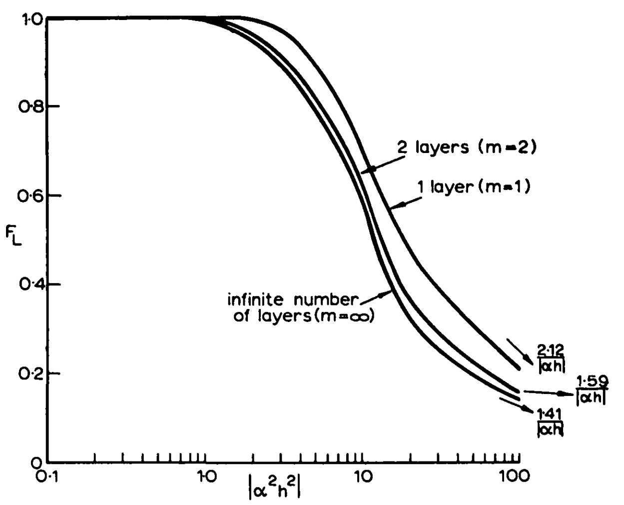

Fig. 10

F_L as a function of $|\alpha^2 h^2|$ for winding portions with integral numbers of layers

For $m > 2$, use curve for $m = \infty$

Before Figs. 8 and 10 can be used to obtain F_R and F_L, it is necessary to calculate $|\alpha^2h^2|$. It can easily be calculated that, for copper conductors at 20°C,

$$|\alpha^2h^2| = 464f\eta h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

At 70°C the numerical coefficient in eqn. 15 becomes 500 and, being linearly temperature dependent, can be obtained at any temperature from the following equation:

$$\text{Numerical coefficient} = 464 + 0{\cdot}72(\theta - 20) \quad . \quad (16)$$

A similar treatment of portions with half layers gives

$R_W = F_R R_{W0}$ (where R_{W0} = half the d.c. resistance of the section of which the portion forms half)

$$\text{where } F_R = \frac{12mM' + 6M'_{1/2} + m(4m^2 + 6m - 1)D'}{12m + 6} \quad (17)$$

($M'_{1/2}$ means the value of M' corresponding to the height of the half layer) and

$$L_W = F_L L_{W0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\text{where } F_L = \frac{12mM'' + 6M''_{1/2} + m(4m^2 + 6m - 1)D''}{4(m + \frac{1}{2})^3|\alpha^2h^2|} \quad . \ (19)$$

The d.c. leakage inductance due to the flux cutting the conductors, L_{W0}, can be seen from Fig. 11 and stored-energy

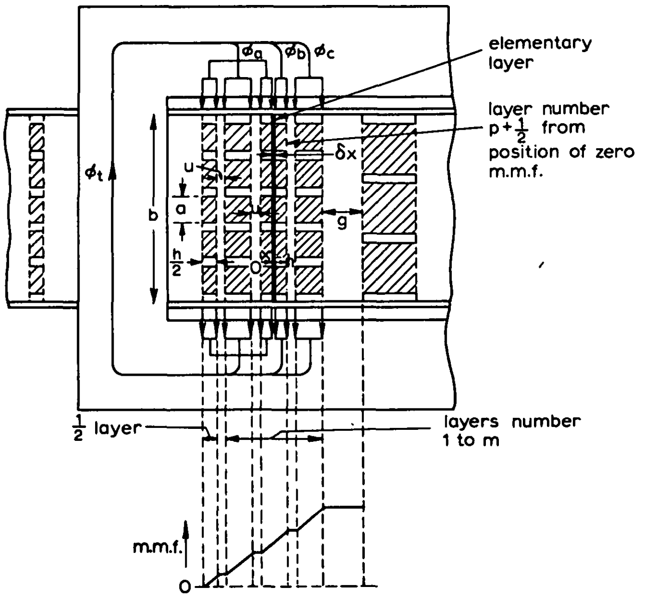

Fig. 11

Winding portion containing a half layer with associated d.c. m.m.f. diagram

ϕ_t = total flux crossing all layers in portion
ϕ_a = flux crossing layers below position x of layer number $(p + \frac{1}{2})$
ϕ_b = flux crossing layer number $(p + \frac{1}{2})$ between position x and h
ϕ_c = flux crossing layers number $(p + \frac{1}{2})$ to $(m + \frac{1}{2})$

considerations to be the same as if there were no gaps between the layers. Thus L_{W0} can be deduced to be given by

$$L_{W0} = \frac{\mu_0(m + \frac{1}{2})^3 N_l^2 h l_T}{3b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

F_R has been calculated for $(m + \frac{1}{2})$ layers, with $m = 0, 1, 2$ and 3, using eqn. 21. The calculated curves of F_R against $|\alpha^2h^2|$ are shown in Fig. 12. If F_R is required for a portion with $m > 3$, it can be calculated using eqn. 17 in conjunction with Fig. 9.

F_L has been calculated for $(m + \frac{1}{2})$ layers, with $m = 0$ and 1 using eqn. 19. It was realised by inspection of eqn. 19 that, for $m \geqslant 2$, the calculated value of F_L would be very close to the value for $m = \infty$. Hence eqn. 19 was used to calculate F_L for $m = \infty$, giving

$$F_{L\infty} = \frac{D''}{|\alpha^2h^2|} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It can be seen that eqn. 21 is identical to eqn. 14. Thus the half layer has no extra effect on portions containing an infinite number of layers; this is, of course, hardly surprising.

Before using Figs. 12 and 13 to obtain F_R and F_L it is

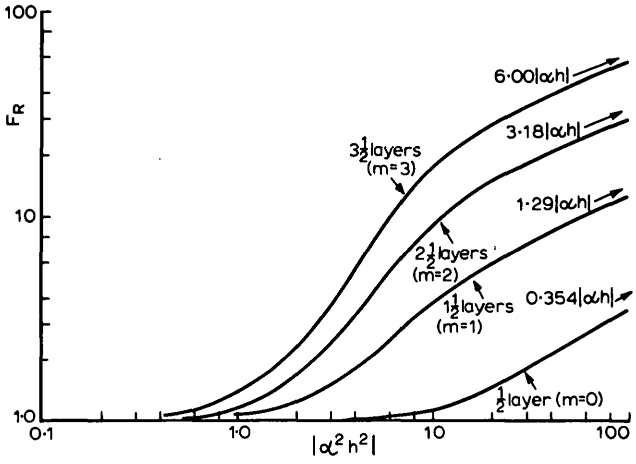

Fig. 12

F_R as a function of $|\alpha^2h^2|$ for winding portions with $(m + \frac{1}{2})$ layers, where m is an integral number

For $m > 3$ use eqn. 17, viz.

$$F_R = \frac{12mM' + 6M'_{1/2} + m(4m^2 + 6m - 1)D'}{12m + 6}$$

M' and D' are given in Fig. 9

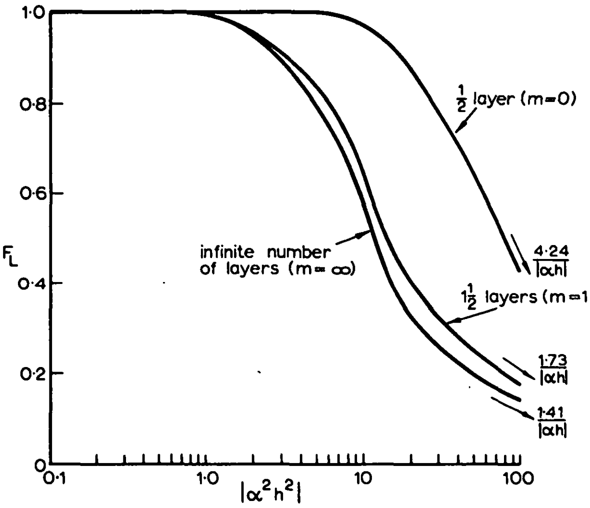

Fig. 13

F_L as a function of $|\alpha^2h^2|$ for winding portions with $(m + \frac{1}{2})$ layers, where m is an integral number

For $m > 1$, use curve for $m = \infty$

necessary to calculate $|\alpha^2h^2|$ using eqn. 15. When using this equation it is important to remember to insert the value of h for the full layers, and not for the half layer.

5 Conclusions

A quick and simple method has been derived for obtaining the a.c. leakage impedance of a transformer, from the calculated values of d.c. winding resistance and d.c. leakage inductance. It is rather unfortunate that calculating the d.c.-leakage-inductance components accurately is a fairly lengthy process, although it mainly consists of substituting numbers in the given formulas. However, in some cases it may be sufficient to calculate the a.c. winding resistance accurately and to obtain a pessimistic estimate of the a.c. leakage inductance by using the classical equation applicable at low frequencies to windings with large numbers of layers. The results have not as yet been thoroughly checked experimentally, but the leakage resistance and leakage inductance of a transformer with single-layer windings have been measured using direct voltage, and up to 10 Mc/s. The results obtained were in good agreement with the calculated values. The dimensions of the transformer were such that the effects of the eddy currents in the windings were very noticeable; e.g. the leakage resistance increased by a factor of about 100 and the component of the leakage inductance due to the flux cutting the conductors fell to very nearly zero.

6 Acknowledgments

The author would like to thank Mullard Research Laboratories for giving permission for this paper to be published, and also Mr. E. C. Snelling for his helpful suggestions.

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8 Appendix

Consider the elementary layer at position x of the pth layer of conductors, as shown in Fig. 7. It can be seen that the layer flux linking the elementary layer is given by

$$\text{layer flux} = \phi_b + \phi_c \qquad (22)$$

Now let the flux passing through the elementary layer of width δx be $|\delta\phi|$. Thus when x goes to $x + \delta x$, the net linking layer flux goes from $\phi_b + \phi_c$ to $\phi_b - |\delta\phi| + \phi_c$, i.e.

$$\frac{d(\phi_b + \phi_c)}{dx} = \frac{|\delta\phi|}{\delta x}$$

If the flux density at position x is B,

$$|\delta\phi| = \delta x B l_T \qquad (23)$$

Now $B = \mu_0 H$, and

$$\oint H\,ds = \Sigma i$$

Thus again assuming constant field strength H and a magnetic-path length equal to the winding breadth b,

$$H = \frac{\Sigma i}{b} = \frac{1}{b}\{IN_l(p-1)\} + \frac{1}{b}\int_0^x \eta b J dx$$

Therefore

$$B = \frac{\mu_0}{b}\{IN_l(p-1)\} + \mu_0\eta\int_0^x J dx$$

Thus from eqn. 23,

$$|\delta\phi| = \mu_0 \delta x l_T\left\{\frac{IN_l(p-1)}{b} + \int_0^x \eta J dx\right\}$$

but it has been shown that

$$\frac{d(\phi_b + \phi_c)}{dx} = \frac{-|\delta\phi|}{\delta x}$$

Therefore

$$\frac{-d(\phi_b + \phi_c)}{dx} = \mu_0 l_T\left\{\frac{IN_l(p-1)}{b} + \int_0^x Jdx\right\} \qquad (24)$$

Now the voltage V_l induced in the elementary-winding layer, considered as a solenoid, is $N_l\frac{d(\phi_b + \phi_c)}{dt}$, i.e. $j\omega(\phi_b + \phi_c)N_l$ (for sine waves). Since the same voltage exists across the whole height of the conductor, $\frac{dV_l}{dx} = 0$. Equating the total voltage across the ends of the layer to the resistive-voltage drop plus the reverse e.m.f. due to layer-flux linkage gives, for the elementary winding layer, $V_l = NJ\rho l_T + j\omega(\phi_b + \phi_c)N_l$

Therefore

$$0 = \frac{dV_l}{dx} = N_l \rho l_T \frac{dJ}{dx} + j\omega N_l\left\{\frac{d(\phi_b + \phi_c)}{dx}\right\}$$

Therefore

$$\frac{dJ}{dx} = \frac{-j\omega}{\rho l_T}\left\{\frac{d(\phi_b + \phi_c)}{dx}\right\}$$

thus from eqn. 24

$$\frac{dJ}{dx} = \frac{j\omega\mu_0}{\rho}\left\{\frac{IN_l(p-1)}{b} + \eta\int_0^x Jdx\right\} \qquad (25)$$

Therefore

$$\frac{d^2J}{dx^2} = \left(\frac{j\omega\mu_0\eta}{\rho}\right)J \quad (I \text{ is independent of } x)$$

or

$$\frac{d^2J}{dx^2} = \alpha^2 J \qquad (26)$$

$$\text{where } \alpha^2 = \frac{j\omega\mu_0\eta}{\rho} \qquad (27)$$

The solution of eqn. 26 is

$$J = P\cosh\alpha x + Q\sinh\alpha x \qquad (28)$$

where P and Q are constants which can be evaluated as follows: Substituting eqn. 28 in eqn. 25 gives

$$Q = \frac{\alpha I N_l(p-1)}{\eta b} \qquad (29)$$

Hence eqn. 28 may be rewritten as

$$J = P\cosh\alpha x + \frac{\alpha I N_l(p-1)}{b}\sinh\alpha x$$

Thus by integrating both sides with respect to x, and eliminating J using

$$\int_0^h \eta J b dx = N_l I \quad \text{(see Fig. 7)}$$

we obtain

$$P = \frac{N\alpha}{\eta b}\left\{\frac{I}{\sinh\alpha h} - I(p-1)\tanh\frac{\alpha h}{2}\right\} \qquad (30)$$

Thus from eqns. 28, 29 and 30,

$$J = \frac{N_l I\alpha}{\eta b}\left\{\frac{\cosh\alpha x}{\sinh\alpha h} - (p-1)\tanh\frac{\alpha h}{2}\cosh\alpha x + (p-1)\sinh\alpha x\right\} \qquad (31)$$

Hence at the tops of the conductors, the current density J_h is given by

$$J_h = \frac{N_l I\alpha}{\eta b}\left\{\coth\alpha h - (p-1)\tanh\frac{\alpha h}{2}\cosh\alpha h + (p-1)\sinh\alpha h\right\}$$

i.e.

$$J_h = \frac{N_l I\alpha}{\eta b}\left\{\coth\alpha h + (p-1)\tanh\frac{\alpha h}{2}\right\}$$

Writing

$$\alpha h \coth \alpha h = M = M' + jM'' \quad \ldots \quad (32)$$

and $$2\alpha h \tanh \frac{\alpha h}{2} = D = D' + jD'' \quad \ldots \quad (33)$$

then $$J_h = \frac{N_l I}{\eta b h}\left\{M + \frac{(p-1)}{2} D\right\} \quad \ldots \quad (6)$$

The total voltage across the ends of a layer of conductors is constant across the height of the conductors, and hence can be obtained by calculating V_h, the voltage across the layer at the tops of the conductors. Now V_h is the sum of the resistive-voltage drop ($V_r = N_l \rho J_h l_T$) and the induced voltage V_i due to the net layer flux linking the tops of the conductors. Now from eqn. 22 and Fig. 7 this linking layer flux can be seen to be equal to ϕ_c; i.e. that part of ϕ_t which crosses the winding space to the right of the top of the pth layer. Thus that part of ϕ_c crossing the winding space via a given layer is responsible for inducing a voltage in each of the lower layers (i.e. in the layers nearer to the position of zero m.m.f.) So, in general terms, the flux ϕ_p in the pth layer induces a voltage V_{ip} in each of the $(p-1)$ lower layers and

$$V_{ip} = jN_l \omega \phi_p \quad \ldots \quad (34)$$

and from eqn. 24 the flux in the pth layer is given by

$$\phi_p = \mu_0 l_T \int_0^h \left\{\frac{IN_l(p-1)}{b} + \eta \int_0^x J dx\right\} dx \quad \ldots \quad (35)$$

$\left(\text{As for a given layer } \frac{d(\phi_b + \phi_c)}{dx} = \frac{d\phi_b}{dx}\right)$

ϕ_p can be found from eqn. 35, using eqn. 31 to give $\int_0^h \int_0^x J dx$, the result being

$$\phi_p = \frac{\mu_0 l_T N_l I}{b \alpha^2 h}\{(p - \tfrac{1}{2})D\} \quad \ldots \quad (7)$$

Hence $V_{ip} = \frac{\rho l_T N_l^2 I}{\eta b h}\{(p - \tfrac{1}{2})D\}$

The resistive-voltage drop at the tops of the conductors of the pth layer can be seen from eqn. 6 to be given by

$$V_{rp} = \frac{\rho l_T N_l^2 I}{\eta b h}\left\{M + \frac{(p-1)D}{2}\right\} \quad \ldots \quad (36)$$

Now the voltage across a layer can always be thought of as the sum of the voltages at the tops of the conductors. Thus the total voltage across a portion can be obtained by summing the resistive-voltage drops and induced voltages at the tops of the layers. For the purpose of summation these induced voltages can be regarded as being associated with the layer which is crossed by the responsible linking flux. Thus the induced voltage to be associated with layer p is $(p-1)V_{ip}$ [i.e. the sum of the induced voltages in the other layers, due to the flux crossing layer p (eqn. 34)]. The resistive-voltage drop to be associated with layer p is the actual resistive-voltage drop at the top of that layer, V_{rp}. Thus the total voltage to be associated with layer p is $(p-1)V_{ip} + V_{rp}$. Hence the total voltage V across a general portion of m layers is given by

$$V = \sum_{p=1}^{p=m} V_{rp} + \sum_{p=1}^{p=m} (p-1)V_{ip}$$

Thus, from eqns. 7 and 36,

$$V = \frac{\rho l_T N_l^2 I}{\eta b h}\left\{mM + \frac{m(m^2-1)D}{3}\right\}$$

So the component of leakage impedance due to the flux cutting the conductors Z_w is given by

$$Z_w = \frac{\rho l_T N_l^2}{\eta b h}\left\{mM + \frac{m(m^2-1)D}{3}\right\} \quad \ldots \quad (8)$$