LRCL modelization

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1 Model description

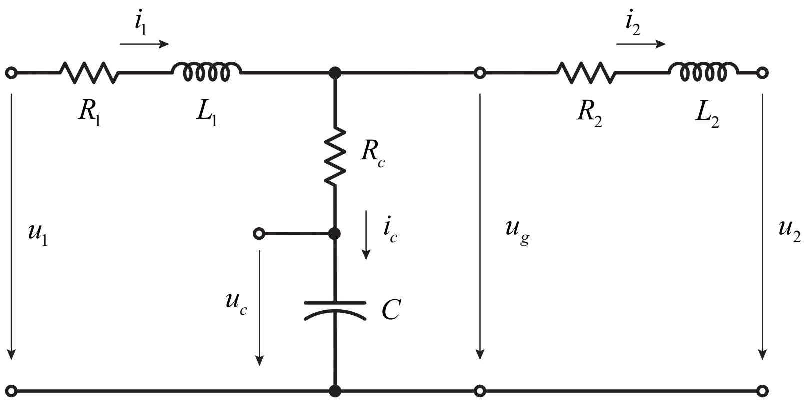

Figure 1: LRCL electrical schematic.

Kirchhoff's equations†:

$$u_1(t) - R_1 i_1(t) - L_1 \frac{di_1(t)}{dt} - u_g(t) = 0 \tag{1.1}$$

$$u_g(t) - R_2 i_2(t) - L_2 \frac{di_2(t)}{dt} - u_2(t) = 0 \tag{1.2}$$

$$i_1(t) - i_2(t) - i_c(t) = 0 \tag{1.3}$$

$$u_g(t) - R_c i_c(t) - u_c(t) = 0 \tag{1.4}$$

$$i_c(t) = C \frac{du_c(t)}{dt} \tag{1.5}$$

Eq. (1.4) and Eq. (1.3) can be merged and Eq. (1.4) can be written as follows

$$u_g(t) = R_c \Big[i_1(t) - i_2(t) \Big] + u_c(t) \tag{1.6}$$

Eq. (1.3) and Eq. (1.4)-Eq. (1.5) can be merged and can be written as follows

$$i_c(t) = i_1(t) - i_2(t) = C \frac{du_c(t)}{dt} \tag{1.7}$$

Combining Eq. (1.6)-Eq. (1.7) with Eq. (1.1)-Eq. (1.2) we obtain the following system

$$u_1(t) - R_1 i_1(t) - L_1 \frac{di_1(t)}{dt} - \Big\{ R_c \Big[i_1(t) - i_2(t) \Big] + u_c(t) \Big\} = 0 \tag{1.8}$$

$$R_c \Big[i_1(t) - i_2(t) \Big] + u_c(t) - R_2 i_2(t) - L_2 \frac{di_2(t)}{dt} - u_2(t) = 0 \tag{1.9}$$

$$C \frac{du_c(t)}{dt} = i_1(t) - i_2(t) \tag{1.10}$$

†Physical relation between voltage variation and relative charge variation of an electrical capacitor:

$CdV(t) = dQ(t) \Rightarrow C\frac{dV(t)}{dt} = i_c(t)$

or

$$L_1 \frac{di_1(t)}{dt} = -\Big(R_1 + R_c\Big) i_1(t) + R_c i_2(t) - u_c(t) + u_1(t) \tag{1.11}$$

$$L_2 \frac{di_2(t)}{dt} = R_c i_1(t) - \Big(R_2 + R_c\Big) i_2(t) + u_c(t) - u_2(t) \tag{1.12}$$

$$C \frac{du_c(t)}{dt} = i_1(t) - i_2(t) \tag{1.13}$$

and

$$\frac{di_1(t)}{dt} = -\frac{R_1 + R_c}{L_1} i_1(t) + \frac{R_c}{L_1} i_2(t) - \frac{1}{L_1} u_c(t) + \frac{1}{L_1} u_1(t) \tag{1.14}$$

$$\frac{di_2(t)}{dt} = \frac{R_c}{L_2} i_1(t) - \frac{R_2 + R_c}{L_2} i_2(t) + \frac{1}{L_2} u_c(t) - \frac{1}{L_2} u_2(t) \tag{1.15}$$

$$\frac{du_c(t)}{dt} = \frac{1}{C}\Big[i_1(t) - i_2(t)\Big] \tag{1.16}$$

To write the system of Eq. (1.14)-Eq. (1.16) in a state space form it is necessary to define the input and the output of the system as follows

$$u(t) = u_1(t) \qquad \textit{this is the input (scalar) of the system} \tag{1.17}$$

$$\vec{y}(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \\ u_g(t) \end{bmatrix} \qquad \textit{this is the output (vector) of the system} \tag{1.18}$$

the system contains also another term $u_2(t)$ which can be treated as a unknown (or partially unknown) disturbance.

Obviously the state vector is represented by the vector

$$\vec{x} = \begin{bmatrix} i_1(t) \\ i_2(t) \\ u_c(t) \end{bmatrix} \tag{1.19}$$

The system Eq. (1.14)-Eq. (1.16) can be now written in a matrix form as follows

$$\dot{\vec{x}}(t) = \tilde{\mathbf{A}}\vec{x}(t) + \tilde{\mathbf{B}}u_1(t) + \tilde{\mathbf{E}}u_2(t) \tag{1.20}$$

$$\vec{y}(t) = \mathbf{C}\vec{x}(t) \tag{1.21}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} -\frac{R_1+R_c}{L_1} & \frac{R_c}{L_1} & -\frac{1}{L_1} \\ \frac{R_c}{L_2} & -\frac{R_2+R_c}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \tag{1.22}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} \qquad \tilde{\mathbf{E}} = \begin{bmatrix} 0 \\ -\frac{1}{L_2} \\ 0 \end{bmatrix} \tag{1.23}$$

and

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ R_c & -R_c & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{bmatrix} \tag{1.24}$$

The transfer function between the input $u_1(t)$ and the output $u_g(t)$ can be represented as follows

$$H_g(s) = \frac{U_g(s)}{U_1(s)} = \mathbf{C}_3 \left[s\mathbf{I}_3 - \tilde{\mathbf{A}} \right]^{-1} \tilde{\mathbf{B}} \tag{1.25}$$

and result is in the form

$$H_g(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{1.26}$$

The transfer function between the input $u_1(t)$ and the output $i_2(t)$ can be represented as follows

$$H_2(s) = \frac{I_2(s)}{U_1(s)} = \mathbf{C}_2 \left[s\mathbf{I}_3 - \tilde{\mathbf{A}} \right]^{-1} \tilde{\mathbf{B}} \tag{1.27}$$

and result is in the form

$$H_2(s) = \frac{b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{1.28}$$

The transfer function between the input $u_1(t)$ and the output $i_1(t)$ can be represented as follows

$$H_1(s) = \frac{I_1(s)}{U_1(s)} = \mathbf{C}_1 \left[s\mathbf{I}_3 - \tilde{\mathbf{A}} \right]^{-1} \tilde{\mathbf{B}} \tag{1.29}$$

and result is in the form

$$H_1(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{1.30}$$

In the following, the $H_2(s)$ transfer function of Eq. (1.28) will be taken into account. This transfer function is derived from the state space system

$$\dot{\vec{x}}(t) = \tilde{\mathbf{A}}\vec{x}(t) + \tilde{\mathbf{B}}u_1(t) \tag{1.31}$$

$$i_2(t) = \mathbf{C}_2\vec{x}(t) \tag{1.32}$$

where the disturbance term $u_2(t)$ has been omitted.

The coefficients relative to Eq. (1.28) are as follows

$$a_2 = \frac{L_1 R_c + L_2 R_c + L_1 R_2 + L_2 R_1}{L_1 L_2} \tag{1.33}$$

$$a_1 = \frac{L_1 + L_2 + R_2 R_c C + R_1 R_2 C + R_1 R_2 C}{L_1 L_2 C} \tag{1.34}$$

$$a_0 = \frac{R_1 + R_2}{L_1 L_2 C} \tag{1.35}$$

$$b_1 = \frac{R_c}{L_1 L_2} \tag{1.36}$$

$$b_0 = \frac{1}{L_1 L_2 C} \tag{1.37}$$

The transfer function of Eq. (1.28) can be rewritten in state space form using the controllable canonical form as follows

$$\dot{\vec{z}}(t) = \tilde{\mathbf{M}}\vec{z}(t) + \tilde{\mathbf{N}}u_1(t) \tag{1.38}$$

$$y(t) = i_2(t) = \mathbf{G}\vec{z}(t) \tag{1.39}$$

where

$$\tilde{\mathbf{M}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad \tilde{\mathbf{N}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.40}$$

and

$$\mathbf{G} = \begin{bmatrix} b_0 & b_1 & 0 \end{bmatrix} \tag{1.41}$$

$$i_2^{(0)}(t) = \mathbf{G}\vec{z}(t) \tag{1.42}$$

$$i_2^{(1)}(t) = \mathbf{G}\tilde{\mathbf{M}}\vec{z}(t) + \mathbf{G}\tilde{\mathbf{N}}u_1(t) \tag{1.43}$$

$$i_2^{(2)}(t) = \mathbf{G}\tilde{\mathbf{M}}^2\vec{z}(t) + \mathbf{G}\tilde{\mathbf{M}}\tilde{\mathbf{N}}u_1(t) \tag{1.44}$$