Shell-model project for Nuclear Talent course - Solutions

Group 1

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We show that the Hamiltonian $\hat{H}_0$ and $\hat{V}$ commute with both the spin projection $\hat{S}_z$ and the total spin $\hat{S}^2$. Recall

$$\hat{H}_0 = \sum_{p\sigma}(p-1)a^\dagger_{p\sigma}a_{p\sigma}$$
$$\hat{V} = -g\frac{1}{4}a^\dagger_{q+}a^\dagger_{q-}a_{s+}a_{s-}$$

Therefore the commutation relation reads

$$\begin{aligned}[\hat{H}_0, \hat{S}_z] &= [\sum_{p\sigma_1}(p-1)a^\dagger_{p\sigma_1}a_{p\sigma_1}, \frac{1}{2}\sum_{q\sigma_2}\sigma_2 a^\dagger_{q\sigma_2}a_{q\sigma_2}] \\ &= \sum_{p\sigma_1}(p-1)\frac{1}{2}\sum_{q\sigma_2}\sigma_2[a^\dagger_{p\sigma_1}a_{p\sigma_1}, a^\dagger_{q\sigma_2}a_{q\sigma_2}].\end{aligned}$$

Let us calculate the last commutation relation, where we will only manipulate the second element

$$\begin{aligned}[a^\dagger_{p\sigma_1}a_{p\sigma_1}, a^\dagger_{q\sigma_2}a_{q\sigma_2}] &= a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2} - a^\dagger_{q\sigma_2}\underbrace{a_{q\sigma_2}a^\dagger_{p\sigma_1}}a_{p\sigma_1} \\ &= a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2} - a^\dagger_{q\sigma_2}(\delta_{\sigma_1,\sigma_2}\delta_{p,q} - a^\dagger_{p\sigma_1}a_{q\sigma_2})a_{p\sigma_1} \\ &= a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2} + \underbrace{a^\dagger_{q\sigma_2}a^\dagger_{p\sigma_1}}\underbrace{a_{q\sigma_2}a_{p\sigma_1}} - a^\dagger_{q\sigma_2}a_{p\sigma_1}\delta_{\sigma_1,\sigma_2}\delta_{p,q} \\ &= a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2} + a^\dagger_{p\sigma_1}\underbrace{a^\dagger_{q\sigma_2}a_{p\sigma_1}}a_{q\sigma_2} - a^\dagger_{q\sigma_2}a_{p\sigma_1}\delta_{\sigma_1,\sigma_2}\delta_{p,q} \\ &= a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2} + a^\dagger_{p\sigma_1}(\delta_{\sigma_1,\sigma_2}\delta_{p,q} - a_{p\sigma_1}a^\dagger_{q\sigma_2})a_{q\sigma_2} - a^\dagger_{q\sigma_2}a_{p\sigma_1}\delta_{\sigma_1,\sigma_2}\delta_{p,q} \\ &= \underbrace{a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2}} + a^\dagger_{p\sigma_1}\delta_{\sigma_1,\sigma_2}\delta_{p,q}a_{q\sigma_2} - \underbrace{a^\dagger_{p\sigma_1}a_{p\sigma_1}a^\dagger_{q\sigma_2}a_{q\sigma_2}} - a^\dagger_{q\sigma_2}a_{p\sigma_1}\delta_{\sigma_1,\sigma_2}\delta_{p,q} \\ &= (a^\dagger_{p\sigma_1}a_{q\sigma_2} - a^\dagger_{q\sigma_2}a_{p\sigma_1})\delta_{\sigma_1,\sigma_2}\delta_{p,q} = 0.\end{aligned}$$

Next we show that $[\hat{V}, S_z] = 0$. We hence show that

$$[a^\dagger_{q+}a^\dagger_{q-}a_{s+}a_{s-}, a^\dagger_{p\sigma}a_{p\sigma}] = a^\dagger_{q+}a^\dagger_{q-}a_{s+}a_{s-}a^\dagger_{p\sigma}a_{p\sigma} - a^\dagger_{p\sigma}a_{p\sigma}a^\dagger_{q+}a^\dagger_{q-}a_{s+}a_{s-}. \tag{1}$$

We concentrate on the first element of Eq. 1, rewriting it in normal order

$$\begin{aligned}a^\dagger_{q+}a^\dagger_{q-}a_{s+}\underbrace{a_{s-}a^\dagger_{p\sigma}}a_{p\sigma} &= a^\dagger_{q+}a^\dagger_{q-}a_{s+}(\delta_{p,s}\delta_{\sigma,-} - a^\dagger_{p\sigma}a_{s-})a_{p\sigma} \\ &= a^\dagger_{q+}a^\dagger_{q-}a_{s+}\delta_{p,s}\delta_{\sigma,-}a_{p\sigma} - a^\dagger_{q+}a^\dagger_{q-}\underbrace{a_{s+}a^\dagger_{p\sigma}}a_{s-}a_{p\sigma} \\ &= a^\dagger_{q+}a^\dagger_{q-}a_{s+}\delta_{p,s}\delta_{\sigma,-}a_{p\sigma} - a^\dagger_{q+}a^\dagger_{q-}(\delta_{p,s}\delta_{\sigma,+} - a^\dagger_{p\sigma}a_{s+})a_{s-}a_{p\sigma} \\ &= a^\dagger_{q+}a^\dagger_{q-}(a_{s+}\delta_{\sigma,-} - a_{s-}\delta_{\sigma,+})a_{p\sigma}\delta_{p,s} + a^\dagger_{p\sigma}a^\dagger_{q+}a^\dagger_{q-}a_{s+}a_{s-}a_{p\sigma}\end{aligned} \tag{2}$$

Next, we concentrate on the last element, rewriting it in normal order as well

$$\begin{aligned}a^\dagger_{p\sigma}\underbrace{a_{p\sigma}a^\dagger_{q+}}a^\dagger_{q-}a_{s+}a_{s-} &= a^\dagger_{p\sigma}(\delta_{p,q}\delta_{\sigma,+} - a^\dagger_{q+}a_{p\sigma})a^\dagger_{q-}a_{s+}a_{s-} \\ &= a^\dagger_{p\sigma}\delta_{p,q}\delta_{\sigma,+}a^\dagger_{q-}a_{s+}a_{s-} - a^\dagger_{p\sigma}a^\dagger_{q+}\underbrace{a_{p\sigma}a^\dagger_{q-}}a_{s+}a_{s-} \\ &= a^\dagger_{p\sigma}\delta_{p,q}\delta_{\sigma,+}a^\dagger_{q-}a_{s+}a_{s-} - a^\dagger_{p\sigma}a^\dagger_{q+}(\delta_{p,q}\delta_{\sigma,-} - a^\dagger_{q-}a_{p\sigma})a_{s+}a_{s-} \\ &= a^\dagger_{p\sigma}\delta_{p,q}\delta_{\sigma,+}a^\dagger_{q-}a_{s+}a_{s-} - a^\dagger_{p\sigma}a^\dagger_{q+}\delta_{p,q}\delta_{\sigma,-}a_{s+}a_{s-} + a^\dagger_{p\sigma}a^\dagger_{q+}a^\dagger_{q-}a_{p\sigma}a_{s+}a_{s-} \\ &= a^\dagger_{p\sigma}(a^\dagger_{q-}\delta_{\sigma,+} - a^\dagger_{q+}\delta_{\sigma,-})a_{s+}a_{s-}\delta_{p,q} + a^\dagger_{p\sigma}a^\dagger_{q+}a^\dagger_{q-}a_{s+}a_{s-}a_{p\sigma}\end{aligned} \tag{3}$$

The second element in Eq. 2 is the same as the second element in Eq. 3 and therefore they be subtracted

[1] P. E. Garrett, K. L. Green, H. Lehmann, J. Jolie, C. A. McGrath, M. Yeh, and S. W. Yates, Phys. Rev. C **75**, 054310 (2007).
[2] P. E. Garrett, K. L. Green, and J. L. Wood, Phys. Rev. C **78**, 44307 (2008).
[3] A. Leviatan, (2016).