# Part 1b

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## 1 TEXT OF THE EXERCISE

Assume now that the effective Hilbert space consists only of the two lowest single-particle states and that we have two particles only. Set up the possible two-particle configurations when we have only two single-particle states, that is $p = 1$ and $p = 2$. Construct thereafter the Hamiltonian matrix using second quantization and for example Wick's theorem for a system with no broken pairs and spin $S = 0$ (with projection $S_z = 0$) for the case of the two lowest single-particle levels and two particles only. This gives you a $2 \times 2$ matrix to be diagonalized. Find the eigenvalues by diagonalizing the Hamiltonian matrix. Vary your results for selected values of $g \in [-1, 1]$ and comment your results.

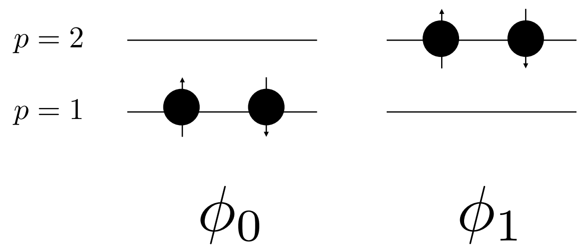
The Hamiltonian can be written as:

$$\hat{H} = \xi \sum_{p\sigma} (p-1) a^{\dagger}_{p\sigma} a_{p\sigma} - g \sum_{pq} \hat{P}^{+}_{p} \hat{P}^{-}_{q} \tag{1.1}$$

where we take $\xi = 1$ and the so-called pair creation and pair annihilation operators are defined as $\hat{P}^{+}_{p} = a^{\dagger}_{p+} a^{\dagger}_{p-}$ and $\hat{P}^{-}_{p} = a_{p-} a_{p+}$ respectively.

# 2 Solution

The two states that we can have in such a system are:




and we can write our Hamiltonian Matrix as:

$$
\begin{array}{ccc}
 & \phi_0 & \phi_1 \\
\begin{array}{c}\phi_0 \\ \phi_1\end{array} & \multicolumn{2}{c}{\begin{pmatrix} \langle\phi_0|\,\hat{H}\,|\phi_0\rangle & \langle\phi_0|\,\hat{H}\,|\phi_1\rangle \\ \langle\phi_1|\,\hat{H}\,|\phi_0\rangle & \langle\phi_1|\,\hat{H}\,|\phi_1\rangle \end{pmatrix}.}
\end{array}
\tag{2.1}
$$

Before computing the matrix elements it is better to define two part of the Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \sum_{p\sigma}(p-1)\,a^{\dagger}_{p\sigma}a_{p\sigma} - g\sum_{pq}\hat{P}^{+}_{p}\hat{P}^{-}_{q}$$

, where $\hat{H}_0$ is the unperturbed part (1−body) and $\hat{H}_1$ is the interaction part (2−body).

## 2.1 Unperturbed term

Let us compute first the 1−body term:

$$\langle\phi_0|\,\hat{H}_0\,|\phi_0\rangle = \langle\phi_0|\sum_{p\sigma}(p-1)\,a^{\dagger}_{p\sigma}a_{p\sigma}\,|\phi_0\rangle = \langle\phi_0|\sum_{\sigma}(1-1)\,|\phi_0\rangle = 0. \tag{2.2}$$

$$\langle\phi_1|\,\hat{H}_0\,|\phi_1\rangle = \langle\phi_1|\sum_{p\sigma}(p-1)\,a^{\dagger}_{p\sigma}a_{p\sigma}\,|\phi_1\rangle = \langle\phi_1|\sum_{\sigma}(2-1)\,|\phi_1\rangle = 2. \tag{2.3}$$

And due to orthogonality:

$$\langle\phi_0|\,\hat{H}_0\,|\phi_1\rangle = \langle\phi_0|\sum_{p\sigma}(p-1)\,a^{\dagger}_{p\sigma}a_{p\sigma}\,|\phi_1\rangle = 0. \tag{2.4}$$

$$\langle\phi_1|\,\hat{H}_0\,|\phi_0\rangle = \langle\phi_1|\sum_{p\sigma}(p-1)\,a^{\dagger}_{p\sigma}a_{p\sigma}\,|\phi_0\rangle = 0. \tag{2.5}$$

Another way to look at the same results is to write the sums of $\hat{H}_0$:

$$\begin{aligned}\hat{H}_0 =&\sum_{p\sigma}(p-1)a^{\dagger}_{p\sigma}a_{p\sigma} = \sum_{\sigma}(1-1)a^{\dagger}_{1\sigma}a_{1\sigma} + \sum_{\sigma}(2-1)a^{\dagger}_{2\sigma}a_{2\sigma}\\ =&0 + a^{\dagger}_{2+}a_{2+} + a^{\dagger}_{2-}a_{2-},\end{aligned}$$

i.e. think in terms of counting operator in the states $p = 2, \sigma = +1$ and $p = 2, \sigma = -1$ and we get the same result.

## 2.2 Interaction term

Let us look at the interaction term. This terms creates ($P^+_p$) and annihilates ($\hat{P}^-_p$) a pair of particles in the same $p$ state. So we can compute the terms:

$$\langle\phi_0|\,\hat{H}_1\,|\phi_0\rangle = \langle\phi_0| - g\sum_{pq}\hat{P}^+_p\hat{P}^-_q\,|\phi_0\rangle\,. \tag{2.6}$$

In order not to have orthogonal states we must have $q = 1$ and $p = 1$. So we get that:

$$\langle\phi_0|\,\hat{H}_1\,|\phi_0\rangle = -g. \tag{2.7}$$

The second term is:

$$\langle\phi_0|\,\hat{H}_1\,|\phi_1\rangle = \langle\phi_0| - g\sum_{pq}\hat{P}^+_p\hat{P}^-_q\,|\phi_1\rangle\,. \tag{2.8}$$

In order not to have orthogonal states we must have $q = 2$ and $p = 1$. So we get that:

$$\langle\phi_0|\,\hat{H}_1\,|\phi_1\rangle = -g. \tag{2.9}$$

The third term is:

$$\langle\phi_1|\,\hat{H}_1\,|\phi_0\rangle = \langle\phi_1| - g\sum_{pq}\hat{P}^+_p\hat{P}^-_q\,|\phi_0\rangle\,. \tag{2.10}$$

In order not to have orthogonal states we must have $p = 2$ and $q = 1$. So we get that:

$$\langle\phi_0|\,\hat{H}_1\,|\phi_1\rangle = -g. \tag{2.11}$$

The final term is:

$$\langle\phi_1|\,\hat{H}_1\,|\phi_1\rangle = \langle\phi_1| - g\sum_{pq}\hat{P}^+_p\hat{P}^-_q\,|\phi_1\rangle\,. \tag{2.12}$$

In order not to have orthogonal states we must have $p = 2$ and $q = 2$. So we get that:

$$\langle\phi_1|\,\hat{H}_1\,|\phi_1\rangle = -g. \tag{2.13}$$

## 2.3 RESULTS

So our Hamiltonian Matrix reads:

$$\begin{pmatrix} -g & -g \\ -g & 2-g \end{pmatrix} \tag{2.14}$$

and we can compute the eigenvalues the usual way:

$$\begin{pmatrix} -g-\lambda & -g \\ -g & 2-g-\lambda \end{pmatrix}, \tag{2.15}$$

and set the determinant to 0, i.e. :

$$det(H-\lambda I) = (-g-\lambda)(2-g-\lambda) - (-g)(-g) = \ldots = \lambda^2 + 2(g-1)\lambda - 2g = 0. \tag{2.16}$$

The solutions of this second order equations are:

$$\lambda_{1,2} = 1 - g \pm \sqrt{g^2+1} \tag{2.17}$$

We plot for the eigenvalues in function of the strength of the interaction $g \in [-1, 1]$.