Traceless Symmetric Tensor Approach to Legendre Polynomials & Spherical Harmonics - II

Devang Bajpai

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From Previous Talk

Using traceless symmetric tensors, we can expand any function of angle as

$$\mathcal{F}(\hat{n}) = C^{(0)} + C_i^{(1)}\hat{n}_i + C_{ij}^{(2)}\hat{n}_i\hat{n}_j + ... + C_{i_1,i_2,...i_\ell}^{(\ell)}\hat{n}_{i_1}\hat{n}_{i_2}\dots\hat{n}_{i_\ell} \quad (1)$$

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where the $C^{(\ell)}_{i_1,i_2,\ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}$ are traceless symmetric tensors, the indices $i_1, i_2, \ldots i_\ell$ are summed from 1 to 3 as Cartesian indices, and

$$\hat{n} = \sin(\theta)\cos(\phi)\hat{e}_1 + \sin(\theta)\sin(\phi)\hat{e}_2 + \cos(\theta)\hat{e}_3$$

In the more standard approach, an arbitrary function of (θ, ϕ) is expanded in spherical harmonics:

$$\mathcal{F}(\hat{n}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

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We have shown that

$$\nabla_\theta^2 \, [C^{(\ell)}_{i_1, i_2, \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}] = -\ell(\ell + 1) \; C^{(\ell)}_{i_1, i_2, \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}, \qquad (2)$$

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In the standard approach one would show that

$$\nabla_\theta^2 Y_{\ell m}(\theta, \phi) = -\ell(\ell+1) Y_{\ell m}(\theta, \phi)$$

That means that there must be some particular traceless symmetric tensor, which we will call $C^{(\ell,m)}_{i_1,\ldots i_\ell}$ which is equivalent to $Y_{\ell m}(\theta,\phi)$. That is,

$$C^{(\ell,m)}_{i_1,\ldots i_\ell}\hat{n}_{i_1}\ldots\hat{n}_{i_\ell} = Y_{\ell m}(\theta,\phi)$$

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The $\mathcal{P}_\ell$ functions are the same as the $Y_{\ell 0}$ functions, except that they are normalized differently:

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \mathcal{P}_\ell(\cos\theta)$$

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$$C_i^{(1)} = \text{const}\ \hat{z}_i$$

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$$\{\hat{z}_{i_1} \hat{z}_{i_2} \ldots \hat{z}_{i_\ell}\} \equiv \text{traceless symmetric part of } \hat{z}_{i_1} \hat{z}_{i_2} \ldots \hat{z}_{i_\ell}$$

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$$\begin{aligned}
\{1\} &= 1 \\
\{\hat{z}_i\} &= \hat{z}_i \\
\{\hat{z}_i \hat{z}_j\} &= \hat{z}_i \hat{z}_j - \frac{1}{3}\delta_{ij} \\
\{\hat{z}_i \hat{z}_j \hat{z}_k\} &= \hat{z}_i \hat{z}_j \hat{z}_k - \frac{1}{5}(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_i \delta_{ij}) \\
\{\hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m\} &= \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \frac{1}{7}(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} \\
&= + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij}) + \frac{1}{35}(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})
\end{aligned} \tag{3}$$

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where the constant is yet to be determined.

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$$
\begin{aligned}
\mathcal{P}_\ell(x) &= \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell [(x^2-1)^\ell] \\
&= \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell [x^{2\ell} + (\text{lower powers})] \\
&= \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^{\ell-1} [(2\ell)x^{2\ell-1} + (\text{lower powers}) \\
&= \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^{\ell-2} [(2\ell)(2\ell-1)x^{2\ell-2} + (\text{lower powers})] \\
&= \frac{1}{2^\ell \ell!}[(2\ell)(2\ell-1)\ldots(\ell+1)x^\ell + (\text{lower powers})] \\
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&= \frac{(2\ell)!}{2^\ell (\ell!)^2} x^\ell + (\text{lower powers}). \qquad (5)
\end{aligned}$$

Matching these coefficients, we see that

$$\mathcal{P}_\ell(\cos\theta) = \frac{(2\ell)!}{2^\ell(\ell!)^2} x^\ell + (\text{lower powers})\{\hat{z}_{i_1}\ldots\hat{z}_{i_\ell}\}\hat{n}_{i_1}\ldots\hat{n}_{i_\ell}$$

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$$v'_x = v_x \cos\psi - v_y \sin\psi$$
$$v'_y = v_x \sin\psi + v_y \cos\psi$$

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for which the solutions are

$$\lambda = \cos\psi \pm \sqrt{\cos^2\psi - 1} = \cos\psi \pm \iota\sin\psi = e^{\pm\iota\psi}$$

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$$\begin{pmatrix} \cos\psi - e^{\pm\iota\psi} & -\sin\psi \\ \sin\psi & \cos\psi - e^{\pm\iota\psi} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0,$$

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which are orthonormal in the sense that

$$\hat{\mathfrak{u}}^{(i)*} \cdot \hat{\mathfrak{u}}^{(j)} = \delta_{ij}$$

$$\int_0^{2\pi} z^*_{m'}(\phi) z_m(\phi) d\phi = 2\delta_{m'm} \quad \text{where, } z_m(\phi) \equiv e^{\iota m\phi}$$

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are in fact equal to zero. This leads to the convenient fact that

$$\hat{\mathfrak{u}}^{(+)}_{i_1} \dots \hat{\mathfrak{u}}^{(+)}_{i_\ell}$$

is both traceless and symmetric.

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$$\mathcal{F}_{\ell m}(\theta, \phi) \equiv \{\hat{\mathfrak{u}}^{(+)}_{i_1} \dots \hat{\mathfrak{u}}^{(+)}_{i_m} \hat{z}_{i_{m+1}} \dots \hat{z}_{i_\ell}\} \hat{n}_{i_1} \dots \hat{n}_{i_\ell} \qquad (6)$$

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To compare this equation with the leading term in the expansion for the standard function $Y_{\ell m}$, we need a formula for $Y_{\ell m}(\theta,\phi)$.

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\mathcal{P}_\ell^m(\cos\theta)e^{\iota m\phi}$$

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$$\mathcal{P}_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

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Matching the coefficients of these leading terms, we find that we can write (for $m \geq 0$)

$$Y_{\ell m}(\theta,\phi) = C^{(\ell,m)}_{i_1,\dots i_\ell}\hat{n}_{i_1}\dots\hat{n}_{i_\ell},$$

where

$$C^{(\ell,m)}_{i_1,\dots i_\ell} = d_{\ell m}\{\hat{\mathfrak{u}}^+_{i_1}\dots\hat{\mathfrak{u}}^+_{i_m}\hat{z}_{i_{m+1}}\dots\hat{z}_{i_\ell}\}$$

with

$$d_{\ell m} = \frac{(-1)^m (2\ell)!}{2^\ell \ell!} \sqrt{\frac{2^m (2\ell + 1)}{4\pi(\ell + m)!(\ell - m)!}}$$

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where to allow for negative m we need to write $d_{\ell m}$ as

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$$\{\hat{\mathfrak{u}}_i^+\hat{\mathfrak{u}}_j^-\} = -\frac{1}{2}\{\hat{z}_i\hat{z}_j\}$$

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\begin{aligned}
\{1\} &= 1 \\
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Up to a normalization convention, these are the Legendre polynomials $\mathcal{P}_\ell^m(\cos\theta)$.

Multipole Expansion:

The most general solution to Laplace's equation, in spherical coordinates, was given as

$$\boxed{\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \left(C^{(\ell)}_{i_1, i_2, \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \right)} \tag{8}$$

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$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1,i_2,\ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}$$

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The first few terms of this series have special names: the $\ell = 0$ term is the *monopole* term, the $\ell = 1$ term is the *dipole*, the $\ell = 2$ term is the *quadrupole*, and the $\ell = 3$ term is the *octupole*.

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where r and r' are the lengths of the vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively, and θ is the angle between these vectors. Next he used the fact that the Legendre polynomials can be defined by the generating function

$$g(x, \lambda) = \frac{1}{\sqrt{1 + \lambda^2 + 2\lambda x}}$$

which means that the Legendre polynomials $\mathcal{P}_\ell(x)$ can be obtained by expanding $g(x, \lambda)$ in a power series in λ:

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Inserting this relation into Eq.9, we find

$$V = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\mathbf{r}') \mathcal{P}_\ell(\cos\theta') d^3x \tag{11}$$

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$$V = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}\int r'^{\ell}\rho(\mathbf{r}')\mathcal{P}_\ell(\cos\theta')d^3x \qquad (11)$$

The standard method of "improving" Eq.11 is to use spherical harmonics, but here I will derive the equivalent relations using the traceless symmetric tensor approach. Instead of expanding $1/\mathscr{r}$

in powers of r', we will think of it as a function of three variables — the components $\mathbf{r}'_i$ of $\mathbf{r}'$, and we will expand it as a Taylor series in 3 variables. To make the formalism clear, I will define the function

$$f(\mathbf{r}') \equiv \frac{1}{\mathscr{r}}$$

The function can then be expanded in a power series using the standard multi-variable Taylor expansion:

$$f(\mathbf{r}') = f(\vec{0}) + \frac{\partial f}{x_i'}\bigg|_{\mathbf{r}'=\vec{0}} x_i' + \frac{1}{2!}\frac{\partial^2 f}{\partial x_i' \partial x_j'}\bigg|_{\mathbf{r}'=\vec{0}} x_i' x_j' + \dots$$

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$$f(\mathbf{r}') = f(\vec{0}) + r'\left(\frac{\partial f}{x_i'}\right)\bigg|_{\mathbf{r}'=\vec{0}} \hat{n}_i' + \frac{r'^2}{2!}\frac{\partial^2 f}{\partial x_i' \partial x_j'}\bigg|_{\mathbf{r}'=\vec{0}} \hat{n}_i' \hat{n}_j' + \dots \quad (12)$$

The notation can now be simplified by noting that since f is a function of $\mathbf{r} - \mathbf{r}'$, the derivatives with respect to x_i' can be replaced by derivatives with respect to x_i with a change of sign:

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$$\frac{\partial f}{\partial x_i'} = \frac{\partial}{\partial x_i'} \left(\frac{1}{\mathscr{r}} \right) \bigg|_{\mathbf{r}'=0} = -\frac{\partial}{\partial x_i'} \left(\frac{1}{\mathscr{r}} \right) \bigg|_{\mathbf{r}'=0}$$

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Combining Eqs.13 with 12, we can write

$$\frac{1}{\mathscr{r}} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell r'^\ell}{\ell!} \left(\frac{\partial^\ell}{\partial x_{i_1} \ldots \partial x_{i_\ell}} \frac{1}{|\mathbf{r}|}\right) \hat{n}_{i_1}' \ldots \hat{n}_{i_1}' \tag{14}$$

Note that the quantity in parentheses in the equation above is traceless, because

$$\left(\frac{\partial^{\ell}}{\partial x_{i_1}\partial x_i \partial x_{i+2}\ldots\partial x_{i_\ell}}\frac{1}{|\mathbf{r}|}\right) = \nabla^2 \frac{\partial^{\ell}}{x_{i+2}\ldots\partial x_{i_\ell}}\frac{1}{|\mathbf{r}|} = 0$$

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Because $\nabla^2(1/|\mathbf{r}|) = 0$ except at $\mathbf{r} = 0$. So we can see the traceless symmetric tensor formalism emerging. To evaluate this quantity, we will work out the first several terms until we recognize the pattern. We write

$$\mathbf{r} \equiv r\hat{n}$$

and adopt the abbreviation

$$\partial_i \equiv \frac{\partial}{x_i}$$

It is useful to start by evaluating the derivatives of the basic quantities r and $\hat{n}_i$:

$$\partial_i r = \partial_i (x_i x_j)^{1/2} = \frac{1}{2}\partial_i (x_j x_j)(x_k x_k)^{-1/2} = \frac{1}{2r} 2x_j \delta_{ij} = \frac{x_i}{r} = \hat{n}_i$$

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It is then straightforward to show that

$$\partial_i \left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{n}_i$$

$$\partial_i \partial_j \left(\frac{1}{r}\right) = \frac{3}{r^3}\hat{n}_i \hat{n}_j$$

$$\partial_i \partial_j \partial_k \left(\frac{1}{r}\right) = -\frac{5 \cdot 3}{r^4}\hat{n}_i \hat{n}_j \hat{n}_k$$

where $\{\}$ denotes the traceless symmetric part. It becomes clear that the general formula, which can be proven by induction, is

$$\frac{\partial^\ell}{\partial x_{i_1} \dots \partial x_{i_\ell}} \frac{1}{|\mathbf{r}|} = \frac{(-1)^\ell (2\ell - 1)!!}{r^{\ell+1}} \{\hat{n}_{i_1} \dots \hat{n}_{i_\ell}\}$$

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where

$$(2\ell - 1)!! \equiv (2\ell - 1)(2\ell - 3)(2\ell - 5) \dots 1 = \frac{(2\ell)!}{2^\ell \ell!}, \text{ with } (-1)!! \equiv 1$$

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Inserting the result into eq.14, we find

$$\frac{1}{\mathscr{r}} = \sum_{\ell=0}^{\infty} \frac{(2\ell - 1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{\hat{n}_{i_1} \dots \hat{n}_{i_\ell}\} \hat{n}'_{i_1} \dots \hat{n}'_{i_1} \qquad (15)$$

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One can write this more symmetrically by writing

$$\frac{1}{\mathscr{r}} = \sum_{\ell=0}^{\infty} \frac{(2\ell - 1)!!}{\ell!} \frac{r'^\ell}{r^{\ell+1}} \{\hat{n}_{i_1} \dots \hat{n}_{i_\ell}\} \{\hat{n}'_{i_1} \dots \hat{n}'_{i_1}\} \tag{16}$$

Inserting this expression for $1/\mathscr{r}$ into eq.9, we have the final result

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}C^{(\ell)}_{i_1,i_2,\dots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\dots\hat{n}_{i_\ell} \tag{17}$$

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Note that the coefficient in the above expression can also be written as

$$\frac{(2\ell-1)!!}{\ell!} = \frac{(2\ell)!}{2^\ell(\ell!)^2}$$

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where

$$p_i = C^{(1)} = \int \rho(\mathbf{r}')\mathbf{r}_i d^3x'$$

Thank you for your attention