Traceless Symmetric Tensor Approach to Legendre Polynomials & Spherical Harmonics

Devang Bajpai

Theoretical Nexus Seminar, June 2024

In this talk I will describe the separation of variable technique for solving Laplace's equation, using spherical polar coordinates. The solutions will involve Legendre polynomials for cases with azimuthal symmetry, and more generally they will involve spherical harmonics. I will construct these solutions using traceless symmetric tensors, in next talk I will describe how the solutions in this form relate to the more standard expressions in terms of Legendre polynomials and spherical harmonics.

Laplace's Equation in Spherical Coordinates:

In Spherical Coordinates Laplace's Equation can be written as,

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\right)}_{\nabla_\theta^2\varphi} = 0 \tag{1}$$

Laplace's Equation in Spherical Coordinates:

In Spherical Coordinates Laplace's Equation can be written as,

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\right)}_{\nabla^2_\theta\varphi} = 0 \tag{1}$$

Now if we use the method of separation of variables, we can seek a solution of the form

$$\varphi = R(r)\mathcal{F}(\theta,\phi)$$

Then Laplace's equation can be written as

$$0 = \frac{r}{R\mathcal{F}}\nabla^2\varphi = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\mathcal{F}}\nabla_\theta^2\mathcal{F}$$

Then Laplace's equation can be written as

$$0 = \frac{r}{R\mathcal{F}}\nabla^2\varphi = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\mathcal{F}}\nabla_\theta^2\mathcal{F}$$

Since the first term on the right-hand side depends only on r, and the second term depends only on θ and φ, the only way that the equation can be satisfied is if each term is a constant. Thus we can write

Then Laplace's equation can be written as

$$0 = \frac{r}{R\mathcal{F}}\nabla^2\varphi = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\mathcal{F}}\nabla^2_\theta\mathcal{F}$$

Since the first term on the right-hand side depends only on r, and the second term depends only on θ and φ, the only way that the equation can be satisfied is if each term is a constant. Thus we can write

$$\frac{1}{\mathcal{F}}\nabla^2_\theta\mathcal{F} = C_\theta \tag{2}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -C_\theta \tag{3}$$

The Expansion of $\mathcal{F}(\theta, \phi)$

Now I wish to find the most general solution to the equation

$$\nabla_\theta^2 \mathcal{F} = C_\theta \mathcal{F}$$

The Expansion of $\mathcal{F}(\theta, \phi)$

Now I wish to find the most general solution to the equation

$$\nabla_\theta^2 \mathcal{F} = C_\theta \mathcal{F}$$

If such a function $\mathcal{F}$ can be found, we say that $\mathcal{F}$ is an *eigenfunction* of the operator ∇_θ^2, with *eigenvalue* C_θ

The Expansion of $\mathcal{F}(\theta, \phi)$

Now I wish to find the most general solution to the equation

$$\nabla_\theta^2 \mathcal{F} = C_\theta \mathcal{F}$$

If such a function $\mathcal{F}$ can be found, we say that $\mathcal{F}$ is an *eigenfunction* of the operator ∇_θ^2, with *eigenvalue* C_θ

A function of angles (θ, ϕ) can equivalently be thought of as a function of the unit vector $\hat{n}$ that points in the direction of θ and ϕ, which can be written explicitly as

$$\hat{n} = \sin(\theta)\cos(\phi)\hat{e}_1 + \sin(\theta)\sin(\phi)\hat{e}_2 + \cos(\theta)\hat{e}_3$$

Claim: The most general function of (θ, ϕ) can be written as a power series in $\hat{n}$, or more precisely as a power series in the components of $\hat{n}$.

Claim: The most general function of (θ, ϕ) can be written as a power series in $\hat{n}$, or more precisely as a power series in the components of $\hat{n}$.I will not prove this, but it is true at least for square-integrable piece-wise continuous functions $\mathcal{F}(\theta, \phi)$. Such a power series can be written as

Claim: The most general function of (θ, ϕ) can be written as a power series in $\hat{n}$, or more precisely as a power series in the components of $\hat{n}$.I will not prove this, but it is true at least for square-integrable piece-wise continuous functions $\mathcal{F}(\theta, \phi)$. Such a power series can be written as

$$\boxed{\mathcal{F}(\hat{n}) = C^{(0)} + C^{(1)}_i \hat{n}_i + C^{(2)}_{ij} \hat{n}_i \hat{n}_j + ... + C^{(\ell)}_{i_1, i_2, ... i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} ... \hat{n}_{i_\ell} + ...} \tag{4}$$

where repeated indices are summed from 1 to 3 (as Cartesian coordinates). Note that $C^{(\ell)}_{i_1,i_2,...i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}...\hat{n}_{i_\ell}$ represents the general term in the series, where the first three terms correspond to $\ell = 0$, $\ell = 1$, and $\ell = 2$. The indices $i_1, i_2, ..., i_\ell$ represent ℓ different indices, like i and j; since they are repeated, they are each summed from 1 to 3.

where repeated indices are summed from 1 to 3 (as Cartesian coordinates). Note that $C^{(\ell)}_{i_1,i_2,...i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}...\hat{n}_{i_\ell}$ represents the general term in the series, where the first three terms correspond to $\ell = 0$, $\ell = 1$, and $\ell = 2$. The indices $i_1, i_2, ..., i_\ell$ represent ℓ different indices, like i and j; since they are repeated, they are each summed from 1 to 3.

The coefficients $C^{(\ell)}_{i_1,i_2,...i_\ell}$ are called tensors, and the number of indices is called the rank of the tensor. Note that $C^{(0)}$, $C^{(1)}_i$ and $C^{(\ell)}_{i_1,i_2,...i_\ell}$ are special cases of tensors, although they can also be considered a scalar, a vector, and a matrix.

1. Since the tensors $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$ are symmetric under any reordering of the indices:

$$C^{(\ell)}_{i_1,i_2,\ldots i_\ell} = C^{(\ell)}_{j_1,j_2,\ldots j_\ell},$$

1. Since the tensors $C^{(\ell)}_{i_1,i_2,\dots i_\ell}$ are symmetric under any reordering of the indices:
$$C^{(\ell)}_{i_1,i_2,\dots i_\ell} = C^{(\ell)}_{j_1,j_2,\dots j_\ell},$$
2. The tensors $C^{(\ell)}_{i_1,i_2,\dots i_\ell}$ are traceless, in the sense that if any two indices are set equal to each other and summed, the result is equal to zero. Since the tensors are already assumed to be symmetric, it does not matter which indices are summed, so we can choose the last two:
$$C^{(\ell)}_{i_1,i_2,\dots i_\ell} = 0$$

To explain why these restrictions on the $C(\ell)'s$ do not impose any restriction on the right hand side of Eq. (4), I will use the example of $C_{ij}^{(2)}$. Suppose that $C_{ij}^{(2)}$ were not traceless. Then we could write

$$C_{ii}^{(2)} = \lambda \neq 0$$

To explain why these restrictions on the $C(\ell)'s$ do not impose any restriction on the right hand side of Eq. (4), I will use the example of $C_{ij}^{(2)}$. Suppose that $C_{ij}^{(2)}$ were not traceless. Then we could write

$$C_{ii}^{(2)} = \lambda \neq 0$$

We could then define a new quantity,

$$\tilde{C}_{ij}^{(2)} = C_{ij}^{(2)} - \frac{1}{3}\lambda\delta_{ij}$$

To explain why these restrictions on the $C(\ell)'s$ do not impose any restriction on the right hand side of Eq. (4), I will use the example of $C_{ij}^{(2)}$. Suppose that $C_{ij}^{(2)}$ were not traceless. Then we could write

$$C_{ii}^{(2)} = \lambda \neq 0$$

We could then define a new quantity,

$$\tilde{C}_{ij}^{(2)} = C_{ij}^{(2)} - \frac{1}{3}\lambda\delta_{ij}$$

It follows that $\tilde{C}_{ij}^{(2)}$ is traceless:

$$\begin{aligned}\tilde{C}_{ii}^{(2)} &= C_{ii}^{(2)} - \frac{1}{3}\lambda\delta_{ii} \\ &= \lambda - \frac{1}{3}\lambda\delta_{ii} \\ &= 0\end{aligned}$$

where we used the fact that $\delta_{ij}\hat{n}_i\hat{n}_j = 1$, since $\hat{n}$ is a unit vector. The extra term, $\frac{1}{3}\lambda$ can then be absorbed into a redefinition of $C^{(0)}$:

$$\tilde{C}^{(0)} = C^{(0)} + \frac{1}{3}\lambda$$

where we used the fact that $\delta_{ij}\hat{n}_i\hat{n}_j = 1$, since $\hat{n}$ is a unit vector. The extra term, $\frac{1}{3}\lambda$ can then be absorbed into a redefinition of $C^{(0)}$:

$$\tilde{C}^{(0)} = C^{(0)} + \frac{1}{3}\lambda$$

Finally, we can write

$$C^{(0)} + C_i^{(1)}\hat{n}_i + C_{ij}^{(2)}\hat{n}_i\hat{n}_j = \tilde{C}^{(0)} + \tilde{C}_i^{(1)}\hat{n}_i + \tilde{C}_{ij}^{(2)}\hat{n}_i\hat{n}_j,$$

So we can insist that the tensor that multiplies $\hat{n}_i\hat{n}_j$ be traceless with no restriction on what functions can be expressed in this form.

Evaluation of $\nabla_\theta^2 \mathcal{F}_\ell(\hat{n})$:

To evaluate $\nabla_\theta^2 \mathcal{F}_\ell(\hat{n})$, we are going to use an convenient trick.

[1]Here $\mathcal{F}_\ell(\hat{n})$ represents the ℓ^{th} term of the expansion

$$\mathcal{F}_\ell(\hat{n}) = C^{(\ell)}_{i_1, i_2, \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}. \tag{5}$$

Evaluation of $\nabla_\theta^2 \mathcal{F}_\ell(\hat{n})$:

To evaluate $\nabla_\theta^2 \mathcal{F}_\ell(\hat{n})$, we are going to use an convenient trick.Instead of dealing directly with $\mathcal{F}_\ell(\hat{n})$[1], we will instead introduce a radial variable r, using it to define a coordinate vector

$$\vec{r} = r\hat{n}$$

$$\vec{r} = x_i\hat{e}_i = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$$

[1]Here $\mathcal{F}_\ell(\hat{n})$ represents the ℓ^{th} term of the expansion

$$\mathcal{F}_\ell(\hat{n}) = C^{(\ell)}_{i_1,i_2,\ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}. \tag{5}$$

Evaluation of $\nabla^2_\theta \mathcal{F}_\ell(\hat{n})$:

To evaluate $\nabla^2_\theta \mathcal{F}_\ell(\hat{n})$, we are going to use an convenient trick.Instead of dealing directly with $\mathcal{F}_\ell(\hat{n})$[1], we will instead introduce a radial variable r, using it to define a coordinate vector

$$\vec{r} = r\hat{n}$$

$$\vec{r} = x_i\hat{e}_i = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$$

Then, given any $\mathcal{F}_\ell(\hat{n})$ of the form given in Eq.5, we can define a function $\tilde{\mathcal{F}}_\ell(\hat{n})$ by

$$\tilde{\mathcal{F}}_\ell(\hat{n}) = \tilde{C}^{(\ell)}_{i_1,i_2,\ldots i_\ell} x_{i_1} x_{i_2} \ldots x_{i_\ell} = r^\ell \mathcal{F}_\ell(\hat{n}).$$

[1]Here $\mathcal{F}_\ell(\hat{n})$ represents the ℓ^{th} term of the expansion

$$\mathcal{F}_\ell(\hat{n}) = C^{(\ell)}_{i_1,i_2,\ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell}. \tag{5}$$

Note that $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$ is the same rank ℓ traceless symmetric tensor used to define $\mathcal{F}_\ell(\hat{n})$, but we are defining $\tilde{\mathcal{F}}_\ell(\vec{r})$ by multiplying $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$ by $x_{i_1}x_{i_2}\ldots x_{i_\ell}$and then summing over indices, instead of multiplying by $\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}$ and then summing.

Note that $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$ is the same rank ℓ traceless symmetric tensor used to define $\mathcal{F}_\ell(\hat{n})$, but we are defining $\tilde{\mathcal{F}}_\ell(\vec{r})$ by multiplying $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$ by $x_{i_1}x_{i_2}\ldots x_{i_\ell}$and then summing over indices, instead of multiplying by $\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}$ and then summing.

Now we can make use of Eq.1, which relates the full Laplacian ∇^2 to the angular Laplacian, ∇^2_θ . We will find that in this case the full Laplacian and the radial derivative piece of Eq.1 will both be simple, so we will be able to determine the angular Laplacian by evaluating the other terms in Eq.1.

To evaluate $\tilde{\mathcal{F}}_\ell(\vec{r})$, we start with $\ell = 0$. Clearly

$$\tilde{\mathcal{F}}_0(\vec{r}) = \nabla^2 C_0 = 0,$$

To evaluate $\tilde{\mathcal{F}}_\ell(\vec{r})$, we start with $\ell = 0$. Clearly

$$\tilde{\mathcal{F}}_0(\vec{r}) = \nabla^2 C_0 = 0,$$

since the derivative of a constant vanishes. Similarly for $\ell = 1$,

$$\tilde{\mathcal{F}}_1(\vec{r}) = \nabla^2 C_i^{(1)} x_i = 0,$$

To evaluate $\tilde{\mathcal{F}}_\ell(\vec{r})$, we start with $\ell = 0$. Clearly

$$\tilde{\mathcal{F}}_0(\vec{r}) = \nabla^2 C_0 = 0,$$

since the derivative of a constant vanishes. Similarly for $\ell = 1$,

$$\tilde{\mathcal{F}}_1(\vec{r}) = \nabla^2 C_i^{(1)} x_i = 0,$$

since the first derivative produces a constant, so the second derivative vanishes. The first nontrivial case is $\ell = 2$:

$$\begin{aligned}
\tilde{\mathcal{F}}_2(\vec{r}) &= \nabla^2 C_{ij}^{(2)} x_i x_j \\
&= C_{ij}^{(2)} \frac{\partial}{\partial m} \frac{\partial}{\partial m} (x_i x_j) \\
&= C_{ij}^{(2)} \frac{\partial}{\partial m} [\delta_{im} x_j + \delta_{jm} x_i] \\
&= C_{ij}^{(2)} [\delta_{im} \delta_{jm}] = 2 C_{ii}^{(2)} = 0
\end{aligned}$$

Where in the last step we used the all-important fact that $C_{ij}^{(2)}$ is traceless.

Where in the last step we used the all-important fact that $C^{(2)}_{ij}$ is traceless.

Thinking about the general term, one can see that after the derivatives are calculated, there are $\ell - 2$ factors of x_i that remain, but there are still ℓ indices on $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$. Since all indices are summed, there are always two indices on $C^{(\ell)}_{i_1,i_2,\ldots i_\ell}$ which are contracted (i.e., set equal to each other) and summed, which causes the result ℓ to vanish by the tracelessness condition. The bottom line, then, is that

$$\nabla^2_\theta \tilde{\mathcal{F}}_\ell(\vec{r}) = 0 \qquad \text{For all } \ell.$$

Where in the last step we used the all-important fact that $C^{(2)}_{ij}$ is traceless.

Thinking about the general term, one can see that after the derivatives are calculated, there are $\ell - 2$ factors of x_i that remain, but there are still ℓ indices on $C^{(\ell)}_{i_1, i_2, \ldots i_\ell}$. Since all indices are summed, there are always two indices on $C^{(\ell)}_{i_1, i_2, \ldots i_\ell}$ which are contracted (i.e., set equal to each other) and summed, which causes the result ℓ to vanish by the tracelessness condition. The bottom line, then, is that

$$\nabla^2_\theta \tilde{\mathcal{F}}_\ell(\vec{r}) = 0 \qquad \text{For all } \ell.$$

To see what this says about $\nabla^2_\theta \mathcal{F}_\ell(\hat{n})$, recall that $\tilde{\mathcal{F}}(\vec{r}) = r^\ell \mathcal{F}_\ell(\hat{n})$ Using Eq.1, we can write

$$
\begin{aligned}
0 = \nabla^2 \tilde{\mathcal{F}}_\ell(\vec{r}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\mathcal{F}}_\ell(\vec{r})}{\partial r}\right) + \frac{1}{r^2}\nabla_\theta^2 \tilde{\mathcal{F}}_\ell(\vec{r}) \\
&= \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dr^\ell}{dr}\right)\mathcal{F}_\ell(\hat{n}) + \frac{1}{r^2} r^\ell \nabla_\theta^2 \tilde{\mathcal{F}}_\ell(\hat{n}) \\
&= r^{\ell-2}[\ell(\ell+1)\mathcal{F}_\ell(\hat{n}) + \nabla_\theta^2 \mathcal{F}_\ell(\hat{n})]
\end{aligned}
$$

$$
\begin{aligned}
0 = \nabla^2 \tilde{\mathcal{F}}_\ell(\vec{r}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\mathcal{F}}_\ell(\vec{r})}{\partial r}\right) + \frac{1}{r^2}\nabla_\theta^2 \tilde{\mathcal{F}}_\ell(\vec{r}) \\
&= \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dr^\ell}{dr}\right)\mathcal{F}_\ell(\hat{n}) + \frac{1}{r^2} r^\ell \nabla_\theta^2 \tilde{\mathcal{F}}_\ell(\hat{n}) \\
&= r^{\ell-2}[\ell(\ell+1)\mathcal{F}_\ell(\hat{n}) + \nabla_\theta^2 \mathcal{F}_\ell(\hat{n})]
\end{aligned}
$$

and therefore

$$
\boxed{\nabla_\theta^2 \mathcal{F}_\ell(\hat{n}) = -\ell(\ell+1)\mathcal{F}_\ell(\hat{n})} \tag{6}
$$

$$
\begin{aligned}
0 = \nabla^2 \tilde{\mathcal{F}}_\ell(\vec{r}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\mathcal{F}}_\ell(\vec{r})}{\partial r}\right) + \frac{1}{r^2}\nabla_\theta^2 \tilde{\mathcal{F}}_\ell(\vec{r}) \\
&= \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dr^\ell}{dr}\right)\mathcal{F}_\ell(\hat{n}) + \frac{1}{r^2} r^\ell \nabla_\theta^2 \tilde{\mathcal{F}}_\ell(\hat{n}) \\
&= r^{\ell-2}[\ell(\ell+1)\mathcal{F}_\ell(\hat{n}) + \nabla_\theta^2 \mathcal{F}_\ell(\hat{n})]
\end{aligned}
$$

and therefore

$$
\boxed{\nabla_\theta^2 \mathcal{F}_\ell(\hat{n}) = -\ell(\ell+1)\mathcal{F}_\ell(\hat{n})} \tag{6}
$$

Thus, we have found the *eigenfunctions* ($\mathcal{F}_\ell(\hat{n}) = C^{(\ell)}_{i_1,i_2,\ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}$) and *eigenvalues* ($-\ell(\ell+1)\mathcal{F}_\ell(\hat{n})$) of the differential operator ∇_θ^2. This is a very useful result!

General Solution of Laplace's Equation in Spherical Coordinates:

Now that we know the eigenfunctions of ∇_θ^2, we can return to the solution to Laplace's equation by the separation of variables in spherical coordinates.

General Solution of Laplace's Equation in Spherical Coordinates:

Now that we know the eigenfunctions of ∇_θ^2, we can return to the solution to Laplace's equation by the separation of variables in spherical coordinates.We now know that in Eq.2, the only allowed values of C_θ are $-\ell(\ell+1)$, where ℓ is an integer. Thus, Eq.3 becomes

General Solution of Laplace's Equation in Spherical Coordinates:

Now that we know the eigenfunctions of ∇^2_θ, we can return to the solution to Laplace's equation by the separation of variables in spherical coordinates.We now know that in Eq.2, the only allowed values of C_θ are $-\ell(\ell+1)$, where ℓ is an integer. Thus, Eq.3 becomes

$$\frac{d}{dr}\left(r^2\frac{dR}{dr} = \ell(\ell+1)R\right)$$

General Solution of Laplace's Equation in Spherical Coordinates:

Now that we know the eigenfunctions of ∇^2_θ, we can return to the solution to Laplace's equation by the separation of variables in spherical coordinates.We now know that in Eq.2, the only allowed values of C_θ are $-\ell(\ell+1)$, where ℓ is an integer. Thus, Eq.3 becomes

$$\frac{d}{dr}\left(r^2\frac{dR}{dr} = \ell(\ell+1)R\right)$$

and we can look for solutions by trying $R(r) = r^p$. We find consistency provided that

$$p(p+1) = \ell(\ell+1)$$

General Solution of Laplace's Equation in Spherical Coordinates:

Now that we know the eigenfunctions of ∇^2_θ, we can return to the solution to Laplace's equation by the separation of variables in spherical coordinates.We now know that in Eq.2, the only allowed values of C_θ are $-\ell(\ell+1)$, where ℓ is an integer. Thus, Eq.3 becomes

$$\frac{d}{dr}\left(r^2\frac{dR}{dr} = \ell(\ell+1)R\right)$$

and we can look for solutions by trying $R(r) = r^p$. We find consistency provided that

$$p(p+1) = \ell(\ell+1)$$

which is a quadratic equation with two roots, $p = \ell$ and $p = -(\ell+1)$. Since we found two solutions to a second order linear differential equation,

we know that any solution can be written as a linear sum of these two. Thus we can write

$$R_\ell = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

we know that any solution can be written as a linear sum of these two. Thus we can write

$$R_\ell = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

allowing for different values of A_ℓ and B_ℓ for each ℓ. The most general solution to Laplace's equation, in spherical coordinates, can then be written as

we know that any solution can be written as a linear sum of these two. Thus we can write

$$R_\ell = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

allowing for different values of A_ℓ and B_ℓ for each ℓ. The most general solution to Laplace's equation, in spherical coordinates, can then be written as

$$\boxed{\Phi(\vec{r}) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)\left(C^{(\ell)}_{i_1,i_2,\ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell}\right)} \tag{7}$$

we know that any solution can be written as a linear sum of these two. Thus we can write

$$R_\ell = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

allowing for different values of A_ℓ and B_ℓ for each ℓ. The most general solution to Laplace's equation, in spherical coordinates, can then be written as

$$\boxed{\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \left(C^{(\ell)}_{i_1, i_2, \ldots i_\ell} \hat{n}_{i_1} \hat{n}_{i_2} \ldots \hat{n}_{i_\ell} \right)} \tag{7}$$

where the A_ℓ's and B_ℓ's are arbitrary constants, and each $C^{(\ell)}_{i_1, i_2, \ldots i_\ell}$ is an arbitrary traceless symmetric tensor.

References:

[1] Mary L. Boas, *"Mathematical Methods in Physical Sciences"*, *3rd edition*, Wiley Publication.

[2] Arfken, Weber, & Harris, *"Mathematical Methods for Physicists"*, *7th edition*, Academic Press.

[3] David J. Griffiths, *"Introduction to Electrodynamics"*, *4th edition*, Cambridge University Press.

[4] J.D. Jackson, *"Classical electrodynamics"*, *3rd edition*, Wiley Publication.

Thank you for listening . . .