Quantitative Risk Management

Group Assignment. Solution

(i) We estimated the parameters of models M2, M3 and M4 using maximum-likelihood estimation for the last 500 observations:

M2: Maximum-likelihood estimators for mean, standard deviation and correlation are the same as sample parameters. Using the formulas 1 – 5:

$$\hat{\mu}_j = \frac{\sum_{i=1}^{n} r_{ij}}{n}, j = 1,2, n = 500 \tag{1}$$

$$\hat{\sigma}_j = \sqrt{\frac{\sum_{i=1}^{n}\left(r_{ij} - \hat{\mu}_j\right)^2}{n}}, j = 1,2, n = 500 \tag{2}$$

$$\hat{\rho} = \frac{\sum_{i=1}^{n}(r_{i1} - \hat{\mu}_1)(r_{i2} - \hat{\mu}_2)}{n\hat{\sigma}_1\hat{\sigma}_2}, n = 500 \tag{3}$$

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} \tag{4}$$

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 \\ \hat{\rho}\hat{\sigma}_1\hat{\sigma}_2 & \hat{\sigma}_2^2 \end{pmatrix} \tag{5}$$

We get the following estimations:

$$\hat{\mu} = \begin{pmatrix} 0.000688 \\ 0.000397 \end{pmatrix}, \hat{\Sigma} = \begin{pmatrix} 0.000131 & 0.000075 \\ 0.000075 & 0.000090 \end{pmatrix}$$

M3: The values $\hat{\mu}_j, \hat{\sigma}_j, j = 1,2$ are the same as in M2. The estimator $\hat{\theta}$ of the multivariate parameter of Gumbel Copula is found by Maximum-likelihood method as follows:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} \ln c\left(\Phi\left(\frac{r_{i1} - \hat{\mu}_1}{\hat{\sigma}_1}\right), \Phi\left(\frac{r_{i2} - \hat{\mu}_2}{\hat{\sigma}_2}\right); \theta\right) \tag{6}$$

where

$$c(u_1, u_2; \theta) = \frac{\partial^2 C(u_1, u_2; \theta)}{\partial u_1 \partial u_2}$$

is a probability density function of Gumbel Copula

$$C(u_1, u_2; \theta) = \exp\left(-\left((-\,ln(u_1))^{\theta} + (-\,ln(u_2))^{\theta}\right)^{\frac{1}{\theta}}\right)$$

VineCopula R package computes the value

$$\hat{\theta} = 1.743254$$

M4: Denote $\epsilon_{ij} = \frac{R_{ij} - \hat{\mu}_i}{\hat{\sigma}_i}, j = 1{,}2$. Degrees of freedoms ν_j are estimated as follows:

$$\hat{\nu}_j = \underset{\nu_j}{\operatorname{argmax}} \sum_{i=1}^{n} \ln f(\epsilon_{ij}; \nu_j),$$

where

$$f(x; \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

is t-distribution density function.

Gaussian copula parameter ρ is estimated by Maximum-likelihood method as

$$\hat{\rho} = \underset{\rho}{\operatorname{argmax}} \sum_{i=1}^{n} \ln c(\mathrm{F}(\epsilon_{i1}; \hat{\nu}_1), \mathrm{F}(\epsilon_{i2}; \hat{\nu}_2); \rho),$$

where $F(\epsilon, \nu)$ is cumulative distribution function of t-distribution with ν degrees of freedom and $c(u_1, u_2; \rho)$ is the density function of Gaussian copula.

The calculated values of the estimator $\hat{\rho}$ and degrees of freedom are:

$$\hat{\rho} = 0.666511$$

$$\hat{\nu}_1 = 17.7420$$

$$\hat{\nu}_2 = 24.9968$$

(ii) Using model M1 for R, we simulated the portfolio distribution (10,000 simulations of the portfolio return) and estimated 1-day portfolio's value-at-risk and expected shortfall at 90%, 95% and 99% confidence levels using formulas 10 and 11:

$$VaR_{\alpha}(L) = -inf\{x \in \mathbb{R} \mid \mathbb{P}[L \leq x] > 1 - \alpha\} \tag{10}$$

$$ES_{\alpha}(X) = \frac{1}{1-\alpha}\{\mathbb{E}_{\mathbb{P}}[-X 1_{\{X \leq -VaR_{\alpha}(X)\}}] - VaR_{\alpha}(X)(\mathbb{P}[X < -VaR_{\alpha}(X)] - (1-\alpha))\} \tag{11}$$

$$VaR_{90\%} = 83624.85, ES_{90\%} = 127727.6$$
$$VaR_{95\%} = 114636.94, ES_{95\%} = 158023.6$$
$$VaR_{99\%} = 185697.30, ES_{99\%} = 223071.7$$

(iii) Using model M2 for R, we simulated the portfolio distribution (10,000 simulations of the portfolio return) and estimated 1-day portfolio's value-at-risk and expected shortfall at 90%, 95% and 99% confidence levels using formulas 5 and 6:

$$VaR_{90\%} = 96740.9, ES_{90\%} = 134326.4$$
$$VaR_{95\%} = 124873.7, ES_{95\%} = 159125.4$$
$$VaR_{99\%} = 181933.3, ES_{99\%} = 208846.9$$

Using model M4 for R, we simulated the portfolio distribution (10,000 simulations of the portfolio return) and estimated 1-day portfolio's value-at-risk and expected shortfall at 90%, 95% and 99% confidence levels using formulas 5 and 6:

$$VaR_{90\%} = 102219.9, ES_{90\%} = 148679.1$$
$$VaR_{95\%} = 136802.3, ES_{95\%} = 179159.9$$
$$VaR_{99\%} = 207654.0, ES_{99\%} = 244623.5$$

(iv) We analyse the impact of the number of historical observations N on the estimated value-at-risk and expected shortfall using the last N = 100, 200, 500 and 1000 observations of 1-day returns, respectively. The estimated parameters are shown in the table below (for N=500 see section (i)):

Estimated parameter	N=100	N=200	N=1000
$\hat{\mu}$	$\begin{pmatrix}0.000664\\0.001250\end{pmatrix}$	$\begin{pmatrix}0.000612\\0.001018\end{pmatrix}$	$\begin{pmatrix}0.000543\\0.000594\end{pmatrix}$
$\hat{\Sigma}$	$\begin{pmatrix}0.000076 & 0.000029\\0.000029 & 0.000036\end{pmatrix}$	$\begin{pmatrix}0.000085 & 0.000029\\0.000029 & 0.000035\end{pmatrix}$	$\begin{pmatrix}0.000111 & 0.000077\\0.000077 & 0.000094\end{pmatrix}$
$\hat{\rho}$	0.417425	0.371775	0.743217

The estimated 1-day portfolio's value-at-risk and expected shortfall for models M1, M2 and M4 at 90%, 95% and 99% confidence levels are the following:

Model	Risk Measure	N=100	N=200	N=500	N=1000
M1	$VaR_{90\%}$	47414.33	50361.00	83624.85	83056.33
	$ES_{90\%}$	76232.44	74542.11	127727.6	128834.3
	$VaR_{95\%}$	71165.77	68676.23	114636.94	115326.05
	$ES_{95\%}$	94443.65	90141.89	158023.6	159465.8
	$VaR_{99\%}$	106070.53	104135.44	185697.30	185717.77
	$ES_{99\%}$	125262.1	119083	223071.7	225015

Model	Risk Measure	N=100	N=200	N=500	N=1000
M2	$VaR_{90\%}$	62497.00	66064.49	96740.9	90053.33
	$ES_{90\%}$	91692.86	98732.95	134326.4	128444.1
	$VaR_{95\%}$	85514.35	91075.56	124873.7	119264.54
	$ES_{95\%}$	110668.7	120426.3	159125.4	152607.9
	$VaR_{99\%}$	124496.51	141023.14	181933.3	173028.92
	$ES_{99\%}$	145615.0	163858.2	208846.9	203966.8
M4	$VaR_{90\%}$	64335.06	69101.95	102219.9	96705.72
	$ES_{90\%}$	98112.54	102196.1	148679.1	143099.4
	$VaR_{95\%}$	87417.76	93641.74	136802.3	130619.86
	$ES_{95\%}$	121854.1	124393.9	179159.9	174341.3
	$VaR_{99\%}$	142549.88	143357.62	207654.0	201276.23
	$ES_{99\%}$	170391.5	168154.6	244623.5	237320.4

(v) Using a rolling window of 200 observations we estimated the 1-day portfolio's value-at-risk at 95% confidence level, under models M2 and M4 and calculated the percentage of violations of value-at risk level. For model M2 the out-of-sample-portfolio profit/loss is below $VaR_{95\%}$ in 6.8% of observations, for model M4 the out-of-sample portfolio profit/loss is below $VaR_{95\%}$ in 5.8% of observations. Both models have percentage of violations close to the confidence level of 5%, which tells about the good quality of the models. Model M4 is more conservative than model M2 and gives a more precise estimation of $VaR_{95\%}$.