CS345 : Design and Analysis of Algorithms

Assignment 6

Ankur Kumar(14109), Tushar Vatsal(14766)

1 An atmospheric science experiment

1.1 Part 1

Construct a directed graph $G = (V, E)$ as follows:

- $V = A \cup B \cup \{s, t\}$
- A is set of m vertices(corresponding to ballons), each numbered from 1 to m.
- B is a set of n vertices(corresponding to conditions), each numbered from 1 to n.
- $E = E_1 \cup E_2 \cup E_3$ *where* $E_1 = \{(s, i)|\forall i \in A\}$, $E_2 = \{(j, t)|\forall j \in B\}$, $E_3 = \{(i, j)|\forall i \in A, \forall j \in S_i\}$
- Capacity of edge e, c(e)= $\begin{cases} 2 & \text{if } e \in E_1 \\ 1 & \text{if } e \in E_2 \\ \text{k} & \text{if } e \in E_3 \end{cases}$

This is the max-flow instance for the given problem.

Theorem 1. *There is a way to measure each condition by k different balloons, while each balloon only measures at most two conditions if and only if the maximum flow from s to t in G has value nk.*

Proof. **Part 1**

Suppose there is a way to measure each condition by k different balloons, while each balloon only measures at most two conditions. We shall construct a flow from s to t in G as follows.

- If balllon i measures condition j, send a flow of 1 unit along edge $(i, j) \in E_2$ *where* $i \in A$ *and* $j \in B$. Capacity constraint is satisfied for this edge as its capacity is 1.
- If balllon i measures condition j, send a flow of 1 unit along edge $(s, i) \in E_1$. Capacity constraint is satisfied for this edge as its capacity is 2 and ballon i can measure atmost two conditions. Also conservation constraint is also satisfied at vertex $i \in A$ as there is flow of 1 unit along as many edges entering and leaving it as the number of conditions it measures.
- If balllon i measures condition j, send a flow of 1 unit along edge $(j, t) \in E_3$. Conservation constraint is satisfied because for each of the k ballons that measure condition j, we send a flow of 1 unit along in-edge (i,j) and out-edge (j,t). Capacity constraint is satisfied for this edge as its capacity is k and number of out-edges is exactly k.
- This flow is a max-flow since value of this flow is nk(as each vertex $\in$ B has has $f_{out} = k$) and the capacity of cut $(\{s\} \cup A \cup B, \{t\})$ is nk.

Part 2

Suppose the maximum flow from s to t in G has value nk. We shall consider integral flow corresponding to the above maximum flow. Then assign conditions measured by ballon as follows.

- If there is a flow along edge $(i, j) \in E_2$, then ballon i shall measure condition j.
- Since the value of this flow is nk, each edge in E_3 will be carrying a flow of value k. This means k different edges will be carrying a flow of 1 unit(as capacity of such edge is 1) to each vertex in B. This means k different ballons will measure any condition j.
- Since each vertex $i \in A$ has only two out-edge, there can be flow along those two edges only. Since capacity of such edge is 1, this means each ballon i can measure atmost two conditions only.

□

1.2 Part 2

To meet the additional constraint, we add a set C containing 3*n vertices and some edges to G to construct new graph $G' = (V', E')$ as follows:

- $V' = V \cup C$
- C contains vertices marked as $c_{1i}, c_{2i}, c_{3i}\ \forall i\ such\ that\ 1 \leq i \leq n$.
- $E' = E_1 \cup E_3 \cup E_4$
- $E_4 = \{(i, c_{kj}), (c_{kj}, j) \mid \forall (i,j) \in E_3$ and ballon i is manufactured by contractor $k \in \{1,2,3\}\}$
- Capacity of edge e, c(e)= $\begin{cases} 2 & \text{if } e \in E_1 \\ \text{k} & \text{if } e \in E_3 \\ 1 & \text{if } e \in E_4\ and\ e = (i, c_{kj}) \\ k-1 & \text{if } e \in E_4\ and\ e = (c_{kj}, j) \end{cases}$

Theorem 2. *There is a way to measure each condition by k different balloons, while each balloon only measures at most two conditions and no condition is measured by all k ballons of same contractor if and only if the maximum flow from s to t in G has value nk.*

Proof. **Part 1**

Suppose there is a way to measure each condition by k different balloons, while each balloon only measures at most two conditions and no condition is measured by all k ballons of same contractor. We shall construct a flow from s to t in G as follows.

- If balllon i(of contractor p) measures condition j, send a flow of 1 unit along edge $(i, c_{pj})\ and\ (c_{pj}, j) \in E_4\ where\ i \in A\ and\ j \in B$. Capacity constraint is satisfied for this edge as its capacity is 1. Conservation constraint is satisfied also for each such flow through vertex c_{pj}, hence it is satisfied for the total flow also through c_{pj}.
- If balllon i(of contractor p) measures condition j, send a flow of 1 unit along edge $(s, i) \in E_1$. Capacity constraint is satisfied for this edge as its capacity is 2 and ballon i can measure atmost two conditions. Also conservation constraint is also satisfied at vertex $i \in A$ as there is flow of 1 unit along as many edges entering and leaving it as the number of conditions it measures.
- If balllon i(of contractor p measures condition j, send a flow of 1 unit along edge $(j, t) \in E_3$. Conservation constraint is satisfied because for each of the k ballons that measure condition j, we send a flow of 1 unit along in-edge (c_{pj}, j) and out-edge (j,t). Capacity constraint is satisfied for this edge as its capacity is k and number of out-edges is exactly k.
- This flow is a max-flow since value of this flow is nk(as each vertex $\in$ B has has $f_{out} = k$) and the capacity of cut $(\{s\} \cup A \cup B \cup C, \{t\})$ is nk.

Part 2

Suppose the maximum flow from s to t in G has value nk. We shall consider integral flow corresponding to the above maximum flow. Then assign conditions measured by ballon as follows.

- If there is a flow along edge $(i, c_{pj}) \in E_4$, then ballon i(of contractor p) will measure condition j.
- Since the value of this flow is nk, each edge in E_3 will be carrying a flow of value k. This means k different edges will be carrying a flow of 1 unit(as capacity of such edge is 1) to each vertex in B. Since there is one to one correspondence betwen in-edge and out-edge of vertices in C, this means k different ballons will measure any condition j. Also since each condition is connected to only three vertices(which denotes three contractors) in C with edge of capacity $(k-1)$, this means any condition can have at most $k-1$ measures from a single contractor.
- Since each vertex $i \in A$ has only two out-edge, there can be flow along those two edges only. Since capacity of such edge is 1, this means each ballon i can measure atmost two conditions only.
- Hence there is a way to measure each condition by k different balloons, while each balloon only measures at most two conditions and no condition is measured by all k ballons of same contractor.

□

2 A Social Networking Problem

2.1 Notations

- $F_{ij} = F_{ji} = 1$ if $(i,j) \in E$ otherwise 0
- F_i = number of friends of i in V
- $q(A) = 2(|E(A)| - 10|A|)$ where $A \subset V$ or $A = V$ and $E(A)$ is same as defined in the problem

2.2 Towards solving the problem

We have to maximize this $q(A)$ and just have to check if $q(A)_{max} \geq 0$

Expression of $q(A)$:
Let

$$F(A, \overline{A}) = \sum_{(i,j) \in E and (i \in A, j \in \overline{A})} F_{ij}$$

$q(A) = 2(|E(A)| - 10|A|) = 2(\frac{\sum_{i \in A} F_i}{2} - F(A, \overline{A}) - 10|A|)$

$= 2(\frac{\sum_{i \in V} F_i - \sum_{i \in \overline{A}} F_i}{2} - F(A, \overline{A}) - 10|A|)$

$= \sum_{i \in V} F_i - (\sum_{i \in \overline{A}} F_i + 2F(A, \overline{A}) + 20|A|)$

$= \sum_{i \in V} F_i - q'(A)$ where q'(A)=$\sum_{i \in \overline{A}} F_i + 2F(A, \overline{A}) + 20|A|$

Since $\sum_{i \in V} F_i$ is a constant so to maximize $q(A)$ we have to minimize $q'(A)$
Now there is a proposed graph $G' = (V', E')$ whose $c(A, \overline{A}) = q'(A)$
Let $V' = V \cup \{s, t\}$, s and t are two new vertices introduced as source and sink in network G'

E' is as follows :

- From s to a vertex $i \in V$ construct directed edge (s, i) with capacity,$c(s, i) = F_i \ \forall i \in V$
- From any vertex $i \in V$ to t construct directed edge (i, t) with capacity,$c(i, t) = 20 \ \forall i \in V$
- For each edge $(i, j) \in E$ construct two directed edges (i, j) and (j, i) in E' with capacity, $c(i, j) = c(j, i) = 2$

This network G' ensures $q'(A) = c(A, \overline{A})$

If $c(A, \overline{A})_{min} \leq \sum_{i \in V} F_i$ then there exists a subset X such that $|E(X)| \geq |X|$ otherwise not.
Note : This $c(A, \overline{A})_{min}$ is nothing but the maximum possible flow from s to t in network G' which can be achieved by running Ford Fulkerson on G' from s to t

2.3 Proof

To prove : $q'(A) = c(A, \overline{A})$ for any $A \subset V' - t$ or $A = V' - t$ such that $s \in A$

Try to compute $c(A, \overline{A})$ in the network $G' = (V', E')$:

The edges leaving from A and entering $\overline{A}$ are listed below :

- All edges (s, i) where $i \in \overline{A}$
- All edges (i, t) where $i \in A$
- All edges (i, j) such that i and j are friends where $i \in A$ and $j \in \overline{A}$

Thus $c(A, \overline{A})$ is nothing but the sum of capacities of these edges listed above

$$\Rightarrow c(A, \overline{A}) = \sum_{i \in \overline{A}} c(s, i) + \sum_{i \in A} c(i, t) + \sum_{i \in A, j \in \overline{A}} c(i, j)$$

$$\Rightarrow c(A, \overline{A}) = \sum_{i \in \overline{A}} F_i + \sum_{i \in A} 20 + \sum_{i \in A, j \in \overline{A}} 2 = \sum_{i \in \overline{A}} F_i + 20|A| + F(A, \overline{A}) = q'(A)$$