SOIL MECHANICS

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Unit 9 - Consolidation

May 28th, 2025

Day	08:00-09:30	09:45-11:15	13:00-14:30	14:45-16:15
19/05/25	Introduction	Programming	Phase Rel.	Tutorial
20/05/25	Classification	Tutorial	LAB	LAB
21/05/25	Compaction	Tutorial		
22/05/25	Groundwater	Tutorial	LAB	LAB
23/05/25	Groundwater	Tutorial		
26/05/25	Effective Str.	Tutorial	Stress Incr.	Tutorial
27/05/25	Compressib.	Tutorial	LAB	LAB
28/05/25	Consolidation	Tutorial		
29/05/25	Shear Str.	Tutorial	Shear.Str.	Tutorial
30/05/25	Shear Str.	Review		

So far we have considered the magnitude of consolidation settlement, that is independent of time.

We have seen how the determination of an appropriate compressibility coefficient from laboratory tests can be used to predict the settlement occurring for a given soil under given stress increment.

In this part of the lecture we consider the time dimension of consolidation settlement because in fine grained soils, the completion of the predicted settlement can take many, possibly tens, of years.

In soil mechanics, the process of volume reduction associated with the expulsion of water from the voids is referred to as *CONSOLIDATION.*

Note the differences in contrast to other similar concepts you have studied so far...

CONSOLIDATION: Volume reduction associated with the expulsion of water and dissipation of pore water pressures *with time*.
COMPRESSION: Volume reduction associated with changes in effective stress.
COMPACTION: Volume reduction associated with expulsion of air.

Contents

Time is the key thing here…

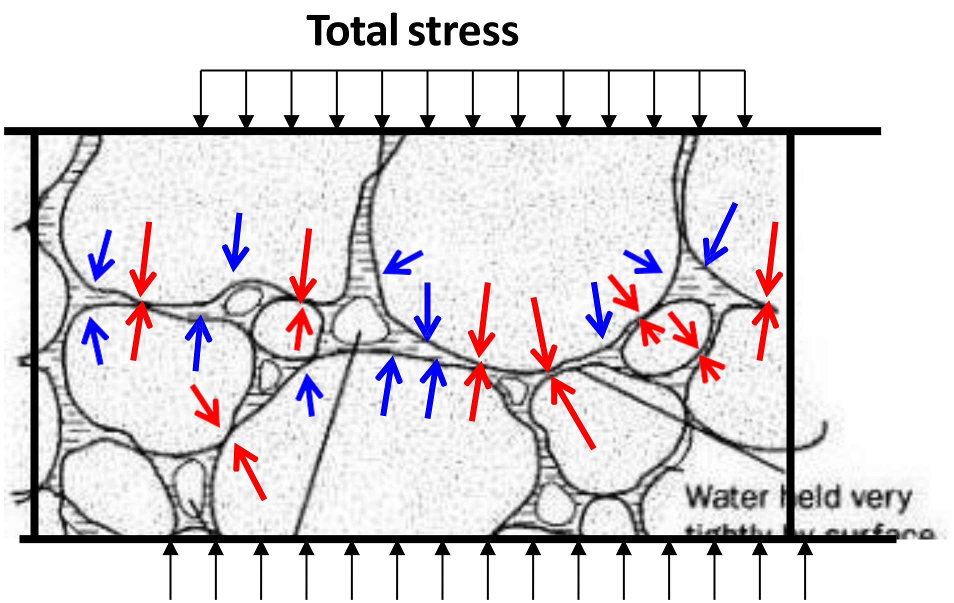
After this lecture you will be able to predict how long does it take a certain soil to settle.

If you remember that time = money, then you can apprecciate very easily how this lecture might be very relevant to you in the future!

Why is this lecture important?

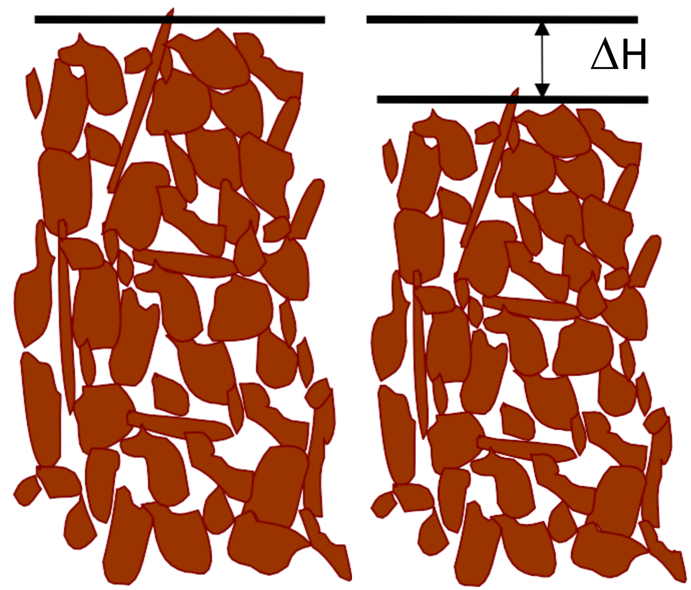
Remember these …

what would happen if the water was trapped in the pores?

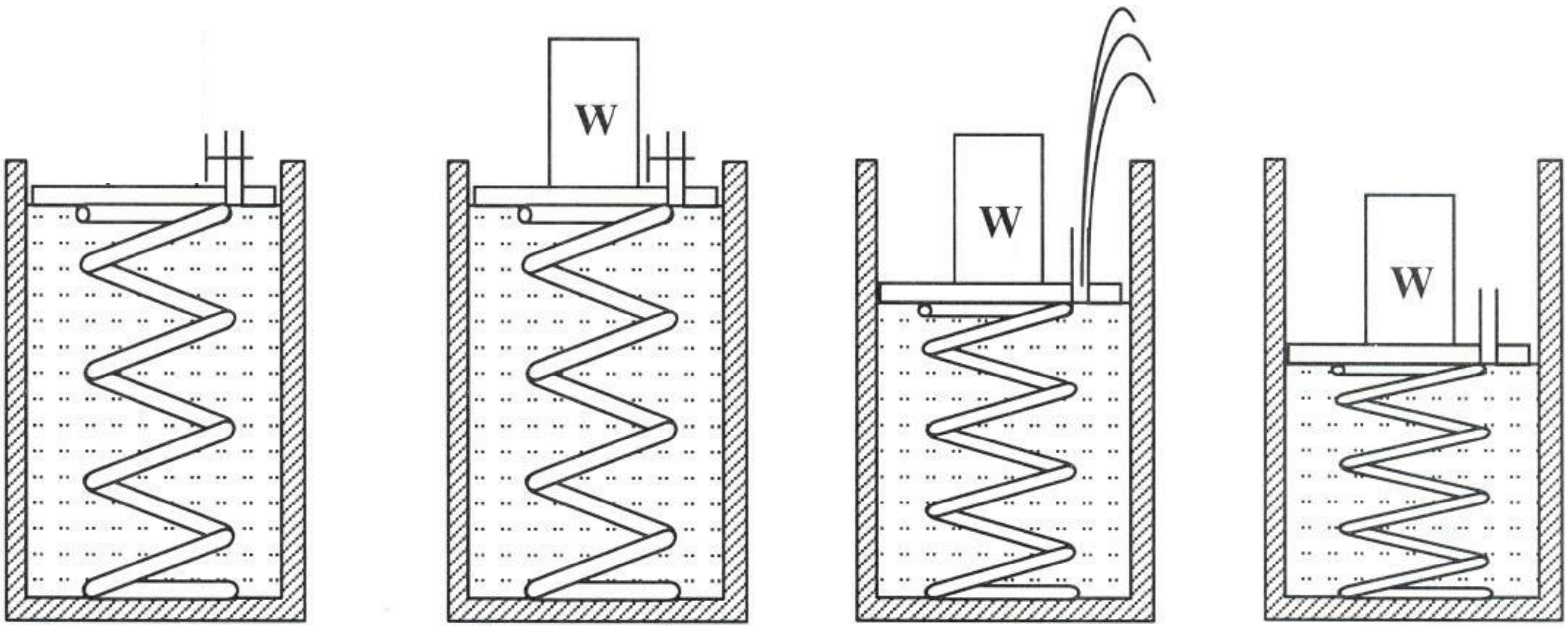


$$\sigma = \sigma' + u$$

Effective stress + Pore water pressure

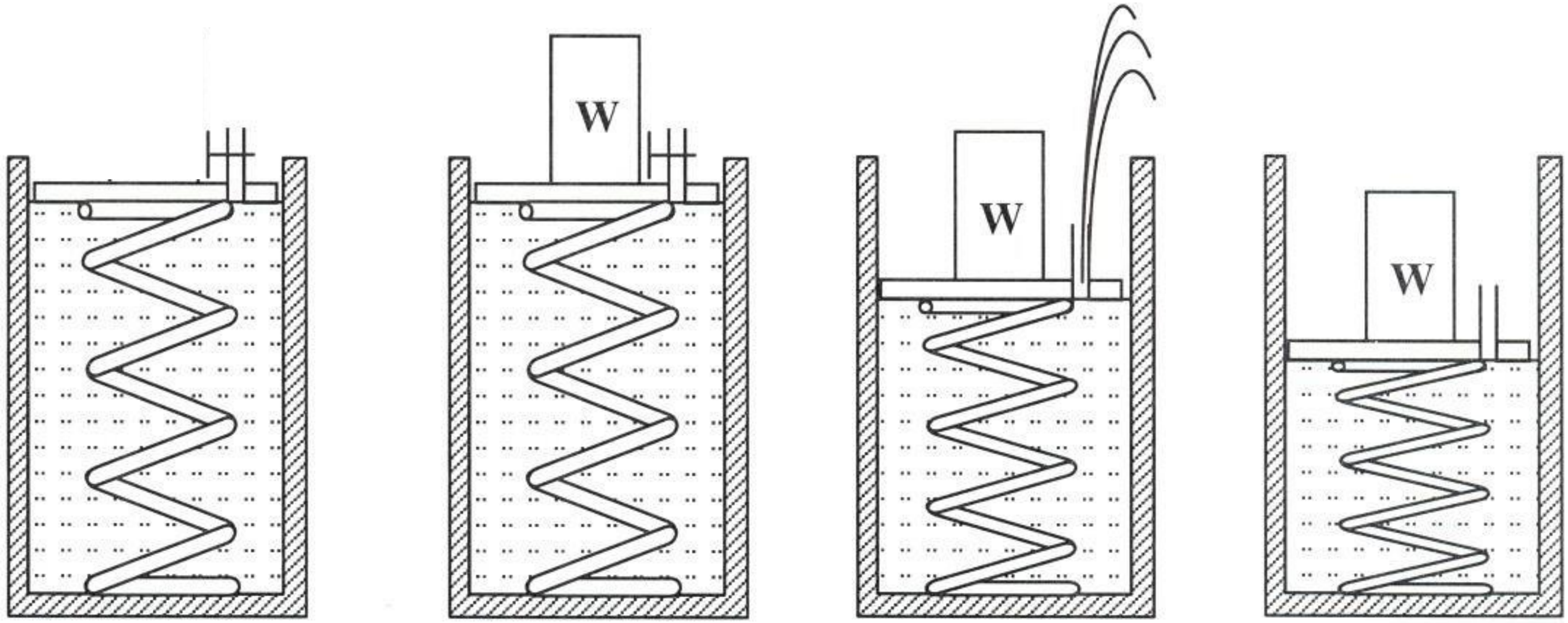


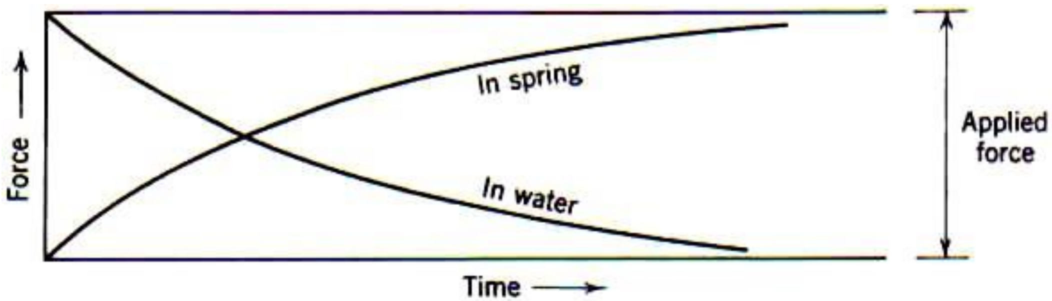
$W = \Delta\sigma$



Consolidation – the physical analogue

$W = \Delta\sigma$





Consolidation – the physical analogue

- Settlement = Piston movement
- Settlement depends on how much water has been squeezed out of the cylinder (voids in the soil)
- How much water has squeezed (change in void ratio) is proportional to the amount of excess pore water pressure that has dissipated
- Rate of settlement is related to rate of excess pore water pressure dissipation

THAT IS CONSOLIDATION!

Now we need an equation or theory that predicts the pore pressure and void ratio at any point in time and space of a soil layer.

Consolidation – the physical analogue

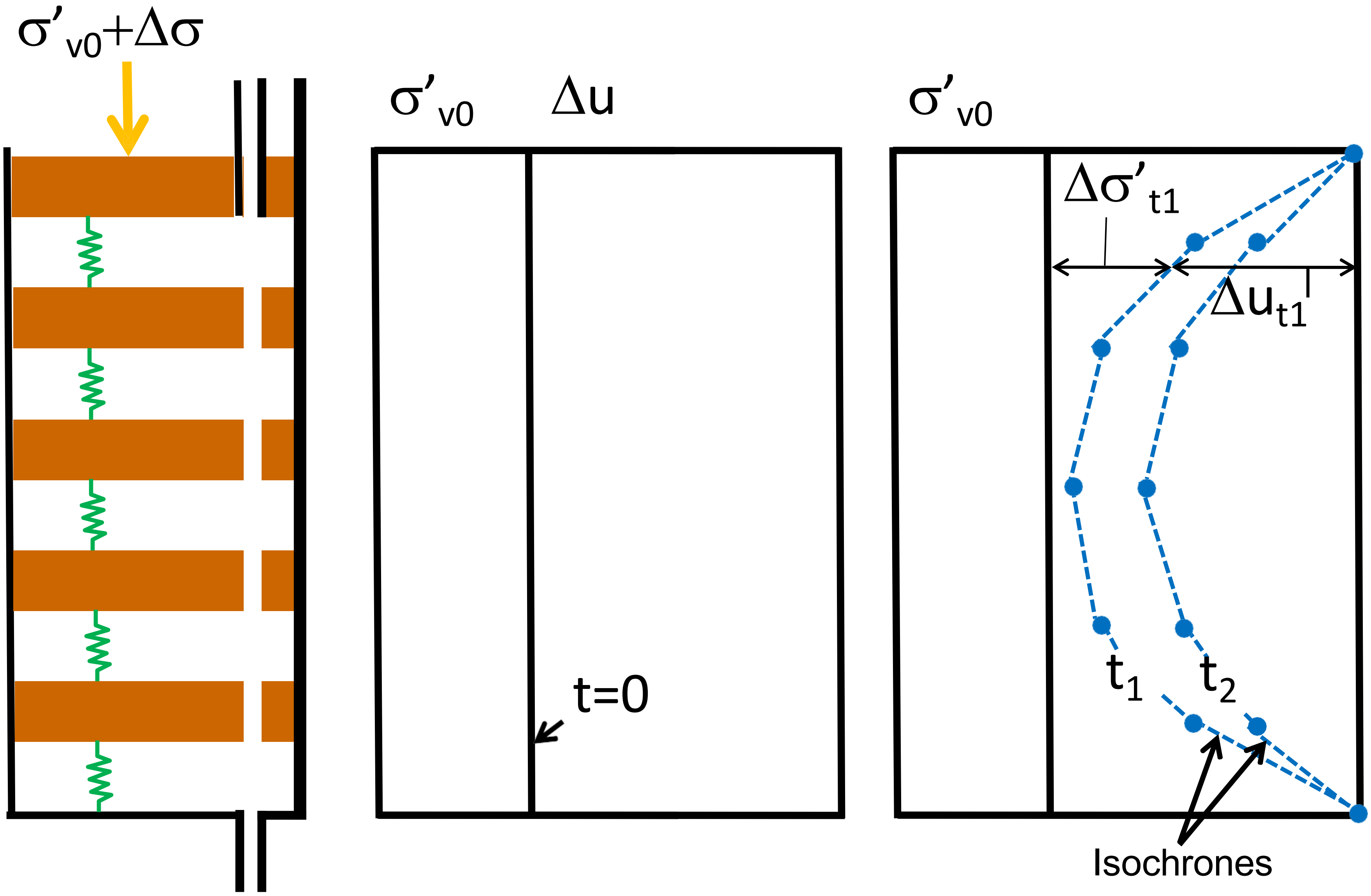
Consolidation and time

The void ratio:load curve is the result of a number of individual loading stages, each lasting 24 hours, during which time there is consolidation (or primary) settlement

At the moment a load increment is applied to a saturated fine-grained soil, it is carried entirely by the pore water as an ***EXCESS*** pore water pressure.

Gradually, excess pore water pressures dissipate and the soil consolidates (strict definition). In so doing, the additional load is transferred to the soil skeleton as an effective (vertical) stress.

The process continues until pore water pressures revert to their equilibrium state.



Consolidation – a more realistic physical analogue

ASSUMPTIONS

- Compressible layer is homogeneous and completely saturated
- Soil grains and water are completely incompressible
- Water flow is governed by Darcy's law

$$v = ki = k\frac{dh}{dl}$$

- Drainage and compression are one-dimensional
- Small strain theory (i.e. k and m_v remain constant during consolidation)
- There is a unique relationship between Δe and $\Delta\sigma'$

Consolidation – Terzaghi's 1D theory assumptions

The instantaneous response to loading is an equivalent increase in pore water pressure. Drainage then commences and in a saturated soil, there is a simultaneous reduction in void ratio.

1	Drainage – Darcy's law	$v = -k\frac{dh_w}{dz} = -\frac{k}{\gamma_w}\frac{du}{dz}$
2	From continuity (mass balance)	$\frac{\partial V_T}{\partial t} = -\frac{\partial v}{\partial z}\delta z\, A_z = -\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}\delta x \delta y \delta z$
3	Compression – m_v	$dV_T = m_v d\sigma'_v V_{T0}$
4	Total stress constant	$d\sigma = d\sigma' + du = 0 \qquad d\sigma' = -du$
5	Rate of compression	$\frac{\partial V_T}{\partial t} = -m_v \frac{\partial u}{\partial t} V_{T0}$
6	Combining 2 & 5	$-m_v \frac{\partial u}{\partial t} V_{T0} = -\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}\delta_x \delta_y \delta_z$

Consolidation – Terzaghi theory

Towards a coefficient of consolidation

The instantaneous response to the loading is an equivalent increase in pore water pressure. Drainage then commences and in a saturated soil there is a reduction in void ratio.

Combining 2 & 5

$$-m_v \frac{\partial u}{\partial t} V_{T0} = -\frac{k}{\gamma_w} \frac{\partial^2 u}{\partial z^2} \delta_x \delta_y \delta_z$$

$$\frac{\partial u}{\partial t} = \frac{k}{\gamma_w m_v} \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$$

From which we define the coefficient of consolidation,

$$c_v = \frac{k}{\gamma_w m_v}$$

Consolidation – the theory

Terzaghi derived an equation describing one-dimensional consolidation in which the rate of settlement in a soil is characterised by a coefficient of consolidation, denoted c_v.

This coefficient is related to a number of soil properties:

$$c_v = \frac{k}{\gamma_w m_v}$$

Coefficient of consolidation	c_v	[m²/yr]
Permeability	k	[m/yr (usually m/s)]
Unit weight of water	γ_w	[kN/m³]
Volume compressibility	m_v	[m²/kN]

Coefficient of consolidation – c_v

$$\frac{\delta u}{\delta t} = c_v \frac{\delta^2 u}{\delta z^2}$$

The theory gives the rate of dissipation of pore water pressure for any depth within the compressible soil layer at any time.

We have already seen that the rate of dissipation of pore water pressures is related to the rate of settlement

So the only problem left is…
How do we solve this equation?

Consolidation – Terzaghi's 1D theory

- Harr (1966) provides an approximate solution using the finite difference method.
- Solution of the equation using finite element methods (i.e. Plaxis) is also possible
- Taylor (1948) gives a mathematically rigorous solution in terms of a Fourier series expansion. This is the solution we will use in these course.

Consolidation – solution

$$\frac{\delta u}{\delta t} = c_v \frac{\delta^2 u}{\delta z^2}$$

Any differential equation requires intial boundary conditions if a unique solution is needed, then...

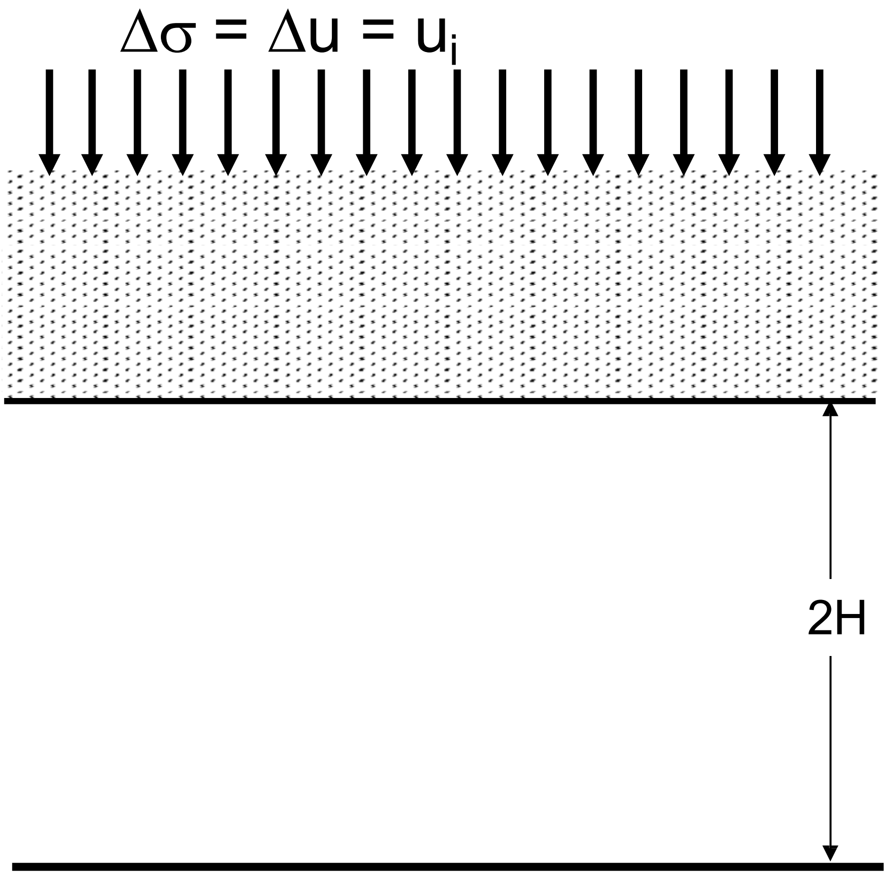
INITIAL BOUNDARY CONDITIONS:

1. There is complete drainage at the top and bottom of the compressible layer
2. The initial excess pore water pressure $\Delta u = u_i$ is equal to the applied increment of stress at the boundary, $\Delta\sigma$.

Consolidation – solution

Consolidation – solution

Initial boundary conditions... A graphical view!



When $\left.\begin{array}{l} z = 0 \\ z = 2H \end{array}\right\} u = 0$

When $t = 0 \;\}\; \Delta u = u_i = \Delta\sigma$

When $t = \infty \;\}\; \Delta u = 0$

$$u = \Delta\sigma \sum_{n=0}^{\infty} f_1(Z) f_2(T)$$

T = Time factor
Z = Geometry parameter
H = Longest drainage path
t = Time
z = Depth
c_v = Coeff. of consolidation

$$Z = z / H$$

$$T = c_v \frac{t}{H^2}$$

Consolidation – Taylor's solution

Application of the consolidation theory & coefficient

So know we know.... The consolidation equation is a second order partial differential equation. It can be solved analytically in terms of *u*, but most easily …

… by defining a degree of consolidation U,

$$U = \frac{e_0 - e}{e_0 - e_1}$$

e = void ratio of interest

The degree of consolidation is a dimensionless measure of the amount of settlement that occurs as a proportion of the total amount of settlement that is predicted to occur.

The degree of consolidation may be defined in terms of displacement, void ratio or excess pore water pressure, each of which vary linearly during the consolidation process. When defined as a proportionate change …

Consolidation – solution

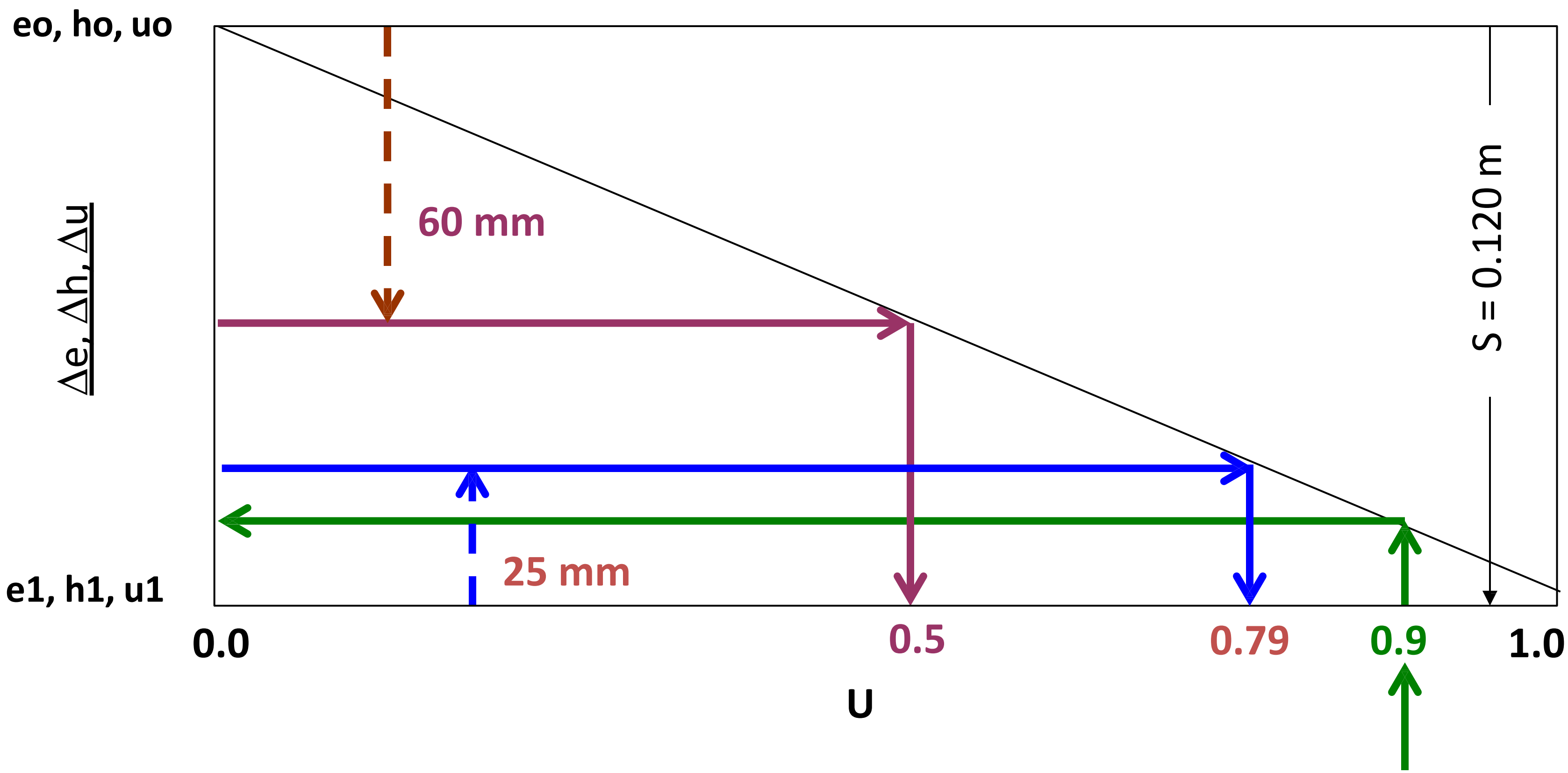
$$U = \frac{e_0 - e}{e_0 - e_1} = \frac{h_0 - h}{h_0 - h_1} = \frac{S}{S_{tot}} = \frac{u_0 - u}{u_0 - u_1} \qquad 0.0 \le U \le 1.0$$

For 120 mm total settlement, when is … ?

A) 90% consolidation [U = 0.9; 108 mm] achieved

B) settlement 60 mm [U = 0.5]

C) remaining settlement less than 25 mm [U = 0.79]



Application of the consolidation theory & coefficient

The consolidation equation is a second order partial differential equation.
It can be solved analytically but most easily …

… by defining a degree of consolidation U,

$$U = \frac{e_0 - e}{e_0 - e_1}$$

e = void ratio of interest

… and a dimensionless time factor T_v,

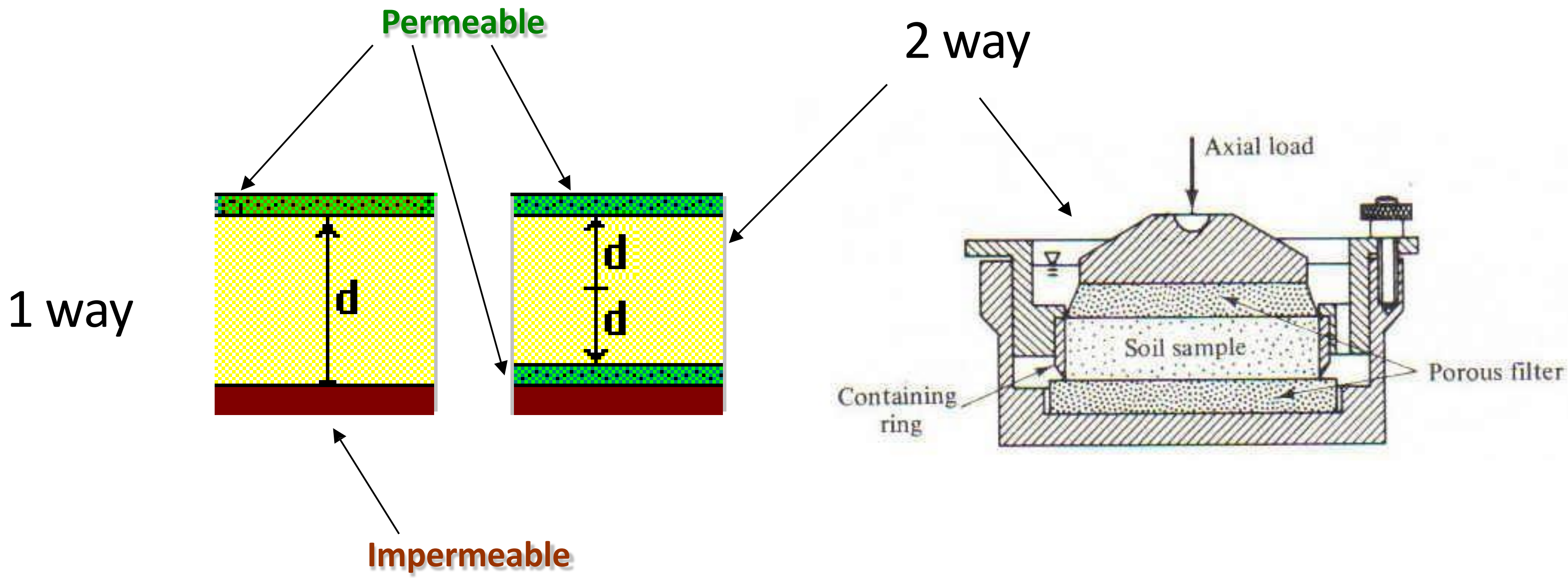
$$T_v = \frac{c_v t}{d^2}$$

t = time of interest
d = drainage path length

It is important to recognise and correctly account for the drainage conditions (i.e. Initial boundary conditions can change).

Drainage path length depends on permeability of soil or rock adjacent to, i.e. over- and under-lying, the compressible layer.

It is maximum distance any particle of water must travel to escape the compressible layer



Drainage path length – 2 way or 1 way?

Application of the consolidation theory & coefficient

The consolidation equation is a second order partial differential equation.
It can be solved analytically but most easily …

… by defining a degree of consolidation U,

$$U = \frac{e_0 - e}{e_0 - e_1}$$

e = void ratio of interest

… and a dimensionless time factor T_v,

$$T_v = \frac{c_v t}{d^2}$$

t = time of interest
d = drainage path length

A solution to the consolidation equation is then obtained using charts (from equations based on parabolic isochrones).

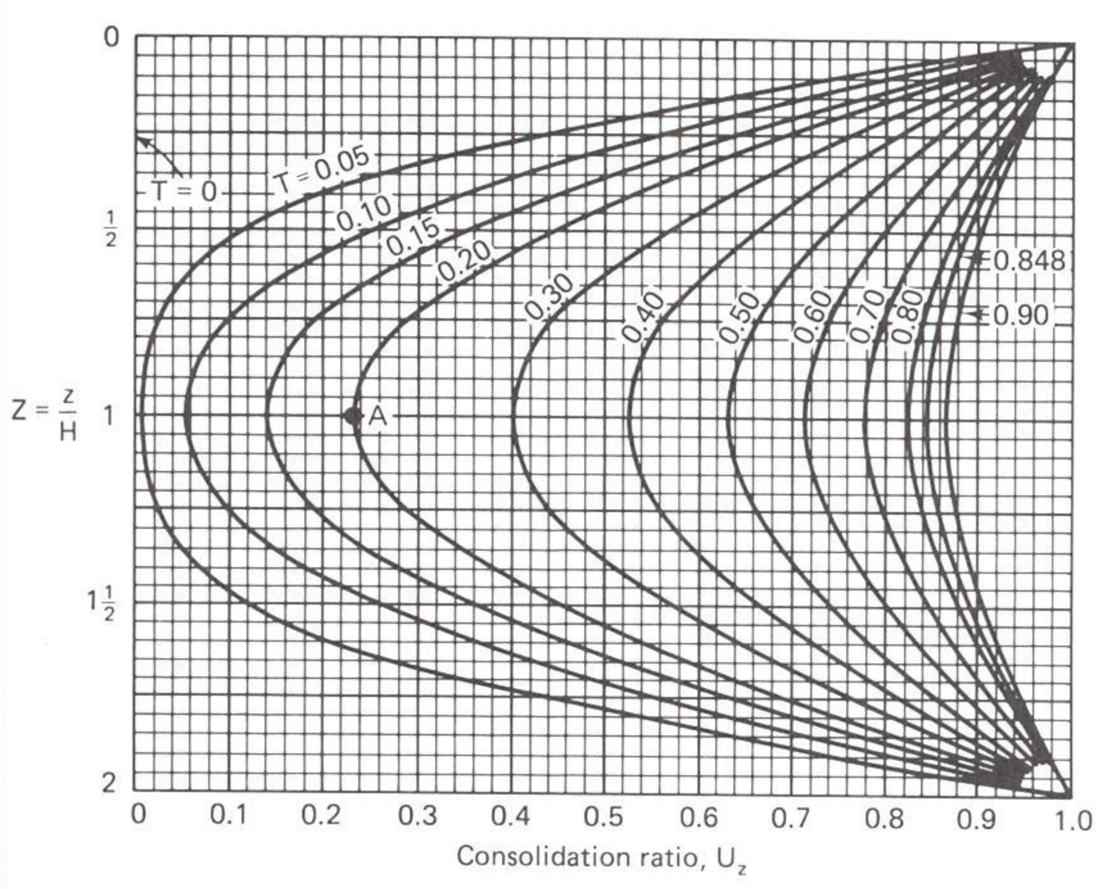
Consolidation – solution

We can then express the degree of consolidation in terms of a similar Fourier series expansion but with *T* (time factor) and *Z* as independent variables.

$$U_z = 1 - \sum_{n=0}^{\infty} f_1(Z) f_2(T)$$

Consolidation – solution

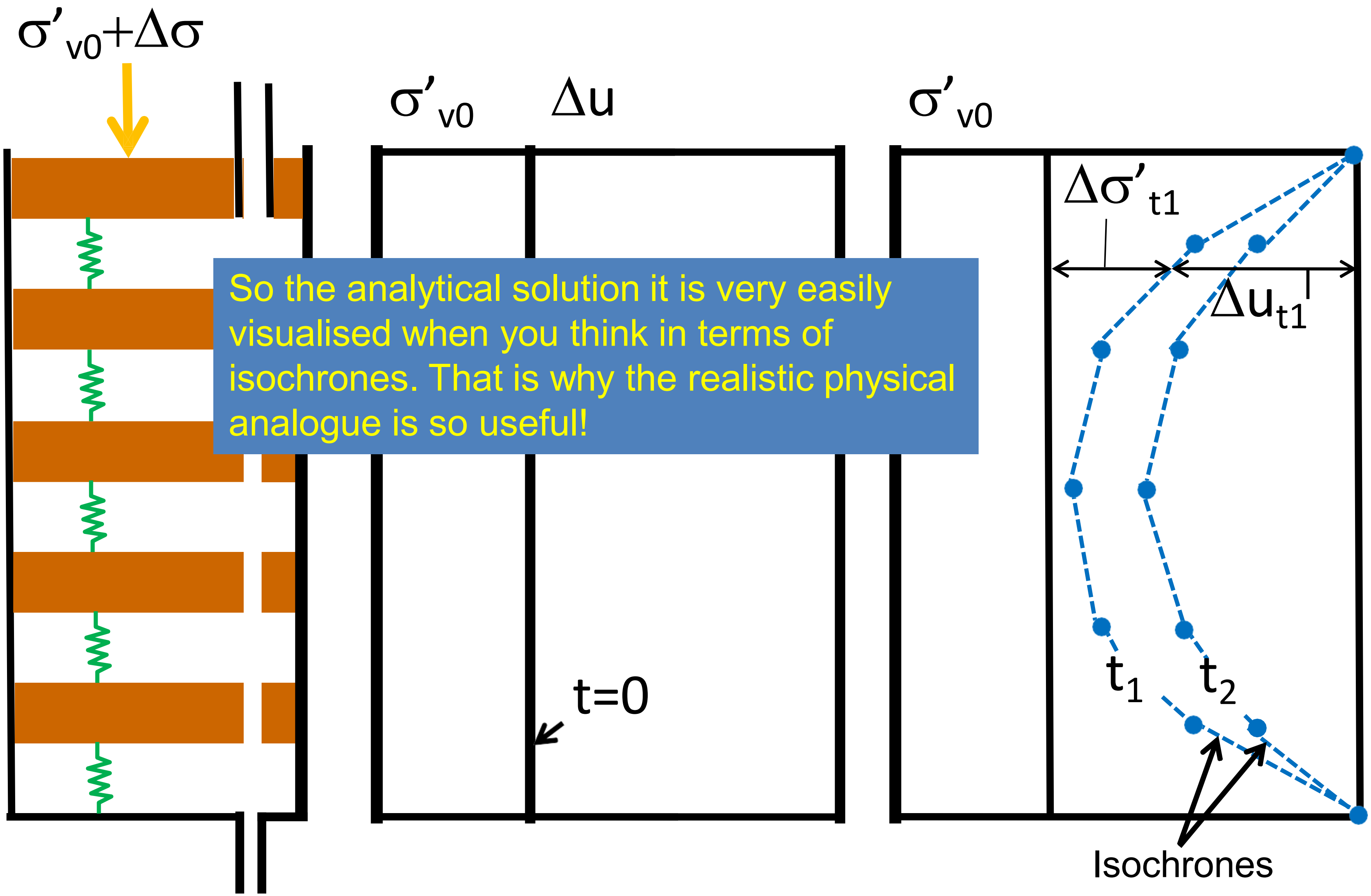
$$U_z = 1 - \sum_{n=0}^{\infty} f_1(Z) f_2(T)$$



NOTE

The degree of consolidation represents conditions at a point in the consolidating layer.

Consolidation – solution



So the analytical solution it is very easily visualised when you think in terms of isochrones. That is why the realistic physical analogue is so useful!

Consolidation – solution

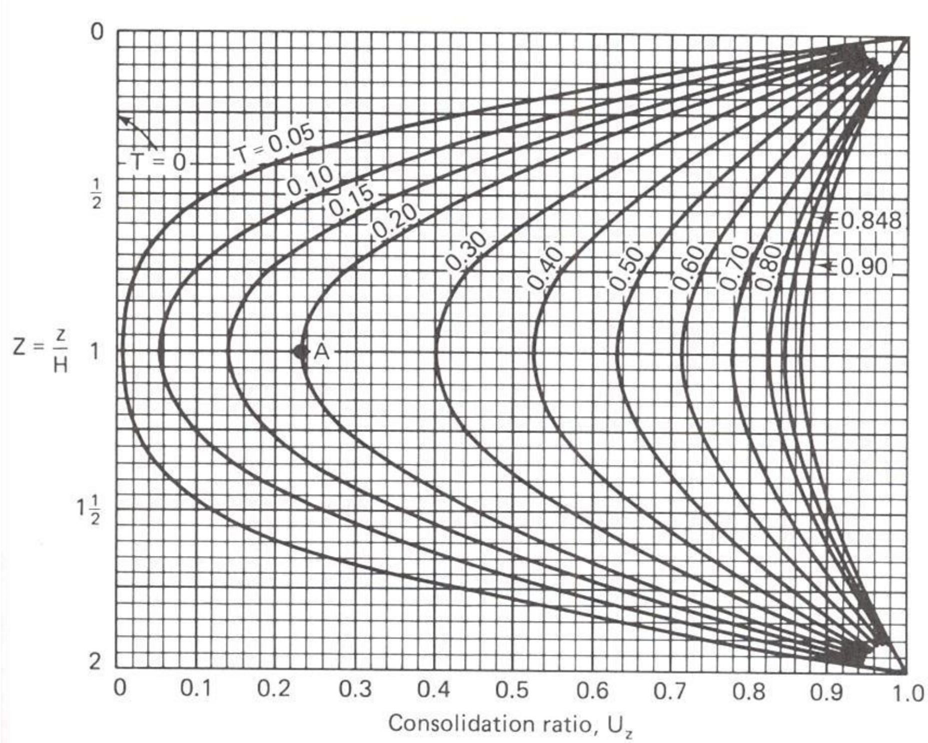
A 12 m thick layer of London clay with double drainage. The coefficient of consolidation is $c_v = 2.1\times10^{-8}$ m²/s . Find the degree of consolidation for the clays 5 years after loading at depths of 0, 3, 6, 9 and 12 m.

Consolidation – example

Compute the time factor (T):

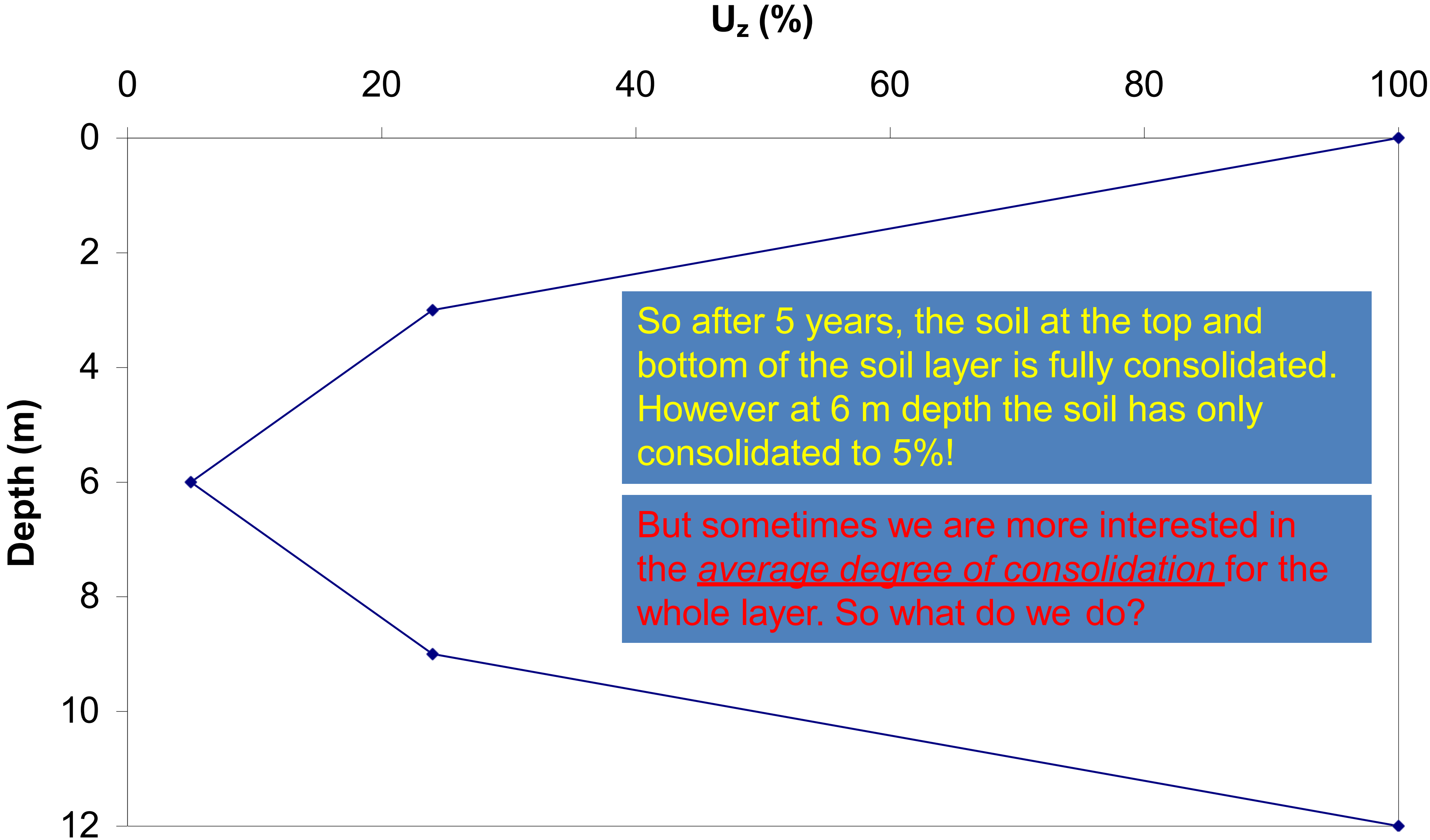
$$T = \frac{c_v t}{H^2} = \frac{2.1\times10^{-8}(5)(365)(24)(60)(60)}{6^2} = 0.092$$

From the chart…



Depth (m)	z/H	U_z
0	0.0	100%
3	0.5	24%
6	1.0	5%
9	1.5	24%
12	2.0	100%

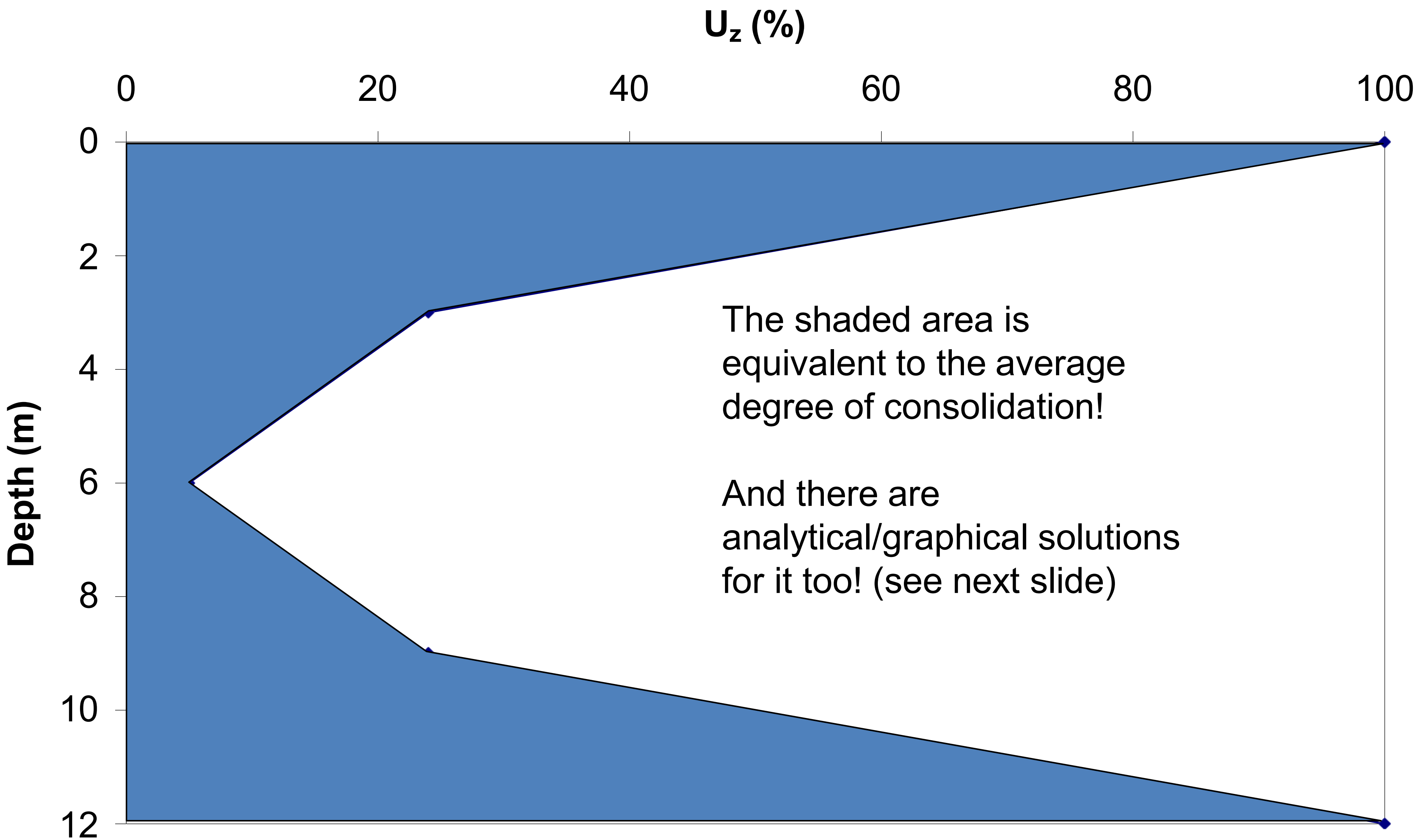
Consolidation – solution



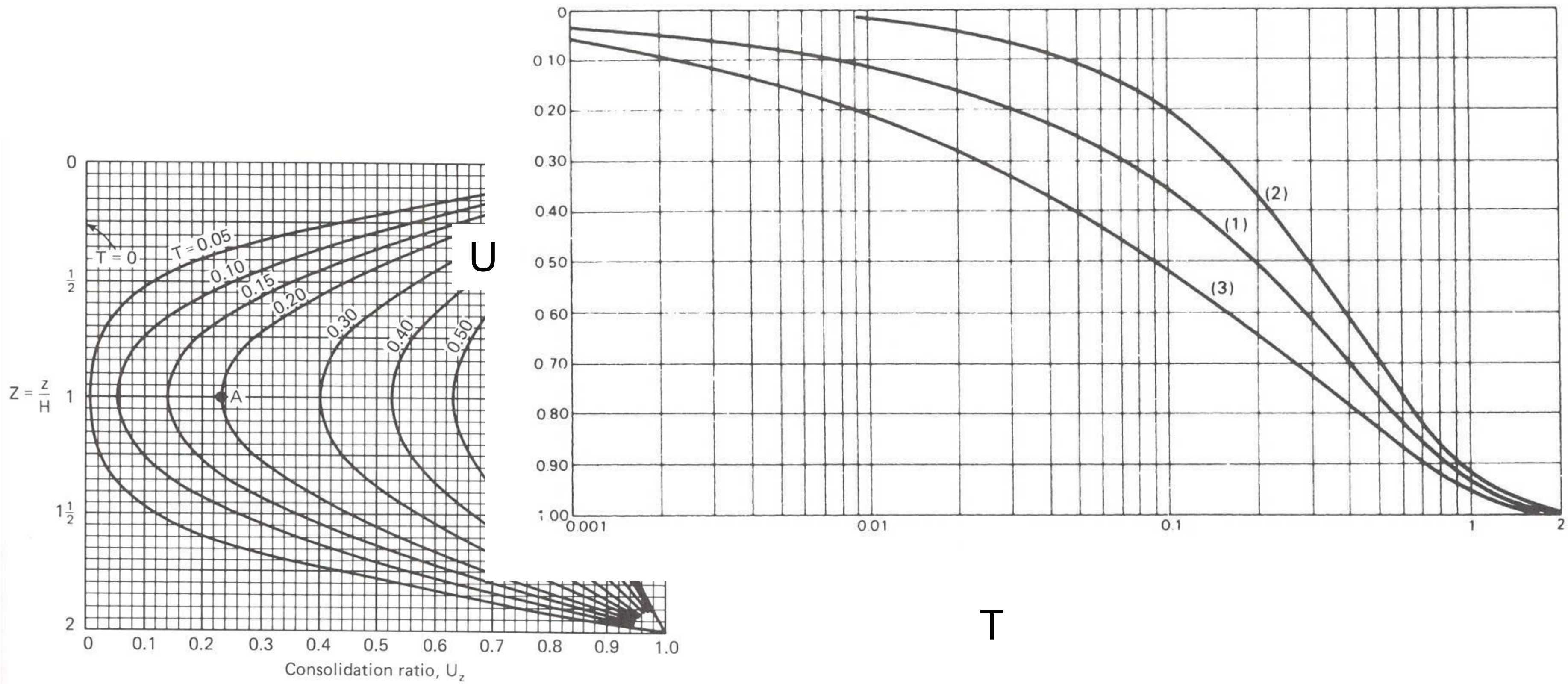
So after 5 years, the soil at the top and bottom of the soil layer is fully consolidated. However at 6 m depth the soil has only consolidated to 5%!

But sometimes we are more interested in the *average degree of consolidation* for the whole layer. So what do we do?

Consolidation – solution



Consolidation – Average degree of consolidation (U)



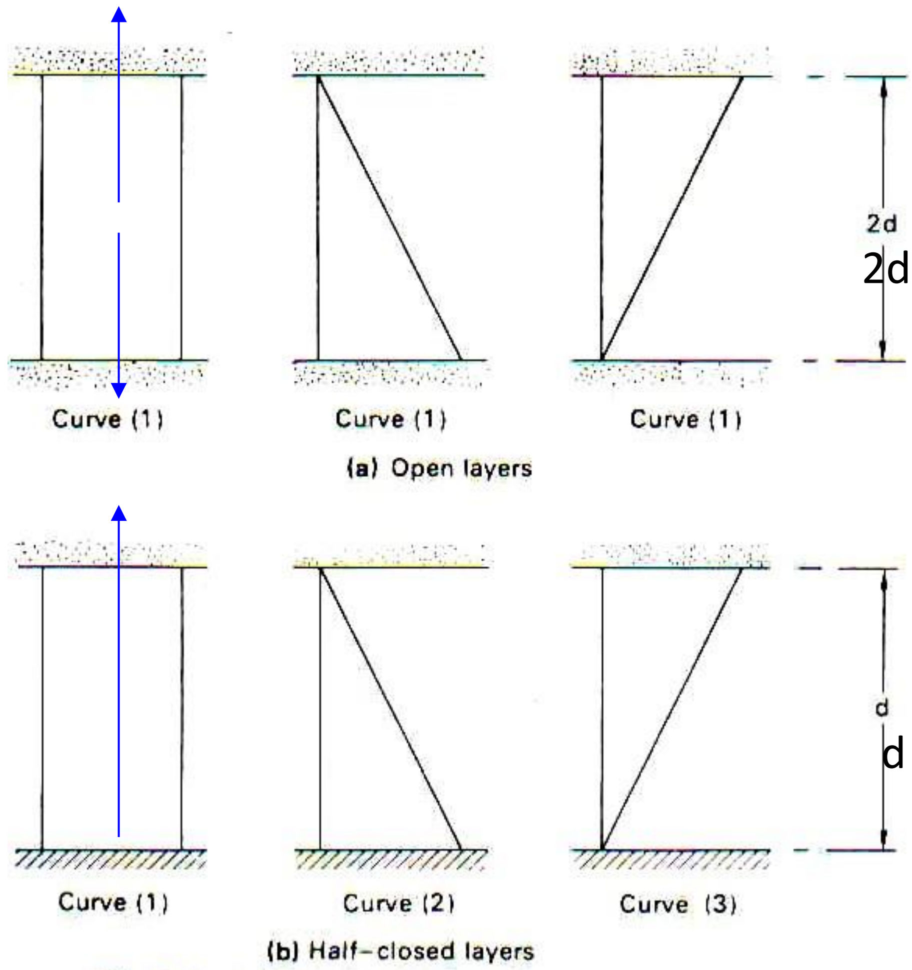
For the example T = 0.092, then from the chart U = 35%

Consolidation – Average degree of consolidation (U) (For the same example)

Application of consolidation theory

Open layers
All curve 1

Half-closed layers
Curves 1, 2, 3



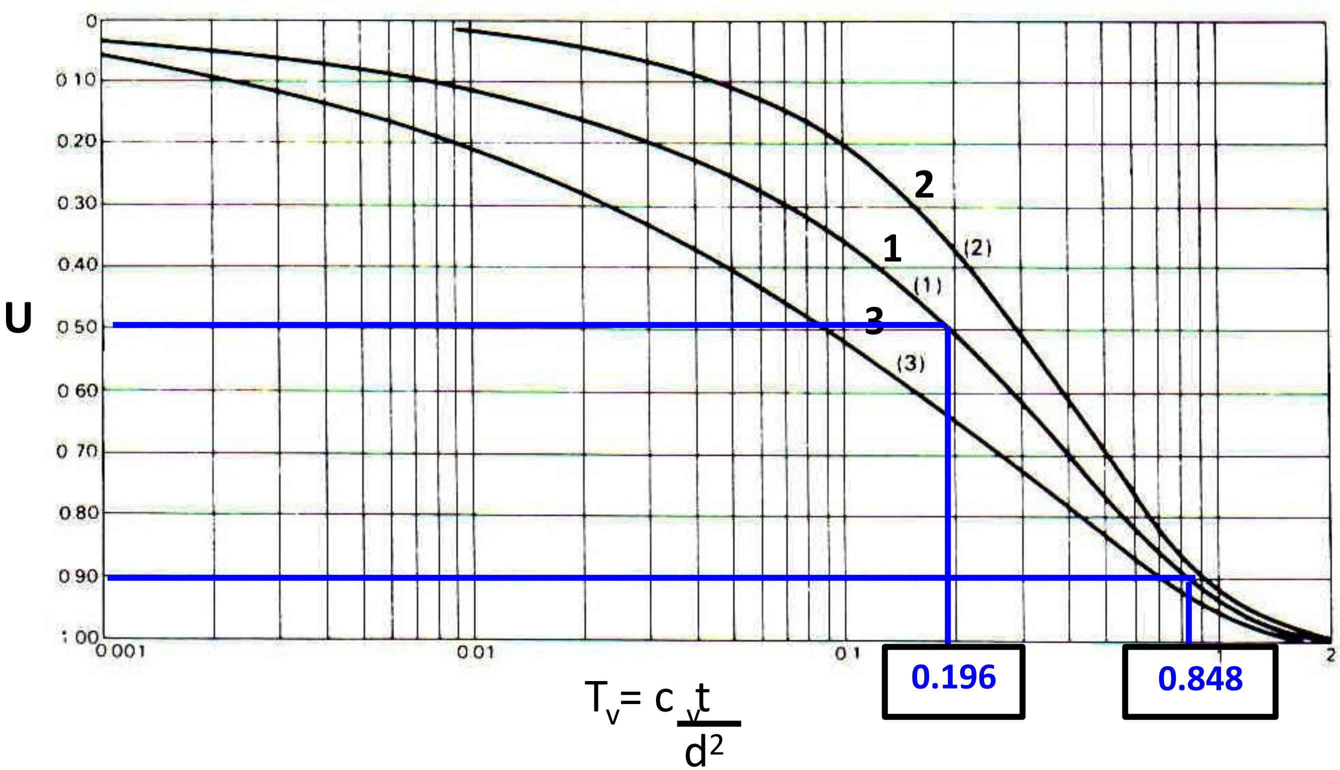
Extensive loading (Embankment)

Self-weight

Point load

Application of the coefficient of consolidation

There is a relationship between degree of consolidation and the dimensionless time factor. Note that these solutions are presented in terms of U (not U_z!)



In oedometer, i.e. two way drainage, degrees of consolidation:

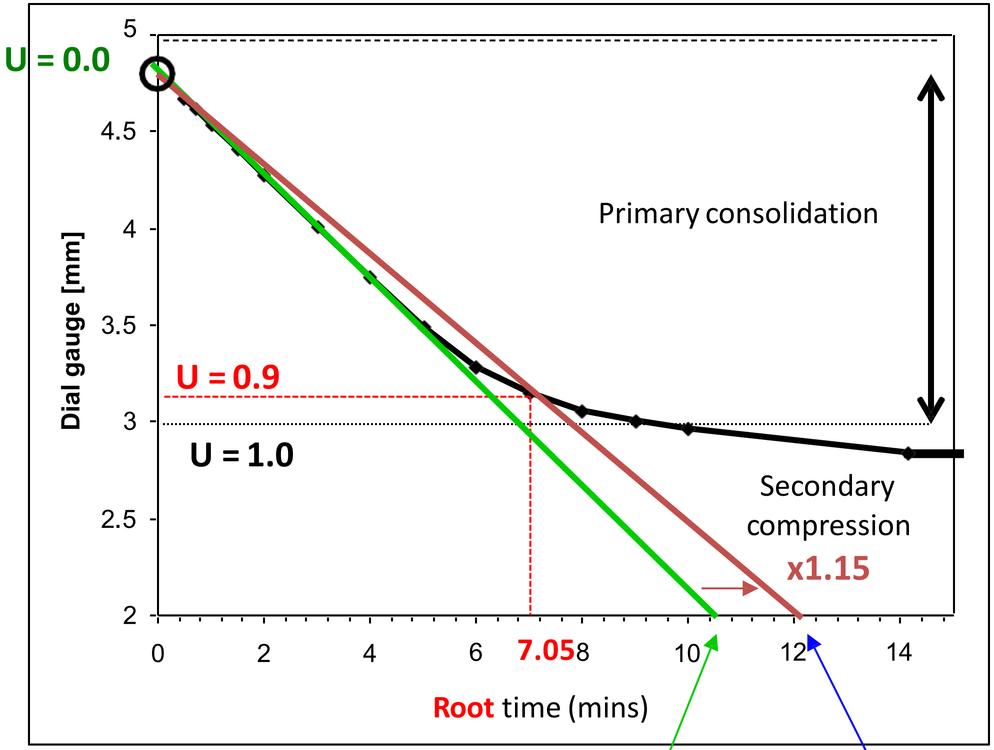
0.9 (90%) $T_V = 0.848$

0.5 (50%) $T_V = 0.196$

Consolidation - solution

Laboratory measurement – a) root time method (Taylor's)

Find t_{90}



10.5 x 1.15 = 12.1

1) Fix U = 0.0: produce back early linear portion of curve to intersect vertical axis.

2) Fix U = 0.9: construct a line having abscissae values 1.15 times that of experimental curve.

Locate intersection of this line with experimental curve.

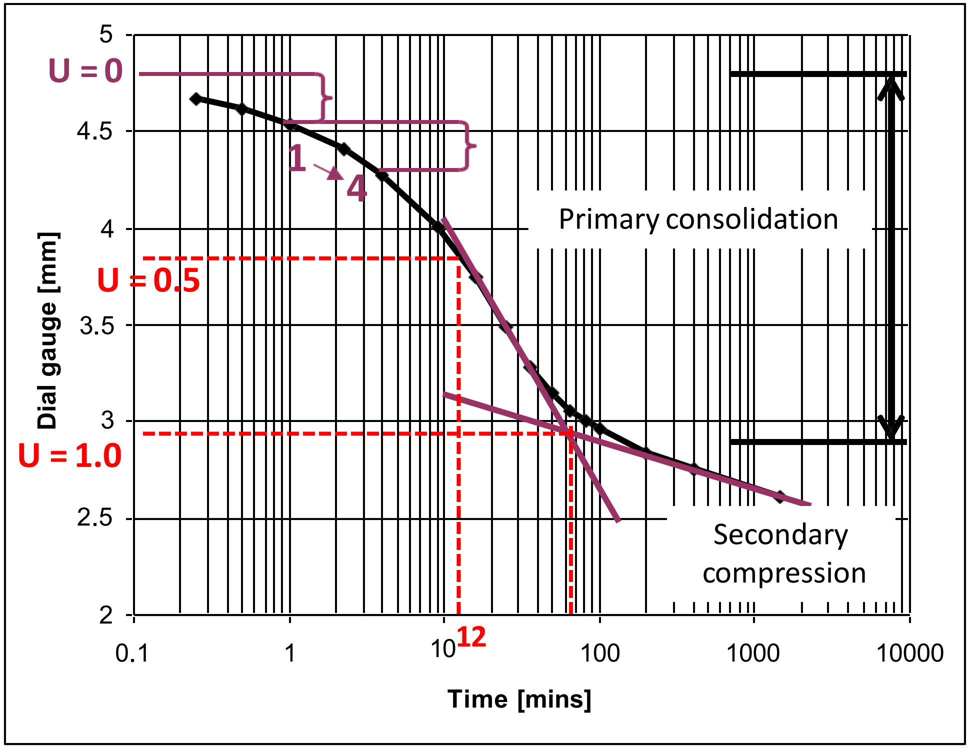
Calculate $t_{U=0.90}$ then

$$Cv = \frac{0.848\, d^2}{t_{90}}$$

First determine your consolidation coefficient …

Laboratory measurement – a) log time method (Casagrande)

Find t_{50}



1) Fix U=0.0: from initial part of curve, set off dial gauge readings for two values of t in ratio of 4:1…

… above earlier reading

2) Fix U=1.0: locate intersection of the two linear portions of the curve. Then determine U=0.5

Calculate $t_{U = 0.50}$ then

$$Cv = \frac{0.196\, d^2}{t_{50}}$$

… preferably twice!

Laboratory measurements: a comparison

Data in previous slides is identical.
Average sample thickness during loading stage was 14.80mm, therefore drainage path length = 7.40mm

Using Taylor's root time method

$$C_v = \frac{0.848\, d^2}{t_{90}} = \frac{0.848 \times 7.40^2}{7.05^2}$$

$$= 0.93\ [mm^2/min]$$

$$\frac{0.93 \times 1440 \times 365}{10^6} = 0.49\ m^2/yr$$

Using Casagrande's log time method

$$C_v = \frac{0.196\, d^2}{t_{50}} = \frac{0.196 \times 7.40^2}{11}$$

$$= 0.98\ [mm^2/min]$$

$$\frac{0.98 \times 1440 \times 365}{10^6} = 0.51\ m^2/yr$$

Note close agreement – fortunate!
Also realise that Taylor's root time method is workable with early stage data alone; Casagrande's log time method requires long term data to complete second linear section.

A road 4m embankment is to be constructed and will exert an additional 75kN/m^2 upon a sequence of soft alluvial soils.
The soil sequence comprises 1m sand underlain by 6m of clay, underlain by highly porous sandstone.
A sample of the clay from mid-height of the layer has been tested in the laboratory, with the following results. The in-situ effective stress of the sample is 35 kN/m^2

Questions

a) What will be the maximum settlement?

b) At what time can the road surface be laid to ensure that settlements between the embankment and a piled bridge abutment are less than 30mm?

Consolidation: worked example

A road 4m embankment is to be constructed and will exert an additional 75kN/m^2 upon a sequence of soft alluvial soils.
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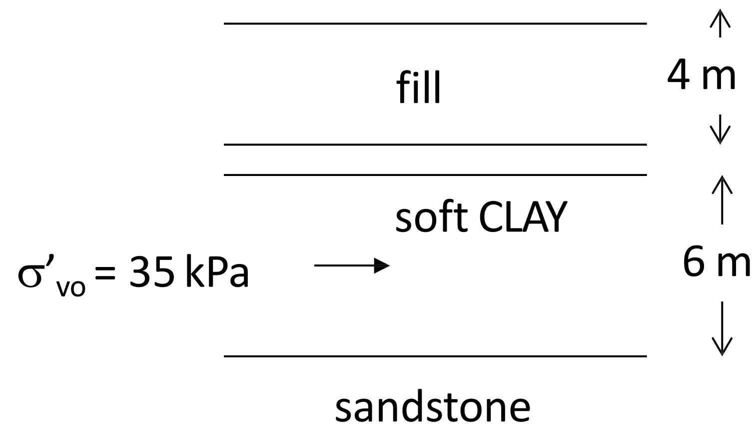


$\sigma'_{vf} = 35 + 75 = 110$ kPa

__Examination note__

In examination questions we often provide information that shortens or simplifies the preliminary work/calculations.

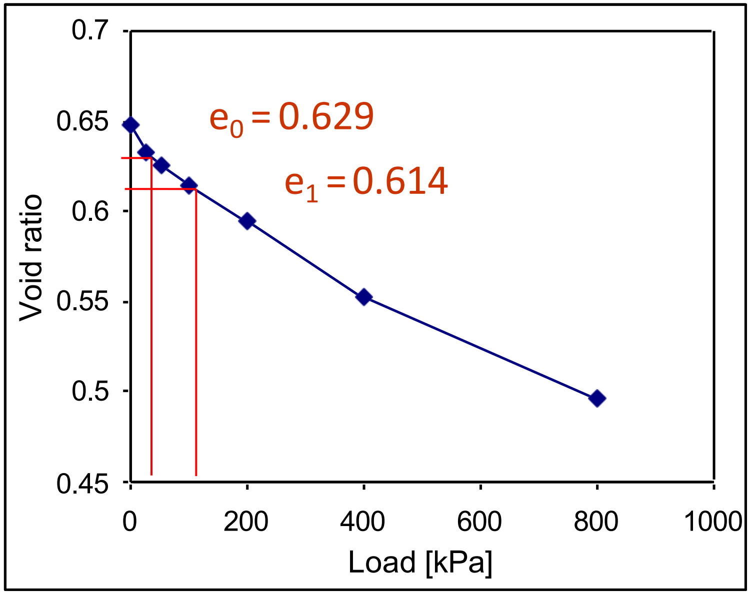
This is to allow sufficient time to probe the more substantive parts of the topic rather than the simple numerics.

Questions

a) What will be the maximum settlement?

But you do need to graph the data

Load	Void ratio
0	0.648
25	0.632
50	0.626
100	0.615
200	0.595
400	0.552
800	0.497



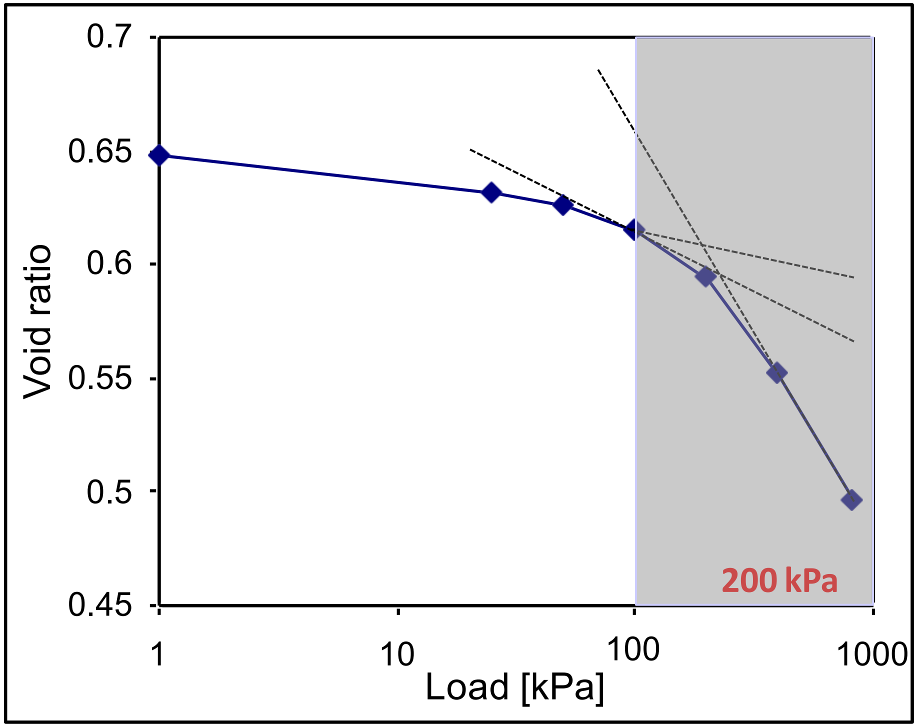
$$m_V = \frac{e_0 - e_1}{(1+e_0)\Delta\sigma_v'} = \frac{0.629 - 0.614}{1.629 \times 75} = 0.123 \times 10^{-3} \frac{m^2}{kN}$$

Settlement $m_V \Delta\sigma_v' H_0$ = $0.123 \times 10^{-3} \times 75 \times 6 = 0.055$m **[55mm]**

Predicted settlement is small, which suggests an overconsolidated sample.

Presentation of the oedometer data on $\log_{10}$ load scale clearly confirms a bilinear plot characteristic of an overconsolidated material.

Casagrande construction suggests a pre-consolidation pressure of 200 kPa and therefore an OCR = 5.7 at initial loading falling to 1.8 on loading



Worked example (contd)

<u>Questions</u>

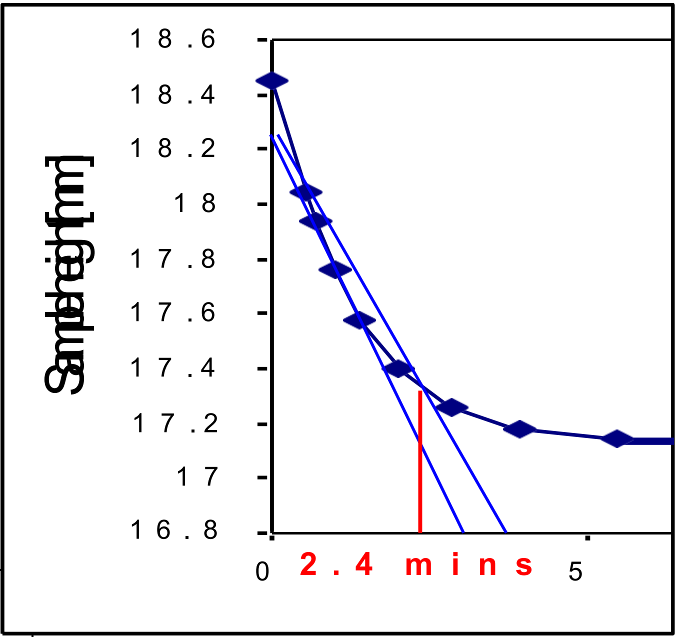
b) At what time can the road surface be laid to ensure that settlements between the embankment and a piled bridge abutment are less than 30mm?

Time	Sample H
0	18.45
.25	18.04
.50	17.94
1.0	17.76
2.0	17.58
4.0	17.40
8.0	17.26
15.0	17.18
30.0	17.14
60	17.10
120	17.05
240	17.00
480	16.94

First find c_v

Plot the data.

Taylors Root time method



$$Cv = \frac{0.848\, d^2}{t_{90}} = \frac{0.848 \times 8.85^2}{2.40^2}$$

$$= 11.53\ [mm^2/min]$$

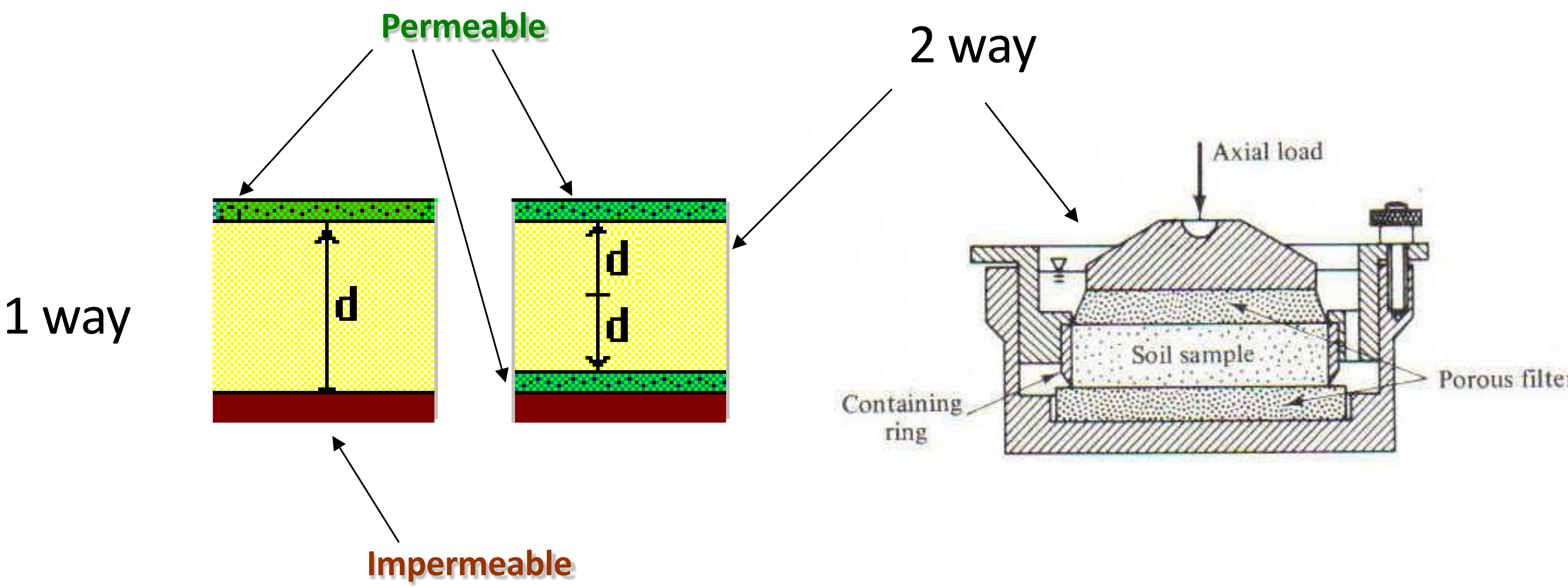
$$\frac{11.53 \times 1440 \times 365}{10^6} = \mathbf{6.06\ m^2/yr}$$

Worked example (contd)

Drainage path length – 2 way or 1 way?

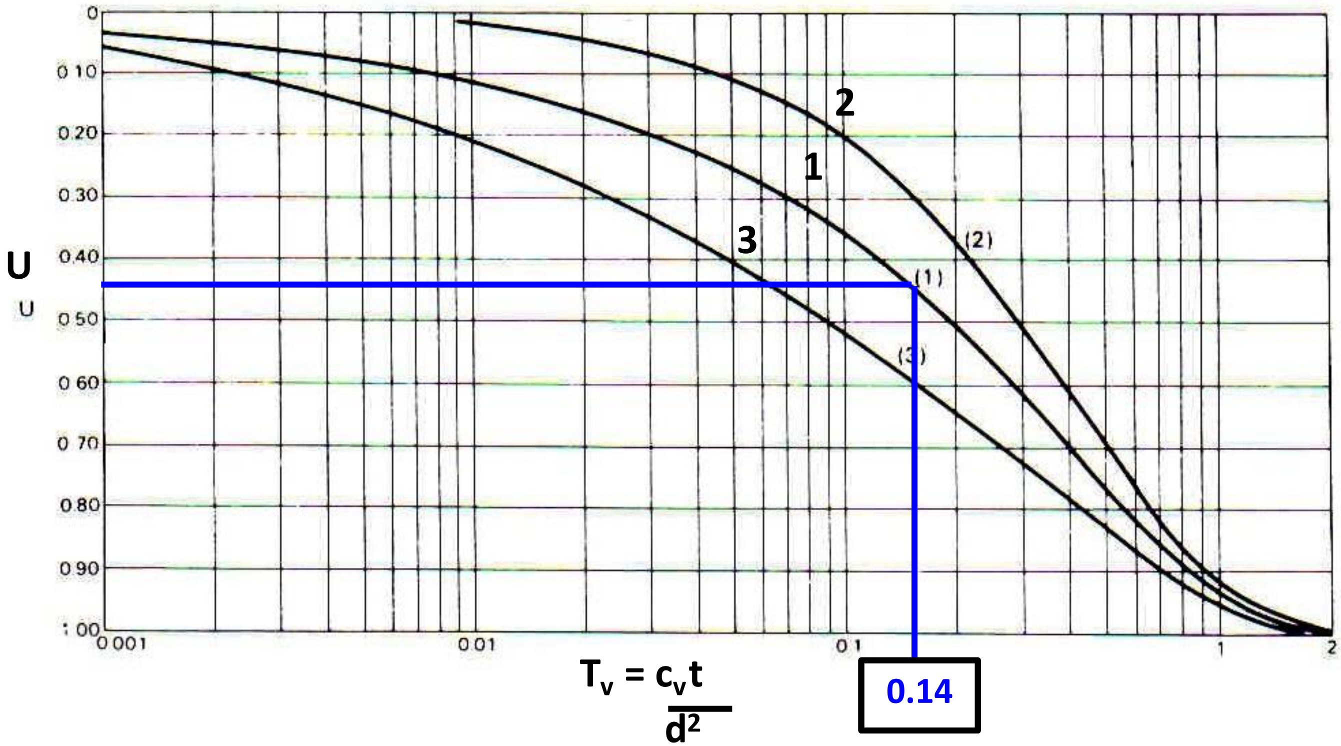
Drainage path length depends on permeability of soil adjacent to, i.e. over- and under-lying, the compressible layer.

It is maximum distance any particle of water must ravel to escape the compressible layer



The degree of consolidation indicates the progress of consolidation settlement as a number between 0 and 1. For a total consolidation settlement of 55mm, when a settlement of 25mm has occurred the remaining allowable settlement will be 30mm. The corresponding degree of consolidation, U = 0.454, i.e.

$$U = \frac{e_0 - e}{e_0 - e_1} = \frac{settlement}{total\ settlement} = \frac{H_0 - H}{H_0 - H_1} = \frac{\delta S}{S} = \frac{55-30}{55} = 0.454$$



Reading from curve 1 (2 way drainage),

$T_V = 0.140$

Next establish U … Tv

The time taken to achieve 45% consolidation settlement, i.e. no more than 30mm settlement remaining is:

$$t = \frac{T_v d^2}{c_v} = \frac{0.14 \times 3^2}{6.06} = 0.21 \text{ years}$$

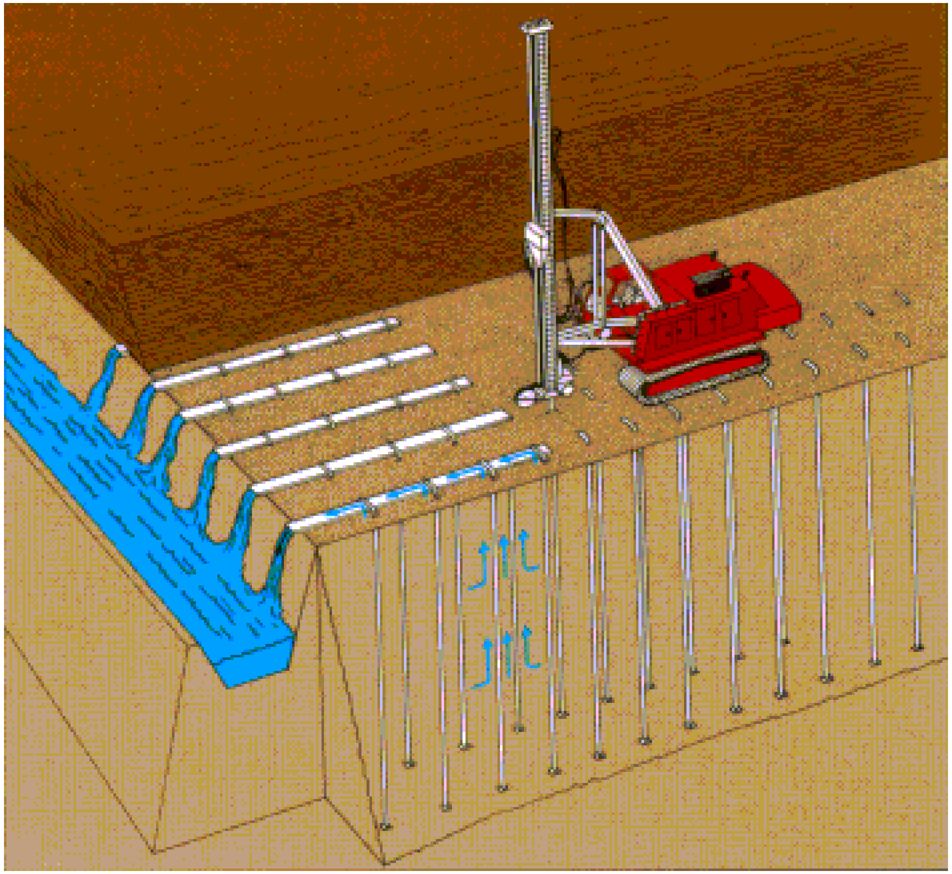
In other words, it will take less than 3 months for the embankment to consolidate and remaining settlements to be no more than 30mm.

What if this were too long a time?

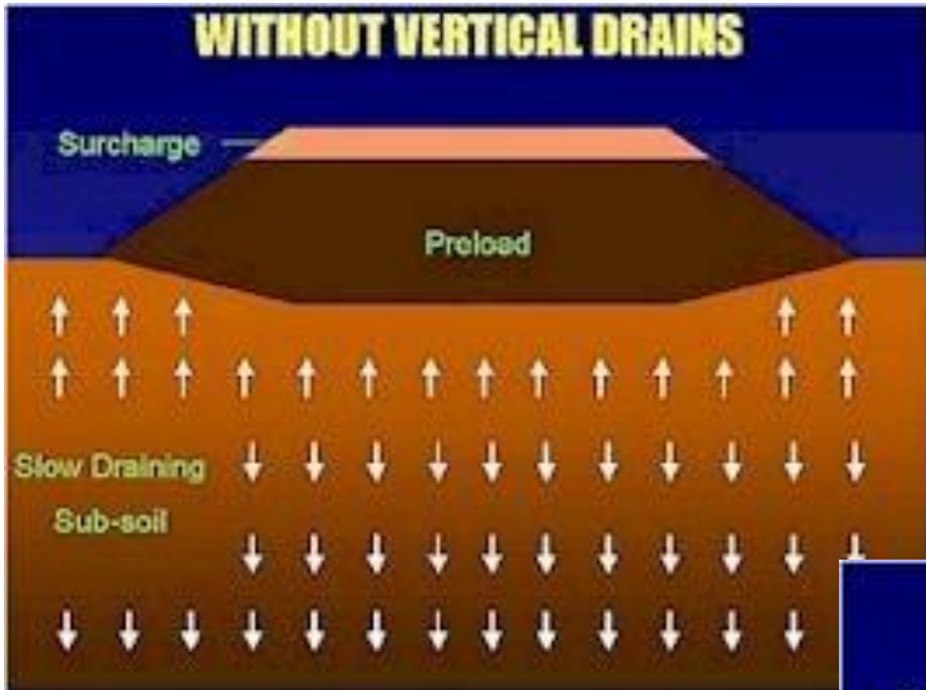
… vertical sand drains?
… pre-loading ?

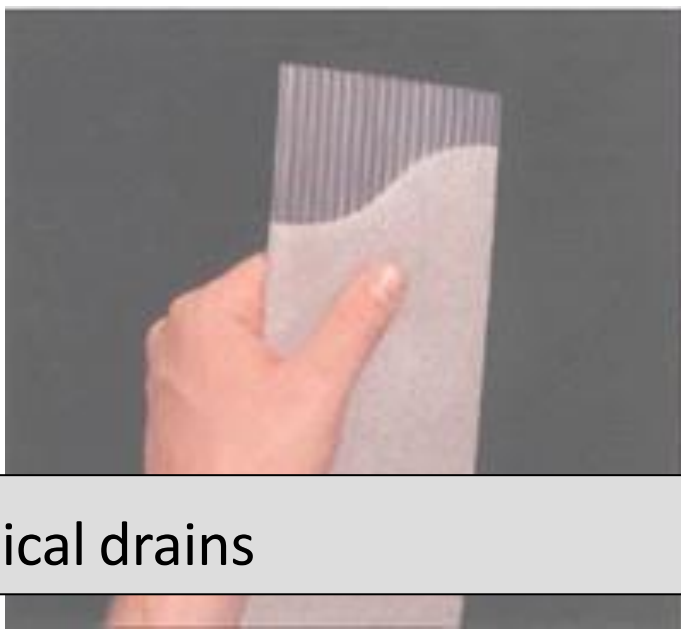
Worked example (contd)

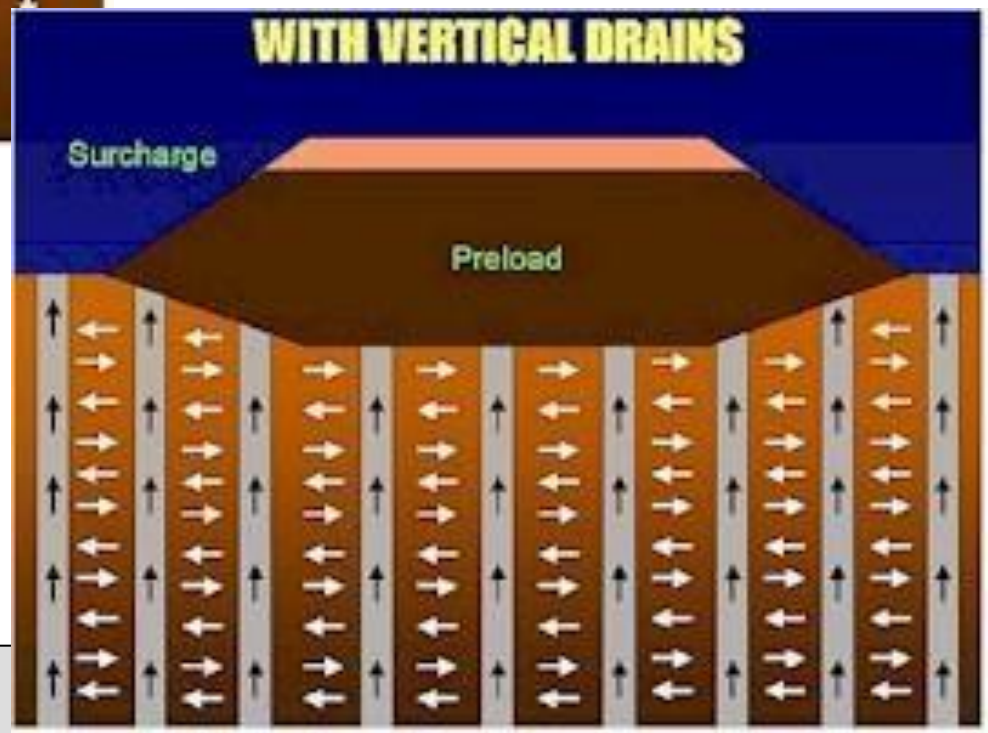
Vertical sand or wick drains are a means of accelerating consolidation settlement by reducing the drainage path length (in horizontal plane).



Geotechnical Engineering – vertical drains

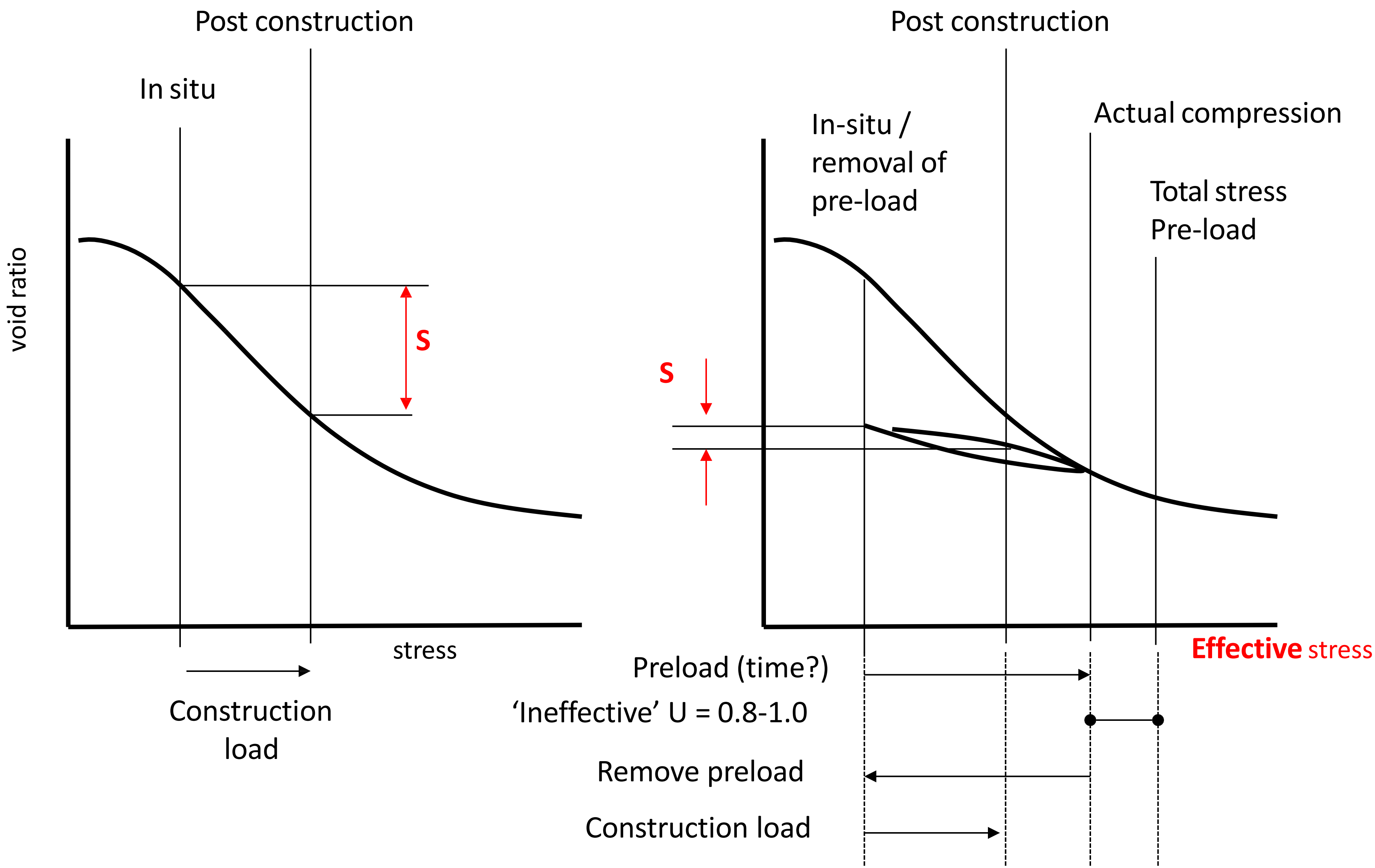






Vertical drains

Surcharging / Preloading



The placement and subsequent removal of a surcharge load means that the soil is overconsolidated. Recompression will be along an unload reload line; settlement will then be relatively small.

Of course the viability of this technique depends on an adequate supply of surcharge material, the magnitude of the surcharge in relation to the planned loading and the time the surcharge is left in place. All of these factors can be explored using the analytical techniques we have just learnt.

Surcharging

After studying this part of the course, you should be able to:

- Predict the compressibility of a soil due to a change in effective stress.
  - ✓ Determine changes in effective stress states
  - ✓ Calculate relevant m_v values
  - ✓ Recognise the compressibility of a soil in relation to its loading history, i.e. determine pre-consolidation pressure and overconsolidation ratio

- Predict the magnitude of consolidation settlement over time
  - ✓ Determine c_v by root time/log time
  - ✓ Apply c_v values to predict settlement at given times and/or predict times at which given settlement will have occurred.
  - ✓ Recommend ground modification methods to reduce long consolidation times.

Summary of what you can now do!

SELF-CHECK QUESTIONS – SOIL COMPRESSIBILITY

1. What is the vertical effective stress at the midpoint of an 18 m surface clay layer, with bulk unit weight = 20.2 kN/m^3 and ground water level 2 m below the clay surface?

2. What is the void ratio of a saturated soil sample with a moisture content of 28%? (Assume G_s = 2.7)

3. A soil sample loaded in an oedometer at 100 kPa has a height of 19.42 mm and a void ratio of 0.604. After loading to 200 kPa, the new sample height is 19.19 mm. Calculate the new void ratio.

4. Calculate the volume compressibility of the soil above.

5. What will be the settlement of a 7m clay layer with m_v = 0.240 x 10^{-3} m^2/kN subjected to a uniform increase in vertical stress of 65 kN/m^2 ?

SELF-CHECK QUESTIONS – CONSOLIDATION

1. The predicted consolidation settlement of a soil layer due to the construction of a slab foundation and other structural loads is 160mm. Precision equipment to be situated on the slab foundation require foundation movement to be no more than 30 mm (maximum allowable settlement). Determine the degree of consolidation at which maximum allowable settlement condition is met.

2. What is the time factor for U = 0.7?

3. A clay layer 6m thick, underlain by permeable sandstone, has a c_v of 0.34 m^2/yr. Determine the time to reach 50% consolidation.

4. What effect would an impermeable layer at the base of the clay described in Q3 have on the predicted 50% consolidation time?

5. If a clay soil achieves 40% consolidation in 15 months, how long will it take to reach 90% consolidation?

SELF-CHECK QUESTIONS – SOIL COMPRESSIBILITY

1) What is the vertical effective stress at the midpoint[1] of an 18 m surface clay layer, with bulk unit weight = 20.2 kN/m³ and ground water level 2 m below the clay surface?

$9m \times 20.2\ kN/m^3 - 7m \times 9.81\ kN/m^3$

$= 113.13\ kN/m^2$

2) What is the void ratio of a saturated soil sample with a moisture content of 28%? (Gs=2.7)

$wG = Se$

$e = 0.28 \times 2.7 / 1.0$

$= 0.756$

3) A soil sample loaded in an oedometer at 100 kPa has a height of 19.42 mm and a void ratio of 0.604. After loading to 200 kPa, the new sample height is 19.19 mm. Calculate the new void ratio.

$\Delta h/ho = \Delta e/(1+e_o)$

$\Delta e = 0.23 / 19.42 \times (1.604) = 0.019$

$e_1 = 0.604 - 0.019$

$= 0.585$

4) Calculate the volume compressibility of the soil above.

$m_v = (e_o - e_1) / [(1+e_o)\, \Delta\sigma'_v]$

$= 0.019 / [1.604 \times 100\ ]$

$= 0.118 \times 10\text{-}3\ m^2/kN$

5) What will be the settlement of a 7m clay layer with $m_v = 0.240 \times 10^{-3}\ m^2/kN$ subjected to a uniform increase in vertical stress of 65 kN/m² ?

$S = m_v.H.\ \Delta\sigma'_v$

$= 0.240 \times 10^{-3} \times 7 \times 65$

$= 109mm$ [say 110mm]

1 What is the justification for making compressibility calculations at the mid-height of a layer?

SELF-CHECK QUESTIONS – CONSOLIDATION

1 The predicted consolidation settlement of a soil layer due to the construction of a slab foundation and other structural loads is 160mm. Precision equipment to be situated on the slab foundation require foundation movement to be no more than 30 mm (maximum allowable settlement). Determine the degree of consolidation at which maximum allowable settlement condition is met.

$U = (160\text{-}30)/160$
$= 0.81$

2 What is the time factor for U = 0.7?

$Tv = 0.405$

3 A clay layer 6m thick, underlain by permeable sandstone, has a c_v of 0.34 m^2/yr. Determine the time to reach 50% consolidation.

cv = Tv.d2/t
$t = 0.196 \times 3^2 / 0.34$
= 5.2 yrs

4 What effect would an impermeable layer at the base of the clay described in Q3 have on the predicted 50% consolidation time?

$t = 0.196 \times 6^2 / 0.34$
= 20.8 yrs
times 4 (2^2)

5 If a clay soil achieves 40% consolidation in 15 months, how long will it take to reach 90% consolidation?

Assume cv constant, then $[Tv.d^2/t]40 = [Tv.d^2/t]90$
$Tv_{40} = 0.11$
d is common, so $0.11/1.25 = 0.848/t_{90}$
$t_{90} = 0.848 \times 1.25 / 0.11$
= 9.6 yrs