An introduction to particle filters

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Outline

- Motivation and ideas
- Algorithm
  - High-level
  - Matlab code
- Practical aspects
  - Resampling
  - Computational complexity
  - Software
  - Terminology
- Advanced topics
  - Convergence
  - Extensions
- References

State Space Models

State Space Models

- Linear Gaussian state space model

$$
\begin{aligned}
x_{t+1} &= Ax_t + Bu_t + w_t \\
y_t &= Cx_t + Du_t + e_t
\end{aligned}
$$

with w_t and e_t Gaussian and $\mathbb{E}\left[w_t w_t^T\right] = Q, \ \mathbb{E}\left[e_t e_t^T\right] = R.$

State Space Models

- Linear Gaussian state space model

$$x_{t+1} = Ax_t + Bu_t + w_t$$
$$y_t = Cx_t + Du_t + e_t$$

with w_t and e_t Gaussian and $\mathbb{E}\left[w_t w_t^T\right] = Q, \; \mathbb{E}\left[e_t e_t^T\right] = R.$

- A more general state space model in different notation

$$x_{t+1} \sim f(x_{t+1} | x_t, u_t)$$
$$y_t \sim g(y_t | x_t, u_t)$$

Filtering - Find $p(x_t|y_{1:t})$

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Linear Gaussian problems:

Filtering - Find $p(x_t | y_{1:t})$

Linear Gaussian problems:

- Kalman filter!

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Nonlinear problems:

Filtering - Find $p(x_t | y_{1:t})$

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Nonlinear problems:

- EKF, UKF, ...
- Particle filter

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Linear Gaussian problems:

- Kalman filter! *"Exact solution to the right problem"*

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Linear Gaussian problems:

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Nonlinear problems:

- EKF, UKF, … *"Exact solution to the wrong problem"*
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Filtering - Find $p(x_t|y_{1:t})$

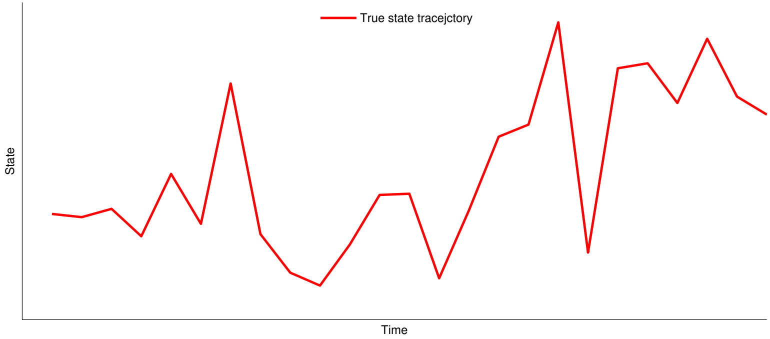
Linear Gaussian problems:

- Kalman filter! *"Exact solution to the right problem"*

Nonlinear problems:

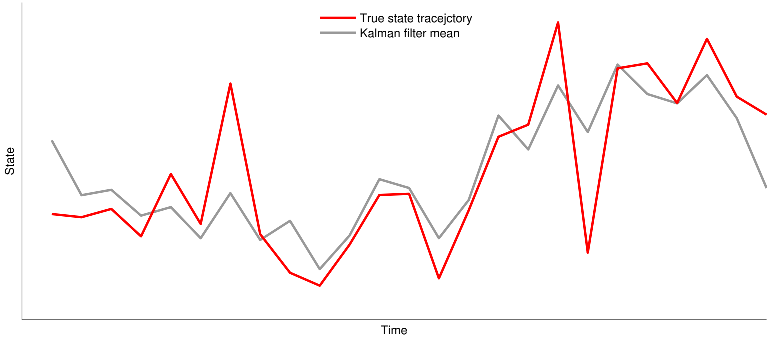
- EKF, UKF, ... *"Exact solution to the wrong problem"*
- Particle filter *"Approximate solution to the right problem"*

Idea



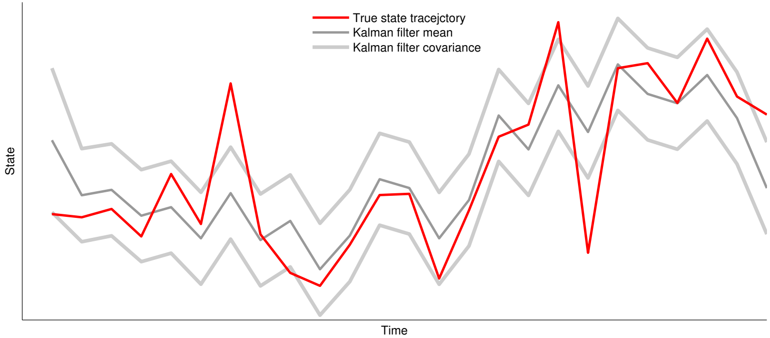
A linear system - Find $p(x_t|y_{1:t})$ (true x_t shown)!

Idea



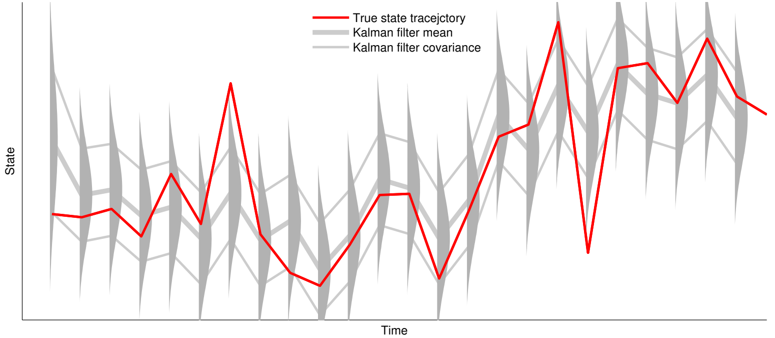
Kalman filter mean

Idea



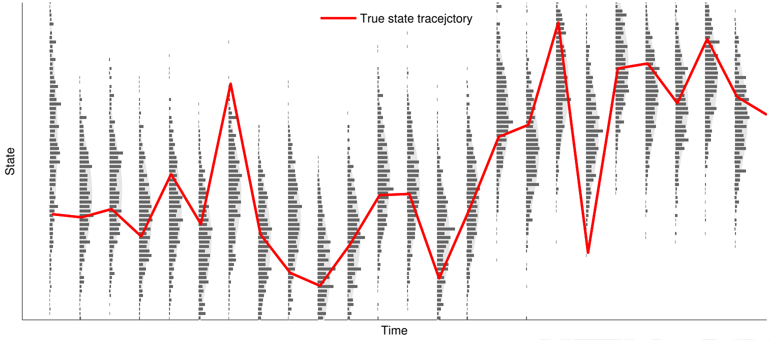
Kalman filter mean and covariance

Idea



Kalman filter mean and covariance defines a Gaussian distribution at each t

Idea



A numerical approximation can be used to describe the distribution - Particle Filter

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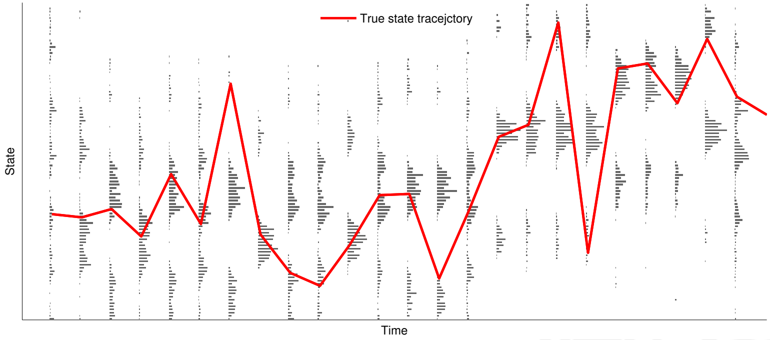
Idea



The Particle Filter can easily handle, e.g., non-gaussian multimodal hypotheses

Idea

Generate a lot of hypotheses about x_t, keep the most likely ones and propagate them further to x_{t+1}. Keep the likely hypotheses about x_{t+1}, propagate them again to x_{t+2}, etc.

Algorithm

0 Initialize $x_1^i \sim p(x_1)$ and $w_i = \frac{1}{N}$ for $i = 1, \dots, N$

for t = 1 to T

1. Evaluate $w_t^i = g(y_t | x_t^i, u_t)$ for $i = 1, \dots, N$
2. Resample $\{x_t^i\}_{i=1}^N$ from $\{x_t^i, w_t^i\}_{i=1}^N$
3. Propagate x_t^i by sampling x_{t+1}^i from $f(\cdot | x_t^i, u_t)$ for $i = 1, \dots, N$

end

N Number of particles,

x_t^i Particles,

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```
0 x(:,1) = random(i_dist,N,1);
  w(:,1) = ones(N,1)/N;

  for t = 1:T

    1 w(:,t) =
      pdf(m_dist,y(t)-g(x(:,t)));
      w(:,t) =
      w(:,t)/sum(w(:,t));
    2 Resample x(:,t)
    3 x(:,t+1) = f(x(:,t),u(t))
      + random(t_dist,N,1);

  end
```

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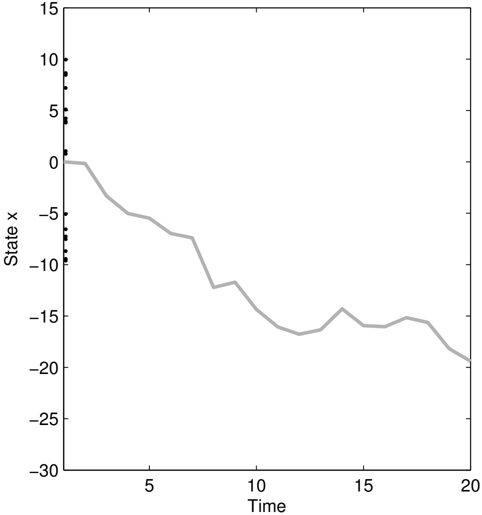

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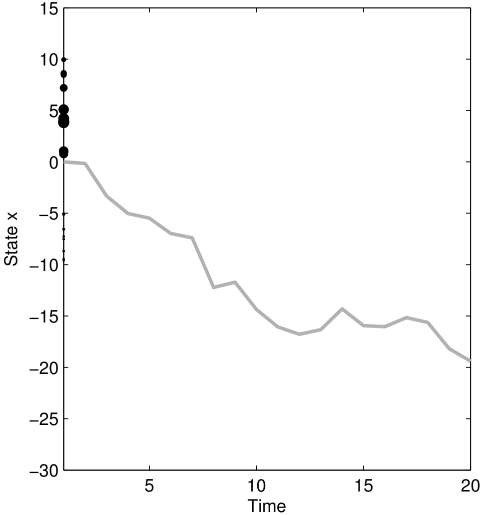

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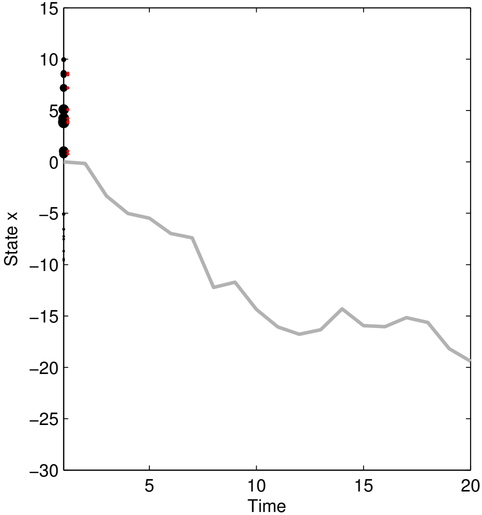

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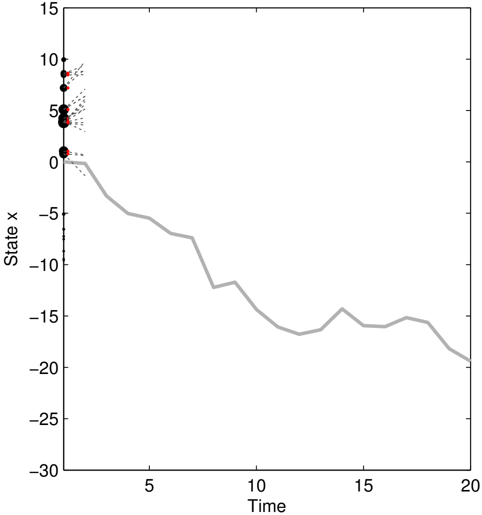

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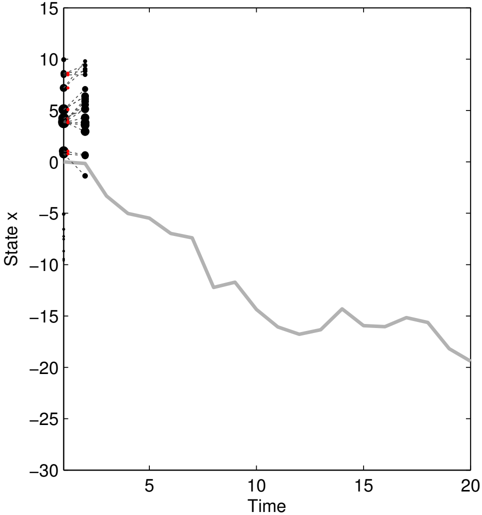

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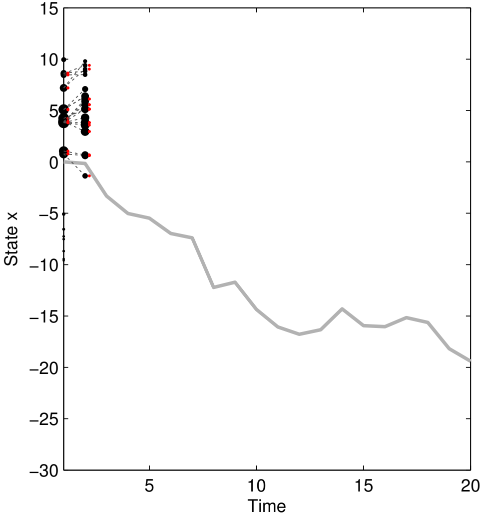

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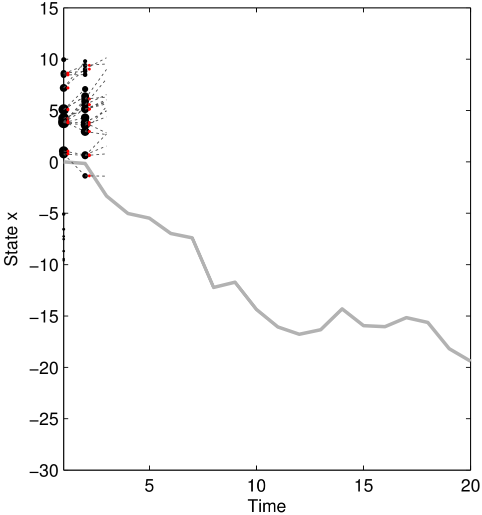

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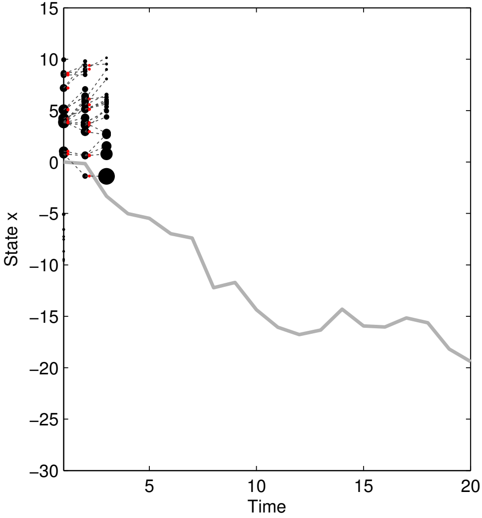

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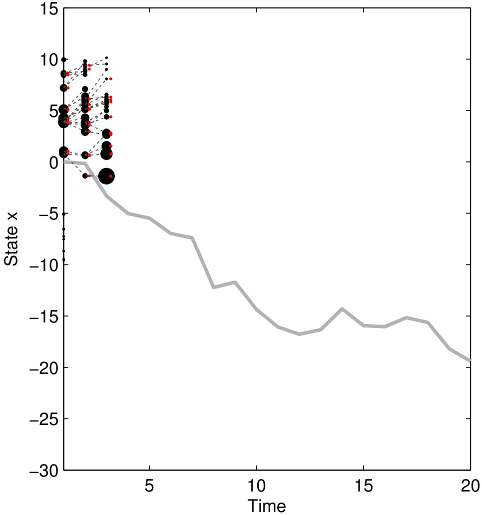

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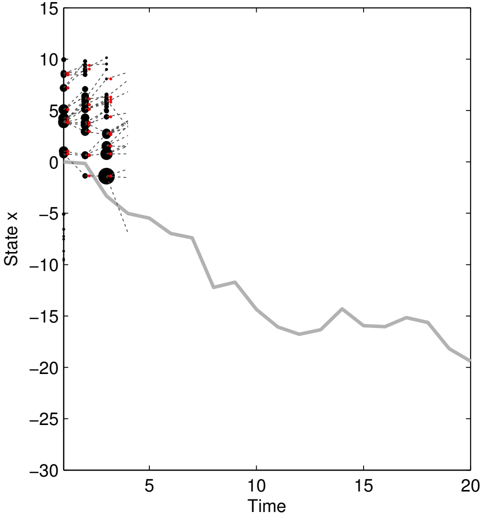

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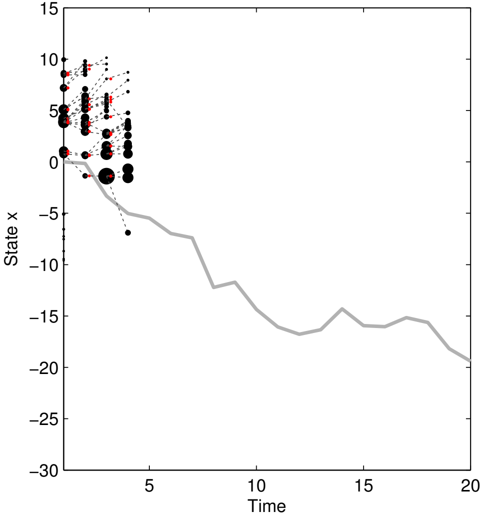

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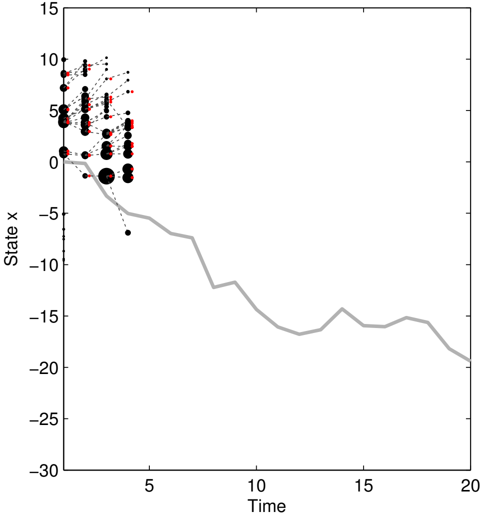

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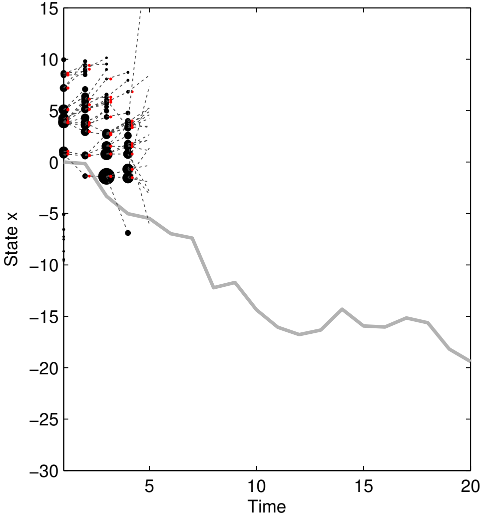

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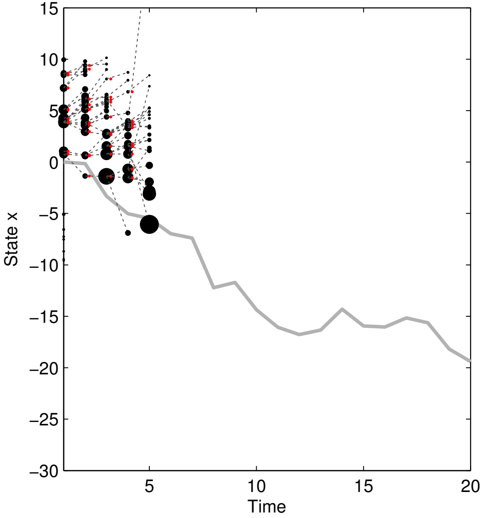

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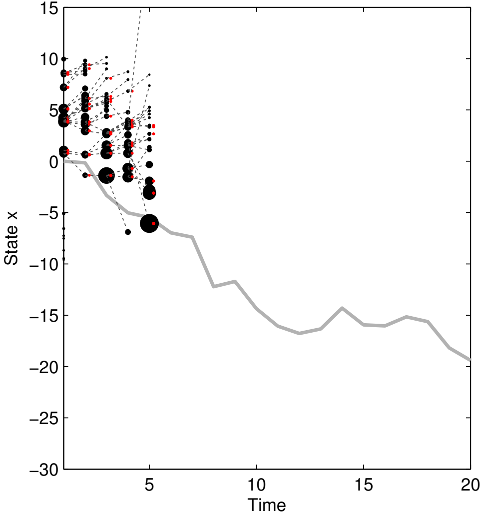

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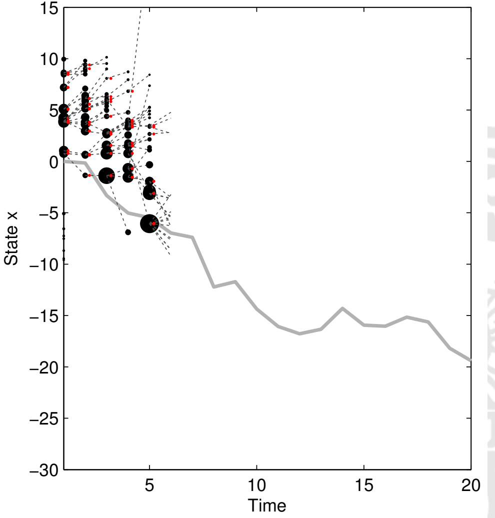

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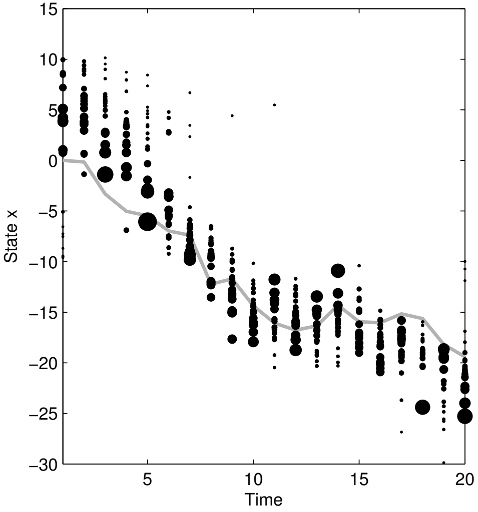
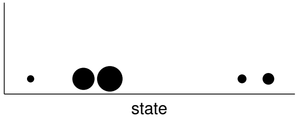


Resampling

Resampling

- Represent the information contained in the N black dots (of different sizes) …



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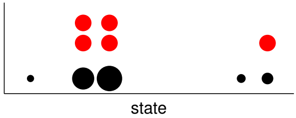

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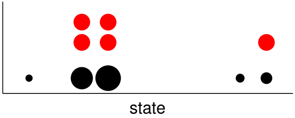

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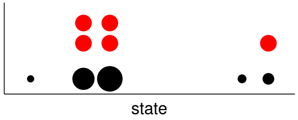
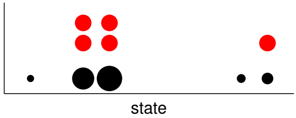


- Can be seen as sampling from a cathegorical distribution
- To avoid particle depletion ("a lot of 0-weights particles")

Resampling

- Represent the information contained in the N black dots (of different sizes) ... with the N red dots (of equal sizes)



- Can be seen as sampling from a cathegorical distribution
- To avoid particle depletion ("a lot of 0-weights particles")
- Some Matlab code:

```
v = rand(N,1); wc = cumsum(w(:,t)); [ ,ind1] = sort([v;wc]);
ind=find(ind1<=N)-(0:N-1)'; x(:,t)=x(ind,t);
w(:,t)=ones(N,1)./N;
```

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- It is sometimes preferred to use the logarithms of the weights for numerical reasons.

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Possible bottlenecks:

- Resampling step
- Likelihood evaluation (for the weight evaluation)
- Sampling from f for the propagation (trick: use proposal distributions!)

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A few relevant existing packages:

- Python: pyParticleEst
- Matlab: PFToolbox, PFLib
- C++: Particle++

Disclaimer: I have not used any of these packages myself

Terminology

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Sequential Monte Carlo (SMC) $\approx$ Particle filter

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If the system "forgets exponentially fast" (e.g. linear systems), and some additional weak assumptions, $p_t = p < \infty$, i.e.,

$\mathbb{E}\left[\hat{g}(x_t) - \mathbb{E}\left[g(x_t)\right]\right]^2 \leq \frac{C}{N}$

Extensions

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Extensions

- Smoothing: Finding $p(x_t|y_{1:T})$ (marginal smoothing) or $p(x_{1:t}|y_{1:T})$ (joint smoothing) instead of $p(x_t|y_{1:t})$ (filtering). Offline ($y_{1:T}$ has to be available). Increased computational load.

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- System identification: PMCMC, SMC^2.

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References

- Gustafsson, F. (2010). **Particle filter theory and practice with positioning applications.** *Aerospace and Electronic Systems Magazine, IEEE,* 25(7), 53-82.
- Schön, T.B., & Lindsten, F. **Learning of dynamical systems - Particle Filters and Markov chain methods.** *Draft available.*
- Doucet, A., & Johansen, A. M. (2009). **A tutorial on particle filtering and smoothing: Fifteen years later.** *Handbook of Nonlinear Filtering*, 12, 656-704.

Homework: Implement your own Particle Filter for any (simple) problem of your choice!