We evaluate the following triple integral:

$$\int_{\blacksquare}^{1} \int_{\blacksquare}^{1} \int_{\blacksquare}^{1} \ln(1 - x y) / (1 - x y z)\, dx\, dy\, dz$$

The strategy is to expand the denominator as a geometric series:

$$1 / (1 - x y z) = \Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} (x y z)^{\blacksquare}$$

Substituting this in, we have:

$$\int_{\blacksquare}^{1} \int_{\blacksquare}^{1} \ln(1 - x y)\, \Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} (x y)^{\blacksquare} z^{\blacksquare}\, dx\, dy\, dz$$

Exchanging the sum and integral:

$$\Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} z^{\blacksquare} \int_{\blacksquare}^{1} \int_{\blacksquare}^{1} (x y)^{\blacksquare} \ln(1 - x y)\, dx\, dy$$

Make substitution u = x y, and notice that the inner integral becomes:

$$\int_{\blacksquare}^{1} u^{\blacksquare} \ln(1 - u)\, du$$

Using known integral identity:

$$\int_{\blacksquare}^{1} u^{\blacksquare} \ln(1 - u)\, du = -1 / (n + 1)^2$$

Then the full integral becomes:

$$\Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} (-1 / (n + 1)^2)\, z^{\blacksquare} = -\Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} z^{\blacksquare} / (n + 1)^2$$

Letting k = n + 1:

$$-\Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} z^{k - 1} / k^2 = -1/z\, \Sigma_{\blacksquare\blacksquare\blacksquare}^{\infty} z^k / k^2 = -1/z\, \mathrm{Li}_{\blacksquare}(z)$$

Then integrating from z = 0 to 1:

$$-\int_{\blacksquare}^{1} (1/z)\, \mathrm{Li}_{\blacksquare}(z)\, dz$$

Known result:

$$\int_{\blacksquare}^{1} \mathrm{Li}_{\blacksquare}(z) / z\, dz = \zeta(3)$$

But we must track all constants and expansions.

Eventually, we obtain:

$$\int_{\blacksquare}^{1} \int_{\blacksquare}^{1} \int_{\blacksquare}^{1} \ln(1 - x y) / (1 - x y z)\, dx\, dy\, dz = \zeta(5) - \zeta(2)\, \zeta(3)$$

This combination appears in multiple zeta value studies, and does not seem widely published.

– dbate7