UNIVERSITÄT BIELEFELD

Fakultät für Wirtschaftswissenschaften

Using subspace algorithms to estimate the factor dynamics in approximate dynamic factor models

Universität Passau,

13.6.2024

Financing from the DFG (project GLASS No. 469278259) is gratefully acknowledged.

Dietmar Bauer

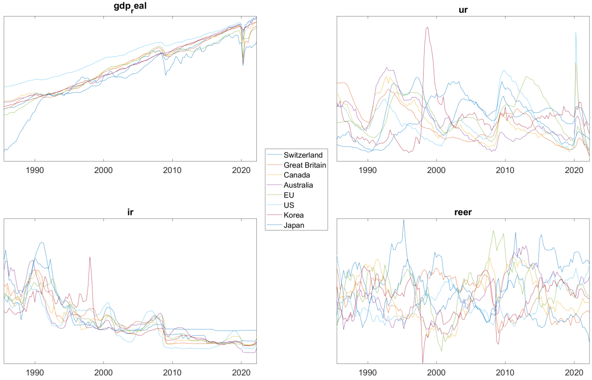
Introduction

1. Introduction

- Often in the analysis of, for example, macro-data or financial data one models a large number of variables jointly.
- Examples include the vectorization of matrix valued time series (MaTS), wherein a number of variables is observed in a number of regions.
- The number of variables typically is smaller but similar to the number of time points.
- To mention just one example, Bai and Ng (2019) use an example with $N = 128$ variables and $T = 676$ (monthly) observations from the FRED-MD data base.
- Joint models for such number of series necessarily contain many parameters and pose problems for specification and estimation.
- In order to cope with the high dimensionality, factor models and generalisations thereof are considered to reduce the dimensionality.

Typical data set

Quarterly data from 8 countries/regions on a number of variables:



Outline

- **Factor models (FM)** reduce the dimensionality by focusing on *common* features that appear in many time series first.
- Common factors typically are **dynamic**, that is not independent over time: state space models.
- **Subspace** procedures like CVA are used to estimate such models.
- **Consistency results** confirm that they provide useful information in the aDFM situation.
- (**Simulations**: specification of integer valued parameters.)

This is a methodological talk, I will not show real world data or modelling!

Specification of Factor Models

Approximate (generalized) Dynamic Factor Models: aDFMs

$$y_{it} = \chi_{it} + \xi_{it} = \underbrace{\lambda_i' F_t}_{\text{factor component}} + \underbrace{\xi_{it}}_{\text{idiosyncratic component}}, i = 1, .., N.$$

$$Y_t^N = \Lambda_N F_t + \Xi_t^N \in \mathbb{R}^N$$

- $Y_t^N = [y_{1t}, ..., y_{Nt}] \in \mathbb{R}^N, t = 1, ..., T$
- $F_t \in \mathbb{R}^r$... static factors, where $r \ll N$.

Assumption (Independence)

The factors F_t and the idiosyncratic component ξ_{is} are independent for all variables i and all times t, s.

Assumption (Factor Loadings)

The factor loadings λ_i are assumed deterministic such that
$N^{-1} \sum_{i=1}^{N} \lambda_i \lambda_i' = N^{-1} \Lambda_N' \Lambda_N \to M_\Lambda > 0$.
$\sup_N \max_i \|\lambda_i\| \leq M_\lambda$.

Stationarity

- In this talk we restrict attention to stationary processes.
- This may involve the need to transform some variables for example via taking temporal differences.

Assumption (Stationarity)

The processes $(F_t)_{t\in\mathbb{Z}}, (\Xi_t^N)_{t\in\mathbb{Z}}$ are jointly wide sense stationary with zero expected value for all N and possess spectral densities.
For each of the processes F_t, Ξ_t^N, y_t^N we have

$$\max_{0\leq k\leq H_T} \max_{i,j} \left\| T^{-1} \sum_{t=1+k}^{T} x_{t,i}x_{t-k,j} - \mathbb{E}\, x_{t,i}x_{t-k,j} \right\| = O(Q_T)$$

where $Q_T := \sqrt{(\log\log T/T)}$ and $H_T = (\log T)^a$ for some integer $a > 1$.

These high level assumptions differ from the literature. Often assumptions on the underlying processes are stated such that the covariance estimates fulfill similar assumptions.

Covariance of observations:

$$\Gamma_{y,N} = \mathbb{E}\, y_t^N (y_t^N)' = \Lambda_N (\mathbb{E}\, F_t F_t') \Lambda_N' + \Gamma_{\Xi,N}.$$

Two issues:

(I) separate common and idiosyncratic parts

(II) identify common factors and loadings from the product $\Lambda_N F_t$.

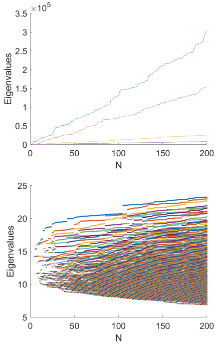
Assumption (Identification)

- $\Gamma_{\Xi,N} = \mathbb{E}\, \Xi_t^N (\Xi_t^N)'$: $\sup_N \lambda_{max}(\Gamma_{\Xi,N}) \leq M_\Xi$.
- $\Gamma_F = \mathbb{E}\, F_t F_t' = I_r$.
- Λ_N is positive lower triangular (for example, it starts with a lower triangular matrix with positive diagonal entries).

2. Specification of Factor Models

It follows that

- $\mathbb{E}\,\chi_t^N(\chi_t^N)' = \Lambda_N(\mathbb{E}\,F_t F_t')\Lambda_N' = \Lambda_N\Lambda_N'$
- $\Lambda_N'\Lambda_N/N \to M_\Lambda > 0$: all eigenvalues of $\Lambda_N\Lambda_N'$ grow essentially linearly as a function of N.
- In $\Gamma_{y,N} = \Lambda_N\Lambda_N' + \Gamma_{\Xi,N}$ the first r eigenvalues are asymptotically (in N) proportional to N.
- The remaining $N - r$ ones remain bounded.
- This suggest to estimate F_t using PCA.



Asymptotic identification in the sense of Chamberlain and Rothschild (1983) $\Rightarrow$ approximate DFM (aDFM).

State Space Modelling

Modelling Dynamics

- Factors typically are not uncorrelated in time.
- Typical models are autoregressive processes:

$$F_t = A_1 F_{t-1} + A_2 F_{t-2} + ... + A_p F_{t-p} + v_t,$$

$$a(z)F_t = v_t, \qquad a(z) = I - A_1 z - A_2 z^2 - ... - A_p z^p.$$

- To understand the dynamics we can rewrite the AR(p) into a higher dimensional AR(1) model:

$$\underbrace{\begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} A_1 & A_2 & \dots & \dots & A_p \\ I & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} F_{t-1} \\ F_{t-2} \\ \vdots \\ F_{t-p} \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{B} v_t$$

$$x_{t+1} = Ax_t + Bv_t,$$

$$F_t = \underbrace{\begin{pmatrix} A_1 & A_2 & \dots & \dots & A_p \end{pmatrix}}_{C} x_t + v_t.$$

From MA(1) to state space

- Rewrite a MA(1) into a state space model:

$$F_t = \underbrace{B_1}_{C} \underbrace{v_{t-1}}_{x_t} + v_t$$

$$x_{t+1} = v_t = 0 * x_t + Bv_t$$

- This leads to general state space system of the form:

$$y_t = Cx_t + v_t$$
$$x_{t+1} = Ax_t + Bv_t$$

- One can show that each ARMA system can be rewritten into a state space system.
- The dynamics are contained in the state equation.
- If the matrix A is stable, stationary solutions are given by

$$F_t = v_t + \sum_{j=0}^{\infty} CA^j Bv_{t-j-1}.$$

Assumption (Rationality; cf. Lippi, Deistler, Anderson (2023))

The static factor process $(F_t)_{t\in\mathbb{Z}}$, $F_t \in \mathbb{R}^r$, does not depend on N and has a minimal state space representation as

$$F_t = Cx_t + Du_t, \quad x_{t+1} = Ax_t + Bu_t, \quad u_t \in \mathbb{R}^q .$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times q}$ and where $(u_t)_{t\in\mathbb{Z}}$ is a stationary, ergodic martingale difference sequence and has expectation zero and variance matrix $\Omega = I_q$.
The transfer function $k(z) = D + zC(I_n - zA)^{-1}B$ has no zeros or poles inside the unit circle.

- Then also $\chi_t^N = \Lambda_N F_t$ has a state space representation.
- $(u_t)_{t\in\mathbb{Z}}$: white noise process of *dynamic factors*.
- The case $q < r$ of tall transfer functions fits in nicely in the state space framework. It is more complicated in the ARMA world.

Tall transfer functions

$k(z) = D + zC(I_n - zA)^{-1}B \in \mathbb{C}^{r\times q}$ where $q < r$ typically: 'tall'.

- Dimension reduction: $Y_t^N \in \mathbb{R}^N \to F_t \in \mathbb{R}^r \to u_t \in \mathbb{R}^q$.
- Deistler et al. (2010): generically there exists polynomial left pseudo-inverses $k^\dagger(z)$ to $k(z)$.
- In these cases we obtain a *singular* autoregressive representation $k^\dagger(z)F_t = v_t = \begin{pmatrix} I \\ 0 \end{pmatrix} u_t$.
- From this we see that u_t can be written as a linear combination of past F_t's.
- The pseudo-inverse $k^\dagger(z)$ is not unique. Not even the lag order. This complicates estimation.

Assumption (factor dynamics)

$k(z) = D + zC(I_n - zA)^{-1}B$ where

- The matrix $D = k(0) \in \mathbb{R}^{r\times q}$ has full column rank q such that the heading $q \times q$ submatrix is lower triangular with positive entries on the diagonal.
- There exists a pseudo inverse $D^\dagger$ such $\underline{A} = A - BD^\dagger C$ is stable. Then the transfer function $c(z) = \sum_{j=0}^{\infty} c_j z^j \in \mathbb{R}^{q\times r}$, $c_j = C\underline{A}^{j-1}BD^\dagger$ is a pseudo left-inverse such that $c(z)k(z) = I_q$.
- There exists a real value $\rho_0 < 1$ such that $\|c_j\| \leq \rho_0^j \mu, \forall j \in \mathbb{N}$ for $0 < \mu < \infty$.
- If $c(z)$ is a polynomial $\rho_0 = 0$. The degree of $c(z)$ then is denoted as p_0.

Then Du_t are the innovations for the process $(F_t)_{t\in\mathbb{Z}}$ and the past spaces for $(u_t)_{t\in\mathbb{Z}}$ and $(F_t)_{t\in\mathbb{Z}}$ coincide.

Canonical Variate Analysis (CVA)

4. Canonical Variate Analysis (CVA)

$$F_t = Cx_t + Du_t, \qquad x_{t+1} = Ax_t + Bu_t.$$

$\Rightarrow$ estimation would be simple, if the state was known!

CVA is based on two facts:

1. The state x_t can be approximated by past observations:

$$x_t(p) = \sum_{j=1}^{p} \mathcal{K}_j(p) F_{t-j} = \mathcal{K}_p F_t^- \overset{p \to \infty}{\to} x_t$$

2. Predictions of $F_{t+h}, h \geq 0$, based on the past of F_t are a function of the state:

$$F_{t+h} = \underbrace{CA^h x_t}_{F_{t+h|t-1}} + \underbrace{Du_{t+h} + \sum_{j=0}^{h-1} CA^j Bu_{t+h-j-1}}_{v_{t+h|t-1}}$$

This holds for $h = 0, 1, ..., f$ and can be seen as a multi-step long VAR approximation.

4. Canonical Variate Analysis (CVA)

Jointly this implies using $x_t(p) \approx \mathcal{K}_p F_t^-$

$$\underbrace{\begin{pmatrix} F_t \\ \vdots \\ F_{t+f} \end{pmatrix}}_{F_t^+} = \mathcal{O}_f \mathcal{K}_p \underbrace{\begin{pmatrix} F_{t-1} \\ \vdots \\ F_{t-p} \end{pmatrix}}_{F_t^-} + V_t(p), \qquad (*)$$

$$\mathcal{O}_f = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^f \end{pmatrix}, \mathcal{K}_p = \begin{pmatrix} \mathcal{K}_1(p) & \mathcal{K}_2(p) & \dots & \mathcal{K}_p(p) \end{pmatrix},$$

$$V_t(p) = \begin{pmatrix} v_{t|t-1} \\ \vdots \\ v_{t+f|t-1} \end{pmatrix} + \mathcal{O}_f(x_t - x_t(p)).$$

$(*)$ is a regression equation, where the matrix $\mathcal{O}_f \mathcal{K}_p$ has low rank $n \ll (f+1)r$.

CVA algorithm in the singular (tall transfer function) case

1. Choose f, p.
2. Perform a rank restricted regression of F_t^+ onto F_t^-. In this step the order n needs to be specified.
3. Use the estimate $\hat{\mathcal{K}}_p$ from the last step to estimate the state $\hat{x}_t(p) = \hat{\mathcal{K}}_p F_t^-$.
4. Estimate C by regressing F_t onto $\hat{x}_t(p)$. This step provides $\hat{v}_t = F_t - \hat{C}\hat{x}_t(p)$.
5. Obtain $\hat{D}$ from a truncated SVD of

$$\hat{\Sigma}_T = T^{-1} \sum_{t=p+1}^{T} \hat{v}_t \hat{v}_t' = \hat{D}\hat{D}' + \hat{R}_q$$

   using the q largest singular values, where $\hat{D}$ is p.l.t.
6. Regress $\hat{x}_{t+1}(p)$ onto $\hat{x}_t(p)$ and $\hat{u}_t = \hat{D}^\dagger \hat{v}_t$ to obtain the estimates $\hat{A}$ and $\hat{B}$.
7. Convert $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ to an overlapping form, for example echelon forms.

CVA has been proposed by W. Larimore (1990).

Issues

- F_t^- typically is a singular vector with rank $n + pq \leq rp \Rightarrow$ regularisation is needed.
- We only use the projection $\hat{x}_t(p) = \hat{\mathcal{K}}_p F_t^-$, such that regularization in the kernel of the variance of F_t^- does not change the projection.
- Due to sampling errors $\hat{v}_t$ typically will have a full rank sample variance matrix. Consistency implies that all but the first q singular values will tend to zero.

Asymptotic Results

Extraction of the static factors F_t

- First step: static PCA on $Y_t^N \in \mathbb{R}^N$ using $\frac{\hat{\Gamma}_Y}{N} = (NT)^{-1} \sum_{t=1}^T Y_t^N (Y_t^N)'$.

$$\frac{\hat{\Gamma}_Y}{N} = \frac{\Lambda_N}{\sqrt{N}} \hat{\Gamma}_F \frac{\Lambda_N}{\sqrt{N}}' + \frac{\hat{\Gamma}_\Xi}{N} + \text{cross terms} = \hat{U}_{N,r} \hat{S}_{N,r} \hat{U}_{N,r}' + \hat{R}_N$$

- We get $\hat{\Lambda}_N = \hat{U}_{N,r} \hat{S}_{N,r}^{1/2} \hat{L}_N$ ($\hat{L}_N$ is introduced to fulfill the identification restrictions)
- Then use ($\hat{\Lambda}_N^\dagger = (\hat{\Lambda}_N' \hat{\Lambda}_N)^{-1} \hat{\Lambda}_N'$)

$$\hat{F}_t = \hat{\Lambda}_N^\dagger y_t^N = \underbrace{(\hat{\Lambda}_N^\dagger \Lambda_N)}_{\Delta_T} F_t + \hat{\Lambda}_N^\dagger \Xi_t^N.$$

- We obtain

$$T^{-1} \sum_{t=1}^T (\hat{F}_t \hat{F}_t' - \Delta_T F_t F_t' \Delta_T') = O(Q_T + 1/N).$$

- Furthermore $\|\hat{\Lambda}_N/\sqrt{N} - \Lambda_N/\sqrt{N}\| = O(Q_T + 1/N)$.
- In this step the dimension r of the static factor needs to be specified.

5. Asymptotic Results

Under the assumption $T/N^2 \to 0$ we have $O(Q_T + 1/N) = O(Q_T)$.

Theorem (Consistency)

- Let the data be generated according to the before mentioned Assumptions.
- Assume that $T/N^2 \to 0$.
- Let $p(T) \leq H_T, f \geq n, p = p(T) \geq -\frac{e \log T}{2 \log \rho_0}, e > 1$, for $\rho_0 > 0$ and $p \geq p_0$ for $\rho_0 = 0$.
- Further let $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ denote the CVA estimators based on the PCA estimates $\hat{F}_t$ for given order n converted to an appropriate overlapping form.

Then

$$\max\{\|\hat{A} - A\|, \|\hat{B} - B\|, \|\hat{C} - C\|, \|\hat{D} - D\|\} = O(Q_T).$$

Hence the transfer function is estimated consistently.

Extensions

Different degrees of persistence:

- We can allow for (co-)integrated factor processes.
- In that case consistency still holds for a slightly larger lower bound for the increase of p.
- No knowledge on the number of common trends = integrated state components is needed.
- Also over-differentiation is possible such that the process F_t has simple (i.e. not higher order) spectral zeros at $z = 1$. In that case the rate of convergence drops from $O(Q_T)$ to $O(p^{2.5} Q_T + p^{-1})$.

Specification of Integers

Illustration

Simulation system:

- $N = 200$, $T = 800$
- $r = 5, q = 2, n = 3$.
- Factor dynamics: $A = \text{diag}(0.8, -0.8, 0.4)$, all other matrices chosen randomly.
- Idiosyncratic terms: each individual series follows an AR(1) with randomly chosen $|\rho_i| \leq 0.7, i = 1, ..., N$ and noise with variance $\sigma^2 = 0.25$.
- $M = 1000$ replications.

Dimension of the static factor r

- There are many methods to choose r.
- To mention just one example, Bai and Ng (2002) suggest to use an information type criterion to select the number of static factors:

$$\widehat{IC_2}(k) = \log SSR_k + k\frac{N+T}{NT}\log(NT/(N+T)), \quad SSR_k = \|Y - \hat{\Lambda}_{N,k}\hat{F}_k'\|_{Fr}^2$$

- This penalizes the fit (in terms of variation not explained by the common component) against complexity (modelled as linear in number of factors).
- $\hat{r}$ minimizes the criterion.

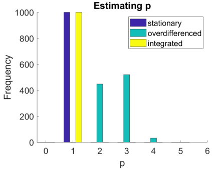
6. Specification of Integers

f, p

- f needs to be large enough such that the observability matrix $\mathcal{O}_f$ is of full rank.
- In the aDFM case, a small value typically is sufficient, as $r \geq n$ is reasonable, where hence $f = 1$ would suffice.
- p should be chosen such that the approximation of x_t by $x_t(p)$ is accurate.
- This is related to the lag selection in a long VAR representation for $\hat{F}_t$.
- Since this in the aDFM setting is a singular process, the information criterion needs to be adapted.
- A simple fix is: $\hat{\Sigma}_T(p)$ denoting the innovation variance estimate for a lag p AR approximation of F_t:

$$IC(p; C_T) = \underbrace{\mathrm{tr}}_{\cancel{\log\det}} \left[\hat{\Sigma}_T(p)\right] + \frac{2r^2 p C_T}{T}.$$

$T = 800, N = 200$



T	200	400	800	1600
stat.	1.00	1.00	1.00	1.00
overdiff.	1.27	2.00	2.58	4.16
integr.	1.00	1.00	1.00	1.00

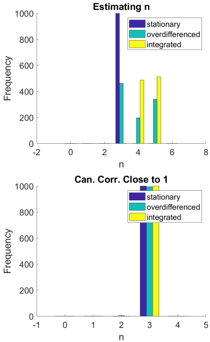
System order n

- Within CVA the system order is often estimated using the singular values in the RRR step.
- Selection can be done using

$$SVC(n) = \sum_{j=1}^{n} \log \hat{\sigma}_j^2 + \frac{2rnC_T}{T}$$

- For sufficiently large penalty term C_T we achieve consistent estimation of the order.
- If $N^2/T \to \infty$ the estimation error in $\hat{F}_t$ is dominated by the sampling error Q_T. Thus the same rates as in the square non-singular case (see Bauer, 2005) arise and $C_T/T \to 0$, $C_T/(fp \log\log T) \to \infty$ is sufficient for consistency.
- Breitung and Pigorsch (2013) show, the canonical correlations are close to one in the singular case.

$T = 800, N = 200$



$N = 50$, percent $\hat{n} = n$

T	200	400	800	1600
stat.	1.00	1.00	1.00	0.95
overdiff.	0.00	0.00	0.05	0.84
integr.	0.00	0.05	0.45	0.99

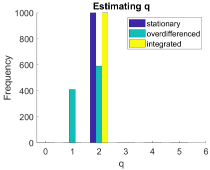
Dynamic factor dimension q

- Typically this would be determined based on dynamic PCA.
- An alternative is to use the estimated innovation variance $\hat{\Sigma}_T \in \mathbb{R}^{r \times r}$.
- The largest q eigenvalues will tend to their non-zero limits, the remaining ones tending to zero.
- Consequently also here information criteria can be used:

$$IC(q; C_T) = \sum_{j=1}^{q} \mu_j(\hat{\Sigma}_T) + \frac{rqC_T}{T}.$$

where $\mu_j(\hat{\Omega}_T)$ denotes the j-th largest eigenvalue.

$T = 800, N = 200$



diff., percent $\hat{q} = q$

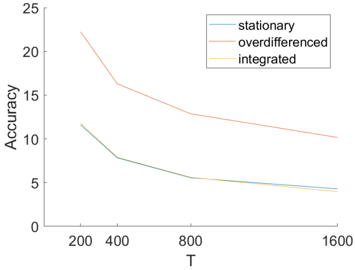
T	200	400	800	1600
$N = 50$	0.00	0.02	0.18	1.00
$N = 100$	0.00	0.00	0.46	1.00
$N = 150$	0.00	0.01	0.56	1.00
$N = 200$	0.00	0.00	0.59	1.00

Estimation Accuracy

- Accuracy measured as

$$\|\hat{D}\hat{D}' - DD'\| + \sum_{j=0}^{10} \|\hat{C}\hat{A}^j\hat{K}(\hat{C}\hat{A}^j\hat{K})' - CA^jK(CA^jK)'\|$$

- Averaged over 1000 replications for $N = 50$.
- Convergence is clearly visible.
- Overdifferenced case leads to worst results.



Conclusions

7. Conclusions

- CVA can be used to obtain consistent estimates in the aDFM setting.
- The main assumption beside the aDFM model structure is that $T/N^2 \to 0$ such that the cross-sectional dimension grows fast enough compared to the time dimension.
- The procedure is simple being based on static PCA and regression methods.
- The method provides information on all required integer parameters, which in all cases lead to consistent estimation.
- The method also works, if the transfer function relating the dynamic factors to the static factors is rank deficient at zero frequency, as would be expected when working with differenced or seasonally adjusted data.

This is the first step. Then the idiosyncratic component needs to be dealt with.

8. Literature

- Bai, J.; Ng, S. (2002): Determining the Number of Factors in Approximate Factor Models. In: Econometrica 70 (1), S. 191–221.
- Bai, J.; Ng, S. (2004): A PANIC Attack on Unit Roots and Cointegration. In: Econometrica (42), S. 1127–1177.
- Bai, J.; Ng, S. (2019): Rank regularized estimation of approximate factor models. In: Journal of Econometrics 212 (1), S. 78–96.
- Bauer, D. (2005): Estimating linear dynamical systems using subspace methods. In: Econ. Theory 21 (01).
- Bauer, D. (2023): Using Subspace Algorithms for the Estimation of Linear State Space Models for Over-Differenced Processes. Paper submitted to Economics Letters, 2023. Available at Github.
- Breitung, J.; Pigorsch, U. (2013). A canonical correlation approach for selecting the number of dynamic factors. Oxford Bulletin of Economics and Statistics, 75(1), 23-36.
- Chamberlain, G., Rothschild, M. (1983). Arbitrage, factor structure, and mean-variance analysis on large asset markets. Econometrica 51 (5), 1281–1304.
- Deistler, M.; Anderson, B. D.O.; Filler, A.; Zinner, Ch.; Chen, W. (2010): Generalized Linear Dynamic Factor Models: An Approach via Singular Autoregressions. In: European Journal of Control 16 (3), S. 211–224. DOI:
- Forni, M.; Hallin, M.; Lippi, M.; Zaffaroni, P. (2017): Dynamic factor models with infinite-dimensional factor space: Asymptotic analysis. In: Journal of Econometrics 199 (1), S. 74–92.
- Forni, M., Lippi, M. (2023) Approximating Singular by Means of NonSingular Structural VARs. EIEF Working Paper 23/1, October 2023.
- Kapetanios, G.; Marcellino, M. (2009). A parametric estimation method for dynamic factor models of large dimensions. Journal of Time Series Analysis, 30(2), 208-238.
- Larimore, W. E. (1990). Canonical variate analysis in identification, filtering, and adaptive control. In 29th IEEE Conference on Decision and Control (pp. 596-604). IEEE.
- Lippi, M.; Deistler, M.; Anderson, B.D.O. (2023): High-Dimensional Dynamic Factor Models: A Selective Survey and Lines of Future Research. In: Econometrics and Statistics 26, S. 3–16.
- Poskitt, D.S., 2007. Autoregressive approximation in nonstandard situations: the fractionally integrated and non-invertible case. Annals of Institute of Statistical Mathematics 59, 697–725.

Thank you for your attention!