14 Example: Pitcher Problem Solver

Program file for this chapter: `pour`

You have probably seen puzzles like this one many times:

> You are at the side of a river. You have a three-liter pitcher and a seven-liter pitcher. The pitchers do not have markings to allow measuring smaller quantities. You need two liters of water. How can you measure two liters?

These puzzles are used in some IQ tests, so many people come across them in schools. To solve the problem, you must pour water from one pitcher to another. In this particular problem, there are six steps in the shortest solution:

1. Fill the three-liter pitcher from the river.
2. Pour the three liters from the three-liter pitcher into the seven-liter pitcher.
3. Fill the three-liter pitcher from the river again.
4. Pour the three liters from the three-liter pitcher into the seven-liter pitcher (which now contains six liters).
5. Fill the three-liter pitcher from the river yet again.
6. Pour from the three-liter pitcher into the seven-liter pitcher until the latter is full. This requires one liter, since the seven-liter pitcher had six liters of water after step 4. This step leaves two liters in the three-liter pitcher.

This example is a relatively hard pitcher problem, since it requires six steps in the solution. On the other hand, it doesn't require pouring water back into the river, and it doesn't have any unnecessary pitchers. An actual IQ test has several such problems, starting with really easy ones like this:

> You are at the side of a river. You have a three-liter pitcher and a seven-liter pitcher. The pitchers do not have markings to allow measuring smaller quantities. You need four liters of water. How can you measure four liters?

and progressing to harder ones like this:

> You are at the side of a river. You have a two-liter pitcher, a five-liter pitcher, and a ten-liter pitcher. The pitchers do not have markings to allow measuring smaller quantities. You need one liter of water. How can you measure one liter?

The goal of this project is a program that can solve these problems. The program should take two inputs, a list of pitcher sizes and a number saying how many liters we want. It will work like this:

```
? pour [3 7] 4
Pour from river to 7
Pour from 7 to 3
Final quantities are 3 4
? pour [2 5 10] 1
Pour from river to 5
Pour from 5 to 2
Pour from 2 to river
Pour from 5 to 2
Final quantities are 2 1 0
```

How do *people* solve these problems? Probably you try a variety of special-purpose techniques. For example, you look at the sums and differences of the pitcher sizes to see if you can match the goal that way. In the problem about measuring four liters with a three-liter pitcher and a seven-liter pitcher, you probably recognized right away that $7 - 3 = 4$. A more sophisticated approach is to look at the remainders when one pitcher size is divided by another. In the last example, trying to measure one liter with pitchers of two, five, and ten liters, you might notice that the remainder of 5/2 is 1. That means that after removing some number of twos from five, you're left with one.

Such techniques might or might not solve any given pitcher problem. Mathematicians have studied ways to solve such problems in general. To a mathematician, a pitcher problem is equivalent to an algebraic equation in which the variables are required to take on integer (whole number) values. For example, the problem at the beginning of this chapter corresponds to the equation

$$3x + 7y = 2$$

In this equation, x represents the number of times the three-liter pitcher is filled and y represents the number of times the seven-liter pitcher is filled. A positive value means that the pitcher is filled from the river, while a negative value means that it's filled from another pitcher.

An equation with two variables like this one can have infinitely many solutions, but not all the solutions will have integer values. One integer-valued solution is $x = 3$ and $y = -1$. This solution represents filling the three-liter pitcher three times from the river (for a total of nine liters) and filling the seven-liter pitcher once from the three-liter pitcher. Since the seven-liter pitcher is bigger than the three-liter pitcher, it has to be filled in stages. Do you see how this analysis corresponds to the sequence of steps I gave earlier?

An equation with integer-valued variables is called a *Diophantine* equation. In general, a Diophantine equation will have infinitely many solutions, but they won't all be practical as solutions to the original problem. For example, another solution to the equation we've been considering is $x = -4$ and $y = 2$. This solution tells us to fill the seven-liter pitcher from the river twice, and the three-liter pitcher from the seven-liter pitcher four times. Here's how that works out as a sequence of steps:

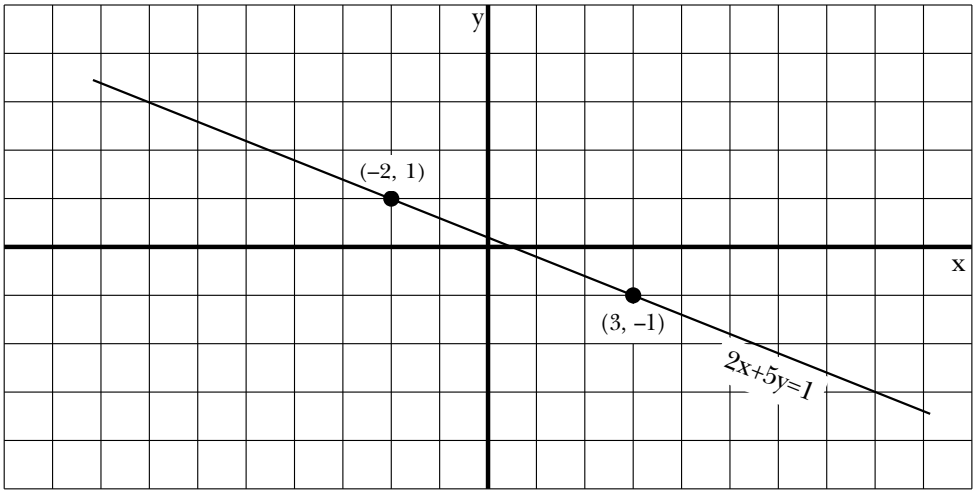
1. Fill the seven-liter pitcher from the river.
2. Fill the three-liter pitcher from the seven-liter pitcher. (This leaves four liters in the seven-liter pitcher.)
3. Empty the three-liter pitcher into the river.
4. Fill the three-liter pitcher from the seven-liter pitcher. (This leaves one liter in the seven-liter pitcher.)
5. Empty the three-liter pitcher into the river.
6. Pour the contents of the seven-liter pitcher (one liter) into the three-liter pitcher.
7. Fill the seven-liter pitcher from the river (for the second and last time).
8. Fill the three-liter pitcher (which already had one liter in it) from the seven-liter pitcher. (This leaves five liters in the seven-liter pitcher.)
9. Empty the three-liter pitcher into the river.
10. Fill the three-liter pitcher from the seven-liter pitcher. This leaves the desired two liters in the seven-liter pitcher.

This solution works, but it's more complicated than the one I used in the first place.

One way to solve Diophantine equations is graphically. For example, consider the problem about measuring one liter of water with pitcher capacities two, five, and ten liters. It turns out that the ten-liter pitcher is not actually needed, so let's forget it for now and consider the simpler but equivalent problem of using just the two-liter and the five-liter pitchers. This problem gives rise to the equation

$$2x + 5y = 1$$

For the moment, never mind that we are looking for integer solutions. Just graph the equation as you ordinarily would. The graph will be a straight line; probably the easiest way to draw the graph is to find the x-intercept (when $y = 0$, $2x = 1$ so $x = 1/2$) and the y-intercept (when $x = 0$, $y = 1/5$).



Once you've drawn the graph, you can look for places where the line crosses the grid points of the graph paper. In this case, two such points of intersection are $(-2, 1)$ and $(3, -1)$. The first of these points represents the solution shown earlier, in which the five-liter pitcher is filled from the river and then used as a source of water to fill the two-liter pitcher twice. The second integer solution represents the method of filling the two-liter pitcher from the river three times, then pouring the water from the two-liter pitcher to the five-liter pitcher each time. (On the third such pouring, the five-liter pitcher fills up after only one liter is poured, leaving one liter in the two-liter pitcher.)

What about the original version of this problem, in which there were three pitchers?

In this case, we have a Diophantine equation with three variables:

$$2x + 5y + 10z = 1$$

The graph of this equation is a plane in a three-dimensional coordinate system. An example of a solution point that uses all three pitchers is $(-2, -1, 1)$. How would you interpret this as a series of pouring steps?

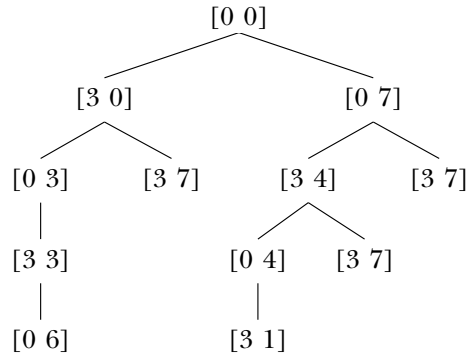
By the way, not all pitcher problems have solutions. For example, how could you measure one liter with a two-liter pitcher and a ten-liter pitcher? The answer is that you can't; since both pitchers hold an even number of liters, any amount of water measurable with them will also be even.*

Tree Search

My program does not solve pitcher problems by manipulating Diophantine equations. Instead, it simply tries every possible sequence of pouring steps until one of the pitchers contains the desired amount of water. This method is not feasible for a human being, because the number of possible sequences is generally quite large. Computers are better than people at doing large numbers of calculations quickly; people have the almost magical ability to notice the one relevant pattern in a problem without trying all the possibilities. (Some researchers attribute this human ability to "parallel processing"—the fact that the human brain can carry on several independent trains of thought all at once. They are beginning to build computers designed for parallel processing, and hope that these machines will be able to perform more like people than traditional computers.)

The possible pouring steps for a pitcher problem form a *tree*. The root of the tree is the situation in which all the pitchers are empty. Connected to the root are as many branches as there are pitchers; each branch leads to a node in which one of the pitchers has been filled from the river. Each of those nodes has several branches connected to it, corresponding to the several possible pouring steps. Here is the beginning of the tree for the case of a three-liter pitcher and a seven-liter pitcher. Each node is represented in the diagram by a list of numbers indicating the current contents of the three-liter pitcher and the seven-liter pitcher; for example, the list `[3 4]` means that the three-liter pitcher is full and the seven-liter pitcher contains four liters.

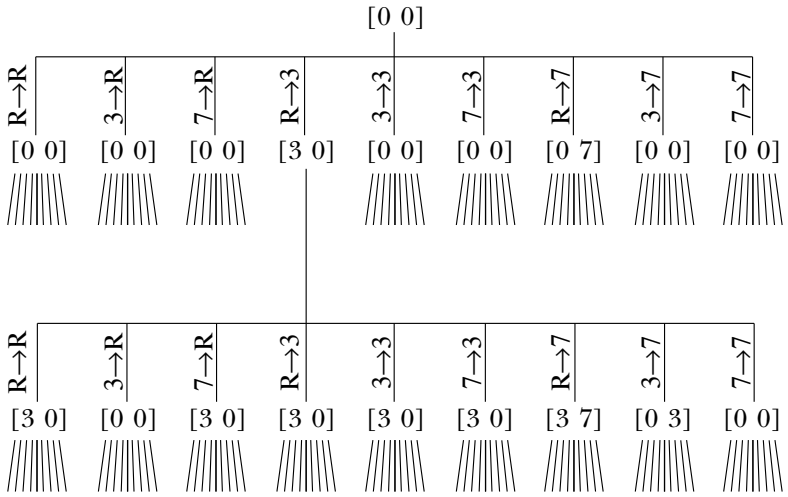
* You can find a computational algorithm to solve (or show that there are no solutions to) any linear Diophantine equation with two variables on page 50 of Courant and Robbins, *What Is Mathematics?* (Oxford University Press, 1941).



Actually, I have simplified this tree by showing only the meaningful pouring steps. The program must consider, and of course reject, things like the sequence

1. Fill the three-liter pitcher from the river.
2. Empty the three-liter pitcher into the river.

and individual meaningless steps like pouring from a pitcher into itself, pouring from an empty pitcher, and pouring into a full pitcher. For a two-pitcher problem there are three possible sources of water (the two pitchers and the river) and three possible destinations, for a total of nine possible pouring steps. Here is the top of the full tree:



At each level of the tree, the number of nodes is multiplied by nine. If we're trying to measure two liters of water, a six-step problem, the level of the tree at which the solution is found will have 531,441 nodes! You can see that efficiency will be an important consideration in this program.

In some projects, a tree is represented within the program by a Logo list. That's not going to be the case in this project. The tree is not explicitly represented in the program at all, although the program will maintain a list of the particular nodes of the tree that are under consideration at a given moment. The entire tree can't be represented as a list because it's infinitely deep! In this project, the tree diagram is just something that should be in your mind as a model of what the program is doing: it's *searching* through the tree, looking for a node that includes the goal quantity as one of its numbers.

Depth-first and Breadth-first Searching

Many programming problems can be represented as searches through trees. For example, a chess-playing program has to search through a tree of moves. The root of the tree is the initial board position; the second level of the tree contains the possible first moves by white; the third level contains the possible responses by black to each possible move by white; and so on.

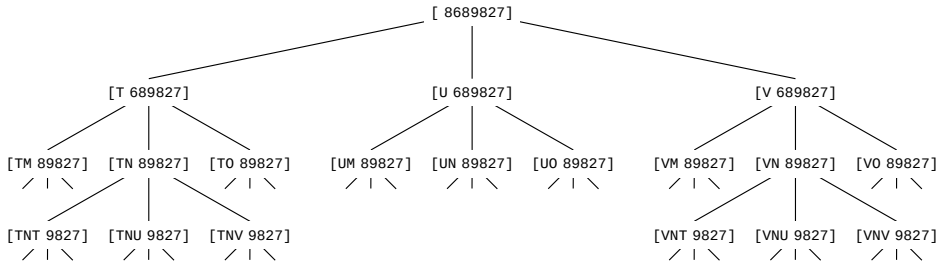
There are two general techniques for searching a tree. These techniques are called *depth-first search* and *breadth-first search*. In the first technique, the program explores all of the "descendents" of a given node before looking at the "siblings" of that node. In the chess example, a depth-first search would mean that the program would explore all the possible outcomes (continuing to the end of the game) of a particular opening move, then go on to do the same for another opening move. In breadth-first search, the program examines all the nodes at a given level of the tree, then goes on to generate and examine the nodes at the next level. Which technique is more appropriate will depend on the nature of the problem.

In a programming language like Logo, with recursive procedures and local variables, it turns out that depth-first search leads to a simpler program structure. Suppose that we are given an operation called `children` that takes a node as input and gives us as its output a list of all the children (one level down) of that node. Suppose we also are given a command called `process` that takes a node as input and does whatever the program needs to do for each node of the tree. (You can just use `print` in place of `process` if you want to see what's in the tree.) Here is how to do a depth-first search:

```
to depth.first :node
process :node
foreach (children :node) "depth.first
end
```

In this program, the structure of the tree is reflected in the structure of recursive invocations of `depth.first`.

It might be worthwhile to consider a specific example of how this program works. One of the suggested activities in Chapter 11 was to write a program that takes a telephone number as input and prints out all possible spellings of that number as letters. (Each digit can represent any of three letters. To keep things simple, I'm going to ignore the problem of the digits zero and one, which don't represent any letters on telephone dials in the United States.) Here is a partial picture of the tree for a particular telephone number. Each node contains some letters and some digits. (In the program, a node will be represented as a Logo list with two members, a word of letters and a word of digits.) The root node is all digits; the "leaf" nodes will be all letters.



The operation `children` must output a list of three nodes, selecting each of the three possible letters for the first remaining digit. If the input to `children` is a leaf node (one with all letters), it must output the empty list to indicate that there are no children for that node.

```
to children :node
if emptyp last :node [output []]
output map [child (first :node) ? (butfirst last :node)] ~
           letters first last :node
end

to letters :digit
output item :digit [[] abc def ghi jkl mno prs tuv wxy]
end

to child :letters :this :digits
output list (word :letters :this) :digits
end

?show children [tnt 9827]
[[tntw 827] [tntx 827] [tnty 827]]
```

The top-level procedure has to turn a number into a root node and invoke a depth-first search:

```
to spell :number
depth.first list " :number
end
```

What about the `process` command? The program wants to print only leaf nodes:

```
to process :node
if emptyp last :node [print :node]
end
```

☞ Try this program. To get the tree illustrated above, use the instruction

```
spell 8689827
```

Then try again, but investigate the order in which the program searches the nodes of the tree by using a different version of `process`:

```
to process :node
print :node
end
```

This will let you see the order in which the program encounters the nodes of the tree.

Writing a breadth-first search is a little more complicated because the program must explicitly arrange to process all the nodes of a given level before processing those at the next level. It keeps track of the nodes waiting to be processed in a *queue,* a list in which new nodes are added at the right and the next node to be processed is taken from the left. Here is the program:

```
to breadth.first :root
breadth.descend (list :root)
end

to breadth.descend :queue
if emptyp :queue [stop]
process first :queue
breadth.descend sentence (butfirst :queue) ~
                         (children first :queue)
end
```

This breadth-first search program uses the same `children` and `process` subprocedures as the depth-first version. You can try a breadth-first listing of telephone number spellings simply by changing the top-level `spell` procedure to invoke `breadth.first` instead of `depth.first`. What you'll find is that (with the version of `process` that only prints leaf nodes) the two versions produce the same results, but the depth-first program trickles the spellings out one by one, while the breadth-first version prints nothing for a long time and then spits out all the spellings at once. If you use the version of `process` that prints all the nodes, you can see why.

The telephone number speller is an unusual example of a tree-search program for two reasons. First, the tree is finite; we know in advance that it extends seven levels below the root node, because a telephone number has seven digits. Second, the goal of the program requires searching the *entire* tree. It's more common that the program is looking for a solution that's "good enough" in some sense, and when a solution is found, the program stops looking. For example, in the pitcher problem program, once we find a sequence of steps to measure the desired amount of water, we don't care if there is also a second way to do it.

For the pitcher problem solver, I decided that a breadth-first search is appropriate. The main reason is that I wanted to present the *shortest* possible solution. To do that, first I see if any one-step sequences solve the problem, then I see if any two-step sequences solve it, and so on. This is a breadth-first order.

Data Representation

At first, I thought that I would represent each node of the tree as a list of numbers representing the contents of the pitchers, as in the diagram I showed earlier. I called this list of quantities a *state*. This information is enough to be able to generate the children of a node. Later, though, I realized that when I find a winning solution (one that has the goal quantity as one of the quantities in the state list) I want to be able to print not only the final quantities but also the sequence of pouring steps used to get there. In a depth-first search, this information is implicitly contained in the local variables of the procedure invocations leading to the winning solution. In a breadth-first search, however, the program doesn't keep track of the sequence of events leading to a given node. I had to remember this information explicitly.

The solution I chose was to have an extra member in the list representing a state, namely a list of *pourings*. A pouring is a list of two numbers representing the source and the destination of the water being poured. Zero represents the river; numbers greater

than zero are pitcher numbers. (A pitcher number is not the same as the size of the pitcher. If you enter the instruction

```
pour [2 5 10] 1
```

then the two-liter pitcher is pitcher number 1, the five-liter is number 2, and the ten-liter is number 3.) The list of pourings is the first member of the expanded state list; pourings are added to that list at the front, with `fput`. For example, in the interaction

```
?pour [3 7] 4
Pour from river to 7
Pour from 7 to 3
Final quantities are 3 4
```

the extended state information for the final solution state is

```
[[[2 1] [0 2]] 3 4]
```

In this list, the sublist `[0 2]` represents pouring water from the river into pitcher number 2, which is the seven-liter pitcher. The sublist `[2 1]` represents pouring water from pitcher number 2 into pitcher number 1.

Abstract Data Types

Up to this point I've continued to call this expanded data structure a *state*. That's what I did in the program, also, until I found that certain procedures needed the new version, while other procedures dealt with what I had originally considered a state, with only the final quantities included in the list. As a result, my program had local variables named `state` in several procedures, some of which contained the old kind of state, and some the new kind. I thought this might be confusing, so I did what I should have done in the first place: I invented a new name for the expanded data structure. It's now called a *path*; when you read the program you can confidently assume that `:state` represents a list like

```
[3 4]
```

while `:path` represents a list like

```
[[[2 1] [0 2]] 3 4]
```

The trouble with using a list of lists of lists in a program is that it can become very complicated to keep track of all the uses of selectors like `first` and constructors like `fput`. For example, suppose the value of the variable `oldpath` is a path, and we decide to pour water from pitcher number `:from` to pitcher number `:to`. We now want to construct a new path, which will include a new state (computed from the old state and the two pitcher numbers) and a new list of moves, with the new move added to the existing ones. We'd end up saying

```
make "newpath fput (fput (list :from :to) first :oldpath) ~
                   (newstate butfirst :oldpath :from :to)
```

assuming that we have a procedure `newstate` that computes the new state. This instruction is hard to read! The two invocations of `fput` have quite different purposes. One adds a new move to a list of moves, while the other connects a list of moves to a state in order to form a path. We can clarify instructions like this one if we make up synonyms for procedures like `first` and `fput` to be used in particular contexts. For example, we make a new path using `fput`, but we'll call it `make.path` when we're using it for that purpose. Just as `fput` is a constructor, and `first` a selector, for lists, we can invent constructors and selectors for *abstract* data types (ones that we make up, rather than ones built into Logo) such as paths:

```
to make.path :moves :state
output fput :moves :state
end

to path.moves :path
output first :path
end

to path.state :path
output butfirst :path
end
```

That unreadable instruction shown earlier would now be written this way:

```
make "newpath make.path (fput (list :from :to) path.moves :oldpath) ~
                        (newstate (path.state :oldpath) :from :to)
```

At first glance this may not seem like much of an improvement, since the new names are longer and less familiar than the old ones. But we can now read the instruction and see that it calls a constructor `make.path` with two inputs, one that seems to have to do with

moves, and the other that seems to have to do with states. If we remember that a path has two parts, a list of moves and a state, this makes sense.

☞ Invent a constructor and selectors for a *move* data type.

`Sentence` as a Combiner

The general breadth-first search program I showed earlier contains this procedure:

```
to breadth.descend :queue
if emptyp :queue [stop]
process first :queue
breadth.descend sentence (butfirst :queue) (children first :queue)
end
```

The most common use of `sentence` is in generating English sentences. In that use, the input and output lists are *sentences* or flat lists. You're supposed to think, "`Sentence` takes two words or sentences as inputs; its output is a sentence containing all the words of the inputs." In this program, we're using `sentence` in a different way, more like what is called `append` in Lisp. Here you're supposed to think, "`Sentence` takes two lists as inputs; its output is a list containing the members of the inputs." Those members could be words or lists, but in this case they'll be lists, namely paths.

Recursive procedures that manipulate non-flat lists generally use `fput` as the combiner. That wouldn't work here for two reasons. First, the queue structure that we need to implement breadth-first search requires that we add new entries at the opposite end of the list from where we look for the next node to process. If we use `first` to select a node and `fput` to add new candidate nodes, then instead of a queue we'd be using a *stack,* in which the newest entries are processed first instead of the oldest ones first. That would give us a depth-first tree search algorithm. We could solve that problem by using `lput` as the combiner, but the second reason for choosing `sentence` is that we don't generate new entries one at a time. Instead, `children` gives us several children to add to the queue at once. That means we must append the list output by `children` to the list that represents the nodes already queued.

Finding the Children of a Node

`Pour` is going to work essentially by invoking `breadth.first` on a root node containing zeros for all the current quantities. But in this case we want to pick a single node that

satisfies the conditions of the problem, so we must modify `breadth.first` to make it an *operation* that outputs the first such node:

```
to breadth.first :root
output breadth.descend (list :root)
end

to breadth.descend :queue
if emptyp :queue [output []]
if winnerp first :queue [output first :queue]
output breadth.descend sentence (butfirst :queue) ~
                                (children first :queue)
end
```

The predicate `winnerp` will output `true` if its input is a node that satisfies the problem conditions:

```
to winnerp :path
output memberp :goal path.state :path
end
```

If `breadth.first` runs out of nodes without finding a solution, it returns an empty list to indicate failure.

Here is a simplified version of `pour`:

```
to pour :sizes :goal
win breadth.first make.path [] all.empty :sizes
end

to all.empty :list
output map [0] :list
end
```

`All.empty` is an operation that outputs a state in which all of the values are zeros. The number of zeros in the list is equal to the number of members in its input, which is the number of pitchers. `Pour` combines this initial state with an empty list of moves to produce the first path.

To allow `breadth.first` to work, we must have an operation called `children` that outputs a list of the children of a node. Starting from a particular state, what are the possible outcomes of a single pouring? As I mentioned earlier, the source of a pouring can be the river or any of the pitchers, and the destination can also be the river or any

of the pitchers. If there are n pitchers, then there are $n+1$ sources, $n+1$ destinations, and therefore $(n+1)^2$ possible pourings. Here is how the program structure reflects this. I'm assuming that we've created (elsewhere in the program) a variable called `pitchers` whose value is a list of all the integers from zero to n.

```
to children :path
output map.se [children1 :path ?] :pitchers
end

to children1 :path :from
output map.se [child :path :from ?] :pitchers
end

to child :path :from :to
output (list make.path (fput (list :from :to) path.moves :path)
                       (newstate (path.state :path) :from :to))
end
```

The version of `child` presented here is simpler than the one in the actual project, but the other procedures are the real versions. We'll see later how `child` is expanded. The immediately important point is to see how `children` and `children1` ensure that every possible source (`:from`) and destination (`:to`) from zero to the number of pitchers are used.

You should be wondering, at this point, why `children1` uses `sentence` as a combiner. (That's what it means to use `map.se` rather than `map`.) It makes sense for `children` to combine using `sentence` because, as I discussed earlier, the things it's combining are lists of nodes, the outputs from invocations of `children1`. But `children1` is not combining lists of nodes; it's combining the outputs from invocations of `child`. Each invocation of `child` computes a single child node. It would be more straightforward to write the program this way:

```
to children1 :path :from                          ;; simplified
output map [child :path :from ?] :pitchers
end

to child :path :from :to                          ;; simplified
output make.path (fput (list :from :to) path.moves :path) ~
                 (newstate (path.state :path) :from :to)
end
```

This also eliminates the use of `list` in `child`, needed in the other version to turn a single node into a singleton (one-member) list of nodes, which is what `sentence` needs to function properly as a combiner.

The reason for the use of `sentence` in `children1` is that we are later going to modify `child` so that sometimes it rejects a possible new node for efficiency reasons. For example, it makes no sense to have nodes for pourings in which the source and the destination are the same. When it wants to reject a node, `child` will output the empty list. Using `sentence` as the combiner, this empty list simply doesn't affect the accumulated list of new nodes. Here is a version of `child` modified to exclude pourings to and from the same place:

```
to child :path :from :to
if equalp :from :to [output []]
output (list make.path (fput (list :from :to) path.moves :path)
                       (newstate (path.state :path) :from :to))
end
```

With this version of `child`, the use of `sentence` in `children1` may seem more sensible to you.

To create the variable `pitchers` we modify the top-level `pour`:

```
to pour :sizes :goal
local "pitchers
make "pitchers fput 0 (map [#] :sizes)
win breadth.first make.path [] all.empty :sizes
end
```

Here we are taking advantage of a feature of `map` that I haven't mentioned earlier. The number sign (`#`) can be used in a `map` template to represent the position in the input, rather than the value, of a member of the input data list. That is, `#` is replaced by 1 for the first member, 2 for the second, and so on. In this example, these position numbers are all we care about; the template does not contain the usual question mark to refer to the values of the data.

Computing a New State

The job of `child` is to produce a new child node, that is to say, a new path. Its inputs are an old path and the source and destination of a new pouring. The new path consists of

a new state and a new list of pourings. The latter is easy; it's just the old list of pourings with the new one inserted. `Child` computes that part itself, with the expression

```
fput (list :from :to) path.moves :path
```

The new state is harder to compute. There are four cases.

1. If the destination is the river, then the thing to do is to empty the source pitcher.
2. If the source is the river, then the thing to do is to fill the destination pitcher to its capacity.
3. If source and destination are pitchers and the destination pitcher has enough empty space to hold the contents of the source pitcher, then the thing to do is to add the entire contents of the source pitcher to the destination pitcher, setting the contents of the source pitcher to zero.
4. If both are pitchers but there is not enough room in the destination to hold the contents of the source, then the thing to do is fill the destination to its capacity and subtract that much water from the source.

Here is the procedure to carry out these computations:

```
to newstate :state :from :to
if riverp :to [output replace :state :from 0]
if riverp :from [output replace :state :to (size :to)]
if (water :from) < (room :to) ~
   [output replace2 :state ~
                    :from 0 ~
                    :to ((water :from)+(water :to))]
output replace2 :state ~
                :from ((water :from)-(room :to)) ~
                :to (size :to)
end
```

Each instruction of this procedure straightforwardly embodies one of the four numbered possibilities.

Helper procedures are used to compute a new list of amounts of water, replacing either one or two old values from the previous list:

```
to replace :list :index :value
if equalp :index 1 [output fput :value butfirst :list]
output fput first :list (replace butfirst :list :index-1 :value)
end
```

```
to replace2 :list :index1 :value1 :index2 :value2
if equalp :index1 1 ~
   [output fput :value1 replace butfirst :list :index2-1 :value2]
if equalp :index2 1 ~
   [output fput :value2 replace butfirst :list :index1-1 :value1]
output fput first :list ~
            replace2 butfirst :list :index1-1 :value1 :index2-1 :value2
end
```

`Replace` takes as inputs a list, a number representing a position in the list, and a value. The output is a copy of the first input, but with the member selected by the second input replaced with the third input. Here's an example:

```
?show replace [a b c d e] 4 "x
[a b c x e]
```

`Replace2` has a similar purpose, but its output has *two* members changed from their values in the input list.

Remember that `newstate` has as one of its inputs a state, that is, a list of numbers representing quantities of water. `Newstate` uses `replace` to change the amount of water in one of the pitchers. The second input to `replace` is the pitcher number, and the third is the new contents of that pitcher. For example, if the destination is the river then we want to empty the source pitcher. This case is handled by the instruction

```
if riverp :to [output replace :state :from 0]
```

If the destination is the river, the output state is the same as the input state except that the pitcher whose number is `:from` has its contents replaced by zero. The other cases are handled similarly, except that two replacements are necessary if both source and destination are pitchers.

More Data Abstraction

The instructions in `newstate` use some procedures I haven't written yet, such as `riverp` to test whether a source or destination is the river, and `room` to find the amount of empty space in a pitcher. If we think of a pitcher as an abstract data type, then these can be considered selectors for that type. Here they are:

```
to riverp :pitcher
output equalp :pitcher 0
end

to size :pitcher
output item :pitcher :sizes
end

to water :pitcher
output item :pitcher :state
end

to room :pitcher
output (size :pitcher)-(water :pitcher)
end
```

To underscore the importance of data abstraction, here is what `newstate` would look like without these selectors. (I actually wrote it this way at first, but I think you'll agree that it's unreadable.)

```
to newstate :state :from :to
if equalp :to 0 [output replace :state :from 0]
if equalp :from 0 [output replace :state :to (item :to :sizes)]
if ((item :from :state) < ((item :to :sizes)-(item :to :state))) ~
   [output replace2 :state ~
                    :from 0 ~
                    :to ((item :from :state)+(item :to :state))]
output replace2 :state ~
                :from ((item :from :state)-
                       ((item :to :sizes)-(item :to :state))) ~
                :to (item :to :sizes)
end
```

Printing the Results

When `breadth.first` finds a winning path, the top-level procedure `pour` invokes `win` with that path as its input. `Win`'s job is to print the results. Since the list of moves is kept in reverse order, `win` uses the Logo primitive operation `reverse` to ensure that the moves are shown in chronological order.

```
to win :path
if emptyp :path [print [Can't do it!] stop]
foreach (reverse path.moves :path) "win1
print sentence [Final quantities are] (path.state :path)
end

to win1 :move
print (sentence [Pour from] (printform first :move)
                [to] (printform last :move))
end

to printform :pitcher
if riverp :pitcher [output "river]
output size :pitcher
end
```

Efficiency: What Really Matters?

The `pour` program as described so far would run extremely slowly. The rest of the commentary in this chapter will be on ways to improve its efficiency. The fundamental problem is one I mentioned earlier: the number of nodes in the tree grows enormously as the depth increases. In a problem with two pitchers, the root level has one node, the next level nine nodes, the third level 81, the fourth level 729, the fifth level 6561, and the sixth level 59049. A six-step problem like

```
pour [3 7] 2
```

would strain the memory capacity of many computers as well as taking forever to run!

When you're trying to make a program more efficient, the easiest improvements to figure out are not usually the ones that really help. The easy things to see are details about the computation within some procedure. For example, the `newstate` procedure described earlier calls the `room` procedure twice to compute the amount of room available in the destination pitcher. Each call to `room` computes the quantity

```
(item :to :sizes)-(item :to :state)
```

This expression represents the amount of empty space in the destination pitcher. Perhaps it would be faster to compute this number only once, and store it in a variable? I haven't bothered trying to decide, because the effect is likely to be small either way. Improving the speed of computing each new node is much less important than cutting down the

number of nodes we compute. The reason is that eliminating one node also eliminates all its descendants, so that the effect grows as the program moves to lower levels of the tree.

The best efficiency improvement is likely to be a complete rethinking of the algorithm. For example, I've mentioned that a numerical algorithm exists for solving two-variable linear Diophantine equations. This algorithm would be a *much* faster way to solve two-pitcher problems than even the best tree search program. I haven't used that method because I wanted a simple program that would work for any number of pitchers, but if I really had to solve such problems in practice, I'd use the Diophantine equation method wherever possible.

Avoiding Meaningless Pourings

We have already modified `child` to avoid one kind of meaningless pouring, namely ones in which the source is the same as the destination. Two other avoidable kinds of meaningless pourings are ones from an empty source and ones to a full destination. In either case, the quantity of water poured will be zero, so the state will not change. Here is a modified version of `child` that avoids these cases:

```
to child :path :from :to
local "state
if equalp :from :to [output []]
make "state path.state :path
if not riverp :from ~
   [if equalp (water :from) 0 [output []]]
if not riverp :to ~
   [if equalp (water :to) (size :to) [output []]]
output (list make.path (fput list :from :to path.moves :path)
                       (newstate :state :from :to))
end
```

The local variable `state` is set up because the procedure `water` needs it. (`Water` relies on Logo's dynamic scope to give it access to the `state` variable provided by its caller.)

The important changes are the two new `if` instructions. The first avoids pouring from an empty pitcher; the second avoids pouring into a full one. In both cases, the test makes sense only for actual pitchers; the river does not have a size or a current contents.

To underscore what I said earlier about what's important in trying to improve the efficiency of a program, notice that these added tests *slow down* the process of computing each new node, and yet the overall effect is beneficial because the number of nodes is dramatically reduced.

Eliminating Duplicate States

It's relatively easy to find individual pourings that are absurd. A harder problem is to avoid *sequences* of pourings, each reasonable in itself, that add up to a state we've already seen. The most blatant examples are like the one I mentioned a while back about filling a pitcher from the river and then immediately emptying it into the river again. But there are less blatant cases that are also worth finding. For example, suppose the problem includes a three-liter pitcher and a six-liter pitcher. The sequence

```
Pour from river to 6
Pour from 6 to 3
```

leads to the same state (`[3 3]`) as the sequence

```
Pour from river to 3
Pour from 3 to 6
Pour from river to 3
```

The latter isn't an absurd sequence of pourings, but it's silly to pursue any of its children because they will have the same states as the children of the first sequence, which is one step shorter. Any solution that could be found among the descendents of the second sequence will be found one cycle earlier among the descendents of the first.

To avoid pursuing these duplicate states, the program keeps a list of all the states found so far. This strategy requires changes to `pour` and to `child`.

```
to pour :sizes :goal
local [oldstates pitchers]
make "oldstates (list all.empty :sizes)
make "pitchers fput 0 (map [#] :sizes)
win breadth.first make.path [] all.empty :sizes
end
```

```
to child :path :from :to
local [state newstate]
if equalp :from :to [output []]
make "state path.state :path
if not riverp :from ~
   [if equalp (water :from) 0 [output []]]
if not riverp :to ~
   [if equalp (water :to) (size :to) [output []]]
make "newstate (newstate :state :from :to)
if memberp :newstate :oldstates [output []]
make "oldstates fput :newstate :oldstates
output (list make.path (fput list :from :to path.moves :path) :newstate)
end
```

The change in `pour` is simply to initialize the list of already-seen states to include the state in which all pitchers are empty. There are two important new instructions in `child`. The first rejects a new node if its state is already in the list; the second adds a new state to the list. Notice that it is duplicate *states* we look for, not duplicate *paths*; it's in the nature of a tree-search program that there can never be duplicate paths.

Stopping the Program Early

The breadth-first search mechanism we're using detects a winning path as it's *removed* from the front of the queue. If we could detect the winner as we're about to *add* it to the queue, we could avoid the need to compute all of the queue entries that come after it: children of nodes that are at the same level as the winning node, but to its left.

It's not easy to do this elegantly, though, because we add new nodes to the queue several at a time, using the procedure `children` to compute them. What we need is a way to let `child`, which constructs the winning node, prevent the computation of any more children, and notify `breadth.first` that a winner has been found.

The most elegant way to do this in Berkeley Logo uses a primitive called `throw` that we won't meet until the second volume of this series. Instead, in this chapter I'll use a less elegant technique, but one that works in any Logo implementation. I'll create a variable named `won` whose value is initially `false` but becomes `true` as soon as a winner is found. Here are the necessary modifications:

```
to pour :sizes :goal
local [oldstates pitchers won]
make "oldstates (list all.empty :sizes)
make "pitchers fput 0 (map [#] :sizes)
make "won "false
win breadth.first make.path [] all.empty :sizes
end

to breadth.descend :queue
if emptyp :queue [output []]
if :won [output last :queue]
op breadth.descend sentence (butfirst :queue) ~
                            (children first :queue)
end

to child :path :from :to
local [state newstate]
if :won [output []]
if equalp :from :to [output []]
make "state path.state :path
if not riverp :from ~
   [if equalp (water :from) 0 [output []]]
if not riverp :to
   [if equalp (water :to) (size :to) [output []]]
make "newstate (newstate :state :from :to)
if memberp :newstate :oldstates [output []]
make "oldstates fput :newstate :oldstates
if memberp :goal :newstate [make "won "true]
output (list make.path (fput list :from :to path.moves :path) :newstate)
end
```

The procedure `winnerp` is no longer used; we are now checking a state, rather than a path, for the goal amount.

Further Explorations

☞ Is it possible to eliminate more pieces of the tree by more sophisticated analysis of the problem? For example, in all of the specific problems I've presented, the best solution never includes pouring from pitcher A to pitcher B and then later pouring from B to A. Is this true in general? If so, many possible pourings could be rejected with an instruction like

```
if memberp list :to :from path.moves :path [output []]
```

in `child`.

☞ Do some research into Diophantine equations and the techniques used to solve them computationally. See if you can devise a general method for solving pitcher problems with any number of pitchers, based on Diophantine equations.

☞ Think about writing a program that would mimic the way people actually approach these problems. The program would, for example, compute the differences and remainders of pairs of pitcher sizes, looking for the goal quantity.

☞ What other types of puzzles can be considered as tree searching problems?

Program Listing

```
;; Initialization

to pour :sizes :goal
local [oldstates pitchers won]
make "oldstates (list all.empty :sizes)
make "pitchers fput 0 (map [#] :sizes)
make "won "false
win breadth.first make.path [] all.empty :sizes
end

to all.empty :list
output map [0] :list
end

;; Tree search

to breadth.first :root
op breadth.descend (list :root)
end

to breadth.descend :queue
if emptyp :queue [output []]
if :won [output last :queue]
op breadth.descend sentence (butfirst :queue) ~
                            (children first :queue)
end
```

```
;; Generate children

to children :path
output map.se [children1 :path ?] :pitchers
end

to children1 :path :from
output map.se [child :path :from ?] :pitchers
end

to child :path :from :to
local [state newstate]
if :won [output []]
if equalp :from :to [output []]
make "state path.state :path
if not riverp :from ~
   [if equalp (water :from) 0 [output []]]
if not riverp :to ~
   [if equalp (water :to) (size :to) [output []]]
make "newstate (newstate :state :from :to)
if memberp :newstate :oldstates [output []]
make "oldstates fput :newstate :oldstates
if memberp :goal :newstate [make "won "true]
output (list make.path (fput list :from :to path.moves :path) :newstate)
end

to newstate :state :from :to
if riverp :to [output replace :state :from 0]
if riverp :from [output replace :state :to (size :to)]
if (water :from) < (room :to) ~
   [output replace2 :state ~
                    :from 0 ~
                    :to ((water :from)+(water :to))]
output replace2 :state ~
                :from ((water :from)-(room :to)) ~
                :to (size :to)
end

;; Printing the result

to win :path
if emptyp :path [print [Can't do it!] stop]
foreach (reverse path.moves :path) "win1
print sentence [Final quantities are] (path.state :path)
end
```

```
to win1 :move
print (sentence [Pour from] (printform first :move)
                [to] (printform last :move))
end

to printform :pitcher
if riverp :pitcher [output "river]
output size :pitcher
end

;; Path data abstraction

to make.path :moves :state
output fput :moves :state
end

to path.moves :path
output first :path
end

to path.state :path
output butfirst :path
end

;; Pitcher data abstraction

to riverp :pitcher
output equalp :pitcher 0
end

to size :pitcher
output item :pitcher :sizes
end

to water :pitcher
output item :pitcher :state
end

to room :pitcher
output (size :pitcher)-(water :pitcher)
end
```

```
;; List processing utilities

to replace :list :index :value
if equalp :index 1 [output fput :value butfirst :list]
output fput first :list (replace butfirst :list :index-1 :value)
end

to replace2 :list :index1 :value1 :index2 :value2
if equalp :index1 1 ~
   [output fput :value1 replace butfirst :list :index2-1 :value2]
if equalp :index2 1 ~
   [output fput :value2 replace butfirst :list :index1-1 :value1]
output fput first :list ~
            replace2 butfirst :list :index1-1 :value1 :index2-1 :value2
end
```